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Necessary and sufficient conditions for stability on finite time horizon

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Abstract

Control theory is often interested in studying stability and stabilization of dynamical systems in an infinite time horizon. However, in many practical situations, focusing on the system behavior in a finite time interval is more important than requiring the system to reach a given equilibrium.

Analysis on finite time horizon can be useful, for example, to study state and output transients due to disturbances or to examine the effects of sudden changes of the state variable due to external or internal system perturbations. Moreover, some systems naturally evolve in a finite time interval, as for example earthquakes or animal and vegetal systems.

Furthermore, only qualitative behavior of dynamical systems are usually taken into account, as in Lyapunov Asymptotic Stability and classic Input-Output $\mathcal{L}_p$-Stability. In many applications though, it is necessary to provide specific bounds to the system state and/or output variables. When dealing with linear models obtained linearizing nonlinear systems around an equilibrium, for example, it is important to keep the state trajectory close to the equilibrium point to avoid the effects of nonlinearity; the presence of variables physical constraints, like actuators saturations, is another example.

Two different concepts of stability over a finite time horizon are dealt with in this thesis, namely Finite-Time Stability (FTS) and Input-Output Finite-Time Stability (IO-FTS). Their aim is to provide quantitative bounds, during a finite time interval, on the state and output trajectories, respectively.

These quantitative bounds can be very useful for hybrid systems, e.g. Switching Linear Systems (SLSs), which present jumps in the state space in particular instants of time called resetting times. Hybrid systems exhibit both continuous-time and discrete-time dynamics and are used to model systems such as a thermostat turning the heat on and off, a server switching between buffers in a queueing networks, the gear shift control in a car.

This thesis extends the FTS results already present in the previous literature, i.e. necessary and sufficient condition to check finite-time stability, to a larger class of hybrid systems, namely SLS. It also copes with the very useful case of uncertainty on the resetting times.

In the context of IO-FTS, this thesis provides necessary and sufficient conditions to check the stability of Linear Time Varying (LTV) systems and sufficient conditions to check the stability of SLSs, also in the case of resetting times not a priori known.

For both FTS and IO-FTS, different stabilization problems are solved for SLSs and LTV systems.

The new concept of Structured IO-FTS is introduced, which makes possible to limit the effort on the actuators by introducing quantitative bounds on the control inputs.

Some applications are considered to demonstrate the effectiveness of the developed control techniques.

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together, she had to change a lot her life to be with me and she never let me feel responsible for that, I was lucky to find her and I will never let her go.

Contents

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Acronyms

AS	Arbitrary Switching
BMI	Bilinear Matrix Inequality
DLE	Differential Lyapunov Equation
D/DLE	Difference/Differential Lyapunov Equation
D/DLMI	Difference/Differential Linear Matrix Inequality
FTS	Finite-Time Stability
IDLS	Impulsive Dynamical Linear System
IO-FTS	Input-Output Finite-Time Stability
KS	Known Switching
LAS	Lyapunov Asymptotic Stability
LMI	Linear Matrix Inequality
LTI	Linear Time Invariant
LTV	Linear Time Varying
SD	State Dependent
SLS	Switching Linear Systems
TD	Time Dependent
US	Unknown Switching

Chapter 1

Introduction

The concept of stability on a finite time horizon is a system property concerning the quantitative behavior of its state/outputs variables over an a priori defined time interval of length T. This concept finds application whenever it is desired that the state/output variables do not exceed a given threshold during the transients, examples can be found in Amato et al. [2011b, 2012b,c]; Chen and Yang [2012].

Two different definitions of stability over a finite time horizon can be given, namely Finite-Time Stability (FTS), and Input-Output Finite-Time Stability (IO-FTS). They will be introduced in this section, where the main results obtained by the author will be highlighted and compared with the previous literature.

The concept of FTS dates back to the mid fifties, when it was introduced in the Russian literature (Lebedev [1954]). During the sixties and seventies, FTS appeared also in the western literature (Dorato [1961], Weiss and Infante [1967], Michel [1972]).

After almost three decades of relatively small interest on these subjects, research in this field experienced a renaissance in the last fifteen years, since some practical methods to check FTS by means of Differential Lyapunov Equations (DLEs) and Linear Matrix Inequalities (LMIs) and Differential LMIs have been proposed (Amato et al. [2001, 2006]; Garcia et al. [2009]).

It is worth noticing that FTS and Lyapunov Asymptotic Stability (LAS) are completely independent concepts, i.e. neither one implies the other, hence a system can be FTS but not LAS and vice versa. As it has recently been recalled in Michel and Hou [2008], Lyapunov stability concerns the qualitative behavior of a dynamical system and usually it does not involve quantitative information (e.g. specific estimates of trajectory bounds), whereas FTS involves specific quantitative information. Indeed, a dynamical system is said to be FTS if all the state trajectories starting within a prespecified set remain confined into a second prespecified set of admissible states, over an assigned finite time interval.

For the sake of completeness, it should be noticed that a different definition of FTS, also called attractive-FTS or finite-time attractiveness, has been defined for nonlinear systems (see for example Ryan [1979], Bhat and Bernstein [2000], Orlov [2005], Nersesov and Perruquetti [2008]). This different concept implies the system state to reach the system equilibrium in a finite time; therefore, by definition, it cannot be applied to linear systems. The same concept has been extended in Hong et al. [2008] to the case of non autonomous systems.

In Amato et al. [2001, 2006] sufficient conditions for FTS and finite-time stabilization of continuous-time linear time-invariant (LTI) systems are provided. A different approach, which is reminiscent of optimal control techniques and is also applicable to linear time-varying (LTV) systems, has been proposed in Amato et al. [2010b]. In Garcia et al. [2009] the authors solves the problem of finite time stabilization for LTV continuous systems by solving a Lyapunov differential matrix equation. In Amato et al. [2011a]; Zhao et al. [2008] the theory of Amato et al. [2010b] has been extended to the class of hybrid systems named impulsive dynamical linear systems (IDLS), defined in Haddad et al. [2006], which are linear continuous-time systems whose state undergoes finite jump discontinuities at discrete instants of time.

The main contribution introduced in this field by the author has been the extension of FTS to a larger class of hybrid systems, namely Switching Linear System (SLS), which are linear continuous-time systems with isolated discrete switching events, and whose state can undergo finite jump discontinuities (more details can be found in Liberzon [2003] and Haddad et al. [2006]). In particular, time-dependent SLS (TD-SLS) have been considered, i.e., the state jump and the change in the continuous dynamic is driven by time. The class of SLS contains the class of IDLS, since the latter admits only state switchings, while the former also allows a change of the system dynamics.

Another contribution, with respect to the previous literature, has been to consider the very important case in which the set of resetting times is unknown; more details are provided in Chapter 3. This allows to tackle real engineering situations where the change of system dynamics is unpredictable and/or is due to an external triggering event.

The concept of IO-FTS is more recent and it has been introduced in Amato et al. [2010a, 2011e] in the context of linear systems and in Amato et al. [2011b] in the context of hybrid systems. A system is said to be IO-FTS if, given a class of norm bounded input signals defined over a specified time interval of length T, the outputs of the system do not exceed an assigned threshold during such time interval.

It is important to remark that the definition of IO-FTS given in Amato et al. [2010a] is fully consistent with the definition of (state) FTS, where the state of an autonomous system is considered, rather than its inputs and outputs.

In order to correctly frame the definition of IO-FTS in the current literature, it should be noticed that the main differences between classic IO stability (IO $\mathcal{L}_p$-stable [Khalil, 1992, Ch. 5]) and IO-FTS are that the latter involves signals defined over a finite time interval, does not necessarily require the inputs and outputs to belong to the same class, and that quantitative bounds on both inputs and outputs must be specified. Therefore, IO stability and IO-FTS are independent concepts. With respect to IO stability, the IO-FTS is a more practical concept, useful to study the behavior of the system within a finite (possibly short) interval, and therefore it finds application whenever it is desired that the output variables do not exceed a given threshold during the transients, given a certain class of input signals.

In Amato et al. [2012b] LTV systems are seen as linear operators between $\mathcal{L}_2$ and $\mathcal{L}_\infty$ spaces, therefore IO-FTS is interpreted as the $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain, on a finite time interval, from the exogenous input to the output. The same approach can be used in the case of inputs of class $\mathcal{L}_\infty$. More details can be found in Chapter 4.

It should be recalled that, in the special case of a scalar output, H_2 control can also be interpreted as the minimization of the $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain (for the different interpretations of H_2 control the interested readers can refer to Paganini and Feron [2000]). However, in the general case, IO-FTS and H_2 control are completely different concepts. Indeed, H_2 control is based on the minimization of a system norm which is not induced by inputs and outputs signal norms.

IO-FTS is not related to H_∞ control either, which is based on the minimization of the root-mean-square gain, or $\mathcal{L}_2$ to $\mathcal{L}_2$ gain (see Khargonekar et al. [1991]).

In the framework of linear systems, in Amato et al. [2010a] two sufficient conditions for IO-FTS of linear systems have been originally provided. In particular, the two classes of inputs $\mathcal{L}_2$ and $\mathcal{L}_\infty$ were considered. Both conditions required the solution of a feasibility problem involving differential linear matrix inequalities (DLMIs).

The IO finite-time stabilization problem has been discussed in Amato et al. [2010a] (state feedback) and Amato et al. [2011f, 2012b] (dynamic output feedback).

One of the main contributions introduced by the author in the field of IO-FTS has been to prove that, in the case of $\mathcal{L}_2$ inputs, the condition given in Amato et al. [2010a] in the form of DLMI is also necessary for IO-FTS. More details are given in Chapter 4. The result has been proven with the help of a machinery based on the Grammian controllability matrix. Moreover, an alternative necessary and sufficient condition requiring that a certain DLE admits a positive definite solution has been found. The solution of the DLE is much more efficient from a computational point of view. Nevertheless, the DLMIs formulation turns out to be useful for controller design purposes.

Another important contribution introduced by the author has been the solution of the state feedback IO-FTS problem with constrained control input,which allows to cope with many practical situations in which the effort on the control variables play an important role. To achieve this goal, a slightly different concept of IO-FTS, namely *Structured* IO-FTS, has been introduced; a fictitious system is built, in which the output vector is augmented with the control input variables, which are conceptually treated in the same way as the actual outputs. The advantages of this new definition are twofold: the output vector can be partitioned and different constraints can be imposed on each group of partitioned outputs; explicit constraints can be enforced on the control inputs. This result is presented in details in Chapter 5.

Contributions have been given also in the context of hybrid systems; Chapter 6 provides sufficient conditions guaranteeing IO-FTS for both Time-Dependent (TD) and State-Dependent (SD) IDLS, for two different input classes. Furthermore the problem of IO finite-time stabilization via static output feedback has been tackled. While the analysis problem has been framed into the LMI framework (see Boyd et al. [1994]), the static output feedback problem involves solving a Bilinear Matrix Inequalities (BMIs, VanAntwerp and Braatz [2000]) feasibility problem. Note that, although the latter problem is well known to be NP-hard, it can be effectively solved by means of off-the-shelf optimization tools (e.g., Henrion et al. [2005]).

Eventually, TD-SLS has been dealt within the practical case of uncertainty on the resetting times for the two classes of inputs $\mathcal{L}_2$ and $\mathcal{L}_\infty$.

The present thesis is organized as follow. Chapter 2 defines the concepts of FTS and IO-FTS and the main differences with other classical control definitions, introduces the main systems considered, namely LTV systems and Time-Dependent Switching Linear Systems (TD-SLS) and the control design problems dealt with in this thesis.

Chapter 3 addresses the FTS problem for the class of TD-SLS. In particular, the assumption that the sequence of resetting times is a priori known is removed. This allows the FTS to be applied to real engineering situations where the change of system dynamics is not predictable. The additional cases in which the resetting times are known with a given uncertainty or completely unknown are analyzed. It is shown that the reduction of the uncertainty intervals reduces the conservatism of the conditions to check FTS.

Chapter 4 provides a necessary and sufficient condition for IO-FTS, for the $\mathcal{L}_2$ class of inputs. It is firstly shown that the condition given in Amato et al. [2010a], which involves a DLMIs feasibility problem, is actually also *necessary*. Afterwards, an alternative necessary and sufficient condition is presented, which requires that a certain DLE admits a positive definite solution. While the condition based on the DLE is more effective from a numerical point of view, the DLMIs formulation turns out to be useful for design purposes. Eventually, the analysis condition based on DLMIs is used to design an output feedback dynamic controller.

Chapter 5 extends the results on the IO finite-time stabilization given in Chapter 4 by introducing the new definition of *Structured* IO-FTS. In particular, this chapter deals with the state feedback IO-FTS problem in the practical situations where the controller needs to be designed with the constraint of limiting the effort of the control variables. A fictitious system is built, in which the output vector is augmented with the control input variables, which are conceptually dealt with in the same way as the actual outputs. A necessary and sufficient condition ($\mathcal{L}_2$ inputs) and a sufficient condition ($\mathcal{L}_\infty$ inputs) for structured IO-FTS (open loop system) is firstly given. Then, a necessary and sufficient condition and a sufficient condition for IO finite-time stabilization with constrained control inputs are stated in the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ context, respectively.

Chapter 6 addresses the IO-FTS control problem in the case of hybrid systems. It firstly introduces the results obtained in the case of TD and SD-IDLS, which are linear continuous-time systems whose state undergoes finite jump discontinuities at discrete instants of time. Both IO-FTS analysis and design control problems are addressed. The second part of the chapter deals with TD-SLS in the very important case in which the set of resetting times is unknown.

Chapter 7 presents a number of applications and examples in order to show the effectiveness and usefulness of the main results introduced in the previous chapters.

Chapter 2

FTS and IO-FTS: definitions and problems statements

The two concepts of Finite-Time Stability (FTS) and Input-Output Finite-Time Stability (IO-FTS) are introduced in this chapter and they will be then applied to the class of Linear Time Varying (LTV) systems and Switching Linear Systems (SLS). The latter is a class of hybrid systems and is introduced in Section 2.3. Section 2.1 defines the FTS and explains the difference with the Lyapunov Asymptotic Stability (LAS). Section 2.2 introduces the IO-FTS concept, its main aim and the difference with the classical IO-Stability and H_2 control.

The definition of FTS and IO-FTS will be given for a LTV system in the form:

$$\dot{x}(t) = A(t)x(t) + G(t)w(t)\,, \quad x(t_0) = x_0 \tag{2.1a}$$

$$y(t) = C(t)x(t)\,, \tag{2.1b}$$

where $A(\cdot) : \Omega \mapsto \mathbb{R}^{n\times n}$, $G(\cdot) : \Omega \mapsto \mathbb{R}^{n\times r}$, $C(\cdot) : \Omega \mapsto \mathbb{R}^{m\times n}$, where $\Omega := [t_0\,, t_0 + T]$. It should be notice that, the same definitions can be used also in the case of SLS.

2.1 Finite time stability

FTS is a system property concerning the quantitative behaviour of its state variables over a finite time interval, which is a priori fixed.

A system is said to be finite-time stable if, given a bound on the initial condition, its state (weighted) norm does not exceed a certain threshold during the specified time interval.

The formal definition of FTS is given below.

Definition 1 (Finite-Time Stability) *Given a time interval* Ω*, a positive definite matrix* R*, and a positive definite matrix-valued function* $\Gamma(\cdot)$ *defined over* Ω*, system 2.1 is said to be FTS with respect to* $\big(R\,,\Gamma(\cdot)\,,\Omega\big)$ *if*

$$x_0^T R x_0 \leq 1 \;\Rightarrow\; x^T(t)\Gamma(t)x(t) < 1, \quad t \;\in\,]t_0\,, t_0 + T]\,,$$

when $w(t) = 0$ *for all* $t \in \Omega$*.* ▲

An interesting interpretation of Definition 1 can be provided using the concept of ellipsoidal domains. Indeed, the set defined by $x_0^T R x_0 \leq 1$ contains all the admissible initial states, and the inequality

$$x^T(t)\Gamma(t)x(t) \leq 1 \tag{2.2}$$

defines a time-varying ellipsoid that bounds the state trajectory over the time interval Ω.

It is worth noticing that FTS and LAS are completely independent concepts. Contrarily to FTS, indeed, Lyapunov stability concerns the qualitative behavior of a dynamical system and usually it does not involve quantitative information (e.g. specific estimates of trajectory bounds), as it has recently been recalled in Michel and Hou [2008].

2.2 Input-Output Finite-Time Stability

IO-FTS deals with the behavior of dynamic systems within a finite (possibly short) interval. Roughly speaking, a system is said to be IO-FTS if, given a class of norm bounded input signals defined over a specified time interval T, the outputs of the system do not exceed an assigned threshold during T.

Before providing the formal definition of IO-FTS, two different classes of input signals will be introduced.

The symbol $\mathcal{L}_p$ is used to denote the space of vector-valued signals whose p-th power is absolutely integrable over $[0,+\infty)$. The restriction of $\mathcal{L}_p$ to Ω is denoted by $\mathcal{L}_p(\Omega)$.

Given the set Ω, a symmetric positive definite matrix-valued function $W(\cdot)$, bounded on Ω, and a vector-valued signal $s(\cdot) \in \mathcal{L}_p(\Omega)$, the weighted signal norm

$$\left(\int_\Omega \left[s^T(\tau)W(\tau)s(\tau)\right]^{\frac{p}{2}} d\tau\right)^{\frac{1}{p}} ,$$

will be denoted by $\|s(\cdot)\|_{p,W}$. If $p = \infty$

$$\|s(\cdot)\|_{\infty,W} = \operatorname*{ess\,sup}_{t\in\Omega}\left[s^T(t)W(t)s(t)\right]^{\frac{1}{2}} .$$

When the weighting matrix $W(\cdot)$ is time-invariant and equal to the identity matrix I, the simplified notation $\|s(\cdot)\|_p$ will be used.

The following two classes of input signals will be dealt with:

i) the set $\mathcal{W}$ coincides with the set of norm bounded square integrable signals over Ω, defined as

$$\mathcal{W}_2\big(\Omega\,, W(\cdot)\big) := \big\{w(\cdot) \in \mathcal{L}_2(\Omega)\ :\ \|w\|_{2,W} \le 1\big\} .$$

ii) The set $\mathcal{W}$ coincides with the set of the uniformly bounded signals over Ω, defined as

$$\mathcal{W}_\infty\big(\Omega\,, W(\cdot)\big) := \big\{w(\cdot) \in \mathcal{L}_\infty(\Omega)\ :\ \|w\|_{\infty,W} \le 1\big\}.$$

In order to simplify the notation, the dependency of the class of input $\mathcal{W}$ on Ω and the weighting matrix $W(\cdot)$ will be dropped whenever this does not lead the readers to confusion.

Given a class of input signals $\mathcal{W}$, the following definition of IO-FTS can be introduced.

Definition 2 (Input-Output Finite-Time Stability) *Given a time interval* Ω*, a class of input signals* $\mathcal{W}$ *defined over* Ω*, a positive definite matrix-valued function* $Q(\cdot)$ *defined over* Ω*, system* (2.1)*, with* $x_0 = 0$ *, is said to be IO-FTS with respect to* $\big(\mathcal{W}\,, Q(\cdot)\,, \Omega\big)$ *if*

$$w(\cdot) \in \mathcal{W} \;\Rightarrow\; y^T(t)Q(t)y(t) < 1, \quad t \;\in\; \Omega\,,$$

when $x_0 = 0$. ▲

An equivalent definition of IO-FTS is given in Chapter 4, where LTV systems are seen as linear operators between $\mathcal{L}_2$ and $\mathcal{L}_\infty$ spaces, and the system norm is the one induced by the $\|\;\cdot\;\|_{2,R}$ norm on exogenous inputs and $\|\;\cdot\;\|_{\infty,Q}$ norm on outputs. Using this definition, IO-FTS can be interpreted as the $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain, on a finite time interval, from the exogenous input to the regulated output. The same interpretation can be given in the case of input of class $\mathcal{L}_\infty$.

It is worth noticing that IO-FTS is a completely different concept with respect to H_2 control, not only because the latter is defined on an infinite time horizon. The H_2 norm of a linear system of transfer function $H(j\omega)$ is defined as:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} Tr\big(H(j\omega)^* H(j\omega)\big)d\omega \tag{2.3}$$

where the (Tr) operator extracts the trace of a matrix and $(^*)$ represent the complex conjugate operator. The H_2 norm represents the steady-state response of the system to white noise. Only in the special case of a scalar output, H_2 control can also be interpreted as the minimization of the $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain (for all the interpretations of H_2 control the interested readers can refer to Paganini and Feron [2000]). Thus, in the general case, IO-FTS and H_2 control are different concepts. Indeed, H_2 control, differently from IO-FTS, is based on the minimization of a system norm which is not induced by inputs and outputs signal norms.

IO-FTS differs also from classic IO stability. A system is said to be IO $\mathcal{L}_p$ stable if for any input of class $\mathcal{L}_p$, the system exhibits a corresponding output which belongs to the same class [Khalil, 1992, Ch. 4]. The main differences are that the IO-FTS involves signals defined over a finite time interval, does not necessarily require the inputs and outputs to belong to the same class, and that quantitative bounds on both inputs and outputs must be specified. Furthermore, while IO stability deals with the behavior of a system within a sufficiently long (in principle infinite) time interval, IO-FTS is a more practical concept, useful to study the behavior of the system within a finite (possibly short) interval, and therefore it finds application whenever it is desired that the output variables do not exceed a given threshold during the transients, given a certain class of input signals.

Starting from Definition 2, the following design control problems can be defined.

Problem 1 (IO FT stabilization via dynamic output feedback) *Consider the LTV system*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)w(t)\,, \quad x(t_0) = 0 \tag{2.4a}$$

$$y(t) = C(t)x(t) \tag{2.4b}$$

where $u(\cdot)$ is the control input and $w(\cdot)$ is the exogenous input. Given the class of signals $\mathcal{W}$, and a continuous positive definite matrix-valued function $Q(\cdot)$ defined over Ω, find a dynamic output feedback controller in the form

$$\dot{x}_c(t) = A_K(t)x_c(t) + B_K(t)y(t)\,, \tag{2.5a}$$

$$u(t) = C_K(t)x_c(t) + D_K(t)y(t) \tag{2.5b}$$

where $x_c(t)$ has the same dimension of $x(t)$, such that the closed-loop system obtained by the connection of (2.4) *and* (2.5) *is IO-FTS with respect to $\big(\mathcal{W}\,, Q(\cdot)\,, \Omega\big)$. In particular, the closed loop system is in the form*

$$\begin{pmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{pmatrix} = \begin{pmatrix} A(t) + B(t)D_K(t)C(t) & B(t)C_K(t) \\ B_K(t)C(t) & A_K(t) \end{pmatrix} \begin{pmatrix} x(t) \\ x_c(t) \end{pmatrix} + \begin{pmatrix} G(t) \\ 0 \end{pmatrix} w(t)$$

$$:= A_{\mathrm{CL}}(t)\, x_{\mathrm{CL}}(t) + G_{\mathrm{CL}}(t)\, w(t)\,, \quad t \notin \mathcal{T} \tag{2.6a}$$

$$y(t) = \begin{pmatrix} C_{\sigma(t)}(t) & 0 \end{pmatrix} x_{\mathrm{CL}}(t) := C_{\mathrm{CL}}(t)\, x_{\mathrm{CL}}(t)\,. \tag{2.6b}$$

▲

2.3 Time-dependent Switching Linear Systems (TD-SLS)

This section introduces a particular class of hybrid systems named Time-Dependent Switching Linear Systems (TD-SLS).

In order to define the class of TD-SLS, the notion of switching signal $\sigma(\cdot)$ is needed (see Liberzon [2003]).

Let $\sigma(\cdot) : \mathbb{R}_0^+ \mapsto \mathcal{P} = \{1, \ldots, l\}$ be a piecewise constant function, where the discontinuities are called *resetting times*. The signal $\sigma(\cdot)$ is called switching signal and it is assumed to be right-continuous everywhere. Furthermore, the set of resetting times is denoted by $\mathcal{T} = \{t_1, t_2, \ldots\} \subset \mathbb{R}_0^+$.

Let now consider the family of linear systems:

$$\dot{x}(t) = A_p(t)x(t) + G_p(t)w(t)\,, \tag{2.7a}$$

$$y(t) = C_p(t)x(t) \tag{2.7b}$$

where $p \in \mathcal{P}$, and $A_p(\cdot) : \mathbb{R}_0^+ \mapsto \mathbb{R}^{n\times n}$, $G_p(\cdot) : \mathbb{R}_0^+ \mapsto \mathbb{R}^{n\times r}$, and $C_p(\cdot) : \mathbb{R}_0^+ \mapsto \mathbb{R}^{m\times n}$ are continuous matrix-valued functions.

Given the family (2.7) and the switching signal $\sigma(\cdot)$, the class of TD-SLS is given by

$$\dot{x}(t) = A_{\sigma(t)}(t)x(t) + G_{\sigma(t)}(t)w(t)\,, \quad x(t_0) = x_0\,, \quad t \notin \mathcal{T} \tag{2.8a}$$

$$x(t^+) = J(t)x(t)\,, \qquad t \in \mathcal{T} \tag{2.8b}$$

$$y(t) = C_{\sigma(t)}(t)x(t) \tag{2.8c}$$

where $J(\cdot) : \mathbb{R}_0^+ \mapsto \mathbb{R}^{n\times n}$ is a matrix-valued function. In particular (2.8a) describes the continuous-time dynamics of the TD-SLS, while (2.8b) represents the resetting law. The switching signal $\sigma(t)$ specifies, at each time instant t, the linear system currently being *active*.

Note that, in the definition given above, all the linear systems in the family (2.7) have the same order. Although this could not be necessarily true, we make this assumption for the sake of simplicity.

