DOTTORATO DI RICERCA

in

SCIENZE MATEMATICHE

Ciclo XXV

SEDE AMMINISTRATIVA: UNIVERSITÀ DI NAPOLI FEDERICO II

ROBERTA SCHIATTARELLA

SOBOLEV AND BOUNDED VARIATION HOMEOMORPHISMS IN $\mathbb{R}^n$

TESI DI DOTTORATO DI RICERCA

TUTORE: Ch.mo Prof. Carlo Sbordone

COORDINATORE: Ch.mo Prof. Francesco de Giovanni

Contents

1 Lusin's condition $\mathcal{N}$ and related results **9**

1.1 Basic properties and definitions 9

1.2 Lusin's condition for Sobolev homeomorphisms 18

2 Sobolev and BV homeomorphism: the regularity of the inverse. **23**

2.1 Bi–Sobolev mappings 23

2.2 Regularity of the inverse of a Sobolev homeomorphism with finite distortion 27

2.3 Regularity of the inverse of a homeomorphism of bounded variation 31

2.4 Composition of bi–Sobolev homeomorphisms 46

3 Some pathological examples of Sobolev homeomorphisms **53**

3.1 Sign of the Jacobian 53

3.2 Sobolev homeomorphism with zero Jacobian almost everywhere 54

3.3 Bi–Sobolev homeomorphism with zero Jacobian almost everywhere 57

4 On the continuity of the Jacobian of orientation preserving mappings **71**

4.1 The integrability of the Jacobian 71

4.2 The continuity of the Jacobian 73

Introduction

In recent years there has been an increasing interest in the theory of deformations in non linear elasticity. Let us quote here the pioneering works of Antman, Ball and Ciarlet ([4], [5], [14]).

The domain $\Omega \subset \mathbb{R}^n$ is view as a solid body in the space and the mapping $f : \Omega \to \mathbb{R}^n$ as a deformation of the body Ω to $f(\Omega)$.

The questions that naturally arise are the following:

- Does f satisfy Lusin $\mathcal{N}$ condition; i. e. does f map null sets to null sets? That means, is a “new material ”created from “nothing ”?
- If f is invertible, what are the properties of the inverse map f^{-1}?
- If f is sufficiently smooth, what are the properties of its Jacobian?

The central concept running through all the thesis is the definition of bi–Sobolev mapping, originally proposed in [48]. We recall that a homeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is called a *bi–Sobolev mapping* if $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ and its inverse $f^{-1} \in W^{1,1}_{\text{loc}}(\Omega', \mathbb{R}^2)$.

The thesis is organized as follows.

In Chapter 1 we recall some definitions and properties of the approximate gradient of a Borel map. Moreover, we focus our attention on the Lusin $\mathcal{N}$

condition and we discuss what kind of mappings satisfy this property. In [78], Y. G. Reshetnyak showed that every homeomorphism of class $W^{1,n}$ satisfies Lusin $\mathcal{N}$ condition. An improvement of this result is due to J. Kauhanen, P. Koskela, and J. Malý ([62]). Using a result of L. Greco ([33]) on the equality between pointwise and distributional Jacobian, they relaxed the regularity assumption $f \in W^{1,n}$ into the setting of the closure of $W^{1,n}$ into the norm of Grand Sobolev space. We also investigate connections between Lusin $\mathcal{N}$ condition and area formula.

The main topic of Chapter 2 is the regularity of the inverse of Sobolev and BV homeomorphisms.

In general, one cannot expect the same regularity for f and its inverse. Indeed, there are examples of Lipschitz homeomorphisms whose inverse are not in $W^{1,1}_{\text{loc}}$ (see Section 2). In the planar case, S. Hencl, P. Koskela and J. Onninen in [44] proved that if $f\colon \Omega \overset{\text{onto}}{\longrightarrow} \Omega'$ is a homeomorphism of bounded variation then so does its inverse map $f^{-1} = (x, y)\colon \Omega' \to \Omega$. We present a different proof giving precise formulae for the total variations of the coordinate functions of f^{-1}. Extensions to higher dimension into the Sobolev case are also considered.

As an application of above results, we study the composition operator of two Sobolev homeomorphisms. Some of these results can be found in [18], [22], [35].

In Chapter 3 we investigate how big can be the zero set of the Jacobian determinant of a homeomorphism f. The null set of the Jacobian determinant of a bi–Sobolev map could have positive measure (see [53], Section 6.2.6). Moreover, we are able to construct for $n \geq 3$ a pathological example of a bi–Sobolev map with $J_f = J_{f^{-1}} = 0$ almost everywhere. Such a pathological homeomorphism cannot exists in dimension $n = 2$ or with higher regularity $f \in W^{1,n-1}$ (see [19]).

In the last Chapter we investigate the continuity in L^1 norm of the Jacobian determinant of orientation preserving maps belonging to the Grand Sobolev space $W^{1,n)}(\Omega, \mathbb{R}^n)$. Let us recall that if $J_f \geq 0$ a.e., then $|Df| \in L^{n)}(\Omega)$ is the weakest assumption to guarantee that J_f is locally integrable (see [55]).

Chapter 1

Lusin's condition $\mathcal{N}$ and related results

The main topic of this chapter is the study of Lusin's condition $\mathcal{N}$. We focus our attention on some of its consequences and then investigate what are the minimal integrability conditions on the partial derivatives of a Sobolev homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ needed to guarantee that f satisfies it.

1.1 Basic properties and definitions

Let Ω be a bounded domain in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^n$ a mapping of the Sobolev class $W^{1,p}(\Omega, \mathbb{R}^n)$. Here, $W^{1,p}(\Omega, \mathbb{R}^n)$ consists of all p-integrable mappings of Ω into $\mathbb{R}^n$ whose coordinate functions have p-integrable distributional derivatives.

We denote by $Df(z)$ the differential matrix of f at the point $z \in \Omega$ and by $J_f(z)$ the Jacobian determinant of Df:

$$J_f(z) = \det Df(z).$$

The norm of $Df(z)$ is defined as follows:

$$|Df(z)| = \sup_{\xi \in \mathbb{R}^n, |\xi|=1} |Df(z)\xi|.$$

It is easy to see that other norms are equivalent.

We need to recall the definition of approximate gradient of a Borel map (see [32]).

For any measurable set $A \subset \mathbb{R}^n$, the *upper density* $\theta^*(A, z)$ of A at z is defined as

$$\limsup_{r \to 0} \frac{|B(z,r) \cap A|}{|B(z,r)|}.$$

Similarly, the lower density $\theta_*(A, z)$ of A at z is given by

$$\liminf_{r \to 0} \frac{|B(z,r) \cap A|}{|B(z,r)|}.$$

The density of A at z is defined whenever $\theta^*(A, z) = \theta_*(A, z)$ as the common value:

$$\theta(A, z) = \theta^*(A, z) = \theta_*(A, z).$$

Let $u : \Omega \longrightarrow \mathbb{R}$ be a measurable function. We recall that l is the *approximate limit* of u at z when ρ tends to z in Ω, and we write

$$l = \operatorname{ap}\lim_{\rho \to z} u(\rho)$$

if for all $\varepsilon > 0$, the set

$$\Omega_\varepsilon = \{\rho \in \Omega \mid |u(\rho) - l| \geq \varepsilon\}$$

has density 0 at z.

The *approximate upper limit* of u at z is defined as the number (eventually $+\infty$ or $-\infty$) given by

$$\operatorname{ap}\limsup_{\rho \to z} u(\rho) = \inf\{t \mid \theta^*(E_t^+, z) = 0\}$$

where $E_t^+ = \{z \in \Omega | u(z) > t\}$.

Similarly, the *approximate lower limit* is given by

$$\operatorname{ap}\liminf_{\rho \to z} u(\rho) = \sup\{t | \theta^*(E_t^-, z) = 0\}$$

where $E_t^- = \{z \in \Omega | u(z) < t\}$.

Of course,

$$\operatorname{ap}\liminf_{\rho \to z} u(\rho) \leq \operatorname{ap}\limsup_{\rho \to z} u(\rho)$$

and whenever

$$\operatorname{ap}\liminf_{\rho \to z} u(\rho) = \operatorname{ap}\limsup_{\rho \to z} u(\rho) = l \in \mathbb{R}$$

the approximate limit exists and we have

$$\operatorname{ap}\lim_{\rho \to z} u(\rho) = l.$$

We say that $u : \Omega \longrightarrow \mathbb{R}$ is approximately continuous at z if

$$u(z) = \operatorname{ap}\lim_{\rho \to z} u(\rho).$$

One can also prove

Proposition 1.1. *If the function $u : \Omega \longrightarrow \mathbb{R}$ is measurable then u is approximately continuous at a.e. $z \in \Omega$.*

The previous Proposition is an easy consequence of the Lebesgue differentation theorem in the case of L^1 maps. We have in fact

Proposition 1.2. *Let $u \in L^1_{\text{loc}}(\Omega)$. If there exists $\tau \in \mathbb{R}$ such that*

$$\lim_{r \to 0} \fint_{B_r(z)} |u(\rho) - \tau| d\rho = 0 \tag{1.1}$$

then,

$$\operatorname{ap}\lim_{\rho \to z} u(\rho) = \tau. \tag{1.2}$$

In particular, as a consequence, the approximate limit exists at each Lebesgue point z.

In the same spirit of approximate limits and approximate continuity, we may now introduce the notion of approximate differential.

Definition 1.1. *Let Ω be a measurable subset of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ be a measurable map. Suppose that $z \in \mathbb{R}^n$ be such that $\theta^*(A, z) > 0$. We say that u is* approximately differentiable *at z if there exists a linear mapping*

$L : \mathbb{R}^n \longrightarrow \mathbb{R}$ *such that:*

$$\operatorname{ap}\limsup_{\rho \to z} \frac{|u(\rho) - u(z) - L(\rho - z)|}{|\rho - z|} = 0.$$

From the definition, it is not difficult to show that the approximate differential, denoted by $\operatorname{ap}Du(z)$ is unique whenever it exists.

Theorem 1.1. *Assume $u \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$. Then u is approximately differentiable a.e. and its approximate derivative equals its weak derivative a.e.:*

$$\operatorname{ap}Du(z) = Du(z) \qquad a.\ e.$$

In geometric function theory we are interested in the study of mappings and their properties. One of the most important properties is the following:

Definition 1.2. *Let $f : \Omega \to \mathbb{R}^n$ be a measurable mapping. We say that f satisfies Lusin's condition $\mathcal{N}$ if for each $E \subset \Omega$,*

$$|E| = 0 \quad \Rightarrow \quad |f(E)| = 0.$$

This property has the following physical interpretation. If we imagine that Ω is a body in the space that is subjected to a deformation f, "new material" cannot be crated from "nothing ".

For the sake of completeness we review some of the standard results together with their proofs.

Let Ω and Ω' be bounded domains in $\mathbb{R}^n$and let us denote by $Hom(\Omega, \Omega')$ the set of all homeomorphisms $f : \Omega \to \Omega' = f(\Omega)$.

Proposition 1.3. *$f \in \mathrm{Hom}(\Omega, \Omega')$ satisfies Lusin's condition $\mathcal{N}$ if, and only if, f takes measurable sets to measurable sets.*

Proof. Let us assume that f satisfies the condition $\mathcal{N}$. If $A \subset \Omega$ is a measurable set, then there exists a Borel set B such that $B \subset A \subset \Omega$ and $|B \backslash A| = 0$. Then we have $|f(B \backslash A)| = |f(B) \backslash f(A)| = 0$ and hence $f(B) \backslash f(A)$ is measurable. Moreover, as f is a homeomorphism, $f(B)$ is a Borel set and hence $f(A)$ is also measurable. Conversely, by contradiction suppose that $E \subset \Omega$ verifies $|E| = 0$ and $|f(E)| > 0$. If $A' \subset f(E)$ is a non measurable set then, $f^{-1}(A')$ is a measurable set of measure zero. By assumption also $A' = f(f^{-1}(A'))$ is measurable as well and there is a contradiction. $\square$

Proposition 1.4. *$f \in \mathrm{Hom}(\Omega, \Omega')$ satisfies Lusin's condition $\mathcal{N}$ if, and only if, $|f(E)| = 0$ whenever $E \subset\subset \Omega$ is a compact set with zero measure.*

Proof. Let E be a compact set with zero measure such that $|f(E)| = 0$. Then, there exists a Borel set $B \supset E$ such that $|B| = 0$. Suppose by contradiction that $|f(B)| > 0$. Hence, there exists a compact set $C' \subset f(B)$ such that $|C'| = |f(B)| > 0$. But f is a homeomorphism and $f^{-1}(C')$ is compact and $|f^{-1}(C')| \leq |B| = 0$. This is not possible by assumption. $\square$

The condition $\mathcal{N}$ is strongly connected with the validity of *the area formula* which is crucial in the next developments. An advanced version of the theorem on change of variables is due to Federer [[25], Theorem 3.2.3]. It states that the *area formula*:

$$\int_{\Omega} \eta(f(z))|J_f(z)|\, dz = \int_{\mathbb{R}^n} \Bigg(\sum_{z \in \Omega:\, f(z) = w} \eta(w) \Bigg)\, dw \tag{1.3}$$

is valid for all measurable functions $\eta : \mathbb{R}^n \longrightarrow [0, +\infty)$, if f is a Lipschitz map.

By Theorem 1.1, the set of points of approximate differentiable of f

$$\mathcal{A}_{\mathcal{D}}(f) = \{z \in \Omega \,|\, f \text{ is approximately differentiable at} z\}$$

is a set of full measure.

Let us consider the set:

$$\mathcal{A}_{\mathcal{L}}(f) = \left\{ z \in \Omega \,\middle|\, \operatorname{ap}\limsup_{w \to z} \frac{|f(w) - f(z)|}{|w - z|} < \infty \right\}.$$

Trivially, $\mathcal{A}_{\mathcal{D}}(f) \subset \mathcal{A}_{\mathcal{L}}(f)$. By Theorem 3.1.8 of [25] , $A_D(f)$ is the union of a countable family of measurable sets such that the restriction of f to each member of the family is Lipschitz and hence the formula (1.3) becomes:

$$\int_{\Omega} \eta(f(z))|J_f(z)|\, dz \leq \int_{\mathbb{R}^n} \left(\sum_{z \in \Omega :\, f(z) = w} \eta(w) \right) dw \tag{1.4}$$

Moreover, in (1.4), there is an equality if f satisfies Lusin's condition $\mathcal{N}$. Notice that the area formula holds on the set where f is approximately differentiable; in fact, the Lusin's condition $\mathcal{N}$ holds on $A_D(f)$.

From (1.4), we can derive the following:

Theorem 1.2 (Area formula for Sobolev homeomorphisms)**.** *Let $f : \Omega \longrightarrow \mathbb{R}^n$ be a Sobolev homeomorphism, $\eta : \mathbb{R}^n \longrightarrow [0, +\infty)$ be a nonnegative Borel measurable function and A be a Borel measurable set. Then:*

$$\int_A \eta(f(z))|J_f(z)|\, dz \leq \int_{f(A)} \eta(w)\, dw. \tag{1.5}$$

The equality:

$$\int_A \eta(f(z))|J_f(z)|\,dz = \int_{f(A)} \eta(w)\,dw \tag{1.6}$$

is verified if f satisfies Lusin's condition $\mathcal{N}$.

When (1.6) occurs we say that the *area formula* holds for f on A.

We note the following consequence of (1.5). If $A' \subset f(A)$ is a Borel subset with $|A'| = 0$, then $J_f(z) = 0$ for a.e. $z \in f^{-1}(A')$. Indeed,

$$\int_{f^{-1}(A')} |J_f(z)|\,dz \leq \int_{A'} dw = |A'| = 0.$$

Moreover, it is well known that there exists a set $\bar{\Omega} \subset \Omega$ of full measure $|\bar{\Omega}| = |\Omega|$, such that:

$$\int_{\bar{\Omega}} \eta(f(z))|J_f(z)|\,dz = \int_{f(\bar{\Omega})} \eta(w)\,dw. \tag{1.7}$$

As a consequence, if f is a Sobolev homeomorphism with f^{-1} satisfying the $\mathcal{N}$, then $J_f(z) > 0$ for almost every $z \in \Omega$. Indeed, by (1.7),

$$|f\left(z \in \bar{\Omega} : J_f(z) = 0\right)| = 0.$$

Hence, by the $\mathcal{N}$ for f^{-1} and since $\bar{\Omega}$ has full measure:

$$|z \in \Omega : J_f(z) = 0| = |\left(z \in \bar{\Omega} : J_f(z) = 0\right) \cup \left(\Omega \setminus \bar{\Omega}\right)| = 0.$$

Thus in particular, as $\mathcal{A}_{\mathcal{D}}(f)$ is a set of full measure, the image of the set of all critical points has zero measure:

$$|f\left(\{z \in \Omega : f \text{ is approximately differentiable at } z \text{ and } J_f(z) = 0\}\right)| = 0.$$

This is a weak version of the classical Sard theorem.

An interesting application of condition $\mathcal{N}$ is the following result on the inverse of an a.e. approximately differentiable homeomorphism.

