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Some Inequalities for Eigenfunctions and Eigenvalues of Certain Elliptic Operators

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INTRODUCTION

The main purpose of this thesis is to illustrate some applications of symmetrization techniques to problems of geometrical and analytical flavor.

Symmetrization is a simple but powerful tool, which enables to gain sharp informations out of many geometric and functional inequalities. It consists in rearranging given sets or functions into new sets or functions which have a more symmetric aspect. This idea dates back to J. Steiner, who used it to give a beautiful (though incomplete) proof of the *plane isoperimetric inequality*: in fact, Steiner invented a method, nowcalled *Steiner symmetrization*, aimed at converting a plane figure into another one having same area, lower perimeter and an *extra* symmetry.
Some years later, H. Schwarz found a way to extended the method of Steiner to functions: Schwarz's aim was to transform both a function and its domain into a new function defined in a new domain, both *more symmetric* than the original ones, in such a way that neither the measure of the domain nor some norm of the function would be changed.
The symmetrization method of Schwarz was lately popularized by Hardy, Littlewood and Polya in the mid-thities and by Polya and Szegö in the fifties.
In particular, Polya and Szegö showed that Schwarz symmetrization could be used gain sharp bounds for the values of some important physical quantities, e.g., the *fundamental tone* of a membrane, the *capacity* of a condenser or the *torsional rigidity* of a rod. For example, they proved to be true a conjecture in Acoustic formulated by Lord Rayleigh, namely that the fundamental tone of a circular membrane is the lowest possible among all membranes having fixed area.
In later years, it was shown that Schwarz symmetrization technique was a useful tool for proving theorems which compare so-

lutions or other quantities associated to different boundary value problems for elliptic (or even parabolic) differential equations.
Typically, this technique can be used to make pointwise comparison between PDE solutions, or to get estimates on some of their norms, or even to compare other quantities associated to a given problem and the corresponding ones associated to an auxiliary *symmetrized* problem.
In the first cases, the basic idea is to get some differential inequality for the distribution function of the solution, which will reduce to an equality on the solution of the symmetrized problem.
In the latter case, one of the basic techniques consists in proving that the considered quantity has a variational nature, then using rearangement inequalities to prove the comparison result.
On the other hand, the geometric symmetrization of Steiner was used by de Giorgi (among other things) to finally settle the general isoperimetric inequality in the fifties.

As stated above, we present some geometric and analytic inequalities related to solutions of certain PDEs. In particular, here we focus on:

- some isoperimetric inequalities satisfied by level sets of functions which satisfy the Euler–Lagrange equation of a variational problem related to some Hardy–Sobolev inequalities;

- two stabilty estimates for the symmetrized first eigenfunction of linear elliptic operators;

- a Faber–Krahn type inequality for the principal weighted eigenvalue of nonlinear elliptic operators obtained by adding an indefinite potential to the classical p-Laplacian.

All these results are obtained by means of symmetrization techniques. In particular, we use Steiner symmetrization in the proof of the isoperimetric inqualities, while we use Schwarz symmetrization machineries for the other two points.

The present work is structured into four chapters. The first one is a general overview onto facts about rearrangements of sets and functions, while the following three contain the results of our researches. The latter chapters could be also read independently, provided the reader is familiar with some notation and properties of rearrangements: in fact, each chapter is equipped with a detailed introduction to problems it deals with.

In chapter 1, we introduce notations and symmetrization techniques which will be used through the paper, namely Steiner symmetrization for sets, onedimensional and Schwarz rearrangements for functions and their main properties.

In chapter 2, based on the work [32], we prove a family of isoperimetric inequalities for bodies of revolution which arise in connection with the problem of finding the extremals in some Hardy-Sobolev inequality. For these inequalities, we are able to prove that they are sharp and that a characterization of the equality case is available, yielding the best constant.
In particular, we prove that for sufficiently smooth bounded bodies of revolution $D \subset \mathbb{R}^N$ with $N \geqslant 3$, the following inequality holds:

$$\Big[\mathrm{Per}(D) - a(N-2)\,\mathrm{Sec}(D)\Big]^N \geqslant 2(N-1)N^N\omega_{N-1}\ \varphi_N(a)\,\mathrm{Vol}^{N-1}(D)\,,$$

depending on the parameter $a \in]0,1]$, where the symbols $\mathrm{Vol}(D)$, $\mathrm{Per}(D)$ and $\mathrm{Sec}(D)$ denote respectively the volume, the perimeter and a weighted measure with respect to a weight which depends only on the distance of the points of D from the rotation axis; and $\varphi_N(a)$ is a suitable nonnegative constant.
Moreover we are able to prove that $2(N-1)N^N\omega_{N-1}\varphi_N(a)$ is the best constant for inequality (2.1) and to characterize the equality case.

Inequalities are proved first for symmetric bodies of revolution; therefore, their validity is extended by means of Steiner symmetrization.

In chapter 3, which is based on our work [33], we prove two stability-type estimates which involve the symmetrized L^∞-normalized first eigenfunction u_1 of problem:

$$\begin{cases} -\operatorname{div}(A(x)\cdot\nabla u) + c(x)\, u = \lambda\, u & \text{, in } \Omega \\ u = 0 & \text{, on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ be a bounded open domain with unit measure, the matrix A satisfies an uniform ellipticity condition and the potential term c is nonnegative.
These estimates apply when the first eigenvalue $\lambda_1 := \lambda_1^{A,c}(\Omega)$ is close to the lowest possible one (i.e., to $\lambda_1^\star$, the first eigenvalue of the Dirichlet Laplacian in the ball $\Omega^\star$ having the same measure of Ω); in particular, they give a rough idea of how fast two quantities related to $u_1^\star$ decay in terms of the distance $\lambda_1 - \lambda_1^\star$ or in terms of the value $u_1^\star$ assumes on a specified set.
To be more precise, we prove that if $\lambda_1 \approx \lambda_1^\star$ then the L^∞-distance between the Schwarz rearrangement $u_1^\star$ and the L^∞-normalized positive first eigenfunction U_1 of the Dirichlet Laplacian in $\Omega^\star$ corresponding to $\lambda_1^\star$ is less than a suitable power of the difference $\lambda_1 - \lambda_1^\star$ times a universal constant, namely that:

$$0 \leqslant \lambda_1 - \lambda_1^\star \leqslant \delta_1 \;\Rightarrow\; \|u_1^\star - U_1\|_{\infty,\Omega^\star} \leqslant C_1\, (\lambda_1 - \lambda_1^\star)^{2/(N+2)}$$

where $C_1, \delta_1 > 0$ are suitable constants depending only on the dimension N.
We also show that the L^∞-distance between the L^∞-normalized positive first eigenfunction of the Dirichlet Laplacian in a ball B whose first eigenvalue equals λ_1 and the rearrangement $u_1^\star$ can be controlled with a power of the value $\varepsilon \approx 0$ assumed by $u_1^\star$ on the boundary ∂B, viz. that:

$$0 \leqslant \varepsilon \leqslant \delta_2 \;\Rightarrow\; \|u_1^\star - V_1\|_{\infty,B} \leqslant C_2\, \varepsilon^{2/(N+2)}\,,$$

where again $C_2, \delta_2 > 0$ are suitable constants depending only on the dimension N and V_1 solves:

$$\begin{cases} -\Delta V_1 = \lambda_1 \ V_1 & \text{, in } B \\ V_1 = 0 & \text{, on } \partial B \end{cases}$$

with $V_1 \geqslant 0$ in B and $\|V_1\|_{\infty,B} = 1$.

In chapter 4, we prove a generalization of the classical Faber-Krahn inequality for the principal weighted eigenvalue of p-Laplace operator plus an indefinite potential.
To be more precise, we consider the nonlinear weighted eigenvalue problem:

$$\begin{cases} \Delta_p u + V(x) \ |u|^{p-2} u = \lambda \ m(x) \ |u|^{p-2} u & \text{, in } \Omega \\ u = 0 & \text{, on } \partial\Omega \text{ ,} \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded open domain, $p \in]1, \infty[$ and the weight m and the potential V are indefinite measurable functions. Such a problem has attracted some interests in the last decade, for it arises as a generalization of the classical eigenvalue problem for the p-Laplacian. In particular, it was recently proved that some principal eigenvalue exists provided m and V satisfy certain summability assumptions and the variational quantity:

$$\alpha(\Omega, V, m) := \inf \left\{ \int_\Omega |\nabla u|^p + V(x) \ |u|^p, \ u \in W_0^{1,p}(\Omega), \ \|u\|_{p,\Omega} = 1 \right. \\ \left. \text{and } \int_\Omega m(x) \ |u|^p = 0 \right\}$$

is positive or nonnegative, depending on the sign of m. In the particular case $m \geqslant 0$ a.e. in Ω, which is the one we are interested in, such principal eigenvalue, call it $\lambda_p(\Omega, V, m)$, is unique. On the other hand, it is woth noticing that principal eigenvalue needs not to be unique, but nonuniqueness happens only when m changes its sign.
Here we show that the unique principal eigenvalue $\lambda_p(\Omega, V, m)$

decreases under Schwarz symmetrization. In particular, we firstly prove that potentials can be chosen in such a way that the problem under investigation and the two symmetrized problems:

$$\begin{cases} \Delta_p v + V_\bullet(x)\ |v|^{p-2}v = \lambda\ m^\star(x)\ |v|^{p-2}v & \text{, in } \Omega^\star \\ v = 0 & \text{, on } \partial\Omega^\star\ , \end{cases}$$
$$\begin{cases} \Delta_p w - (V_-)^\star(x)\ |w|^{p-2}w = \lambda\ m^\star(x)\ |w|^{p-2}w & \text{, in } \Omega^\star \\ w = 0 & \text{, on } \partial\Omega^\star\ , \end{cases}$$

(where $\Omega^\star$ is the unique open ball centered in o having the same measure of Ω, and $m^\star$, $V_\bullet$, $(V_-)^\star$ are suitable Schwarz symmetral of m, V and V_-) simultaneously have a unique principal eigevalue; then we demonstrate that the three principal eigenvalues $\lambda_p(\Omega, V, m)$, $\lambda_p(\Omega^\star, V_\bullet, m^\star)$ and $\lambda_p(\Omega^\star, -(V_-)^\star, m^\star)$ satisfy the following inequalities:

$$\lambda_p(\Omega, V, m) \geqslant \lambda_p(\Omega^\star, V_\bullet, m^\star) \geqslant \lambda_p(\Omega^\star, -(V_-)^\star, m^\star)\ .$$

Moreover, in the spirit of the original Faber–Krahn inequality, we prove that if $\lambda_p(\Omega^\star, -(V_-)^\star, m^\star) \geqslant 0$ then equality between the rightmost and the leftmost sides is attained only in the *radially symmetric setting*, i.e. when $\Omega = \Omega^\star$, $V = -(V_-)^\star$ and $m = m^\star$ modulo translations.
While both chapter 2 and 3 are based on published results, this final chapter is based on the *work in progress* paper [34], therefore it is more sketchy than the previous ones.

Anyway, we refer the reader to the introductions of chapters 2, 3 and 4 for more informations on the technical matters there treated.

1

REARRANGEMENTS OF MEASURABLE SETS AND FUNCTIONS

INDEX

1.1 INTRODUCTION

1.1.1 *Some Hystorical Remarks*

The idea of rearrange "wild", irregular sets into nicer, more symmetric ones dates back to J. Steiner (cfr. [59]), who used it to give a beautiful (though incomplete) proof of the *plane isoperimetric inequality*:

$$L^2 \geqslant 4\pi A$$

(where L and A are, respectively, the perimeter and the area of a plane figure): in fact Steiner invented a method, nowcalled

Steiner symmetrization, aimed at converting a plane figure into another having same area, lower perimeter and an *extra* symmetry, i.e. an axial symmetry with respect to a chosed straighline.
Some years later, H. Schwarz found a way to extended the method of Steiner to functions: Schwarz's aim was to transform both a function and its domain into a new function exhibiting some extra symmetry defined in a new domain more symmetric than the original one, in such a way that neither the measure of the domain nor some norm of the function would be changed. In particular, he invented the method known as *Schwarz symmetrization*, which enables to transform the domain into a ball with equal measure and the function into a radially symmetric decreasing function having the same L^p-norm.
The symmetrization method of Schwarz was lately popularized by Hardy, Littlewood and Polya [42] in the mid-thities and by Polya and Szegö [55] in the fifties.
In particular, Polya and Szegö showed that Schwarz symmetrization was a powerful tool to gain sharp bounds for the values of some important physical quantities, e.g., the *fundamental tone* of a membrane, the *capacity* of a condenser or the *torsional rigidity* of a rod. For example, they gave an alternative proof of a theorem of Faber [36] and Krahn [46] which answered in the positive a conjecture in Acoustic formulated by Lord Rayleigh [56], namely that the fundamental tone of a circular membrane is the lowest possible among all membranes having fixed area; and they proved a conjecture of Poincaré in Electrostatic, namely that the spherical condenser is the one having least capacity among all condenser having prescribed volume.
On the other hand, Steiner symmetrization was used by de Giorgi [28] to prove the *isoperimetric property of the ball*, i.e. that in space of arbitrary (finite) dimension the ball, and the ball alone, has the lowest perimeter among all the set sharing the same measure.

1.1.2 *Organization*

The present chapter gives an overview of the most basic symmetrization techniques, namely Steiner and Schwarz symmetrization of measurable sets and onedimensional and Schwarz rearrangements of measurable functions functions.
In the fisrt two sections we give some definitions and illustrate the most basic properties of rearrangements.
In the third section, we state some well-known rearrangement inequalities, as the Perimeter Inequality or Hardy–Littlewood or Polya–Szegö Inequalities, all of which will be used in the following chapters.
In the latter section, we state and give short proofs of three basic theorems in the theory of elliptic PDEs which can be obtained using symmetrization techniques, namely the Faber–Krahn and Talenti Inequalities and Chiti Comparison Lemma, which we will be referring to in chapters 3 and 4.

1.2 REARRANGEMENTS OF MEASURABLE SETS

For notations and proofs we refer to [23] and [37] and to the references therein.

Let $E \subseteq \mathbb{R}^N$ be a measurable subset with respect to the Lebesgue measure $|\cdot|$.

DEFINITION 1.1: The *Schwarz symmetral of* E is the unique open ball $E^\star$ centered in 0 having the same measure of E.

Now let $u \in S^{N-1}$ be any direction and Π any hyperplane othogonal to u.

DEFINITION 1.2: The *Steiner symmetral of* E *with respect to* Π is the unique open set E^s having the following property: for any straightline r orthogonal to Π, the (possibly degenerate) segment $r \cap E^s$ is symmetric about Π and has length equal to the 1-dimensional measure of the segment $r \cap E$.

More precisely, let us label the axis in such a way that $u = (\underbrace{0, \dots, 0}_{N-1}, 1)$ and Π is the hyperplane of equation $x_N = 0$, let (x, y) denote a point in $\mathbb{R}^N = \mathbb{R}^{N-1}_x \times \mathbb{R}_y$ and set:

$$\begin{aligned} E_x &:= \{y \in \mathbb{R} : \ (x, y) \in E\} \\ \ell(x) &:= m(E_x) \\ \pi(E)^+ &:= \{x \in \mathbb{R}^{N-1} : \ \ell(x) > 0\}, \end{aligned}$$

where $m(\cdot)$ is the 1-dimensional Lebesgue measure; then the set E^s is defined as:

$$E^s := \{(x, y) \in \mathbb{R}^N : \ x \in \pi(E)^+ \text{ and } |y| < \ell(x)\}.$$

Remark 1.1:
Using Fubini theorem it is not difficult to prove that also E^s satisfies $|E^s| = |E|$. ♢

1.3 REARRANGEMENTS OF MEASURABLE FUNCTIONS

For notations and proofs we refer to [8, 43, 45, 48, 49, 53] and to the references therein.

1.3.1 *Distribution Function; One-dimensional Rearrangements*

Let $\Omega \subseteq \mathbb{R}^N$ be a measurable set with $|\Omega| < \infty$ and $f : \Omega \to [0, \infty]$ be a measurable function.
For each fixed $t > 0$, the *level set* $\{f > t\} := \{x \in \Omega : \ f(x) > t\}$ is measurable, thus it is possible to set:

$$\mu_f(t) := |\{f > t\}|.$$

The function $\mu_f : [0,\infty[\to [0,\infty[$ is called *distribution function of* f. Such a function is decreasing, right-continuous and satisfies:

$$\lim_{t\to 0} \mu_f(t) = |\operatorname{supp} f| = \mu_f(0)\ ,$$
$$\operatorname{supp} \mu_f = [0, \operatorname{esssup}_\Omega f]\ ;$$

moreover, $\mu_f(t)$ is continuous at t if and only if $|\{f = t\}| = 0$, i.e. if the graph of f has no nonnegligible flat parts at level t, and:

$$\lim_{t\to\infty} \mu_f(t) = 0\ .$$

DEFINITION 1.3 (One-dimensional Rerrangements): Let Ω, f and μ_f be as above.
The function $f^* : [0,\infty[\to [0,\infty]$ defined by:

$$\begin{aligned} f^*(s) &:= \inf\{t \geqslant 0 :\ \mu_f(t) \leqslant s\} \\ &= \sup\{t \geqslant 0 :\ \mu_f(t) > s\} \end{aligned} \tag{1.1}$$

is called *decreasing rearrangement of* f, while the function $f_* :]0,\infty[\to [0,\infty]$ defined by:

$$f_*(s) := f^*(|\Omega| - s)$$

is called *increasing rearrangement of* f.

REMARK 1.2:
The function f^* *is the socalled* generalized inverse *of* μ_f. *In fact, if* μ_f *is strictly monotone then for all* $t_0 \in \operatorname{supp} \mu_f, s_0 \in \operatorname{supp} f^*$ *we have* $f^*(\mu_f(t_0)) = t_0$ *and* $\mu_f(f^*(s_0)) = 0$.
On the other hand, f^* *fails to be a proper inverse of* μ_f *when the latter function is discontinuous: assume that the graph of* μ_f *has a discontinuity jump in* t_0, *then* f^* *is constant in the nondegenerate interval* $I_0 = [\mu_f(t_0), \mu_f(t_0^-)]$, *and for* $s \in I_0^\circ$ *we only get* $\mu_f(f^*(s)) = \mu_f(t_0) < s$. $\diamondsuit$

It is possible to prove that: f^* is decreasing and right-continuous; $f^*(0) = \operatorname{esssup}_\Omega f$; $\mu_{f^*} = \mu_f$, thus f and f^* are *equidistributed*.
Using Fubini theorem we can easily prove that decreasing and increasing rearrangement preserves the L^p-norm for any $p \in$

$[1,\infty]$, that is f^* is an $L^p(0,|\Omega|)$ function if $f \in L^p(\Omega)$ and that $\|f^*\|_{p,[0,|\Omega|[} = \|f\|_{p,\Omega}$. Actually, more is true [48, Thm. 6.15]:

THEOREM 1.1
Let $\Omega \subseteq \mathbb{R}^N$ *be measurable,* $f : \Omega \to [0,\infty[$ *be a* weakly vanishing at infinity[1] *function and* $\Phi : [0,\infty[\to [0,\infty[$ *be a Borel function. Then:*

$$\int_0^{|\Omega|} \Phi(f^*(t))\,dt \leqslant \int_\Omega \Phi(f(x))\,dx\,. \tag{1.2}$$

Equality holds in (1.2) *if* $\Phi(0) = 0$*, or* $m(\{f > 0\}) < \infty$ *or both* $|\{f > 0\}| = \infty$ *and* $|\{f = 0\}| = 0$*.*

Moreover, the pointwise equality:

$$(\psi(f))^* = \psi(f^*) \tag{1.3}$$

holds a.e. in $[0,|\Omega|]$ for any nondecreasing function $\psi : \mathbb{R} \to \mathbb{R}$.

1.3.2 *Schwarz Rearrangements*

From now on, we let $\omega_N = \pi^{N/2}/\Gamma(1 + N/2)$, thus ω_N is the volume of the unit ball of $\mathbb{R}^N$.

DEFINITION 1.4 (Schwarz Rearrangements): Let Ω and f be as above.
The function $f^\star : \Omega^\star \to [0,\infty]$ defined by setting:

$$f^\star(x) := f^*(\omega_N\,|x|^N) \tag{1.4}$$

1 A measurable function f is said to vanish weakly at infinity iff each level set $\{f > t\}$ has finite measure; the latter condition ensures that μ_f is finite for every $t > 0$, hence f^* can be defined as in the Definition above.

is called *Schwarz decreasing rearrangement* (or *radially symmetric and decreasing rearrangement*) *of* f, while the function $f_\star : \Omega^\star \to [0, \infty]$ defined by:

$$f_\star(x) := f_*(\omega_N\, |x|^N)$$

is called *Schwarz increasing rearrangement* (or *radially symmetric and increasing rearrangement*) *of* f.

Functions f, $f^\star$ and $f_\star$ are equimeasurable and therefore, by Theorem 1.2, $f^\star, f_\star \in L^p(\Omega^\star)$ if and only if $f \in L^p(\Omega)$ (for $p \in [1, \infty]$); moreover, f and both its Schawarz rearrangements share the same value of the L^p-norm, i.e.:

$$\|f^\star\|_{p,\Omega} = \|f^\star\|_{p,\Omega^\star} = \|f_\star\|_{p,\Omega^\star}\ .$$

1.3.3 *Signed Rearrangements*

In the previous section we defined both onedimensional and Schwarz rearrangements only for nonnegative measurable functions.
A possible way to provide suitable generalizations of Definitions 1.3 & 1.4 to a sign-changing measurable function f consists in replacing the distribution of f in §1.3.1 with the distribution of the absolute value of f, i.e. $\mu_{|f|}$; then one can set by definition $f^* := |f|^*$ and $f^\star = |f|^\star$ to define the descreasing and the Schwarz rearrangement of f.
If we keeps this way, all the informations concerning the sign of the original function f would be destroyed. Hence, in many situations, it is useful to consider a *signed rearrangement* of a measurable function.

Definition 1.5: Let $\Omega \subseteq \mathbb{R}^N$ be a bounded measurable set and $f : \Omega \to [-\infty, \infty]$ be a measurable function.
Then the function $f^\circ : [0, \infty[\to [-\infty, \infty]$ defined by setting:

$$f^\circ(s) := \inf\{t \in \mathbb{R} :\ |\{f > t\}| \leqslant s\} \tag{1.5}$$

is called *signed decreasing rearrangement of* f; analogously, the function:

$$f_\circ(s) := f^\circ(|\Omega| - s)$$

is called *signed increasing rearrangement of* f.

It is easy to see that the following representations:

$$f^\circ(s) = (f_+)^*(s) - (f_-)^*(|\Omega| - s)$$
$$f_\circ(s) = (f_+)^*(|\Omega| - s) - (f_-)^*(s) \ ,$$

where f_+ and f_- are the positive and the negative part of f, holds a.e. in $[0, |\Omega|]$.
Using the signed decreasing rearrangement we can also build the socalled *signed Schwarz decreasing* and *increasing rearrangements* simply by setting:

$$f^\bullet(x) := f^\circ(\omega_N \ |x|^N)$$
$$f_\bullet(x) := f_\circ(\omega_N \ |x|^N) \ .$$

1.4 REARRANGEMENT INEQUALITIES

1.4.1 *Isoperimetric and Perimeter Inequalities*

One of the main geometric features of Steiner symmetrization is the following:

THEOREM 1.2 (Perimeter Inequality)
Let $E \subseteq \mathbb{R}^N$ *be a set of* finite perimeter.
Then the Steiner symmetral E^s *(with respect to any hyperplane) has finite perimeter and* $\mathrm{Per}(E^s) \leqslant \mathrm{Per}(E)$.