Remark 1 *Without loss of generality, it is assumed that the first resetting time* $t_1 \in \Upsilon$ *is such that* $t_1 > t_0$*. Indeed, the case* $t_1 = t_0$ *is equivalent to a change of the initial state. Furthermore, since only the behavior of TD-SLS in a given time interval is taken into account, it can be assumed that*

$$]t_0\,, t_0 + T] \cap \Upsilon = \{t_1\,, t_2\,, \dots\,, t_h\}\,,$$

i.e., only a finite number of switches occurs in (2.8)*. This also prevents the TD-SLS* (2.8) *from exhibiting Zeno behavior (Ames et al. [2006]).* ▲

Remark 2 *It is worth to notice that TD-SLSs include the case of time-dependent impulsive dynamical linear systems (TD-IDLS, Haddad et al. [2006]; Amato et al. [2011a]), where a single continuous dynamic is considered. Indeed, for TD-IDLS, the switching signal* $\sigma(t)$ *is constant for all* t *and can be discarded, while the resetting times set* Υ *corresponds to the set of times where the state jumps. The advantage of the SLS definition is that two different dynamics can be defined at the same time. This is useful when the resetting times are not known; this case will be dealt with in Chapter 3 and Section 6.2.* ▲

It should be noticed that Definition 1 and Definition 2 still apply when TD-SLSs are considered. Furthermore, they can be used when considering TD-SLS composed by linear systems of different orders. In this case, the weighting matrix is still square but it changes dimension after the resetting times in order to keep the same dimension as the linear system currently active.

For the case of FTS, a graphical interpretation is given in Fig. 2.1 by using the ellipsoidal domain related definition. The state trajectory starts inside the ellipsoid defined by the matrix R in the 3-dimensional space and it is limited by a 3-dimensional weighting matrix $\Gamma(t)$. After the first resetting time t_1, the system become of the second order, and a second order $\Gamma(t)$ has to be defined accordingly. Note that in this case $\Gamma(t)$ defines an ellipse. The TD-SLS is FTS with respect to $\big(R\,, \Gamma(\cdot)\,, \Omega\big)$ if, given initial conditions inside the ellipsoid defined by $x_0^T R x_0 \leq 1$, the state trajectory remains inside the time varying ellipsoid defined by $x^T(t)\Gamma(t)x(t) \leq 1$ for all $t \in \Omega$.

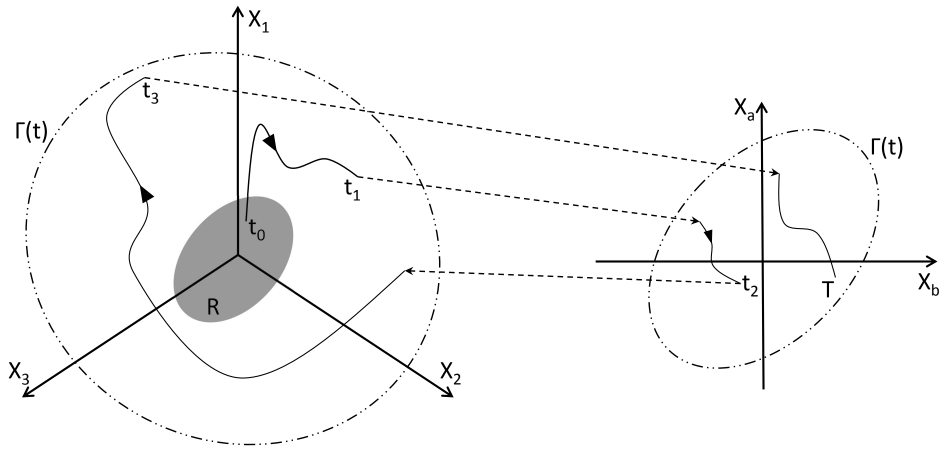


Figure 2.1: Example of finite-time stable trajectory of a TD-SLS.

Using Definition 1, the related problem of finite-time stabilize a TD-SLS via state-feedback can be identified.

Problem 2 (FT Stabilization via state feedback) *Consider the TD-SLS*

$$\dot{x}(t) = A_{\sigma(t)}(t)x(t) + B_{\sigma(t)}(t)u(t)\,, \quad x(t_0) = x_0\,, \quad t \notin \mathcal{T} \tag{2.9a}$$
$$x(t^+) = J(t)x(t)\,, \quad t \in \mathcal{T} \tag{2.9b}$$

where $u(\cdot)$ is the control input. Given a time interval Ω, a positive-definite matrix R, and a positive definite matrix-valued function $\Gamma(\cdot)$ defined over Ω, find a state feedback control law $u(t) = K(t)x(t)$, where $K(\cdot)$ is a piecewise continuous matrix-valued function defined over Ω, such that the closed-loop system

$$\dot{x}(t) = \hat{A}_{\sigma(t)}(t)x(t)\,, \quad t \notin \mathcal{T} \tag{2.10a}$$
$$x(t^+) = J(t)x(t)\,, \quad t \in \mathcal{T} \tag{2.10b}$$

with

$$\hat{A}_{\sigma(t)}(t) = A_{\sigma(t)}(t) + B_{\sigma(t)}(t)K(t)\,,$$

is FTS with respect to $\big(R\,,\Gamma(\cdot)\,,\Omega\big)$. ▲

In the case of IO-FTS, using definition 2, the following design control problem can be defined:

Problem 3 (IO FT stabilization via state feedback) *Consider the TD-SLS*

$$\begin{aligned}
\dot{x}(t) &= A_{\sigma(t)}(t)x(t) + B_{\sigma(t)}(t)u(t) + G_{\sigma(t)}(t)w(t)\,, \quad x(t_0) = 0 && \text{(2.11a)}\\
x(t^+) &= J(t)x(t)\,, \quad t \in \mathcal{T} && \text{(2.11b)}\\
y(t) &= C_{\sigma(t)}(t)x(t) && \text{(2.11c)}
\end{aligned}$$

where $u(\cdot)$ is the control input. Given a time interval Ω, a positive-definite matrix R, and a positive definite matrix-valued function $Q(\cdot)$ defined over Ω, find a state feedback control law $u(t) = K(t)x(t)$, where $K(\cdot)$ is a piecewise continuous matrix-valued function defined over Ω, such that the closed-loop system

$$\begin{aligned}
\dot{x}(t) &= \hat{A}_{\sigma(t)}(t)x(t)\,, \quad t \notin \mathcal{T} && \text{(2.12a)}\\
x(t^+) &= J(t)x(t)\,, \quad t \in \mathcal{T} && \text{(2.12b)}\\
y(t) &= C_{\sigma(t)}(t)x(t) && \text{(2.12c)}
\end{aligned}$$

with

$$\hat{A}_{\sigma(t)}(t) = A_{\sigma(t)}(t) + B_{\sigma(t)}(t)K(t)\,,$$

is IO-FTS with respect to $\big(R\,, Q(\cdot)\,, \Omega\big)$. ▲

Problem 4 (IO FT stabilization via static output feedback) *Consider the TD-SLS* (2.11)*, where $u(\cdot)$ is the control input. Given a time interval Ω, a positive-definite matrix R, and a positive definite matrix-valued function $Q(\cdot)$ defined over Ω, find a static output feedback control law $u(t) = K(t)y(t)$, where $K(\cdot)$ is a piecewise continuous matrix-valued function defined over Ω, such that the closed-loop system* (2.12) *with*

$$\hat{A}_{\sigma(t)}(t) = A_{\sigma(t)}(t) + B_{\sigma(t)}(t)K(t)C_{\sigma(t)}(t)\,,$$

is IO-FTS with respect to $\big(R\,, Q(\cdot)\,, \Omega\big)$. ▲

Problem 5 (IO FT stabilization via dynamic output feedback) *Consider the TD-SLS* (2.11)*, where $u(\cdot)$ is the control input and $w(\cdot)$ is the exogenous input. Given the class of signals $\mathcal{W}$, and a continuous positive definite matrix-valued* *function $Q(\cdot)$ defined over Ω, find a dynamic output feedback controller in the form*

$$
\begin{aligned}
\dot{x}_c(t) &= A_K(t)x_c(t) + B_K(t)y(t)\,, \quad t \notin \mathcal{T} && \text{(2.13a)}\\
x_c(t^+) &= J_K(t)x_c(t) + H_K(t)y(t)\,, \quad t \in \mathcal{T} && \text{(2.13b)}\\
u(t) &= C_K(t)x_c(t) + D_K(t)y(t) && \text{(2.13c)}
\end{aligned}
$$

where $x_c(t)$ has the same dimension of $x(t)$, such that the closed-loop system obtained by the connection of (2.11) *and* (2.13) *is IO-FTS with respect to* $\big(\mathcal{W}\,, Q(\cdot)\,, \Omega\big)$. *In particular, the closed loop system is in the form*

$$
\begin{aligned}
\begin{pmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{pmatrix} &= \begin{pmatrix} A_{\sigma(t)}(t) + B_{\sigma(t)}(t)D_K(t)C_{\sigma(t)}(t) & B_{\sigma(t)}(t)C_K(t) \\ B_K(t)C_{\sigma(t)}(t) & A_K(t) \end{pmatrix} \begin{pmatrix} x(t) \\ x_c(t) \end{pmatrix} \\
&\quad + \begin{pmatrix} G_{\sigma(t)}(t) \\ 0 \end{pmatrix} w(t) := A_{\mathrm{CL}}(t)\, x_{\mathrm{CL}}(t) + G_{\mathrm{CL}}(t)\, w(t)\,, \quad t \notin \mathcal{T} && \text{(2.14a)}\\
\begin{pmatrix} x(t^+) \\ x_c(t^+) \end{pmatrix} &= \begin{pmatrix} J(t) & 0 \\ H_K(t)C_{\sigma(t)}(t) & J_K(t) \end{pmatrix} \begin{pmatrix} x(t) \\ x_c(t) \end{pmatrix} := J_{\mathrm{CL}}(t)\, x_{\mathrm{CL}}(t)\,, \quad t \in \mathcal{T} && \text{(2.14b)}\\
y(t) &= \begin{pmatrix} C_{\sigma(t)}(t) & 0 \end{pmatrix} x_{\mathrm{CL}}(t) := C_{\mathrm{CL}}(t)\, x_{\mathrm{CL}}(t)\,. && \text{(2.14c)}
\end{aligned}
$$

Chapter 3

Necessary and sufficient conditions for FTS of TD-SLS

Finite-time stability and stabilization of Time-Dependent Switching Linear Systems (TD-SLS) is dealt with in this chapter. The innovative contribution introduced by the author with respect to the previous literature goes in several direction. First of all, the results achieved for Impulsive Dynamical Linear System (IDLS) are extended to the more general class of TD-SLS. Furthermore, the assumption that the sequence of resetting times is a priori known is removed. Preliminary results on this topic can be found in Amato et al. [2011c]. This makes the technique more appealing, allowing to tackle real engineering situations where the change of system dynamics is unpredictable and/or is due to an external triggering event. A first result provided in this chapter is a sufficient condition for FTS when the resetting times are known with a certain degree of uncertainty. Such condition requires the solution of a suitable feasibility problem based on coupled difference/differential linear matrix inequalities (D/DLMIs). It will be shown that, the reduction of the uncertainty intervals reduces the conservatism of this condition, which becomes *necessary* and sufficient in the certain case (i.e. the resetting instants are perfectly known). In this case it will be shown that the D/DLMIs based condition is in turn equivalent to the existence of a certain positive definite solution of a coupled difference/differential Lyapunov *equation* (D/DLE), the latter being more efficient from the computational point of view. Eventually, the conceptually different situation in which the resetting times are totally unknown, namely the arbitrary switching case is considered. The analysis results are then used to derive sufficient conditions for the existence of state feedback controllers that finite-time stabilize the closed loop system, in the three cases mentioned above.

To summarize, as for the knowledge of the resetting set $\mathcal{T}$ is concerned, three different cases will be considered:

Known Switching (KS): the resetting times in the time interval $[t_0\,, t_0+T]$ are perfectly known without uncertainty;

Arbitrary Switching (AS): no information about the resetting times is available, i.e., the resetting times are totally unknown;

Uncertain Switching (US): the resetting times are known with a given uncertainty. It is assumed that $t_j \in [\bar{t}_j - \Delta T_j\,, \bar{t}_j + \Delta T_j]$, without loss of generality; $\bar{t}_j$ is the *nominal* value of the j-th resetting time t_j.

Since the switching signal $\sigma(\cdot)$ is piecewise constant with discontinuities in correspondence of the resetting times, in the US case it is useful to introduce the following notation

$$\begin{aligned}
\Psi_1 &= \big]t_0\,, \bar{t}_1 + \Delta T_1\big[\,,\\
\Psi_j &= \big]\bar{t}_{j-1} - \Delta T_{j-1}\,, \bar{t}_j + \Delta T_j\big[\,,\ j = 2\,, \dots\,, h\\
\Psi_{h+1} &= \big]\bar{t}_h - \Delta T_h\,, t_0 + T\big]\\
\Phi_j &= \big[\bar{t}_j - \Delta T_j, \bar{t}_j + \Delta T_j\big]\,,\ j = 1\,, \dots\,, h
\end{aligned}$$

The following assumption is made.

Assumption 1 *The resetting times order is perfectly known, i.e.*

$$\bigcap_{j=1}^{h} \Phi_j = \emptyset\,, \tag{3.1}$$

▲

3.1 Finite-time stability of uncertain SLSs

This section presents sufficient conditions for finite-time stability of TD-SLS, when the three different levels of knowledge of the resetting times introduced in the previous section are considered. Firstly, a sufficient condition to check the FTS in the case of US is introduced; then, two alternative necessary and sufficient conditions for the FTS in the KS case are stated. Finally the AS case is derived as corollary of the US theorem.

The following theorem introduces a sufficient condition to check FTS when the j-th resetting time is known with a given uncertainty $\pm\Delta T_j$, i.e., when the US case is considered.

Theorem 1 (FTS in US case) *If there exist $h+1$ differentiable matrix-valued functions $P_j(\cdot)$, $j = 1\,, \ldots\,, h+1$, that satisfy the following D/DLMI*

$$\dot{P}_j(t) + A^T_{\sigma(t_{j-1})}(t)P_j(t) + P_j(t)A_{\sigma(t_{j-1})}(t) < 0\,, \quad t \in \Psi_j\,,\ j = 1\,, \ldots\,, h+1 \tag{3.2a}$$

$$J^T(t)P_{j+1}(t)J(t) - P_j(t) < 0\,, \quad t \in \Phi_j\,,\ j = 1\,, \ldots\,, h \tag{3.2b}$$

$$P_j(t) \geq \Gamma(t)\,, \quad t \in \Psi_j\,,\ j = 1\,, \ldots\,, h+1 \tag{3.2c}$$

$$P_1(t_0) < R \tag{3.2d}$$

then the SLS (2.8) is FTS wrt $\big(R\,, \Gamma(\cdot)\,, \Omega\big)$, under the Uncertain Switching assumption.

Proof. Before starting the proof, it should be noticed that, exploiting the knowledge of the resetting times order implied by (3.1) (see Assumption 1), it is possible to *assign* a single optimization matrix $P_j(\cdot)$ and the corresponding active linear system enabled by $\sigma(t)$ to each time interval Ψ_j, with $j = 1\,, \ldots\,, h+1$.

Suppose now that $\Delta T_j = 0$ for $j = 1\,, \ldots\,, h$; it turns out that the intervals Φ_j become equal to $\{\bar{t}_j\}$ and

$$\bigcap_{j=1}^{h+1} \Psi_j = \emptyset\,.$$

Hence, it is possible to define the following piecewise differentiable positive definite matrix-valued function $P(\cdot)$

$$P(t) = P_j(t) \quad \text{when} \quad t \in \Psi_j \, .$$

Let

$$V(t,x) = x^T(t)P(t)x(t) \, ,$$

given a time instant $t \notin \mathcal{T}$, the derivative with respect to time reads [1]

$$\frac{d}{dt}\left(x^T P x\right) = x^T \left(\dot{P} + A_\sigma^T P + P A_\sigma\right) x \, ,$$

which is negative definite in view of (3.2a). At the discontinuity point $\bar{t}_j$ it is

$$\begin{aligned} V(t_j^+, x) - V(t_j, x) &= x^T(t_j^+)P(t_j^+)x(t_j^+) - x^T(t_j)P(t_j)x(t_j) \\ &= x^T(t_j)\left(J^T(t_j)P(t_j^+)J(t_j) - P(t_j)\right)x(t_j) \, , \end{aligned}$$

which is negative definite by virtue of (3.2b). We can conclude that $V(t,x)$ is strictly decreasing along the state trajectories of system (2.8). Hence, given x_0 such that $x_0^T R x_0 \leq 1$, it is

$$\begin{aligned} x^T(t)\Gamma(t)x(t) &\leq x^T(t)P(t)x(t) \\ &< x_0^T P(t_0) x_0 \\ &< x_0^T R x_0 \leq 1 \, , \end{aligned}$$

where the first inequality is guaranteed by (3.2c) and the third one by (3.2d).

Now consider the case of $\Delta T_j \neq 0$. First notice that, although unknown, a resetting times set exists. Furthermore, knowing the number of the resetting times and their order, it is also known the time interval in which every switch can occur. It turns out that the previous proof still applies when considering the time interval Φ_j in place of the time instant $\bar{t}_j$.

Indeed, conditions (3.2a) and (3.2c) still have to be verified in Ψ_j, i.e. in the time interval in which the corresponding linear system is potentially active. Moreover, in the uncertain case, condition (3.2b) has to be checked in Φ_j, i.e. in

[1] The time argument is omitted for brevity.

the time interval in which the state jump could occur. ♦

The conditions in Theorem 1 become less severe as the uncertainty intervals reduce in size; to this regard, the next theorem shows that, in the limit case of no uncertainty (KS case), the conditions stated in Theorem 1 becomes also necessary for FTS. Also, an alternative necessary and sufficient condition requiring the solution of coupled D/DLE it is provided.

Theorem 2 (FTS in KS case) *The following statements are equivalent:*

i) *The SLS (2.8a-2.8b) is FTS wrt* $(R\,,\Gamma(\cdot)\,,\Omega)$ *under the Known Switching assumption.*

ii) *The piecewise differentiable matrix-valued solution* $W(\cdot)\ :\ \Omega \mapsto\ \mathbb{R}^{n\times n}$ *of the D/DLE.*

$$-\dot{W}(t) + A_{\sigma(t)}(t)W(t) + W(t)A^T_{\sigma(t)}(t) = 0\,, \quad t \in \Omega\,, t \notin \mathcal{T} \tag{3.3a}$$

$$W^+(t_i) = J(t_i)W^-(t_i)J^T(t_i)\,, \quad t_i \in \mathcal{T} \tag{3.3b}$$

$$W(t_0) = R^{-1}\,, \tag{3.3c}$$

is positive definite and satisfies

$$C(t)W(t)C^T(t) < I\,, \quad \forall\ t \in \Omega\,, \tag{3.4}$$

where $C(\cdot)$ *is a nonsingular matrix-valued function such that* $\Gamma(t) = C^T(t)C(t)$ *in* Ω*.*

iii) *There exists a piecewise differentiable positive definite matrix-valued function* $P(\cdot) : \Omega \mapsto\ \mathbb{R}^{n\times n}$ *that satisfies the following D/DLMI*

$$\dot{P}(t) + A^T_{\sigma(t)}(t)P(t) + P(t)A_{\sigma(t)}(t) < 0\,, \quad \forall\ t \in \Omega\,, t \notin \mathcal{T} \tag{3.5a}$$

$$J^T(t_i)P^+(t_i)J(t_i) - P^-(t_i) < 0\,, \quad t_i \in \mathcal{T} \tag{3.5b}$$

$$P(t) \geq \Gamma(t)\,, \forall\ t \in \Omega \tag{3.5c}$$

$$P(t_0) < R \tag{3.5d}$$

Proof. The equivalence of the three statements will be proven by showing first that **i)** $\Leftrightarrow$ **ii)**; then that **ii)** $\Rightarrow$ **iii)** and that **iii)** $\Rightarrow$ **i)**.
$\big[$**i)** $\Leftrightarrow$ **ii)**$\big]$. Note that, given the symmetric and positive definite matrix-valued function $\Gamma(\cdot) \in \mathbb{R}^n$, it is always possible to find a nonsingular matrix-valued function $C(\cdot): \ \Omega \ \mapsto \ \mathbb{R}^{n\times n}$ such that $\Gamma(t) = C^T(t)C(t)$, $t \in \Omega$.

First, from Amato et al. [2008] it follows that system (2.8) is FTS if and only if

$$\begin{aligned}
&\Phi^T(t\,,t_0)C^T(t)C(t)\Phi(t\,,t_0) - R < 0 \\
&\qquad\qquad \Leftrightarrow R^{-\frac{1}{2}}\Phi^T(t\,,t_0)C^T(t)C(t)\Phi(t\,,t_0)R^{-\frac{1}{2}} - I < 0 \\
&\qquad\qquad\qquad\qquad \Leftrightarrow C(t)\Phi(t\,,t_0)R^{-1}\Phi^T(t\,,t_0)C^T(t) - I < 0 \qquad (3.6)
\end{aligned}$$

for all $t \in \Omega$, where $\Phi(t\,,t_0)$ is the state transition matrix of (2.8). If we let

$$W(t) = \Phi(t\,,t_0)R^{-1}\Phi^T(t\,,t_0)\,, \qquad (3.7)$$

then it turns out that $W(t)$ is positive definite and that (3.6) can be rewritten as (3.4).

In order to prove that $W(\cdot)$ is the solution of (3.3), consider the following properties of the transition matrix

$$\Phi(t_0\,,t_0) = I\,, \qquad (3.8a)$$
$$\dot{\Phi}(t\,,t_0) = A_{\sigma(t)}(t)\Phi(t\,,t_0)\,, \quad t \notin \mathcal{T} \qquad (3.8b)$$
$$\Phi^+(t_i\,,t_{i-1}) = J(t_i)\Phi^-(t_i\,,t_{i-1})\,, \quad t_i \in \mathcal{T} \qquad (3.8c)$$

Hence (3.3c) readily follows from (3.8a), while (3.3a) and (3.3b) can be obtained computing $\dot{W}(t)$ and $W^+(t_i)$ from (3.7) and exploiting (3.8b) and (3.8c), respectively.

$\big[$**ii)** $\Rightarrow$ **iii)**$\big]$. It has already been shown that if $W(\cdot)$ satisfies (3.3) and (3.4), then the SLS (2.8) is FTS. Now, by continuity arguments, if system (2.8) is FTS, there exist two real scalars $\varepsilon_1, \varepsilon_2 > 0$ such that the following SLS

$$\dot{z}(t) = \left(A_{\sigma(t)}(t) + \frac{\varepsilon_1}{2}\right) z(t)\,, \quad z(t_0) = x_0\,, \quad t \notin \mathcal{T} \tag{3.9a}$$

$$z^+(t_i) = J(t_i)\left(1 + \frac{\varepsilon_2}{2}\right) z^-(t_i)\,, \quad t_i \in \mathcal{T} \tag{3.9b}$$

is also FTS. Taking again into account the equivalence between the two conditions **i)** and **ii)**, $W_\varepsilon(\cdot)$ denotes the piecewise differentiable positive definite matrix-valued solution of

$$-\dot{W}_\varepsilon(t) + A_{\sigma(t)}(t)W_\varepsilon(t) + W_\varepsilon(t)A^T_{\sigma(t)}(t) + \varepsilon_1 W_\varepsilon(t) = 0\,, \quad \forall\, t \in \Omega\,, t \notin \mathcal{T} \tag{3.10a}$$

$$W^+_\varepsilon(t_i) = J(t_i)W^-_\varepsilon(t_i)J^T(t_i) + \frac{\varepsilon_2^2}{4}J(t_i)W^-_\varepsilon(t_i)J^T(t_i) + \varepsilon_2 J(t_i)W^-_\varepsilon(t_i)J^T(t_i) \quad t_i \in \mathcal{T} \tag{3.10b}$$

$$W_\varepsilon(t_0) = R^{-1} \tag{3.10c}$$

which also satisfies $C(t)W_\varepsilon(t)C^T(t) < I$, for all t in Ω. Note that the D/DLE (3.10) readily follows from (3.3) when we consider system (3.9). Exploiting continuity arguments once more, it turns out that there exists a real scalar $\alpha > 1$ such that

$$\alpha C(t)W_\varepsilon(t)C^T(t) < I\,, \quad t \in \Omega\,. \tag{3.11}$$

Let $X(t) = \alpha W_\varepsilon(t)$ for all t in Ω, inequality (3.11) reads

$$C(t)X(t)C^T(t) < I\,, \quad t \in \Omega\,. \tag{3.12}$$

Since $\dot{X}(t) = \alpha\dot{W}_\varepsilon(t)$, from (3.10a) it is obtained

$$-\dot{X}(t) + A_{\sigma(t)}(t)X(t) + X(t)A^T_{\sigma(t)}(t) + \varepsilon_1 X(t) = 0\,,$$

in Ω and for $t \notin \mathcal{T}$; taking into account the positive definitiveness of $X(t)$ it follows that

$$-\dot{X}(t) + A_{\sigma(t)}X(t) + X(t)A^T_{\sigma(t)}(t) < 0\,. \tag{3.13}$$

for all $t \notin \mathcal{T}$. Furthermore, from (3.10b) it readily follows that

$$X^{+}(t_i) > J(t_i)X^{-}(t_i)J^{T}(t_i)\,, \tag{3.14}$$

while taking into account (3.10c) it is obtained

$$X(t_0) > R^{-1}\,. \tag{3.15}$$

Eventually, letting $P(t) = X^{-1}(t)$, $t \in \Omega$, and applying Schur complements to (3.14), inequalities (3.5) can be easily obtained from (3.12)-(3.15).

$\big[$**iii)** $\Rightarrow$ **i)**$\big]$. The same arguments exploited in the proof of Theorem 1 can be used, providing that the $h+1$ different $P_j(\cdot)$ can be replaced by a single matrix-valued function $P(\cdot)$.

Indeed, when $\Delta T_j = 0$ for $j = 1, \dots, h$ no uncertainty is considered, which corresponds to the KS case.