Proposition 1.5. *Let $f \in \mathrm{Hom}(\Omega; \Omega')$ be approximately differentiable a.e.. If f verifies Lusin's condition $\mathcal{N}$, then the inverse f^{-1} is approximately differentiable a.e.*

Proof. We decompose the set $\mathcal{A}_{\mathcal{D}}(f)$ of points of approximately differentiable of f as follows:

$$\mathcal{A}_{\mathcal{D}}(f) = \mathcal{R}_f \cup \mathcal{Z}_f$$

where

$$\mathcal{R}_f = \{z \in \Omega : f \text{ is approximately differentiable at } z \text{ and } J_f(z) \neq 0\}$$

and

$$\mathcal{Z}_f = \{z \in \Omega : f \text{ is approximately differentiable at } z \text{ and } J_f(z) = 0\}.$$

Moreover, we consider

$$\mathcal{E}_f = \{z \in \Omega : f \text{ is not approximately differentiable at } z\}$$

and hence

$$\Omega = \mathcal{Z}_f \cup \mathcal{R}_f \cup \mathcal{E}_f.$$

By the weak version of the classical Sard Lemma, $|f(\mathcal{Z}_f)| = 0$. Since f^{-1} is approximately differentiable a.e., $\mathcal{E}_f$ has zero measure and by condition $\mathcal{N}$, $f(\mathcal{E}_f)$ has zero measure.

We notice that f^{-1} is approximately differentiable in $f(\mathcal{R}_f)$ which is a subset of full measure of $f(\Omega)$; indeed,

$$f(\Omega) \setminus f(\mathcal{R}_f) \;=\; f(\mathcal{Z}_f) \cup f(\mathcal{E}_f)$$

has zero measure. $\square$

1.2 Lusin's condition for Sobolev homeomorphisms

We study conditions under which a map f could satisfy Lusin's condition $\mathcal{N}$.

Firstly, we consider the case of real- valued functions. A function f is called absolutely continuous in a set E if the following condition is satisfied: for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\sum_{i=1}^{N} |f(b_i) - f(a_i)| < \varepsilon$$

for every collection $\{(a_i, b_i)\}$ of pairwise disjoint intervals with

$$\sum_{i=1}^{N}(b_i - a_i) < \delta.$$

An absolutely continuous function clearly satisfies Lusin's condition $\mathcal{N}$ and hence also every Lipschitz function sends sets of measure zero to sets of measure zero. We note that by Rademacher's theorem, locally Lipschitz functions are differentiable almost everywhere. However, differentiability a.e. is not enough to conclude Lusin's condition $\mathcal{N}$ (see for example the Cantor function).

In dimension $n \geq 2$, the situation is much more complicated. M. Marcus and V. Mizel, in [67], proved that Lusin's condition $\mathcal{N}$ holds if f is a continuous map in $W^{1,p}$ provided $p > n$. The Lusin condition may fail for continuous mappings in $W^{1,n}$. L. Cesari ([13]) demonstrated that there exists a continuous map $f \in W^{1,n}([-1,1]^n, [-1,1]^n)$ with $n \geq 2$ such that

$$f\left([-1,1] \times \{0\}^{n-1}\right) = [-1,1]^n$$

and hence f does not satisfy the Lusin's condition $\mathcal{N}$. Moreover, in the paper [68], O. Martio and W. Ziemer investigated how condition $\mathcal{N}$ is related to mappings in the Sobolev space $W^{1,n}(\Omega, \mathbb{R}^n)$ with nonnegative Jacobians. In particular, they showed that if f is a continuous map in $W^{1,n}(\Omega, \mathbb{R}^n)$ with $J_f > 0$ a.e. on Ω then, f satisfies Lusin's condition $\mathcal{N}$.

For a homeomorphism, less regularity is needed: it suffices to assume that $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. This is due to Reshetnyak ([77]). This result is sharp in the scale of $W^{1,p}(\Omega, \mathbb{R}^n)$ - homeomorphisms. In fact, in the case $p < n$, Ponomarev showed in [76], that for every $n > 1$ there exists a homeomorphism $f : [0,1]^n \longrightarrow [0,1]^n$ such that:

- the restriction of f to the boundary of the cube $[0,1]^n$ is the identity mapping;
- for all $0 < p < n$, f belongs to the class $W^{1,p}([0,1]^n, [0,1]^n)$;
- f does not have property $\mathcal{N}$;
- the inverse map f^{-1} belongs to the class $W^{1,p}([0,1]^n, [0,1]^n)$ with any $1 < p < \infty$.

Another example has been constructed in [62] of a homeomorphism $f \in \cap_{1 \le p < n} W^{1,p}(\Omega, \mathbb{R}^n)$ satisfying the condition

$$\sup_{0<\varepsilon\le n-1} \varepsilon \int_\Omega |Df|^{n-\varepsilon} < \infty \tag{1.8}$$

and such a mapping f does not satisfy condition $\mathcal{N}$.

Let us define the Grand Lebesgue space $L^{n)}(\Omega)$ as the collection of all measurable functions u with

$$\|u\|_{n)} = \sup_{0<\varepsilon\le n-1} \left(\varepsilon \int_\Omega |u(z)|^{n-\varepsilon}\, dz \right)^{\frac{1}{n-\varepsilon}} < \infty.$$

This Banach space was introduced in [55] for mappings $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ (not necessarily homeomorphisms) in the study of the integrability of nonnegative Jacobians under minimal integrability assumptions for $|Df|$.

We also mention the article [62] by J. Kauhanen, P. Koskela and J. Malý, where it is shown that the sharp regularity assumption to rule out the failure of the condition $\mathcal{N}$ can be the following one:

$$\lim_{\varepsilon\to 0^+} \varepsilon \int_\Omega |Df|^{n-\varepsilon} = 0 \tag{1.9}$$

under the condition that the Jacobian determinant is non negative a.e.. When (1.9) occurs, we write $|Df| \in L_b^{n)}(\Omega)$. This function space is the closure of bounded functions in $L^{n)}$ (see [33]). Moreover, the following inclusions hold:

$$L_b^{n)}(\Omega) \subset L^{n)}(\Omega) \subset \bigcap_{p<n} L^p(\Omega).$$

We want underline that if $|Df| \in L_b^{n)}$ then, the weak Jacobian of the mapping f coincides with the pointwise Jacobian by a result of L. Greco (see [33]).

Let us now introduce a space slightly larger than $L^n(\Omega)$ and then we shall examine some relations between $L_b^{n)}$ and other classes of functions.

Our main source here will be [53, Section 4.12]. We shall need to consider the Zygmund space $L^p \log^\alpha L(\Omega)$, for $1 \leqslant p < \infty$, $\alpha \in \mathbb{R}$ ($\alpha \geqslant 0$ for $p = 1$), and $\Omega \subset \mathbb{R}^n$. This is the Orlicz space generated by the function

$$\Phi(t) = t^p \log^\alpha(a+t)\,, \qquad t \geqslant 0\,,$$

where $a > 0$ is a suitably large constant, so that Φ is increasing and convex on $[0,\infty[$. The choice of a will be immaterial, as we shall always consider these spaces on bounded domains. Thus, more explicitly, for a measurable function u on Ω, $u \in L^p \log^\alpha L(\Omega)$ simply means that

$$\int_\Omega |u|^p \log^\alpha(a + |u|)\, dx < \infty\,.$$

As an example, for $\alpha = 0$ we have the ordinary Lebesgue spaces. We shall consider in $L^p \log^\alpha L(\Omega)$ the Luxemburg norm

$$\|u\|_{L^p \log^\alpha L} = \inf\left\{\lambda > 0 \ : \ \int_\Omega \Phi(|u|/\lambda)\, dx \leqslant 1\right\}.$$

The following Hölder type inequality for Zygmund spaces will be important to us:

$$\|u_1 \cdots u_k\|_{L^p \log^\alpha L} \leqslant C\, \|u_1\|_{L^{p_1} \log^{\alpha_1} L} \cdots \|u_k\|_{L^{p_k} \log^{\alpha_k} L} \tag{1.10}$$

where $p_i > 1$, $\alpha_i \in \mathbb{R}$, for $i = 1, \ldots, k$, and

$$\frac{1}{p} = \frac{1}{p_1} + \cdots \frac{1}{p_k}\,, \qquad \frac{\alpha}{p} = \frac{\alpha_1}{p_1} + \cdots \frac{\alpha_k}{p_k}$$

The positive constant C in (1.10) is independent of u_i.

We shall write $u \in L^p \log^\alpha L_{\mathrm{loc}}(\Omega)$ if $u \in L^p \log^\alpha L(E)$, for every compact subset E of Ω.

For the Zygumund space $\dfrac{L^n}{\log L}$, the following inclusions hold:

$$L^n \subset \frac{L^n}{\log L} \subset L^n_b \subset L^{n)}.$$

(see [33]).

Chapter 2

Sobolev and BV homeomorphism: the regularity of the inverse.

This chapter is concerned with the regularity of the inverse of a homeomorphism f of bounded variation in the planar case. We give precise formulae for the total variations of the coordinate functions of f^{-1}. Extensions to higher dimension are also given.

2.1 Bi–Sobolev mappings

Let Ω and Ω' be domains in $\mathbb{R}^2$. Recently, the relation between a homeomorphism $f : \Omega \longrightarrow \Omega'$ and its inverse has been intensively studied (see [43], [45], [48]).

In the class of planar homeomorphisms a crucial role is played by bi–

Sobolev mappings, originally proposed in [48]. We recall that a homeomorphism $f : \Omega \xrightarrow{\text{onto}} \Omega'$ is called a *bi–Sobolev mapping* if $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ and its inverse $f^{-1} \in W^{1,1}_{\text{loc}}(\Omega', \mathbb{R}^2)$.

A first interesting property of a bi–Sobolev map $f = (u, v)$ in the plane is that u and v have the same critical points.

Theorem 2.1. *Let $f = (u, v) : \Omega \xrightarrow{onto} \Omega'$ be a bi–Sobolev map. Then u and v have the same critical points:*

$$\{z \in \Omega \mid \nabla u(z) = 0\} \;=\; \{z \in \Omega \mid \nabla v(z) = 0\} \qquad a.\ e.$$

The same result holds also for the inverse.

The connection between the planar elliptic PDE's and Function Theory has been known since pioneering works of .B. Morrey [71], R. Caccioppoli [11], L. Bers and L. Nirenberg [10], I.N. Vekua [81] and B. Bojarski [7] (see also ([3],[65]).

A homeomorphism $f = (u, v) : \Omega \longrightarrow \mathbb{R}^2$ is *K–quasiconformal mapping* for a constant $K \geq 1$, if $f \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^2)$ and

$$|Df(z)|^2 \leq KJ_f(z) \qquad \text{a.e. } z \in \Omega. \tag{2.1}$$

The smallest constant K for which (2.1) holds almost everywhere is called the *distortion* of the mapping f.

Quasiconformal mappings possess many interesting properties, as embodied in the following

Theorem 2.2. *Let $f : \Omega \longrightarrow \Omega'$ be a K– quasiconformal mapping and let*

$g : \Omega' \longrightarrow \mathbb{R}^2$ a K'– quasiconformal mapping. Then

- *$f^{-1} : \Omega' \to \Omega$ is K– quasiconformal*
- *$g \circ f : \Omega \to \mathbb{R}^2$ is KK'– quasiconformal*
- *f satisfy the Lusin's condition $\mathcal{N}$*
- *the Jacobian determinant $J_f(z) > 0$ almost everywhere in Ω.*

Some results hold also when the distortion $K = K(z)$ is not bounded. A mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ is said to have *finite distortion* if there exists a measurable function $K\colon \Omega \longrightarrow [1, \infty)$ such that

$$|Df(z)|^n \leqslant K(z)\, J_f(z)\,. \tag{2.2}$$

Moreover, we assume that $J_f \in L^1_{\mathrm{loc}}(\Omega)$.

We note that the existence of a measurable function K finite a.e. and satisfying (2.2) amounts to saying that

$$J_f(z) = 0 \quad \Longrightarrow \quad Df(z) = 0 \qquad \text{a.e.} \tag{2.3}$$

This condition makes it possible to consider the *distortion quotient*

$$\frac{|Df(z)|^n}{J_f(z)} \qquad \text{for a.e. } z \in \Omega. \tag{2.4}$$

Hereafter the undetermined ratio $\frac{0}{0}$ is understood to be equal to 1 for z in the zero set of the Jacobian:

$$K_f(z) = \begin{cases} \dfrac{|Df(z)|^n}{J_f(z)}\,, & \text{if } J_f(z) > 0\,; \\ \\ 1\,, & \text{otherwise.} \end{cases} \tag{2.5}$$

In other words, K_f is the smallest function greater than or equal to 1 for which (2.2) holds a.e.

It is well known (see [57]) that if $f = (u, v)$ is a K–quasiconformal map then u and v lie in $W^{1,2}(\Omega)$ and satisfy the same elliptic equation

$$\text{div}A(z)\nabla u = 0 \qquad \text{and} \qquad \text{div}A(z)\nabla v = 0 \tag{2.6}$$

where $A = A(z) \in L^\infty(\Omega, \mathbb{R}^2 \times \mathbb{R}^2)$ is a symmetric matrix with $\det A = 1$ satisfying the uniform ellipticity bounds

$$\frac{|\xi|^2}{K} \leq \langle A(z)\xi, \xi\rangle \leq K|\xi|^2.$$

If $f = (u, v)$ is only a bi–Sobolev homeomorphism, there is also an interplay with the solutions of (2.6) in the sense of distributions. Indeed, it was shown in [43] and [15] that each bi-Sobolev mapping in dimension n = 2 has finite distortion and in [48], it was shown that for any such $f = (u, v)$ there corresponds a degenerate elliptic matrix $A = A_f(z)$ with eigenvalues in the interval $\left[\dfrac{1}{K(z)}, K(z)\right]$ such that u and v are very weak solutions to (2.6), i.e. $u, v \in W^{1,1}$ satisfy (2.6) in the sense of distributions.

Theorem 2.3. *To each bi–Sobolev mapping $f = (u, v) : \Omega \to \Omega'$, there* *corresponds a measurable function $A = A(z)$ valued in symetric matrices with* $\det A(z) = 1$ *that for a.e.* $z \in \Omega$ *and for all* $\xi \in \mathbb{R}^2$ *we have*

$$\frac{|\xi|^2}{K(z)} \leq \langle A(z)\xi, \xi\rangle \leq K(z)|\xi|^2.$$

where $K(z)$ denotes the distortion function of f. The components of f are very weak solutions of equation (2.6) *with finite energy, i.e.*

$$\int_\Omega \langle A(z)\nabla u, \nabla u\rangle < \infty \quad \text{and} \quad \int_\Omega \langle A(z)\nabla v, \nabla v\rangle < \infty.$$

Another important property of bi-Sobolev mappings is the following (see [43], [48])

Theorem 2.4. *Let $f\colon \Omega \to \Omega'$ be a bi-Sobolev mapping. Then*

$$\int_\Omega |Df(z)|\, dz = \int_{\Omega'} |Df^{-1}(w)|\, dw. \tag{2.7}$$

2.2 Regularity of the inverse of a Sobolev homeomorphism with finite distortion

A part of the study of mappings of finite distortion which is vital to us is the regularity of the inverse of a Sobolev homeomorphism, see [15, 43, 44, 48, 74, 28].

The bi-Sobolev assumption rules out a large class of homeomorphisms. In fact, it is possible to construct a homeomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that f is Lipschitz, but the inverse f^{-1} does not belong to $W^{1,1}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$. For example, the mapping

$$f_0 \colon (0,2) \times (0,1) \xrightarrow{\text{onto}} (0,1) \times (0,1) \qquad f_0(x,y) = (h(x), y) \tag{2.8}$$

where $h^{-1}(t) = t + c(t)$ and $c\colon (0,1) \to (0,1)$ is the usual Cantor ternary function, is a Lipschitz homeomorphism whose inverse f_0^{-1} does not belong to $W^{1,1}_{\mathrm{loc}}$.

Notice that $J_{f_0}(z)$ vanishes in a set of positive area. In fact, a sufficient condition under which the inverse of a Sobolev homeomorphism belongs to $W^{1,1}_{\mathrm{loc}}$ is that $J_f > 0$ a.e. (see theorem 1.1 of [43]). But, it is not a necessary condition. Indeed, it is possible to construct a bi–Sobolev homeomorphism such that $J_f = 0$ in a set of positive area (see [53], Section 6.5.6).

A necessary and sufficient condition for $f^{-1} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ is that $Df(z) = 0$ almost everywhere in the set $\{z : J_f(z) = 0\}$, i. e. that f has finite distortion (see [43] and [41]).

Theorem 2.5. *Let Ω, Ω' be planar domains and $f : \Omega \to \Omega'$ be a Sobolev homeomorphism. Then, the following conditions are equivalent:*

- $f^{-1} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$
- *f has finite distortion*
- *f^{-1} has finite distortion.*

A suitable integrability condition on the distortion function K guarantees a better regularity for the inverse.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^2$ be a domain. If $f : \Omega \to \mathbb{R}^2$ is a Sobolev homeomorphism of finite distortion $K \in L^1(\Omega)$, then f^{-1} belongs to $W^{1,2}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^2)$ and is a mapping of finite distortion.*

We want underline that an integrability assumption of $K^{1-\delta}$ for $0 < \delta < 1$ does not give any better than $W^{1,1}$–regularity of f^{-1}. Hereafter, we use the notation $Q_0 = [0,1] \times [0,1]$ for the unit square in $\mathbb{R}^2$.