Here the perimeter of a set is defined in the sense of Caccioppoli as the total variation of its characteristic function, that is:

$$\mathrm{Per}(E) := \sup_{\Phi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N)} \frac{1}{\|\Phi\|_\infty} \int_E \mathrm{div}\Phi \ \ dx \ .$$

As remarked in the introduction, Theorem 1.2 was used by de Giorgi to prove the following:

THEOREM 1.3 (Classical isoperimetric inequality)
Let $N \geqslant 2$ *and* $E \subseteq \mathbb{R}^N$ *be a bounded measurable set.*
Then:

$$\mathrm{Per}^N(E) \geqslant N^N \omega_N \, |E|^{N-1} \; . \tag{1.6}$$

Moreover, E *satisfies equality in* (1.6) *if and only if* E *is a ball (up to a negligible set).*

REMARK 1.3:
Inequality (1.6) *can be stated in a slightly different form: in fact, since the dimensional constant* $N^N \omega_N$ *can be rewritten as* $\mathrm{Per}(E^\star)/|E^\star|^{N-1}$*, we get:*

$$\mathrm{Per}(E) \geqslant \mathrm{Per}(E^\star) \; . \tag{1.7}$$

The equality condition then implies that the perimeter of E *equals that of* $E^\star$ *if and only if* $E = E^\star$ *(up to a null set) modulo translations.*
As $|E^s| = |E|$*, we also have* $(E^s)^\star = E^\star$ *and therefore if* E *has finite perimeter then* $\mathrm{Per}(E^\star) \leqslant \mathrm{Per}(E^s) \leqslant \mathrm{Per}(E)$. ♢

We also remark that Theorem 1.3 is false in the case $N = 1$, for $E = [0, \infty[$ has finite perimeter but infinite measure.

Equality condition in the classical isoperimetric inequality gives sharp informations on the shape of the set E. Hence we may wonder if it is possilble to recover analogous informations on E when the set satisfies equality in Theorem 1.2, i.e. when $\mathrm{Per}(E^s) \leqslant \mathrm{Per}(E)$.
It turns out that $\mathrm{Per}(E^s) = \mathrm{Per}(E)$ implies $E = E^s$ (modulo translations) provided (i) the boundary of E^s does not contain "large" parts which are flat in the direction orthogonal to the symmetrization hyperplane and (ii) E is connected in a "proper way".
To be more precise, the following holds (for notations and proof we refer to [23]):

THEOREM 1.4
Let E *be a set of finite perimeter in* $\mathbb{R}^N$, $N \geqslant 2$, *satisfying* $\mathrm{Per}(E^s) = \mathrm{Per}(E)$.
Assume that:

$$\mathcal{H}^{N-1}\Big(\{(x,y) \in \partial^* E^s : \nu_y^{E^s}(x,y) = 0\} \times (\Omega \times \mathbb{R})\Big) = 0$$

$$\tilde{\ell}(x) > 0 \qquad \textit{for } \mathcal{H}^{N-2}\textit{-a.e. } x \in \Omega$$

(where: ∂^* *is the essential boundary,* ν_y *is the outer normal in the* y *direction and* $\tilde{\ell}$ *il the Lebesgue representative of* ℓ*) are fulfilled for some connected open subset* $\Omega \subseteq \mathbb{R}^{N-1}$ *such that* $\pi(E)^+ \setminus \Omega$ *is negligible with respect to the* $(N-1)$*-dimensional Lebesgue measure.*
Then E *is equivalent to* E^s *(modulo translations along the* y*-axis).*

1.4.2 *Hardy–Littlewood Inequality*

The following inequalities are classical and go back to [42]:

THEOREM 1.5 (Hardy–Littlewood Inequality)
Let $\Omega \subseteq \mathbb{R}^N$ *be a bounded measurable set and* $f, g : \Omega \to [0, \infty]$ *be measurable functions.*
If $f \in L^p(\Omega)$ *and* $g \in L^{p'}(\Omega)$, *with* $1 \leqslant p \leqslant \infty$, *then:*

$$\begin{aligned}\int_{\Omega^\star} f^\star(x)\, g_\star(x)\, \mathrm{d}x &= \int_0^{|\Omega|} f^*(s)\, g_*(s)\, \mathrm{d}s \\ &\leqslant \int_\Omega f(x)\, g(x)\, \mathrm{d}x \\ &\leqslant \int_0^{|\Omega|} f^*(s)\, g^*(s)\, \mathrm{d}s = \int_{\Omega^\star} f^\star(x)\, g^\star(x)\, \mathrm{d}x\,. \end{aligned} \tag{1.8}$$

REMARK 1.4:
The assumptions on the summability of f *and* g, *which implies that all integrals in* (1.8) *are finite, may also be suppressed. See [49].* ♢

REMARK 1.5:
We explicitly remark that Hardy–Littlewood inequalities also hold when functions f *and* g *are allowed to change sign in* Ω, *with the only dif-*

ference that signed rearrangements are to be used. To be more precise, we have:

$$\begin{aligned}\int_{\Omega^\star} f^\bullet(x)\, g_\bullet(x)\, dx &= \int_0^{|\Omega|} f^\circ(s)\, g_\circ(s)\, ds \\ &\leqslant \int_\Omega f(x)\, g(x)\, dx \\ &\leqslant \int_0^{|\Omega|} f^\circ(s)\, g^\circ(s)\, ds = \int_{\Omega^\star} f^\bullet(x)\, g^\bullet(x)\, dx\,. \end{aligned} \tag{1.9}$$

The proof, which can be found in [45, Theorem 1.2.2], relies on a suitable layer–cake representation formula *for* f *and* g*; nevertheless, when only one functions changes sign, while the other remains non-negative, a simpler argument can be used.*
In fact, let g *be the sign-changing function and let* $f \geqslant 0$ *in* Ω*; then:*

$$g^\circ(s) = (g_+)^*(s) - (g_-)^*(|\Omega| - s)$$

and:

$$\begin{aligned}\int_\Omega f(x)\, g_+(x)\, dx &\leqslant \int_0^{|\Omega|} f^*(s)\, (g_+)^*(s)\, ds \\ \int_\Omega f(x)\, g_-(x)\, dx &\geqslant \int_0^{|\Omega|} f^*(s)\, (g_-)_*(s)\, ds \\ &= \int_0^{|\Omega|} f^*(s)\, (g_-)^*(|\Omega| - s)\, ds \end{aligned}$$

hence:

$$\begin{aligned}\int_\Omega f(x)\, g(x)\, dx &= \int_\Omega f(x)\, g_+(x)\, dx - \int_\Omega f(x)\, g_-(x)\, dx \\ &\leqslant \int_0^{|\Omega|} f^*(s)\, (g_+)^*(s)\, ds \\ &\quad - \int_0^{|\Omega|} f^*(s)\, (g_-)^*(|\Omega| - s)\, ds \\ &= \int_0^{|\Omega|} f^*(s)\, g^\circ(s)\, ds \\ &= \int_{\Omega^\star} f^\star(x)\, g^\bullet(x)\, dx\,; \end{aligned}$$

in the same way one can prove the reverse inequality with the increasing rearrangement $g_\bullet$ *replacing the decreasing one.* $\diamondsuit$

For our later purposes, we will need a characterization of the equality cases in (1.8) and (1.9).
The equality problem for (1.8) was investigated, among others, in [5, 24]: in those papers it was shown that nonnegative functions which attains equality in the rightmost Hardy–Littlewood inequality, i.e. which satisfy:

$$\int_\Omega f(x)\ g(x)\ \ dx = \int_0^{|\Omega|} f^*(s)\ g^*(s)\ \ ds\,, \tag{1.10}$$

have mutually nested level sets. In other words, the following holds:

THEOREM 1.6
Let Ω, f *and* g *be as in Theorem 1.5.*
If $f, g \geqslant 0$ *a.e. in* Ω *and if equality* (1.10) *holds, then for every* $t, \tau > 0$*:*

$$\textit{either}\ \{f > t\} \subseteq \{g > \tau\} \qquad \textit{or} \qquad \{g > \tau\} \subseteq \{f > t\}$$

up to a negligible set.

On the other hand, functions attaining equality (1.10) need not to be fully characterized by Theorem 1.6. In fact, if we fix a function g whose graph has a flat part at some level τ, then we may find (infinitely) many equidistributed functions f yielding equality (1.10).
Therefore, as far as uniqueness of functions yielding equality (1.10) for fixed g is concerned, we have to make some suitable "non-flatness" assumption on g. It turns out that the strict monotonicity of g^* can get the job done: in fact, a more general and stronger result than Theorem 1.6 was recently obtained in [25] (after [57]). We restate it here in lesser generality:

THEOREM 1.7
Let Ω, f *and* g *be as above.*

Assume g^* *is strictly decreasing in* $[0, |\Omega|]$.
Then equality (1.10) *holds if and only if:*

$$f(x) = f^*(\mu_g(g(x))) \qquad \textit{a.e. in } \Omega \, .$$

1.4.3 *Polya–Szegö Inequality*

We have seen that a measurable function f belongs to $L^p(\Omega)$ if and only if its Schwarz decreasing rearrangement $f^\star$ belongs to $L^p(\Omega^\star)$ and that those functions share the same L^p-norm.
Actually, more is true: in fact, if f is sufficiently "smooth" in its domain, then also $f^\star$ is "smooth" in $\Omega^\star$: this is the socalled *Polya–Szegö Principle*, which is based onto the following theorem

THEOREM 1.8
Let $\Omega \subseteq \mathbb{R}^N$ *be a bounded domain,* $p \in [1, \infty]$ *and* $f \in W_0^{1,p}(\Omega)$.
Then $f^\star \in W_0^{1,p}(\Omega)$ *and:*

$$\|\nabla f^\star\|_{p,\Omega} \leqslant \|\nabla f\|_{p,\Omega} \, . \tag{1.11}$$

Moreover, the same conclusions holds if Ω *is replaced by* $\mathbb{R}^N$ *and if* $f \in W^{1,p}(\Omega)$.

Therefore the L^p-norm of the gradient decreases under Schwarz symmetrization. In general, it turns out that many other types of functionals depending on the gradient descrease under Schwarz symmetrization and that they decrease *strictly* unless the setting is not already radial: almost classical results in this direction are the following, due to Brothers and Ziemer [17].

THEOREM 1.9
Let $p \in [1, \infty[$, Ω *a bounded domain,* $f \in W_0^{1,p}(\Omega)$ *be a nonnegative function and* $A : [0, \infty[\to [0, \infty[$ *be a* C^2 *function with* $A^{1/p}$ *convex and* $A(0) = 0$.
Then:

$$\int_{\Omega^\star} A(\nabla f^\star(x)) \, dx \leqslant \int_\Omega A(\nabla f(x)) \, dx \, . \tag{1.12}$$

The same conclusion holds also in the case $\Omega = \mathbb{R}^N$ *and* $f \in W^{1,p}(\mathbb{R}^N)$.

THEOREM 1.10
Let p, Ω, f *and* A *be as above.*
If $p > 1$ *and if the distribution* $\mu_f = \mu_{f^\star}$ *is* absolutely continuous, *i.e. if:*

$$|\{x \in \Omega : \ 0 < f^\star(x) < \operatorname{esssup} f \textit{ and } |\nabla f^\star(x) = 0|\}| = 0 \quad (1.13)$$

then equality holds in (1.12) *if and only if* $\Omega = \Omega^\star$ *and* $f = f^\star$ *(modulo translations).*
The same conclusions hold also if $\Omega = \mathbb{R}^N$ *and* $f \in W^{1,p}(\mathbb{R}^N)$.

If condition (1.13) does not hold, in general the claim of the latter Theorem does not hold: for a simple counterexample see [44].

1.5 REARRANGEMENTS AND ELLIPTIC EQUATIONS

Symmetrization is a useful tool for proving theorems which compare solutions or other quantities associated to different boundary value problems for elliptic (or even parabolic) differential equations. Typically, we may want to make pointwise comparison between solutions, or to get estimates on some of their norms, or even to compare other quantities associated to a given problem and the corresponding ones associated to an auxiliary *symmetrized* problem.
In the first cases, the basic idea is to get a differential inequality for the distribution function of the solution, which will reduce to an equality when the solution of the symmetrized problem is considered, and then to deduce from that the comparison result or the estimate.
In the latter case, one of the basic techniques consists in proving that the considered quantity has a variational nature, then using rearangement inequalities to prove the comparison result. When this approach does not work, several other alternatives are available; but the core of this kind of techniques remains the use of rearrangment inequalities.

1.5.1 *Faber–Krahn Inequality*

In this section a generalization of the result of Faber and Krahn is provided as an important example of comparison result. Moreover, this result will be needed later, in chapter 3.

Let us consider the differential operator:

$$\mathcal{L} := -\operatorname{div}(A(x)\ \nabla) + c(x)$$

acting in the weak sense onto functions $u \in W_0^{1,2}(\Omega)$, where $\Omega \subseteq \mathbb{R}^N$ is an open bounded domain and A, c satisfy the following assumptions:

(H1) $A := (a_{ij}) \in L^\infty(\Omega; \mathbb{R}^{N\times N})$ is a symmetric uniformly elliptic matrix such that $\sum_{i,j} a_{i,j}(x)\ \xi_i \xi_j \geqslant |\xi|^2$ for all $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$ and a.e. $x \in \Omega$,

(H2) $c \in L^\infty(\Omega)$ is a.e. nonnegative.

The *eigenvalue problem* for $\mathcal{L}$ requires to seek all the possible couples $(\lambda, u) \in \mathbb{R} \times W_0^{1,2}(\Omega)$ whose second coordinate solves the boundary value problem:

$$\begin{cases} -\operatorname{div}(A(x)\ \nabla u) + c(x)\ u = \lambda\ u & , \text{in } \Omega \\ u = 0 & , \text{on } \partial\Omega\ , \end{cases} \tag{1.14}$$

in the weak sense.
If there exists any couple $(\tilde\lambda, \tilde u)$ of the aforementioned type, then the value $\tilde\lambda$ is called *eigenvalue of* $\mathcal{L}$ (or an *eigenvalue of problem* (1.14)) and $\tilde u$ is called *eigenfunction of* $\mathcal{L}$ (or *eigenfunction of problem* (1.14)) *associated to* $\tilde\lambda$.
Using standard Functional Analytic tools, one can see that there exists a nondecreasing, positive-diverging sequence $(\lambda_n^{A,c}(\Omega))$ of eigenvalues of $\mathcal{L}$. Moreover, any eigenvalue $\lambda_n^{A,c}(\Omega)$ admits

a variational characterization in terms of the socalled *Rayleigh quotient*:

$$\mathcal{R}[u] := \frac{\int_\Omega \langle A(x)\nabla u, \nabla u\rangle \, dx + \int_\Omega c(x)\, u^2 \, dx}{\int_\Omega u^2 \, dx} \ ;$$

in particular:

$$\lambda_n^{A,c}(\Omega) = \min\left\{\mathcal{R}[u], \ u \in W_0^{1,2}(\Omega), u \neq 0, \ u \perp V_{n-1}\right\} \quad (1.15)$$

where $V_0 = \{0\}$ and:

$$V_n = \text{span}\left\{\text{eigenfunctions associated to } \lambda_1^{A,c}, \dots, \lambda_{n-1}^{A,c}\right\}$$

for $n \geqslant 1$.
It then follows that problem (1.14) has a smallest eigenvalue, namely $\lambda_1^{A,c}(\Omega)$.
This eigenvalue always possesses some interesting features: besides being *variational* because of (1.15), it is also *isolated*, *simple* (i.e., the eigenspace associated to $\lambda_1^{A,c}$ is one-dimensional) and *principal* (i.e., the nontrivial eigenfunctions associated to it do not change their sign in Ω). In particular, it turns out that principality characterizes the smallest eigenvalue of (1.14), in the sense that if $\lambda_n^{A,c}$ is a principal eigenvalue of $\mathcal{L}$, then $\lambda_n^{A,c}(\Omega) = \lambda_1^{A,c}(\Omega)$.

The variational characterization of $\lambda_1^{A,c}(\Omega)$ allows to use symmetrization techniques to prove the following generalization of the aforementioned theorem of Faber and Krahn:

THEOREM 1.11 (generalized Faber–Krahn inequality)
Let $\lambda_1^{A,c}(\Omega)$ be the smallest eigenvalue of problem (1.14) *and let $\lambda_1^{I,0}(\Omega^\star)$ be the smallest eigenvalue of the problem:*

$$\begin{cases} -\Delta u = \lambda\, u & , \textit{in } \Omega^\star \\ u = 0 & , \textit{on } \partial\Omega^\star \ . \end{cases} \quad (1.16)$$

Then $\lambda_1^{I,0}(\Omega^\star) \leqslant \lambda_1^{A,c}(\Omega)$.
Moreover, equality is attained if and only if $\Omega = \Omega^\star$ *(modulo translations),* $c = 0$ *a.e. in* Ω *and the matrix* A *satisfies the condition:*

$$\sum_{j=1}^{N} a_{i,j}(x)\, x_j = x_i \,.$$

Proof. Let u_1 be a nonnegative nontrivial eigenfunction associated to $\lambda_1^{A,c}(\Omega)$. Using the sign assumption on c, the ellipticity condition on A, the invariance of the L^2-norm under Schwarz rearrangement and the Polya–Szegö Principle, we get:

$$\begin{aligned}
\lambda_1^{A,c}(\Omega) &= \mathcal{R}[u_1] \\
&\geqslant \frac{\int_\Omega \langle A(x)\nabla u_1, \nabla u_1\rangle \, dx}{\int_\Omega u_1^2 \, dx} \\
&\geqslant \frac{\int_\Omega |\nabla u_1|^2 \, dx}{\int_\Omega u_1^2 \, dx} \\
&\geqslant \frac{\int_{\Omega^\star} |\nabla u_1^\star|^2 \, dx}{\int_{\Omega^\star} (u_1^\star)^2 \, dx} \\
&\geqslant \lambda_1^{I,0}(\Omega^\star)
\end{aligned} \tag{1.17}$$

which is the desidered inequality.
If equality $\lambda_1^{I,0}(\Omega^\star) = \lambda_1^{A,c}(\Omega)$ holds, then it does through (1.17). In particular, we have equality in the Polya–Szegö Principle, thus the theorem of Brothers–Ziemer applies (because of the strict monotony of u_1^*) and it gives $\Omega = \Omega^\star$, $u_1 = u_1^\star$ (modulo translations). Moreover, equality holds between the second and the third member of (1.17), hence $\int_\Omega c(x)u_1(x) \, dx = 0$; from this we infer $c(x) = 0$ a.e. in Ω, because $u_1 > 0$ inside Ω by *Harnack Inequality* (cfr. [40]). Finally, equality holds also between the third and the fourth member of (1.17), implying $\sum_{j=1}^N a_{i,j}(x)\, x_j = x_i$ as in [45, §3.2]. $\square$

Of course, there are may variants and refinements of the result cited above. For the linear case, for example, there is the one in [61, §5]; while for the nonlinear case, e.g. operators modelled onto the p-Laplacian, see [3, 2].

1.5.2 *Talenti Inequality and Chiti Comparison Lemma*

In this section we give two exemples of pointwise comparison results for solutions which can be proved by symmetrization: the first one due to Talenti [61] and the second due to Chiti [21].
In particular, Talenti's result gives a pointwise comparison and a sharp estimate of the norms for the solution of a Poisson equation with homogeneous Dirichlet BCs. On the other hand, Chiti's one gives a pointwise comparison between the decreasing rearrangement of the first eigenfunction of a second order linear operator with homogeneous Dirichlet BCs and the radial solution of a suitable symmetric problem.
Both results are *isoperimetric*, in the sense that equality is attained only in the symmetric setting, i.e. when the base domain is a ball and the second member of the PDE (if needed) is radial and decreasing (cfr. [5, 45]).
These results will be needed later in chapter 3.

Let us consider the problem of finding $u \in W_0^{1,p}(\Omega)$ which solves in the weak sense the problem:

$$\begin{cases} -\operatorname{div}(A(x)\ \nabla u) + c(x)\ u = f(x) & \text{, in } \Omega \\ u = 0 & \text{, on } \partial\Omega\ , \end{cases} \tag{1.18}$$

where: $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, $A = (a_{i,j}) \in L^\infty(\Omega; \mathbb{R}^{N\times N})$ is a uniformly elliptic matrix in Ω, i.e.:

$$\forall x \in \Omega,\ \sum_{i,j=1}^{N} a_{i,j}(x)\ \xi_i \xi_j \geqslant |\xi|^2 \quad \text{for all } \xi = (\xi_n) \in \mathbb{R}^N\ ,$$

$c(x) \geqslant 0$ a.e. in Ω, and $f \in L^r(\Omega)$ with $r = 2N/(N+2)$ if $N > 2$ or $r > 1$ if $N = 2$. Such a problem has unique solution and the following comparison result holds:

THEOREM 1.12 (Talenti)
Let u *be the solution of problem* (1.18) *and* v *be the solution of the symmetrized problem:*

$$\begin{cases} -\Delta v(x) = |f|^\star(x) & \text{, in } \Omega^\star \\ v = 0 & \text{, on } \partial\Omega^\star \ . \end{cases}$$

Then $v(x) \geqslant |u|^\star(x)$ *in* Ω *and therefore:*

$$\|u\|_{p,\Omega} \leqslant \|v\|_{p,\Omega^\star}$$

for each $p \in]0, \infty]$.

In particular, if problem (1.18) admits a nonnegative solution, then $|u|^\star$ can be replaced with $u^\star$.
As mentioned above, the proof of Theorem 1.12 is based on a differential inequality for the distribution function of $|u|$, namely the following:

$$\mu_{|u|}^{2-2/N}(t) \leqslant \frac{1}{N^2 \omega_N^{2/N}} \left(-\mu'_{|u|}(t)\right) \int_0^{\mu_{|u|}(t)} |f|^*(s) \ ds \ . \quad (1.19)$$

which holds for a.e. $t > 0$.
An interesting feature of the proof of inequality (1.19) is that it can be rewritten almost verbatim when the second member f is replaced with $\lambda_1^{A,c} u$, i.e., when we consider the eigenvalue problem (1.14) associated to the smallest eigenvalue $\lambda_1^{A,c}(\Omega)$.
In such a case, inequality (1.19) becomes:

$$\mu_{|u|}^{2-2/N}(t) \leqslant \frac{\lambda_1^{A,c}}{N^2 \omega_N^{2/N}} \left(-\mu'_{|u|}(t)\right) \int_0^{\mu_{|u|}(t)} |u|^*(s) \ ds \ ;$$

thus, if one considers the L^∞-normalized positive first eigenfunction u_1 and applies Fubini's theorem to evaluate the integral in the right-hand side, the inequality rewrites:

$$\frac{c_N^2}{\lambda_1^{A,c}} \ \mu_{u_1}^{2-2/N}(t) \leqslant \left(-\mu'_{u_1}(t)\right) \left(t \ \mu_{u_1}(t) + \int_t^1 \mu^*_{u_1}(\tau) \ d\tau \right) \ , \quad (1.20)$$

(where $c_N := N\omega_N^{1/N}$) which is the socalled *Talenti inequality*.