The only thing which needs to be shown is that a single matrix-valued function $P(\cdot)$ is sufficient to check the conditions (3.2) when $\Delta T_j = 0$. To this aim, note that if $\Delta T_j = 0$, then the time intervals Ψ_j are separated by the resetting times; it follows that considering different $P_j(\cdot)$ defined on Ψ_j is equivalent to consider a single $P(\cdot)$ with *jumps* on the resetting times ♦

Remark 3 *As expected, a trade-off clearly appears when comparing conditions* (3.2) *given by Theorem 1 with conditions* (3.5) *in Theorem 2. In particular, the less is the uncertainty on the resetting times, the less restrictive are the constraints to be verified.* ▲

Remark 4 *The condition based on the coupled D/DLE turns out to be much more efficient from the computational point of view (e.g. see Amato et al. [2012b]). However, as it will be shown in the next section, the D/DLMI based approach can be used to derive design conditions, while the D/DLE condition can not.* ▲

The next corollary extends the result given in Theorem 1 to the AS case.

Corollary 1 (FTS in AS case) *If there exist l differentiable matrix-valued functions $P_i(\cdot)$, $i = 1\,, \dots\,, l$, that satisfy the following D/DLMI*

$$\dot{P}_i(t) + A_i^T(t)P_i(t) + P_i(t)A_i(t) < 0\,, \quad t \in \,]t_0, t_0 + T]\,, \quad i \in \mathcal{P} \tag{3.16a}$$

$$J^T(t)P_i(t)J(t) - P_j(t) < 0\,, \quad t \in \,]t_0\,, t_0 + T]\,, \quad i\,, j \in \mathcal{P} \tag{3.16b}$$

$$P_i(t) \geq \Gamma(t)\,, \quad t \in \,]t_0\,, t_0 + T]\,, \quad i \in \mathcal{P} \tag{3.16c}$$

$$P_i(t_0) < R\,, \quad i \in \mathcal{P} \tag{3.16d}$$

then the SLS (2.8) *is FTS wrt* $\big(R\,, \Gamma(\cdot)\,, \Omega\big)$ *under the AS assumption.*

Proof. The case of AS can be seen as a special case of US, in which the uncertainty covers all the considered time interval Ω. If this is the case, the intervals Ψ_j and $\Phi_j\,, j \in \mathcal{P}$ coincide with Ω. As a consequence, for all i in $\mathcal{P}$, conditions (3.2a) and (3.2c) have to be verified in $]t_0\,, t_0 + T]$, while (3.2d) must be verified at t_0. It turns out that (3.16a), (3.16c) and (3.16d) hold.

Furthermore, condition (3.2b) has to be verified for all $i\,, j \in \mathcal{P}$ since, at each time instant, the system can switch between any linear dynamics defined in the family (2.7). Hence (3.16b) holds, and since the resetting times are not a priori known, $P_j(\cdot)\,, j \in \mathcal{P}$, need to be differentiable. ♦

The AS case turns out to introduce the maximum level of conservatism. In particular, since condition (3.16b) has to be verified for all $i\,, j \in \mathcal{P}$, $J(\cdot)$ needs to be Schur for all t in $]t_0\,, t_0 + T]$. In other words, due to the lack in the resetting times knowledge, it is necessary to have *stable* resetting laws in order to check the FTS of SLS. Furthermore, each linear system in the family (2.7) has to be FTS in order to make FTS the uncertain SLS.

Remark 5 *As said, when family* (2.7) *is composed of systems with different dimensions, Theorems 1 and 2, as well as Corollary 1, still hold. Indeed, if this is the case, the single matrix-valued function $\Gamma(\cdot)$ should be replaced by $h + 1$ functions $\Gamma_j(\cdot)$, defined on the different subintervals $\Psi_j(\cdot)$. These matrix-valued functions are square but do not necessarily have the same dimension. It follows that inequality* (3.2b) *can still be defined by choosing $P_{j+1}(\cdot)\,, j = 1\, \dots\, h$, with the same dimension of $A_{\sigma(t_j)}(\cdot)$ for all t, and by noticing that, in general, $J(t_k)$ is not a square matrix.* ▲

3.2 Finite-time stabilization via state feedback

Starting from the analysis conditions provided in the previous section, Problem 2 can be now addressed. In particular only the US case is presented, since the results for the KS and AS cases can be derived from the next theorem exploiting similar arguments as for the analysis. As noticed in Remark 4, in the KS case the D/DLMI (3.5) allows to state a necessary and sufficient condition also for the solution of the finite-time stabilization problem via state feedback.

Theorem 3 (FT stabilization in the US case) *In the case of uncertain switchings, Problem 2 is solvable via state-feedback control if there exist $h+1$ differentiable positive definite matrix-valued functions $\Pi_j(\cdot)$ and $h+1$ matrix-valued functions $L_j(\cdot)$, $j = 1, \dots, h+1$, such that*

$$-\dot{\Pi}_j(t) + A_{\sigma(t_{j-1})}(t)\Pi_j(t) + \Pi_j(t)A^T_{\sigma(t_{j-1})}(t) + L_j^T(t)B^T_{\sigma(t_j)}(t) + B_{\sigma(t_j)}(t)L_j(t) < 0\,, \quad t \in \Psi_j\,,\ j = 1, \dots, h+1 \tag{3.17a}$$

$$\begin{pmatrix} -\Pi_{j+1}(t) & J(t)\Pi_j(t) \\ \Pi_j(t)J^T(t) & -\Pi_j(t) \end{pmatrix} < 0\,, \quad t \in \Phi_j\,,\ j = 1, \dots, h \tag{3.17b}$$

$$\Pi_j(t) \le \Gamma^{-1}(t)\,, \quad t \in \Psi_j\,,\ j = 1, \dots, h+1 \tag{3.17c}$$

$$\Pi_1(t_0) > R^{-1} \tag{3.17d}$$

$K_j(t) = L_j(t)\Pi_j(t)^{-1}$ is the controller which solves Problem 2 via state-feedback.

Proof. The proof readily follows applying Theorem 1 to the closed-loop system (2.10), letting $\Pi(t) = P^{-1}(t)$, and applying the Schur complements to (3.2b) (see [Amato et al., 2011a, Theorem 4]). ♦

Remark 6 *Output feedback could be used as an alternative control technique to solve Problem 2. The application of this technique in the case of known resetting times can be found in Theorem 3 of Amato et al. [2011a]. The extension to the US case is not reported for the sake of brevity, but it can be obtained considering different controllers, and then different optimization functions, for each time interval Φ_i, with $i = 1, \cdots, h$. The proof can be attained by exploiting the knowledge of the resetting times order using similar argument as in the proof of Theorem 1.*

Chapter 4

Necessary and sufficient conditions for IO-FTS of LTV systems

A necessary and sufficient condition for IO-FTS, for the class of $\mathcal{W}_2$ inputs introduced in Section 2.2, is presented in this chapter. The results contained in this chapter can be found in Amato et al. [2012b]. First it is shown that the condition given in Amato et al. [2010a], which requires the existence of a positive definite solution to an optimization problem involving DLMIs, is actually also *necessary*. Afterwards, an alternative necessary and sufficient condition is presented, which requires that a certain Differential Lyapunov Equation (DLE) admits a positive definite solution. It will be shown in Section 7.3.2 that the condition based on the DLE is more effective from a numerical point of view. On the other hand, the DLMIs formulation turns out to be useful for design purposes.

To conclude the chapter, the analysis condition based on DLMIs is used in the synthesis context. A necessary and sufficient condition is presented which guarantees the existence of an output feedback dynamic controller which renders the closed loop system IO-FTS.

Before presenting the main results of the chapter, some notations and preliminary results are introduced.

4.1 Notation and preliminary results

The systems taken into consideration in this chapter are LTV systems in the form

$$\Lambda : \begin{cases} \dot{x}(t) = A(t)x(t) + G(t)w(t)\,, & x(t_0) = 0 \\ y(t) = C(t)x(t) \end{cases} \tag{4.1}$$

where $A(\cdot) : \Omega \mapsto \mathbb{R}^{n\times n}$, $G(\cdot) : \Omega \mapsto \mathbb{R}^{n\times r}$, and $C(\cdot) : \Omega \mapsto \mathbb{R}^{m\times n}$, are continuous matrix-valued functions; Λ can be viewed as a linear operator mapping input signals ($w(\cdot)$'s) into output signals ($y(\cdot)$'s).

The state transition matrix of system (4.1) will be referred to as $\Phi(t\,,\tau)$, and its impulsive response as

$$H(t\,,\tau) = C(t)\Phi(t\,,\tau)G(\tau)\delta_{-1}(t-\tau)\,,$$

where $\delta_{-1}(t)$ is the Heaviside step function.

Given a vector $v \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$, $|v|$ denotes the euclidian norm of v, and $|A|$ the induced matrix norm

$$|A| = \sup_{v\neq 0} \frac{|Av|}{|v|}\,.$$

Given the two constants $t_0 \geq 0$ and $T > 0$, the bounded interval $\Omega = [t_0, t_0 + T]$ is defined.

Some well known results concerning the reachability Gramian of LTV systems are now recalled, more details can be found in Callier and Desoer [1991].

Definition 3 (Reachability Gramian) *The reachability Gramian of system* (4.1) *is defined as*

$$W_r(t\,,t_0) \triangleq \int_{t_0}^{t} \Phi(t\,,\tau)G(\tau)G^T(\tau)\Phi^T(t\,,\tau)d\tau\,.$$

Note that $W_r(t\,,t_0)$ is symmetric and positive semidefinite for all $t \geq t_0$.

Remark 7 *If the pair $\big(A\,,G\big)$ is controllable, then $W_r(t\,,t_0)$ is positive definite for all $t > t_0$.*

Furthermore, if system (4.1) *is time-invariant, then* $W_r(t\,,t_0)=W_r(t-t_0)$ *and*

$$W_r(t_2-t_0)\geq W_r(t_1-t_0)\,,\quad t_2\geq t_1\geq t_0\,.$$

▲

Lemma 1 (Callier and Desoer [1991]) *Given system* (4.1)*,* $W_r(t\,,t_0)$ *is the unique solution of the matrix differential equation*

$$\dot{W}_r(t\,,t_0)=A(t)W_r(t\,,t_0)+W_r(t\,,t_0)A^T(t)+G(t)G^T(t)\,, \tag{4.2a}$$

$$W_r(t_0\,,t_0)=0\,. \tag{4.2b}$$

▲

An equivalent definition of IO-FTS will be used in this chapter, it can be easily derived considering the LTV system (4.1) as a linear operator that maps signals from the space $\mathcal{L}_2(\Omega)$ to the space $\mathcal{L}_\infty(\Omega)$, i.e.:

$$\Lambda: w(\cdot)\in\mathcal{L}_2(\Omega)\mapsto y(\cdot)\in\mathcal{L}_\infty(\Omega)\,. \tag{4.3}$$

Moreover, equipping the $\mathcal{L}_2(\Omega)$ and $\mathcal{L}_\infty(\Omega)$ spaces with the weighted norms $\|\cdot\|_{2,R}$ and $\|\cdot\|_{\infty,Q}$, respectively, the induced norm of the linear operator (4.3) is given by

$$\|\Lambda\|=\sup_{\|w(\cdot)\|_{2,R}=1}\Big[\|y(\cdot)\|_{\infty,Q}\Big]\,,$$

i.e. the norm of Λ is computed considering the input signals in $\mathcal{W}_2$. Hence, requiring system (4.1) to be IO-FTS wrt $\big(\mathcal{W}_2\,,Q(\cdot)\,,\Omega\big)$ is equivalent to require that $\|\Lambda\|<1$; the following theorem holds.

Theorem 4 *Given a time interval* Ω*, the class of input signals* $\mathcal{W}_2$*, and a continuous positive definite matrix-valued function* $Q(\cdot)$*, system* (4.1) *is IO-FTS with respect to* $\big(\mathcal{W}_2\,,Q(\cdot)\,,\Omega\big)$ if and only if Λ *is a bounded linear operator with* $\|\Lambda\|<1$. ▲

Given the linear operator (4.3), its dual operator is

$$\overline{\Lambda}: z(\cdot)\in\mathcal{L}_1(\Omega)\mapsto v(\cdot)\in\mathcal{L}_2(\Omega)\,,$$

corresponding to the dual system [Callier and Desoer, 1991, pg. 236]

$$\overline{\Lambda} : \begin{cases} \dot{\tilde{x}}(t) = -A^T(t)\tilde{x}(t) - C^T(t)z(t) \\ v(t) = G^T(t)\tilde{x}(t) \end{cases} . \tag{4.4}$$

According to duality, the norm of $\overline{\Lambda}$ is defined as

$$\|\overline{\Lambda}\| = \sup_{\|z(\cdot)\|_{1,Q}=1} \Big[\|v(\cdot)\|_{2,R} \Big] ;$$

moreover, by definition of dual operator (Yosida [1980]), given $z(\cdot) \in \mathcal{L}_1(\Omega)$ and $w(\cdot) \in \mathcal{L}_2(\Omega)$, it is

$$\langle z \,, \Lambda w \rangle = \langle \overline{\Lambda} z \,, w \rangle \,, \tag{4.5}$$

where, given two signals $u(\cdot)$ and $v(\cdot)$, it is

$$\langle u \,, v \rangle = \int_\Omega u^T(t) v(t) dt \,.$$

Therefore (4.5) reads

$$\begin{aligned} \langle z \,, \Lambda w \rangle = \int_\Omega z^T(t) \int_\Omega H(t\,,\tau) w(\tau) d\tau dt \\ = \int_\Omega \left(\int_\Omega z^T(t) H(t\,,\tau) dt \right) w(\tau) d\tau \\ = \int_\Omega \left(\int_\Omega z^T(t) \overline{H}^T(\tau\,,t) dt \right) w(\tau) d\tau = \langle \overline{\Lambda} z \,, w \rangle \,, \end{aligned}$$

where

$$\overline{H}(t\,,\tau) = H^T(\tau\,,t) = G^T(t)\Phi^T(\tau\,,t) C^T(\tau) \delta_{-1}(\tau - t) \,, \tag{4.6}$$

is the impulsive response of the dual system (4.4).

Furthermore it holds that (see [Yosida, 1980, p. 195]):

$$\|\Lambda\| = \|\overline{\Lambda}\| \,. \tag{4.7}$$

Lemma 2 *If*

$$v(t) \triangleq \int_\Omega f(t\,,\tau)\; d\tau\,, \quad t \in \Omega$$

with $f(\cdot\,,\tau)$ $\mathcal{L}_2$-integrable, then the following inequality hold

$$\|v(\cdot)\|_2 \leq \int_\Omega \|f(\cdot\,,\tau)\|_2\; d\tau\,. \tag{4.8}$$

▲

For the proof of Lemma 2, the interested readers can refer to Amato et al. [2012b]

The next theorem is a generalization of a result given in Wilson [1989] to the case of LTV systems, and it allows to compute the norm of Γ as a function of the spectral radius of the reachability Gramian.

Theorem 5 *Given the LTV system* (4.1)*, the norm of the corresponding linear operator* (4.3) *is given by*

$$\|\Lambda\| = \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\Big(Q^{\frac{1}{2}}(t)C(t)W(t\,,t_0)C^T(t)Q^{\frac{1}{2}}(t)\Big)\,, \tag{4.9}$$

for all $t \in \Omega$, where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue, and $W(t\,,t_0)$ is the positive semidefinite matrix-valued solution of

$$\dot{W}(t\,,t_0) = A(t)W(t\,,t_0) + W(t\,,t_0)A^T(t) + G(t)R(t)^{-1}G^T(t) \tag{4.10a}$$

$$W(t_0\,,t_0) = 0\,. \tag{4.10b}$$

Proof. For the sake of simplicity, the weighting matrices will be firstly considered equal to the identity, that is

$$R(t) = I \text{ and } Q(t) = I\,, \forall\; t \in \Omega\,.$$

Note that, given this assumption, the solution of (4.10) is given by the reachability gramian $W_r(t\,,t_0)$; how to take into account the weighting matrices will be discussed at the end of the proof.

First note that, in view of (4.7), proving (4.9) is equivalent to show that

$$\|\overline{\Lambda}\| = \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\Big(C(t)W_r(t\,,t_0)C^T(t)\Big)\,,$$

where $\overline{\Lambda}$ is the dual operator of Λ. Taking into account the definition of $\overline{\Lambda}$, letting

$$\Upsilon(t) = \int_{\Omega} H(t\,,\sigma)H^T(t\,,\sigma)d\sigma\,, \tag{4.11}$$

and denoting by $v(\cdot)$ the output of system (4.4), it is

$$\begin{aligned}
\|v(\cdot)\|_2 &= \| \textstyle\int_{\Omega} \overline{H}(\cdot\,,\tau)z(\tau)d\tau\|_2 \leq \int_{\Omega} \|\overline{H}(\cdot\,,\tau)z(\tau)\|_2 d\tau \\
&= \textstyle\int_{\Omega}\Big(z^T(\tau)\int_{\Omega}\overline{H}^T(t\,,\tau)\overline{H}(t\,,\tau)dt\,z(\tau)\Big)^{\frac{1}{2}}d\tau \\
&= \textstyle\int_{\Omega}\Big(z^T(\tau)\Upsilon(\tau)z(\tau)\Big)^{\frac{1}{2}}d\tau && \text{by (4.6) and (4.11)} \\
&= \textstyle\int_{\Omega} |\Upsilon^{\frac{1}{2}}(\tau)z(\tau)|d\tau \\
&\leq \textstyle\int_{\Omega} |\Upsilon^{\frac{1}{2}}(\tau)|\cdot|z(\tau)|d\tau \\
&= \textstyle\int_{\Omega} \lambda_{\max}^{\frac{1}{2}}\big(\Upsilon(\tau)\big)\cdot|z(\tau)|\;d\tau
\end{aligned}$$

where the first inequality comes from Lemma 2 and the last equality holds since the matrix-valued function $\Upsilon(\cdot)$ is positive semidefinite.

Now, since all system and weighting matrices are bounded, it follows that the impulsive response of system (4.1) is bounded, and so it is also $\Upsilon(\cdot)$; therefore

$$\begin{aligned}
\|v(\cdot)\|_2 &\leq \operatorname{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\big(\Upsilon(t)\big)\cdot \textstyle\int_{\Omega}|z(\tau)|\;d\tau \\
&= \operatorname{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\big(\Upsilon(t)\big)\cdot\|z(\cdot)\|_1\,,
\end{aligned}$$

thus

$$\|\overline{\Lambda}\| \leq \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\big(\Upsilon(t)\big)\,. \tag{4.12}$$

From Definition 3 the matrix-valued function $\Upsilon(t)$ is equal to

$$\Upsilon(t) = C(t)W_r(t\,,t_0)C^T(t)\,;$$

hence (4.12) reads

$$\|\overline{\Lambda}\| \leq \operatorname*{ess\,sup}_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\big(C(t)W_r(t\,,t_0)C^T(t)\big)\,. \tag{4.13}$$

The last part of the proof is devoted to show that (4.13) is actually an equality. To this end, denote by γ the right hand side in equation (4.13); therefore (4.13) can be rewritten

$$\|\overline{\Lambda}\| \leq \gamma\,. \tag{4.14}$$

In the following a sequence of inputs to system (4.4) with unit norm in $\mathcal{L}_1(\Omega)$ will be built, such that the sequence of the norms of the corresponding output signals converges to γ.

To this end consider a subset $\Omega' \subset \Omega$, such that, for all $t \in \Omega'$,

$$\lambda_{\max}^{\frac{1}{2}}\big(C(t)W_r(t\,,t_0)C^T(t)\big) \geq \gamma - \varepsilon\,,$$

with $\varepsilon > 0$. Now let $\sigma \in \Omega'$ and consider the sequence of inputs

$$z_{\varepsilon\,,\alpha}(t) = h(\sigma)u_\alpha(t)\,,$$

where $h(\sigma)$ is the unit eigenvector corresponding to the maximum eigenvalue of $C(\sigma)W_r(\sigma\,,t_0)C^T(\sigma)$, and u_α is a sequence of positive scalar functions with unit norm in $\mathcal{L}_1(\Omega)$, which approach the Dirac delta function applied in σ as $\alpha \mapsto 0$. Let

$$v_{\varepsilon\,,\alpha}(t) = \overline{\Lambda} z_{\varepsilon,\alpha}(t) = \int_\Omega \overline{H}(t\,,\tau) z_{\varepsilon\,,\alpha}(\tau)\;d\tau\,.$$

It is simple to recognize that, as $\alpha \to 0$, it is

$$v_{\varepsilon,\alpha}(\cdot) \to \int_\Omega \overline{H}(t,\tau)h(\sigma)\delta(\tau-\sigma)d\tau = \overline{H}(t,\sigma)h(\sigma) \quad \text{in } \mathcal{L}_2(\Omega)\,.$$

Therefore

$$\begin{aligned}\lim_{\alpha\to 0}\|v_{\varepsilon\,,\alpha}(\cdot)\|_2^2 &= \int_\Omega h^T(\sigma)\overline{H}^T(t,\sigma)\overline{H}(t,\sigma)h(\sigma)dt \\ &= h^T(\sigma)\int_\Omega H(\sigma,t)H^T(\sigma,t)dt\ h(\sigma) \\ &= h^T(\sigma)C(\sigma)W_r(\sigma,t_0)C^T(\sigma)h(\sigma)\,.\end{aligned}$$

It turns out that

$$\lim_{\alpha\mapsto 0}\|v_{\varepsilon\,,\alpha}(\cdot)\|_2 = \lambda_{\max}^{\frac{1}{2}}\big(C(\sigma)W_r(\sigma\,,t_0)C^T(\sigma)\big) \geq \gamma-\varepsilon\,;$$

therefore, given $\eta > 0$, it is possible to choose a sufficiently small α such that

$$\|v_{\varepsilon\,,\alpha}(\cdot)\|_2 \geq \gamma-\varepsilon-\eta\,.$$

Taking into account (4.14), that the scalars ε and η can be chosen arbitrarily small, and that the set of the signals $z_{\varepsilon,\alpha}$ is a subset of the set of the unit norm signals in $\mathcal{L}_1(\Omega)$, it can be concluded that

$$\begin{aligned}\gamma \geq \|\overline{\Lambda}\| &= \sup_{\|z(\cdot)\|_1=1}\|v(\cdot)\|_2 \\ &\geq \sup_{z_{\varepsilon,\alpha}(\cdot)}\|v_{\varepsilon,\alpha}(\cdot)\|_2 = \gamma\,.\end{aligned}$$

From the last chain of inequality the proof follows.

Eventually, note that when the weighting matrices are taken into account the proof still holds by modifying the model matrices as follows

$$\widetilde{G}(t) = G(t)R(t)^{-\frac{1}{2}}\,,\quad \widetilde{C}(t) = Q^{\frac{1}{2}}(t)C(t)\,,$$

and replacing $W_r(t\,,t_0)$ by $W(t\,,t_0)$. ♦

Remark 8 *It is worth to notice that, since all the system matrices in* (4.1) *and the weighting matrices* $R(\cdot)$ *and* $Q(\cdot)$ *are assumed to be continuous, in the closed* *time interval* Ω *the condition* (4.9) *is equivalent to*

$$\|\Lambda\| = \max_{t\in\Omega} \lambda_{\max}^{\frac{1}{2}}\Big(Q^{\frac{1}{2}}(t)C(t)W(t\,,t_0)C^T(t)Q^{\frac{1}{2}}(t)\Big)\,.$$

▲

Lemma 3 *Given* $\epsilon > 0$*, the solution of the matrix differential equation*

$$\dot{W}_\epsilon(t\,,t_0) = A(t)W_\epsilon(t,t_0) + W_\epsilon(t,t_0)A^T(t) + G(t)R(t)^{-1}G^T(t) + \epsilon I\,, \tag{4.15a}$$

$$W_\epsilon(t_0\,,t_0) = \epsilon I \tag{4.15b}$$

is the positive definite matrix

$$W_\epsilon(t,t_0) = W(t\,,t_0) + \epsilon\Phi(t,t_0)\Phi^T(t,t_0) + \epsilon\int_{t_0}^{t}\Phi(t\,,\tau)\Phi^T(t\,,\tau)d\tau\,, \tag{4.16}$$

where $W(\cdot,\cdot)$ *is the solution of equations* (4.10)*.*

Proof. The proof follows from direct substitution of $W_\epsilon(\cdot,\cdot)$ in (4.15), and by the fact that the matrix $\Phi(t,t_0)\Phi^T(t,t_0)$ is positive definite. ♦

4.2 Necessary and sufficient conditions for IO-FTS of LTV systems

It is now possible to introduce the following theorem stating two necessary and sufficient conditions for the IO-FTS of system (4.1).

Theorem 6 *Given system* (4.1)*, the class of inputs* $\mathcal{W}_2$*, a continuous positive definite matrix-valued function* $Q(\cdot)$*, and the time interval* Ω*, the following statements are equivalent:*

i) *System* (4.1) *is IO-FTS with respect to* $\big(\mathcal{W}_2\,,Q(\cdot)\,,\Omega\big)$*.*

ii) *The inequality*

$$\lambda_{\max}\Big(Q^{\frac{1}{2}}(t)C(t)W(t\,,t_0)C^T(t)Q^{\frac{1}{2}}(t)\Big) < 1 \tag{4.17}$$

holds for all $t \in \Omega$, where $W(\cdot,\cdot)$ is the positive semidefinite solution of the DLE (4.10).

iii) *The coupled DLMI/LMI*

$$\begin{pmatrix} \dot{P}(t) + A^T(t)P(t) + P(t)A(t) & P(t)G(t) \\ G^T(t)P(t) & -R(t) \end{pmatrix} < 0 \tag{4.18a}$$

$$P(t) > C^T(t)Q(t)C(t)\,, \tag{4.18b}$$

admits a positive definite solution $P(\cdot)$ over Ω.

Proof. The equivalence of the three statements will be proved by showing that **i)** $\Rightarrow$ **ii)**, **ii)** $\Rightarrow$ **iii)**, and **iii)** $\Rightarrow$ **i)**.

$[$**i)** $\Rightarrow$ **ii)**$]$. The proof readily follows from Theorems 4 and 5, and from Remark 8.