Example 2.1. Let $0 < \delta < 1$. There is a homeomorphism $f : Q_0 \to Q_0$ of finite distortion such that $f \in W^{1,1}_{\mathrm{loc}}(Q_0, \mathbb{R}^2)$, $K^{1-\delta} \in L^1(Q_0)$ but $f^{-1} \notin W^{1,1}_{\mathrm{loc}}(Q_0, \mathbb{R}^2)$.

In higher dimension, besides the outer distortion already introduced in Section 2.1, we shall need to consider the *inner distortion.* A mapping $f \in W^{1,1}_{\mathrm{loc}}(\Omega; \mathbb{R}^n)$ has finite inner distortion if its Jacobian $J_f \in L^1_{\mathrm{loc}}(\Omega)$, $J_f \geq 0$ a.e. and

$$J_f(z) = 0 \quad \Longrightarrow \quad |\operatorname{adj} Df(z)| = 0 \quad \text{a.e. },$$

where $\operatorname{adj} Df$ is the adjugate of the differential matrix Df of f.

For such a map, we call inner distortion of f the smallest function $K^I_f \geqslant 1$ such that

$$|\operatorname{adj} Df(z)|^n \leqslant K^I_f(z)\, J_f(z)^{n-1}\,, \tag{2.9}$$

for a.e. $z \in \Omega$. Clearly, a map of finite outer distortion has also finite inner distortion and $K^I_f \leqslant (K_f)^{n-1}$, as a consequence of Hadamard's inequality ($|\operatorname{adj} Df| \leq |Df|^{n-1}$). In dimension $n = 2$ the two notions coincide.

Let us give the statement analogous to planar case examined in Theorem 2.5.

Theorem 2.7 ([28]). *Let* $f : \Omega \xrightarrow{onto} \Omega'$ *be a homeomorphism in* $W^{1,n-1}_{loc}(\Omega, \mathbb{R}^n)$ *with finite inner distortion. Then,* f^{-1} *is a* $W^{1,1}_{loc}(\Omega', \mathbb{R}^n)$ *mapping of finite distortion. Moreover,*

$$|Df^{-1}(w)|^n \leq K(f^{-1}(w))J_{f^{-1}}(w) \qquad \text{for a.e. } w \in \Omega', \tag{2.10}$$

and

$$\int_{\Omega'} |Df^{-1}(w)|dw = \int_{\Omega} |\operatorname{adj} Df(z)|dz.$$

The same conclusion was known by M. Csörnyei, S. Hencl, J. Malý [15] under the strong assumption that f has finite outer distortion.

The regularity assumption $|Df| \in L^{n-1}_{\mathrm{loc}}$ cannot be weakened in the scale of Sobolev space. In fact, it is possible to construct a homeomorphism $f \in W^{1,n-1-\varepsilon}$ where $0 < \varepsilon < 1$ such that $|\operatorname{adj} Df| \in L^1(\Omega)$ but $f^{-1} \notin W^{1,1,}_{\mathrm{loc}}$. The above theorem is also sharp in the setting of Orlicz space (see [39]).

Regarding the higher dimensional setting, in [48] the authors showed:

Theorem 2.8. *Let* $n \geq 2$ *and let* $f : \Omega \to \mathbb{R}^n$ *be a bi–Sobolev map. Suppose that for a measurable set* $E \subset \Omega$ *we have* $J_f = 0$ *almost evrywhere on* E*. Then,* $|\operatorname{adj} Df| = 0$ *a.e. on* E*. If we moreover assume that* $J_f \geq 0$*, it follows that* f *has finite inner distortion.*

For $n > 2$ a similar result of Theorem 2.6 has been established in [72].

Theorem 2.9. *Let* Ω, Ω' *be bounded domains in* $\mathbb{R}^n$ *and* $f \in W^{1,n-1}$ *be a homeomorphism with finite inner distortion such that* $K^I_f \in L^1(\Omega)$*. Then,*

$|Df^{-1}| \in L^n(\Omega')$ *and*

$$\int_{\Omega'} |Df^{-1}(w)|^n dw = \int_{\Omega} K_f^I(z)dz.$$

In [72], it is worth pointing out that the regularity of the distortion influences the regularity of the inverse mapping also in the scale of Orlicz space.

2.3 Regularity of the inverse of a homeomorphism of bounded variation

Let Ω be a domain in $\mathbb{R}^2$. A function $u \in L^1(\Omega)$ is of bounded variation, $u \in \mathrm{BV}(\Omega)$, if the distributional partial derivatives of u are measures with finite total variation in Ω: there exist Radon signed measures D_1u, D_2u in Ω such that for $i = 1, 2$, $|D_iu|(\Omega) < \infty$ and

$$\int_{\Omega} uD_i\phi(z)\,dz = -\int_{\Omega} \phi(z)\,dD_iu(z) \qquad \forall \phi \in C_0^1(\Omega).$$

The gradient of u is then a vector-valued measure with finite total variation

$$|\nabla u|(\Omega) = \sup\left\{ \int_{\Omega} u\mathrm{div}\varphi(z)\,dz \ : \ \varphi \in C_0^1(\Omega, \mathbb{R}^2), \|\varphi\|_\infty \leq 1 \right\} < \infty.$$

By $|\nabla u|$ we denote the total variation of the signed measure Du.

The Sobolev space $W^{1,1}(\Omega)$ is contained in $\mathrm{BV}(\Omega)$; indeed for any $u \in W^{1,1}(\Omega)$ the total variation is given by $\int_\Omega |\nabla u| = |\nabla u|(\Omega)$.

We say that $f \in L^1(\Omega;\mathbb{R}^n)$ belongs to $\mathrm{BV}(\Omega;\mathbb{R}^n)$ if each component of f is a function of bounded variation. Finally we say that $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ if $f \in \mathrm{BV}(A;\mathbb{R}^n)$ for every open $A \subset\subset \Omega$. In the following, for $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ we will denote the total variation of f by:

$$|Df|(\Omega) = \sup\left\{\sum_{i=1}^{n}\int_{\Omega} f_i \mathrm{div}\varphi_i(z)\,dz :\ \varphi_i \in C_0^1(\Omega;\mathbb{R}^n),\ \|\varphi_i\|_\infty \le 1, i = 1,\dots,n\right\}.$$

The space $\mathrm{BV}(\Omega,\mathbb{R}^n)$ is endowed with the norm

$$\|f\|_{\mathrm{BV}} := \int_{\Omega} |f(z)|\,dz + |Df|(\Omega).$$

There are equivalent ways to define a norm for BV maps. For example,

$$|f|_{\mathrm{BV}(\Omega)} = \sup\left|\int_{\Omega}[D^T\varphi(z)]f(z)\,dz\right|$$

and the supremum runs over all test mappings $\varphi \in C_0^\infty(\Omega,\mathbb{R}^n)$ with $\|\varphi\|_\infty = 1$.

We will need the definition of sets of finite perimeter (see [1]).

Definition 2.1. *Let E be a Lebesgue measurable subset of $\mathbb{R}^n$. For any open set $\Omega \subset \mathbb{R}^n$ the perimeter of E in Ω, denoted by $P(E,\Omega)$, is the total variation of χ_E in Ω, i.e.*

$$P(E,\Omega) = \sup\left\{\int_E div\varphi\,dz\ :\ \varphi \in C_0^1(\Omega;\mathbb{R}^n),\ \|\varphi\|_\infty \le 1\right\}.$$

We say that E is a set of finite perimeter in Ω if $P(E,\Omega) < \infty$.

Lemma 2.1. *For any open set $\Omega' \subset \mathbb{R}^n$ and $x_i \in L^1_{\mathrm{loc}}(\Omega')$ we have*

$$|\nabla x_i|(\Omega') = \int_{-\infty}^{+\infty} P\left(\{w \in \Omega' : x_i(w) > t\}, \Omega'\right) dt. \tag{2.11}$$

It is a well known fact (see e.g. [1], Section 3.11) that a function $g \in L^1_{\mathrm{loc}}(\Omega)$ is in $BV_{\mathrm{loc}}(\Omega)$ (or in $W^{1,1}_{\mathrm{loc}}(\Omega)$) if and only if there is a representative which has bounded variation (or is an absolutely continuous function) on almost all lines parallel to coordinate axes and the variation on these lines is integrable.

We define the variation $\left|\frac{\partial f}{\partial x_i}\right|(\Omega)$ along the direction x_i as follows:

$$\left|\frac{\partial f}{\partial x_i}\right|(\Omega) = \sup\left\{\int_\Omega f\frac{\partial \phi}{\partial x_i}dx : \phi \in C^1_c(\Omega), \|\phi\|_\infty \leq 1\right\}.$$

In one dimensional setting, each monotone function $f \in L^1(\Omega)$ belongs to BVand hence also the inverse $f^{-1} \in \mathrm{BV}$. The same holds in two dimensions.

The first paper dealing with properties of the inverse of BV-homeomorphisms was [45], where the authors proved the following:

Theorem 2.10. *Let $\Omega, \Omega' \subset \mathbb{R}^2$ be open sets and suppose that $f \in \mathrm{Hom}(\Omega, \Omega')$. Then $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ if, and only if, $f^{-1} \in \mathrm{BV}_{\mathrm{loc}}(\Omega', \mathbb{R}^2)$.*

We shall use from now on the following notation:

$$f(x,y) = (u(x,y), v(x,y)) \qquad \text{for } (x,y) \in \Omega$$

$$f^{-1}(u,v) = (x(u,v), y(u,v)) \qquad \text{for } (u,v) \in \Omega'.$$

It is possible to improve the result of Theorem 2.10, finding how the variations of the components of the inverse map are related to the variation of f. More precisely:

Theorem 2.11. *Let* $f = (u, v) \in \mathrm{BV}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap \mathrm{Hom}(\Omega, \Omega')$, *with inverse* $f^{-1} = (x, y)$. *Then* $x, y \in \mathrm{BV}_{\mathrm{loc}}(\Omega')$ *and*

$$|\nabla x|(\Omega') = \left|\frac{\partial f}{\partial y}\right|(\Omega) \tag{2.12}$$

$$|\nabla y|(\Omega') = \left|\frac{\partial f}{\partial x}\right|(\Omega)\,. \tag{2.13}$$

The statement of Theorem 2.11 is actually contained in [18] with the additional assumption for mappings $f \in W^{1,1}_{\mathrm{loc}}$. However, with some technical tools, it is possible to prove identities for the wider class of mappings with bounded variation. Note that under the assumptions of previous Theorem, we cannot expect a better regularity for the inverse, as it is shown by the mapping f_0 considered in Section 2.2.

Without loss of generality, we can consider $\Omega = (-1, 1)^2$ and $f = (u, v) \colon \Omega \to \Omega' \subset \mathbb{R}^2$. We denote by:

$$f(x, \cdot) \colon t \in (-1, 1) \to (u(x, t), v(x, t)) \in \Omega'$$

$$f(\cdot, y) \colon s \in (-1, 1) \to (u(s, y), v(s, y)) \in \Omega'.$$

the one dimensional restrictions of f along the coordinate axes.

We recall the following result that describes a first link between the variation along a direction and one-dimension sections.

Lemma 2.2. *Let* $f \in \mathrm{BV}(\Omega, \mathbb{R}^2)$*. Then*

$$\left|\frac{\partial f}{\partial x}\right|(\Omega) = \int_{-1}^{1} \left|\frac{d}{dx} f(\cdot, y)\right| ((-1,1))\ dy. \tag{2.14}$$

$$\left|\frac{\partial f}{\partial y}\right|(\Omega) = \int_{-1}^{1} \left|\frac{d}{dy} f(x, \cdot)\right| ((-1,1))\ dx. \tag{2.15}$$

(see [1] Theorem 3.103).

We are in a position to prove Theorem 2.11.

Proof of Theorem 2.11. We start to prove the equality (2.13).

For a general $f \in \mathrm{BV} \cap Hom(\Omega, \Omega')$ it is well known that the area formula (1.5) fails. However, being $f(\cdot, y)$ continuous, the length of the image of the parametrized curve is the total variation of the restriction of f along the coordinate axes. In this way, for the restriction of f along the line $y = t$ a counter part of area formula holds (see [25] Theorem 2.10.13):

$$\left|\frac{d}{dx} f(\cdot, t)\right| (-1,1) = \mathscr{H}^1\big(f\left((-1,1) \times \{t\}\right)\big).$$

Since f belongs to $\mathrm{BV}(\Omega, \mathbb{R}^2) \cap Hom(\Omega, \Omega')$, f has bounded variation on almost all lines parallel to coordinate axes and the variation on these lines is integrable.

Integrating with respect to t , by Lemma 2.2, we obtain:

$$\left|\frac{\partial f}{\partial x}\right|(\Omega) = \int_{-1}^{1} \mathscr{H}^1\big(f\left((-1,1) \times \{t\}\right)\big)\, dt.$$

Since it is clear that

$$f\left((-1,1)\times\{t\}\right)=\{w\in\Omega' : y(w)=t\}$$

then

$$\left|\frac{\partial f}{\partial x}\right|(\Omega)=\int_{-1}^{1}\mathscr{H}^1\left(\{w\in\Omega'\ :\ y(w)=t\}\right)\,dt.$$

As y is continuous then the set $\{w\in\Omega' : y(w)=t\}$ is the boundary of the level set $\{w\in\Omega' : y(w)>t\}$.

By assumptions we know that for a.e. t, $\mathscr{H}^1\left(\{w\in\Omega' : y(w)=t\}\right)<\infty$ and from [1] (p. 209) we have:

$$\mathscr{H}^1\left(\{w\in\Omega'\ :\ y(w)=t\}\right)=P\left(\{w\in\Omega' : y(w)>t\},\Omega'\right)$$

a.e.$t\in(-1,1)$.

Using Coarea Formula from Lemma 2.1, we obtain:

$$|\nabla y|(\Omega')=\left|\frac{\partial f}{\partial x}\right|(\Omega)$$

and we deduce that $y\in \mathrm{BV}_{\mathrm{loc}}(\Omega')$.

The equality (2.12) is proved using the same technique.

$\square$

As an application of Theorem 2.11, we are able to connect the weak* BV convergence of a sequence of homeomorphisms with the weak* BV convergence of their inverse.

We recall the weak* convergence in BV that is useful for its compactness properties.

Definition 2.2. *Let f, f_j with $j = 1, 2, \dots \in \mathrm{BV}(\Omega, \mathbb{R}^n)$. We say that $\{f_j\}$ weakly* converges in* BV *to f if $\{f_j\}$ converges to f in $L^1(\Omega, \mathbb{R}^n)$ and*

$$\lim_{h\to\infty} \int_\Omega \phi dDf_j = \int_\Omega \phi dDf \qquad \forall \phi \in C_0(\Omega)$$

An useful criterion for the weak* convergence in BV is the following.

Proposition 2.1. *Let $f_j \in \mathrm{BV}(\Omega, \mathbb{R}^n)$. Then $\{f_j\}$ weakly* converges to f in* BV *if, and only if, $\{f_j\}$ is bounded in $\mathrm{BV}(\Omega, \mathbb{R}^n)$ and converges to f in $L^1(\Omega, \mathbb{R}^n)$.*

Now we are able to show the connection between the weak* BV convergence of a sequence of homeomorphisms with the weak* BV convergence of their inverses.

More precisely,

Corollary 2.1. *Let $f_j \in \mathrm{Hom}(\Omega, \Omega') \cap \mathrm{BV}(\Omega, \mathbb{R}^2)$ with $j = 1, 2, \dots$. The sequence f_j is locally weakly* compact in* BV *if, and only if, f_j^{-1} is locally weakly* compact in* BV.

Moreover, if $f_j \to f \in Hom(\Omega, \Omega')$ uniformly in Ω, then f_j^{-1} converge weakly in* BV *and locally uniformly to f^{-1}.*

Proof. By Theorem 2.11, f_j^{-1} belongs to $\mathrm{BV}(\Omega', \mathbb{R}^2)$ and there exists a constant $C \geq 1$ such that

$$
\begin{aligned}
|Df_j^{-1}|(\Omega') &\leq |\nabla x|(\Omega') + |\nabla y|(\Omega') \\
&= \left|\frac{\partial f_j}{\partial x}\right|(\Omega) + \left|\frac{\partial f_j}{\partial y}\right|(\Omega) \\
&\leq C|Df_j|(\Omega)
\end{aligned}
\tag{2.16}
$$

Being f_j locally weakly* compact in BV, f_j admits a subsequence $f_{j(k)}$ that weakly* converges to a map $f \in \mathrm{BV}(\Omega, \mathbb{R}^2)$ and $f_{j(k)}$ is bounded in BV.

By the criterion of compactness in BV, we have to prove that

$$
\sup_{j(k)} \|f_{j(k)}^{-1}\|_{\mathrm{BV}} = \sup \left\{ \int_{A'} |f_{j(k)}^{-1}| \, dx + |Df_{j(k)}^{-1}|(A') : j(k) \in \mathbb{N} \right\} < \infty
$$

for all open set $A' \subset\subset \Omega'$.

Since $f_{j(k)}^{-1}$ are homeomorphism between bounded domains, we need to control uniformly only the total variation $|Df_{j(k)}^{-1}|(A')$. This comes from the inequality (2.16) and the boundedness of the subsequence $f_{j(k)}$.

The other implication follows by symmetry.