REMARK 1.6:
A cheap way to obtain (1.20) *in the case when* (1.14) *reduces to the eigenvalue problem for the Dirichlet Laplacian in* Ω *(i.e., when* $A(x) = I$ *and* $c(x) = 0$ *everywhere in* Ω*) is the following.*
Classical regularity results imply that the first eigenfunction u_1 *is of class* $C^\infty(\Omega)$*, hence the set* $\{x \in \Omega : |\nabla u(x)| = 0\}$ *is negligible by Sard's theorem and the distribution function of* u_1*, say* μ_1*, is continuous in* $[0, 1]$.
Cauchy-Schwarz inequality yields:

$$\left[\mathcal{H}^{N-1}(\partial\{u_1 > t\})\right]^2 \leqslant \int_{\partial\{u_1>t\}} |\nabla u_1| \, d\,\mathcal{H}^{N-1} \times \\ \times \int_{\partial\{u_1>t\}} |\nabla u_1|^{-1} \, d\,\mathcal{H}^{N-1} \quad \textit{for a.e. } t \in [0, 1]. \tag{1.21}$$

Applying divergence theorem *to equation* $-\Delta u = \lambda_1^{I,0}(\Omega) u$ *over the level set* $\{u_1 > t\}$ *we find:*

$$\begin{aligned} \int_{\partial\{u_1>t\}} |\nabla u_1| \, d\,\mathcal{H}^{N-1} &= \int_{\partial\{u_1>t\}} \langle \nabla u_1, |\nabla u_1|^{-1} \, \nabla u_1 \rangle \, d\,\mathcal{H}^{N-1} \\ &= -\int_{\{u_1>t\}} \Delta u_1 \, d\,x \\ &= \lambda_1^{I,0}(\Omega) \int_{\{u_1>t\}} u_1 \, d\,x\,, \end{aligned}$$

so, recalling relation:

$$\mu_1'(t) = -\int_{\partial\{u_1>t\}} |\nabla u_1|^{-1} \, d\,\mathcal{H}^{N-1} \quad \textit{for a.e. } t \in [0, 1]$$

(see [45, Theorem 2.2.3]) and the isoperimetric inequality*:*

$$\mathcal{H}^{N-1}(\partial\{u_1 > t\}) \geqslant c_N \, |\{u_1 > t\}|_N^{1-1/N} \,,$$

from (1.21) *we infer:*

$$\frac{c_N^2}{\lambda_1^{I,0}(\Omega)}\, \mu_1^{2-2/N}(t) \leqslant -\mu_1'(t) \int_{\{u_1>t\}} u_1 \, dx\,. \tag{1.22}$$

Finally, Fubini's theorem *yields:*

$$\int_{\{u_1>t\}} u_1 \, dx = t\, \mu_1(t) + \int_t^1 \mu_1(\tau)\, d\tau \quad \textit{for a.e. } t \in [0,1], \tag{1.23}$$

ergo plugging the righthand side of (1.23) *in* (1.22) *we find* (1.20). ◇

In the same spirit of Talenti's comparison theorem, there are several other results which allow pointwise comparison between the rearrangement of the first eigenfunction of problem (1.14) and the first eigenfunction of a suitable symmetrized problem. For example, there are the almost classical results of Chiti [21, 22]: in particular, they yield that the Schwarz decreasing rearrangement of the first nonnegative eigenfunction u_1 of (1.14) can be pointwise compared with the first nonnegative eigenfunction V_1 of the problem:

$$\begin{cases} -\Delta V(x) = \lambda\, V(x) & \text{, in } B \\ V(x) = 0 & \text{, on } \partial B \end{cases}$$

where B is the unique open ball centered in the origin such that $\lambda_1^{I,0}(B) = \lambda_1^{A,c}(\Omega)$. For sake of precision we have:

THEOREM 1.13 (Chiti's comparison lemma)
Let u_1 be a nonnegative, nontrivial eigenfunction associated to the first eigenvalue $\lambda_1^{A,c}(\Omega)$, let B be the ball centered in the origin such that $\lambda_1^{I,0}(B) = \lambda_1^{A,c}(\Omega)$ and let $V_1 \in W_0^{1,p}(B)$ be an eigenfunction associated to $\lambda_1^{I,0}(B)$.
If $\|V_1\|_{\infty,B} = \|u_1\|_{\infty,\Omega}$, then:

$$u_1^\star(x) \geqslant V_1(x) \tag{1.24}$$

for all $x \in B$.

Using a simple scaling argument, one can prove that the ball B has measure not exceeding $|\Omega|$, for its radius equals:

$$r_B = \sqrt{\frac{\lambda_1^{I,0}(\Omega^\star)}{\lambda_1^{A,c}(\Omega)}} \left(\frac{|\Omega|}{\omega_N}\right)^{1/N} \leqslant \left(\frac{|\Omega|}{\omega_N}\right)^{1/N} = r_{\Omega^\star} \;;$$

therefore, extending V_1 to zero in $\Omega^\star \setminus B$ one can make inequality (1.24) hold in the whole of $\Omega^\star$. Consequently one gets the comparison also for the norms, i.e.:

$$\|V_1\|_{p,\Omega^\star} \leqslant \|u_1\|_{p,\Omega}$$

for $p \in [1, \infty]$.

2

A FAMILY OF SHARP ISOPERIMETRIC INEQUALITIES FOR BODIES OF REVOLUTION

INDEX

2.1 INTRODUCTION

In this chapter we prove a family of sharp isoperimetric inequalities for sufficiently smooth bounded bodies of revolution $D \subset \mathbb{R}^N$ with $N \geqslant 3$, namely:

$$\Big[\mathrm{Per}(D) - a(N-2)\,\mathrm{Sec}(D)\Big]^N \geqslant 2(N-1)N^N\omega_{N-1}\ \varphi_N(a)\,\mathrm{Vol}^{N-1}(D)\,, \tag{2.1}$$

depending on the parameter $a \in]0,1]$.

In (2.1) the symbols $\mathrm{Vol}(D)$, $\mathrm{Per}(D)$ and $\mathrm{Sec}(D)$ denote respectively the volume, the perimeter and the weighted measure with respect to a weight which depends only on the distance of the points of D from the rotation axis; and $\varphi_N(a)$ is a suitable non-negative constant.

Moreover we are able to prove that $2(N-1)N^N\omega_{N-1}\varphi_N(a)$

is the best constant for inequality (2.1) and to characterize the equality case.

2.1.1 *Motivations*

We were led to inequalities (2.1) while looking for a symmetrization method to be employed in finding the best constant in the *Hardy-Sobolev inequality*:

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx \, dy \geqslant c \left(\int_{\mathbb{R}^N} \frac{|u|^{p^*(q)}}{|x|^q} \, dx \, dy \right)^{p/p^*(q)} , \quad (2.2)$$

where: $\mathbb{R}^N = \mathbb{R}^k_x \times \mathbb{R}^{N-k}_y$; k, p, q satisfy $2 \leqslant k \leqslant N$, $1 < p < N$, $0 \leqslant q \leqslant p$, $q < k$, with $p^*(q) = p\frac{N-q}{N-p}$; $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, which is the closure of $\mathcal{D}(\mathbb{R}^N) = C^\infty_c(\mathbb{R}^N)$ with respect to the norm $\|u\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} := \|\nabla u\|_{L^p}$.
In particular, inequalities of the type (2.1) seem to play a role in the case $q = 1$.

Inequalities of the type (2.2) with $k = N$ were estabilished as particular cases in [18], where a more general class of inequalities with weights was proved as interpolation between the classical Sobolev and Hardy inequalities.
The results of Caffarelli, Kohn & Nirenberg were extended in various directions: for instance, the full case $2 \leqslant k \leqslant N$ was considered in [7], where it was proved that (2.2) holds and that the best constant is achieved when $q < p$.
The shape of the solutions of the variational problem associated to (2.2) was determined in [4] in the special case $N = 3$, $k = 2$, $p = 2$, $q = 1$ combining an inequality satisfied by the Grushin operator (proved in [10]) and the classical Polya-Szegö principle for the Steiner rearrangement of a function in $\mathcal{D}^{1,p}(\mathbb{R}^N)$.
Even if the question of the shape of the minimizer in (2.2) for general values of N, k, p, q was left open, the authors were able to give a two parameters family of positive solutions of the Euler-

Lagrange equation associated to the variational problem (which involves the p-Laplacian operator) in the case $q = 1$:

$$u(x, y; \alpha, \beta) := \alpha \left[(1 + \beta|x|)^2 + \beta^2|y|^2\right]^{-(N-p)/(2(p-1))} \quad (2.3)$$

where $\alpha, \beta > 0$; moreover, they pointed out that the level sets of those functions satisfy equality in a geometric inequality of type (2.1).
Some symmetry properties of the solutions of Euler-Lagrange equation associated to problem (2.4) in the case $p = 2, q = 1$, as well as their connections with other interesting geometric questions, were estabilished in the series of articles [52], [51], [19] and [20].

When we want to find the best constant in (2.2) by symmetrization, we have to solve the problem in two steps: the first one, said *symmetrization result*, consists in proving that we can restrict the analysis to functions having particular symmetry properties; the second step consists in applying known techniques of Calculus of Variations to solve a constrained minimum problem.
For instance, this method works when we want to find the best constant in the classical Sobolev inequality (e.g. [60]), for we can reduce to a typical one-dimensional problem of the Calculus of Variations.
In our case, even if we can find the way to restrict the analysis to functions exploiting the same kind of symmetry of the ones in (2.3) , the minimum problem reduces to a two-dimensional problem whose solution is not easy.

However, when we want to prove a symmetrization result, isoperimetric inequalities play a key role: in fact, they can force the level sets of extremal functions to have a shape that minimize/maximize some of the terms we are dealing with.

If the inequality we are looking for has to play a role in minimizing the ratio:

$$\frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx\, dy}{\left(\int_{\mathbb{R}^N} \frac{|u|^{p^*(1)}}{|x|} \, dx\, dy\right)^{p/p^*(1)}}, \tag{2.4}$$

with $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and $p^*(1) = p\frac{N-1}{N-p}$, it has to be stated in terms of the right quantities.
If we take a function u sufficiently regular with compact support, Hölder inequality and the classical isoperimetric inequality imply that:

$$\int_{\mathbb{R}^N} |\nabla u|^p \, dx\, dy \geqslant \int_0^\infty \frac{\mathcal{H}^p_{N-1}(\{u = s\})}{(-\mu'(s))^p} \, ds$$

where $\mu(s)$ is the N-dimensional Lebesgue measure of the level set $\{u > s\}$ (hence it is a volume), $\mathcal{H}_{N-1}(\{u = s\})$ is the $(N-1)$-dimensional Hausdorff measure of $\{u = s\} = \partial\{u > s\}$ (hence it is a perimeter). On the other hand, an application of Fubini's theorem shows that:

$$\int_{\mathbb{R}^N} \frac{u^{p^*(1)}}{|x|} \, dx\, dy = p \int_0^\infty s^{p-1} \mu_1(s) \, ds$$

where $\mu_1(s) = \int_{\{u>s\}} \frac{1}{|x|} \, dx\, dy$ is the weighted measure of the level set $\{u > s\}$ with respect to the weight $\frac{1}{|x|}$.
Hence the ratio in (2.4) can be decreased in a natural way using geometric quantities related to the shape of the level sets of u; therefore the isoperimetric inequality we are looking for has to estabilish a relation between the volume, the perimeter and the weighted measure μ_1 of those level sets.

2.1.2 *Organization*

We prove at first that isoperimetric inequalities (2.1) hold for a special class of bodies of revolution, namely the symmetric ones. We also study the problem of the equality case, giving a complete characterization of the optimal bodies when equality is achieved in (2.1). Moreover, we prove that the constant $\varphi_N(a)$

has an explicit elementary form as a function of a and that it satisfies a differential recurrence relation.
We are able to extend inequalities (2.1) keeping their sharpness to the larger class of bodies of revolution in $\mathbb{R}^N$ using *Steiner symmetrization*.
Finally, in the last section we consider the case of sets which are radially simmetric with respect to a h-dimensional affine subspace, with $2 \leqslant h \leqslant N$: in particular, we are able to prove a family of inequalities similar to (2.1) and we conjecture both a value for the best constant and the shape of the optimal sets.

2.2 SHARP ISOPERIMETRIC INEQUALITIES FOR BODIES OF REVOLUTION: THE SYMMETRIC CASE

From now on, we set $N \in \mathbb{N}$ fixed and greater than 2; a point in $\mathbb{R}^N$ will be denoted by (x, y), with $x \in \mathbb{R}^{N-1}$ and $y \in \mathbb{R}$; the Lebesgue measure of the unit ball in $\mathbb{R}^N$ will be ω_N.

Let us consider the set:

$$\begin{aligned}\mathcal{C}_0 := \Big\{ f\colon [0,+\infty[\to [0,+\infty[\colon\ & f \text{ is nonincreasing,} \\ & \text{smooth and satisfies } f(0) > 0 \Big\}, \end{aligned} \tag{2.5}$$

where *smooth* means that $f \in C_c([0,+\infty[) \cap C^{0,1}([0,+\infty[)$.

Definition 2.1: When we choose a function $f \in \mathcal{C}_0$ and a point $(x_0, y_0) \in \mathbb{R}^N$, the set:

$$\begin{aligned} D := \Big\{ (x,y) \in \mathbb{R}^N : \ & |x - x_0| < f(|y - y_0|) \\ & \text{and } |y - y_0| \in (\operatorname{supp} f)^\circ \Big\} \end{aligned} \tag{2.6}$$

will be called *symmetric body of revolution described by* f *around* (x_0, y_0).

Remark 2.1:
A symmetric body of revolution around a point (x_0, y_0) *is axially-*

symmetric about the straight line r_0 *of equation* $x = x_0$ *and also symmetric about the hyperplane* Π_0 *of equation* $y = y_0$.
The point (x_0, y_0) *is the "center of mass" of* D. $\diamondsuit$

The volume $\mathrm{Vol}(D)$ (i.e. the N-dimensional Lebesgue measure of D) and the perimeter $\mathrm{Per}(D)$ of a symmetric body of revolution D described by a function $f \in \mathcal{C}_0$ around (x_0, y_0) can be easily computed in cylindrical coordinates:

$$\mathrm{Vol}(D) = 2\omega_{N-1} \int_0^{\infty} f^{N-1}(t)\,dt\,, \tag{2.7}$$

$$\mathrm{Per}(D) = 2(N-1)\omega_{N-1} \int_0^{\infty} \sqrt{1 + |f'(t)|^2} f^{N-2}(t)\,dt\,. \tag{2.8}$$

The inequality that we are going to prove involves also the weighted measure $\mathrm{Sec}(D)$ (with respect to the weight $W(x) := \frac{1}{|x-x_0|}$) of the body D: like $\mathrm{Vol}(D)$ and $\mathrm{Per}(D)$, the value of $\mathrm{Sec}(D)$ can be computed in cylindrical coordinates:

$$\mathrm{Sec}(D) = 2\frac{N-1}{N-2}\omega_{N-1} \int_0^{\infty} f^{N-2}(t)\,dt \quad . \tag{2.9}$$

REMARK 2.2:
$\mathrm{Sec}(D)$ *is proportional by the factor* $2\frac{(N-1)\omega_{N-1}}{(N-2)\omega_{N-2}}$ *to the* $(N-1)$*-dimensional Lebesgue measure of the sections of* D *determined by intersection with hyperplanes containing the rotation axis: owing to this, we can call* $\mathrm{Sec}(D)$ section measure of D. $\diamondsuit$

REMARK 2.3:
Because $\mathrm{Vol}(D)$, $\mathrm{Per}(D)$ *and* $\mathrm{Sec}(D)$ *are translation invariant, from now on we assume* $(x_0, y_0) = 0$ *without any loss of generality.* $\diamondsuit$

2.2.1 *Inequalities*

From (2.7)-(2.9) it follows that $\mathrm{Per}(D) - a(N-2)\,\mathrm{Sec}(D) \geqslant 0$ for each $a \in]0, 1]$.
But more is true: actually, the *classical isoperimetric inequality*

$\mathrm{Per}^N(D) \geqslant N^N\omega_N \ \mathrm{Vol}^{N-1}(D)$ can be used to show that an inequality of the type:

$$[\mathrm{Per}(D) - a(N-2)\,\mathrm{Sec}(D)]^N \geqslant c\,\mathrm{Vol}^{N-1}(D) \tag{2.10}$$

makes sense for some constant $c > 0$ and to get a rough lower bound for the so called *best constant*, i.e.:

$$\begin{aligned} C(N, a) &:= \sup\{c \geqslant 0 : \ (2.10) \text{ holds}\} \\ &= \inf_D \frac{[\mathrm{Per}(D) - a(N-2)\,\mathrm{Sec}(D)]^N}{\mathrm{Vol}^{N-1}(D)} \ . \end{aligned} \tag{2.11}$$

In fact, since:

$$\begin{aligned} \mathrm{Per}(D) - a(N-2)\,\mathrm{Sec}(D) &\geqslant (1-a)\,\mathrm{Per}(D) \\ &\geqslant (1-a)\left(N^N\omega_N\,\mathrm{Vol}^{N-1}(D)\right)^{1/N} \end{aligned}$$

we also have:

$$[\mathrm{Per}(D) - a(N-2)\,\mathrm{Sec}(D)]^N \geqslant N^N\omega_N(1-a)^N\,\mathrm{Vol}^{N-1}(D) \ ;$$

hence (2.10) holds with $c = N^N\omega_N(1-a)^N$ and the best constant $C(N, a)$ is greater than or equal to $N^N\omega_N(1-a)^N$.

The following is a generalization of [4, Theorem 3.1] and it gives an explicit value for the constant in (2.10):

Theorem 2.1 (Isoperimetric inequalities)
For $a \in]0, 1]$ *there exists a constant* $\varphi_N(a) \geqslant 0$ *such that inequality:*

$$\begin{aligned} &\Big[\mathrm{Per}(D) - a(N-2)\,\mathrm{Sec}(D)\Big]^N \\ &\qquad\qquad \geqslant 2(N-1)N^N\omega_{N-1}\varphi_N(a)\cdot\mathrm{Vol}^{N-1}(D) \quad , \end{aligned} \tag{2.12}$$

holds for all symmetric body of revolution D. *Moreover:*

$$\varphi_N(a) = \int_0^{1-a} u^{N-2}\sqrt{1-(u+a)^2}\,du \quad . \tag{2.13}$$

REMARK 2.4:
When $a \searrow 0$, (2.12) *approaches the* classical isoperimetric inequality *thus we can expect equality in:*

$$\lim_{a \searrow 0} 2(N-1)N^N \omega_{N-1} \varphi_N(a) \leqslant N^N \omega_N \ ,$$

instead of a strict inequality.
This is actually true, because using [41, 3.197-4] and recalling the definition of the beta function $B(t,s) = \Gamma(t)\Gamma(s)/\Gamma(t+s)$, *we find:*

$$\begin{aligned}
\lim_{a \searrow 0} 2(N-1)N^N \omega_{N-1} \varphi_N(a) = 2(N-1)N^N \omega_{N-1} \times& \\
\times \int_0^1 u^{N-2}\sqrt{1-u^2}\ du& \\
= (N-1)N^N \omega_{N-1}\ B\left(\frac{N-1}{2}, \frac{3}{2}\right)& \\
= (N-1)N^N \omega_{N-1}\ \frac{\Gamma(\frac{N-1}{2})\sqrt{\pi}}{2\Gamma(\frac{N}{2}+1)}& \\
= N^N \omega_{N-1}\ \frac{\Gamma(\frac{N-1}{2}+1)\sqrt{\pi}}{\Gamma(\frac{N}{2}+1)}& \\
= N^N \omega_N \ .&
\end{aligned}$$

$\diamondsuit$

PROOF . If $a = 1$, Theorem 2.1 becomes trivial because (2.12) and (2.13) give $\operatorname{Per}(D) - (N-2)\operatorname{Sec}(D) \geqslant 0$ which is true in virtue of the very definition of $\operatorname{Per}(D)$ and $\operatorname{Sec}(D)$. Hence we can limit ourselves to give the proof in the case $a \in]0,1[$.
It follows from (2.7)-(2.9) that in order to get (2.12) we have to prove:

$$(2(N-1)\omega_{N-1})^{1/N} N\ \varphi_N^{1/N}(a) \leqslant \inf_{f \in \mathcal{C}_0} \mathcal{J}_a[f] \ , \tag{2.14}$$

where $\mathcal{J}_a[\cdot]$ is the functional:

$$\begin{aligned}
\mathcal{J}_a[f] :=& (2\omega_{N-1})^{1/N}(N-1) \times \\
& \times \frac{\int_0^\infty \left\{\sqrt{1+|f'(t)|^2} - a\right\} f^{N-2}(t)\, dt}{\left(\int_0^\infty f^{N-1}(t)\, dt\right)^{\frac{N-1}{N}}} \ .
\end{aligned} \tag{2.15}$$

We divide the proof into two steps.
Step 1. Let $f \in \mathcal{C}_0$ be normalized as follows:

$$\|f\|_{L^{N-1}} = 1 \,, \tag{2.16}$$

let $\beta = \frac{f(0)}{1-a} > 0$ and let us define the auxiliary functional $\mathcal{J}_a[f]$:

$$\begin{aligned} \mathcal{J}_a[f] := \int_0^\infty \left\{ \sqrt{1+|f'(t)|^2} - a \right\} f^{N-2}(t)\, d\, t \\ - \frac{1}{\beta} \int_0^\infty f^{N-1}(t)\, d\, t \,. \end{aligned} \tag{2.17}$$

In view of the convexity of $\sqrt{1+z^2}$, for all $\zeta \in \mathbb{R}$ we have:

$$\sqrt{1+|f'(t)|^2} \geqslant \sqrt{1+\zeta^2} + \frac{\zeta}{\sqrt{1+\zeta^2}}(f'(t)-\zeta) \,; \tag{2.18}$$

in particular, if in the previous inequality we choose:

$$\zeta(f) = -\frac{\sqrt{\beta^2-(f+a\beta)^2}}{f+a\beta} \tag{2.19}$$

we deduce that the following inequality:

$$\begin{aligned} \sqrt{1+|f'(t)|^2} \geqslant \frac{\beta}{f(t)+a\beta} - \frac{1}{\beta}\sqrt{\beta^2-(f(t)+a\beta)^2} \times \\ \times \left(f'(t) + \frac{\sqrt{\beta^2-(f(t)+a\beta)^2}}{f(t)+a\beta} \right) \\ = \frac{1}{\beta}\left(f(t)+a\beta - \sqrt{\beta^2-(f(t)+a\beta)^2} f'(t) \right) \,. \end{aligned} \tag{2.20}$$

holds for a.e. $t \in [0, +\infty[$.
Owing to (2.20) we can decrease $\mathcal{J}_a[f]$ as follow:

$$\begin{aligned} \mathcal{J}_a[f] &\geqslant -\frac{1}{\beta}\int_0^\infty f^{N-2}(t)\sqrt{\beta^2-(f+a\beta)^2} f'(t)\, d\, t \\ &= \frac{1}{\beta}\int_0^{\beta(1-a)} f^{N-2}\sqrt{\beta^2-(f+a\beta)^2}\, d\, f \\ &= \beta^{N-1}\int_0^{1-a} u^{N-2}\sqrt{1-(u+a)^2}\, d\, u \\ &= \varphi_N(a)\, \beta^{N-1} \,. \end{aligned} \tag{2.21}$$

Recalling (2.16) and (2.17), from (2.21) we infer:

$$\int_0^\infty \left\{\sqrt{1+|f'(t)|^2}-a\right\} f^{N-2}(t)\,dt \geqslant \frac{1}{\beta}+\varphi_N(a)\,\beta^{N-1}\,.$$

With classical tools of Differential Calculus we can evaluate the minimum of the righthand side as a function of β: this leads to:

$$\begin{aligned}\int_0^\infty \left\{\sqrt{1+|f'(t)|^2}-a\right\} f^{N-2}(t)\,dt \\ \geqslant N(N-1)^{1/N-1}\varphi_N^{1/N}(a)\end{aligned} \tag{2.22}$$

which, after some algebra, gives our claim.

Step 2. If $f \in \mathcal{C}_0$ has L^{N-1}-norm different from 1, we can obtain our claim from **Step 1** using a suitable scaling argument: in fact, putting:

$$\hat{f}(t) := \frac{1}{\sigma}\, f(\sigma t)$$

with $\sigma > 0$ chosen such that (2.16) holds for $\hat{f}$, one can verify that:

$$\mathcal{J}_a[f] = \mathcal{J}_a[\hat{f}] \geqslant \gamma_N^{\frac{1}{N}}(a)\,.$$

Thus our Theorem is completely proved. □

Furthermore we can prove that $2(N-1)N^N\omega_{N-1}\varphi_N(a)$ is in fact the best constant in (2.10):

Proposition 2.1 (Best constant in (2.10))
Let $0 < a \leqslant 1$ *and* $C(N,a)$ *be the best constant in* (2.10).