$[$**ii)** $\Rightarrow$ **iii)**$]$. Given $\epsilon > 0$, consider the DLE (4.15), whose solution $W_\epsilon(\cdot,\cdot)$, given by (4.16), is positive definite and satisfies the DLMI

$$-\dot{W}_\epsilon(t,t_0) + A(t)W_\epsilon(t,t_0) + W_\epsilon(t,t_0)A^T(t) + G(t)R(t)^{-1}G^T(t) < 0\,. \tag{4.19}$$

Now letting

$$W_\epsilon(t,t_0) = P^{-1}(t)\,,$$

it follows that $\dot{W}_\epsilon(t,t_0) = -P^{-1}(t)\dot{P}(t)P^{-1}(t)$, and inequality (4.19) reads

$$P^{-1}(t)\dot{P}(t)P^{-1}(t) + A(t)P^{-1}(t) + P^{-1}(t)A^T(t) + G(t)R^{-1}(t)G^T(t) < 0\,, \tag{4.20}$$

for all $t \in \Omega$. By pre- and post-multiply (4.20) by $P(t)$ it is obtained

$$\dot{P}(t) + P(t)A(t) + A^T(t)P(t) + P(t)G(t)R^{-1}(t)G^T(t)P(t) < 0\,, \tag{4.21}$$

and (4.18a) readily follows by applying Schur complements[1].

[1]The matrix $\begin{pmatrix} J & K \\ K^T & L \end{pmatrix}$ is positive definite *if and only if* L is positive definite and $J - KL^{-1}K^T$ is positive definite. The matrix $J - KL^{-1}K^T$ is called the *Schur complement* of L.

In order to prove that (4.18b) holds, first note that $W_\epsilon(\cdot,\cdot) \overset{\epsilon\mapsto 0}{\longrightarrow} W(\cdot,\cdot)$, hence, by continuity arguments, there exists a sufficiently small ϵ such that

$$\lambda_{\max}\Big(Q^{\frac{1}{2}}(t)C(t)W_\epsilon(t,t_0)C^T(t)Q^{\frac{1}{2}}(t)\Big) < 1\,. \tag{4.22}$$

Furthermore, condition (4.22) is equivalent to

$$I - Q^{\frac{1}{2}}(t)C(t)P^{-1}(t)C^T(t)Q^{\frac{1}{2}}(t) > 0\,, \tag{4.23}$$

that, by applying Schur complements, reads

$$\begin{pmatrix} I & Q^{\frac{1}{2}}(t)C(t) \\ C^T(t)Q^{\frac{1}{2}}(t) & P(t) \end{pmatrix} > 0\,. \tag{4.24}$$

From [Amato, 2006, Lemma 5.3] inequality (4.24) is equivalent to

$$\begin{pmatrix} P(t) & C^T(t)Q^{\frac{1}{2}}(t) \\ Q^{\frac{1}{2}}(t)C(t) & I \end{pmatrix} > 0\,,$$

which yields (4.18b) by applying again Schur complements.

$\big[$**iii) $\Rightarrow$ i)**$\big]$. It has been already mentioned that, by applying Schur complements, condition (4.18a) is equivalent to (4.21). Now, let us consider the quadratic function $V(t,x) = x^T(t)P(t)x(t)$; the derivative with respect to time reads

$$\begin{aligned}\frac{d}{dt}\big(x^T(t)P(t)x(t)\big) &= x^T(t)\dot{P}(t)x(t) + \dot{x}^T(t)P(t)x(t) + x^T(t)P(t)\dot{x}(t) \\ &= x^T(t)\Big(\dot{P}(t) + A^T(t)P(t) + P(t)A(t)\Big)x(t) \\ &\qquad + w^T(t)G^T(t)P(t)x(t) + x^T(t)P(t)G(t)w(t)\,.\end{aligned}$$

Thus condition (4.21) implies that

$$\begin{aligned}\frac{d}{dt}\big(x^T(t)P(t)x(t)\big) < w^T(t)G^T(t)P(t)x(t) + x^T(t)P(t)G(t)w(t) \\ - x^T(t)P(t)G(t)R^{-1}(t)G^T(t)P(t)x(t)\,.\end{aligned}$$

Let $v(t) = R^{1/2}(t)w(t) - R^{-1/2}(t)G^T(t)P(t)x(t)$. Then

$$v^T(t)v(t) = w^T(t)R(t)w(t) + x^T(t)P(t)G(t)R^{-1}(t)G^T(t)P(t)x(t) \\ - w^T(t)G^T(t)P(t)x(t) - x^T(t)P(t)G(t)w(t)\,.$$

It follows that

$$\frac{d}{dt}\big(x^T(t)P(t)x(t)\big) < w^T(t)R(t)w(t) - v^T(t)v(t) \leq w^T(t)R(t)w(t)\,. \tag{4.26}$$

Integrating (6.6) between t_0 and $t \in \Omega$, taking into account that $x(t_0) = 0$ and that $w(\cdot)$ belongs to $\mathcal{W}_2$, it is obtained

$$x^T(t)P(t)x(t) \leq \int_{t_0}^{t} w^T(\sigma)R(\sigma)w(\sigma)d\sigma \leq \|w\|_{2,R}^2 \leq 1\,.$$

By exploiting condition (4.18b), it follows that

$$y^T(t)Q(t)y(t) = x^T(t)C^T(t)Q(t)C(t)x(t) < x^T(t)P(t)x(t) \leq 1\,,$$

for all $t \in \Omega$, hence system (4.1) is IO-FTS wrt $\big(\mathcal{W}_2\,, Q(\cdot)\,, \Omega\big)$. ♦

The following two corollaries deal with the special case in which the linear system (4.1) is time-invariant and the weighting matrices R and Q are constant. In this case conditions (4.17) and (4.18b) in Theorem 6 need to be checked only for $t = T$.

Corollary 2 *Given the time interval $\Omega := [0, T]$, two positive definite matrices $R \in \mathbb{R}^{r\times r}$ and $Q \in \mathbb{R}^{m\times m}$, assume that system* (4.1) *is time-invariant; then system* (4.1) *is IO-FTS with respect to* $\big(\mathcal{W}_2\,, Q\,, \Omega\big)$ if and only if

$$\lambda_{\max}\big(Q^{\frac{1}{2}}CW(T)C^TQ^{\frac{1}{2}}\big) < 1\,, \tag{4.27}$$

where W is the positive semidefinite solution of the differential matrix equation

$$-\dot{W}(t) + AW(t) + W(t)A^T + GR^{-1}G^T = 0\,, \quad W(0) = 0\,.$$

Proof. The proof readily follows from Theorem 6 taking into account the monotonicity of the reachability Gramian in the LTI case (see Remark 7). Indeed, if condition (4.27) is satisfied then

$$\lambda_{\max}\big(Q^{\frac{1}{2}}CW(t)C^TQ^{\frac{1}{2}}\big) < 1\,, \quad \forall\, t \in \Omega\,.$$

♦

Corollary 2 can be exploited to prove the following result.

Corollary 3 *Given the time interval* $\Omega := [0,T]$*, two positive definite matrices* $R \in \mathbb{R}^{r\times r}$ *and* $Q \in \mathbb{R}^{m\times m}$*, assume that system* (4.1) *is time-invariant. System* (4.1) *is IO-FTS with respect to* $\big(\mathcal{W}_2\,, Q\,, \Omega\big)$ if and only if *the DLMI with terminal condition*

$$\begin{pmatrix} \dot{P}(t) + A^TP(t) + P(t)A & P(t)G \\ G^TP(t) & -R \end{pmatrix} < 0\,, \qquad t \in \Omega \tag{4.28a}$$

$$P(T) > C^TQC\,, \tag{4.28b}$$

admits a positive definite solution $P(\cdot)$ *over* Ω*.* ▲

Remark 9 *Note that, even when the system is time-invariant, the solution of a DLMI is required in order to check IO-FTS of the given system. This is due to the finite time nature of the problem we are dealing with (see e.g. the optimal control problem defined over a finite horizon Anderson and Moore [1989]).* ▲

4.3 IO Finite-Time Stabilization via dynamic output feedback

In this section Theorem 6 is exploited to solve Problem 1 for LTV systems. In particular, a necessary and sufficient condition for the IO finite-time stabilization of system (4.1) via dynamic output feedback is provided in terms of a DLMI/LMI feasibility problem.

Theorem 7 *Problem 5 is solvable* if and only if *there exist two symmetric matrix-valued functions $S(\cdot)$, $T(\cdot)$, and four matrix-valued functions $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$ and $D_K(\cdot)$ such that the following DLMIs are satisfied*

$$\begin{pmatrix} \Theta_{11}(t) & \Theta_{12}(t) & 0 \\ \Theta_{12}^T(t) & \Theta_{22}(t) & T(t)G(t) \\ 0 & G^T(t)T(t) & -R(t) \end{pmatrix} < 0\,, \quad t \in \Omega \tag{4.29a}$$

$$\begin{pmatrix} \Psi_{11}(t) & \Psi_{12}(t) & 0 \\ \Psi_{12}^T(t) & S(t) & S(t)C^T(t) \\ 0 & C(t)S(t) & Q^{-1}(t) \end{pmatrix} > 0\,, \quad t \in \Omega \tag{4.29b}$$

where

$$\begin{aligned}
\Theta_{11}(t) &= -\dot{S}(t) + A(t)S(t) + S(t)A^T(t) + B(t)\hat{C}_K(t) \\
&\qquad + \hat{C}_K^T(t)B^T(t) + G(t)R^{-1}(t)G^T(t) \\
\Theta_{12}(t) &= A(t) + \hat{A}_K^T(t) + B(t)D_K(t)C(t) + G(t)R^{-1}(t)G^T(t)T(t) \\
\Theta_{22}(t) &= \dot{T}(t) + T(t)A(t) + A^T(t)T(t) + \hat{B}_K(t)C(t) + C^T(t)\hat{B}_K^T(t) \\
\Psi_{11}(t) &= T(t) - C^T(t)Q(t)C(t) \\
\Psi_{12}(t) &= I - C^T(t)Q(t)C(t)S(t)\,.
\end{aligned}$$

Proof. From Theorem 6 it readily follows that system (2.14a, 2.14c) is IO-FTS wrt $(\mathcal{W}_2\,, Q(\cdot)\,, \Omega)$ if and only if there exists a symmetric matrix-valued function $P(\cdot)$ such that, for all $t \in \Omega$

$$\dot{P}(t) + A_{\mathrm{CL}}^T(t)P(t) + P(t)A_{\mathrm{CL}}(t) + P(t)G_{\mathrm{CL}}(t)R(t)^{-1}G_{\mathrm{CL}}^T(t)P(t) < 0\,, \tag{4.30a}$$

$$P(t) > C^T(t)Q(t)C(t)\,. \tag{4.30b}$$

Given two symmetric matrix-valued functions $S(\cdot)$ and $T(\cdot)$, according to [Amato, 2006, Lemma 5.1], consider a symmetric matrix-valued function $U(\cdot)$ and two nonsingular matrix-valued functions $M(\cdot)$ and $N(\cdot)$ such that

$$P(t) = \begin{pmatrix} T(t) & M(t) \\ M^T(t) & U(t) \end{pmatrix}, \; P^{-1}(t) = \begin{pmatrix} S(t) & N(t) \\ N^T(t) & \star \end{pmatrix}.$$

Furthermore, the following matrices are defined

$$\Pi_1(t) = \begin{pmatrix} S(t) & I \\ N^T(t) & 0 \end{pmatrix}, \quad \Pi_2(t) = \begin{pmatrix} I & T(t) \\ 0 & M^T(t) \end{pmatrix}.$$

Note that, by definition,

$$T(t)S(t) + M(t)N^T(t) = I \tag{4.31a}$$

$$S(t)\dot{T}(t)S(t) + N(t)\dot{M}^T(t)S(t) + S(t)\dot{M}(t)N^T(t) + N(t)\dot{U}(t)N^T(t) = -\dot{S}(t) \tag{4.31b}$$

$$P(t)\Pi_1(t) = \Pi_2(t) \tag{4.31c}$$

where equality (4.31b) can be easily applied since $\dot{P}^{-1}(t) = -P^{-1}(t)\dot{P}(t)P^{-1}(t)$.

It is now proved that, with the given choice of $P(t)$, conditions (4.30) are equivalent to (4.29). Indeed, by pre- and post-multiplying (4.30a)-(4.30b) by $\Pi_1^T(t)$ and $\Pi_1(t)$, respectively, and taking into account (4.31) and Lemma 5.1 in Amato [2006], the proof follows once we let

$$\begin{pmatrix} S(t) & I \\ I & T(t) \end{pmatrix} > 0 \tag{4.32a}$$

$$\hat{B}_K(t) = M(t)B_K(t) + T(t)B(t)D_K(t) \tag{4.32b}$$

$$\hat{C}_K(t) = C_K(t)N^T(t) + D_K(t)C(t)S(t) \tag{4.32c}$$

$$\hat{A}_K(t) = \dot{T}(t)S(t) + \dot{M}(t)N^T(t) + M(t)A_K(t)N^T(t) + T(t)B(t)C_K(t)N^T(t) + M(t)B_K(t)C(t)S(t) + T(t)\big(A(t) + B(t)D_K(t)C(t)\big)S(t)\,. \tag{4.32d}$$

Note that (4.32a) does not need to be explicitly imposed since it is implied by (4.29b). ♦

Remark 10 (Controller design) *Assuming that the hypotheses of Theorem 7* *are satisfied, in order to design the controller, the following steps have to be followed:*

i) Find $S(\cdot)$, $T(\cdot)$, $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$ *and* $D_K(\cdot)$ *such that* (4.29) *are satisfied.*

ii) Let $N(\cdot)$ *be any nonsingular matrix-valued function (e.g.,* $N(t) = I$ *for all* $t \in \Omega$*), and let*

$$M(t) = \big[I - T(t)S(t)\big]N^{-T}(t)\,.$$

iii) Obtain $A_K(\cdot)$, $B_K(\cdot)$ *and* $C_K(\cdot)$ *by inverting* (4.32). *It is important to remark that, in order to invert* (4.32), *we need to preliminarily choose* $N(\cdot)$. *The only constraint for* $N(\cdot)$ *is to be a non singular matrix.* ▲

As it has been done in Section 4.2, starting from Theorem 7 it is possible to derive the following necessary and sufficient condition for the solution of Problem 5 when the case of linear time-invariant systems and constant weighting matrices is considered.

Corollary 4 *Given the time interval* $\Omega := [0, T]$, *two positive definite matrices* $R \in \mathbb{R}^{r\times r}$ *and* $Q \in \mathbb{R}^{m\times m}$, *assume that system* (4.1) *is time-invariant. Problem 5 is solvable* if and only if *there exist two symmetric matrix-valued functions* $S(\cdot)$, $T(\cdot)$, *and four matrix-valued functions* $\hat{A}_K(\cdot)$, $\hat{B}_K(\cdot)$, $\hat{C}_K(\cdot)$ *and* $D_K(\cdot)$ *such that the following DLMI* (6.26a) *with terminal condition is satisfied*

$$\begin{pmatrix} \widehat{\Theta}_{11}(t) & \widehat{\Theta}_{12}(t) & 0 \\ \widehat{\Theta}_{12}^T(t) & \widehat{\Theta}_{22}(t) & T(t)G \\ 0 & G^T T(t) & -R \end{pmatrix} < 0\,, \quad t \in \Omega$$

$$\begin{pmatrix} \widehat{\Psi}_{11}(T) & \widehat{\Psi}_{12}(T) & 0 \\ \widehat{\Psi}_{12}^T(T) & S(T) & S(T)C^T \\ 0 & CS(T) & Q^{-1} \end{pmatrix} > 0\,,$$

where

$$
\begin{aligned}
\widehat{\Theta}_{11}(t) &= -\dot{S}(t) + AS(t) + S(t)A^T + B\hat{C}_K(t) + \hat{C}_K^T(t)B^T + GR^{-1}G^T \\
\widehat{\Theta}_{12}(t) &= A + \hat{A}_K^T(t) + BD_K(t)C + GR^{-1}G^T T(t) \\
\widehat{\Theta}_{22}(t) &= \dot{T}(t) + T(t)A + A^T T(t) + \hat{B}_K(t)C + C^T \hat{B}_K^T(t) \\
\widehat{\Psi}_{11}(t) &= T(t) - C^T QC \\
\widehat{\Psi}_{12}(t) &= I - C^T QCS(t)\,.
\end{aligned}
$$

Proof. The proof readily follows from Theorem 7 and Corollary 3.

Chapter 5

IO finite-time stabilization with constraint on the control input

This chapter extends the results on the IO finite-time stabilization given in Chapter 4 by introducing the possibility to insert constraints on the control input. Preliminary results on the same topic has been presented in Amato et al. [2012a].

In practical situations, the controller should be designed with the constraint of limiting the effort of the control variables. Starting from this motivating example, this chapter deals with the state feedback IO-FTS problem with constrained control inputs. To achieve this goal a fictitious system is built, in which the output vector is augmented with the control input variables, which are conceptually dealt with in the same way as the actual outputs. However, since outputs and control inputs need to be constrained separately, the definition given in Amato et al. [2010a] is extended to that one of *structured* IO-FTS. As a by-product, this new definition gives the possibility of imposing different constraints on distinct groups of output variables.

A necessary and sufficient condition ($\mathcal{L}_2$ inputs) and a sufficient condition ($\mathcal{L}_\infty$ inputs) for structured IO-FTS (open loop system) will be firstly given. Then, a necessary and sufficient condition and a sufficient condition for IO finite-time stabilization with constrained control inputs will be stated in the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ context, respectively. In the last case the situation in which the feedthrough matrix between the disturbance and the output is non-zero will be also discussed.

5.1 Structured IO-FTS and problem statement

The IO finite-time stabilization problem, as defined in Chapter 2, does not allow to effectively deal with constraints on the control variables. This section shows how to modify the definition of IO-FTS in order to take into account, during the design phase, such kind of control requirements, for LTV systems in the form

$$\dot{x}(t) = A(t)x(t) + G(t)w(t)\,, \quad x(t_0) = 0 \tag{5.1a}$$
$$y(t) = C(t)x(t) + F(t)w(t)\,, \tag{5.1b}$$

where $A(\cdot) : \Omega \mapsto \mathbb{R}^{n\times n}$, $G(\cdot) : \Omega \mapsto \mathbb{R}^{n\times r}$, $C(\cdot) : \Omega \mapsto \mathbb{R}^{m\times n}$, $F(\cdot) : \Omega \mapsto \mathbb{R}^{m\times r}$.

To this end, the concept of *structured* IO-FTS is introduced, which generalizes the original definition of IO-FTS given in Amato et al. [2010a]. Given an α-tuple of integer numbers $m_1, \ldots, m_\alpha$, where $\sum_{i=1}^{\alpha} m_i = m$, the output vector can be partitioned as

$$y(t) = \left(y_1^T(t) \cdots y_\alpha^T(t)\right)^T \,, \quad t \in \Omega\,. \tag{5.2}$$

Note that the output partition (5.2) induces a partition of the output equation matrices

$$C(t) = \left(C_1^T(t) \cdots C_\alpha^T(t)\right)^T$$
$$F(t) = \left(F_1^T(t) \cdots F_\alpha^T(t)\right)^T \,.$$

In the original definition of IO-FTS given in Amato et al. [2010a] the output weighting is a symmetric, positive definite matrix belonging to the space $\mathbb{R}^{m\times m}$. Here, the weighting matrix is defined as

$$Q(t) := \mathrm{diag}(Q_1(t), \ldots, Q_\alpha(t))\,, \tag{5.3}$$

where $Q_i(t) \in \mathbb{R}^{m_i\times m_i}$, $i = 1, \ldots, \alpha$ are positive definite matrices.

The following definition of *structured* IO-FTS of LTV systems can be now introduced.

Definition 4 (Structured IO-FTS) *Given a positive scalar T, a class of input* *signals* $\mathcal{W}$ *defined over* Ω*, the weighting matrix* $Q(\cdot)$ *defined in* (5.3)*, system* (5.1) *is said to be structured IO-FTS with respect to* $(\mathcal{W}\,, Q(\cdot)\,, \Omega)$ *if*

$$w(\cdot) \in \mathcal{W} \;\Rightarrow\; y_i^T(t)Q_i(t)y_i(t) < 1\,, \quad t \in \Omega\,, \quad i = 1, \dots, \alpha\,.$$

▲

Given Definition 4, it is straightforward to note that the classical definition of IO-FTS given in Amato et al. [2010a] can be obtained letting $\alpha = 1$.

The first results of this paper, namely some conditions guaranteeing that a given system is structured IO-FTS, will be given in Section 5.2.

The related design problem, i.e. the structured IO finite-time stabilization via state feedback, will be dealt with in Section 5.3. To state precisely this problem, consider system (5.1) and correspondingly, given a β-tuple of integer numbers $q_1, \dots, q_\beta$, where $\sum_{i=1}^{\beta} q_i = q$, partition the control input vector as

$$u(t) = \left(u_1^T(t) \cdots u_\beta^T(t)\right)^T\,, \quad t \in \Omega\,;$$

correspondingly consider β positive definite weighting matrices $T_i(t) \in \mathbb{R}^{q_i \times q_i}$, $i = 1, \dots, \beta$; define

$$T(t) := \mathrm{diag}(T_1(t), \dots, T_\beta(t))\,. \tag{5.4}$$

The following partition of $D(\cdot)$ is induced by (5.2)

$$D(t) = \left(D_1^T(t) \cdots D_\alpha^T(t)\right)^T\,.$$

Problem 6 (Structured IO FT stabilization) *Given a positive scalar* T*, the class of signals* $\mathcal{W}$*, and the weighting matrices* $Q(\cdot)$*,* $T(\cdot)$ *defined in* (5.3) *and* (5.4) *respectively, find a state feedback control law*

$$u(t) = K(t)x(t)\,,$$

where $K(\cdot):\ \Omega \mapsto \mathbb{R}^{q\times n}$*, such that the system*

$$\begin{aligned}
\dot{x}(t) &= \left(A(t)+B(t)K(t)\right)x(t)+G(t)w(t) \\
&=: A_{cl}x(t)+G(t)w(t)\,, \quad x(t_0)=0 \qquad (5.5a)\\
\begin{pmatrix} y(t) \\ u(t) \end{pmatrix} &= \begin{pmatrix} C(t)+D(t)K(t) \\ K(t) \end{pmatrix} x(t) + \begin{pmatrix} F(t) \\ 0 \end{pmatrix} w(t) \\
&= \begin{pmatrix} C_1(t)+D_1(t)K(t) \\ \vdots \\ C_\alpha(t)+D_\alpha(t)K(t) \\ K_1(t) \\ \vdots \\ K_\beta(t) \end{pmatrix} x(t) + \begin{pmatrix} F_1(t) \\ \vdots \\ F_\alpha(t) \\ 0 \end{pmatrix} w(t) \qquad (5.5b)
\end{aligned}$$

is structured IO-FTS with respect to $(\mathcal{W}, \operatorname{diag}\left(Q(\cdot),T(\cdot)\right), \Omega)$. ▲

Finally note that the partition (5.4) induces the following structure for the controller gain

$$K(t) = \left(K_1^T(t)\cdots K_\beta^T(t)\right)^T\,, \quad t\in\Omega\,. \qquad (5.6)$$

Remark 11 *As already mentioned, the concept of structured IO-FTS allows imposing amplitude constraints on both output and the control variables when solving the synthesis problem. In particular, starting from the definition given above it is possible to limit the state and the control variables inside a polyhedral set, that is*

$$w\in\mathcal{W} \Rightarrow \|H_1(t)x(t)\|_\infty < 1 \text{ and } \|H_2(t)u(t)\|_\infty < 1\,, \forall\, t\in\Omega\,. \qquad (5.7)$$

where $H_1(\cdot)\in\mathbb{R}^{m\times n}$ *and* $H_2(\cdot)\in\mathbb{R}^{p\times q}$*. Letting* $y(t) = H_1(t)x(t)$ *and* $v(t) = H_2(t)u(t)$*, the inequalities in* (5.7) *are equivalent to*

$$w\in\mathcal{W} \Rightarrow |y_i(t)|<1\,,\ i=1,\dots,m \quad \text{and } |v_j(t)|<1\,, j=1,\dots,p\,, \quad \forall\, t\in\Omega\,.$$

▲

5.2 Structured IO-FTS analysis

The case in which the feedthrough matrix is null will be dealt with firstly, then the case of a generic feedthrough matrix will be addressed.

5.2.1 The case $F(\cdot) = 0$

This section extends two results originally given in Amato et al. [2010a] and Amato et al. [2012b] to the case of structured IO-FTS given in Definition 4, when $F(\cdot) = 0$. The former states a necessary and sufficient condition to check structured IO-FTS of system (5.1) when $\mathcal{W}_2$ signals are considered, while the latter is a sufficient condition for structured IO-FTS when dealing with inputs that belong to $\mathcal{W}_\infty$.

Theorem 8 *Given system* (5.1) *with* $F(\cdot) = 0$*, the class of inputs* $\mathcal{W}_2$*, a continuous positive definite matrix-valued function* $Q(\cdot) := \mathrm{diag}(Q_1(\cdot), \dots, Q_\alpha(\cdot))$*, and the time interval* Ω*, system* (5.1) *is structured IO-FTS with respect to* $\big(\mathcal{W}_2\,, Q(\cdot)\,, \Omega\big)$ if and only if *the coupled DLMI/LMI*

$$\begin{pmatrix} \dot{P}(t) + A^T(t)P(t) + P(t)A(t) & P(t)G(t) \\ G^T(t)P(t) & -R(t) \end{pmatrix} < 0 \tag{5.8a}$$

$$P(t) \geq C_i^T(t)Q_i(t)C_i(t)\,, \quad i = 1\,, \dots\,, \alpha\,, \tag{5.8b}$$

admits a positive definite solution $P(\cdot)$ *over* Ω.