If we assume in addition that $f_j \to f$ uniformly, the local uniform convergence of f_j^{-1} to f^{-1} follows from Lemma 3.1 of [28]. Moreover, f_j^{-1} converges to f^{-1} in $L^1_{\mathrm{loc}}(\Omega', \mathbb{R}^2)$ because Ω, Ω' are bounded domains. Hence, the weak* convergence in $\mathrm{BV}(\Omega', \mathbb{R}^2)$ easily follows from the Proposition 2.1. $\square$

Corollary 2.1 fails in the setting of bi-Sobolev mappings, despite of the identities of type (2.7) for f_j; i.e. the equi-integrability of $\{Df_j\}$ is not enough to guarantee the equi-integrability of $\{Df_j^{-1}\}$. In [28] the authors consider a sequence of homeomorphisms of finite distortion whose distortion K_j has spherically rearrangement K_j^* satisfying $K_j^* \leq K$ for a fixed Borel function K. Under this assumption they prove that if $\{Df_j\}$ is equi-integrable in Ω, then $\{Df_j^{-1}\}$ is equi-integrable in Ω'. The following example shows that the assumption on the rearrangement K_j^* cannot be removed.

Example 2.2. There exists a sequence $\{f_j\}$ of bi-Sobolev mappings such that $\{Df_j\}$ is equi-integrable in Ω, but $\{D(f_j^{-1})\}$ is not equi-integrable in Ω' and $K_j = K_j^*$ are not uniformly bounded by any Borel function.

We consider the one dimensional approximating sequence $\{c_j\}$ of the usual Cantor ternary function c on the interval $(0, 1)$. Let us now set $g_j(t) = t + c_j(t)$ and $g(t) = t + c(t)$. We note that g fails to be absolutely continuous. On the other hand, we consider the inverse of $g_j^{-1} = h_j$ and $g^{-1} = h$. We observe that h is a Lipschitz function mapping homeomorphically $(0, 2)$ onto $(0, 1)$.

Define $f_j(x, y) = (h_j(x), y)$. For each positive integer j, let us indicate by $[\alpha_j, \beta_j]$ one of the 2^j intervals of length $\left(\frac{1}{3}\right)^j$ which remain after the jth step in the usual construction of Cantor set.

Since $g(\beta_j) - g(\alpha_j) = \beta_j - \alpha_j + c_j(\beta_j) - c_j(\alpha_j) = \left(\frac{1}{3}\right)^j + \left(\frac{1}{2}\right)^j$, then the union E_j' of the 2^j intervals $[g(\alpha_j), g(\beta_j)]$ has measure $1 + \left(\frac{2}{3}\right)^j$.

By standard calculation we have:

$$h_j'(x_1) = \begin{cases} 1 & \text{if } x \in (0, 2) \setminus E_j' \\ \\ \dfrac{1}{1 + \left(\frac{3}{2}\right)^j} & \text{if } x \in E_j' \end{cases}$$

It easy to check that $|Df_j| = \sqrt{(h_j')^2 + 1}$ are equi-bounded and also equi-integrable.

On the other hand, the inverse mappings $f_j^{-1}(u,v) = (g_j(u), v)$ converge to $f^{-1}(u,v) = (g(u), v)$ only weakly* in $\mathrm{BV}\left((0,1)^2, \mathbb{R}^2\right)$.

We observe that the distortions K_j of f_j are:

$$K_j(z) = \frac{|Df_j(z)|^2}{J_{f_j}(z)} = \begin{cases} 2 & z \in \left((0,2) \setminus E'_j\right) \times (0,1) \\ \\ 1 + \left(\frac{3}{2}\right)^j + \dfrac{1}{1 + \left(\frac{3}{2}\right)^j} & z \in E'_j \times (0,1) \end{cases}$$

The set $E'_j \times (0,1) = A'_j$ has positive measure, in particular $\left|A'_j\right| \geq 1$, hence there exists a set

$$A' = \bigcap_{k=1}^{\infty} \bigcup_{j \geq k} A'_j$$

of positive measure ($|A'| \geq 1$) such that

$$\limsup_j K_j(z) = +\infty \qquad \forall z \in A'.$$

We remark that Corollary 2.1 can be applied to the sequence $\{f_j\}$ of previous example.

In general, the limit of a converging sequence of homeomorphisms $f_j : \Omega \xrightarrow{\text{onto}} \Omega'$ may loose injectivity. Iwaniec and Onninen in [54], proved that the weak $W^{1,n}-$ limit of a sequence of $W^{1,n}-$ homeomorphisms admits a right inverse everywhere. On this subject, it is possible to extend Theorem 1.4 of [54] in the following sense.

Theorem 2.12. *Let* $f_j \in W^{1,2}_{\mathrm{loc}} \cap \mathrm{Hom}(\Omega, \Omega')$ *be such that* $\{f_j\}$ *converges weakly in* $W^{1,1}_{\mathrm{loc}}$ *and uniformly to* $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$*. Then* f *admits a right* *inverse* $h \in \mathrm{BV}_{\mathrm{loc}}(\Omega', \mathbb{R}^2)$ *a. e., that is* $f(h(w)) = w$ *a. e. and*

$$\|h\|_{\mathrm{BV}} \leq C \int_{\Omega} |Df|. \tag{2.17}$$

The following example shows that under the assumptions of Theorem 2.12 , the existence of a right inverse everywhere is not guaranteed. We can not expect more than the existence almost everywhere of the weak limit of Sobolev homeomorphisms.

Example 2.3. There exists a sequence $\{f_j\}$ of $W^{1,2}$-bisobolev mappings converging weakly in $W^{1,p}$ for all $1 \leq p < 3/2$ and uniformly, whose limit map does not admit right inverse everywhere.

We consider $\mathbb{D} = \{(x,y) : x^2 + y^2 \leq 1, y \geq 0\}$ and denote by $\mathbb{B}_a(c)$ the ball of center $(0, c)$ and radius a with $a < \frac{1}{2}$.

Let $f_j : \mathbb{D} \xrightarrow{\text{onto}} \mathbb{D}$ be a homeomorphism that maps the ball $\mathbb{B}_a(a + \varepsilon_j)$ into the ball $\mathbb{B}_a\left(\frac{1}{2}\right)$ by a vertical translation. We need to define f_j outside the ball $\mathbb{B}_a(a + \varepsilon_j)$. Let $P \in \mathbb{D} \setminus \mathbb{B}_a(a + \varepsilon_j)$ and θ be the angle between the second coordinate axis and the segment with endpoints P and the center of $\mathbb{B}_a(a + \varepsilon_j)$; moreover, let ρ be the distance of P from the center of $\mathbb{B}_a(a + \varepsilon_j)$. Then, using polar coordinates

$$P = ((0, a + \varepsilon_j) + \rho(\sin\theta, -\cos\theta))$$

We construct f_j that sends P into a point $P' \in \mathbb{D} \setminus \mathbb{B}_a\left(\frac{1}{2}\right)$, preserving θ.

More precisely:

$$f_j\left((0, a+\varepsilon_j) + \rho\left(\sin\theta, -\cos\theta\right)\right) = \left(0, \frac{1}{2}\right) + \left[\alpha(\theta)\rho + \beta(\theta)\right]\left(\sin\theta, -\cos\theta\right).$$

To be consistent with the construction of f_j, we have to consider three different cases, depending for which angles θ the lines r and s intersect the coordinate axis, where r joints P and $(0, a+\varepsilon_j)$ and s joints P' and $\left(0, \frac{1}{2}\right)$.

If $-\theta_0 = -\arctan 2 \leq \theta \leq \arctan 2 = \theta_0$, then

$$\alpha(\theta) = \frac{1 - 2a\cos\theta}{2\left(a(1-\cos\theta) + \varepsilon_j\right)}$$

$$\beta(\theta) = a\left(1 - \alpha\right).$$

If $-\arctan\frac{1}{a+\varepsilon_j} \leq \theta \leq -\theta_0$ and $\theta_0 \leq \theta \leq \arctan\frac{1}{a+\varepsilon_j}$, then

$$\alpha(\theta) = \frac{\cos\theta\left(\cos\theta - 2a + \sqrt{3+\cos^2\theta}\right)}{2\left(a(1-\cos\theta) + \varepsilon_j\right)}$$

$$\beta(\theta) = a\left(1 - \alpha\right).$$

If $-\pi \leq \theta \leq -\arctan\frac{1}{a+\varepsilon_j}$ and $\arctan\frac{1}{a+\varepsilon_j} \leq \theta \leq \pi$

$$\alpha(\theta) = \frac{\cos\theta - 2a + \sqrt{3+\cos^2\theta}}{2\left(a+\varepsilon_j\right)\cos\theta - a + \sqrt{1 - \left(a+\varepsilon_j\right)^2\sin^2\theta}}$$

$$\beta(\theta) = a\left(1 - \alpha\right).$$

We consider the L^p norm of the gradient of these mappings using polar coordinates:

$$\int d\theta \int \rho \left[|D_\rho f_j|^p + \frac{|D_\theta f_j|^p}{\rho^p} \right] d\rho. \tag{2.18}$$

By the expressions of $\alpha(\theta)$ and $\beta(\theta)$, it is clear that the most critical case is when $-\theta_0 \leq \theta \leq \theta_0$. We consider separately the radial derivatives and the angular derivatives.

For the radial derivative we observe that

$$|D_\rho f_j| = \alpha(\theta) = \frac{1 - 2a\cos\theta}{2\,(a(1-\cos\theta) + \varepsilon_j)}$$

and

$$\int_{-\theta_0}^{\theta_0} d\theta \int_a^{\frac{a+\varepsilon_j}{\cos\theta}} \rho |D_\rho f_j|^p d\rho =$$
$$\frac{1}{2}\int_{-\theta_0}^{\theta_0} d\theta \left(\frac{1-2a\cos\theta}{2a\,(1-\cos\theta)+\varepsilon_j} \right)^p \left[\left(\frac{a+\varepsilon_j}{\cos\theta} \right)^2 - a^2 \right] =$$
$$\frac{1}{2}\int_{-\theta_0}^{\theta_0} d\theta \left(\frac{1-2a\cos\theta}{2a\,(1-\cos\theta)+\varepsilon_j} \right)^p \left[\left(\frac{a^2+\varepsilon_j^2 - a^2\cos^2\theta}{\cos^2\theta} \right) + \frac{2a\varepsilon_j}{\cos^2\theta} \right]$$

The term

$$\int_{-\theta_0}^{\theta_0} \left(\frac{1-2a\cos\theta}{2a\,(1-\cos\theta)+\varepsilon_j} \right)^p \left(\frac{a^2+\varepsilon_j^2 - a^2\cos^2\theta}{\cos^2\theta} \right)$$

is finite for $p < 3/2$.

Indeed, when θ is close to 0, the other term has the same behaviour of $\varepsilon_j^{2-p} \arctan\left(\frac{\theta_0}{\sqrt{\varepsilon_j}}\right)^p$; that goes to 0 when $\varepsilon_j \to 0$.

For the derivatives respect to θ, we observe that

$$\alpha'(\theta) = \frac{a\sin\theta}{a(1-\cos\theta)+\varepsilon_j} - \frac{a\sin\theta(1-2a\cos\theta)}{2(a(1-\cos\theta)+\varepsilon_j)^2}.$$

So that

$$\int_{-\theta_0}^{\theta_0} d\theta \int_a^{\frac{a+\varepsilon_j}{\cos\theta}} |D_\theta f_j|^p d\rho =$$

$$\int_{-\theta_0}^{\theta_0} d\theta \int_a^{\frac{a+\varepsilon_j}{\cos\theta}} \left((\alpha'(\theta))^2(\rho-a)^2 + (\alpha(\theta)\rho + a(1-\alpha(\theta)))^2\right)^{p/2} d\rho$$

We observe that the second term on the left hand side is easily controlled and the first term has the following behaviour near to 0:

$$\int_{-\theta_0}^{\theta_0} d\theta \int_a^{\frac{a+\varepsilon_j}{\cos\theta}} (\alpha'(\theta))^p\,(\rho-a) =$$

$$\int_{-\theta_0}^{\theta_0} (\alpha'(\theta))^p \left[\frac{a^2+\varepsilon_j^2+2a\varepsilon_j}{2\cos^2\theta} - a\frac{a+\varepsilon_j}{\cos\theta} + \frac{a^2}{2}\right] d\theta.$$

Arguing as the radial case, it is just routine to prove that the last integral is finite when $\varepsilon_j \to 0$.

We observe that f_j satisfies all the assumptions of Theorem 2.12, then the limit f admits right inverse h a.e..

We emphasize that the "right inverse"h is not injective everywhere as sends the segment of extremal $(0,0)$ and $(0,\frac{1}{2}-a)$ into the origin. The central point is that the sequence f_j does not converge in $W^{1,2}$, hence the Theorem of Iwaniec and Onninen does not apply.

The dimension $n=2$ in Theorem 2.11 is crucial. For $n \geq 3$, to guarantee that f^{-1} has bounded variation, we need stronger assumptions on f (see [44],

[45],[74]).

It was recently proved in [15] the following:

Theorem 2.13. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a homeomorphism. Then $f^{-1} \in \mathrm{BV}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$.*

In previous Theorem the regularity assumption on f, that is $f \in W^{1,n-1}_{\mathrm{loc}}$, is optimal, in fact for each $n \geq 3$ and $0 < \epsilon < 1$, there exists a homeomorphism $f \in W^{1,n-1-\epsilon}_{\mathrm{loc}}((-1,1)^n, \mathbb{R}^n)$ such that $f^{-1} \notin \mathrm{BV}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ (see [45], Example 2).

Also in higher dimension, it is possible to give the explicit value of the variations of the components of the inverse map in terms of minors of Jacobian of f.

To state the main in his generality, we follow the notation of [66].

Let $I(n, n-1)$ be the set of all increasing multindices from $\{1, \ldots, n\}^{n-1}$, i.e. $\alpha = (\alpha_1 \ldots \alpha_{n-1}) \in I(n, n-1)$ if α, are integers $1 \leq \alpha_1 < \ldots < \alpha_{n-1} \leq n$. If $\alpha \in I(n, n-1)$, we define the partial Jacobian

$$\frac{\partial \left(f_{\alpha_1}, \ldots, f_{\alpha_{n-1}}\right)}{\partial(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n)} = \det\left(\frac{\partial f_{\alpha_i}}{\partial x_j}\right)$$

with $i = 1, \ldots, n-1$, $j = 1, \ldots, n$, $j \neq l$.

We define the $(n-1)-$ dimensional partial Jacobian as:

$$J^{(n-1)}_{f^{x_i}} = \sqrt{\sum_{\alpha \in I(n,n-1)} \left(\frac{\partial \left(f_{\alpha_1}, \ldots, f_{\alpha_{n-1}}\right)}{\partial(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)}\right)^2} \tag{2.19}$$

We are in position to state:

Theorem 2.14. *Let $f \in W^{1,n-1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n) \cap \mathrm{Hom}(\Omega, \Omega')$ whose inverse is $f^{-1} = (x_1, \dots, x_n)$. Then $x_i \in \mathrm{BV}_{\mathrm{loc}}(\Omega')$ $\forall i = 1, \dots, n$ and*

$$|\nabla x_i|(\Omega') = \int_\Omega J^{(n-1)}_{f^{x_i}}. \tag{2.20}$$

A key ingredient for the proof of Theorem 2.14 is the following result which prove that if $f \in W^{1,n-1}$ is a homeomorphism, then the area formula holds on almost all hyperplanes (see [15]).

Lemma 2.3. *Let $f \in W^{1,n-1}_{\mathrm{loc}}((-1,1)^n, \mathbb{R}^n)$ be a homeomorphism. Then for almost every $y \in (-1,1)$ the mapping $f_{|(-1,1)^{n-1}\times\{y\}}$ satisfies the Lusin (N) condition, i.e., for every $A \subset (-1,1)^{n-1} \times \{y\}$, $\mathscr{H}^{n-1}(A) = 0$ implies $\mathscr{H}^{n-1}(f(A)) = 0$.*

To prove that the inverse is in BV, in [43] and [45], the authors used a characterization of BV functions (see [71]). The technique in Theorem 2.11 and Theorem 2.14 are completely different; in fact we slice homeomorphism along coordinate directions. Theorem 2.14 generalizes the result of [18].

2.4 Composition of bi–Sobolev homeomorphisms

In this section the following question has been raised: when does a composition $g \circ f$ of two homeomorphisms

$$f : \Omega \longrightarrow \Omega' \qquad \text{and} \qquad g : \Omega' \longrightarrow \Omega''$$

of finite distortion also have finite distortion?

The major difficulty lies in the fact that, also if we assume that f is a bi-Sobolev map, the map f^{-1} need not satisfy the N-condition. In other words, the image of a null set under f^{-1} may fail to be measurable. This poses serious problems concerning measurability of the composition $g \circ f \colon \Omega \xrightarrow{\text{onto}} \Omega''$ and forces us to assume that f^{-1} satisfies the N-condition. In fact, it is well known ([65] p. 121) that the N-condition on f^{-1} guarantees that $g \circ f$ is measurable.

Concerning the composition map, the following result can be deduced by [43]. Let $f \colon \Omega \xrightarrow{\text{onto}} \Omega'$ and $g \colon \Omega' \xrightarrow{\text{onto}} \Omega''$ be homeomorphisms, with $f^{-1} \in W^{1,n}(\Omega', \Omega)$ with finite distortion and $g \in W^{1,n}(\Omega', \Omega'')$. Then $g \circ f$ belongs to $W^{1,1}_{\text{loc}}$.