(i) *If* $0 < a < 1$ *then there is attainment into inequality:*

$$\frac{[\mathrm{Per}(D) - a(N-2)\,\mathrm{Sec}(D)]^N}{\mathrm{Vol}^{N-1}(D)} \geqslant 2(N-1)N^N\omega_{N-1}\varphi_N(a)$$

when D *is the body of revolution generated by a function of the type:*

$$w_a(t;b) := \begin{cases} \sqrt{b^2-t^2}-ab & , \textit{if } t \in [0, b\sqrt{1-a^2}] \\ 0 & , \textit{otherwise,} \end{cases} \tag{2.23}$$

where b *is a positive parameter.*

(ii) If $a = 1$ *there exists a family* $\{D_\varepsilon\}_{\varepsilon>0}$ *of symmetric bodies of revolution such that:*

$$\begin{aligned} &\inf_{\varepsilon>0} \frac{[\mathrm{Per}(D_\varepsilon) - a(N-2)\,\mathrm{Sec}(D_\varepsilon)]^N}{\mathrm{Vol}^{N-1}(D_\varepsilon)} \\ &\quad = 0 \\ &\quad = 2(N-1)N^N \omega_{N-1} \varphi_N(1)\ . \end{aligned}$$

Therefore $C(N,a) = 2(N-1)N^N\omega_{N-1}\varphi_n(a)$ *for each* $a \in]0,1]$.

Proof . *(i)* Assume $0 < a < 1$ and let D be generated by a function of the type $w_a(\cdot;b)$. By means of the substitution $u = \frac{1}{b}(\sqrt{b^2-t^2}-ab)$ and of integration by parts, we find:

$$\begin{aligned} \mathrm{Vol}(D) &= 2\omega_{N-1} \int_0^\infty w_a^{N-1}(t;b)\,dt \\ &= 2\omega_{N-1} \int_0^{b\sqrt{1-a^2}} \left(\sqrt{b^2-t^2}-ab\right)^{N-1} dt \\ &= 2\omega_{N-1}\ b^N \int_0^{1-a} u^{N-1} \frac{u+a}{\sqrt{1-(u+a)^2}}\,du \\ &= 2\omega_{N-1}\ (N-1)b^N \int_0^{1-a} u^{N-2}\sqrt{1-(u+a)^2}\,du \\ &= 2(N-1)\omega_{N-1}\ b^N\ \varphi_N(a)\ ; \end{aligned}$$

analogous computations prove that:

$$
\begin{aligned}
\mathrm{Per}(D) - a(N-2)\,\mathrm{Sec}(D) &= 2(N-1)\omega_{N-1}\times \\
&\quad \times \int_0^\infty w_a^{N-2}(t;b)\left(\sqrt{1+|w_a'(t;b)|^2}-a\right)dt \\
&= 2(N-1)\omega_{N-1}\; b^{N-1}\times \\
&\quad \times \int_0^{1-a} u^{N-2}\frac{1-a(u+a)}{\sqrt{1+(u+a)^2}}\,du \\
&= 2(N-1)\omega_{N-1}\; b^{N-1}\times \\
&\quad \left(\int_0^{1-a} u^{N-1}\frac{u+a}{\sqrt{1-(u+a)^2}}\,du\right. \\
&\qquad \left. + \int_0^{1-a} u^{N-2}\sqrt{1-(u+a)^2}\,du\right) \\
&= 2(N-1)\omega_{N-1}\; Nb^{N-1}\int_0^{1-a} u^{N-2}\sqrt{1-(u+a)^2}\,du \\
&= 2(N-1)N\omega_{N-1}\; b^{N-1}\;\varphi_N(a)\,,
\end{aligned}
$$

hence:

$$
\frac{[\mathrm{Per}(D) - a(N-2)\,\mathrm{Sec}(D)]^N}{\mathrm{Vol}^{N-1}(D)} = 2N^N(N-1)\;\varphi_N(a)\,.
$$

(ii) Assume now $a=1$ and let D_ε be the symmetric double cone generated by:

$$
g_\varepsilon(t) := \begin{cases} -\frac{1}{\varepsilon}\,(t-\varepsilon) & \text{, if } t\in[0,\varepsilon] \\ 0 & \text{, otherwise.} \end{cases}
$$

Explicit computations show that:

$$
\begin{aligned}
\mathrm{Vol}(D_\varepsilon) &= \frac{2\omega_{N-1}}{N}\;\varepsilon \\
\mathrm{Per}(D_\varepsilon) &= 2\omega_{N-1}\;\sqrt{1+\varepsilon^2} \\
\mathrm{Sec}(D_\varepsilon) &= \frac{2\omega_{N-1}}{N-2}\;\varepsilon
\end{aligned}
$$

thus:

$$\begin{aligned}
&\inf_{\varepsilon>0} \frac{[\mathrm{Per}(D_\varepsilon) - a(N-2)\,\mathrm{Sec}(D_\varepsilon)]^N}{\mathrm{Vol}^{N-1}(D_\varepsilon)} \\
&\qquad = \inf_{\varepsilon>0} 2N^{N-1}\omega_{N-1}[\sqrt{1+\varepsilon^2} - \varepsilon]^N\ \varepsilon^{1-N} \\
&\qquad \leqslant 2N^{N-1}\omega_{N-1} \lim_{\varepsilon\to\infty} [\sqrt{1+\varepsilon^2} - \varepsilon]^N\ \varepsilon^{1-N} \\
&\qquad = 0
\end{aligned}$$

as we claimed. □

2.2.2 *The case of equality in* (2.12)

Once we have proved that $2(N-1)N^N\omega_{N-1}\varphi_N(a)$ is the best constant in (2.12), we can address the problem of *characterizing the equality case in* (2.12), i.e. the problem of finding all the symmetric bodies of revolution which satisfy (2.12) with the equal sign.

It turns out that in the case $0 < a < 1$ there is only one class of nontrivial body of revolution satisfying equality in (2.12), which elements are related by scaling.
On the other hand, in the case $a = 1$ it turns out that equality cannot occur in (2.12).

In what follows we are going to fix the value $2\omega_{N-1}$ for the volume of the bodies of revolution we will be dealing with, because this volume constraint simplifies our computations.
We explicitly remark that there is no loss of generality: in fact, a standard *scaling argument* shows that a symmetric body D satisfies equality in (2.12) if and only if all of its dilated bodies λD do.

In order to make our arguments more clear, we state the following:

Theorem 2.2 (Equality in (2.12) for $0 < a < 1$)
Let $0 < a < 1$.
Let D *be a body revolution satisfying equality in* (2.12) *and* $f \in \mathcal{C}_0$ *its*

generatig function.
Then $f(\cdot) = w_a(\cdot;(1-a)^{-1}\sup f)$*, where* $w_a(\cdot;\cdot)$ *is a function defined in Proposition 2.1.*

PROOF . Because of the volume constraint we have $\|f\|_{N-1} = 1$. Retracing the steps in the proof of Theorem 2.1, we find that if equality holds in (2.12) then we have equality in (2.18) with $\zeta = \zeta(t)$ given by (2.19); since $\sqrt{1+z^2}$ is strictly convex, equality occurs in (2.18) only if $f'(t) = \zeta(t)$, hence f solves the following problem:

$$\begin{cases} f'(t) = -\frac{\sqrt{\beta^2-(f(t)+a\beta)^2}}{f(t)+a\beta} \\ f(0) = \beta(1-a) \\ f(t) \geqslant 0 \end{cases} \tag{2.24}$$

in the weak sense inside its support. Moreover, equality has to hold in (2.22), hence we have:

$$\beta = \left(\frac{1}{N\varphi_N(a)}\right)^{1/N} \tag{2.25}$$

where $\beta = (1-a)^{-1}\sup f$.
We explictly remark that uniqueness fails for problem (2.24): in fact the righthand side fails to be Lipschitz in any neighbourhood of the initial condition $(0, \beta(1-a))$, so that a *Peano phenomenon* occurs.
Neverthless we can state that there exists a nonnegative t_0 such that:

$$f(t) = \begin{cases} \beta(1-a) & \text{, if } 0 \leqslant t \leqslant t_0 \\ w_a(t-t_0;\beta) & \text{, if } t \geqslant t_0. \end{cases}$$

Routine computations yield:

$$\begin{aligned}
\|f\|_{L^{N-1}}^{N-1} &= \beta^{N-1}(1-a)^{N-1}t_0 + \int_0^{\beta\sqrt{1-a^2}} w_a^{N-1}(t;\beta)\ dt \\
&= \beta^{N-1}(1-a)^{N-1}t_0 \\
&\quad + (N-1)\beta^N \int_0^{1-a} u^{N-2}\sqrt{1-(u+a)^2}\ du \\
&= (1-a)^{N-1}\beta^{N-1}t_0 + (N-1)\varphi_N(a)\ \beta^N
\end{aligned}$$

therefore t_0 has to satisfy the normalization condition:

$$(1-a)^{N-1}\beta^{N-1}t_0 + (N-1)\varphi_N(a)\ \beta^N = 1\ . \tag{2.26}$$

Owing to (2.25) equation (2.26) implies $t_0 = 0$, hence our claim.

□

REMARK 2.5:
If we try to visualize things in the tridimensional space, then the normalized optimal body for (2.12) *resembles a rugby ball or, say, a spindle. It becomes rounder as* $a \searrow 0$ *for it approaches a ball, the optimal set for the classical isoperimetric inequality. On the other hand, it shrinks to* $\{o\}$ *when* $a \nearrow 1$. ♢

REMARK 2.6:
We also note that a function $w_a(\cdot;b)$ *describes the boundary of the level set* $\{u > s\}$ $(s > 0)$ *of a function in the family* (2.3) *if and only if we choose the parameters* a, b *as follows:*

$$a = (\alpha s)^{-\frac{p-1}{N-p}} \qquad \text{and} \qquad b = \frac{1}{\beta}(\alpha s)^{\frac{p-1}{N-p}}\ .$$

♢

PROPOSITION 2.2 (Equality in (2.12) for $a = 1$)
Equality never occurs in (2.12) *when* $a = 1$.

Proof. Assume *by contradiction* that there exists a function $f \in \mathcal{C}_0$ with $\|f\|_{N-1} = 1$ such that equality occurs in (2.12) for the body

of revolution generated by f.
Thus we have:

$$\int_0^{\infty} \left(\sqrt{1+|f'(t)|^2}-1\right) f^{N-2}(t)\ dt = 0$$

and this implies $(\sqrt{1+|f'(t)|^2}\ -1)\ f^{N-2}(t) = 0$ for a.e. $t \in [0,\infty[$.
For t close to 0 we have $f(t) > 0$, hence it has to be $f'(t) = 0$ a.e. and $f(t) = f(0) > 0$ in a neighbourhood of 0; on the other hand, for all sufficiently large t it is $f(t) = 0$, because f is compactly supported.
Let:

$$t_1 := \sup\{t \geqslant 0 :\ f \text{ is constant and positive in } [0,t]\}$$
$$t_2 := \inf\{t > 0 :\ f \text{ equals zero in } [t,\infty[\}\,;$$

obviously $0 < t_1 \leqslant t_2 < \infty$. We claim $t_1 = t_2$: if this were not the case then f should be positive in $[t_1, t_2[$, hence f' should be a.e. equal to zero in the same interval; but then f should be constant in $[t_1, t_2]$, against the fact that $f(t_1) = f(0) > 0 = f(t_2)$. Equality $t_1 = t_2$ implies that f has a discontinuity jump in t_1, which is a contradiction. □

2.2.3 *Properties of the best constant as a function of* a

Proposition 2.1 says that $2(N-1)N^N\omega_{N-1}\varphi_N(a)$ defined in (2.13) is indeed the best constant in (2.10), hence it could be interesting to investigate in details some properties of such a number.
Since the value of the constant depends on the value of the "mysterious" term $\varphi_N(a)$, we are interested into highlighting some properties of the map $[0,1] \ni a \mapsto \varphi_N(a) \in [0,\infty[$ and the sequence of functions $\mathbb{N} \ni N \mapsto \varphi_N(\cdot) \in C(]0,1])$; in particular, we address the following questions:

1. is it possible to characterize $a \mapsto \varphi_N(a)$ as solution of some differential problem?
2. is it possible to find some kind of recurrence relation for $N \mapsto \varphi_N(\cdot)$?

3. is it possible to give $\varphi_N(a)$ an *explicit elementary form*? That is, is it possible to write down an explicit expression for $\varphi_N(a)$ in terms of elementary functions?

We are going to prove that questions 1-3 can be answered in the positive.

PROPOSITION 2.3
The function $\varphi_N(\cdot)$ is the unique solution in $[0,1]$ of the $(N-2)$-th order ODE:

$$\varphi_N^{(N-2)}(a) = (-1)^{N-2}(N-2)! \cdot \left(\arccos a - a\sqrt{1-a}\right) \quad (2.27)$$

satisfying the homogeneous conditions:

$$\begin{cases} \varphi_N(1) = 0 \\ \varphi_N'(1) = 0 \\ \quad\vdots \\ \varphi_N^{(N-3)}(1) = 0 \end{cases}$$

which is positive in $[0,1[$, strictly decreasing and convex.

PROOF . First of all, note that differentiating the integral:

$$\varphi_N(a) := \int_0^{1-a} u^{N-2}\sqrt{1-(u+a)^2}\ du \quad (2.28)$$

in (2.13) with respect to a yields an elementary integral in u, which can be easily computed by parts: in fact:

$$\begin{aligned}
\varphi_N'(a) &= -u^{N-2}\sqrt{1-(u+a)^2}\Big|_{u=1-a} \\
&\quad + \int_0^{1-a} u^{N-2}\frac{-(u+a)}{\sqrt{1-(u+a)^2}}\,du \\
&= u^{N-2}\sqrt{1-(u+a)^2}\Big|_0^{1-a} \\
&\quad -(N-2)\int_0^{1-a} u^{N-3}\sqrt{1-(u+a)^2}\,du \\
&= -(N-2)\int_0^{1-a} u^{N-3}\sqrt{1-(u+a)^2}\,du \\
&= -(N-2)\,\varphi_{N-1}(a)\,;
\end{aligned} \tag{2.29}$$

in complete analogy, if we differentiate a second time we find:

$$\varphi_N''(a) = \begin{cases} \sqrt{1-a^2} & \text{, if } N=3 \\ (N-2)(N-3)\varphi_{N-2}(a) & \text{, if } N \geqslant 4. \end{cases} \tag{2.30}$$

Now it is easy to see that if $N \geqslant 4$ we can differentiate $\varphi_N(a)$ for $k = 3, \ldots, N-2$ times to obtain:

$$\begin{aligned}
\varphi_N^{(k)}(a) &= (-1)^k\frac{(N-2)!}{(N-2-k)!}\times \\
&\quad \times\int_0^{1-a} u^{N-2-k}\sqrt{1-(u+a)^2}\,du\,,
\end{aligned}$$

and in particular:

$$\begin{aligned}
\varphi_N^{(N-2)}(a) &= (-1)^{N-2}(N-2)!\int_0^{1-a}\sqrt{1-(u+a)^2}\,du \\
&= (-1)^{N-2}(N-2)!\left(\arccos a - a\sqrt{1-a}\right)\,.
\end{aligned}$$

Hence $\varphi_N(a)$ solves:

$$\begin{cases} \varphi_N^{(N-2)}(a) = (-1)^{N-2}(N-2)! \ \left(\arccos a - a\sqrt{1-a}\right) & \text{, in }]0,1[\\ \varphi_N(1) = 0 \\ \varphi'_N(1) = 0 \\ \quad \vdots \\ \varphi_N^{(N-3)}(1) = 0 \end{cases}$$

which is (2.27).
Solution of problem (2.27) is obviously unique; moreover, from (2.29) it follows that $\varphi_N(a)$ is strictly decreasing and convex in $[0,1]$, hence it is positive in $[0,1[$. □

For $N = 3$ problem (2.27) has the solution:

$$\varphi_3(a) = \frac{1}{6}\left((a^2+2)\sqrt{1-a^2} - 3a \arccos a\right)$$

which was already found in [4].
From formula (2.30) and equation (2.27), after some algebra, we obtain:

Proposition 2.4
The sequence $\varphi_N(a)$ satisfies the differential recurrence relation:

$$\begin{gathered} \varphi_3(a) = \frac{1}{6}\left((a^2+2)\sqrt{1-a^2} - 3a \arccos a\right) \ , \\ \begin{cases} \varphi'_{N+1}(a) = -(N-1)\ \varphi_N(a) \\ \varphi_{N+1}(1) = 0 \end{cases} \end{gathered} \tag{2.31}$$

Finally, we prove that $\varphi_N(a)$ is an elementary function of a:

Proposition 2.5
For each $N \geqslant 3$ there exist two polynomials P_N, Q_N, respectively of degree $\lfloor (N-1)/2 \rfloor$ and $\lfloor (N-2)/2 \rfloor$ (here $\lfloor \cdot \rfloor$ is the floor function*), such that:*

$$\begin{aligned} \varphi_N(a) = (-1)^{N-1} a^{\chi(N)} P_N(a^2)\sqrt{1-a^2} \\ + (-1)^N a^{1-\chi(N)} Q_N(a^2) \arccos a \ , \end{aligned} \tag{2.32}$$

where:

$$\chi(N) := \begin{cases} 1 & \text{, } \textit{if } N \textit{ is even} \\ 0 & \text{, } \textit{otherwise.} \end{cases}$$

Proof. Using recurrence relation (2.31) we can compute:

$$\begin{aligned}
\varphi_3(a) &= \frac{1}{6}\Big((a^2+2)\sqrt{1-a^2} - 3a\arccos a\Big) \\
\varphi_4(a) &= \frac{1}{48}\pi\Big(-a(26+4a^2)\sqrt{1-a^2} \\
&\quad + (6+24a^2)\arccos a\Big) \\
\varphi_5(a) &= \frac{1}{120}\pi^2\Big((16+83a^2+6a^4)\sqrt{1-a^2} \\
&\quad - a(45+60a^2)\arccos a\Big) \\
\varphi_6(a) &= \frac{1}{480}\pi^2\Big(-a(226+388a^2+16a^4)\sqrt{1-a^2} \\
&\quad + (30+360a^2+240a^4)\arccos a\Big)\ ,
\end{aligned}$$

hence formula (2.32) holds for $N = 3, \ldots, 6$.
We now use induction. Let us assume (2.32) holds for $N \geqslant 3$: using [41, 2.260-1], we compute:

$$\varphi_{N+1}(a) = \frac{N-2}{N+1}\,(1-a^2)\,\varphi_{N-1}(a) - \frac{2N-1}{N+1}\,a\,\varphi_N(a)\ . \quad (2.33)$$

Plugging (2.32) into (2.33) gives:

$$\begin{aligned}
\varphi_{N+1}(a) = \sqrt{1-a^2}(-1)^N &\Big(\frac{N-2}{N+1}a^{1-\chi(N)}P_{N-1}(a) \\
&- \frac{N-2}{N+1}a^{3-\chi(N)}P_{N-1}(a) + \frac{2N-1}{N+1}a^{1+\chi(N)}P_N(a)\Big) \\
+ \arccos a(-1)^{N+1} &\Big(\frac{N-2}{N+1}a^{1-\chi(N+1)}Q_{N-1}(a) \\
&- \frac{N-2}{N+1}a^{3-\chi(N+1)}Q_{N-1}(a) + \frac{2N-1}{N+1}a^{2-\chi(N)}Q_N(a)\Big)
\end{aligned}$$

which, with some algebra, turns into:

$$\begin{aligned}
\varphi_N(a) &= (-1)^N a^{\chi(N+1)}P_{N+1}(a^2)\sqrt{1-a^2} \\
&\quad + (-1)^{N+1}a^{1-\chi(N+1)}Q_{N+1}(a^2)\arccos a
\end{aligned}$$

as we claimed. $\square$

2.3 SHARP ISOPERIMETRIC INEQUALITIES FOR BODIES OF REVOLUTION: THE GENERAL CASE

An application of a standard symmetrization technique yields that inequalities (2.12) hold also for bodies of revolution in $\mathbb{R}^N$ which are not symmetric.

Let us put:

$$\mathcal{C} := \Big\{ f\colon \mathbb{R} \to [0, +\infty[\colon f \text{ is smooth and positive at some point} \Big\}\,, \tag{2.34}$$

where, as in the previous section, "smooth" means Lipschitz and compactly supported.

DEFINITION 2.2: When we choose a function $f \in \mathcal{C}$, a straight line $r \subset \mathbb{R}^N$ with direction $\nu \in \mathcal{S}^{N-1}$ and a point $(x_0, y_0) \in r$, the set:

$$D := \big\{ (x, y) \in \mathbb{R}^N : \operatorname{dist}((x, y), r) < f(\operatorname{proj}_r(x, y)) \text{ and } \operatorname{proj}_r(x, y) \in (x_0, y_0) + \nu\,(\operatorname{supp} f)^\circ \big\} \tag{2.35}$$

will be called *body of revolution described by* f *around the axis* r *and the point* (x_0, y_0).

REMARK 2.7:
It's easily seen that if we take $f \in \mathcal{C}$, *the even extension of* f *to the whole real line is in the class* $\mathcal{C}_0$. *Therefore symmetric bodies of revolution are particular cases of Definition 2.2.* $\diamondsuit$

A computation in cylindrical coordinates gives the following expression for the volume $\operatorname{Vol}(D)$, the perimeter $\operatorname{Per}(D)$ and the weighted measure $\operatorname{Sec}(D)$ (with respect to the weight $W(x) :=$

$\frac{1}{\mathrm{dist}((x,y),r)}$) of the body of revolution D described by $f \in \mathcal{C}$ around the axis r:

$$\mathrm{Vol}(D) = \omega_{N-1} \int_{-\infty}^{\infty} f^{N-1}(t)\,dt\,, \tag{2.36}$$

$$\mathrm{Per}(D) = (N-1)\omega_{N-1} \int_{-\infty}^{\infty} \sqrt{1+|f'(t)|^2}\, f^{N-2}(t)\,dt\,, \tag{2.37}$$

$$\mathrm{Sec}(D) = \frac{N-1}{N-2}\omega_{N-1} \int_{-\infty}^{\infty} f^{N-2}(t)\,dt\,, \tag{2.38}$$

which are completely analogous to (2.7)-(2.9).

REMARK 2.8:
Note that $\mathrm{Vol}(D)$ *and* $\mathrm{Sec}(D)$ *are proportional to the* L^{N-1} *and* L^{N-2} *norms of* f *respectively.*
Moreover, it holds for the weighted measure of a body of revolution what we wrote in Remark 2.2 about the weighted measure of a symmetric revolution body; hence we can still call $\mathrm{Sec}(D)$ section measure of D. ♢

Next we give the aforementioned generalization of Theorem 2.1:

THEOREM 2.3
Inequalites (2.12) *hold true even if* D *is a body of revolution as in Definition 2.2 instead of a symmetric body of revolution.*
The constant $2(N-1)N^N\omega_{N-1}\varphi_N(a)$ *is the best one for each* $a \in]0,1]$.
Equality is attained only in the case $a \in]0,1[$ *and the optimal bodies are the symmetric ones generated by the functions* $w_a(\cdot;b)$.

In order to prove our Theorem we need to point out the close connection between Steiner symmetrization of a body of revolution described by $f \in \mathcal{C}$ and the Schwarz rearrangement of the function f:

PROPOSITION 2.6
If D *is a body of revolution described by* $f \in \mathcal{C}$ *around the axis* r *and the point* (x_0, y_0) *then, for all hyperplanes* Π *orthogonal to* r, *the*

Steiner symmetral D^s *of* D *with respect to* Π *is the symmetric body of revolution described by the function:*

$$f^s := f^\star|_{[0,\infty[}$$

around the intersection point (x_1, y_1) *of* r *and* Π.