Proof. Given the output partition (5.2), system (5.1) can be considered as a collection of α fictitious systems with the same state equation (5.1a), and output equation given by

$$y_i(t) = C_i(t)x(t)\,,$$

for each $i = 1\,, \dots\,, \alpha$. The proof of the theorem readily follows by considering the result given in Theorem 6, for each one of the α fictitious systems. It is worth noticing that condition (5.8a) is not affected by the output partition, since it involves only the state equation matrix-valued functions. ♦

Theorem 9 *Given the class of inputs $\mathcal{W}_\infty$, the time interval Ω, consider the weighting matrix $\widetilde{Q}_i(t) = (t - t_0)Q_i(t)$, $i = 1, \ldots, \alpha$, and assume that the coupled DLMI/LMI*

$$\begin{pmatrix} \dot{P}(t) + A^T(t)P(t) + P(t)A(t) & P(t)G(t) \\ G^T(t)P(t) & -R(t) \end{pmatrix} < 0 \tag{5.9a}$$

$$P(t) \geq C_i^T(t)\widetilde{Q}_i(t)C_i(t)\,, \quad i = 1, \ldots, \alpha \tag{5.9b}$$

admits a positive definite solution $P(\cdot)$ over Ω, then system (5.1)*, with $F(\cdot) = 0$, is structured IO-FTS with respect to $(\mathcal{W}_\infty, Q(\cdot), \Omega)$.*

Proof. In Amato et al. [2010a] it has been proved that condition (5.9a) implies

$$x^T(t)P(t)x(t) < t - t_0\,.$$

Furthermore, from (5.9b) it follows that

$$\begin{aligned} y_i^T(t)Q_i(t)y_it) &= x^T(t)C_i^T(t)Q_i(t)C_i(t)x(t) \\ &\leq \frac{1}{t-t_0}x^T(t)P(t)x(t) < 1\,, \end{aligned}$$

for $i = 1, \ldots, \alpha$, hence (5.1) is structured IO-FTS with respect to $(\mathcal{W}_\infty, Q(\cdot), \Omega)$.
♦

5.2.2 Extension to the case $F(\cdot) \neq 0$

This section addressed the case of $F(\cdot) \neq 0$. It should be noticed that only the case of $\mathcal{W}_\infty$ exogenous signals can be dealt with, since the concept of IO-FTS with respect to $\mathcal{W}_2$ is ill posed. Indeed, it is straightforward that $\mathcal{W}_2$ includes signals that are unbounded on a zero measure interval included in Ω. When $F(\cdot) \neq 0$, in presence of such input signals, it readily follows that there exists at least one time instant where the output would be unbounded. Hence, when $F(\cdot) \neq 0$, system (5.1) cannot be IO-FTS with respect to $\mathcal{W}_2$.

In order to solve Problem 6 when $F(\cdot) \neq 0$ for $\mathcal{W}_\infty$ signals, Theorem 9 has to be extended to non-strictly proper systems. To do that, the following lemma will be used.

Lemma 4 *Given system* (5.1)*, the weighting matrix* $Q(\cdot)$ *defined in* (5.3)*,* $t \in \Omega$*, the condition*

$$w(\cdot) \in \mathcal{W} \Rightarrow y_i^T(t)Q_i(t)y_i(t) < 1\,,\; i = 1, \dots, \alpha$$

is satisfied if *there exist a positive definite matrix-valued function* $P(\cdot)$ *and* α *scalar functions* $\theta_i(\cdot) > 1$*,* $i = 1, \dots, \alpha$*, such that*

$$\dot{P}(\tau) + A^T(\tau)P(\tau) + P(\tau)A(\tau) + P(\tau)G(\tau)R^{-1}(\tau)G^T(\tau)P(\tau) < 0\,, \quad \tau \in]t_0\,, t] \tag{5.10a}$$

$$\theta_i(t)R(t) - R(t) \geq 2\,\theta_i(t)F_i^T(t)Q_i(t)F_i(t)\,, \quad i = 1\,, \dots\,, \alpha \tag{5.10b}$$

$$P(t) \geq 2\,\theta_i(t)C_i^T(t)\widetilde{Q}_i(t)C_i(t)\,, \quad i = 1\,, \dots\,, \alpha\,, \tag{5.10c}$$

where $\widetilde{Q}_i(t) = (t - t_0)\,Q_i(t)$*.*

Proof. The case $t > t_0$ will be considered first; in Amato et al. [2010a] it has been proven that (5.10a) implies

$$x^T(t)P(t)x(t) < t - t_0\,. \tag{5.11}$$

Given t in Ω, it is

$$\begin{aligned} y_i(t)^T Q_i(t)y_i(t) = {} & x^T(t)C_i^T(t)Q_i(t)C_i(t)x(t) + w^T(t)F_i^T(t)Q_i(t)F_i(t)w(t) \\ & + x^T(t)C_i^T(t)Q_i(t)F_i(t)w(t) + w^T(t)F_i^T(t)Q_i(t)C_i(t)x(t)\,, \end{aligned} \tag{5.12}$$

for all $i \in \{1\,, \dots\,, \alpha\}$. Now let

$$v_i(t) = \left(Q_i(t)^{-\frac{1}{2}}C_i(t)x(t) - Q_i(t)^{-\frac{1}{2}}F_i(t)w(t)\right)\,,$$

then (the time argument is omitted for brevity)

$$v_i^T v_i = x^T C_i^T Q_i C_i x + w^T F_i^T Q_i F_i w - x^T C_i^T Q_i F_i w - w^T F_i^T Q_i C_i x\,,$$

which can be rewritten as

$$x^T C_i^T Q_i F_i w + w^T F_i^T Q_i C_i x = x^T C_i^T Q_i C_i x + w^T F_i^T Q_i F_i w - v_i^T v_i \,. \tag{5.13}$$

Replacing (5.13) in (5.12), it holds

$$\begin{aligned} y_i^T Q_i y_i &= 2x^T C_i^T Q_i C_i x + 2w^T F_i^T Q_i F_i w - v_i^T v_i \\ &< 2\left(x^T C_i^T Q_i C_i x + w^T F_i^T Q_i F_i w\right) \,, \end{aligned}$$

and conditions (5.10b) and (5.10c) imply that

$$y_i^T Q_i y_i < \left(\frac{1}{\theta_i}\frac{x^T P x}{t-t_0} + \frac{\theta_i - 1}{\theta_i} w^T R w\right) \,.$$

Now exploiting (5.11), and recalling that $w(\cdot) \in \mathcal{W}_\infty$ implies that $\|w\|_{\infty,R} \leq 1$, it is

$$y_i^T(t) Q_i(t) y_i(t) < 1 \,, \quad t \in]t_0\,, t_0 + T] \,, \quad \forall\, i \in \{1\,, \dots\,, \alpha\} \,.$$

In the case that $t = t_0$, since the initial state $x(t_0)$ is zero, it is straightforward to prove that condition (5.10b) is sufficient to conclude that

$$y_i^T(t_0) Q_i(t_0) y_i(t_0) < 1$$

for $i = 1\,, \dots\,, \alpha$. ♦

In order to check structured IO-FTS of system (5.1), Lemma 4 would require to check the feasibility of infinitely many optimization problems (one for each t in Ω), which is obviously an impossible task. However, by exploiting similar arguments as in Amato et al. [2010a], it is possible to prove the following theorem, which requires to check a single DLMI feasibility problem.

Theorem 10 *Let $\widetilde{Q}_i(t) = (t - t_0) Q_i(t)$; if there exist a positive definite and continuously differentiable matrix-valued function $P(\cdot)$ and α scalar functions $\theta_i(\cdot)$,*

$i = 1, \ldots, \alpha$, *such that the coupled DLMI/LMI*

$$\begin{pmatrix} \dot{P}(t) + A^T(t)P(t) + P(t)A(t) & P(t)G(t) \\ G^T(t)P(t) & -R(t) \end{pmatrix} < 0\,, \tag{5.14a}$$

$$\theta_i(t)R(t) - R(t) \geq 2\,\theta_i(t)F_i^T(t)Q_i(t)F_i(t)\,, \quad i = 1\,, \ldots\,, \alpha\,, \tag{5.14b}$$

$$P(t) \geq 2\,\theta_i(t)C_i(t)^T\widetilde{Q}_i(t)C_i(t)\,, \;\; i = 1\,, \ldots\,, \alpha\,, \tag{5.14c}$$

are fulfilled over Ω, *then system* (5.1) *is IO-FTS with respect to* $(\mathcal{W}_\infty\,, Q(\cdot)\,, \Omega)$. ▲

5.3 IO finite-time stabilization with control input contraint

This section proposes a number of results to solve Problem 6. First of all, the case in which $F(\cdot) = 0$ in system (5.1) is considered for both $\mathcal{W}_2$ and $\mathcal{W}_\infty$ disturbances. Eventually, the case in which $F(\cdot) \neq 0$ is handled, and a sufficient condition to solve Problem 6 with respect to $\mathcal{W}_\infty$ disturbances is given.

It is worth to mention that it can always be considered $D(\cdot) \neq 0$, since the control action $u(t) = K(t)x(t)$ is bounded in Ω.

If $F(\cdot) = 0$ in Ω, then the following necessary and sufficient condition holds.

Theorem 11 *Given the class of disturbances* $\mathcal{W}_2$ *and* $F(\cdot) = 0$, *Problem 6 is solvable* if and only if *there exist a positive definite and continuously differentiable matrix-valued function* $\Pi(\cdot)$, *and* β *continuously differentiable matrix-valued functions* $L_1(\cdot)\,, \ldots\,, L_\beta(\cdot)$ *such that,*

$$\begin{pmatrix} \Theta(t) & G(t) \\ G^T(t) & -R(t) \end{pmatrix} < 0\,, \tag{5.15a}$$

$$\begin{pmatrix} \Pi(t) & \Pi(t)C_i^T(t) + \left(L_1^T(t)\cdots L_\beta^T(t)\right) D_i^T(t) \\ C_i(t)\Pi(t) + D_i(t)\left(L_1^T(t)\cdots L_\beta^T(t)\right)^T & \Xi_i(t) \end{pmatrix} \geq 0\,, \quad i = 1\,, \ldots\,, \alpha \tag{5.15b}$$

$$\begin{pmatrix} \Pi(t) & L_j^T(t) \\ L_j(t) & \Upsilon_j(t) \end{pmatrix} \geq 0\,, \qquad j = 1\,, \ldots\,, \beta \tag{5.15c}$$

for all $t \in \Omega$*, with*

$$\Theta(t) := -\dot{\Pi}(t) + \Pi(t)A^T(t) + A(t)\Pi(t) + B(t)\left(L_1^T(t)\cdots L_\beta^T(t)\right)^T + \left(L_1^T(t)\cdots L_\beta^T(t)\right)B^T(t)\,,$$

$\Xi_i(t) := Q_i^{-1}(t)$*, and* $\Upsilon_j(t) := T_j^{-1}(t)$*. The controller gain which solves Problem 6 for the input class* $\mathcal{W}_2$ *is given by* (5.6) *with* $K_j(t) = L_j(t)\Pi^{-1}(t)$*,* $j = 1\,,\ldots\,,\beta$*.*

Proof. Conditions (5.8) for the augmented output closed-loop system (5.5) read

$$\begin{pmatrix} \dot{P}(t) + A_{cl}^T(t)P(t) + P(t)A_{cl}(t) & P(t)G(t) \\ G^T(t)P(t) & -R(t) \end{pmatrix} < 0\,, \tag{5.16a}$$

$$P(t) \geq \left(C_i^T(t) + K^T(t)D_i^T(t)\right)Q_i(t)\left(C_i(t) + D_i(t)K(t)\right)\,,\ i = 1\,,\ldots\,,\alpha \tag{5.16b}$$

$$P(t) \geq K_j^T(t)T_j(t)K_j(t)\,,\quad j = 1\,,\ldots\,,\beta\,. \tag{5.16c}$$

Let $\Pi(t) = P^{-1}(t)$. Pre- and post-multiplying (5.16a) by $\begin{pmatrix} \Pi(t) & 0 \\ 0 & I \end{pmatrix} > 0$, and pre- and post-multiplying (5.16b) and (5.16c) by $\Pi(t)$, it is

$$\begin{pmatrix} -\dot{\Pi}(t) + \Pi(t)A_{cl}^T(t) + A_{cl}(t)\Pi(t) & G(t) \\ G^T(t) & -R(t) \end{pmatrix} < 0\,, \tag{5.17a}$$

$$\begin{pmatrix} \Pi(t) & \Pi(t)C_i^T(t) + \Pi(t)K^T(t)D_i^T(t) \\ C_i(t)\Pi(t) + D_i(t)K(t)\Pi(t) & \Xi_i(t) \end{pmatrix} \geq 0\,, \quad i = 1\,,\ldots\,,\alpha \tag{5.17b}$$

$$\begin{pmatrix} \Pi(t) & \Pi(t)K_j^T(t) \\ K_j(t)\Pi(t) & \Upsilon_j(t) \end{pmatrix} \geq 0\,,\quad j = 1\,,\ldots\,,\beta \tag{5.17c}$$

where (5.17b) and (5.17c) are obtained by applying the Schur complements. The proof of the theorem then readily follows by letting $L_j(t) = K_j(t)\Pi(t)$ for $j = 1\,,\ldots\,,\beta$. ♦

Exploiting similar arguments as in the previous proof, starting from Theorem 9 it is possible to derive the following sufficient condition to solve Problem 6 in the case of $\mathcal{W}_\infty$ disturbances.

Theorem 12 *Given the class of disturbances* $\mathcal{W}_\infty$ *and* $F(\cdot) = 0$*, Problem 6 is solvable* if *there exist a positive definite and continuously differentiable matrix-valued function* $\Pi(\cdot)$*, and* β *continuously differentiable matrix-valued functions* $L_1(\cdot), \ldots, L_\beta(\cdot)$ *such that* (5.15a) *and*

$$\begin{pmatrix} \Pi(t) & \Pi(t)C_i^T(t) + \left(L_1^T(t) \cdots L_\beta^T(t)\right) D_i^T(t) \\ C_i(t)\Pi(t) + D_i(t)\left(L_1^T(t) \cdots L_\beta^T(t)\right)^T & \widetilde{\Xi}_i(t) \end{pmatrix} \geq 0\,,$$
$$i = 1, \ldots, \alpha, \quad t \in]t_0, t_0 + T] \tag{5.18a}$$
$$\begin{pmatrix} \Pi(t) & L_j^T(t) \\ L_j(t) & \widetilde{\Upsilon}_j(t) \end{pmatrix} \geq 0\,, \qquad j = 1, \ldots, \beta, \quad t \in]t_0, t_0 + T] \tag{5.18b}$$

hold, with $\widetilde{\Xi}_i(t) := ((t-t_0)Q_i(t))^{-1}$*, and* $\widetilde{\Upsilon}_j(t) := ((t-t_0)T_j(t))^{-1}$*. A controller gain which solves Problem 6 for the input class* $\mathcal{W}_\infty$ *is given by* (5.6) *with* $K_j(t) = L_j(t)\Pi^{-1}(t)$*,* $j = 1, \ldots, \beta$*.* ▲

Finally, Theorem 10 can be exploited to solve Problem 6 when $F(\cdot) \neq 0$. Indeed, the following results holds.

Theorem 13 *Given the class of disturbances* $\mathcal{W}_\infty$*, Problem 6 is solvable* if *there exist a positive definite and continuously differentiable matrix-valued function* $\Pi(\cdot)$*,* β *continuously differentiable matrix-valued functions* $L_1(\cdot), \ldots, L_\beta(\cdot)$*, and* α *strictly positive functions* $\lambda_1(\cdot), \ldots, \lambda_\alpha(\cdot) < 1$ *such that* (5.15a) *and*

$$R(t) - \lambda_i(t)R(t) \geq 2\,F_i^T(t)Q_i(t)F_i(t)\,, \qquad i = 1, \ldots, \alpha \tag{5.19a}$$
$$\begin{pmatrix} \Pi(t) & \Pi(t)C_i^T(t) + \left(L_1^T(t) \cdots L_\beta^T(t)\right) D_i^T(t) \\ C_i(t)\Pi(t) + D_i(t)\left(L_1^T(t) \cdots L_\beta^T(t)\right)^T & \frac{\lambda_i(t)}{2}\,\widetilde{\Xi}_i(t) \end{pmatrix} \geq 0$$
$$i = 1, \ldots, \alpha \tag{5.19b}$$
$$\begin{pmatrix} \Pi(t) & L_j^T(t) \\ L_j(t) & \widetilde{\Upsilon}_j(t) \end{pmatrix} \geq 0\,, \qquad j = 1, \ldots, \beta \tag{5.19c}$$

hold, when $t \in \Omega$*, with* $\widetilde{\Xi}_i(t) := ((t-t_0)Q_i(t))^{-1}$*, and* $\widetilde{\Upsilon}_j(t) := ((t-t_0)T_j(t))^{-1}$*. A controller gain which solves Problem 6 for the input class* $\mathcal{W}_\infty$ *is given by* (5.6) *with* $K_j(t) = L_j(t)\Pi^{-1}(t)\,, j = 1, \ldots, \beta$*.*

Proof. The proof can be derived exploiting the same arguments used in Theorem 11, letting $\lambda_i(\cdot) = \theta_i^{-1}(\cdot)$ in Ω and noticing that there is not a direct link between the disturbance vector $w(\cdot)$ and the fictitious outputs $u_j(\cdot)$, $j = 1, \dots, \beta$.

Chapter 6

IO-FTS in the context of hybrid systems

This chapter presents the results obtained in the context of IO-FTS for hybrid systems. The first section introduces the results obtained in the case of Time-Dependent (TD) and State-Dependent Impulsive Dynamical Linear Systems (SD-IDLS), which has been extracted from Amato et al. [2011b]. IDLS are linear continuous-time systems whose state undergoes finite jump discontinuities at discrete instants of time. For this class of hybrid systems, first the conditions guaranteeing the IO-FTS are presented, then the same conditions are exploited for the design of state and static output feedback controllers. Section 6.2, whose results have been presented in Amato et al. [2011d], deals with the class of Time-Dependent Switching Linear Systems (TD-SLS). In this case, the very important case in which the set of resetting times is unknown is addressed. The main advantage of this result is the possibility to tackle real engineering problems, where the change of system dynamics is unpredictable and/or is due to an external triggering event.

6.1 IO Finite Time Stabilization for Time and State-Dependent IDLS

The main results of this section are sufficient conditions which guarantee that a given IDLS is IO-FTS over a specified time interval, for the two different input classes introduced in Section 2.2. Furthermore, the problem of IO finite-time stabilization via static output feedback is also tackled. As it will be shown, the analysis problem can be framed into the Linear Matrix Inequalities (LMIs Boyd et al. [1994]) framework, while the static output feedback problem can be tackled by solving a Bilinear Matrix Inequalities (BMIs, VanAntwerp and Braatz [2000]) feasibility problem. Although the latter problem is well known to be NP-hard, it can be effectively solved by means of off-the-shelf optimization softwares (e.g., Henrion et al. [2005]).

The formal definition of IDLS is firstly given in Section 6.1.2; then sufficient conditions for the analysis of IO-FTS are provided in Section 6.1.3, for both time-dependent and state-dependent IDLS. In Section 6.1.4 sufficient conditions to solve the stabilisation problem via static output feedback are presented. Before introducing the main results, the so called S-procedure (Jakubovic [1977]) is recalled in the next section, as it will be used in this chapter.

6.1.1 S-procedure

S-procedure will be used to check whether, given a connected and closed set $\mathcal{X} \subseteq \mathbb{R}^n$ and a symmetric matrix $E \in \mathbb{R}^{n\times n}$, the inequality

$$x^T E x < 0\,, \quad x \in \mathcal{X} \setminus \{0\}\,, \tag{6.1}$$

is satisfied. S-procedure allows to recast inequality (6.1) in the LMIs framework.

Lemma 5 ($\mathcal{S}$-procedure) *Let $E\,, F_1\,, \dots\,, F_p \in \mathbb{R}^{n\times n}$ be $p+1$ symmetric matrices. Consider the following condition on $E\,, F_1\,, \dots\,, F_p$*

$$x^T E x < 0 \quad \forall x : x \neq 0 \wedge x^T F_i x \leq 0\,, \quad i = 1, \dots, p\,. \tag{6.2}$$

If

$$\exists\, c_i \in \mathbb{R}\,, c_i \geq 0\,,\ i = 1\,, \ldots\,, p \text{ such that } E - \sum_{i=1}^{p} c_i F_i < 0\,, \tag{6.3}$$

it readily follows that condition (6.2) *holds.*

Furthermore if $p = 1$ *and*

$$\exists\, \tilde{x} \quad \textit{such that} \quad \tilde{x}^T F_1 \tilde{x} < 0\,,$$

then the two conditions (6.2) *and* (6.3) *are equivalent.* ▲

6.1.2 Impulsive Dynamical Linear Systems

An IDLS is described by

$$\dot{x}(t) = A(t)x(t) + G(t)w(t)\,, \quad x(t_0) = 0\,, \quad \big(t, x(t)\big) \notin \mathcal{S} \tag{6.4a}$$

$$x(t^+) = J(t)x(t)\,, \quad \big(t, x(t)\big) \in \mathcal{S} \tag{6.4b}$$

$$y(t) = C(t)x(t) \tag{6.4c}$$

where $A(\cdot)\,, J(\cdot)\ :\ \mathbb{R}_0^+ \mapsto \mathbb{R}^{n\times n}$, $G(\cdot)\ :\ \mathbb{R}_0^+ \mapsto \mathbb{R}^{n\times r}$, and $C(\cdot)\ :\ \mathbb{R}_0^+ \mapsto \mathbb{R}^{m\times n}$ are piecewise continuous matrix-valued functions, and $\mathcal{S} \subset \mathbb{R}_0^+ \times \mathbb{R}^n$ is called the *resetting set* (Haddad et al. [2006]).

Equation (6.4a) describes the continuous-time dynamics of the IDLS, (6.4b) represents the resetting law.

Given a particular exogenous input $w(\cdot)$ and the correspondent trajectory $x(\cdot)$, we denote with t_k, $k \in \mathbb{N}^+$, the k-th instant of time at which $\big(t, x(t)\big)$ intersects $\mathcal{S}$, and we call t_k, $k \in \mathbb{N}^+$, *resetting times*. According to the resetting law (6.4b), system (6.4) exhibits a finite jump from $x(t_k)$ to $x(t_k^+)$, at each resetting time t_k, since in general $x(t_k^+) \neq x(t_k)$.

Depending on the definition of the resetting set $\mathcal{S}$, IDLS can be classified as follows (see also Haddad et al. [2006]):

i) *Time-dependent IDLS (TD-IDLS)*: given a set $\mathcal{T} := \big\{t_1\,, t_2\,, \ldots\big\}$, $\mathcal{S}$ is defined as $\mathcal{S} = \mathcal{T} \times \mathbb{R}^n$. The resetting set is defined by a prescribed sequence of time instants, which are independent of the state $x(\cdot)$ and input $w(\cdot)$;

ii) *State-dependent IDLS (SD-IDLS)*: given a set $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{S}$ is defined as $\mathcal{S} = \mathbb{R}_0^+ \times \mathcal{X}$. Here the resetting set is defined by a region in the state space, which does not depend on the time.

It should be noticed that the definition of TD-IDLS is equivalent to the one of TD-SLS if the resetting times are known (see Remark 2).

Remark 12 *Without loss of generality it is assumed that* $\big(t_0\,,0\big) \notin \mathcal{S}$*. Indeed if the first resetting time* t_1 *were equal to* t_0*, since in* (6.4a) *it is* $x(t_0) = 0$*, it would yield* $x(t_0^+) = 0$*. Hence, an initial jump would not have any effect on the system dynamics.* ▲

The following two assumptions will be made to assure the well-posedness of the resetting times, i.e., that given $i \neq j$ the two resetting times t_i and t_j are different, and to prevent system (6.4) from exhibiting Zeno behavior (Ames et al. [2006]).

Assumption 2 *For all* $t \in \big[0, +\infty\big[$ *such that* $\big(t, x(t)\big) \in \mathcal{S}$*,* $\exists\, \varepsilon > 0 : \big(t+\delta, x(t+\delta)\big) \notin \mathcal{S}$*,* $\forall \delta \in \,]0, \varepsilon]$. ▲

Assumption 3 *Given a compact interval* $[t_0\,, t_0+T]$*, it includes only a finite number of resetting times. It follows that the resetting set to be considered in the time interval* $[t_0\,, t_0+T]$ *is given by*

$$\mathcal{S} = \mathcal{T} \times \mathcal{D} \subset [t_0\,, t_0+T] \times \mathbb{R}^n\,, \quad \text{with } \mathcal{T} = \big\{t_1\,, t_2\,, \ldots\,, t_r\big\}\,.$$

▲

6.1.3 Input-output finite-time stability for IDLS

Sufficient conditions for IO-FTS of time-dependent and state-dependent IDLS are given in this section, for the two classes of input signals $\mathcal{W}_2(t_0\,, T\,, R)$ and $\mathcal{W}_\infty(t_0\,, T\,, R)$.

Two lemmas are now introduced that hold for both TD-IDLS and SD-IDLS. This two lemmas provide sufficient conditions for IO-FTS of a generic IDLS in the case of input signals of class $\mathcal{W}_2$ and $\mathcal{W}_\infty$, respectively. These preliminary results are not stated in terms of LMIs feasibility problems. In order to recast these conditions in the LMI framework, the two different cases listed in Section 6.1.2 will be dealt with separately. In particular, taking advantage of the peculiar definition of the resetting set $\mathcal{S}$ for TD-IDLS and SD-IDLS, it is possible to cast the provided sufficient condition as a single Differential-Difference Linear Matrix Inequality (D/DLMI, Shaked and Suplin [2001]).