In [80] it has been observed that under the above assumptions also $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ belongs to $W^{1,1}$, that is $g \circ f$ is a bi–Sobolev mapping. Hence, by Theorem 2.5 for $n = 2$, the composition has finite distortion.

In [35] we give sharp regularity conditions on f and g in the setting of Zygmund spaces, under which $g \circ f \in W^{1,1}$ (see Section 1.2 for definitions). Our setting is more general and we recover the previous result when $\alpha = 0$.

Theorem 2.15. *Let $f \colon \Omega \xrightarrow{onto} \Omega'$ and $g \colon \Omega' \xrightarrow{onto} \Omega''$ be homeomorphisms, with f^{-1} and g of finite distortion. If $|Df^{-1}| \in L^n \log^\alpha L_{\text{loc}}$ and $|Dg| \in L^n \log^{-\alpha(n-1)} L_{\text{loc}}$, with $\alpha \geqslant 0$, then $h = g \circ f \in W^{1,1}_{\text{loc}}$ and has finite distortion. Moreover*

$$K_h(z) \leqslant K_g(f(z))\, K_f(z)\,, \qquad \text{for a.e. } z \in \Omega\,. \tag{2.21}$$

Proof. Consider first the case $g \in C^\infty(f(\Omega), \mathbb{R}^n)$. Then, g being locally

Lipschitz continuous and f continuous, we have $g \circ f \in W^{1,1}_{\mathrm{loc}}$ and

$$D(g \circ f)(z) = Dg(f(z))\, Df(z)\,.$$

Moreover, f has finite distortion ([44, 74]), and hence

$$J_f(z) = 0 \quad \Longrightarrow \quad D(g \circ f)(z) = 0\,.$$

Let us prove that for any ball $B \Subset \Omega$, we have

$$\int_B |D(g \circ f)|\, dz \leqslant \int_{f(B)} |Dg(w)|\, |Df^{-1}(w)|^{n-1}\, dw\,. \tag{2.22}$$

We decompose the domain Ω' as follows

$$\Omega' = \mathcal{R}_{f^{-1}} \cup \mathcal{Z}_{f^{-1}} \cup \mathcal{E}_{f^{-1}}$$

where

$$\begin{aligned}
\mathcal{R}_{f^{-1}} &= \left\{ w \in \Omega' : f^{-1} \text{ is differentiable at } w \text{ and } J_{f^{-1}}(w) \neq 0 \right\}, \\
\mathcal{Z}_{f^{-1}} &= \left\{ w \in \Omega' : f^{-1} \text{ is differentiable at } w \text{ and } J_{f^{-1}}(w) = 0 \right\}, \\
\mathcal{E}_{f^{-1}} &= \left\{ w \in \Omega' : f^{-1} \text{ is not differentiable at } w \right\}.
\end{aligned}$$

Recall [81] that f^{-1} is differentiable a.e. in Ω', that is $|\mathcal{E}_{f^{-1}}| = 0$. Using the area formula we see that

$$\int_{f^{-1}(\mathcal{E}_{f^{-1}})} J_f(z) dz \leq |\mathcal{E}_{f^{-1}}| = 0.$$

Therefore, $J_f = 0$ a.e. in $f^{-1}(\mathcal{E}_{f^{-1}})$. Furthermore, by Sard's lemma, $|f^{-1}(\mathcal{Z}_{f^{-1}})| = 0$ and therefore $D(g \circ f)(z) = 0$ for a.e. $z \in \Omega \setminus f^{-1}(\mathcal{R}_{f^{-1}})$. On the other hand, for all $w \in \mathcal{R}_{f^{-1}}$, we have

$$J_f(f^{-1}(y)) = \frac{1}{J_{f^{-1}}(y)}\,, \qquad Df(f^{-1}(y)) = (Df^{-1}(y))^{-1}\,. \tag{2.23}$$

Now, defining the Borel set $A = B \cap f^{-1}(\mathcal{R}_{f^{-1}})$, by area formula (1.5) and (2.23) we compute

$$\begin{aligned}
\int_B |D(g \circ f)|\,dz &\leqslant \int_A |Dg(f(z))|\,\frac{|Df(z)|}{J_f(z)}\,J_f(z)\,dz \\
&\leqslant \int_{f(A)} |Dg(w)|\,\frac{|Df(f^{-1}(w))|}{J_f(f^{-1}(w))}\,dw \\
&\leqslant \int_{f(B)} |Dg(w)|\,|\operatorname{adj} Df^{-1}(w)|\,dw\,,
\end{aligned} \tag{2.24}$$

which implies (2.22).

Using Hölder's inequality in Zygmund spaces (1.10), we deduce from (2.22)

$$\int_B |D(g \circ f)|\,dz \leqslant C\,\|Dg\|_{L^n \log^{-\alpha(n-1)} L(f(B))}\,\|Df^{-1}\|^{n-1}_{L^n \log^\alpha L(f(B))}\,.$$

Let now g be an arbitrary function in $W^{1,1}_{\mathrm{loc}}(f(\Omega), \mathbb{R}^n)$ satisfying the assumptions. As in [38], let $\{g_j\}$ be a sequence of smooth mappings which approximate g by standard mollification. We take two indices i, j and we apply (2.22) to $g_i - g_j$. We see that $\{D(g_j \circ f)\}$ is a Cauchy sequence in L^1. Since

g_j converges to g almost everywhere and f^{-1} satisfies the Lusin $\mathcal{N}$ then $g_j \circ f$ converges to $g \circ f$ almost everywhere ($g \circ f$ does not depend on the represenative of g). Since $g_j \circ f$ is Cauchy in L^1_{loc}, we obtain that $g_j \circ f$ converges to $g \circ f$ in L^1_{loc}. This, toghether with the fact that $\{D(g_j \circ f)\}$ is a Cauchy sequence in L^1 implies that $h = g \circ f \in W^{1,1}_{\text{loc}}$.

We only need to prove that h has finite distortion and that (2.21) holds. The map h is differentiable at every point z in the set of full measure $E = f^{-1}(\mathcal{R}_{f^{-1}} \cap (\mathcal{R}_g \cup \mathcal{Z}_g))$, and we have:

$$Dh(z) = Dg(f(z))\, Df(z)\,, \qquad J_h(x) = J_g(f(z))\, J_f(z)\,. \tag{2.25}$$

From these formulas we can deduce that

$$J_h(z) = 0 \quad \Longrightarrow \quad Dh(z) = 0\,, \tag{2.26}$$

for a.e. $z \in \Omega$, that is, the composition map h has finite distortion. To this end, recall that since g has finite distortion, there exists a set $E' \subset \Omega'$ such that $|\Omega' \setminus E'| = 0$ and

$$J_g(w) = 0 \quad \Longrightarrow \quad Dg(w) = 0\,, \qquad \text{for every } w \in E'\,.$$

By (2.25), $J_h(z) = 0$ can happen on E only for $z \in f^{-1}(\mathcal{R}_{f^{-1}} \cap \mathcal{Z}_g)$, so that $J_g(f(z)) = 0$, hence also $Dg(f(z)) = 0$ and in turn $Dh(z) = 0$, if $f(z) \in E'$. Therefore, (2.26) holds at every point z in the set of full measure $f^{-1}(\mathcal{R}_{f^{-1}} \cap (\mathcal{R}_g \cup \mathcal{Z}_g) \cap E')$.

The above argument also gives

$$K_h(z) = \frac{|Dh(z)|^n}{J_h(z)} \leqslant \frac{|Dg(f(z))|^n}{J_g(f(z))} \cdot \frac{|Df(z)|^n}{J_f(z)} = K_g(f(z))\, K_f(z)$$

on $f^{-1}(\mathcal{R}_{f^{-1}} \cap \mathcal{R}_g)$, and $K_h(z) = 1$ a.e. on the complementary, thus inequality (2.21) follows. $\square$

Another question concernig the regularity of the composition is when does a composition $u \circ f$ of a map $f : \Omega \to \mathbb{R}^n$ and a function u has bounded variation?

In the paper [24], we give some conditions on f^{-1} for which the composition operator $u \circ f$ maps one Sobolev space to the space BV.

Chapter 3

Some pathological examples of Sobolev homeomorphisms

In this chapter we investigate when the Jacobian determinant does not change sign; i.e. it is either non–negative almost everywhere or non–positive almost everywhere. We also construct a bi–Sobolev homeomorphism f such that $J_f = J_{f^{-1}} = 0$ almost everywhere.

3.1 Sign of the Jacobian

In geometric function theory the non negativity of the Jacobian is ofen taken as an assumption. We shall examine that this is not a real restriction for homeomorphisms.

First, we investigate the planar case.

Let $\Omega \subset \mathbb{R}^2$ be a domain and $f : \Omega \to \mathbb{R}^2$ be a Sobolev homeomorphism. By Gehring–Lehto theorem (see for example [3]), every Sobolev homeomorphism is differentiable almost everywhere and moreover the following result holds:

Theorem 3.1. *Let $f : \Omega \subset \mathbb{R}^2 \to \Omega' \subset \mathbb{R}^2$ be a $W^{1,1}_{\mathrm{loc}}$ homeomorphism. Then, the Jacobian determinant J_f does not change sign; that is either $J_f \geq 0$ a.e. or $J_{f^{-1}} \leq 0$ a.e. in Ω.*

In general, for $n > 2$, the Jacobian determinant does not change sign if f is a homeomorphism differentiable a.e. in the classical sense.

Recalling that for $n > 2$, each homeomorphism $f \in W^{1,p}$ with $p > n - 1$ is differentiable a.e., we have:

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Suppose that $f \in W^{1,p}_{\mathrm{loc}}$ with $p > n - 1$. Then, either $J_f \geq 0$ a.e. or $J_{f^{-1}} \leq 0$ a.e. in Ω.*

This result was improved in [47]. Indeed, when $n \leq 3$ the authors showed that the Jacobian of a homeomorphism f in $W^{1,1}_{\mathrm{loc}}$ does not change sign and moreover, in higher dimension, the following statement hold:

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain, $n \geq 2$. If $f : \Omega \to \mathbb{R}^n$ is a $W^{1,p}$–homeomorphism for some $p > \left[\frac{n}{2}\right]$. Then, either $J_f \geq 0$ a.e. or $J_{f^{-1}} \leq 0$ a.e. in Ω.*

3.2 Sobolev homeomorphism with zero Jacobian almost everywhere

In this section we examine how big can be the zero set of the Jacobian of a homeomorphism of the Sobolev class $W^{1,p}(\Omega, \mathbb{R}^n)$, $p \geq 1$. We recall that it is possible to construct a Lipschitz homeomorphism with zero Jacobian on a set of positive measure (see [53], Section 6.2.6). The construction is based on the classical "Sierpinski sponge ". Initially, the holes of the sponge form a Cantor set of positive measure. Then, they are squeezed down to a set of positive measure. The homeomorphism constructed lies in every Sobolev class $W^{1,p}$ for $1 \leq p \leq \infty$, has finite distortion $K \in L^p$ for all $p < \frac{1}{n-1}$ but not for $p = \frac{1}{n-1}$, its Jacobian vanishes on a set S of positive measure and as a consequence, $f(S)$ has zero measure.

For a Lipschitz homeomorphism and in general for a homeomorphism that satisfies the Lusin $\mathcal{N}$ condition, the Jacobian cannot vanishes almost everywhere. In fact, in this case the area formula (1.6) holds as an equality and

$$0 = \int_\Omega J_f(x)\, dx = |f(\Omega)|$$

Thus, we have a contradiction.

When is it possible to construct a Sobolev homeomorphism f with $J_f = 0$ almost everywhere? In one dimension, a homeomorphism u in $W^{1,1}$ cannot satisfy $u' = 0$ almost everywhere. In fact, we know that for any increasing homeomorphism that belongs to $W^{1,1}$, the restriction to each compact interval is absolutely continuous and for any measurable set $A \subset \mathbb{R}$

$$0 = \int_A u'(x)dx = |u(A)|.$$

This gives us a contradiction.

Moreover, it is easy to see that for $n \geq 2$ a homeomorphism of finite distortion cannot satisfy $J_f = 0$ a.e. Otherwise, $|Df| = 0$ a.e. and the absolutely continuity on almost all lines gives a contradiction.

A mapping with zero Jacobian almost everywhere has the following consequences. The area formula for Sobolev mappings holds up to a set Z of zero measure and hence

$$0 = \int_{\Omega \setminus Z} J_f(x)dx = |f(\Omega \setminus Z)|.$$

Thus, such a mapping f sends a set of measure zero to a set of full measure:

$$|Z| = 0 \qquad \text{and} \qquad |f(Z)| = |f(\Omega) \setminus f(\Omega \setminus Z)| = |f(\Omega)|.$$

Conversely, f sends a set of full measure to a null set

$$|\Omega \setminus Z| = |\Omega| \qquad \text{and} \qquad |f(\Omega \setminus Z)| = 0.$$

S. Hencl showed in [40] that such pathological homeomorphism exists for any $1 \le p < n$.

Theorem 3.4. *Let $1 \le p < n$. There is a homeomorphism $f \in W^{1,p}\left((0,1)^n, (0,1)^n\right)$ such that $J_f(x) = 0$ almost everywhere.*

Let us note that this example cannot be obtained as a simple iteration of the Sierpinski's sponge construction, because the Sobolev norm of such mapping would grow too fast.

Later, Cerný in [12], with finer choice of parameters and estimates, obtained the best possible integrability of $|Df|$ for a homeomorphism with zero Jacobian almost everywhere.

Theorem 3.5. *Let $n \geq 2$. There is a homeomorphism $f \in W^{1,1}\left((0,1)^n,(0,1)^n\right)$ such that $J_f = 0$ almost everywhere in $[0,1]^n$, f restricted to the boundary is the identity mapping and $|Df|$ belongs to $L^{n)}\left((0,1)^n\right)$.*

This examples are based on the following steps. It is essential to construct sequence of homeomorphisms F_j from some rhomboid onto the same rhomboid (not from the unit cube onto the unit cube for technical reasons), that converges uniformly to a homeomorphism f. Every F_j has $J_{F_j} = 0$ on a particular Cantor set C_j of positive measure.

For $n = 2$, the mapping F_j for j odd, squeezes the set C_j in the horizontal direction and the derivative in the vertical direction will be non–zero. On the other hand, F_j for j even, squeezes the set C_j in the vertical direction and the derivative in the horizontal direction will be non–zero. At the end they need to estimate the derivatives of F_j and since they are constructed as a composition of many mappings the derivative is computed using the chain rule. The key ingredient of the construction is that all the matrices are almost diagonall. This means that the stretching in the horizontal and vertical direction do not multiply and thus the derivative is not big and the norm is finite.

3.3 Bi–Sobolev homeomorphism with zero Jacobian almost everywhere

In this section we focus our attention on the Sobolev regularity of the inverse of a Sobolev homeomorphism with zero Jacobian almost everywhere.

In the previous construction there was no attention on the regularity of the inverse map. The homeomorphism f with zero Jacobian almost everywhere belonged to the Grand Sobolev space $W^{1,n)}$ and hence its inverse was a mapping of bounded variation (see Theorem 2.13).

Here, we would like to underline that there is a "pathological "bi–Sobolev homeomorphism.

Theorem 3.6. *Let $n \geq 3$. There is a bi-Sobolev homeomorphism $f : (0,1)^n \xrightarrow{onto} (0,1)^n$ such that $J_f(x) = 0$ and $J_{f^{-1}}(y) = 0$ almost everywhere.*

Let us note that such a pathological homeomorphism cannot exists in dimension $n = 2$. Otherwise, by the strategic characterization of bi–Sobolev homeomorphism (see Theorem 2.5) we have

$$J_f(x) = 0 \quad \Longrightarrow \quad Df(x) = 0.$$

and hence $Df = 0$ a.e. This gives us a contradiction.

In higher dimension cannot exists a bi–Sobolev homeomorphism such that $J_f = 0$ a.e. with $W^{1,n-1}$–regularity.

Theorem 3.7. *Let $n \geq 2$ and let $f \in W^{1,n-1}((0,1)^n, \mathbb{R}^N)$ be a bi–Sobolev homeomorphism. Then $J_f(x) \neq 0$ on a set of positive measure.*

Proof. Suppose for contrary that there is a bi-Sobolev homeomorphism $f \in W^{1,n-1}$ such that $J_f = 0$ a.e. By Theorem 2.8 we know that each bi-Sobolev mapping is a mapping of finite inner distortion, i.e. for almost every x we have

$$J_f(x) = 0 \quad \Rightarrow \quad \operatorname{adj} Df(x) = 0 \qquad \text{a.e..}$$

Since $J_f(x) = 0$ a.e. we obtain that $\operatorname{adj} Df(x) = 0$ a.e.

By Theorem 2.7 we know that each $W^{1,n-1}$ homeomorphism of finite inner distortion satisfies $f^{-1} \in W^{1,1}$ and we have the following identity

$$\int_{(0,1)^n} |\operatorname{adj} Df(x)|\, dx = \int_{f((0,1)^n)} |Df^{-1}(y)|\, dy\ .$$

Since the left hand side equals to zero we obtain that $Df^{-1}(y) = 0$ a.e. Using the absolute continuity of f^{-1} on almost all lines it is not difficult to deduce that f^{-1} maps everything to a point which clearly contradicts the fact that f is a homeomorphism. □

The construction of the mapping f in Theorem 3.6 is essentially more complicated with respect to Theorem 3.4 and Theorem 3.5. It requires several new ideas and improvements. Moreover, to obtain a map with zero Jacobian in the previous constructions it was enough to squeeze certain Cantor type set only in one direction, but we have to squeeze these sets in two directions to obtain mapping with $\operatorname{adj} Df = 0$ a.e.