PROOF . Without loss of generality, we can assume for sake of semplicity that r coincides with the y-axis, that Π is the coordinate plane $y = 0$ and therefore that $(x_1, y_1) = o$; it then follows:

$$D = \{(x, y) \in \mathbb{R}^N : \ |x| < f(y) \text{ and } y \in (\operatorname{supp} f)^\circ\}\,.$$

Let $(\xi, 0) \in \Pi$ and consider the straightline r_ξ of equation $x = \xi$ which meets D in $D_\xi := \{y \in \mathbb{R} : \ (\xi, y) \in D\} \neq \varnothing$. From the definition of D we infer:

$$D_\xi = \{y \in \mathbb{R} : \ f(y) > |\xi|\}\,,$$

hence D_ξ is a level set of f and $|D_\xi| = \mu_f(|\xi|)$.
On the other hand, if we call f^s the function in $\mathcal{C}_0$ which generates D^s, proceeding in the same way as before we infer $|D^s_\xi| = 2\mu_{f^s}(|\xi|) = \mu_g(|\xi|)$ where we have set $g(t) = f^s(t/2)$.
Since $|D_\xi| = |D^s_\xi|$, functions g and f are equidistributed, and g is a continuous decreasing function in $[0, \infty[$; a classical uniqueness result implies that g equals the decreasing rearrangement f^*, hence f^s equals the restriction of the *Schwarz rearrangement* $f^\star$ to $[0, \infty[$. □

REMARK 2.9:
Since $f \in \mathcal{C}$ *implies* $f^\star$ *is Lipschitz, we also have* f^s *Lipschitz therefore* $\mathrm{Per}(D^s)$ *can be evaluated by means of* (2.9). ♢

In view of the properties of Steiner symmetrization and Schwarz rearrangement stated in chapter 1, of Proposition 2.6, of Remarks 2.8 and 2.9, we can state that the following relations:

$$\mathrm{Vol}(D) = \mathrm{Vol}(D^s)\,, \tag{2.39}$$

$$\mathrm{Per}(D) \geqslant \mathrm{Per}(D^s)\,, \tag{2.40}$$

$$\mathrm{Sec}(D) = \mathrm{Sec}(D^s)\ . \tag{2.41}$$

hold true for each body of revolution D and its Steiner symmetral D^s with respect to any hyperplane orthogonal to its axis. Relations (2.39)-(2.41) lead to a simple proof of Theorem 2.3, as we now show.

PROOF *(of Theorem 2.3).* Owing to (2.39)-(2.41) and (2.12), we have:

$$\begin{aligned}
&[\mathrm{Per}(D) - a(N-2)\,\mathrm{Sec}(D)]^N \\
&\qquad \geqslant [\mathrm{Per}(D^s) - a(N-2)\,\mathrm{Sec}(D^s)]^N \\
&\qquad \geqslant 2(N-1)N^N \omega_{N-1}\varphi_N(a)\ \mathrm{Vol}^{N-1}(D^s) \\
&\qquad = 2(N-1)N^N \omega_{N-1}\varphi_N(a)\ \mathrm{Vol}^{N-1}(D)
\end{aligned}$$

where D^s is the Steiner symmetral of D with respect to, say, the hyperplane othogonal to the axis r through the point (x_0, y_0).

If $0 < a < 1$ and D satisfies equality in (2.12) then also D^s does. In particular, D^s satisfies equality in $\mathrm{Per}(D) = \mathrm{Per}(D^s)$ and we infer D^s is a symmetric body of revolution generated by $w_a(\cdot; b)$ for some value of b from Theorem 2.2.
Therefore D^s is a bounded Lipschitz set which meets condition (i) in [23, Proposition 1.2] with Ω equal to the ball of radius $w_a(0; b) = b(1-a) > 0$: in fact the set:

$$\begin{aligned}
\{(x,y) \in \partial^* D^s :\ \nu_y^{E^s} = 0\} \cap (B(o_x; b(1-a)) \times \mathbb{R}) \\
= \partial B(o_x; b(1-a)) \times \{0\}
\end{aligned}$$

has zero $(N-1)$-dimensional Hausdorff measure. Hence [23, Theorem 1.3] applies and we can infer $D = D^s$. The complete characterization of the equality case follows, together with the value of the best constant.
On the other hand, if $a = 1$ strict inequality holds for D^s hence equality is never attained. □

2.4 REMARKS ON A MORE GENERAL FAMILY OF INEQUALITIES

In this section, we want to point out that inequality of type (2.12) also holds for symmetric bodies which feature a more general kind of symmetry.

In fact, starting with a function $f \in \mathcal{C}_0$ and a point $(x_0, y_0) \in \mathbb{R}^k \times \mathbb{R}^h$ (with $k + h = N$) we can build sets of the type:

$$D := \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^h : |x - x_0| < f(|y - y_0|) \\ \text{and } |y - y_0| \in (\operatorname{supp} f)^\circ\}$$

which are symmetric about the k-dimensional affine subspace of equations $y = y_0$ and radially symmetric about the h-dimensional affine subspace of equations $x = x_0$. We call a set of the previous type *cylindrically symmetric set described by* f *around* (x_0, y_0) *of codimension* h.
Volume, perimeter and weighted measure with respect to the weight $\frac{1}{|x - x_0|}$ of a cylindrically symmetric set D described by f can be easily computed in cylindrical coordinates:

$$\begin{aligned} \mathrm{Vol}(D) &= h\omega_h\omega_k \int_0^\infty f^k(t)\, t^{h-1}\, dt\,, \\ \mathrm{Per}(D) &= hk\omega_h\omega_k \int_0^\infty \sqrt{1 + |f'(t)|^2}\, f^{k-1}(t)\, t^{h-1}\, dt\,, \\ \mathrm{Sec}(D) &= h\frac{k}{k-1}\omega_h\omega_k \int_0^\infty f^{k-1}(t)\, t^{h-1}\, dt\,. \end{aligned}$$

Considering that we have $\mathrm{Per}(D) - a(k-1)\ \mathrm{Sec}(D) \geqslant 0$ for each $a \in]0, 1]$, we can use the classical isoperimetric inequality as in Remark 2.3 to write:

$$\mathrm{Per}(D) - a(k-1)\,\mathrm{Sec}(D) \geqslant (1-a)N\omega_N^{1/N}\,\mathrm{Vol}^{(N-1)/N}(D)\,;$$

this means that volume, perimeter and weighted measure of cylindrically symmetric bodies are involved in some isoperimetric inequalities completely analogous to (2.12): hence we can state:

THEOREM 2.4
For each $a \in]0,1]$ *there exists at least a constant* $c > 0$ *such that inequality:*

$$[\mathrm{Per}(D) - a(k-1)\,\mathrm{Sec}(D)]^N \geqslant c\ \mathrm{Vol}^{N-1}(D)\ , \tag{2.42}$$

holds for all cylindrically symmetric bodies.

Therefore it makes sense to consider the problem of finding the best constant $C(k,h,a)$ and the shape of the optimal cylindrically symmetric bodies (if any!) for (2.42).

For what concerns the value of the best constant in (2.42), we notice what follows.
For $a = 1$ we have $C(k,h,1) = 0$: in fact a direct calculation with $f = g_\varepsilon$ (with g_ε as in the proof of Proposition 2.1-*(ii)*) shows that:

$$\begin{aligned} C(k,h,a) &= \inf_{\varepsilon>0} \frac{[\mathrm{Per}(D_\varepsilon) - (k-1)\,\mathrm{Sec}(D_\varepsilon)]^N}{\mathrm{Vol}^{N-1}(D_\varepsilon)} \\ &\leqslant \lim_{\varepsilon\to\infty} \frac{[\mathrm{Per}(D_\varepsilon) - (k-1)\,\mathrm{Sec}(D_\varepsilon)]^N}{\mathrm{Vol}^{N-1}(D_\varepsilon)} \\ &= 0\ . \end{aligned}$$

On the other hand, when $a \searrow 0$ inequality (2.42) approaches the classical isoperimetric inequality, hence we can reasonably expect that $\lim_{a\searrow 0} C(k,h,a) = N^N\omega_N$ for every k,h and that the optimal sets approach the balls.
Nevertheless we have no clues what to expect when $a \in]0,1[$, except that inequality $C(k,h,a) \geqslant N^N\omega_N\ (1-a)^N$ has to hold.

About the shape of optimal sets in (2.42) in the case $a \in]0,1[$, we remark that the functions (2.23) solve Euler-Lagrange equation relative to the constrained minimum problem associated to (2.42): in fact the equation is:

$$\begin{aligned} &\frac{d}{dt}\left[\frac{f'(t)}{\sqrt{1+|f'(t)|^2}}\, f^{k-1}(t)\, t^{h-1}\right] - (k-1)\{\sqrt{1+|f'(t)|^2} - a\}\times \\ &\quad \times f^{k-2}(t)\, t^{h-1} + \lambda k\, f^{k-1}(t)\, t^{h-1} = 0\ , \end{aligned}$$

and $w_a(\cdot; b)$ solves it with $\lambda = \frac{k+h-1}{bk}$. The lack of convexity of the integrand generating the previous equation doesn't allow us to claim that functions $w_a(\cdot; b)$ actually solve our minimum problem for $a \in]0, 1[$.
Moreover, for each fixed $b > 0$, the cylindrically symmetric body D generated by the profile $w_a(\cdot; b)$ has:

$$\frac{[\mathrm{Per}(D) - a(k-1)\,\mathrm{Sec}(D)]^N}{\mathrm{Vol}^{N-1}(D)} = kN^N \omega_h \omega_k \times \\ \times \int_0^{1-a} u^{k-1} \left[1 - (u+a)^2\right]^{h/2} d\,u$$

and letting $a \searrow 0$ we find:

$$\begin{aligned} \lim_{a \searrow 0} kN^N \omega_h \omega_k \int_0^{1-a} u^{k-1} \left[1 - (u+a)^2\right]^{h/2} d\,u \\ &= N^N \omega_h \omega_k \int_0^1 u^{k-1} (1-u^2)^{h/2} \,d\,u \\ &= kN^N \omega_h \omega_k \, B(k/2, 1 + h/2) \\ &= N^N \omega_N \,, \end{aligned}$$

hence the value of the ratio $[\mathrm{Per}(D) - a(k-1)\,\mathrm{Sec}(D)]^N \ \mathrm{Vol}^{1-N}(D)$ approaches the isoperimetric constant when a becomes small.

Thus we were led to make the following:

Conjecture: when $a \in]0, 1[$ the best constant in (2.42) is:

$$C(k, h, a) = kN^N \omega_h \omega_k \int_0^{1-a} u^{k-1} \left[1 - (u+a)^2\right]^{h/2} d\,u$$

and the functions $w_a(\cdot; b)$ give the profiles of the optimal bodies in (2.42).

Unfortunately we were not able to prove such a claim.

3

STABILITY ESTIMATES FOR THE SYMMETRIZED FIRST EIGENFUNCTION OF CERTAIN ELLIPTIC OPERATORS

INDEX

3.1 INTRODUCTION

3.1.1 *Motivations*

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open domain with unit measure and let us consider the eigenvalue problem:

$$\begin{cases} -\operatorname{div}(A(x)\cdot\nabla u) + c(x)\, u = \lambda\, u & \text{, in } \Omega \\ u = 0 & \text{, on } \partial\Omega \end{cases} \tag{3.1}$$

where the matrix A and the potential term c satisfy assumptions from §1.4.1, i.e.:

(H1) $A := (a_{ij}) \in L^\infty(\Omega;\mathbb{R}^{N\times N})$ is a symmetric uniformly elliptic matrix such that $\sum_{i,j} a_{i,j}(x)\ \xi_i\xi_j \geqslant |\xi|^2$ for all $\xi = (\xi_1,\ldots,\xi_N) \in \mathbb{R}^N$ and a.e. $x \in \Omega$, (H2) $c \in L^\infty(\Omega)$ is a.e. nonnegative.

By the classical results recalled in §1.4.1, there exists only one nonnegative eigenfunction u_1 corresponding to $\lambda_1^{A,c}(\Omega)$ such that $\|u_1\|_{\infty,\Omega} = 1$: in what follows we call u_1 the *first eigenfunction of the problem* (3.1).

Moreover, let $\Omega^\star$ be the ball centered in the origin with the same measure of Ω and let $\lambda_1^\star := \lambda_1^{I,0}(\Omega^\star)$, $U_1 \in W_0^{1,2}(\Omega^\star)$ be the first eigenvalue and the first eigenfunction of the Dirichlet Laplacian in $\Omega^\star$, i.e. the solution of:

$$\begin{cases} -\Delta U_1 = \lambda_1^\star \, U_1 & \text{, in } \Omega^\star \\ U_1 = 0 & \text{, on } \partial\Omega^\star \end{cases} \tag{3.2}$$

normalized in such a way that $\|U_1\|_{\infty,\Omega} = 1$.
It is well known that $\lambda_1^\star = \omega_N^{2/N} \; j^2_{N/2-1,1}$, where $j_{N/2-1,1}$ is the first nontrivial zero of the Bessel function $J_{N/2-1}$, and that U_1 is spherically symmetric and radially decreasing. On the other hand, the Faber-Krahn inequality of section §1.3.1 states that $\lambda_1^{A,c}(\Omega) \geqslant \lambda_1^\star$.

Finally, let B be the ball centered in the origin such that the first eigenvalue of the Dirichlet Laplacian in B coincides with $\lambda_1^{A,c}$, i.e., $\lambda_1^{I,0}(B) = \lambda_1^{A,c}(\Omega)$, and let $V_1 \in W_0^{1,2}(B)$ be the first eigenfunction corresponding to $\lambda_1^{I,0}(B)$, so that V_1 solves:

$$\begin{cases} -\Delta V_1 = \lambda_1^{A,c}(\Omega) \, V_1 & \text{, in } B \\ V_1 = 0 & \text{, on } \partial B \end{cases} . \tag{3.3}$$

As remarked in section §1.4.2, dimensional analysis shows that $B = (\lambda_1^\star/\lambda_1^{A,c}(\Omega))^{1/2} \; \Omega^\star$, therefore $B \subseteq \Omega^\star$ and $u_1^\star \geqslant 0$ on ∂B, with equality if and only if $\lambda_1^{A,c} = \lambda_1^\star$; moreover, V_1 is related to U_1 by scaling.

In the present chapter we prove two stability-type theorems for the symmetrized first eigenfunction of problem (3.1).
In our first result, we show that the difference between the Schwarz

rearrangement of such a function and the first eigenfunction of problem (3.2) can be estimated in terms of the difference between the corresponding eigenvalues; more precisely, if we denote the Schwarz rearrangement of u_1 with $u_1^\star$ (see §2), we have the following:

THEOREM 3.1
Let $\Omega \subseteq \mathbb{R}^N$ *be a bounded domain with unit measure,* $\lambda_1^{A,c}$*,* u_1 *and* $\lambda_1^\star$*,* U_1 *be the first eigenvalue and the first eigenfunction of* (3.1) *and* (3.2) *respectively.*
There exist two positive constants $\delta_1 = \delta_1(N)$ *and* $C_1 = C_1(N)$ *depending only on* N *such that:*

$$\lambda_1^{A,c} - \lambda_1^\star \leqslant \delta_1 \;\Rightarrow\; \|u_1^\star - U_1\|_{\infty,\Omega^\star} \leqslant C_1 \,(\lambda_1^{A,c} - \lambda_1^\star)^{2/(N+2)} \,. \tag{3.4}$$

The second result gives an estimate for the difference between the Schwarz rearrangement $u_1^\star$ and the first eigenfunction V_1 of the Dirichlet Laplacian in B in terms of the value of $u_1^\star$ on the boundary of B; more precisely:

THEOREM 3.2
Let Ω*,* $\lambda_1^{A,c}$ *and* u_1 *be as in Theorem 3.1 and let* B*,* V_1 *be as in* (3.3).
Assume that $u_1^\star = \varepsilon > 0$ *on* ∂B.
There exist two positive constants $\delta_2 = \delta_2(N)$ *and* $C_2 = C_2(N)$ *depending only on* N *such that:*

$$\varepsilon \leqslant \delta_2 \;\Rightarrow\; \|u_1^\star - V_1\|_{\infty,B} \leqslant C_2\, \varepsilon^{2/(N+2)} \,. \tag{3.5}$$

In the $N = 2$ case the above results have both already been proved in [13] and our Theorems 3.1 and 3.2 reduce to [13, Theorems 4.1 and 5.1] when the bidimensional eigenvalue problem is considered.

Stability properties of the first eigenvalue of Dirichlet elliptic operators with respect to variations of the domain has been studied, among others, in [12] and [38]; while stability of the eigenfunctions of (not necessarily linear) elliptic operators with differ-

ent kinds of boundary conditions has been recently addressed in some papers, as [15], [16], [35] and [9], and also in [54].
We also observe that results like Theorem 3.1 can be applied in different contexts; for example, in [14] authors used them to prove the sharpness of some Payne-Rayner type inequality for the solution of a Neumann eigenvalue problem in the plane.

The proofs of both Theorems rely on some classical symmetrization results, i.e. the comparison lemma by Chiti and the inequality by Talenti stated in §1.3.
In particular, Talenti's inequality can be succesfully used to find bounds for the L^∞ distance between the Schwarz rearrangements of u_1, U_1 and V_1: in order to do this we use the *method of maximal solutions* developed in [13] with some modifications.
Originally, i.e. in the case $N = 2$, this method consisted in building continuous decreasing functions z as solutions of a suitable IVP for a parametric integro-differential equation derived from Talenti's inequality and in proving suitable estimates for them. Such estimates were used to find upper bounds for the differences $u_1^\star - U_1$ and $u_1^\star - V_1$ via some elementary inequalities for the generalized inverse of the socalled maximal solution.
In the case $N > 2$, which is the one we mainly address here, we replace some bounds for the z with analogous bounds for their generalized inverses. Of course, this modification can be used also in the case $N = 2$.

3.1.2 *Organization*

This chapter is organized as follows.
In §3.2 we analyse two integro-differential boundary value problems arising from Talenti inequality: in particular, we prove existence result for a more general class of problems and then derive some fundamental properties and estimates for the functions z and their inverses.
These estimates, whose proofs rely onto the linear structure of the integro-differential equations, will be used in the proofs of

both Theorems 3.1 and 3.2.
Finally, in §3.3 we provide the proofs of our main results.

3.2 AN INTEGRO-DIFFERENTIAL PROBLEM

From Talenti inequality of chapter 1, we infer that the distribution function μ_1 of u_1 is an a.e. subsolution of the integro-differential initial value problem:

$$\begin{cases} \frac{c_N^2}{\lambda_1^{A,c}}\, z^{2-2/N}(t) = -z'(t)\ \left(t\, z(t) + \int_t^1 z(\tau)\, d\,\tau\right) & \text{, in } [0,1] \\ z(0) = 1 & . \end{cases} \tag{3.6}$$

On the other hand, it is easy to prove that the distribution function of U_1 does solve the problem:

$$\begin{cases} \frac{c_N^2}{\lambda_1^{\star}}\, z^{2-2/N}(t) = -z'(t)\ \left(t\, z(t) + \int_t^1 z(\tau)\, d\,\tau\right) & \text{, in } [0,1] \\ z(0) = 1 \end{cases}$$

in the classical sense and it is also a classical subsolution of (3.6), because of the Faber-Krahn inequality (Theorem 1.11 of chapter 1); using Talenti's inequality, we can also prove that the distribution function of V_1 solves in the classical sense the problem:

$$\begin{cases} \frac{c_N^2}{\lambda_1^{A,c}}\, z^{2-2/N}(t) = -z'(t)\ \left(t\, z(t) + \int_t^{|B|_N} z(\tau)\, d\,\tau\right) & \text{, in } [0,|B|_N] \\ z(0) = 1 & , \end{cases} \tag{3.7}$$

and Chiti's lemma states that the two aforementioned solutions of problems (3.6) and (3.7) are pointwise comparable and the second is less than the first.
Therefore it seems reasonable to introduce a parameter $\lambda \geqslant 0$ in place of $\lambda_1^{A,c}$ in (3.6) with the aim of studying the relations between solutions of the parametric problem corresponding to different values of λ.
In what follows we prove the existence of solutions for a more general class of parametric problems, as well as some properties

of the solutions which will be useful to prove our main results.

3.2.1 *Existence of positive solutions*

We are led to consider the integro-differential problem:

$$\begin{cases} \frac{c_N^2}{\lambda}\, z^{2-2/N}(t) = -z'(t)\ \left(t\, z(t) + \int_t^1 z(\tau)\, d\tau\right) & \text{, for } t \in [0,1[\\ z(0) = 1 \end{cases}, \tag{3.8}$$

hence we have to provide a suitable definition of solution for (3.8):

DEFINITION 3.1: We say that a function $z : [0,1] \to \mathbb{R}$ is a *positive solution of* (3.8) if it is continuous in $[0,1]$, positive in $[0,1[$, of class $C^1([0,1[)$ and satisfies (3.8).

REMARK 3.1:
As a matter of fact, a positive solution z *of* (3.8) *satisfies also the integral equation:*

$$\ln z(t) = -\frac{4\pi}{\lambda}\int_0^t \left(\tau z(\tau) + \int_\tau^1 z(\vartheta)\, d\vartheta\right)^{-1} d\tau$$
$$\frac{\lambda}{c_N^2}\ \frac{N}{N-2} z^{2/N-1}(t) = \frac{\lambda}{c_N^2}\ \frac{N}{N-2} + \int_0^t \left(\tau z(\tau) + \int_\tau^1 z(\vartheta)\, d\vartheta\right)^{-1} d\tau \tag{3.9}$$

(the former if $N = 2$*, the latter if* $N \geqslant 3$*) which is obtained integrating* (3.8) *over* $[0,t]$*; on the other hand, if* $z \in C([0,1])$ *is positive in* $[0,1[$ *and satisfies* (3.9)*, then* $z \in C^1([0,1[)$ *and it is a positive solution of* (3.13)*.* ♢

Remark 3.1 motivates the following general existence result, from which we can recover the existence of positive solution of (3.8) and other related problems:

THEOREM 3.3
Let β, μ, t_0 *and* ζ_0 *satisfy the following assumptions:*

(i) $\beta : [0, +\infty[\to [0, +\infty[$ *and* $\varphi : [0, +\infty[\to]0, +\infty[$ *continuous functions;*

(ii) $\mu : [0, 1] \to [0, +\infty[$ *a continuous decreasing function such that.:*

$$(1 + \beta(t))\ \varphi(\mu(t)) \leqslant -\mu'(t) \left(t\mu(t) + \int_t^1 \mu(\tau)\,d\tau \right) \quad \textit{a.e. in } [0, 1];$$

(iii) $t_0 \in [0, 1[$ *and* $\zeta_0 \in [\mu(t_0), +\infty[$.

Then there exist functions z *defined in* $[t_0, 1]$ *which are positive in* $[t_0, 1[$*, of class* $C([t_0, 1]) \cap C^1([t_0, 1[)$ *and solve problem:*

$$\begin{cases} (1 + \beta(t))\ \varphi(z(t)) = -z'(t) \left(tz(t) + \int_t^1 z(\tau)\,d\tau \right) & \textit{, in } [t_0, 1[\\ z(t_0) = \zeta_0 & . \end{cases} \tag{3.10}$$

Moreover, there exists a unique solution $Z(t)$ *of* (3.10)*, called the* maximal solution*, s.t.* $z(t) \leqslant Z(t)$ *in* $[t_0, 1]$ *for each other solution* $z(t)$ *of* (3.10).

Proof. The proof is based on properties of monotone functions and on an iterative method of Picard type, whose convergence to a nonnegative function is guaranteed by (ii).
Define:

$$\Phi(z) := \int_{\zeta_0}^{z} \frac{du}{\varphi(u)}\,,$$

thus Φ is a positive increasing function.
Arguing as in Remark 3.1, a positive continuous function z is a solution of equation (3.10) if and only if it solves:

$$\Phi(z(t)) = \Phi(\zeta_0) - \int_{t_0}^{t} (1 + \beta(\tau)) \left(\tau z(\tau) + \int_\tau^1 z(\vartheta)\,d\vartheta \right)^{-1} d\tau \tag{3.11}$$

for $t \in [t_0, 1]$, therefore it suffices to prove that equation (3.11) has a solution in $C([t_0, 1]) \cap C^1([t_0, 1[)$ positive in $[t_0, 1[$ to have the claim.