Lemma 6 *Given system* (6.4)*, a positive definite matrix-valued function* $Q(\cdot)$ *defined over* $[0, T]$*, and* $t \in]t_0, t_0 + T]$*, the condition*

$$w(\cdot) \in \mathcal{W}_2 \Rightarrow y^T(t)Q(t-t_0)y(t) < 1$$

is satisfied if there exists a piecewise continuously differentiable symmetric solution $P(\cdot)$ *defined over the interval* $]t_0\,, t]$ *such that the following conditions are satisfied*

$$\dot{P}(\tau) + A(\tau)^T P(\tau) + P(\tau)A(\tau) + P(\tau)G(\tau)R^{-1}G(\tau)^T P(\tau) < 0\,, \quad \tau \in]t_0, t]\,, \tau \notin \mathcal{T} \tag{6.5a}$$

$$x^T(t_k)\big(J^T(t_k)P(t_k^+)J(t_k) - P(t_k)\big)x(t_k) \le 0\,, \quad \big(t_k\,, x(t_k)\big) \in \mathcal{S} \tag{6.5b}$$

$$P(t) \ge C^T(t)Q(t-t_0)C(t)\,. \tag{6.5c}$$

Proof. Consider the quadratic function $V(\tau, x) = x^T(\tau)P(\tau)x(\tau)$. Given a time instant $\tau \notin \mathcal{T}$, the derivative with respect to time reads[1]

$$\frac{d}{d\tau}\left(x^T P x\right) = x^T \dot{P} x + \dot{x}^T P x + x^T P \dot{x} = x^T \left(\dot{P} + A^T P + PA\right) x + w^T G^T P x + x^T P G w\,.$$

Condition (6.5a) implies that

$$\frac{d}{d\tau}\left(x^T P x\right) < w^T G^T P x + x^T P G w - x^T P G R^{-1} G^T P x\,.$$

Let $v = \left(R^{1/2} w - R^{-1/2} G^T P x\right)$, then

$$v^T v = w^T R w + x^T P G R^{-1} G^T P x - w^T G^T P x - x^T P G w\,.$$

[1] Time argument is omitted for brevity.

It follows that

$$\frac{d}{d\tau}\left(x^T P x\right) < w^T R w - v^T v < w^T R w \,, \quad \forall\, \tau \notin \mathfrak{T} \,. \tag{6.6}$$

First remember that $t_0 \notin \mathfrak{T}$ (see Remark 12), then assume that in the time interval $]t_0\,, t]$ the state jumps h times, i.e.

$$]t_0\,, t] \cap \mathfrak{T} = \left\{t_1\,, t_2\,, \dots\,, t_h\right\}.$$

Integrating (6.6) between t_0 and t_1, taking into account that $x(t_0) = 0$ it is obtained

$$x(t_1)^T P(t_1) x(t_1) < \int_{t_0}^{t_1} w^T(\sigma) R w(\sigma) d\sigma \,. \tag{6.7a}$$

Similarly it is

$$x(t_2)^T P(t_2) x(t_2) - x^T(t_1^+) P(t_1^+) x(t_1^+) < \int_{t_1^+}^{t_2} w^T(\sigma) R w(\sigma) d\sigma \tag{6.7b}$$

$$\dots$$

$$x(t)^T P(t) x(t) - x^T(t_h^+) P(t_h^+) x(t_h^+) < \int_{t_h^+}^{t} w^T(\sigma) R w(\sigma) d\sigma \,, \tag{6.7c}$$

with $t \leq t_0 + T$. From (6.7) it readily follows that

$$x^T(t) P(t) x(t) + \sum_{i=1}^{h}\left(x^T(t_i) P(t_i) x(t_i) - x^T(t_i^+) P(t_i^+) x(t_i^+)\right) < \int_{t_0}^{t} w^T(\sigma) R w(\sigma) d\sigma \,. \tag{6.8}$$

Since

$$x^T(t_k) P(t_k) x(t_k) - x^T(t_k^+) P(t_k^+) x(t_k^+) = x^T(t_k) P(t_k) x(t_k) - x^T(t_k) J^T(t_k) P(t_k^+) J(t_k) x(t_k) \,,$$

for all $t_k \in \mathfrak{T}$, condition (6.5b) implies that

$$\sum_{i=1}^{h}\left(x^T(t_i) P(t_i) x(t_i) - x^T(t_i^+) P(t_i^+) x(t_i^+)\right) \geq 0 \,,$$

hence taking into account that $w(\cdot)$ belongs to $\mathcal{W}_2$, it holds

$$x^T(t)P(t)x(t) < \int_{t_0}^{t} w^T(\sigma)Rw(\sigma) = \|w\|^2_{[t_0,t],R} \le \|w\|^2_{[t_0,t_0+T],R} \le 1\,.$$

Exploiting the terminal condition (6.5c) it follows that

$$y(t)^T Q(t-t_0)y(t) = x^T(t)C^T(t)Q(t-t_0)C(t)x(t) \le x^T(t)P(t)x(t) < 1\,,$$

for all $t \in]t_0\,, t_0+T]$. ♦

Lemma 7 *Given system* (6.4)*, a positive definite matrix-valued function* $Q(\cdot)$ *defined over* $[0,T]$*, and* $t\ \in\ ]t_0, t_0+T]$*, the condition*

$$w(\cdot)\ \in\ \mathcal{W}_\infty\ \Rightarrow\ y^T(t)Q(t-t_0)y(t) < 1$$

is satisfied if there exists a piecewise continuously differentiable symmetric solution $P(\cdot)$*, defined over the interval* $]t_0\,, t]$*, such that conditions* (6.5a)-(6.5b) *and*

$$P(t) \ge C^T(t)\widetilde{Q}(t-t_0)C(t)\,, \tag{6.9}$$

with $\widetilde{Q}(t-t_0) = \big(t-t_0\big)Q(t-t_0)$*, are satisfied.*

Proof. By using the same arguments exploited in Lemma 6, it turns out that inequality (6.6) holds. Since $w(\cdot) \in \mathcal{W}_\infty$, it follows that

$$\frac{d}{d\tau}\left(x^T P x\right) < 1\,. \tag{6.10}$$

As for Lemma 6 let us suppose that in the time interval $]t_0\,, t]$ there are h state jumps. Integrating (6.10) between t_0 and t with $x(t_0) = 0$, and exploiting condition (6.5b), it is obtained

$$x(t)^T P(t)x(t) < t - t_0\,.$$

Hence, taking into account that $\widetilde{Q}(t-t_0) = \big(t-t_0\big)Q(t-t_0)$, the terminal condition (6.9) guarantees that $y(t)^T Q(t-t_0)y(t) < 1$. ♦

In principle, in order to assess IO-FTS of system (6.4) when $\mathcal{W}_2$ ($\mathcal{W}_\infty$) signals are considered, the hypotheses of Lemma 6 (Lemma 7) should be checked for any $t \in]t_0, t_0+T]$. In other words the feasibility of infinitely many optimization problems should be checked, which is obviously an impossible task. However, as stated at the beginning of this section, in the next we exploit Lemmas 6 and 7 to introduce sufficient conditions for IO-FTS that requires to check the feasibility of a single D/DLMI with terminal condition.

It is worth to notice that the sufficient conditions for IO-FTS provided for $\mathcal{W}_2$ signals differ from the one provided for $\mathcal{W}_\infty$ signals only for the condition involving the matrix-valued function $Q(\cdot)$. For this reason, from now on, only the results for the class of $\mathcal{W}_2$ inputs will be given, and the technique to extend them to the case of $\mathcal{W}_\infty$ inputs will be briefly discussed.

6.1.3.1 Time-dependent IDLS

In this section, Lemma 6 will be used to derive a sufficient condition for IO-FTS of TD-IDLS when the class of $\mathcal{W}_2$ inputs.

Theorem 14 *Assume that the following D/DLMI with terminal condition*

$$\begin{pmatrix} \dot{P}(\tau)+A(\tau)^T P(\tau)+P(\tau)A(\tau) & P(\tau)G(\tau) \\ G(\tau)^T P(\tau) & -R \end{pmatrix} < 0\,, \qquad \forall\, \tau \in]t_0, t_0+T]\,, \tau \notin \mathcal{T} \tag{6.11a}$$

$$J^T(t_k)P(t_k^+)J(t_k) - P(t_k) \le 0\,, \quad \forall\, t_k \in \mathcal{T} \tag{6.11b}$$

$$P(t) \ge C(t)^T Q(t-t_0)C(t)\,, \quad \forall\, t \in]t_0, t_0+T] \tag{6.11c}$$

admits a piecewise continuously differentiable positive definite solution $P(\cdot)$, then the time-dependent IDLS (6.4) *is IO-FTS with respect to $\big(\mathcal{W}_2, Q(\cdot), t_0, T\big)$.*

Proof. First note that for all t in $]t_0, t_0+T]$, by using Schur complements [Boyd et al., 1994, p. 7], inequality (6.11a) is equivalent to

$$\dot{P}(t)+A(t)^T P(t)+P(t)A(t)+P(t)G(t)R^{-1}G(t)^T P(t) < 0\,.$$

Furthermore, it is straightforward to see that condition (6.11b) implies (6.5b). Indeed, for TD-IDLS condition (6.5b) must be checked for any possible state, since the resetting set $\mathcal{S} = \mathcal{T} \times \mathbb{R}^n$ (see Section 6.1.2). It turns out that a matrix function $P(\cdot)$ satisfying (6.11) also satisfies (6.5) for all t in $]t_0, t_0+T]$, which proofs the theorem. ♦

Starting from Lemma 7 and exploiting similar arguments as in the proof of Theorem 14, a sufficient condition for IO-FTS of TD-IDLS can be easily derived when dealing with $\mathcal{W}_\infty$ input signals. In particular, considering $\widetilde{Q}(t) = tQ(t)$, if (6.11a) - (6.11b), together with

$$P(t) \geq C(t)^T \widetilde{Q}(t-t_0) C(t)\,, \quad \forall\, t \in]t_0, t_0+T] \tag{6.12}$$

admits a piecewise continuously differentiable solution $P(\cdot) > 0$, then the TD-IDLS (6.4) is IO-FTS with respect to $\big(\mathcal{W}_\infty, Q(\cdot), t_0, T\big)$.

Remark 13 *Although for a sufficiently large value of T the condition $\widetilde{Q}(t) = tQ(t)$ may lead to ill-conditioned problems, it is worth to notice that using a finite-time stability approach makes sense especially when dealing with time horizons that are less then the settling time of the considered system. It turns out that typically T does not assume large values. Accordingly, the values of t and Q will be scaled by similar amounts, thus avoiding ill-conditioning. If it is needed to deal with time horizons much larger than the settling time of the system, then it is probably more opportune to rely on infinite time horizon approaches.* ▲

6.1.3.2 State-dependent IDLS

By exploiting similar arguments as in the proof of Section 6.1.3.1, it is straightforward to state the following two theorems, for the case of $\mathcal{W}_2$ and $\mathcal{W}_\infty$ input signals, respectively.

Theorem 15 *Assume that the following difference/differential inequalities with* *terminal condition*

$$\begin{pmatrix} \dot{P}(t) + A(t)^T P(t) + P(t)A(t) & P(t)G(t) \\ G(t)^T P(t) & -R \end{pmatrix} < 0\,, \quad \forall\, t \in]t_0, t_0 + T] \tag{6.13a}$$

$$x^T(t)\big(J^T(t)P(t)J(t) - P(t)\big)x(t) \leq 0\,, \quad \forall\, t \in]t_0, t_0 + T]\,,\ \forall\, x \in \mathcal{X} \tag{6.13b}$$

$$P(t) \geq C(t)^T Q(t - t_0)C(t)\,, \quad \forall\, t \in]t_0, t_0 + T] \tag{6.13c}$$

admits a continuously differentiable positive definite solution $P(\cdot)$, then the state-dependent IDLS (6.4) *is IO-FTS with respect to $\big(\mathcal{W}_2, Q(\cdot), t_0, T\big)$.* ▲

As it has been seen in Section 6.1.3.1, starting from Theorem 15 a similar result can be derived in the case of $\mathcal{W}_\infty$ signals when the terminal condition (6.12) is considered.

The main difference with respect to the case of TD-IDLS is that, since for SD-IDLS the resetting times are not a priori known, in Theorem 15 the conditions (6.13a) and (6.13b) have to be checked for all t in $]t_0\,, t_0 + T]$. Furthermore, note that condition (6.13b) is not a LMI.

In order to recast these conditions in the LMIs framework, without loss of generality, firstly it has to be assumed that the resetting states set $\mathcal{X}$ is given by

$$\mathcal{X} = \bigcup_{i=1}^{N} \mathcal{X}_i \setminus \{0\}\,.$$

Note that the origin does not belong to $\mathcal{X}$ according to Remark 12. Hence, it is possible to exploit S-procedure (see Lemma 5) to immediately derive the following theorem in the case of $\mathcal{W}_2$ input signals.

Theorem 16 *Given a set of symmetric matrices $F_{i,j}$, with $i = 1\,, \ldots\,, N$ and $j = 1\,, \ldots\,, p_i$, satisfying*

$$x^T F_{i,j} x \leq 0\,, \quad x \in \mathcal{X}_i\,, \quad i = 1, \ldots, N\,, \quad j = 1, \ldots, p_i \tag{6.14}$$

assume there exist a continuously differentiable symmetric matrix function $P(\cdot)$ and nonnegative scalar functions $c_{i,j}(\cdot)$, $i = 1\,, \ldots\,, N$, $j = 1\,, \ldots\,, p_i$, such that (6.13a), (6.13c), *and*

$$J^T(t)P(t)J(t) - P(t) - \sum_{j=1}^{p_i} c_{i,j}(t)F_{i,j} \leq 0\,, \quad \forall\, t \in]t_0, t_0+T]\,,\ \forall\, i = 1\,,\ldots\,,N \tag{6.15}$$

are satisfied, then the SD-IDLS (6.4) *is IO-FTS with respect to* $\big(\mathcal{W}_2, Q(\cdot), t_0, T\big)$.

▲

Although in Theorem 16 the use of the S-procedure may introduce additional conservatism, in Ambrosino et al. [2009] it has been shown that for the case of resetting sets in $\mathbb{R}^2$ and ellipsoidal resetting sets no conservatism is added. Exploiting again the S-procedure, a similar result can be derived also for the class of $\mathcal{W}_\infty$ inputs.

6.1.4 IO Finite Time Stabilisation of IDLS

The results presented in the previous section are now exploited to solve Problem 4, i.e., to IO finite-time stabilise system (6.4) by means of static output feedback. It will be shown that solution to Problem 4 can be stated in terms of a BMIs feasibility problem. Furthermore, it is shown that, when TD-IDLS and state feedback are considered (i.e. Problem 3), the solution to the IO finite-time stabilization problem can be casted into an LMIs feasibility problem.

6.1.4.1 Time-dependent IDLS

When TD-IDLSs are considered, a solution to Problem 4 is given finding two matrix-valued functions $P(\cdot)$ and $K(\cdot)$, that satisfy conditions of Theorem 14, for the closed-loop system (2.10). In both cases, condition (6.11a) becomes

$$\begin{pmatrix} \dot{P}(\tau) + A_{cl}(\tau)^T P(\tau) + P(\tau)A_{cl}(\tau) & P(\tau)G(\tau) \\ G(\tau)^T P(\tau) & -R \end{pmatrix} < 0\,, \forall\, \tau \in]t_0, t_0+T]\,, \tau \notin \mathcal{T} \tag{6.16}$$

Since $A_{cl}(t) = A(t) + B(t)K(t-t_0)C(t)$, it readily follows that inequality (6.16) is a BMI. Hence, the solution of the stabilisation problem is given in terms of Difference-Differential Bilinear Matrix Inequality (D/DBMI) feasibility problem.

The next theorems provide sufficient conditions for the solution of Problem 4 which are equivalent to the ones given in Theorem 14. However, the usefulness of these results is fully exploited when Problem 3 is considered. Indeed, this equivalent formulation allows to recast the solution of Problem 3 into a D/DLMI problem. Let first consider the case of disturbances $w(\cdot)$ in the class $\mathcal{W}_2$.

Theorem 17 *Given the class of disturbances $\mathcal{W}_2$, Problems 3 and 4 are solvable if there exist a positive definite and piecewise continuously differentiable matrix-valued function $\Pi(\cdot)$, and a matrix-valued function $K(\cdot)$ such that the following D/DBMI with terminal condition*

$$\begin{pmatrix} \Upsilon(\tau) & G(\tau) \\ G(\tau)^T & -R \end{pmatrix} < 0\,, \quad \forall\, \tau \in]t_0\,, t_0+T]\,, \tau \notin \mathfrak{T} \tag{6.17a}$$

$$\begin{pmatrix} \Pi(t_k) & \Pi(t_k)J^T(t_k) \\ J(t_k)\Pi(t_k) & \Pi(t_k^+) \end{pmatrix} \geq 0\,, \quad \forall\, t_k \in \mathfrak{T} \tag{6.17b}$$

$$\begin{pmatrix} \Pi(t) & \Pi(t)C(t)^T \\ C(t)\Pi(t) & \Xi(t) \end{pmatrix} \geq 0\,, \quad \forall\, t \in]t_0\,, t_0+T] \tag{6.17c}$$

is satisfied, where

$$\begin{aligned} \Upsilon(t) &= -\dot{\Pi}(t) + \Pi(t)A_{cl}(t)^T + Acl(t)\Pi(t)\,, \\ \Xi(t) &= Q(t-t_0)^{-1}\,. \end{aligned} \tag{6.18}$$

Proof. Let $\Pi(t) = P^{-1}(t)$. By pre- and post-multiply (6.16) by $\begin{pmatrix} \Pi(\tau) & 0 \\ 0 & I \end{pmatrix} > 0$, it is

$$\begin{pmatrix} -\dot{\Pi}(\tau) + \Pi(t)A_{cl}(\tau)^T + A_{cl}(\tau)\Pi(\tau) & G(\tau) \\ G(\tau)^T & -R \end{pmatrix} < 0\,,$$

for all $\tau \in]t_0\,, t_0+T]$ and $\tau \notin \mathfrak{T}$.

Furthermore, by pre- and post-multiply (6.11b) by $\Pi(t_k)$ it is

$$\Pi(t_k)J^T(t_k)\Pi^{-1}(t_k^+)J(t_k)\Pi(t_k) - \Pi(t_k) \leq 0\,. \tag{6.19}$$

Condition (6.17b) readily follows from (6.19) by applying the Schur complements.

Eventually, inequality (6.17c) is obtained pre- and post-multiplying (6.11c) by $\Pi(t)$, and by applying again the Schur complements. ♦

As it has been done in Section 6.1.3 a similar result can be stated for the class of inputs $\mathcal{W}_\infty$ considering

$$\widetilde{\Xi}(t) = \Big((t-t_0)Q(t-t_0) \Big)^{-1} ,$$

in place of $\Xi(t)$ in condition (6.17c).

Remark 14 *It is worth to point out that, in the case of state feedback, i.e., when* $u(t) = K(t - t_0)x(t)$, (6.18) *reads*

$$\Upsilon_{sf}(t) = -\dot{\Pi}(t) + \Pi(t)A(t)^T + A(t)\Pi(t) + B(t)L(t) + L(t)^T B(t)^T ,$$

where $L(t) = K(t-t_0)\Pi(t)$*. Hence the controller gain is given by*

$$K(t-t_0) = L(t)\Pi(t)^{-1} ,$$

and conditions (6.17) *can be casted into D/DLMIs, which will computationally benefit the problem solution.* ▲

6.1.4.2 State-dependent IDLS

By exploiting similar arguments as for the case of TD-IDLS in Section 6.1.4.1, it turns out that, in order to solve Problem 4 when dealing with SD-IDLS, the following BMI, together with conditions (6.13b) and (6.13c), must be satisfied by the matrix-valued functions $P(\cdot)$ and $K(\cdot)$

$$\begin{pmatrix} \dot{P}(t) + A_{cl}(t)^T P(t) + P(t)A_{cl}(t) & P(t)G(t) \\ G(t)^T P(t) & -R \end{pmatrix} < 0 \,, \forall\, t \in]t_0, t_0+T] \,. \tag{6.20}$$

Note that, since for SD-IDLS the resetting times are not a priori known, conditions (6.20) and (6.13b) have to be checked for all t in $]t_0\,, t_0+T]$.

Differently of what it has been done for TD-IDLS, when dealing with SD-IDLS the IO finite-time stabilisation problem must be solved in terms of D/DBMIs also when the state feedback is considered. Indeed, without loss of generality, let consider the case of $\mathcal{W}_2$ input signals. In Theorem 16 it has been shown how to turn conditions (6.13b) into LMIs by means of the S-procedure. As in Theorem 17, it is also possible to recast (6.20) into an LMIs, introducing the matrix-valued function $\Pi(\cdot)$, which is equal to $P^{-1}(\cdot)$. However, when $\Pi(\cdot)$ is considered, condition (6.15), becomes equal to

$$\Pi(t)J^T(t)\Pi^{-1}(t)J(t)\Pi(t) - \Pi(t) - \Pi(t)\left(\sum_{j=1}^{p_i} c_{i,j}(t)F_{i,j}\right)\Pi(t) \leq 0\,,$$

that turns out unavoidably to be a BMI.

It is well known that optimization problems expressed in terms of BMIs are non-convex. Nevertheless, many control problems of interest, that cannot be written in terms of LMIs, can be tackled in the BMIs framework. Furthermore, the use of BMIs may represent a good strategy to face problems with no solution in the literature (some recent examples can be found in Borges et al. [2010] and Ambrosino et al. [2011]). Computations over BMI constraints are fundamentally more difficult than those over LMI constraints, since it is well known that these computations are NP-hard (Toker and Ozbay [1995]; VanAntwerp and Braatz [2000]); this implies that it is highly unlikely that there exists a polynomial-time algorithm for solving these problems. NP-hardness is an attribute of the problem itself, not of any particular algorithm, therefore also for an NP-hard problem practical algorithms can exist (Fukuda and Kojima [2001]; Goh et al. [1995]; Kawanishi et al. [1997]; Tuan and Apkarian [2002]; Zheng et al. [2002]). Furthermore, these algorithms have been recently made available in off-the-shelf software packages (Henrion et al. [2005]).

6.2 IO-FTS and stabilization of TD-SLS with uncertainties on the resetting times

This section extends to TD-SLS the concept of IO-FTS introduced in the previous section in the context of IDLS. Moreover, the three levels of knowledge on the resetting times listed in Section 2.3 are considered. Eventually, conditions to solve Problem 5, i.e. IO finite-time stabilization via dynamic output-feedback, are also presented.

6.2.1 Analysis

Before introducing the theorem providing a sufficient condition to be used to check IO-FTS of TD-SLS, it is worth noting that a similar result as the one stated in Lemma 6 can be obtained.

Lemma 8 *Given the system* (6.4)*, a positive definite matrix-valued function* $Q(\cdot)$ *defined over* Ω*, and* $t \in \Omega$*, the condition*

$$w(\cdot) \in \mathcal{W}_2 \Rightarrow y^T(t)Q(t)y(t) < 1$$

is satisfied if there exists $g{+}1$ *piecewise differentiable positive definite matrix-valued functions* $P_j(\cdot)$*, defined over the interval* $]t_0\,, t]$*, such that the following conditions are satisfied*

$$\begin{aligned}
&\dot{P}_j(\tau) + A^T_{\sigma(t_{j-1})}(\tau)P_j(\tau) + P_j(\tau)A_{\sigma(t_{j-1})}(\tau) \\
&\qquad + P_j(\tau)G_{\sigma(t_{j-1})}(\tau)W^{-1}(t)G^T_{\sigma(t_{j-1})}(\tau)P_j(\tau) < 0\,, \\
&\qquad\qquad \tau \in \Psi_j \cap]t_0, t]\,, j = 1\,, \dots\,, g+1 \qquad (6.21a)
\end{aligned}$$

$$x^T(\tau)\big(J^T(\tau)P_{j+1}(\tau)J(\tau) - P_j(t\tau)\big)x(\tau) \le 0\,, \qquad \tau \in \Phi_j \cap]t_0\,, t]\,, j = 1\,, \dots\,, g \qquad (6.21b)$$

$$P_j(t) \ge C^T_{\sigma(t)}(t)Q(t)C_{\sigma(t)}(t)\,, \quad j = 1\,, \dots\,, g+1 \qquad (6.21c)$$

where $g < h$ *is the number of state jumps in the time interval* $]t_0\,, t]$ *(see Remark 1).*

Proof. The case of $\Delta T_j = 0$ has already been proven in the proof of Lemma 6 for IDLS; the extension to TD-SLS is trivial. The case of $\Delta T_j \neq 0$ has to be dealt with to complete the proof. By exploiting similar arguments as in Theorem 1, it is straightforward to recognize that the proof can be extended to the case in which $\Delta T_j \neq 0$, by substituting the time instant t_j with the interval $\Phi_j\,, j = 1\,, \dots\,, g$.

In this case, conditions (6.21a) and (6.21c) have to be verified in Ψ_j, that is in the time interval in which the correspondent linear system is potentially active, while condition (6.21b) has to be checked in Φ_j, i.e. in the time interval in which the state jump could occur. ♦

Lemma 8 still holds when the family (2.7) is made of systems with different dimensions. Indeed, inequality (6.21b) can still be defined by choosing $P_j(\cdot)$ with the same dimension of $A_{\sigma(t_{j-1})}(\cdot)$ for all t, and by noticing that, in general, $J(\cdot)$ is a rectangular matrix.

Although the previous lemma provides sufficient conditions to assess IO-FTS of system (6.4) wrt $\big(\mathcal{W}_2, Q(\cdot), \Omega\big)$, those should be checked for any $t \in \Omega$. In other words infinitely many optimization problems should be solved, hence the result cannot be useful in practice. The difficulty is overcame by next theorem, that can be proved in a straightforward way by means of Lemma 8. Indeed, it requires to check the feasibility of a single D/DLMI.