In what follows, we will use the usual convention that c denotes a generic constant whose value may change at each occurrence.

We give the sketch of the construction $f = (f_1, f_2, f_3)$ in dimension $n = 3$. In general dimension it is possible to use for example the mapping

$$f(x_1, x_2, \ldots, x_n) = (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3), x_4, \ldots, x_n)$$

which is again a bi–Sobolev homeomorphism with zero Jacobian a.e.

BASIC BUILDING BLOCKS

For $0 < w$ and $s \in (0,1)$, we denote the diamond of width w by

$$Q^z(w) = \{(x,y,z) \in \mathbb{R}^3 : |x| + |y| < w(1-|z|)\}.$$

We will often work with the inner smaller diamond and the outer annular diamond defined as

$$I^z(w,s) = Q^z(ws) \text{ and } O^z(w,s) = Q^z(w) \setminus Q^z(ws).$$

Given parameters $s \in [\frac{1}{2}, 1), s' \in [\frac{1}{4}, 1)$, we will repeatedly employ the mapping $\phi^z_{w,s,s'} \colon Q^z(w) \to Q^z(w)$ defined by

$$\phi^z_{w,s,s'} = \begin{cases} \left(\frac{1-s'}{1-s}x + (1-|z|)w\frac{s'-s}{1-s}\frac{x}{|x|+|y|}, \frac{1-s'}{1-s}y + (1-|z|)w\frac{s'-s}{1-s}\frac{y}{|x|+|y|}, z\right) & (x,y,z) \in O^z, \\ \left(\frac{s'}{s}x, \frac{s'}{s}y, z\right) & (x,y,z) \in I^z. \end{cases}$$

If $s' < s$, then this homeomorphism horizontally compresses $I^z(w,s)$ onto $I^z(w,s')$, while stretching $O^z(w,s)$ onto $O^z(w,s')$. Note that $\phi^z_{w,s,s'}$ is the identity on the boundary of $Q^z(w)$.

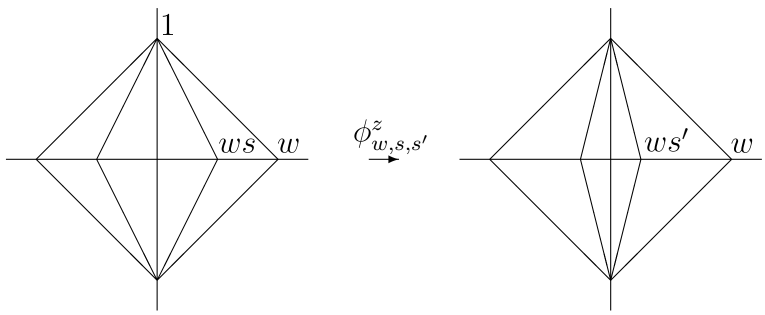

Fig. 1. The restriction of the mapping $\phi^z_{w,s,s'}$ to the x,z-plane

If (x, y, z) is an interior point of $I^z(w, s)$, then

$$D\phi^z_{w,s,s'}(x, y, z) = \begin{pmatrix} \frac{s'}{s} & 0 & 0 \\ 0 & \frac{s'}{s} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.1}$$

If (x, y, z) is an interior point of $O^z(w, s)$ and $z \neq 0$, then

$$D\phi^z_{w,s,s'}(x, y, z) = \begin{pmatrix} \frac{1-s'}{1-s} + 4c\frac{s'-s}{1-s} & 2c\frac{s'-s}{1-s} & cw\frac{s'-s}{1-s} \\ 2c\frac{s'-s}{1-s} & \frac{1-s'}{1-s} + 4c\frac{s'-s}{1-s} & cw\frac{s'-s}{1-s} \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.2}$$

using the convention that c denotes an expression which may depend on x, y, z but we know that $|c| \leq 1$. This expression may have a different value at each occurrence.

Note that by choosing w sufficiently small we can make the first two terms in the last column arbitrarily small. Later we will rotate this matrix in the first two coordinates and we obtain almost upper triangular matrix.

We will need also to estimate the derivative of the inverse mapping and using the same convention as in (3.2), we can write:

$$D(\phi^z_{w,s,s'})^{-1}(\phi^z_{w,s,s'}(x, y, z)) = \begin{pmatrix} \frac{1-s}{1-s'} + 8c\frac{s-s'}{1-s'} & 4c\frac{s-s'}{1-s'} & cw\frac{s-s'}{1-s'} \\ 4c\frac{s-s'}{1-s'} & \frac{1-s}{1-s'} + 8c\frac{s-s'}{1-s'} & cw\frac{s-s'}{1-s'} \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.3}$$

CHOICE OF PARAMETERS

Let C_1 and C_2 be absolute constants whose exact value we will specify later.

We can clearly fix $t > 1$ such that

$$C_1 C_2 \Big(\frac{\pi^2}{6}\Big)^6 \frac{1}{t} < \frac{1}{2} \ . \tag{3.4}$$

For $k \in \mathbb{N}$, we set

$$w_k = \frac{k+1}{tk^2-1}, \ \ s_k = 1 - \frac{1}{tk^2} \text{ and } s'_k = s_k \frac{k}{k+1}. \tag{3.5}$$

In this case,

$$\frac{1-s'_k}{1-s_k} = \frac{tk^2+k}{k+1} \text{ and } \frac{s_k - s'_k}{1-s_k} w_k = \frac{tk^2-1}{k+1} w_k = 1. \tag{3.6}$$

It is also easy to check that $0 < s_k < 1$ and

$$\prod_{i=1}^{\infty} s_i > 0.$$

We will construct a sequence of homeomorphisms F_j which will eventually converge to f.

CONSTRUCTION OF F_1

Let us denote $Q_0 := Q^z(w_1)$. We will construct a sequence of bi-Lipschitz mappings

$$f_{k,1} : Q_0 \xrightarrow{\text{onto}} Q_0$$

and our mapping $F_1 \in W^{1,1}(Q_0, \mathbb{R}^3)$ will be later defined as:

$$F_1(x) = \lim_{k\to\infty} f_{k,1}(x).$$

We will also construct a Cantor-type set $\mathcal{C}_1$ of positive measure such that

$$J_{F_1}(x) = 0 \text{ almost everywhere on } \mathcal{C}_1.$$

We define $f_{1,1}\colon Q_0 \overset{\text{onto}}{\longrightarrow} Q_0$ by

$$f_{1,1}(x,y,z) = \phi^z_{w_1,s_1,s_1'}(x,y,z).$$

Clearly $f_{1,1}$ is a bi-Lipschitz homeomorphism. Now each $f_{k,1}$ will equal to $f_{1,1}$ on the set $O^z(w_1,s_1)$ and it remains to define it on $R_{1,1} := I^z(w_1,s_1)$. Let $\mathcal{Q}_{2,1}$ be any collection of disjoint, scaled and translated copies of $Q^z(w_2)$ which covers $f_{1,1}(I^z(w_1,s_1)) = I^z(w_1,s_1')$ up to a set of measure zero. That is any two elements of $\mathcal{Q}_{2,1}$ have disjoint interiors, and there is a set $E_{2,1} \subset I^z(w_1,s_1')$ of measure 0 such that

$$I^z(w_1,s_1') \setminus E_{2,1} \subset \bigcup_{Q^z \in \mathcal{Q}_{2,1}} Q^z \subset I^z(w_1,s_1').$$

We define $f_{2,1}\colon Q_0 \to Q_0$ by

$$f_{2,1}(x,y,z) = \begin{cases} \phi^{Q^z}_{w_2,s_2,s_2'} \circ f_{1,1}(x,y,z) & f_{1,1}(x,y,z) \in Q^z \in \mathcal{Q}_{2,1}, \\ f_{1,1}(x,y,z) & \text{otherwise.} \end{cases}$$

It is not difficult to check that $f_{2,1}$ is a bi-Lipschitz homeomorphism. From now on each $f_{k,1}$ will equal to $f_{2,1}$ on

$$O^z(w_1,s_1) \cup f_{1,1}^{-1}\Big(\bigcup_{Q^z \in \mathcal{Q}_{2,1}} O^{s_2}_{Q^z}\Big) \cup (f_{1,1})^{-1}(E_{2,1})$$

and it remains to define it on

$$R_{2,1} := f_{1,1}^{-1}\Big(\bigcup_{Q\in\mathcal{Q}_{2,1}} I_{Q^z}^{s_2}\Big).$$

We continue inductively. Assume that $\mathcal{Q}_{k,1}$, $f_{k,1}$ and $R_{k,1}$ have already been defined. We find a family of disjoint scaled and translated copies of $Q^z(w_{k+1})$ that cover $f_{k,1}(R_{k,1})$ up to a set of measure zero $E_{k+1,1}$. Define $\phi_{k+1,1}\colon Q_0 \to Q_0$ by

$$\phi_{k+1,1}(x,y,z) = \begin{cases} \phi^{Q^z}_{w_{k+1},s_{k+1},s'_{k+1}}(x,y,z) & (x,y,z)\in Q^z \in \mathcal{Q}_{k+1,1}, \\ (x,y,z) & \text{otherwise.} \end{cases}$$

The mapping $f_{k+1,1}\colon Q_0 \to Q_0$ is now defined by $\phi_{k+1,1}\circ f_{k,1}$. Clearly each mapping $f_{k+1,1}$ is a bi-Lipschitz homeomorphism. We further define the set

$$R_{k+1,1} := f_{k,1}^{-1}\Big(\bigcup_{Q^z\in\mathcal{Q}_{k+1,1}} I_{Q^z}^{s_{k+1}}\Big).$$

It follows that the resulting Cantor type set

$$\mathcal{C}_1 := \bigcap_{k=1}^{\infty} R_{k,1}$$

satisfies

$$|\mathcal{C}_1| > 0.$$

It is clear from the construction that $f_{k,1}$ converge uniformly and hence the limiting map $F_1(x) := \lim_{k\to\infty} f_{k,1}(x)$ exists and is continuous. It is not difficult to check that F_1 is a one-to-one mapping of Q_0 onto Q_0. Since Q_0 is compact and F_1 is continuous we obtain that F_1 is a homeomorphism.

WEAK DIFFERENTIABILITY OF F_1

Let us estimate the derivative of our functions $f_{m,1}$. Let us fix $m, k \in \mathbb{N}$ such that $m \geq k$. If $Q^z \in \mathcal{Q}_{k,1}$ and $(x, y, z) \in \operatorname{int}(f_{k,1})^{-1}(I_{Q^z}^{s'_k})$, then we have squeezed our diamond k-times. Using (3.1), (3.5) and the chain rule we obtain

$$Df_{k,1}(x,y,z) = \prod_{i=1}^{k} \begin{pmatrix} \frac{i}{i+1} & 0 & 0 \\ 0 & \frac{i}{i+1} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{k+1} & 0 & 0 \\ 0 & \frac{1}{k+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.7}$$

Moreover, if $(x, y, z) \in \operatorname{int}(f_{m,1})^{-1}(O_{Q^z}^{s'_k})$, then we have squeezed our diamond $k-1$ times and then we have stretched it once. It follows from (3.1), (3.5), (3.2), (3.6) and the chain rule that

$$\begin{aligned} Df_{m,1}(x,y,z) &= \begin{pmatrix} \frac{tk^2+k}{k+1} + 4c\frac{tk^2-1}{k+1} & 2c\frac{tk^2-1}{k+1} & c \\ 2c\frac{tk^2-1}{k+1} & \frac{tk^2+k}{k+1} + 4c\frac{tk^2-1}{k+1} & c \\ 0 & 0 & 1 \end{pmatrix} \left(\prod_{i=1}^{k-1} \begin{pmatrix} \frac{i}{i+1} & 0 & 0 \\ 0 & \frac{i}{i+1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) \\ &= \begin{pmatrix} \frac{tk+1}{k+1} + \frac{4c}{k}\frac{tk^2-1}{k+1} & \frac{2c}{k}\frac{tk^2-1}{k+1} & c \\ \frac{2c}{k}\frac{tk^2-1}{k+1} & \frac{tk+1}{k+1} + \frac{4c}{k}\frac{tk^2-1}{k+1} & c \\ 0 & 0 & 1 \end{pmatrix} =: A_k. \end{aligned} \tag{3.8}$$

It is easy to see that the norm of this matrix can be estimated by Ct.

It is possible to check that $f_{k,1}$ forms a Cauchy sequence in $W^{1,1}$. Since $f_{k,1}$ converge to F_1 uniformly we obtain that $F_1 \in W^{1,1}$. From (3.7) we obtain that the derivative of $f_{k,1}$ on $R_{k,1}$ and especially on $\mathcal{C}_1$ equals to

$$Df_{k,1}(x,y,z) = \begin{pmatrix} \frac{1}{k+1} & 0 & 0 \\ 0 & \frac{1}{k+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $Df_{k,1}$ converge to DF_1 in L^1 we obtain that for almost every $(x,y,z) \in \mathcal{C}_1$ we have

$$DF_1(x,y,z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and therefore $J_{F_1}(x,y,z) = 0$.

Moreover, also $\{f_{k,1}^{-1}\}$ forms a Cauchy sequence in $W^{1,1}_{\mathrm{loc}}$ and since $f_{k,1}$ converges uniformly to F_1, $\{f_{k,1}^{-1}\}$ converges to F_1^{-1} uniformly (see Lemma 3.1 [28]). Hence, F_1 is a bi–Sobolev mapping.

From now on each F_k will equal to F_1 on $\mathcal{C}_1$ and we need to define it only on $Q_0 \setminus \mathcal{C}_1$. let us underline that, from the construction,

$$J_{F_1} \neq 0 \text{ a.e. on } Q_0 \setminus \mathcal{C}_1.$$

CONSTRUCTION OF F_2 AND F_3

As before, we construct a sequence of homeomorphisms

$$f_{k,2} \colon Q_0 \xrightarrow{\text{onto}} Q_0$$

and $F_2 \in W^{1,1}(Q_0, \mathbb{R}^3)$ will be later defined as

$$F_2(x) = \lim_{k\to\infty} f_{k,2}(x).$$

We construct a Cantor–type set $\mathcal{C}_2 \subset Q_0 \backslash \mathcal{C}_1$ of positive measure such that $J_{F_2} = 0$ a.e. on $\mathcal{C}_2$. This time we recover $F_1(Q_0 \setminus \mathcal{C}_1)$ up to a set of measure zero by a collection of disjoint, scaled, traslated and “rotated ”copies of the diamond $Q^y(w_1)$ where

$$Q^y(w) = \{(x, y, z) \in \mathbb{R}^3 : |x| + |z| < w(1 - |y|)\}.$$

In Theorem 3.4 and Theorem 3.5, it was essential that all the matrices involved are almost diagonal and thus we can make better estimates of the norm of their product than simply estimate norm of each matrix. After squeezing in two directions our mappings are no longer almost diagonal (for example the matrix from (3.8)) but we repair this using the QR decomposition.

Proposition 3.1. *For every $n \times n$ matrix A we can find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.*

This linear transformation allows us to make some of the matrices almost upper triangular which will be sufficient for our estimates.

The mapping F_3 is constructed in a similar way using traslated and scaled copies of $Q^x(w_1)$ where

$$Q^x(w) = \{(x, y, z) \in \mathbb{R}^3 : |y| + |z| < w(1 - |x|)\}.$$

CONSTRUCTION OF F_4

We will construct a sequence of homeomorphisms $f_{k,4}^{-1} : Q_0 \to Q_0$ and our mapping $F_4 \in W^{1,1}(Q_0, \mathbb{R}^3)$ will be later defined as $F_4(x) = \lim_{k\to\infty} f_{k,4}(x)$. So far we have constructed disjoint Cantor type sets such that $J_{F_1} = 0$ a.e. on $\mathcal{C}_1$, $J_{F_2} = 0$ a.e. on $\mathcal{C}_2$ and $J_{F_3} = 0$ a.e. on $\mathcal{C}_3$. Then, we construct a Cantor type set $\tilde{\mathcal{C}}_4$ of positive measure in the image so that

$$J_{F_4^{-1}} = 0 \quad \text{a.e. on } \tilde{\mathcal{C}}_4$$

and so that $|F_4^{-1}(\tilde{\mathcal{C}}_4)| = 0$.

Thus, the sequence of homeomorphisms F_j which will eventually converge to f is such that

- for $j \in \bigcup_{k\in\mathbb{N}}\{6k+1, 6k+2, 6k+3\}$, $J_{F_j} = 0$ a.e. on $\mathcal{C}_j$ with $|\mathcal{C}_j| > 0$
- for $j \in \bigcup_{k\in\mathbb{N}}\{6k+4, 6k+5, 6k+6\}$, $J_{F_j^{-1}} = 0$ a.e. on $F_j(\mathcal{C}_j)$ and $|F_j(\mathcal{C}_j)| > 0$.

The mappings F_{6k+1} and F_{6k+4}^{-1} are squeezing the Cantor type set in the direction of x and y axes, F_{6k+2} and F_{6k+5}^{-1} are squeezing after rotation in the directions x and z and finally F_{6k+3} and F_{6k+6}^{-1} are squeezing after rotation in the directions y and z.