Set $z_0(t) := \zeta_0$ in $[t_0, 1]$ and, for each $n \in \mathbb{N}$, let z_{n+1} be defined by:

$$\Phi(z_{n+1}(t)) := \Phi(\zeta_0) - \int_{t_0}^{t} (1 + \beta(\tau)) \left(\tau z_n(\tau) + \int_{\tau}^{1} z_n(\vartheta)\, d\vartheta \right)^{-1} d\tau\,.$$

By induction we see that $z_n(t) \geqslant \mu(t)$ for each $n \geqslant 0$ and $t \in [t_0, 1[$. In fact, $z_0(t) = \zeta_0 \geqslant \mu(t_0) \geqslant \mu(t)$ for (iii). Now, assume that for some n we have $z_n(t) \geqslant \mu(t)$ for all $t \in [t_0, 1]$; then from (ii) and known properties of monotone functions we derive:

$$\begin{aligned}
\Phi(\mu(t_0)) - \Phi(\mu(t)) &\geqslant - \int_{t_0}^{t} [\Phi(\mu(\tau))]'\, d\tau \\
&= \int_{t_0}^{t} \frac{\mu'(\tau)}{\varphi(\mu(\tau))}\, d\tau \\
&\geqslant \int_{t_0}^{t} (1 + \beta(\tau)) \left(\tau\mu(\tau) + \int_{\tau}^{1} \mu(\vartheta)\, d\vartheta \right)^{-1} d\tau\,;
\end{aligned}$$

since we are assuming $z_n(t) \geqslant \mu(t)$ we have:

$$\left(\tau\mu(\tau) + \int_{\tau}^{1} \mu(\vartheta)\, d\vartheta \right)^{-1} \geqslant \left(\tau z_n(\tau) + \int_{\tau}^{1} z_n(\vartheta)\, d\vartheta \right)^{-1}$$

thus from the definition of z_{n+1} we infer:

$$\begin{aligned}
\Phi(\mu(t)) &\leqslant \Phi(\mu(t_0)) - \int_{t_0}^{t} (1 + \beta(\tau)) \left(\tau\mu(\tau) + \int_{\tau}^{1} \mu(\vartheta)\, d\vartheta \right)^{-1} d\tau \\
&\leqslant \Phi(\zeta_0) - \int_{t_0}^{t} (1 + \beta(\tau)) \left(\tau z_n(\tau) + \int_{\tau}^{1} z_n(\vartheta)\, d\vartheta \right)^{-1} d\tau \\
&= \Phi(z_{n+1}(t))\,,
\end{aligned}$$

therefore $z_{n+1}(t) \geqslant \mu(t)$ as we claimed. In particular, $z_n(t) > 0$ in $[0, 1[$.

Using the same argument, one can prove that $z_n(t_0) = \zeta_0$, that $z_n(t)$ is continuous and decreasing in $[t_0, 1]$ and that $z_n(t) \geqslant$

$z_{n+1}(t)$ in $[t_0, 1]$ for each index n.
Since z_n is monotone decreasing we can set $Z(t) := \lim_n z_n(t)$: function Z decreases and satisfies $Z(t) \geqslant \mu(t)$ in $[t_0, 1[$, thus Z is positive and satisfies equation (3.11); a bootstrap argument shows that Z is in fact of class $C^\infty([t_0, 1[)$; moreover $Z(t_0) = \lim_n z_n(t_0) = \zeta_0$ and $Z(t)$ is continuous up to 1.

Now, let z be any solution of (3.10) of class $C([t_0, 1]) \cap C^1([t_0, 1[)$ positive in $[t_0, 1[$. Replacing $\mu(t)$ with $z(t)$ in the previous argument, we have $Z(t) \geqslant z(t)$ in $[t_0, 1]$; thus $Z(t)$ is also the maximal solution of (3.10). □

REMARK 3.2:
Theorem 3.3 proves the existence of both positive solutions and maximal solution for problem (3.8). *In fact, it suffices to set* $\beta(t) := 0$, $\varphi(t) := \frac{c_N^2}{\lambda} t^{2-2/N}$ *to recover* (3.8) *from* (3.10)*; therefore we have to take* $\mu(t) = \mu_1(t)$, $t_0 = 0$, $\zeta_0 = 1$ *in order to apply the theorem to* (3.8).
From now on the maximal solution of problem (3.8) *corresponding to the parameter* λ *will be denoted by* Z_λ. ♢

REMARK 3.3:
In what follows we will consider, together with solutions z *and maximal solution* Z_λ *of problem* (3.8)*, also the maximal solution* G_ε *of the problem:*

$$\begin{cases} \frac{c_N^2}{\lambda_1^{A,c}} g^{2-2/N}(t) = -g'(t)\left(tg(t) + \int_t^1 g(\tau)\,d\tau\right) & , \textit{in } [\varepsilon, 1[\\ g(\varepsilon) = (\lambda_1^\star/\lambda_1^{A,c})^{N/2} \end{cases}, \tag{3.12}$$

because estimates for solutions of (3.12) *will be needed in the proof of Theorem 3.2.*
Existence of G_ε *is a straightforward consequence of Theorem 3.3, for it suffices to set* $t_0 = \varepsilon$, $\zeta_0 = (\lambda_1^\star/\lambda_1^{A,c})^{N/2}$ *and* $\varphi(t) := \frac{c_N^2}{\lambda_1^{A,c}} t^{2-2/N}$.
♢

3.2.2 *Some properties and estimates of positive solutions*

All positive solutions of (3.8) share some good properties. The first ones are listed in the following:

PROPOSITION 3.1
Let $z(t)$ *be positive solution of* (3.8).
Then $z(t)$ *is strictly decreasing and strictly convex in* $[0,1]$ *and it is* C^∞ *in* $[0,1[$. *Moreover:*

$$z'(1^-) = \begin{cases} -\frac{4\pi}{\lambda} & \text{, if } N = 2 \\ -\frac{c_N^2}{\lambda}\, z^{1-2/N}(1) & \text{, if } N \geqslant 3. \end{cases}$$

Proof. For $N = 2$ the result is proved in [13, Theorem 2.1], hence here we focus on the case $N \geqslant 3$.
From the equation in problem (3.8) it follows $z'(t) < 0$ in $[0,1[$, thus z decreases strictly in $[0,1]$.
Bootstrapping we see that z is of class $C^\infty([0,1[)$.
Moreover $t\, z(t) + \int_t^1 z(\tau)\, d\tau > 0$ for $t \in [0,1[$ and we can write equation (3.8) as:

$$-z'(t) = \frac{c_N^2}{\lambda}\, z^{2-2/N}(t)\, \left(t\, z(t) + \int_t^1 z(\tau)\, d\tau \right)^{-1} \quad ;$$

differentiating both sides, previous equation gives:

$$\begin{aligned} -z''(t) = \frac{c_N^2}{\lambda} z'(t)\, z^{1-2/N}(t)\, \Big(t\, z(t) + \int_t^1 z(\tau)\, d\tau \Big)^{-2} \\ \Big((1 - \tfrac{2}{N})\, t\, z(t) + 2(1 - \tfrac{1}{N}) \int_t^1 z(\tau)\, d\tau \Big)\,, \end{aligned}$$

therefore $z''(t) > 0$ for $t \in]0,1[$ and z is strictly convex in $[0,1]$.
Finally, assume $N \geqslant 3$. If $z(1) \neq 0$, then equality $z'(1^-) = \lim_{t \to 1^-} z'(t) = -\frac{c_N^2}{\lambda}\, z^{1-2/N}(1)$ follows straightforwardly from (3.8). On the other hand, if $z(1) = 0$ we compute:

$$\lim_{t \to 1^-} \frac{z'(t)}{-\frac{c_N^2}{\lambda}\, z^{1-2/N}(t)} = \lim_{t \to 1^-} \frac{z(t)}{tz(t) + \int_t^1 z(\tau)\, d\tau} = \lim_{t \to 1^-} \frac{z'(t)}{tz'(t)} = 1\,,$$

hence relation $z'(1^-) = -\frac{c_N^2}{\lambda}\, z^{1-2/N}(1) = 0$ follows. □

REMARK 3.4:
The same hold for positive solutions of problem (3.12)*: actually, if* g *is a positive solution of* (3.12)*, then it is strictly decreasing and convex in* $[\varepsilon, 1]$ *and* $g'(1^-) = -\frac{c_N^2}{\lambda_1^{\Lambda,c}} g(1)$. ♢

The following theorem generalizes [13, Theorem 3.1]:

THEOREM 3.4
Let $z(t)$ *be a positive solution of* (3.8)*. Then:*

$$z^{(N+2)/(2N)}(1)\; J_{\frac{N}{2}+1}\left(\sqrt{\lambda}\,\omega_N^{-1/N}\, z^{1/N}(1)\right) = -J_{\frac{N}{2}-1}\left(\sqrt{\lambda}\,\omega_N^{-1/N}\right) \int_0^1 z(t)\,dt\,. \tag{3.13}$$

Proof. Let $\alpha > 0$. Multiplying the equation in (3.8) by $z^\alpha(t)$ and integrating both sides we get:

$$\begin{aligned}
\frac{c_N^2}{\lambda}\int_0^1 & z^{2-2/N+\alpha}(t)\,dt \\
&= -\int_0^1 z^\alpha(t)\, z'(t)\, \left(t\, z(t) + \int_t^1 z(\tau)\,d\tau\right) dt \\
&= -\frac{1}{\alpha+1}\left(\left[z^{\alpha+1}(t)\left(t\, z(t) + \int_t^1 z(\tau)\,d\tau\right)\right]_0^1 \right.\\
&\qquad \left. - \int_0^1 t\, z^{\alpha+1}(t) z'(t)\,dt\right) \\
= -\frac{1}{\alpha+1}&\left(z^{\alpha+2}(1) - \int_0^1 z(t)\,dt - \frac{1}{\alpha+2}\left[t\, z^{\alpha+2}(t)\right]_0^1 \right.\\
&\left. + \frac{1}{\alpha+2}\int_0^1 z^{\alpha+2}(t)\,dt\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\alpha+1}\left(z^{\alpha+2}(1) - \int_0^1 z(t)\,dt - \frac{1}{\alpha+2} z^{\alpha+2}(1)\right.\\
&\qquad \left. + \int_0^1 z^{\alpha+2}(t)\,dt\right)\\
&= \frac{1}{\alpha+1}\int_0^1 z(t)\,dt - \frac{1}{\alpha+2} z^{\alpha+2}(1)\\
&\qquad - \frac{1}{(\alpha+1)(\alpha+2)}\int_0^1 z^{\alpha+2}(t)\,dt\,,
\end{aligned}
$$

hence, setting $\mathcal{I}_\alpha := \int_0^1 z^{2-2/N+\alpha}(t)\,dt$, previous equality reads:

$$
\mathcal{I}_\alpha = \frac{\lambda}{c_N^2}\left(\frac{1}{\alpha+1}\,\mathcal{I}_{\frac{2}{N}-1} - \frac{1}{\alpha+2}\,z^{\alpha+2}(1) - \frac{1}{(\alpha+1)(\alpha+2)}\,\mathcal{I}_{\alpha+\frac{2}{N}}\right). \tag{3.14}
$$

We now use (3.14) recursively. We start setting $\alpha = \alpha_1 := \frac{2}{N} - 1$ in (3.14): rearranging we write:

$$
\left(1 - \frac{\lambda}{c_N^2 A_1}\right)\mathcal{I}_{\alpha_1} = -\frac{\lambda}{c_N^2 B_1}\,z^{\alpha_1+2}(1) - \frac{\lambda}{c_N^2 A_1 B_1}\,\mathcal{I}_{\alpha_2}\,, \tag{3.15}
$$

where $A_1 := \alpha_1 + 1$, $B_1 := \alpha_1 + 2$ and $\alpha_2 := \alpha_1 + \frac{2}{N}$; setting $\alpha = \alpha_2$ in (3.14) yields:

$$
\mathcal{I}_{\alpha_2} = \frac{\lambda}{c_N^2}\left(\frac{1}{\alpha_2+1}\,\mathcal{I}_{\alpha_1} - \frac{1}{\alpha_2+2}\,z^{\alpha_2+2}(1) - \frac{1}{(\alpha_2+1)(\alpha_2+2)}\,\mathcal{I}_{\alpha_3}\right), \tag{3.16}
$$

where $\alpha_3 := \alpha_2 + \frac{2}{N} = \alpha_1 + \frac{4}{N}$, hence plugging the lefthand side of (3.16) in (3.15) we get:

$$
\begin{aligned}
\left(1 - \frac{\lambda}{c_N^2 A_1} + \frac{\lambda^2}{c_N^4 A_2 B_1}\right)\mathcal{I}_{\alpha_1} &= -\frac{\lambda}{c_N^2 B_1}\,z^{\alpha_1+2}(1)\\
&\quad + \frac{\lambda^2}{c_N^4 A_1 B_2}\,z^{\alpha_2+2}(1) + \frac{\lambda^2}{c_N^4 A_2 B_2}\,\mathcal{I}_{\alpha_3}\,,
\end{aligned} \tag{3.17}
$$

with $A_2 := A_1(\alpha_2+1)$, $B_2 := B_1(\alpha_2+2)$. Setting $\alpha = \alpha_3$ in (3.14) we get:

$$\mathcal{J}_{\alpha_3} = \frac{\lambda}{c_N^2}\left(\frac{1}{\alpha_3+1}\,\mathcal{J}_{\alpha_1} - \frac{1}{\alpha_3+2}\,z^{\alpha_3+2}(1) - \frac{1}{(\alpha_3+1)(\alpha_3+2)}\,\mathcal{J}_{\alpha_4}\right) \tag{3.18}$$

where $\alpha_4 := \alpha_3 + \frac{2}{N} = \alpha_1 + \frac{6}{N}$, therefore plugging the lefthand side of (3.18) into (3.17) and rearranging, we find:

$$\begin{aligned} &\left(1 - \frac{\lambda}{c_N^2 A_1} + \frac{\lambda^2}{c_N^4 A_2 B_1} - \frac{\lambda^3}{c_N^6 A_3 B_2}\right)\mathcal{J}_{\alpha_1} \\ &= -\frac{\lambda\, z^{\alpha_1+2}(1)}{c_N^2 B_1} + \frac{\lambda^2\, z^{\alpha_2+2}(1)}{c_N^4 A_1 B_2} \\ &\qquad - \frac{\lambda^3\, z^{\alpha_3+2}(1)}{c_N^6 A_2 B_3} - \frac{\lambda^3}{c_N^6 A_3 B_3}\,\mathcal{J}_{\alpha_4}\,, \end{aligned} \tag{3.19}$$

with $A_3 := A_2(\alpha_3+1)$ and $B_3 := B_2(\alpha_3+2)\ldots$
After n steps we get:

$$\begin{aligned} \sum_{k=0}^{n}(-1)^k \frac{\lambda^k}{c_N^{2k} A_k B_{k-1}}\,\mathcal{J}_{\alpha_1} &= \sum_{k=1}^{n}(-1)^k \frac{\lambda^k z^{\alpha_k+2}(1)}{c_N^{2k} A_{k-1} B_k} \\ &\quad + (-1)^n \frac{\lambda^n}{c_N^{2n} A_n B_n}\,\mathcal{J}_{\alpha_{n+1}} \end{aligned} \tag{3.20}$$

where:

$$A_n := \begin{cases} 1 & \text{, if } n \leqslant 0 \\ \prod_{h=1}^{n}(\alpha_h+1) & \text{, if } n \geqslant 1 \end{cases}, \tag{3.21}$$

$$B_n := \begin{cases} 1 & \text{, if } n \leqslant 0 \\ \prod_{h=1}^{n}(\alpha_h+2) & \text{, if } n \geqslant 1 \end{cases}. \tag{3.22}$$

In fact, (3.20) holds when $n = 1, 2, 3$ by (3.15), (3.17) and (3.19). If we assume (3.20) holds for n then from (3.20) and (3.14) with $\alpha = \alpha_{n+1} = \alpha_1 + 2(n+1)/N$, i.e.:

$$\mathcal{J}_{\alpha_{n+1}} = \frac{\lambda}{c_N^2}\left(\frac{1}{\alpha_{n+1}+1}\,\mathcal{J}_{\alpha_1} - \frac{1}{\alpha_{n+1}+2}\,z^{\alpha_{n+1}+2}(1) - \frac{1}{(\alpha_{n+1}+1)(\alpha_{n+1}+2)}\,\mathcal{J}_{\alpha_{n+2}}\right) \tag{3.23}$$

where $\alpha_{n+2} = \alpha_{n+1} + 2/N$, we get:

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\frac{\lambda^k}{c_N^{2k}A_kB_{k-1}}\,\mathcal{J}_{\alpha_1} &= \sum_{k=1}^{n}(-1)^k\frac{\lambda^k z^{\alpha_k+2}(1)}{c_N^{2k}A_{k-1}B_k}\\ &\quad+(-1)^n\frac{\lambda^{n+1}}{c_N^{2(n+1)}A_nB_n}\times\\ &\quad\times\left(\frac{1}{\alpha_{n+1}+1}\mathcal{J}_{\alpha_1} - \frac{1}{\alpha_{n+1}+2}z^{\alpha_{n+1}+2}(1)\right.\\ &\quad\left. - \frac{1}{(\alpha_{n+1}+1)(\alpha_{n+1}+2)}\mathcal{J}_{\alpha_{n+2}}\right)\end{aligned}$$

which, after some algebra, becomes (3.20) for $n+1$. This completes the justification by induction of (3.20).
Now, it is easy to see that:

$$\alpha_n = \frac{2n}{N} - 1 \quad \text{for } n = 1, 2, \ldots \tag{3.24}$$

hence:

$$A_n = \prod_{h=1}^{n}\frac{2h}{N} = \frac{2^n}{N^n}\,n! \qquad \text{for } n = 1, 2, \ldots \tag{3.25}$$

$$B_n = \prod_{h=1}^{n}\left(\frac{2h}{N}+1\right) = \frac{2^n}{N^n}\,\frac{\Gamma(n+1+N/2)}{\Gamma(1+N/2)} \quad \text{for } n = 1, 2, \ldots \tag{3.26}$$

therefore (3.20) reads:

$$\left(1 + \sum_{k=0}^{n}(-1)^k\frac{\lambda^k N^{2k-1}\Gamma(1+N/2)}{c_N^{2k}2^{2k-1}k!\,\Gamma(k+N/2)}\right)\,\mathcal{J}_{\frac{2}{N}-1}$$

$$= \sum_{k=1}^{n} (-1)^k \frac{\lambda^k N^{2k-1} \Gamma(1+N/2)}{c_N^{2k} 2^{2k-1}(k-1)!\,\Gamma(1+k+N/2)} z^{1+2k/N}(1)$$
$$+ (-1)^n \frac{\lambda^n N^{2n} \Gamma(1+N/2)}{c_N^{2n} 2^{2n} n!\,\Gamma(1+n+N/2)} \mathcal{I}_{\frac{2(n+1)}{N}-1}\,.$$

Remembering that $c_N = N\omega_N^{1/N}$, we obtain:

$$\left(1 + \sum_{k=1}^{n} (-1)^k \frac{\Gamma(1+N/2)}{k!\,\frac{N}{2}\Gamma(k+N/2)} \left(\frac{\sqrt{\lambda}}{2\omega_N^{1/N}}\right)^{2k}\right) \mathcal{I}_{\frac{2}{N}-1}$$
$$= \sum_{k=1}^{n} (-1)^k \frac{\lambda^k \Gamma(1+N/2)}{\omega_N^{2k/N} 2^{2k-1} \frac{N}{2}(k-1)!\,\Gamma(1+k+N/2)} z^{1+2k/N}(1)$$
$$+ (-1)^n \frac{\lambda^n \Gamma(1+N/2)}{\omega_N^{2n/N} 2^{2n} n!\,\Gamma(1+n+N/2)} \mathcal{I}_{\frac{2(n+1)}{N}-1} \quad (3.27)$$

and, since $y\Gamma(y) = \Gamma(1+y)$, after some algebra we find:

$$\left(\sum_{k=0}^{n} (-1)^k \frac{\Gamma(1+N/2)}{k!\,\Gamma(1+k+N/2)} \left(\frac{\sqrt{\lambda}}{2\omega_N^{1/N}}\right)^{2k}\right) \mathcal{I}_{\frac{2}{N}-1}$$
$$= \sum_{k=1}^{n} (-1)^k \frac{\lambda^k \Gamma(1+N/2)}{\omega_N^{2k/N} 2^{2k-1}(k-1)!\,\Gamma(2+k+N/2)} z^{1+2k/N}(1)$$
$$+ (-1)^n \frac{\lambda^n \Gamma(1+N/2)}{\omega_N^{2n/N} 2^{2n} n!\,\Gamma(1+n+N/2)} \mathcal{I}_{\frac{2(n+1)}{N}-1}\,. \quad (3.28)$$

Because $0 \leqslant z(t) \leqslant 1$ in $[0,1]$, using *dominated convergence* and the asymptotic $\Gamma(n+1+N/2) \approx \sqrt{2\pi}\, e^{-n} n^{n+(N+1)/2}$ ([1], §6.1.39), we can pass to the limit for $n \to \infty$ in (3.28) to get:

$$\left(\sum_{k=0}^{+\infty} (-1)^k \frac{\Gamma(1+N/2)}{k!\,\Gamma(1+k+N/2)} \left(\frac{\sqrt{\lambda}}{2\omega_N^{1/N}}\right)^{2k}\right) \mathcal{I}_{\frac{2}{N}-1}$$
$$= \sum_{k=1}^{+\infty} (-1)^k \frac{\lambda^k \Gamma(1+N/2)}{\omega_N^{2k/N} 2^{2k-1}(k-1)!\,\Gamma(2+k+N/2)} z^{1+2k/N}(1)\,; \quad (3.29)$$

remembering the series expansions for Bessel functions of the first kind [65], (3.29) reads:

$$\frac{\Gamma(1+N/2)2^{N/2}}{N}\left(\frac{\sqrt{\lambda}}{\omega_N^{1/N}}\right)^{1-N/2} J_{\frac{N}{2}-1}\left(\sqrt{\lambda}\,\omega_N^{-1/N}\right)\mathcal{I}_{\frac{2}{N}-1}$$

$$= -\frac{\Gamma(1+N/2)2^{N/2}}{N}\left(\frac{\sqrt{\lambda}}{\omega_N^{1/N}}\right)^{1-N/2} z^{(N+2)/(2N)}(1)\times$$

$$\times J_{\frac{N}{2}+1}\left(\frac{\sqrt{\lambda}z^{1/N}(1)}{\omega_N^{1/N}}\right), \quad (3.30)$$

which, after some more algebra, is the claim. □

Relation (3.13) can be generalized further: for example, if in Theorem 3.4 we integrate from a variable endpoint t to 1 (instead of from 0 to 1) we obtain:

THEOREM 3.5
Let λ and $z(t)$ satisfy the hypotheses of Theorem 3.4. Then:

$$t\,z(t)\;J_{\frac{N}{2}+1}\left(\sqrt{\lambda}\,\omega_N^{-1/N}\;z^{1/N}(t)\right)$$
$$= J_{\frac{N}{2}-1}\left(\sqrt{\lambda}\,\omega_N^{-1/N}\;z^{1/N}(t)\right)\int_t^1 z(\tau)\,d\tau$$
$$- z^{1/2-1/N}(t)\,J_{\frac{N}{2}-1}\left(\sqrt{\lambda}\,\omega_N^{-1/N}\right)\int_0^1 z(\tau)\,d\tau\,. \quad (3.31)$$

We explicitly remark that equalities of this type can be proved in a more general setting, e.g. for the positive solutions of problem (3.10) when $\varphi(t)$ is a power function with exponent greater than 1.
Formula (3.13) implies:

PROPOSITION 3.2
If $\lambda < \lambda_1^\star$ then (3.8) has no positive solutions.
Moreover, if z is a positive solution of (3.8) such that $z(1) = 0$ then $\lambda = \lambda_1^\star$ and $z = Z_{\lambda_1^\star}$.