Theorem 18 (IO-FTS wrt $\mathcal{W}_2$ in the US case) *If there exist $h+1$ differentiable matrix-valued functions $P_j(\cdot)$, $j = 1, \dots, h+1$, that satisfy the following D/DLMI*

$$\begin{pmatrix} \dot{P}_j(t) + A^T_{\sigma(t_{j-1})}(t)P_j(t) + P_j(t)A_{\sigma(t_{j-1})}(t) & P_j(t)G_{\sigma(t_{j-1})}(t) \\ G^T_{\sigma(t_{j-1})}(t)P_j(t) & -W(t) \end{pmatrix} < 0\,, \qquad t \in \Psi_j\,,\ j = 1, \dots, h+1 \tag{6.22a}$$

$$J^T(t)P_{j+1}(t)J(t) - P_j(t) \le 0\,, \quad t \in \Phi_j\,,\ j = 1, \dots, h \tag{6.22b}$$

$$P_j(t) \ge C^T_{\sigma(t_{j-1})}(t)Q(t)C_{\sigma(t_{j-1})}(t)\,, \quad t \in \Psi_j\,,\ j = 1, \dots, h+1 \tag{6.22c}$$

then system (6.4) is IO-FTS wrt $\big(\mathcal{W}_2, Q(\cdot), \Omega\big)$ in the Uncertain Switching case.

Proof. First note that for all $t \in \Psi_j\,, j = 1, \dots, h+1$, by using Schur complements, inequality (6.22a) is equivalent to

$$\begin{aligned} &\dot{P}_j(\tau) + A^T_{\sigma(t_{j-1})}(\tau)P_j(\tau) + P_j(\tau)A_{\sigma(t_{j-1})}(\tau) \\ &\qquad + P_j(\tau)G_{\sigma(t_{j-1})}(\tau)W^{-1}(t)G^T_{\sigma(t_{j-1})}(\tau)P_j(\tau) < 0 \end{aligned}$$

Furthermore, it is straightforward to see that condition (6.22b) implies (6.21b). Indeed, for TD-SLS condition (6.21b) must be checked for any possible $x(\cdot)$, since the state trajectory is not a priori known. It turns out that $h+1$ matrix functions $P_j(\cdot)$ satisfying (6.22) also satisfy (6.21), which proofs the theorem. ♦

Theorem 19 (IO-FTS wrt $\mathcal{W}_\infty$ in the US case) *If there exist $h+1$ differentiable matrix-valued functions $P_j(\cdot)$, $j = 1, \dots, h+1$, that satisfy the following D/DLMI*

$$\begin{pmatrix} \dot{P}_j(t) + A^T_{\sigma(t_{j-1})}(t)P_j(t) + P_j(t)A_{\sigma(t_{j-1})}(t) & P_j(t)G_{\sigma(t_{j-1})}(t) \\ G^T_{\sigma(t_{j-1})}(t)P_j(t) & -W(t) \end{pmatrix} < 0\,, \quad t \in \Psi_j\,,\ j = 1\,, \dots, h+1 \tag{6.23a}$$

$$J^T(t)P_{j+1}(t)J(t) - P_j(t) \le 0\,, \quad t \in \Phi_j\,,\ j = 1\,, \dots, h \tag{6.23b}$$

$$P_j(t) \ge C^T_{\sigma(t_{j-1})}(t)\widetilde{Q}(t)C_{\sigma(t_{j-1})}(t)\,, \quad t \in \Psi_j\,,\ j = 1\,, \dots, h+1 \tag{6.23c}$$

where $\widetilde{Q}(t) = (t - t_0)Q(t)$, then system (6.4) *is IO-FTS wrt $\big(\mathcal{W}_\infty, Q(\cdot), \Omega\big)$ in the Uncertain Switching case.*

Proof. The proof readily follow by using same argument as the proof of Lemma 7. ♦

The following corollaries provide two sufficient conditions to check IO-FTS for the KS and AS case, respectively. The proofs of these further results can be readily derived from Theorem 18 exploiting similar arguments as in item iii) of Theorem 6 and Corollary 1.

Corollary 5 (IO-FTS wrt $\mathcal{W}_2$ signals in KS case) *Assume that the following D/DLMI*

$$\begin{pmatrix} \dot{P}(t) + A^T_{\sigma(t)}(t)P(t) + P(t)A_{\sigma(t)}(t) & P(t)G_{\sigma(t)}(t) \\ G^T_{\sigma(t)}(t)P(t) & -W(t) \end{pmatrix} < 0\,, \quad \forall\ t \in\ ]t_0, t_0+T]\,, t \notin \mathcal{T} \tag{6.24a}$$

$$J^T(t_k)P(t_k^+)J(t_k) - P(t_k) \le 0\,, \quad \forall\ t_k \in\ \mathcal{T} \tag{6.24b}$$

$$P(t) \ge C^T_{\sigma(t)}(t)Q(t)C_{\sigma(t)}(t)\,, \quad \forall\ t \in\ ]t_0, t_0+T] \tag{6.24c}$$

admits a piecewise differentiable positive definite matrix-valued function $P(\cdot)$, *then system* (6.4) *is IO-FTS with respect to* $\big(\mathcal{W}_2, Q(\cdot), \Omega\big)$ *in the case of Known Switching.* ▲

Corollary 6 (IO-FTS wrt $\mathcal{W}_2$ in the AS case) *If there exist* l *differentiable positive definite matrix-valued functions* $P_i(\cdot)\,,\ i = 1\,,\ldots\,,l$, *that satisfy thefollowing D/DLMI*

$$\begin{pmatrix} \dot{P}_i(t) + A_i^T(t)P_i(t) + P_i(t)A_i(t) & P_i(t)G_i(t) \\ G_i^T(t)P_i(t) & -W(t) \end{pmatrix} < 0\,, t \in\,]t_0\,, t_0 + T]\,,\ i \in \mathcal{P} \tag{6.25a}$$

$$J^T(t)P_i(t)J(t) - P_j(t) \le 0\,, \quad t \in\,]t_0\,, t_0 + T]\,,\ i\,, j \in \mathcal{P} \tag{6.25b}$$

$$P_i(t) \ge C_i^T(t)Q(t - t_0)C_i(t)\, t \in\,]t_0\,, t_0 + T]\,,\ i \in \mathcal{P} \tag{6.25c}$$

then system (6.4) *is IO-FTS wrt* $\big(\mathcal{W}_2, Q(\cdot), \Omega\big)$ *in the case of Arbitrary Switching.* ▲

It is worth noticing that, when the input signal $w(\cdot)$ belongs to the $\mathcal{W}_\infty$ class, sufficient conditions for IO-FTS in the KS and AS case can be obtained by replacing the matrix $Q(t)$ with $\widetilde{Q}(t) = (t - t_0)Q(t)$ in (6.24c) and (6.25c), respectively.

6.2.2 IO Finite-time stabilization via output-feedback

The following result states a sufficient condition to solve Problem 5. Its proof can be derived from the one of Theorem 7 and [Amato et al., 2011a, Theorem 3].

Theorem 20 *Problem 5 is solvable* if and only if *there exist* $2 \cdot (h+1)$ *symmetric matrix-valued functions* $S_j(\cdot)$, $T_j(\cdot)$, $h + 1$ *nonsingular matrix-valued functions* $N_j(\cdot)$ *and* $6 \cdot (h + 1)$ *matrix-valued functions* $\hat{A}_{K,j}(\cdot)$, $\hat{B}_{K,j}(\cdot)$, $\hat{C}_{K,j}(\cdot)$, $D_{K,j}(\cdot)$,

$\hat{J}_{K,j}(\cdot)$, and $\hat{H}_{K,j}(\cdot)$ such that the following D/DLMIs are satisfied

$$\begin{pmatrix} \Theta_{11,j}(t) & \Theta_{12,j}(t) & 0 \\ \Theta_{12,j}^T(t) & \Theta_{22,j}(t) & T_j(t)G_{\sigma(t_{j-1})}(t) \\ 0 & G_{\sigma(t_{j-1})}^T(t)T_j(t) & -W(t) \end{pmatrix} < 0\,, \quad t \in \Psi_j\,, \quad j = 1\,, \dots\,, h+1 \tag{6.26a}$$

$$\begin{pmatrix} \Lambda_{11,j} & \Lambda_{12,j} \\ \Lambda_{12,j}^T & \Lambda_{22,j} \end{pmatrix} \leq 0\,, \quad t \in \Phi_j\,, \quad j = 1\,, \dots\,, h \tag{6.26b}$$

$$\begin{pmatrix} \Xi_{11,j}(t) & \Xi_{12,j}(t) & 0 \\ \Xi_{12,j}^T(t) & S_j(t) & S_j(t)C_{\sigma(t_{j-1})}^T(t) \\ 0 & C_{\sigma(t_{j-1})}(t)S_j(t) & Q^{-1}(t) \end{pmatrix} \geq 0\,, \quad t \in \Psi_j\,, \quad j = 1\,, \dots\,, h+1 \tag{6.26c}$$

where (time argument is omitted for sake of brevity)

$$\begin{aligned}
\Theta_{11,j} &= -\dot{S}_j + A_{\sigma(t_{j-1})}S_j + S_jA_{\sigma(t_{j-1})}^T + B_{\sigma(t_{j-1})}\hat{C}_{K,j} \\
&\qquad + \hat{C}_{K,j}^T B_{\sigma(t_{j-1})}^T + G_{\sigma(t_{j-1})}W^{-1}G_{\sigma(t_{j-1})}^T \\
\Theta_{12,j} &= A_{\sigma(t_{j-1})} + \hat{A}_{K,j}^T + B_{\sigma(t_{j-1})}D_{K,j}C_{\sigma(t_{j-1})} + G_{\sigma(t_{j-1})}W^{-1}G_{\sigma(t_{j-1})}^T T_j \\
\Theta_{22,j} &= \dot{T}_j + T_jA_{\sigma(t_{j-1})} + A_{\sigma(t_{j-1})}^T T_j + \hat{B}_{K,j}C_{\sigma(t_{j-1})} + C_{\sigma(t_{j-1})}^T \hat{B}_{K,j}^T \\
\Lambda_{11,j} &= -\begin{pmatrix} S_j & I \\ I & T_j \end{pmatrix} \\
\Lambda_{12,j} &= \begin{pmatrix} S_jJ^T & \hat{J}_{K,j}^T \\ J^T & J^T T_{j+1} + C_{\sigma(t_{j-1})}^T \hat{H}_{K,j}^T \end{pmatrix} \\
\Lambda_{22,j} &= \begin{pmatrix} S_{j+1} & I \\ I & T_{j+1} \end{pmatrix} \\
\Xi_{11,j} &= T_j - C_{\sigma(t_{j-1})}^T Q C_{\sigma(t_{j-1})} \\
\Xi_{12,j} &= I - C_{\sigma(t_{j-1})}^T Q C_{\sigma(t_{j-1})} S_j
\end{aligned}$$

Chapter 7

Applications

This chapter presents a number of applications and examples in order to show the effectiveness and usefulness of the main results introduced in the previous chapters.

7.1 Level control of interconnected reservoirs

This section deals with the level control of three interconnected reservoirs R-1, R-2 and R-3, depicted in Fig. 7.1. The two states (ON/OFF) isolation valves V-1 and V-2, when open, let the liquid flow under the effect of gravity. It is assumed that these valves are alternatively open for given amounts of time and they are activated by an external controller. The two pumps P-1 and P-2 move the liquid from R-3 to R-1 and R-2, respectively. Moreover, the pump P-j, $j = 1, 2$, can be controlled only when V-j is open, and it is stopped otherwise. The aim of the controller is to regulate the amount of liquid transferred by the bottom reservoir R-3 towards the other two reservoirs through the pumps P-1 and P-2.

If this is the case, the system can be modeled as a SLSs as in (2.8), where the resetting law $J(t)$ is constant and equal to the identity matrix.

Considering a linear dependency of the outflow from the amount of liquid

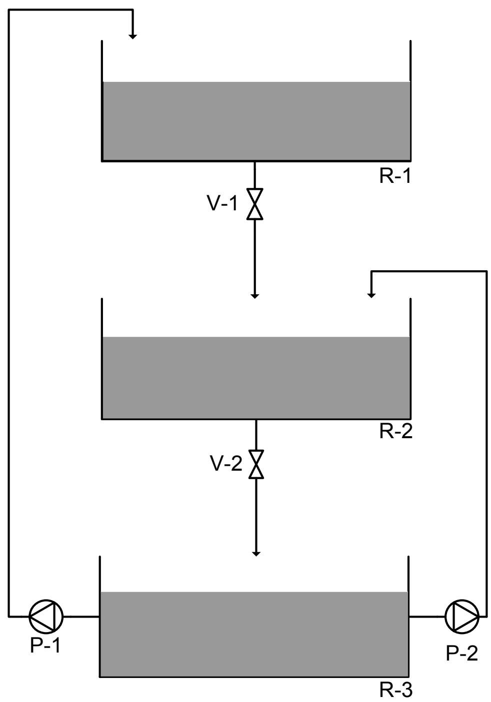


Figure 7.1: Schematic representation of a connected reservoirs system.

contained in the reservoirs, the system can be represented by:

$$A_1 = \begin{pmatrix} -\frac{k_1}{a_1} & 0 & 0 \\ \frac{k_1}{a_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; B_1 = \begin{pmatrix} \frac{1}{a_1} & 0 \\ 0 & 0 \\ -\frac{1}{a_3} & 0 \end{pmatrix} \tag{7.1a}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{k_2}{a_2} & 0 \\ 0 & \frac{k_2}{a_3} & 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{a_2} \\ 0 & -\frac{1}{a_3} \end{pmatrix} \tag{7.1b}$$

where, for $i = 1, \ldots, 3$,

- x_i is the height of the liquid in the reservoir i;
- a_i is the area of the reservoir i;
- k_i is the storage coefficient of the reservoir i;
- u_i is the amount of liquid transferred by P-i from R-3 towards R-i.

Table 7.1: Model and design parameters used in the example of the connected reservoirs system.

Parameter	Value
a_i	$1\,m$
k_i	$10^{-3}\,m^2/s$
T	$60\,s$
T_s	$1.5\,s$
ΔT	$3\,s$

The values of the model parameters are summarized in Table 7.1.

Let now suppose that the liquid is initially stored in R-3, and consider the goal of equally distributing it among the three reservoirs in 60 seconds. This problem can be casted in the finite-time stabilization framework, by considering the weighting matrix $\Gamma(t)$ shown in Fig. 7.2(a). Indeed, since $\Gamma(t)$ is diagonal, each element on the main diagonal weights only one state variable. As shown in Fig. 7.2(a), the weight of the third state $\Gamma(3,3) = 0$ when $t = 0\,s$ and it increases until reaching the final value of 0.4 when $t = 60\,s$. The weights of the other two states follow the opposite time evolution starting from the value of 0.7. Since the three weights converge towards the same value, the controller tries to distribute the liquid equally among the three reservoirs in order to minimize the state weighted norm.

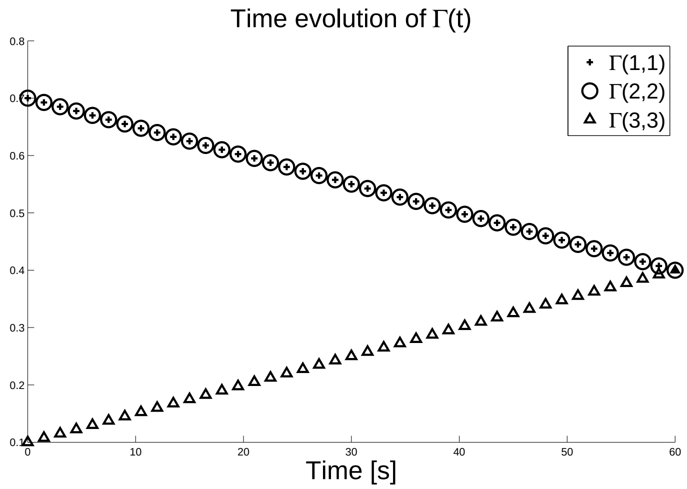
The switching signal $\sigma(t)$ is shown in Fig. 7.2(b) and an uncertainty on the switching signal of ± 3 seconds has been considered, due to possible delays in the valves actuation.

The problem can be solved by applying the results obtained in Theorem 3. The correspondent feasibility problem can be recasted into LMIs by means of $h+1$ piecewise linear matrix-valued functions $\Pi_k(\cdot)$, with $k = 1, \dots, h+1$. In particular, consider the $(k+1)$-th time interval between two resetting times t_k and t_{k+1}. In this time interval the matrix valued function $\Pi_{k+1}(\cdot)$ is assumed to have the following structure

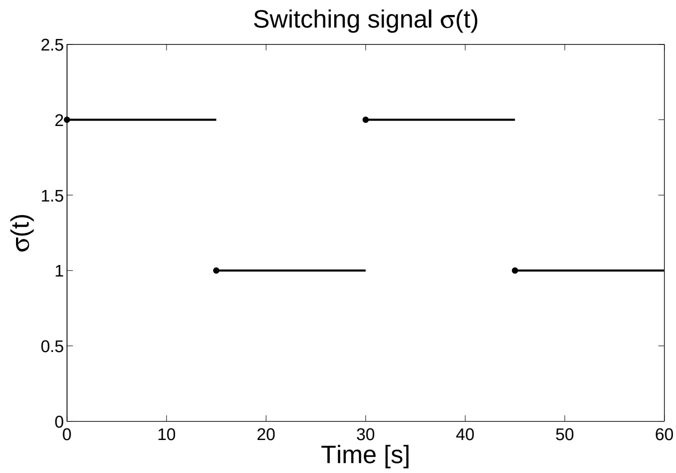
$$\Pi_{k+1}(t) = \begin{cases} \Pi_{k+1}(t_k) + \Theta_{k,1}\,(t - t_k)\ , & t \in [t_k, t_k + T_s]\ , \\ \Pi_{k+1}(t_k) + \sum_{h=1}^{j} \Theta_{k,h} T_s + \Theta_{k,j+1}\,(t - jT_s - t_k)\ , \\ \quad t \in \,]t_k + jT_s, t_k + (j+1)T_s]\ , \quad j = 1, \ldots, J_k \end{cases}$$

where $J_k = \max\{j \in \mathbb{N} : j < (t_{k+1} - t_k)/T_s\}$, $T_s \ll T$ and Π_k, $\Theta_{k,j}$, are the optimization variables.
It is straightforward to recognize that such a piecewise linear matrix function can approximate a general continuous matrix function with adequate accuracy, provided that the length of T_s is sufficiently small.

Fig. 7.3 shows the main outcomes of the proposed example. As can be seen in Fig. 7.3(a) and 7.3(b) the controller manages to obtain the expected result by keeping the state weighted norm always below 1. Since the controller cannot act on the first reservoir during the first 15 seconds, a big control action is taken right after the first switching instant reducing the state weighted norm of a large amount, as shown in Fig. 7.3(b) at $t = 15\,s$.

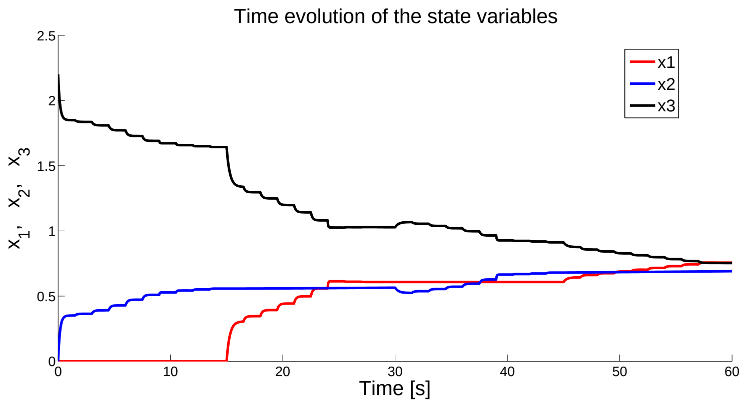


(a) Time evolution of the weighting matrix $\Gamma(t)$. Since $\Gamma(t)$ is a diagonal matrix, only the elements on the diagonal are reported.

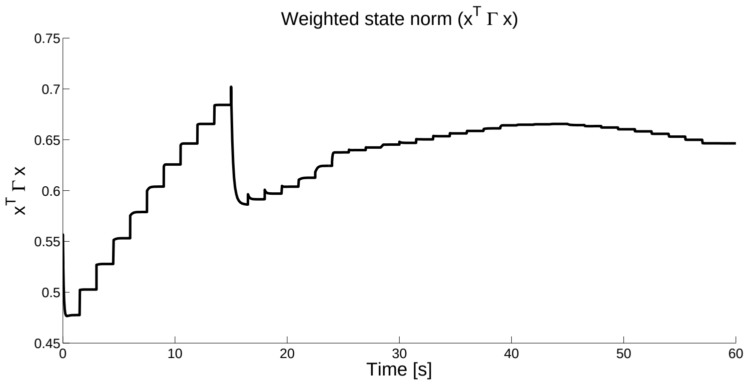


(b) Switching signal $\sigma(t)$. An uncertainty of $1.5s$ has been considered on each resetting time instant.

Figure 7.2: Weighting matrix-valued function $\Gamma(\cdot)$ and switching signal $\sigma(\cdot)$ considered in the example of the connected reservoirs system.



(a) Time trace of the state variables. The time evolution of the state variable is driven by the time evolution of the weighting matrix-valued function $\Gamma(t)$.



(b) Time trace of the state variables weighted norm. This norm has to be less than one for the system to be FTS.

Figure 7.3: Application of the controller design technique proposed in Theorem 3 on System (7.1).

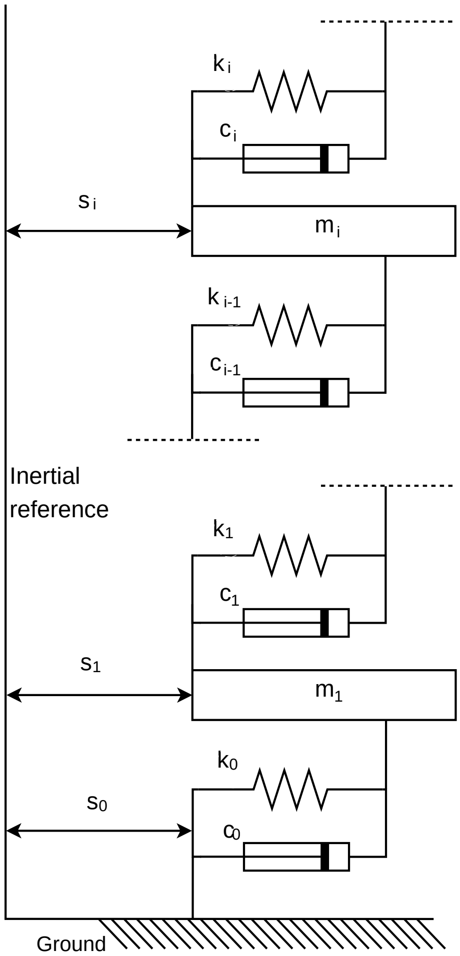


Figure 7.4: Lumped parameters model of a N-storeys building.

7.2 Seismic control of buildings during earthquakes

The vibration control of an N-storeys building subject to an earthquake is presented in this section. The building lumped parameters model is reported in Figure 7.4. The control system is made by a base isolator together with an actuator that generates a control force on the base floor.

The aim of the isolator is to produce a dynamic decoupling of the structure from its foundation. If this is the case, the inter-storeys drifts are reduced and the building behavior can be approximated by the one of a rigid body (Pozo et al.

[2009]). Furthermore, the description of the system in terms of absolute coordinates, i.e., when the displacement is defined with respect to an inertial reference, ensures that the disturbances act only at the base floor (Kelly et al. [1987]).

It turns out that it is sufficient to provide an actuator only on the base floor in order to keep the displacement and velocity of the structure under a specified boundary. Indeed, the goal of the control system is to overcome the forces generated by the isolation system at the base floor, in order to minimize the absolute displacement and velocity of the structure.

The state-space model of the considered system is

$$\dot{x}(t) = Ax(t) + Bu(t) + Gw(t) \tag{7.2a}$$

$$y(t) = Cx(t) \tag{7.2b}$$

If $s_0(\cdot)$ and $\dot{s_0}(\cdot)$ denote the displacement and the velocity of the ground and $s_i(\cdot)$ and $\dot{s}_i(\cdot)$ denote the displacement and the velocity of the i-th floor, then the state vector can be defined as $x(\cdot) = [x_1(\cdot)\, x_2(\cdot)\, \dots\, x_{2N}(\cdot)]^T$, where $x_i(\cdot) = \dot{s}_i(\cdot)$ and $x_{i+N}(\cdot) = s_i(\cdot)$, $i = 1, \dots, N$. The vector $w(\cdot) = [s_0(\cdot)\ \dot{s_0}(\cdot)]^T$ represents the exogenous input and $u(t)$ is the control force applied to the base floor. The model matrices in (7.2) are

$$A = \begin{pmatrix} A_1 & A_2 \\ I & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1/m_1 \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} k_0/m_1 & c_0/m_1 \\ 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{-(c_0+c_1)}{m_1} & \frac{c_1}{m_1} & \underbrace{0\ \dots\ 0}_{N-2} & \frac{-(k_0+k_1)}{m_1} & \frac{k_1}{m_1} & \underbrace{0\ \dots\ 0}_{N-2} \end{pmatrix},$$

Table 7.2: Model parameters for the considered six storeys building (N=6).

Floor #	Mass [kg]	Spring coefficient [kN/m]	Damping coefficient [kNs/m]
0	–	k_0=1200	c_0=2.4
1	m_1=6800	k_1=33732	c_1=67
2	m_2=5897	k_2=29093	c_2=58
3	m_3=5897	k_3=28621	c_3=57
4	m_4=5897	k_4=24954	c_4=50
5	m_5=5897	k_5=19059	c_5=38
6	m_6=5897	–	–

where A_1 and A_2 are $N \times N$ tridiagonal matrices defined as

$$A_1 = \begin{pmatrix} -\frac{(c_0+c_1)}{m_1} & \frac{c_1}{m_1} & 0 & & & \dots & & 0 \\ & & & \dots\dots & & & & \\ 0 & \dots & \frac{c_{i-1}}{m_i} & -\frac{(c_{i-1}+c_i)}{m_i} & \frac{c_i}{m_i} & & \dots & 0 \\ & & & \dots\dots & & & & \\ 0 & & \dots & & 0 & & \frac{c_{N-1}}{m_N} & -\frac{c_{N-1}}{m_N} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -\frac{(k_0+k_1)}{m_1} & \frac{k_1}{m_1} & 0 & & & \dots & & 0 \\ & & & \dots\dots & & & & \\ 0 & \dots & \frac{k_{i-1}}{m_i} & -\frac{(k_{i-1}+k_i)}{m_i} & \frac{k_i}{m_i} & & \dots & 0 \\ & & & \dots\dots & & & & \\ 0 & & \dots & & 0 & & \frac{k_{N-1}}{m_N} & -\frac{k_{N-1}}{m_N} \end{pmatrix}.$$

The model parameters of the considered six storeys building are reported in Table 7.2.