PROPERTIES OF f

Now we define $f(x) = \lim_{j\to\infty} F_j(x)$. Since F_j converges uniformly, f is a homeomorphism. Moreover, DF_j and DF_j^{-1} is Cauchy in L^1 and thus f is bi-Sobolev. Moreover, it results,

$$\left|\bigcup_{j=1}^{\infty} \mathcal{C}_j\right| = |Q_0|.$$

This together with $J_{F_j} = 0$ on C_j for each $j \in \bigcup_{l\in\mathbb{N}}\{6l+1, 6l+2, 6l+3\}$ and $F_k = F_j$ on $\mathcal{C}_j$ for each $k > j$, imply that $J_f = 0$ almost everywhere on Q_0. Analogously we will deduce that $J_{f^{-1}} = 0$ a.e. on Q_0.

Chapter 4

On the continuity of the Jacobian of orientation preserving mappings

In several situations it is necessary to integrate the Jacobian. The usual hypothesis ensuring this integrability has been $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. Here we are concerned on the minimal condition on the regularity of the map that ensures the local integrability and the continuity property for the Jacobian determinant. The right setting will be the Grand Sobolev space $W^{1,n)}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.

4.1 The integrability of the Jacobian

Let Ω be a domain of $\mathbb{R}^n$ and let $f = (f^1, \dots, f^n) : \Omega \to \mathbb{R}^n$ be a locally integrable map whose distributional differential Df is locally integrable. Then, its Jacobian determinant is defined point-wise at almost every point $x \in \Omega$.

In what follows, in the language of differential forms, we write

$$J_f(x)dx = df^1 \wedge \ldots \wedge df^n(x).$$

From the Hadamard's inequality:

$$|J_f(x)| \leq \prod_{i=1}^{n} |Df^i(x)|, \tag{4.1}$$

follows that $|J_f(x)| \leq |Df(x)|^n$ and hence, the natural assumption to guarantee the integrability of the Jacobian is $f \in W^{1,n}$.

Without any restriction, there is no reason to expect that the degree of integrability of J_f is different from that of $|Df|^n$. S. Muller in [73] was the first to observe that the assumption that J_f does not change sign in Ω, implies higher integrability of the Jacobian.

A map f is said *orientation preserving* if $J_f \geq 0$ almost everywhere. S. Muller in [73] proved that, if $|Df| \in L^n(\Omega)$ and $J_f \geq 0$ then $J_f \in L \log L(K)$ for any compact set K such that $2K \subset \Omega$. The result can be rephrased as follows:

$$\int_K J_f(x) \log\left(e + \frac{J_f(x)}{J_K}\right) dx \leq C(K) \int_\Omega |Df|^n dx$$

where J_K is the integral mean of the Jacobian over K.

T. Iwaniec and C. Sbordone in [55] relaxed the natural assumption on $|Df|$, proving that if f is an orientation preserving map, that is $J_f(x) \geq 0$ a.e., and $|Df| \in L^{n)}(\Omega)$, then the Jacobian of f is locally integrable. The key estimate for their proof was the following:

$$\fint_B J_f(x)dx \leq c(n) \sup_{0<\varepsilon\leq 1} \left[\varepsilon \fint_{2B} |Df|^{n-\varepsilon}\right]^{\frac{1}{n-\varepsilon}}$$

for concentric balls $B \subset 2B \subset \Omega$.

At this point it became clear that the improved integrability property of the Jacobian could be observed also in spaces slightly larger than $W^{1,n}(\Omega)$. As a matter of fact, under the hypothesis $|Df| \in L^n \log^{-1} L(\Omega)$ of G. Moscariello, ([71]), the Jacobian of f is even slightly more integrable; indeed, J_f lies in $L \log\log L_{loc}(\Omega)$.

A result that interpolate between Müller's result and [55] is given in [9]; namely, if $|Df| \in L^n(\log L)^{-s}(\Omega)$, $0 \leq s \leq 1$, then $J_f \in L(\log L)^{1-s}(K)$ for any K such that $2K \subset \Omega$, see also L. Greco ([33]).

4.2 The continuity of the Jacobian

Until now we have considered the weakest assumption to guarantee that the Jacobian determinant is integrable and its integrability properties. A question that naturally arises is the following: what are the sharpest hypotheses under which we may guarantee that if $f_j \in W^{1,1}(\Omega, \mathbb{R}^n)$ converges to f then $J_{f_j} \to J_f$ in some sense? It is well known that if $f_j \rightharpoonup f$ in $W^{1,n}$ then $J_{f_j} \to J_f$ in the sense of distributions. This is a classical result due to Morrey ([70]) and Reshetnyak ([78]). The key ingredient is that in $W^{1,n}$ the point-wise Jacobian agrees with the distributional one. In this setting, in [55], it was proved that if f_j is a sequence of orientation preserving mapping weakly converging to f in $W^{1,n}$ then,

$$\lim_j \int_\Omega \varphi J_{f_j} = \int_\Omega \varphi J_f \qquad \forall \varphi \in \mathrm{Exp}(\Omega).$$

We recall that the exponential class $\mathrm{Exp}(\Omega)$ is formed by measurable functions u on Ω for which there exists $\lambda = \lambda(u) > 0$ such that

$$\exp(\lambda |u|) \in L^1(\Omega)\,.$$

T. Iwaniec and A. Verde in [60], under the strong convergence in $W^{1,n}$ of a sequence f_j of orientation preserving mappings, proved that:

$$\|J_{f_j} - J_f\|_{L \log L(\Omega)} \longrightarrow 0.$$

Our aim here is to prove a similar result in the case that f_j and f are orientation preserving mappings in the Grand Sobolev space $W^{1,n)}(\Omega, \mathbb{R}^n)$ (see Section 2 for definition). We observe that in this case the distributional Jacobian is not equal in general to the point-wise one. More precisely,

Theorem 4.1. *Let Ω be a bounded domain of $\mathbb{R}^n$. If f_j and f are orientation preserving mappings in $W^{1,n)}(\Omega, \mathbb{R}^n)$ such that*

$$f_j \longrightarrow f \;\; in \; W^{1,n)}(\Omega, \mathbb{R}^n)$$

then

$$J_{f_j} \longrightarrow J_f \quad in \; L^1_{\mathrm{loc}}(\Omega).$$

The key tool is the following stability result for the Hodge decomposition, first established in [49],[55] and [57].

Lemma 4.1. *For every vector field ω of class $L^{r(1-\varepsilon)}(\Omega,\mathbb{R}^n)$ with $r>1$ and $-\infty<\varepsilon<1-\frac{1}{r}$, we can find function η and Du such that*

$$|\omega|^{-\varepsilon}\omega=\eta+Du$$

with $u\in W_0^{1,r}(\Omega)$ and $div\eta=0$. This is called a Hodge decomposition and the following estimates hold:

- *If $\omega=Dv$ is a gradient field with $v\in W_0^{1,r(1-\varepsilon)}(\Omega)$, then*

$$\|\eta\|_{L^r(\Omega)}\leq c(n,r)|\varepsilon|\|\omega\|^{1-\varepsilon}_{L^{r(1-\varepsilon)}} \tag{4.2}$$

- *If ω is divergence free, that is $div\omega=0$, then*

$$\|Du\|_{L^r(\Omega)}\leq c(n,r)|\varepsilon|\|\omega\|^{1-\varepsilon}_{L^{r(1-\varepsilon)}} \tag{4.3}$$

From the Hadamard's inequality (4.1) follows one of the key pointwise estimate for the Jacobian: for $0<\varepsilon<1$,

$$\begin{aligned}|df^1|^{-\varepsilon}|J_f(x)dx|&\leq\left(1+|df^1|^{-1}\right)|J_f(x)dx|\\&\leq|J_f(x)dx|+|df^2|\dots|df^n|\leq|J_f(x)dx|+|Df|^{n-1}.\end{aligned} \tag{4.4}$$

An important application of the L^p-theory of differential form is:

Proposition 4.1. *Let Ω be a bounded domain of $\mathbb{R}^n$, $u\in W_0^{1,\frac{n-\varepsilon}{1-\varepsilon}}(\Omega)$ and*

$g \in W^{1,\frac{n-\varepsilon}{n-1}}(\Omega, \mathbb{R}^{n-1})$, *then*

$$\int_{\Omega} du \wedge \left(dg^1 \wedge \ldots dg^{n-1}\right) = 0 \tag{4.5}$$

If $u \in C_0^\infty(\Omega)$ and $g \in C^\infty(\Omega, \mathbb{R}^{n-1})$ by Stokes' theorem

$$\begin{aligned}\int_{\Omega} du \wedge \left(dg^1 \wedge \ldots dg^{n-1}\right) &= \int_{\Omega} d\left(u \wedge dg^1 \wedge \ldots dg^{n-1}\right) \\ &= \int_{\partial\Omega} u \wedge \left(dg^1 \wedge \ldots dg^{n-1}\right) = 0.\end{aligned}$$

As $\frac{n-\varepsilon}{1-\varepsilon}$ and $\frac{n-\varepsilon}{n-1}$ are Hölder conjugate then, $du \wedge (dg^1 \wedge \ldots dg^{n-1}) \in L^1(\Omega)$ and (4.5) follows via an approximation argument (see [55], page 138).

Proof of Theorem 4.1. It is sufficient to prove the theorem for a subsequence. Hence, without loss of generality, by assumptions, we can assume that

$$Df_j(x) \to Df(x) \qquad \text{a.e. } x \in \Omega. \tag{4.6}$$

Firstly, we prove the following:

$$\lim_{j\to\infty} \int_{\Omega} \varphi J_{f_j} = \int_{\Omega} \varphi J_f \tag{4.7}$$

for every nonnegative test functions $\varphi \in C_0^1(\Omega)$. For technical reason, we shall consider an auxiliary test function $\psi \in C_0^1(\Omega)$ which is equal to 1 in the support of φ.

Let $f_j = (f_j^1, \ldots, f_j^n)$. We consider the compactly supported maps: $g_j = (\psi f_j^1, \ldots, \psi f_j^{n-1}, \varphi f_j^n) = (g_j^1, \ldots, g_j^n)$ and $g = (\psi f^1, \ldots, \psi f^{n-1}, \varphi f^n) = (g^1, \ldots, g^n)$.

Using the calculus of differential forms, the Jacobian determinant of g_j is:

$$J_{g_j} dx = dg_j^1 \wedge \ldots \wedge dg_j^n = \begin{cases} d(f_j^1) \wedge \ldots \wedge d(f_j^{n-1}) \wedge d(\varphi f_j^n) & \text{on } supp\varphi \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

The telescopic decomposition of the Jacobian ([53], Section 8) leads to:

$$\begin{aligned} \left(J_{g_j} - J_g\right) dx = \left(dg_j^1 \wedge \ldots \wedge dg_j^n\right) - \left(dg^1 \wedge \ldots \wedge dg^n\right) = \\ =&(dg_j^1 - dg^1) \wedge dg_j^2 \wedge \ldots \wedge dg_j^n + \\ &+ dg^1 \wedge \left(dg_j^2 - dg^2\right) \wedge dg_j^3 \wedge \ldots \wedge dg_j^n + \ldots + \\ &+ dg^1 \wedge \ldots \wedge dg^{n-1} \wedge \left(dg_j^n - dg^n\right). \end{aligned} \tag{4.9}$$

Applying the Hodge decomposition to $Dg_j - Dg$, we have:

$$|Dg_j - Dg|^{-\varepsilon}(Dg_j - Dg) = DG_j + L_j \tag{4.10}$$

with $DG_j \in L^{\frac{n-\varepsilon}{1-\varepsilon}}$, $L_j \in L^{\frac{n-\varepsilon}{1-\varepsilon}}$ and

$$\|L_j\|_{\frac{n-\varepsilon}{1-\varepsilon}} \leq c(n,\varepsilon)\varepsilon \|Dg_j - Dg\|_{n-\varepsilon}^{1-\varepsilon}. \tag{4.11}$$

It is important to realize that L_j becomes small as ε goes to zero.

(4.10) can be rewritten as:

$$|Dg_j - Dg|^{-\varepsilon}(Dg_j^k - Dg^k) = DG_j^k + l_j^k \qquad k = 1, \ldots, n \tag{4.12}$$

where l_j^k is the 1-form whose coefficients are the entries of the k-th column of L_j.

By (4.9) we have:

$$|Dg_j - Dg|^{-\varepsilon}\left(J_{g_j} - J_g\right) = |Dg_j - Dg|^{-\varepsilon}\Big[(dg_j^1 - dg^1) \wedge dg_j^2 \wedge \ldots \wedge dg_j^n + \\ + dg^1 \wedge \left(dg_j^2 - dg^2\right) \wedge dg_j^3 \wedge \ldots \wedge dg_j^n + \ldots + \\ + dg^1 \wedge \ldots \wedge dg^{n-1} \wedge \left(dg_j^n - dg^n\right)\Big]. \tag{4.13}$$

We focus our attention on the first term of the previous formula; using (4.12), we have:

$$|Dg_j - Dg|^{-\varepsilon}(dg_j^1 - dg^1) \wedge dg_j^2 \wedge \ldots \wedge dg_j^n = \\ = dG_j^1 \wedge dg_j^2 \wedge \ldots \wedge dg_j^n + l_j^1 \wedge dg_j^2 \wedge \ldots \wedge dg_j^n \tag{4.14}$$

Let us estimate both terms in the right hand side of the last formula.

By Proposition 4.1, we find that:

$$\int_\Omega dG_j^1 \wedge dg_j^2 \wedge \ldots \wedge dg_j^n = \int_\Omega d\left(G_j^1 dg_j^2 \wedge \ldots \wedge dg_j^n\right) = 0.$$

Applying Hölder inequality and (4.11), we can estimate the second term in (4.14):

$$
\begin{aligned}
\int_\Omega |l_j^1 \wedge \left(dg_j^2 \wedge \ldots \wedge dg_j^n\right)| \leq& \\
\leq& \left(\int_\Omega |l_j^1|^{\frac{n-\varepsilon}{1-\varepsilon}}\right)^{\frac{1-\varepsilon}{n-\varepsilon}} \left(\int_\Omega |dg_j^2 \wedge \ldots \wedge dg_j^n|^{\frac{n-\varepsilon}{n-1}}\right)^{\frac{n-1}{n-\varepsilon}} \\
\leq& c(n,\varepsilon)\varepsilon \|Dg_j - Dg\|_{n-\varepsilon}^{1-\varepsilon} \|Dg_j\|_{n-\varepsilon}^{n-1} \\
\leq& c(n,\varepsilon)\|Dg_j - Dg\|_{L^{n)}}^{1-\varepsilon} \|Dg_j\|_{L^{n)}}^{n-1}.
\end{aligned}
\tag{4.15}
$$

Analogously, we can estimate the k-th term in (4.13) as before, to obtain:

$$
\begin{aligned}
&\int_\Omega |Dg_j - Dg|^{-\varepsilon} |dg^1 \wedge \ldots \wedge dg^{k-1} \wedge d(g_j^k - dg^k) \wedge dg_j^{k+1} \wedge \ldots \wedge dg_j^n| \\
\leq& c(n,\varepsilon)\varepsilon \|Dg_j - Dg\|_{n-\varepsilon}^{1-\varepsilon} \|dg^1 \wedge \ldots \wedge dg^{k-1} \wedge dg_j^{k+1} \wedge \ldots \wedge dg_j^n\|_{\frac{n-\varepsilon}{n-1}} \\
\leq& c(n,\varepsilon)\varepsilon \|Dg_j - Dg\|_{n-\varepsilon}^{1-\varepsilon} \; \||Dg^{k-1}| \; |Dg_j|^{n-k}\|_{\frac{n-\varepsilon}{n-1}} \\
\leq& c(n,\varepsilon)\|Dg_j - Dg\|_{L^{n)}}^{1-\varepsilon} \|Dg\|_{L^{n)}}^{k-1} \|Dg_j\|_{L^{n)}}^{n-k} \\
\leq& c(n,\varepsilon)\|Dg_j - Dg\|_{L^{n)}}^{1-\varepsilon} \left(\|Dg\|_{L^{n)}}^{n-1} + \|Dg_j\|_{L^{n)}}^{n-1}\right).
\end{aligned}
$$

Thus, for every j we get:

$$
\begin{aligned}
&\left| \int_{\Omega} |Dg_j - Dg|^{-\varepsilon} \left(J_{g_j} - J_g\right) dx \right| \\
&\qquad \leq n\, c(n,\varepsilon) \|Dg_j - Dg\|_{L^{n)}}^{1-\varepsilon} \left(\|Dg\|_{L^{n)}}^{n-1} + \|Dg_j\|_{L^{n)}}^{n-1} \right).
\end{aligned}
\tag{4.16}
$$

Using telescopic decomposition of the Jacobian and (4.4), we can pass to the limit under the integral sign by the dominated convergence theorem. Since, Dg_j converges to Dg in $L^{n)}$ on the support of φ, we have:

$$
\limsup_{j\to\infty} \left(\left| \int_{\mathrm{supp}\varphi} (J_{g_j} - J_g) dx \right| \right) = 0. \tag{4.17}
$$

By (4.8), (4.17) is equivalent to:

$$
\begin{aligned}
&\limsup_{j\to\infty} \left(\left| \int_{\mathrm{supp}\varphi} \varphi J_{f_j} - \varphi J_f \right.\right. \\
&\left.\left. + \int_{\mathrm{supp}\varphi} f_j^n (df_j^1 \wedge \ldots \wedge df_j^{n-1} \wedge d\varphi) - f^n (df^1 \wedge \ldots \wedge df^{n-1} \wedge d\varphi) \right| \right) = 0.
\end{aligned}
$$

Now, we can apply the imbedding of Grand Sobolev space $W^{1,n)}$ into $\mathrm{Exp}(\Omega)$ (see [27]). It means that f_j^n, f^n belong to L^p for every $1 < p < \infty$ and by assumptions it follows that f_j^n converges to f^n in L^p for every $1 < p < \infty$. This remark together with the telescopic decomposition of the Jacobian and

Hölder's inequality gives:

$$
\begin{aligned}
&\limsup_{j\to\infty}\left(\left|\int_{\mathrm{supp}\varphi} f_j^n(df_j^1\wedge\ldots\wedge df_j^{n-1}\wedge d\varphi)-f^n(df^1\wedge\ldots\wedge df^{n-1}\wedge d\varphi)\right|\right)\\
&=\limsup_{j\to\infty}\left(\left|\int_{\mathrm{supp}\varphi}(f_j^n-f^n)(df_j^1\wedge\ldots\wedge df_j^{n-1}\wedge d\varphi)\right.\right.\\
&\qquad\left.\left.+\int_{\mathrm{supp}\varphi} f^n(df_j^1\wedge\ldots\wedge df_j^{n-1}-df^1\wedge\ldots\wedge df^{n-1})\wedge d\varphi)\right|\right)=0.
\end{aligned}
\tag{4.18}
$$

Therefore,

$$
\limsup_{j\to+\infty}\left(\left|\int_{supp\varphi}\varphi J_{f_j}-\varphi J_f\right|\right)=0.
$$

Since f_j and f are orientation preserving maps, we get:

$$
\lim_{j\to+\infty}\int_{supp\varphi}\varphi J_{f_j}=\int_{supp\varphi}\varphi J_f. \tag{4.19}
$$

for every non negative $\varphi\in C_0^1(\Omega)$.