The proof can be worked using the interlacing property of the zeros of Bessel functions $J_{N/2-1}$ and $J_{N/2+1}$ as in the proofs of [13, Theorem 3.3 and Corollary 3.1].

Equality (3.13) yields the following estimates that will be fundamental in the next section:

PROPOSITION 3.3
There exist two positive constants $\delta_1 = \delta_1(N), M_1 = M_1(N)$ *depending only on* N *such that:*

$$\lambda_1^\star \leqslant \lambda \leqslant \lambda_1^\star + \delta_1 \;\Rightarrow\; z(1) \leqslant M_1\,(\lambda-\lambda_1^\star)^{N/(N+2)}\,. \tag{3.32}$$

for each positive solution z *of* (3.8) *corresponding to* λ.

Proof. Let Z_λ be the maximal solution to (3.8) corresponding to λ; then $z(1) \leqslant Z_\lambda(1)$, hence it suffices to prove (3.32) with Z_λ replacing z.
By [11, Lemmata 3.1-3.3], the function $\lambda \mapsto Z_\lambda(1)$ decreases to zero when $\lambda \searrow \lambda_1^\star$.
Remembering the series expansion of $J_{\frac{N}{2}+1}$ and that $\sqrt{\lambda_1^\star}\,\omega_N^{-1/N}$ equals the first nontrivial zero of $J_{\frac{N}{2}-1}$, we obtain:

$$\lim_{\lambda\searrow\lambda_1^\star} \frac{Z_\lambda^{(N+2)/(2N)}(1)\; J_{\frac{N}{2}+1}\left(\sqrt{\lambda}\,\omega_N^{-1/N}\; Z_\lambda^{1/N}(1)\right)}{Z_\lambda^{(N+2)/N}(1)\,\left(\sqrt{\lambda}\,\omega_N^{-1/N}\right)^{(N+2)/2}} = 1\,,$$

therefore there exists $\delta_1 > 0$ such that:

$$\begin{aligned} Z_\lambda^{(N+2)/(2N)}(1)\; J_{\frac{N}{2}+1}&\left(\sqrt{\lambda}\,\omega_N^{-1/N}\; Z_\lambda^{1/N}(1)\right) \\ &\geqslant \frac{1}{2}\, Z_\lambda^{(N+2)/N}(1)\,\left(\sqrt{\lambda}\,\omega_N^{-1/N}\right)^{(N+2)/2}. \end{aligned}$$

on the other hand:

$$\begin{aligned} \left|J_{\frac{N}{2}-1}\left(\sqrt{\lambda}\,\omega_N^{-1/N}\right)\right| &= \left|J_{\frac{N}{2}-1}\left(\sqrt{\lambda}\,\omega_N^{-1/N}\right) - J_{\frac{N}{2}-1}\left(\sqrt{\lambda_1^\star}\,\omega_N^{-1/N}\right)\right| \\ &\leqslant c\,(\lambda-\lambda_1^\star)\,, \end{aligned}$$

where the positive constant:

$$c = \sup_{\lambda\geqslant\lambda_1^\star}\left|\frac{d}{d\lambda}\left[J_{\frac{N}{2}-1}\left(\sqrt{\lambda}/\omega_N^{1/N}\right)\right]\right| < +\infty$$

depends only on N.
Therefore using (3.13) we infer there exists $M_1 = 2c > 0$ such that inequality:

$$Z_\lambda^{(N+2)/(N)}(1) \leqslant M_1 \left(\sqrt{\lambda_1^\star}\ \omega_N^{-1/N}\right)^{-(N+2)/2} (\lambda - \lambda_1^\star)$$

holds when $\lambda \approx \lambda_1^\star$, which is our claim. □

Remark 3.5:
Using the same iterative scheme as in the proof of Theorem 3.4 we can prove that an equality of the same type of (3.13) *holds also for the positive solutions of* (3.12) *(and in particular for the maximal solution* G_ε*). Namely, we find:*

$$\begin{aligned}\left(\sqrt{\lambda_1^{A,c}}\ \omega_N^{-1/N}\zeta_0^{1/N}\right)^{1-N/2} & J_{\frac{N}{2}-1}\left(\sqrt{\lambda_1^{A,c}}\ \omega_N^{-1/N}\zeta_0^{1/N}\right)\times \\ & \times\int_\varepsilon^1 g(\tau)\,d\tau \\ & = \varepsilon\ \zeta_0\left(\sqrt{\lambda_1^{A,c}}\ \omega_N^{-1/N}\zeta_0^{1/N}\right)^{1-N/2}\times \\ & \qquad J_{\frac{N}{2}+1}\left(\sqrt{\lambda_1^{A,c}}\ \omega_N^{-1/N}\zeta_0^{1/N}\right) \\ & \quad - g(1)\left(\sqrt{\lambda_1^{A,c}}\ \omega_N^{-1/N}g^{1/N}(1)\right)^{1-2/N}\times \\ & \qquad J_{\frac{N}{2}+1}\left(\sqrt{\lambda_1^{A,c}}\ \omega_N^{-1/N}g^{1/N}(1)\right);\end{aligned} \tag{3.33}$$

and, remembering that $\zeta_0 = (\lambda_1^\star/\lambda_1^{A,c})^{N/2}$ *as in Remark 3.3, equality* (3.33) *yields:*

$$\begin{aligned}\varepsilon\ (\lambda_1^\star/\lambda_1^{A,c})^{N/2}\left(\sqrt{\lambda_1^\star}\ \omega_N^{-1/N}\right)^{1-N/2} & J_{\frac{N}{2}+1}\left(\sqrt{\lambda_1^\star}\ \omega_N^{-1/N}\right) \\ & = g(1)\left(\sqrt{\lambda_1^{A,c}}\ \omega_N^{-1/N}g^{1/N}(1)\right)^{1-N/2}\times \\ & \quad \times J_{\frac{N}{2}+1}\left(\sqrt{\lambda_1^{A,c}}\ \omega_N^{-1/N}g^{1/N}(1)\right),\end{aligned} \tag{3.34}$$

where $\sqrt{\lambda_1^\star}\omega_N^{-1/N} = j_{N/2-1,1}$ *is the first nonzero zero of* $J_{N/2-1}$. *Equation* (3.34) *is a generalization of the one in [13, Lemma 5.1].*
Thus, from the interlacing properties of the zeros of $J_{N/2-1}$ *and* $J_{N/2+1}$,

the formulas for the derivatives of $z^{N/2+1} J_{N/2+1}(z)$ *and the asymptotic of* $J_{N/2+1}$ *in* 0*, from* (3.34) *we infer* $\lim_{\varepsilon\searrow 0} g(1) = 0$ *and that there exist* $\delta_2 = \delta_2(N), M_2 = M_2(N) > 0$ *such that:*

$$0 \leqslant \varepsilon \leqslant \delta_2 \qquad \Rightarrow \qquad g(1) \leqslant G_\varepsilon(1) \leqslant M_2\, \varepsilon^{N/(N+2)} \,. \tag{3.35}$$

♢

3.2.3 *Estimates for the inverse functions of maximal solutions of* (3.8)

Let $\lambda \geqslant \lambda^\star$ and z a positive solution of (3.8) corresponding to λ; we denote ξ the *generalized inverse function* of z, i.e.:

$$\xi(s) := \begin{cases} \|z\|_\infty & , \text{if } s = 0 \\ \inf\{t \in [0,1] : \; z(t) < s\} & , \text{if } s \in]0,1], \end{cases} \tag{3.36}$$

and set $\bar{s} := z(1)$.

Function ξ is constant in the whole interval $[0,\bar{s}]$, which reduces to $\{0\}$ if and only if $z(1) = 0$, i.e. if and only if $\lambda = \lambda_1^\star$ and $z = Z_{\lambda_1^\star}$.

Furthermore, ξ is of class $C([0,1])$ and piecewise C^∞ in $]0,1[$, its only singular point being $\bar{s}$, and ξ is a positive decreasing classical solution of the following problem:

$$\begin{cases} s^{2-2/N}\, \xi'(s) = -\frac{\lambda}{c_N^2} \int_0^s \xi(\sigma)\, d\,\sigma & , \text{for } s \in]\bar{s},1[\\ \xi(s) = 1 & , \text{for } t \in [0,\bar{s}] \\ \xi(1) = 0 & , \end{cases} \tag{3.37}$$

which can be deduced from (3.8) via a change of variable; moreover Theorem 3.1 implies that ξ is strictly convex in $[\bar{s},1]$.

Obviously we can read Theorems 3.4 and 3.5 in terms of ξ instead of z, hence we have:

$$\begin{aligned} \bar{s}^{(N+2)/(2N)}\; J_{\frac{N+2}{2}}\left(\sqrt{\lambda}\, \omega_N^{-1/N}\, \bar{s}^{1/N}\right) & \qquad (3.38) \\ = -J_{\frac{N-2}{2}}\left(\sqrt{\lambda}\, \omega_N^{-1/N}\right) \int_0^1 \xi(s)\, d\,s\,, & \\ s\, \xi(s)\; J_{\frac{N}{2}+1}\left(\sqrt{\lambda}\, \omega_N^{-1/N}\, s^{1/N}\right) & \qquad (3.39) \end{aligned}$$

$$
\begin{aligned}
&= J_{\frac{N}{2}-1}\left(\sqrt{\lambda}\,\omega_N^{-1/N}\,s^{1/N}\right)\int_0^s \xi(\sigma)\,d\sigma \\
&\quad - s\,\xi(s)\;J_{\frac{N}{2}-1}\left(\sqrt{\lambda}\,\omega_N^{-1/N}\,s^{1/N}\right) \\
&\quad - s^{1/2-1/N}\;J_{\frac{N}{2}-1}\left(\sqrt{\lambda}\,\omega_N^{-1/N}\right)\int_0^1 \xi(\sigma)\,d\sigma\,;
\end{aligned}
$$

on the other hand Theorem 3.3 reads:

$$
\lambda_1^\star \leqslant \lambda \leqslant \lambda_1^\star + \delta_1 \qquad \Rightarrow \qquad \bar{s} \leqslant M_1 (\lambda - \lambda^\star)^{\frac{N}{N+2}}\,. \tag{3.40}
$$

Proposition 3.4

Let $\Lambda \geqslant \lambda \geqslant \lambda^\star$, ξ the generalized inverse of a positive solution of (3.8) corresponding to λ and Ξ the generalized inverse of the maximal solution Z_Λ; finally, let $\bar{s} < \bar{S} \in [0,1[$ be the discontinuity points of the derivatives of ξ and Ξ respectively.
Then:

$$
0 \leqslant \Xi(s) - \xi(s) \leqslant 1 - \xi(\bar{S}) \quad \textit{for } s \in [0,1]. \tag{3.41}
$$

Proof. By construction we have $\Xi(s) \geqslant \xi(s)$ in $[0,1]$, hence $\Xi(s) - \xi(s) \geqslant 0$.
By (3.37) we have $\Xi(s) = 1 = \xi(s)$ in $[0,\bar{s}]$ and $\xi(\bar{S}) \leqslant 1$ in $[0,1]$, therefore $\Xi(s) - \xi(s) = 0 \leqslant 1 - \xi(\bar{S})$.
If $s \in]\bar{s},\bar{S}]$ then $\Xi(s) = 1$ and $\xi(s) \geqslant \xi(\bar{S})$ (for $\xi(s)$ decreases), hence $\Xi(s) - \xi(s) \leqslant 1 - \xi(\bar{S})$ again.
Finally, if $s \in]\bar{S},1]$ then:

$$
\begin{aligned}
0 &\leqslant \frac{\lambda}{c_N^2}\int_{\bar{S}}^s \frac{1}{\sigma^{2-2/N}}\int_0^\sigma [\Xi(\vartheta) - \xi(\vartheta)]\,d\vartheta\,d\sigma \\
&\leqslant \frac{\Lambda}{c_N^2}\int_{\bar{S}}^s \frac{1}{\sigma^{2-2/N}}\int_0^\sigma \Xi(\vartheta)\,d\vartheta\,d\sigma - \frac{\lambda}{c_N^2}\int_{\bar{S}}^s \frac{1}{\sigma^{2-2/N}}\int_0^\sigma \xi(\vartheta)\,d\vartheta\,d\sigma \\
&= -\int_{\bar{S}}^s \Xi'(\sigma)\,d\sigma + \int_{\bar{S}}^s \xi(\sigma)\,d\sigma \\
&= [\Xi(\bar{S}) - \xi(\bar{S})] - [\Xi(s) - \xi(s)] \\
&= [1 - \xi(\bar{S})] - [\Xi(s) - \xi(s)]\,,
\end{aligned}
$$

which concludes the proof. $\square$

REMARK 3.6:
An analogous result holds when we consider the generalized inverse T *of the maximal positive solution* G_ε *of problem* (3.12) *and the rearrangement of the first eigenfunction* V_1 *of problem* (3.3). *In fact, the same argument applies with minor modifications and it yields:*

$$0 \leqslant T(s) - V_1^*(s) \leqslant 1 - V_1^*(G_\varepsilon(1)) \,.$$

$\diamondsuit$

Now, if we let $\lambda = \lambda_1^\star$ in Proposition 3.4 then ξ becomes the decreasing rearrangement of the first eigenfunction of the Laplacian in the ball $\Omega^\star$, say U_1^*; therefore we can write an explicit expression of $U_1^*(s)$ in terms of the Bessel function of the first kind, i.e.:

$$U_1^*(s) = k\, s^{1/N-1/2}\; J_{\frac{N}{2}-1}\left(\sqrt{\lambda_1^\star}\; \omega_N^{-1/N}\; s^{1/N}\right)$$

(where $k > 0$ is a suitable normalization constant) from which we infer that $U_1^*(s)$ is a Hölder continuous function with exponent $2/N$: in fact the derivative $(U_1^*)'$ is continuous in $]0,1]$ and behaves like $s^{2/N-1}$ around 0.
Hence we have:

PROPOSITION 3.5
Let $\Xi(s)$ *be the generalized inverse of the maximal solution* $Z_\lambda(t)$ *corresponding to* $\lambda \geqslant \lambda_1^\star$.
Then there exists a constant $\gamma_1 = \gamma_1(N) > 0$ *depending only on* N *such that inequality:*

$$\Xi(s) - U_1^*(s) \leqslant \gamma_1\; \bar{S}^{2/N} \tag{3.42}$$

holds for all $s \in [0,1]$

Proof. By Proposition 3.4 we have $\Xi(s) - U_1^*(s) \leqslant 1 - U_1^*(\bar{S})$. Now $1 = U_1^*(0)$ and $U_1^*(s)$ is Hölder continuous, hence $1 - U_1^*(\bar{S}) \leqslant \gamma_1\; \bar{S}^{2/N}$ for a suitable positive constant γ_1 depending only on N, and the claim follows. $\square$

REMARK 3.7:
Again, analogous result holds when we deal with the rearrangement of V_1^* *and the generalized inverse* T *of the maximal positive solution* G_ε *of problem* (3.12)*: in fact, we find:*

$$T(s) - V_1^*(s) \leqslant \gamma_2 \; G_\varepsilon^{2/N}(1)$$

where $\gamma_2 = \gamma_2(N) > 0$ *is a suitable constant depending only on* N. $\diamondsuit$

3.3 PROOFS OF THE MAIN RESULTS

3.3.1 *Proof of Theorem 3.1*

Proof. Let $\lambda_1^{A,c} - \lambda_1^\star \leqslant \delta_1$, with $\delta_1 > 0$ as in Theorem 3.3.
Let V be the generalized inverse of the maximal solution $Z_{\lambda^{A,c}}$ of (3.8).
Chiti's comparison lemma and the construction of $Z_{\lambda_1^{A,c}}$ imply that $V_1^*(s) \leqslant u_1^*(s) \leqslant V(s)$ in $[0,1]$ ($V^*(s)$ is extended to 0 outside $[0, |B|_N]$), hence:

$$\begin{aligned} \|u_1^\star - U_1\|_{\infty,\Omega^\star} &= \|u_1^* - U_1^*\|_{\infty,[0,1]} \\ &\leqslant \max\left\{\|V_1^* - U_1^*\|_{\infty,[0,1]}, \|V - U_1^*\|_{\infty,[0,1]}\right\} . \end{aligned} \tag{3.43}$$

Thus, in order to prove our claim it suffices to find suitable estimates for the two norms in the rightmost side of (3.43).

Let us consider $V - U_1^*$: since V is the generalized inverse of the maximal solution $Z_{\lambda^{A,c}}$, Proposition 3.5 and (3.40) apply and we get:

$$\|V - U_1^*\|_{\infty,[0,1]} \leqslant \gamma_1 \; M_1 \; (\lambda_1^{A,c} - \lambda_1^\star)^{2/(N+2)} . \tag{3.44}$$

Now we turn to $V_1^* - U_1^*$. Since V_1^* and U_1^* are both rearrangements of first eigenfunctions of the Laplacian in concentric balls, they are related via scaling:

$$V_1^*(s) = \begin{cases} U_1^*(hs) & , \text{if } s \in [0, h^{-1}[\\ 0 & , \text{if } s \in [h^{-1}, 1] \end{cases}$$

with $h := (\lambda_1^{A,c}/\lambda_1^\star)^{N/2} = |B|_N^{-1} > 1$; thus:

$$V_1^*(s) - U_1^*(s) = \begin{cases} U_1^*(hs) - U_1^*(s) & , \text{if } s \in [0, h^{-1}[\\ -U_1^*(s) & , \text{if } s \in [h^{-1}, 1] \end{cases} .$$

Now let $s \in [0, h^{-1}]$: using Hölder continuity of U_1^*, the elementary inequality:

$$(1-\tau^\alpha)^{1/\alpha} \leqslant \alpha^{1/\alpha}\,(1-\tau)^{1/\alpha} \qquad \text{for } 0 \leqslant \tau \leqslant 1 \text{ and } \alpha \geqslant 1 \quad (3.45)$$

and the Faber–Krahn inequality we find:

$$\begin{aligned} |V_1^*(s) - U_1^*(s)| &\leqslant \gamma_1\ (hs)^{2/N}|1-h^{-1}|^{2/N} \\ &\leqslant \gamma_1\ \left|1-(\lambda_1^\star/\lambda_1^{A,c})^{N/2}\right|^{2/N} \\ &\leqslant \gamma_1\ \left(\frac{N}{2\lambda_1^\star}\right)^{2/N}\ (\lambda_1^{A,c}-\lambda_1^\star)^{2/N} . \end{aligned}$$

On the other hand, if $s \in [h^{-1}, 1]$, then $V_1^*(s) - U_1^*(s) = -U_1^*(s) = U_1^*(1) - U_1^*(s)$ and using again Hölder continuity, (3.45) and the Faber–Krahn inequality we obtain:

$$\begin{aligned} V_1^*(s) - U_1^*(s) &\leqslant \gamma_1\ |1-s|^{2/N} \\ &\leqslant \gamma_1\ \left|1-\frac{1}{h}\right|^{2/N} \\ &\leqslant \gamma_1\ \left(\frac{N}{2\lambda_1^\star}\right)^{2/N}\ (\lambda_1^{A,c}-\lambda_1^\star)^{2/N} ; \end{aligned}$$

a comparison of the last two inequalities yields:

$$\|V_1^* - U_1^*\|_{\infty,[0,1]} \leqslant \gamma_1\ \left(\frac{N}{2\lambda_1^\star}\right)^{2/N}\ (\lambda_1^{A,c}-\lambda_1^\star)^{2/N} . \quad (3.46)$$

Thus inequalities (3.43), (3.44) and (3.46) imply:

$$\|u_1^\star - U_1\|_{\infty,\Omega} \leqslant C_1 \ (\lambda_1^{A,c} - \lambda_1^\star)^{2/(N+2)} \tag{3.47}$$

for some $C_1 > 0$ when $\lambda_1^{A,c} - \lambda_1^\star \leqslant \delta_1$, which was our claim. □

REMARK 3.8:
From Theorem 3.1 we deduce:

$$\|u_1^\star - U_1\|_{\infty,\Omega^\star} = O\left((\lambda_1^{A,c} - \lambda_1^\star)^{2/(N+2)}\right)$$

as $\lambda_1^{A,c} \searrow \lambda_1^\star$. ♢

3.3.2 *Proof of Theorem 3.2*

Proof. Let $\varepsilon \leqslant \delta_2$, with $\delta_2 > 0$ as in *Remark* 3.5.
Reasoning as in the previous proof, we find that the norm $\|u_1^\star - V_1\|_{\infty,B}$ is less than $\|T - V_1^*\|_{\infty,[0,\lambda_1^\star/\lambda_1^{A,c}]}$, which in turn can be controlled by $G_\varepsilon^{2/N}(1)$ times a suitable universal constant for small values of $G_\varepsilon(1)$ as stated in Remark 3.7.
Since $G_\varepsilon(1) \leqslant M_2 \ \varepsilon^{N/(N+2)}$ for small values of ε, as in Remark 3.5 (3.35), we have the claim. □

REMARK 3.9:
From Theorem 3.2 we deduce:

$$\|u_1^\star - V_1\|_{\infty,B} = O\left(\varepsilon^{2/(N+2)}\right)$$

as $\varepsilon = u_1^*((\lambda_1^\star/\lambda_1^{A,c})^{1/2}) \searrow 0$. ♢

4

A FABER–KRAHN INEQUALITY FOR THE FIRST WEIGHTED EIGENVALUE OF THE p-LAPLACIAN PLUS AN INDEFINITE POTENTIAL

INDEX

4.1 INTRODUCTION

4.1.1 *Motivations*

In the present chapter we deal with the following weighted eigenvalue problem for the p-Laplacian operator:

$$\begin{cases} \Delta_p u + V(x)\ |u|^{p-2}u = \lambda\ m(x)\ |u|^{p-2}u & , \text{in } \Omega \\ u = 0 & , \text{on } \partial\Omega \end{cases} \tag{4.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded open domain, $p \in]1, \infty[$ and the weight m and the potential V are measurable functions in Ω such that:

(H1) $m, V \in L^r(\Omega)$ with:

$$r \begin{cases} > N/p & , \text{if} 1 < p \leqslant N \\ = 1 & \text{otherwise;} \end{cases}$$

(H2) $m \geqslant 0$ a.e. in Ω.

Eigenvalue problems of the type (4.1) have attracted some interests in the last decade, for they arise as generalization of the classical p-Laplacian eigenvalue problem which have been extensively studied from the eighties (see [30, 39, 6, 50], just to mension a few) until today. In particular, the case $V = 0$ and m indefinite was studied in [26], where principality, simplicity, isolation in the spectrum properties of the first *positive* weighted eigenvalue of problem (4.1) were estabilished. More recently, in [27] the same properties were studied for the first eigenvalue in the *fully indefinite* case, i.e. the case with $m, V \neq 0$ both indefinite in sign. In the latter paper authors proved that, for general indefinite weights and potentials, problem (4.1) possesses some principal eigenvalues provided a certain variational quantity associated to m and V has positive or nonnegative infimum (depending on the sign of m) over a manifold contained in $W_0^{1,p}(\Omega)$; moreover, the first eigenvalue is variational, principal, simple and isolated. It is worth noticing that principal eigenvalue needs not to be unique; but nonuniqueness may arise only when the weight m is indefinite, which is *not* addressed here.