Taking into account the presence of the isolator and given the choice of the C matrix, the controlled output is related to the acceleration at the base floor. Concerning the choice of the IO-FTS parameters, for a given geographic area these can be chosen starting from the worst earthquake on record. Indeed, from the time trace of the ground acceleration, velocity and displacement of the *El Centro* earthquake (May 18, 1940) reported in Figure 7.5, the following IO-FTS parame-

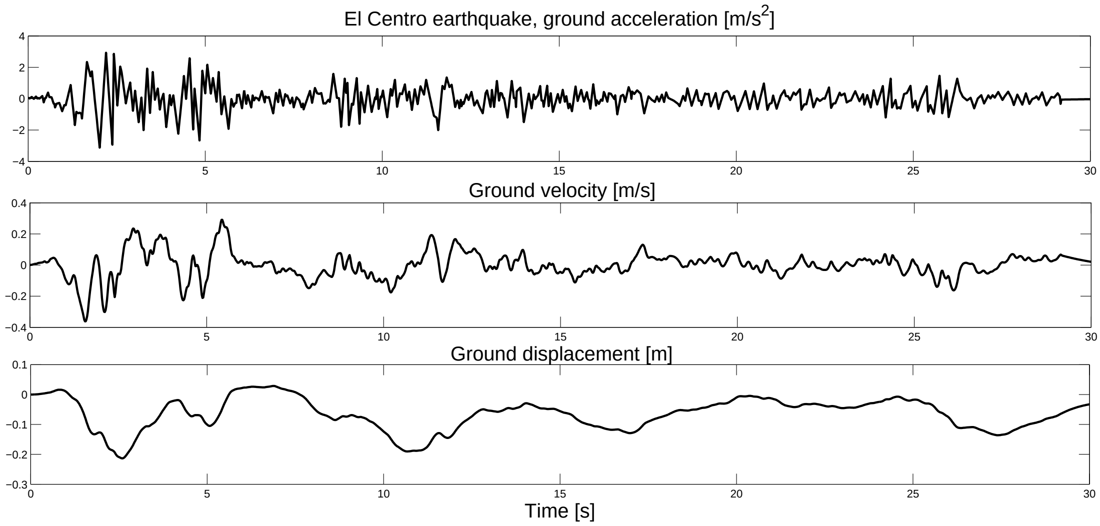


Figure 7.5: Ground acceleration, velocity and displacement of El Centro earthquake (California, May 18, 1940).

ters have been considered

$$R = I\,,\; Q = 0.1\,,\; \Omega = \big[0\,,35\big]\,. \tag{7.3}$$

Indeed, given the considered output of the building model, the values for the matrices W and Q should assure that

$$|s_1(t)| \leq 10 \text{ cm and } |\dot{s}_1(t)| \leq 1.5 \text{ cm/s}\,, \tag{7.4}$$

when an earthquake having a magnitude and a duration similar to the *El Centro* occurs.

By means of simulation it can be verified that the considered system is not open-loop IO-FTS with respect to the chosen parameters; hence it does not meet the constraints (7.4), as shown in Figure 7.6.

Exploiting Theorem 7, and assuming for $S(\cdot)$ and $T(\cdot)$ the same structure foreseen for the $P(\cdot)$ matrix-valued function in Section 7.2, it is possible to find the controller matrix-valued functions $A_k(\cdot)$, $B_k(\cdot)$, $C_k(\cdot)$, and $D_k(\cdot)$ that make system (7.2) IO-FTS with respect to the parameters given in (7.3), when $\mathcal{W}_2$ disturbances are considered.

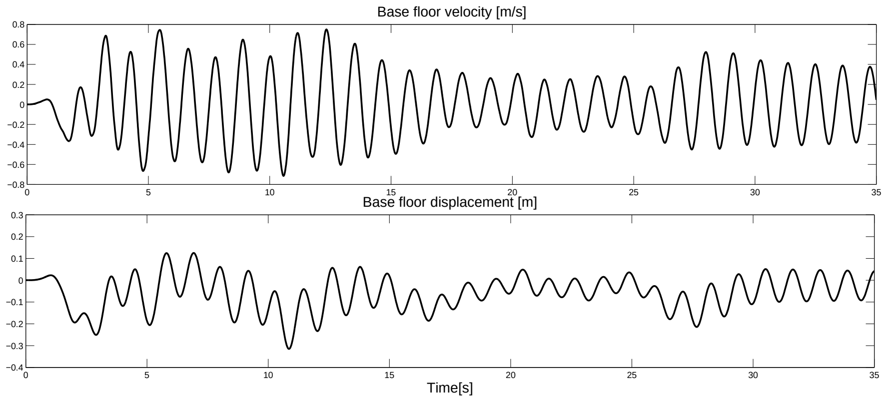


Figure 7.6: Uncontrolled base floor velocity and displacement.

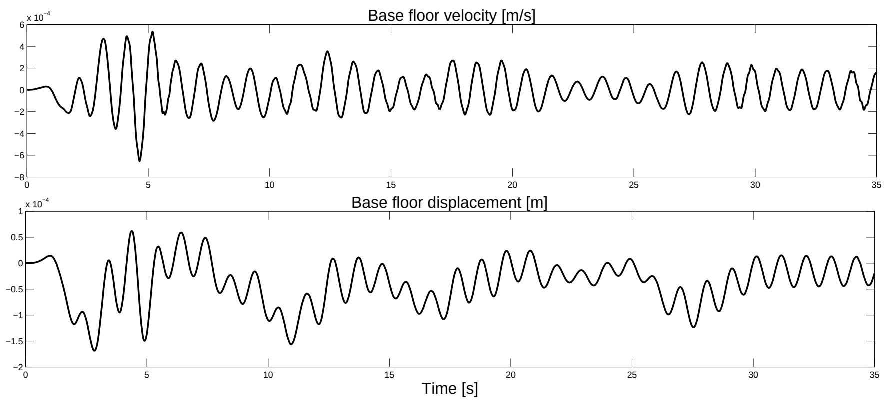


Figure 7.7: Controlled base floor velocity and displacement.

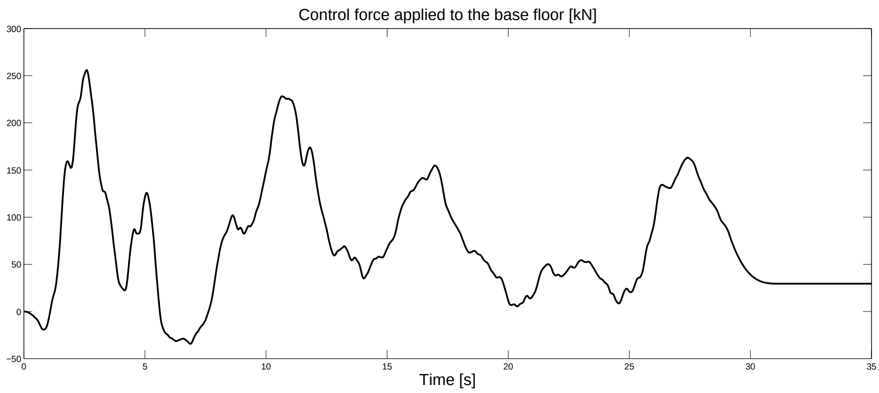


Figure 7.8: Control force applied to the base floor.

Figure 7.6 shows the base floor velocity and displacement time traces for the uncontrolled building with base isolation system, under the assumed earthquake excitation. As it can be seen in Figure 7.7, the control system manages to keep very small both the velocity and the displacement of the structure. The relative control force is depicted in Figure 7.8.

7.3 Vehicle active suspension

The problem of controlling the active suspension of a vehicle is firstly faced in this section. Afterwards, the same system model is used to show that the DLE condition of Theorem 6 is computationally more efficient with respect to the correspondent DLMI when addressing the FTS analysis of LTV systems.

7.3.1 Active suspension control

The scheme of a two-degree-of-freedom quarter-car model is reported in Figure 7.9: the system comprises the sprung mass, m_s, the unsprung mass, m_u, the suspension damper with damping coefficient b_s, the suspension spring with elastic coefficient k_s, the elastic effect caused by the tire deflection, modeled by means of a spring with elastic coefficient k_u, the hydraulic actuator S, generating a scalar active force u_f. The system model has been taken from Chen and Guo [2005], where the chosen state variables are the suspension stroke $x_s - x_u$, the vertical velocity of the sprung mass, the tire deflection $x_u - x_o$ and the vertical velocity of the unsprung mass, that is

$$\begin{aligned} x_1 &= x_s - x_u \\ x_2 &= \dot{x}_s \\ x_3 &= x_u - x_o \\ x_4 &= \dot{x}_u \end{aligned}$$

where x_s and x_u are the vertical displacement of the sprung and unsprung masses, respectively, and x_o is the vertical ground displacement caused by road unevenness.

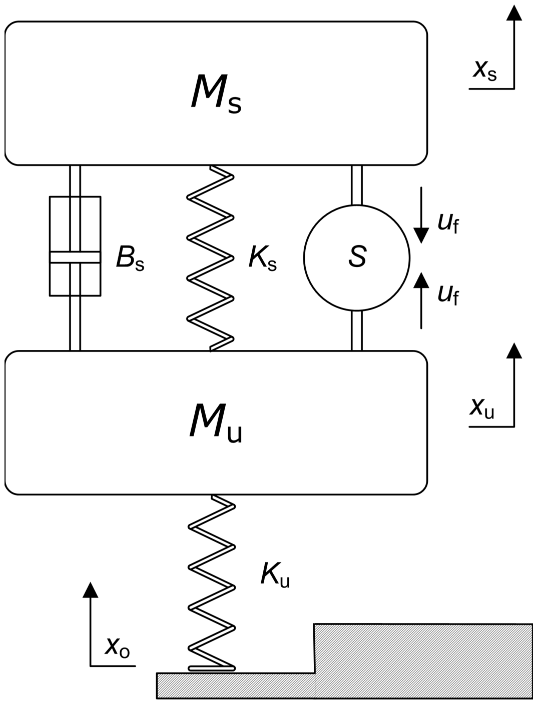


Figure 7.9: Schematic representation of the active suspension system.

The resulting open-loop dynamical model reads

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -\frac{k_s}{m_s} & -\frac{b_s}{m_s} & 0 & \frac{b_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & -\frac{k_u}{m_u} & -\frac{b_s}{m_u} \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{u_{\max}}{m_s} \\ 0 \\ -\frac{u_{\max}}{m_u} \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} w(t), \quad (7.5)$$

where the normalized active force $u(\cdot) = u_f(\cdot)/u_{\max}$ is the control input and the exogenous input $w(\cdot) = \dot{x}_o(\cdot)$ represents the disturbance caused by the road roughness.

When designing a controller for an active suspension system a number of constraints should be considered (Chen and Guo [2005]). In particular, in order to ensure a firm uninterrupted contact of wheels to road, the dynamic tire load should not exceed the static one, that is

$$k_u|x_3(t)| < (m_s + m_u)\, g \quad \forall\, t \geq 0\,, \quad (7.6)$$

Table 7.3: Model and design parameters used in the example of the active suspension system.

Parameter	Value
m_s	$320\,kg$
k_s	$18\,kN/m$
b_s	$1\,kN \cdot s/m$
k_u	$200\,kN/m$
m_u	$40\,kg$
$u_{\max}$	$1.5\,kN$
SS	$0.08\,m$

and the suspension stroke should fulfill the following constraint

$$|x_1(t)| \leq SS\,, \quad \forall\, t \geq 0\,. \tag{7.7}$$

Therefore, in order to cast the control design problem in the IO-FTS framework, the following system outputs are considered

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} \dot{x}_2(t) \\ \frac{x_1(t)}{SS} \\ \frac{k_u x_3(t)}{g(m_s+m_u)} \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} x(t) + \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} u\,, \tag{7.8}$$

where

$$\begin{aligned} C_1 &= \begin{pmatrix} -\frac{k_s}{m_s} & -\frac{b_s}{m_s} & 0 & \frac{b_s}{m_s} \end{pmatrix}\,, & D_1 &= \frac{u_{\max}}{m_s}\,, \\ C_2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}\,, & D_2 &= 0\,, \\ C_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}\,, & D_3 &= 0\,. \end{aligned}$$

The values for the model parameters (Gordon et al. [1991]; Chen and Guo [2005]) are summarized in Table 7.3.

In a real control problem one has to take into account that, due to actuator saturation, the active force is bounded by $u_{\max}$, i.e. the normalized force has to satisfy

$$|u(t)| \leq 1\,, \quad \forall\, t \geq 0\,. \tag{7.9}$$

Eventually, the objective of the active suspension is to keep as small as possible the body acceleration $\ddot{x}_s(\cdot) = \dot{x}_2(\cdot)$ on a finite-time interval.

The problem of designing the active suspension control system can be tackled in the context of structured IO finite-time stabilization. According to (5.5b), the output equation can be rewritten as

$$\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} \dot{x}_2(t) \\ \frac{x_1(t)}{SS} \\ \frac{k_u x_3(t)}{g(m_s+m_u)} \\ K(t)x(t) \end{pmatrix} = \begin{pmatrix} C_1 + D_1 K(t) \\ C_2 + D_2 K(t) \\ C_3 + D_3 K(t) \\ K(t) \end{pmatrix} x(t)\,. \tag{7.10}$$

The time-varying controller $K(t)$ will be designed trying to optimize the response to an *isolated bump* modeled as the $\mathcal{W}_2$ disturbance

$$w(t) = \begin{cases} \frac{M}{2}\left(1 - \cos\left(\frac{2\pi V}{L} t\right)\right)\,, & 0 \le t \le \frac{L}{V} \\ 0\,, & t > \frac{L}{V} \end{cases} \tag{7.11}$$

where $M = 0.1\,m$, $L = 5\,m$ are the bump height and width, respectively, while $V = 45\ km/h$ is the vehicle forward velocity.

In particular, given the bump (7.11) the aim of the controller is to minimize the body acceleration $y_1(t) = \dot{x}_2(t)$ fulfilling the constraints (7.6)-(7.9). It turns out that the following IO-FTS parameters must be chosen

$$T = 2\ s\,, \quad R = 8\,.$$

Furthermore, given the selected outputs (7.10), the two outputs weighting matrices

$$Q_2 = Q_3 = 1\,,$$

allows to take into account the constraints (7.6) and (7.7), while the input weighting matrix is

$$T_1 = 0.15\,,$$

which allows to exploit the full scale of the control input when the exogenous input (7.11) is considered.

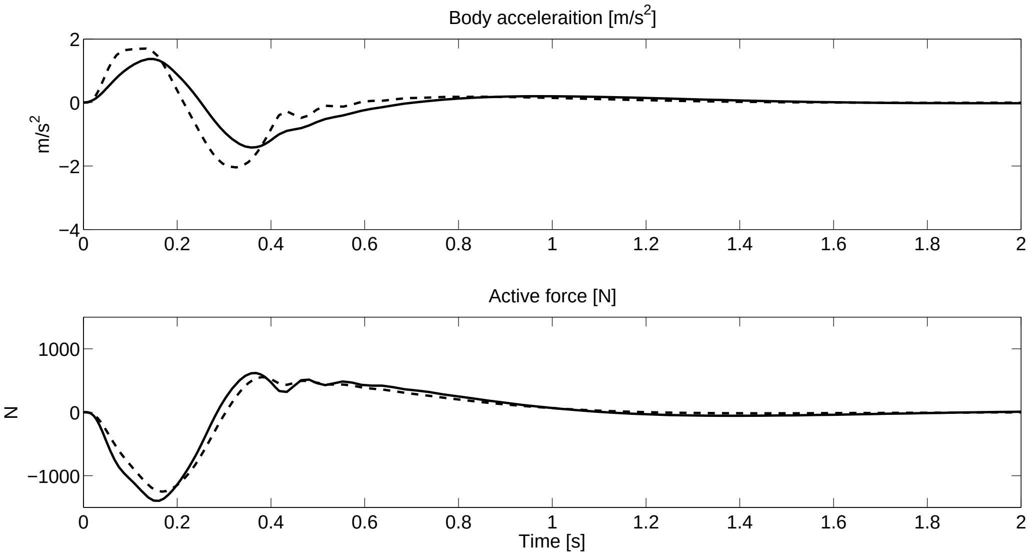


Figure 7.10: Bump response: IO-FTS time-varying controller (–), constrained $\mathcal{H}_\infty$ controller (- -).

In order to minimize the body acceleration it is possible to exploit Theorem 11 and solve the following optimization problem

$$\begin{aligned} &\text{minimize } \Xi_1 \\ &\text{subject to (5.15)} \end{aligned} \tag{7.12}$$

where $\Xi_1 = Q_1^{-1}$.

Assuming the two matrix-valued functions $\Pi(\cdot)$ and $L(\cdot)$ to be piecewise linear, it is possible to recast problem (7.12) in the LMIs framework (see Section 7.1), and hence solve it by using off-the-shelf optimization tools such as the Matlab LMI Toolbox® (Gahinet et al. [1995]).

By solving (7.12), the two feasible matrix-valued functions $\Pi(\cdot)$ and $L(\cdot)$ are obtained; moreover the minimum value of Ξ_1 has been found as $\Xi_{1_{\min}} = 7.22$. The time-varying controller $K(t)$ is then given by $K(t) = L(t)\Pi(t)^{-1}$.

Figure 7.10 shows the comparison between the proposed time-varying controller $K(t)$ and the constrained $\mathcal{H}_\infty$ controller proposed in Chen and Guo [2005]. It can be seen that, although it slightly increases the active control force, the structured IO-FTS controller reduces the maximum body acceleration for the con-

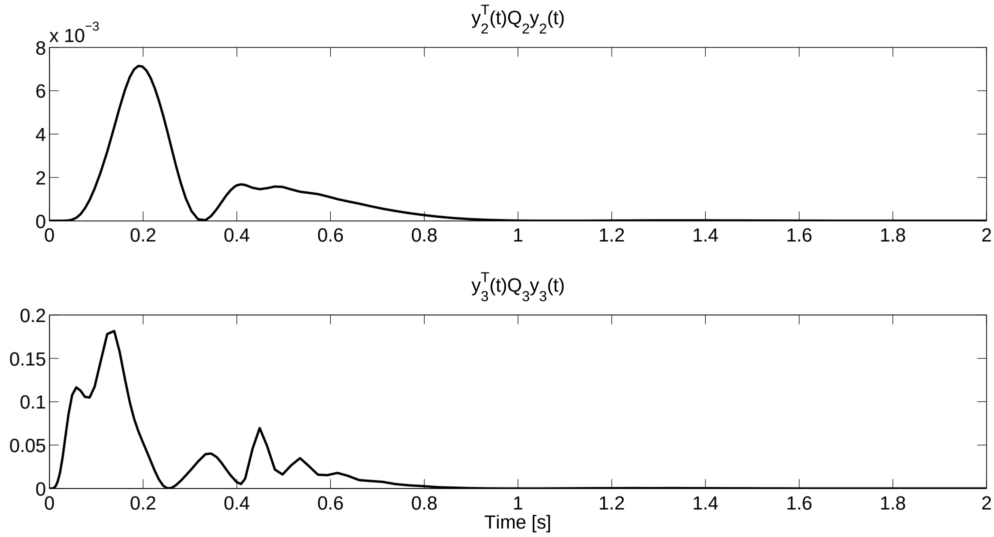


Figure 7.11: Bump response: time behavior of the weighted outputs $y_2(t)^T Q_2 y_2(t)$ and $y_3(t)^T Q_3 y_3(t)$ when the IO-FTS time-varying controller is considered.

sidered bump. Furthermore, Figure 7.11 shows the time behavior of the constraints on $y_2(\cdot)$ and $y_3(\cdot)$.

7.3.2 Comparison of DLE and DLMI conditions to check IO-FTS

The conditions stated in Theorem 6 are, in principle, necessary and sufficient. However, due to the time-varying nature of the involved matrices, the numerical implementation of such conditions introduces some conservativeness.

The two distinct but equivalent conditions given in Theorem 6 will be used in this section to check the IO-FTS of the quarter-car model introduced in section 7.3.1. The aim is to compare the DLE and DLMI conditions given in Theorem 6, from the computational point of view. In order to do that, the output weighting matrix is left as a free parameter. More precisely, the parameter Q_{max} is defined as the maximum value of the matrix Q such that system (7.5), when only $y_1(t)$ is considered as output, is IO-FTS.

The conditions stated in Theorem 6 are used to obtain an estimate of Q_{max}.

To recast the DLMI condition (4.18) in terms of LMIs, the matrix-valued func-

Table 7.4: Maximum values of Q satisfying Theorem 6 for the LTV system (7.5).

IO-FTS condition	Sample Time (T_s)	Estimate of Q_{max}	Computation time [s]
DLMI (6.11)	0.05	$1.93 \cdot 10^{-4}$	12.5
	0.025	$1.99 \cdot 10^{-4}$	106
	0.0125	$2.04 \cdot 10^{-4}$	1150
	0.01	$2.05 \cdot 10^{-4}$	2005
	0.008	$2.06 \cdot 10^{-4}$	4915
Solution of (4.10) and inequality (4.17)	$2 \cdot 10^{-5}$	$2.12 \cdot 10^{-4}$	9

tions $P(\cdot)$ has been assumed piecewise linear; the solution of the feasibility problem (4.18) has been found by exploiting standard optimization tools such as the Matlab LMI Toolbox® (Gahinet et al. [1995]).

Since the equivalence between IO-FTS and condition (6.11) holds when $T_s \mapsto 0$, the maximum value of Q satisfying condition (4.18), namely Q_{max}, has been evaluated for different values of T_s. The obtained estimates of Q_{max}, the corresponding values of T_s and of the computation time are shown in Table 7.4. These results have been obtained by using a PC equipped with an Intel®i7-720QM processor and 4 GB of RAM.

The problem of finding the maximum value of Q satisfying condition (4.17), where $W(\cdot,\cdot)$ is the positive semidefinite solution of (4.10) has been considered. In particular, equation (4.10) has been firstly integrated, with a sample time $T_s = 2 \cdot 10^{-5}\,s$, by using the Euler forward method, and then the maximum value of Q satisfying condition (4.17) has been evaluated by means of a linear search. As a result, it has been found the estimate $Q_{max} = 2.12 \cdot 10^{-4}$, with a computation time of about 9 s, as it is shown in the last row of Table 7.4.

This example shows that the condition based on the reachability Gramian is much more efficient with respect to the solution of the DLMI when considering the IO-FTS analysis problem; however, as said, the DLMI feasibility problem is necessary in order to solve the stabilization problem discussed in Section 7.2.

Chapter 8

Conclusions

The problem of stability on a finite-time horizon has been dealt with in this thesis by considering the two different concepts of Finite-Time Stability (FTS) and Input-Output Finite-Time Stability (IO-FTS).

The previous literature on both FTS and IO-FTS has been analyzed in Chapter 1, highlighting the main contributions introduced by the author.

Chapter 2 introduced the formal concepts of FTS and IO-FTS and the main differences with other classical control definitions. The main systems considered in this thesis have also been introduced, namely Linear Time Varying (LTV) systems and Time-Dependent Switching Linear Systems (TD-SLSs), together with the control design problems dealt with.

The FTS problem for the class of TD-SLSs has been addressed in Chapter 3. In order to apply the FTS to more real engineering situations where the change of system dynamics is not predictable, the assumption on the knowledge of the resetting times has been removed. The additional cases in which the resetting times are known with a given uncertainty or completely unknown have been analyzed, showing that the reduction of the uncertainty intervals reduces the conservatism of the conditions to check FTS.

Necessary and sufficient conditions have been also provided to check IO-FTS in the case of exogenous input of class $\mathcal{L}_2$. Chapter 4 showed that the condition given in Amato et al. [2010a], which involves a DLMIs feasibility problem, is actually also *necessary*. Moreover, the same chapter proposed an alternative necessary and sufficient condition involving a Differential Lyapunov Equation (DLE). While the condition based on the DLE is more effective from a numerical point of view, the DLMIs formulation turns out to be useful for design purposes. Eventually, the analysis condition based on DLMIs was used to design an output feedback dynamic controller.

In order to consider also the practical situations where the controller needs to be designed with the constraint of limiting the effort on the control inputs, the new definition of *Structured* IO-FTS has been introduced in Chapter 5. A fictitious system is built, in which the output vector is augmented with the control input variables, which are conceptually dealt with in the same way as the actual outputs. A necessary and sufficient condition ($\mathcal{L}_2$ inputs) and a sufficient condition ($\mathcal{L}_\infty$ inputs) for structured IO-FTS (open loop system) were firstly given. Then, a necessary and sufficient condition and a sufficient condition for IO finite-time stabilization with constrained control inputs were stated in the $\mathcal{L}_2$ and $\mathcal{L}_\infty$ context, respectively.

Some of the results just mentioned in the context of IO-FTS were applied to particular classes of hybrid systems. Chapter 6 firstly addressed the IO-FTS analysis and design control problem in the case of Time-Dependent (TD) and State-Dependent Impulsive Dynamical Linear Systems (SD-IDLS), which are linear continuous-time systems whose state undergoes finite jump discontinuities at discrete instants of time. The second part of the chapter extended the same results to the TD-SLS, in the very important case in which the set of resetting times is unknown.

Eventually, Chapter 7 presented a number of applications and examples to show the effectiveness and usefulness of the main theoretical results introduced in this thesis.

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Curriculum vitae

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Automation engineering, University of Naples Federico II

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Publications

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