Now, we are in position to prove the thesis.

We recall, by a Theorem of Iwaniec and Sbordone ([55]), that J_{f_j} and J_f belong to $L^1_{\mathrm{loc}}(\Omega)$. On the other hand, by assumptions, J_{f_j} converges to J_f a.e. Hence, an application of Fatou's Lemma gives, for every measurable set $E\subset\subset\Omega$:

$$
\begin{aligned}
\int_E J_f\le\liminf_{j\to+\infty}\int_E J_{f_j}&\le\limsup_{j\to+\infty}\int_E J_{f_j}\\
&\le\lim_{j\to+\infty}\int_E\varphi J_{f_j}\\
&=\int_E\varphi J_f
\end{aligned}
$$

where $\varphi \in C_0^1(\Omega)$ can be nonnegative function which is equal to one on E.

Taking infimum with respect to all such functions, we get that:

$$\lim_{j \to +\infty} \int_E J_{f_j} = \int_E J_f, \tag{4.20}$$

that means:

$$\|J_{f_j}\|_{L^1(E)} \longrightarrow \|J_f\|_{L^1(E)} \qquad E \subset\subset \Omega. \tag{4.21}$$

(4.21) together with the convergence almost everywhere of $J_{f_j}(x)$ to $J_f(x)$ completes the proof.

$\square$

Bibliography

[1] L. Ambrosio, N. Fusco, D. Pallara: *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, (2000).

[2] K. Astala,T. Iwaniec, P. Koskela, and G. Martin: *Mappings of BMO-bounded distortion*, Math. Ann. **317** (2000), 703–726.

[3] K. Astala, T. Iwaniec, G. Martin. *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton Mathematical Series vol. 24, (2008).

[4] S. Antman, *Fundamental mathematical problems in the theory of non-linear elasticity. Numerical solution of partial differential equations*, III (Proc. Third Sympos. (SYNSPADE), Univ. Maryland, College Park, Md., 1975), pp. 35–54. Academic Press, New York, 1976.

[5] J. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. 63 (1976/77), no. 4, 337–403.

[6] L. Bers, L. Nirenberg: *On linear and nonlinear elliptic boundary value problems in the plane*, Convegno Internazionale sulle Equazioni Lineari

alle Derivate Parziali, Trieste, 1954, Edizioni Cremonese, Roma, 1955, 141– 167.

[7] B. Bojarski: *Generalized solutions of systems differential equations of first order and of elliptic type with discontinuous coefficients*, Math. Sb. (N.S.) 43 (85) (1957), 451– 503.

[8] B. Bojarski and T.Iwaniec: *Analytical foundations of the theory of quasiconformal mappings in* $\mathbb{R}^n$. Ann. Acad. Sci. Fenn. Ser. A I Math. 8 (1983), 257–324.

[9] H. Brezis, N. Fusco and C.Sbordone, *Integrability for the Jacobian of orientation preserving mappings*, J. Funct. Anal. 115 (1993), no. 2, 425–431.

[10] H. Brezis and H. Nguyen, *The Jacobian determinant revisited*, Invent. Math. 185 (2011), no. 1, 17–54.

[11] R. Caccioppoli: *Fondamenti per una teoria generale delle funzioni pseudoanalitiche di una variabile complessa* , Rend. Accad. Naz. Lincei 13 (1952), 197–204.

[12] R. Cerný, *Homeomorphism with zero Jacobian: Sharp integrability of the derivative*, J. Math. Anal. Appl, 373, (2011), 161–174.

[13] L. Cesari: *Sulle trasformazioni continue*. Ann. Mat. Pura Appl. (4) 21, (1942), 157–188.

[14] P. Ciarlet, *Mathematical elasticity.* Vol. II. Theory of plates. Studies in Mathematics and its Applications, 27. North-Holland Publishing Co., Amsterdam, 1997.

[15] M. Csörnyei, S. Hencl, J. Malý: *Homeomorphisms in the Sobolev space* $W^{1,n-1}$. J. Reine Angew. Math. **644** (2010), 221-235.

[16] S. Daneri and A. Pratelli, *Smooth approximation of bi–Lipschitz orientation–preserving homeomorphisms*, preprint (2011), arXiv:1106.1192.

[17] G. David: *Solution de l'equation de Beltrami avec* $\|\mu\|_\infty = 1$, Ann. Acd. Sci. Fenn. Ser. A I Math. **13:1** (1988), 25-70.

[18] P. Di Gironimo, L. D'Onofrio, C. Sbordone, R. Schiattarella: *Anisotropic Sobolev homeomorphisms*, Ann. Acad. Sci. Fenn. Math., 36, (2011), 593–602.

[19] L. D'Onofrio, S. Hencl and R. Schiattarella, *Bi-Sobolev homeomorphism with zero Jacobian almost everywhere*, Preprint MATH-KMA-2012/408.

[20] L. D'Onofrio, C. Sbordone and R. Schiattarella, *The Grand Sobolev homeomorphisms and their measurability properties*, Advanced Nonlinear Studies, 12 (2012), no. 4, 767–782.

[21] L. D'Onofrio and R. Schiattarella, *On the continuity of Jacobian of orientation preserving mappings in the Grand Sobolev space*, preprint.

[22] L. D'Onofrio and R. Schiattarella, *On the total variations for the inverse of a BV-homeomorphism*, to appear in Adv. Calc.Var, DOI 10.1515/acv-2012-0103.

[23] D. Edmunds, P. Gurka and B. Opic, *On embeddings of logarithmic Bessel potential spaces*, J. Funct. Anal. 146 (1997), no. 1, 116–150.

[24] F. Farroni, R. Giova, G. Moscariello and R. Schiattarella: *Homeomorphisms of finite inner distortion: composition operators on Zygmund-Sobolev and Lorentz–Sobolev spaces*, preprint 2012.

[25] H. Federer: *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer - Verlag, New York (1969) (Second edition 1996).

[26] I. Fonseca, G. Leoni and J. Malý, *Weak continuity and lower semicontinuity results for determinants*, Arch. Ration. Mech. Anal. 178 (2005), no. 3, 411–448.

[27] N. Fusco, P. Lions and C. Sbordone, *Sobolev imbedding theorems in borderline cases*, Proc. Amer. Math. Soc. 124 (1996), no. 2, 561–565.

[28] N. Fusco, G. Moscariello, C. Sbordone: *The limit of $W^{1,1}$-homeomorphisms with finite distortion*, Calc. Var. **33** (2008), 377-390.

[29] F. W. Gehring and O. Lehto: *On the total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fenn. Ser. A I 272, (1959), 1–9.

[30] F. Giannetti, L. Greco and A. Passarelli di Napoli: *Regularity of mappings of finite distortion*, Funct. Approx. Comment. Math. **40** (2009), no. 1, 91–103.

[31] F. Giannetti, T. Iwaniec, J. Onninen and A. Verde: *Estimates of Jacobians by subdeterminants*, J. Geom. Anal. 12 (2002), no. 2, 223–254.

[32] M. Giaquinta, G. Modica and J. Souček: *Area and the area formula.* Proceedings of the Second International Conference on Partial Differential Equations (Italian) (Milan, 1992). Rend. Sem. Mat. Fis. Milano 62 (1992), 53-87 (1994).

[33] L. Greco, *A remark on the equality det Df = Det Df*, Differential Integral Equations 6 (1993), 1089–1100.

[34] L. Greco, T. Iwaniec, G. Moscariello: *Limits of the improved integrability of the volume forms*, Indiana Univ. Math. J. **44** (1995), 305-339.

[35] L. Greco, C. Sbordone and R. Schiattarella, *Composition of bi-Sobolev homeomorphisms*, Proc. Roy. Soc. Edinburgh Sect. A 142 (2012), no. 1, 61–80.

[36] L. Greco, C. Sbordone, C. Trombetti: *A note on planar homeomorphisms*, Rend. Accad. Sci. Fis. Mat. Napoli (4) **75** (2008), 53-59.

[37] L. Greco and A. Verde, *On some nonlinear expressions of the Jacobian.* Diff. Int. Eq. 13 (2000), no. 10-12, 1569–1582.

[38] S. Hencl: *On the weak differentiability of $u \circ f^{-1}$*, to appear in Math. Scand. 107 (2010), no. 2, 198–208.

[39] S. Hencl: *Sharpness of the assumptions for the regularity of a homeomorphism,* Michigan Math. J. 59 (2010), no. 3, 667–678.

[40] S. Hencl: *Sobolev homeomorphism with zero Jacobian almost everywhere*, J. Math. Pures Appl. (9) 95 (2011), no. 4, 444-458.

[41] S. Hencl, P. Koskela: *Lectures on Mappings of finite distortion*, Monograph, (2012), preprint.

[42] S. Hencl, P. Koskela: *Mappings of finite distortion: composition operator*, Ann. Acd. Sci. Fenn. Math. **33** (2008), 65-80.

[43] S. Hencl, P. Koskela: *Regularity of the inverse of a planar Sobolev homeomorphism*, Arch. Ration. Mech. Anal. **180** (2006), 75-95.

[44] S. Hencl, P. Koskela and J. Malý: *Regularity of the inverse of a Sobolev homeomorphism in space*, Proc. Roy. Soc. Edinburgh Sect. A **136:6** (2006), 1267-1285.

[45] S. Hencl, P. Koskela and J. Onninen: *Homeomorphisms of Bounded Variation*, Arch. Ration. Mech. Anal. 186, 2007, 351-360.

[46] S. Hencl, P. Koskela and X. Zhong, *Mappings of finite distortion: reverse inequalities for the Jacobian.* J. Geom. Anal. 17 (2007), no. 2, 253–273.

[47] S. Hencl, J. Malý: *Jacobians of Sobolev homeomorphisms*, Calc. Var. Partial Differential Equations 38 (2010), no. 1–2, 233–242.

[48] S. Hencl, G. Moscariello, A. Passarelli di Napoli and C. Sbordone: *Bi-Sobolev mappings and elliptic equations in the plane*, J. Math. Anal. Appl. **355** (2009), 22-32.

[49] T. Iwaniec : *p -harmonic tensors and quasiregular mappings.* Ann. of Math. (2) 136 (1992), no. 3, 589–624.

[50] T. Iwaniec, P. Koskela and G. Martin: *Mappings of BMO-distortion and Beltrami-type operators*. Dedicated to the memory of Tom Wolff. J. Anal. Math. **88** (2002), 337-381.

[51] T. Iwaniec, P. Koskela, G. Martin and C. Sbordone, *Mappings of finite distortion:* $L^n \log^\gamma L$ *-integrability*, J. London Math. Soc. (2) 67 (2003), no. 1, 123–136.

[52] T. Iwaniec, L. Kovalev and J. Onninen *Diffeomorphic approximation of Sobolev homeomorphisms*, Arch. Ration. Mech. Anal. 201 (2011), no. 3, 1047–1067.

[53] T. Iwaniec and G. Martin, *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2001.

[54] T. Iwaniec, J. Onninen: *Deformations of finite conformal energy: Boundary behavior and limit theorems.*, Trans. Amer. Math. Soc. 363 (2011), no. 11, 5605–5648.

[55] T. Iwaniec and C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses*, Arch. Rational Mech. Anal. 119 (1992), no. 2, 129–143.

[56] T. Iwaniec and C. Sbordone, *Quasiharmonic fields*, Ann. Inst. H. Poincaré Anal. Non Linéaire 18 (2001), no. 5, 519–572.

[57] T. Iwaniec and C. Sbordone, *Weak minima of variational integrals*, J. Reine Angew. Math. 454 (1994), 143–161.

[58] T. Iwaniec and V. Šverák: *On mappings with integrable dilatation.* Proc. Amer. Math. Soc. **118** (1993), 181–188.

[59] T. Iwaniec and V. Šverák: *On mappings with integrable dilatation.* Proc. Amer. Math. Soc. **118** (1993), 181–188.

[60] T. Iwaniec and A. Verde: *On the Operator* $\mathcal{L}(f) = f \log |f|$, Journ. of Funct Anal. 169 (1999) 391–420.

[61] J. Kauhanen, P. Koskela and J. Malý: *On functions with derivatives in a Lorentz space*, Manuscripta Math. **100** (1999), 87-101.

[62] J. Kauhanen, P. Koskela and J. Malý: *Mappings of finite distortion: condition N*, Michigan Math. J. **49** (2001), 169-181.

[63] P. Koskela and J. Malý: *Mappings of finite distortion: the zero set of the Jacobian*, J. Eur. Math. Soc. **5:2** (2003), 95-105.

[64] P. Koskela and J. Onninen: *Mappings of finite distortion: decay of the Jacobian in the plane*, Adv. Calc. Var. **1** (2008), 3, 309-321.

[65] O. Lehto, K. Virtanen: *Quasiconformal Mappings in the Plane*, Springer- Verlag, Berlin, (1971).

[66] J. Malý: *Lectures on change of variables in integral*, Graduate School in Helsinki, (2001).

[67] M. Marcus and V. Mizel: *Transformations by functions in Sobolev spaces and lower semicontinuity for parametric variational problems*, Bull. Amer. Math. Soc. 79 (1973), 790–795.

[68] O. Martio and W. Ziemer: *Lusin's condition (N) and mappings with nonnegative Jacobians*, Michigan Math. J. 39 (1992), no. 3, 495–508.

[69] C.B. Morrey: *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. 43 (1938) 126–166.

[70] C. B. Morrey, *Multiple integrals in the calculus of variations*, Springer Verlag, New York, 1966.

[71] G. Moscariello, *On the integrability of the Jacobian in Orlicz spaces*, Math. Japon. 40 (1994), no. 2, 323–329.

[72] G. Moscariello and A. Passarelli di Napoli, *The regularity of the inverses of Sobolev homeomorphisms with finite distortion*,to appear on J. Geom. Anal..

[73] S. Müller, *Higher integrability of determinants and weak convergence in L^1*, J. Reine Angew. Math. 412 (1990), 20–34.

[74] J. Onninen: *Regularity of the inverse of spatial mappings with finite distortion*, Calc. Var. **26** (3) (2006), 331-341.

[75] S. P. Ponomarev: *An example of an ACTLp homeomorphism that is not absolutely continuous in the sense of Banach.* Dokl. Akad. Nauk SSSR 201 (1971), 1053–1054.

[76] S. P. Ponomarev: *Property N of homeomorphism in the class* $W^{1,p}$, Transl. Sibirskii Math., **28** (2) (1987), 140-148.

[77] Yu. G. Reshetnyak: *Some geometrical properties of functions and mappings with generalized derivatives*, Sibirsk. Math. Zh. **7** (1966), 886-919.

[78] Y. G. Reshetnyak, *Stability theorems for mappings with bounded excursion,* Sib. Math. J. 9 (1968), 499–512.

[79] V. Ryazanov, U. Srebro and E. Yakubov: *Finite mean oscillation and the Beltrami equation*, Israel J. Math. **153** (2006), 247-266.

[80] R. Schiattarella: *Composition of bisobolev mappings*, Rend. Accad. Sci. Fis. Mat. Napoli (4) 77 (2010), 7–14.

[81] J. Väisälä: *Two new characterizations for quasiconformality*, Ann. Acad. Sci. Fenn. Ser. A I Math., 362, (1965), 1-12.

[82] S. K. Vodop'janov and V. M. Gol'dšteĭn: *Quasiconformal mappings, and spaces of functions with first generalized derivatives.* (Russian) Sibirsk. Mat. Ž. **17** (1976), no. 3, 515-531.

[83] W.P. Ziemer: *Change of variables for absolutely continuous functions*, Duke Math. J. ,36:1, (1969), 171–178.