Under assumptions (H1) and (H2), the general result of [27] can be used to prove that problem (4.1) has a principal eigenvalue if and only if inequality:

$$\alpha_p(\Omega, V, m) = \inf\left\{ \int_\Omega \left(|\nabla u|^p + V(x)\,|u|^p\right)\, dx,\ \|u\|_{p,\Omega}^p = 1 \right.$$
$$\left. \text{and } \int_\Omega m(x)\,|u|^p\, dx = 0 \right\} > 0 \tag{4.2}$$

holds (see [27, Thm. 1 & 3-(i)]).
Assuming condition (4.2) is fulfilled, let $\lambda_p(\Omega, V, m)$ denote the first eigenvalue of problem (4.1). As said above, such an eigen-

value is also simple, isolated and variational, i.e. it is the minimum of a nonlinear Rayleigh quotient:

$$\lambda_p(\Omega, V, m) = \min \left\{ \frac{\int_\Omega |\nabla u|^p + V(x)\,|u|^p}{\int_\Omega m(x)\,|u|^p},\ u \in W_0^{1,p}(\Omega) \right.$$
$$\left. \text{and} \int_\Omega m(x)\,|u|^p > 0 \right\}. \tag{4.3}$$

Moreover, $\lambda_p(\Omega, V, m)$ is the only principal value of (4.1) and any other eigenvalue of (4.1) is strictly greater than it: hence we will usually refer to $\lambda_p(\Omega, V, m)$ as to *first weighted eigenvalue* of the operator $\mathcal{L}[u] = -\Delta_p u + V(x)|u|^{p-2}u$.

Since the situation is very similar to the one arising in the linear case (see chapter 1), in the spirit of the classical Faber-Krahn inequality we can ask whether or not there exist domains, potentials and weights which minimize the functional $\lambda_p(\cdot,\cdot,\cdot)$ under suitable "geometric–analytical" constraints.

This kind of minimization problem was firstly studied in [3] for the first eigenvalue of the p-Laplacian (i.e., in the case $m = 1$ and $V = 0$): in their paper, authors proved that, among domains sharing the same measure, the first eigenvalue is minimized by the ball alone.
Then, in [2] the weighted eigenvalue case with $1 < p < N$, $m > 0$ and $V = 0$ was addressed: under the aforementioned assumptions, authors showed that, among domains having the same measure and among equidistributed weights, the first weighted eigenvalue of the p-Laplacian is minimized in the *radially decreasing setting*, viz. when the domain Ω is a ball and the weight m (if not constant) is a radially symmetric and decreasing function.
More recently, work in another direction appeared in [29], in which authors proved that the first eigenvalue of problem with $m = 1$ a.e. in Ω can be maximize or minimized over a convex set of potentials $\mathcal{B} \subset L^r(\Omega)$ by choosing suitable potentials $V^*, V_* \in \mathcal{B}$.

Our aim is to extend the results of [3] and [2] to problem of the type (4.1).
In particular, we prove that, among domains sharing the same measure and among equidistributed weights and potentials, the eigenvalue $\lambda_p(\Omega, V, m)$ is minimized in the *radially symmetric setting*, i.e. when Ω is a ball and m, V are radially symmetric functions, the first decreasing and the latter increasing.
We stress that the main point here is that we work with a (possibly) *indefinite potential*. In other words, we do not assume $V \geqslant 0$ in order to get rid of the term $V(x)|u|^{p-2}u$ in the symmetrized problem (as we did in the prototype of Faber–Krahn inequality of chapter 1); instead, we prefer to keep memory of the sign of V in the symmetrization process.

For sake of precision, in what follows we will make a comparison between the first eigenvalue $\lambda_p(\Omega, V, m)$ of (4.1) and the first eigenvalues $\lambda_p(\Omega^\star, V_\bullet, m^\star)$ and $\lambda_p(\Omega^\star, -(V_-)^\star, m^\star)$ of the symmetrized problems:

$$\begin{cases} \Delta_p v + V_\bullet(x)\ |v|^{p-2}v = \lambda\ m^\star(x)\ |v|^{p-2}v & \text{, in } \Omega^\star \\ v = 0 & \text{, on } \partial\Omega^\star\ , \end{cases} \tag{4.4}$$

(where $V_\bullet$ and $m^\star$ are, respectively, the increasing signed radially symmetric rearrangement of the potential V and the classical Schwarz rearrangement of the weight m) and:

$$\begin{cases} \Delta_p w - (V_-)^\star(x)\ |w|^{p-2}w = \lambda\ m^\star(x)\ |w|^{p-2}w & \text{, in } \Omega^\star \\ w = 0 & \text{, on } \partial\Omega^\star\ , \end{cases} \tag{4.5}$$

(where $(V_-)^\star$ is the decreasing Schwarz rearrangement of the negative part of V and m^star is as above).
The main reason which leads us to consider such symmetrized problems is that they arise quite naturally when symmetrization inequalities are applied to the minimization of the Rayleigh quotient in (4.3).
Of course, in order to $\lambda_p(\Omega^\star, V_\bullet, m^\star)$ and $\lambda_p(\Omega^\star, -(V_-)^\star, m^\star)$ to

exist, problems (4.4) and (4.5) have to satisfy $\alpha_p(\Omega^\star, V_\bullet, m^\star) > 0$ and $\alpha_p(\Omega^\star, -(V_-)^\star, m^\star) > 0$, where α_p is defined as in the equation (4.2) above.

4.1.2 *Organization*

Our plan is the following.
We firstly give some motivations for the choice of the symmetrized problems (4.4) and (4.5). Then we prove that there actually exist potentials such that the three quantities $\alpha_p(\Omega, V, m)$, $\alpha_p(\Omega^\star, V_\bullet, m^\star)$ and $\alpha_p(\Omega^\star, -(V_-)^\star, m^\star)$ are positive, thus problems (4.1), (4.4) and (4.5) have unique principal eigenvalue.
Finally, we prove the Faber–Krahn inequalities and discuss the equality case.

4.2 ABOUT THE SYMMETRIZED PROBLEMS

4.2.1 *Construction of the Symmetrized Problems*

As we said above, the first eigenvalue of (4.1) is variational and it coincides with the minimum of the nonlinear Rayleigh quotient associated with (4.1), viz.:

$$\lambda_p(\Omega, V, m) = \min\left\{ \frac{\int_\Omega |\nabla u|^p + V(x)\,|u|^p}{\int_\Omega m(x)\,|u|^p},\ u \in W_0^{1,p}(\Omega) \right. \\ \left. \text{and } \int_\Omega m(x)\,|u|^p > 0 \right\}.$$

Aiming to minimize the functional $\lambda_p(\cdot,\cdot,\cdot)$ under the measure constraint for the domain and the equimeasurability constraint for both potential and weight, we can apply Hardy–Littlewood and Polya–Szegö inequalities to get:

$$\begin{aligned}\frac{\int_\Omega |\nabla u|^p + V(x)\ u^p}{\int_\Omega m(x)\ u^p} &\geqslant \frac{\int_{\Omega^\star} |\nabla u^\star|^p + V_\bullet(x)\ (u^\star)^p}{\int_\Omega^\star m^\star(x)\ (u^\star)^p} \\ &\geqslant \frac{\int_{\Omega^\star} |\nabla u^\star|^p - (V_-)^\star(x)\ (u^\star)^p}{\int_\Omega^\star m^\star(x)\ (u^\star)^p}\end{aligned}$$

for each positive eigenfunction $u \in W_0^{1,p}(\Omega)$; recalling that $u^\star \in W_0^{1,p}(\Omega^\star)$, previous inequalities entail:

$$\begin{aligned}\lambda_p(\Omega, V, m) \geqslant \inf &\left\{ \frac{\int_{\Omega^\star} |\nabla v|^p + V_\bullet(x)\ |v|^p}{\int_{\Omega^\star} m^\star(x)\ |v|^p},\ v \in W_0^{1,p}(\Omega^\star) \right. \\ &\quad \left. \text{and } \int_{\Omega^\star} m^\star(x)\ |v|^p > 0 \right\} \\ \geqslant \inf &\left\{ \frac{\int_{\Omega^\star} |\nabla w|^p - (V_-)^\star(x)\ |w|^p}{\int_{\Omega^\star} m^\star(x)\ |w|^p},\ w \in W_0^{1,p}(\Omega^\star) \right. \\ &\quad \left. \text{and } \int_{\Omega^\star} m^\star(x)\ |w|^p > 0 \right\}.\end{aligned}$$

The two latter members in previous inequalities are the infima of the nonlinear Rayleigh quotients associated to problems (4.4) and (4.5), hence they are finite and coincide with the principal eigenvalues $\lambda_p(\Omega^\star, V_\bullet, m^\star)$ and $\lambda_p(\Omega^\star, -(V_-)^\star, m^\star)$ provided both $\alpha_p(\Omega^\star, V_\bullet, m^\star)$ and $\alpha_p(\Omega^\star, -(V_-)^\star, m^\star)$ are positive.
Therefore, it seems natural to make a comparison between the values of $\lambda_p(\Omega, V, m)$, $\lambda_p(\Omega^\star, V_\bullet, m^\star)$ and $\lambda_p(\Omega^\star, -(V_-)^\star, m^\star)$, assuming the latter quantities exist.

4.2.2 *The Choice of the Potential*

Once we found suitable symmetrized problems, we need to prove that there actually exists at least one potential V such that (4.1), (4.4) and (4.5) have a principal eigenvalue, i.e such that $\alpha_p(\Omega, V, m)$, $\alpha_p(\Omega^\star, V_\bullet, m^\star)$ and $\alpha_p(\Omega^\star, -(V_-)^\star, m^\star)$ are positive.

From the definitions, it follows that $\alpha_p(\Omega, V, m) = \alpha_p(\Omega^\star, V_\bullet, m^\star)$ $= \alpha_p(\Omega^\star, -(V_-)^\star, m^\star) = +\infty > 0$ whenever $m > 0$ a.e. in Ω (for in such a case the set in the rightmost side of (4.2) is always empty).

On the other hand, when the zero-level sets of m (and of $m^\star$) are not negligible, we cannot say "a priori" that $\alpha_p(\Omega, V, m)$, $\alpha_p(\Omega^\star, V_\bullet, m^\star)$ and $\alpha_p(\Omega^\star, -(V_-)^\star, m^\star)$ are positive, neither that the positiveness of one of them implies the positiveness of the others.
Neverthless, we can use Sobolev or Poincaré inequalities to prove that the positivity of the three values of $\alpha_p(\cdot,\cdot,\cdot)$ can be obtained under suitable smallness conditions on V_-: in fact, we have the following:

Proposition 4.1
Let V and m satisfy assumptions (H1) and (H2) and let $|\{m = 0\}| > 0$. If $1 < p < N$, then there exists a constant $C(p, N, \Omega) > 0$ such that $\alpha_p(\Omega, V, m)$, $\alpha_p(\Omega^\star, V_\bullet, m^\star)$ and $\alpha_p(\Omega^\star, -(V_-)^\star, m^\star) > 0$ when $\|V_-\|_{N/p,\Omega} < C(p, N, \Omega)$.
If $p \geqslant N$, then there exists a constant $C(p, N, \Omega) > 0$ such that $\alpha_p(\Omega, V, m)$, $\alpha_p(\Omega^\star, V_\bullet, m^\star)$ and $\alpha_p(\Omega^\star, -(V_-)^\star, m^\star) > 0$ when $\|V_-\|_{\infty,\Omega} < C(p, N, \Omega)$ a.e. in Ω.

Proof. We have:

$$\alpha_p(\Omega, V, m) = \inf\left\{ \int_\Omega |\nabla u|^p + V(x)\,|u|^p,\ u \in W_0^{1,p}(\Omega),\right.$$
$$\left. \|u\|_{p,\Omega} = 1 \text{ and } \int_\Omega m(x)\,|u|^p = 0 \right\}.$$

Assuming $1 < p < N$, Sobolev inequality $\|\nabla u\|_p \geqslant \gamma\ \|u\|^p_{p^*,\Omega}$ (with $\gamma = \gamma(p, N, \Omega) > 0$) applies and it yields:

$$\alpha_p(\Omega, V, m) \geqslant \inf\left\{ \left(\gamma - \|V_-\|_{N/p,\Omega}\right)\ \|u\|^p_{p^*,\Omega},\ u \in W_0^{1,p}(\Omega),\right.$$
$$\left. \|u\|_{p,\Omega} = 1 \text{ and } \int_\Omega m(x)\,|u|^p = 0 \right\}$$

thus $\alpha_p(\Omega, V, m) > 0$ if $\|V_-\|_{N/p,\Omega} < \gamma$; analogous arguments imply that $\alpha_p(\Omega^\star, V_\bullet, m^\star) > 0$ and $\alpha_p(\Omega^\star, -(V_-)^\star, m^\star) > 0$ if $\|(V_\bullet)_-\|_{N/p,\Omega}$, $\|(V_-)^\star\|_{N/p,\Omega^\star} < \gamma^\star = \gamma(p, N, \Omega^\star)$. Owing to equimeasurability, we have $\|(V_\bullet)_-\|_{N/p,\Omega^\star} = \|(V_-)^\star\|_{N/p,\Omega^\star} = \|V_-\|_{N/p,\Omega}$ and the claim follows.
On the other hand, if $p \geqslant N$, Poincaré inequality $\|\nabla u\|_p \geqslant \vartheta \; \|u\|^p_{p,\Omega}$ (with $\vartheta = \vartheta(p, N, \Omega) > 0$) applies and it yields:

$$\alpha_p(\Omega, V, m) \geqslant \inf\left\{ \int_\Omega (\vartheta - V_-(x)) \; |u|^p, \; u \in W_0^{1,p}(\Omega), \right.$$
$$\left. \|u\|_{p,\Omega} = 1 \text{ and } \int_\Omega m(x) \; |u|^p = 0 \right\}$$

thus $\alpha_p(\Omega, V, m) > 0$ if $\|V_-\|_{\infty,\Omega} < \vartheta$; analogous arguments yield $\alpha_p(\Omega^\star, V_\bullet, m^\star) > 0$ and $\alpha_p(\Omega^\star, -(V_-)^\star, m^\star) > 0$ if $\|(V_\bullet)_-\|_{\infty,\Omega^\star}$, $\|(V_-)^\star\|_{\infty,\Omega^\star} < \vartheta^\star = \vartheta(p, N, \Omega^\star)$. Since $\|(V_\bullet)_-\|_{\infty,\Omega^\star} = \|(V_-)^\star\|_{\infty,\Omega^\star} = \|V_-\|_{\infty,\Omega}$, the claim follows. □

4.3 FABER–KRAHN TYPE INEQUALITIES

Proposition (4.1) above shows that the class $\mathcal{P}$ of measurable potentials satisfying (H1) & (H2) for which the comparisons suggested in §4.1 make sense is nonempty.
From now on we assume $V \in \mathcal{P}$, thus (4.1), (4.4) and (4.5) have a principal eigenvalue.

The argument in the previous section alone would suffice in proving the following:

THEOREM 4.1 (Faber–Krahn inequalities)
Let Ω, p, V and m satisfy all the assumptions above and let $V \in \mathcal{P}$. Then:

$$\lambda_p(\Omega, V, m) \geqslant \lambda_p(\Omega^\star, V_\bullet, m^\star) \geqslant \lambda_p(\Omega^\star, -(V_-)^\star, m^\star)\;. \quad (4.6)$$

Anyway, for sake of completeness, we write a one-line proof of the stated inequalities here.

Proof. For any positive eigenfunction $u \in W_0^{1,p}(\Omega)$ associated with $\lambda_p(\Omega, V, m)$ we have $u^\star \in W_0^{1,p}(\Omega^\star)$ and:

$$\begin{aligned} \lambda_p(\Omega, V, m) &= \frac{\int_\Omega |\nabla u|^p + V(x)|u|^p}{\int_\Omega m(x)|u|^p} \\ &\geqslant \frac{\int_{\Omega^\star} |\nabla u^\star|^p + V_\bullet(x)|u^\star|^p}{\int_{\Omega^\star} m^\star(x)|u^\star|^p} \\ &\geqslant \lambda_p(\Omega^\star, V_\bullet, m^\star) \end{aligned} \tag{4.7}$$

by Hardy–Littlewood and Polya–Szegö inequalities; thus the leftmost inequality follows.
Getting rid of the positive part of $V_\bullet$ in the nonlinear Rayleigh quotient, i.e. $(V_+)_\star$, we obtain the rightmost inequality. □

Obviously, equality holds in the leftmost inequality (4.6) if we are in the *radially symmetric setting*, i.e. if $\Omega = \Omega^\star$, $V = V_\bullet$ and $m = m^\star$ (modulo translations); on the other hand, equality holds in the rightmost inequality if $V_\bullet = -(V_-)^\star$ and it holds through (4.6) if $\Omega = \Omega^\star$, $V = -(V_-)^\star$ and $m = m^\star$ (modulo translations). In the spirit of [36, 46, 3, 2], we may wonder if the latter is the only setting which gives equality through (4.6).

The characterization of the equality case in (4.6) will be proved in a while, under a sign condition on $\lambda_p(\Omega^\star, -(V_-)^\star, m^\star)$.
We have:

THEOREM 4.2 (Equality case in Faber–Krahn inequality)
Let Ω, p, V and m be as in the Theorem above.
Assume further that $\lambda_p(\Omega^\star, -(V_-)^\star, m^\star) \geqslant 0$.
Then equality holds in (4.6) *if and only if $\Omega = \Omega^\star$, $V = -(V_-)^\star$ and $m = m^\star$ (modulo translations).*

Proof. For sake of simplicity, let λ_p and $\lambda_p^\star$ denote respectively the leftmost and the rightmost side of (4.6).
If $\lambda_p = \lambda_p^\star$, then equality holds also through (4.7). Thus, if u is a

positive eigenfunction associated to λ_p, we have equality in each of the following:

$$\int_\Omega |\nabla u|^p \geqslant \int_{\Omega^\star} |\nabla u^\star|^p \tag{4.8}$$

$$\int_\Omega V_+(x) u^p \geqslant \int_{\Omega^\star} (V_+)_\star(x)(u^\star)^p \geqslant 0 \tag{4.9}$$

$$\int_\Omega V_-(x) u^p \leqslant \int_{\Omega^\star} (V_-)^\star(x)(u^\star)^p \tag{4.10}$$

$$\int_\Omega m(x) u^p \leqslant \int_{\Omega^\star} m^\star(x)(u^\star)^p \tag{4.11}$$

(because otherwise we would get a contradiction) and $u^\star$ is a positive eigenfunction of (4.5) associated with $\lambda_p^\star$, viz. it satisfies:

$$\begin{cases} -\Delta_p u^\star - (V_-)^\star(x)\ (u^\star)^{p-1} = \lambda_p^\star\ m^\star(x)\ (u^\star)^{p-1} & \text{, in } \Omega^\star \\ u^\star = 0 & \text{, on } \partial\Omega^\star\ . \end{cases} \tag{4.12}$$

Standard regularity theorems [47, 58] apply and they yield $u^\star \in L^\infty(\Omega^\star) \cap C^{0,\alpha}_{loc}(\Omega^\star)$ (though stronger $C^{1,\alpha}$-regularity can be obtained in some cases using [31, 63]).
Moreover, using Harnack inequality [64, 40] we infer $u^\star > 0$ in $\Omega^\star$.

Step 1. $\Omega = \Omega^\star$ *and* $u = u^\star$ *(modulo translations).*
Since $u^\star(x) = u^*(\omega_N |x|^N)$ and since $u^\star$ is positive in $\Omega^\star$, then u^* does also.
Moreover, standard computations (e.g., [62]) can be used to prove that the onedimensional decreasing rearrangement u^* satisfies the following integro–differential problem written with respect to the "measure coordinate" $s = \omega_N |x|^N$:

$$\begin{cases} \dot{u}^*(s) = -\frac{1}{(c_N\ s^{1-1/N})^{p/(p-1)}}\ \left(\int_0^s (\lambda_p^\star m^* + (V_-)^*) u^{p-1}\, d\,\sigma\right)^{\frac{1}{p-1}} \\ u^*(0) = \|u\|_{\infty,\Omega} \\ u^*(|\Omega|) = 0\ . \end{cases} \tag{4.13}$$

We will now prove that the graph of u^* has no flat parts at level $\bar{u} \in]0, \|u\|_{\infty,\Omega}[$ inside $[0, |\Omega|]$.
By contradiction, we assume there exists an interval $[s_1, s_2] \subset]0, |\Omega|[$ in which $u^*(s) = \bar{u}$ for some $0 < \bar{u} < \|u\|_{\infty,\Omega}$; hence $\dot{u}^*(s) = 0$ in $]s_1, s_2[$ and:

$$0 = \int_0^s (\lambda_p^\star m^* + (V_-)^*) \, d\sigma$$

in the same interval.
Setting:

$$\begin{aligned} \mathcal{M}(s) &:= \int_0^s \lambda_p^\star \, m^*(\sigma) \, d\sigma \\ \mathcal{V}(s) &:= -\int_0^s (V_-)^*(\sigma) \, d\sigma \end{aligned}$$

for each $s \in [0, |\Omega|]$, previous equality reads:

$$\mathcal{M}(s) = \mathcal{V}(s) \text{ for } s \in [s_1, s_2] \,. \tag{4.14}$$

If $\lambda_p^\star = 0$, then $\mathcal{M}(s) = 0$ everywhere in $[0, |\Omega|]$ and (4.14) yields $\mathcal{V}(s) = 0$ in $[s_1, s_2]$; on the other hand, $\mathcal{V}$ is convex in $[0, |\Omega|]$ and $\mathcal{V}(0) = 0$, therefore $\mathcal{V}(s) = 0$ in the whole of $[0, s_2]$. Thus from (4.13) we infer $\dot{u}^*(s) = 0$ in $]0, s_2[$ and $u^*(s) = u^*(0) = \|u\|_{\infty,\Omega}$ in $[0, s_2]$.
If $\lambda_p^\star > 0$, owing to the concavity of $\mathcal{M}$ and the convexity of $\mathcal{V}$ in $[0, |\Omega|]$, from (4.14) we infer that both $\mathcal{M}$ and $\mathcal{V}$ are linear in $[s_1, s_2]$, i.e. that there exist $\alpha, \beta \in \mathbb{R}$ such that:

$$\mathcal{M}(s) = \alpha \, s + \beta = \mathcal{V}(s) \text{ for } s \in [s_1, s_2] \,. \tag{4.15}$$

Since $\mathcal{V}$ is convex and $\mathcal{M}$ is concave, we have $\mathcal{M}(s) \leqslant \alpha \, s + \beta \leqslant \mathcal{V}(s)$ in $[0, s_2]$, hence $0 = \mathcal{M}(0) \leqslant \beta \leqslant \mathcal{V}(0) = 0$; therefore (4.15) rewrites:

$$\mathcal{M}(s) = \alpha \, s = \mathcal{V}(s) \text{ for } s \in [s_1, s_2] \,, \tag{4.16}$$

and condition $\mathcal{M}(0) = 0 = \mathcal{V}(0)$ implies that (4.16) holds in the whole of $[0, s_2]$. Thus from (4.13) we infer again $\dot{u}^*(s) = 0$ in

$]0, s_2[$, hence $u^*(s) = u^*(0) = \|u\|_{\infty,\Omega}$.
In both cases, the conclusion is in sound contradiction with the assumption $u^*(s) = \bar{u} < \|u\|_{\infty,\Omega}$.
It follows that u^* decreases strictly in $[0, |\Omega|]$, thus $u^\star$ is strictly radially decreasing and the set:

$$\{x \in \Omega^\star : \ u^\star(x) \in]0, \|u\|_{\infty,\Omega}[\text{ and } |\nabla u^\star(x)| = 0\}$$

is a null set. Therefore, Brothers & Ziemer theorem applies in (4.8), yielding $\Omega = \Omega^\star$ and $u = u^\star$ (modulo translations).

Step 2. $m = m^\star$ *and* $V = -(V_-)^\star$ *(modulo translations)*.
Equality in the rightmost inequality of (4.9) and $u > 0$ in Ω yield $(V_+)_\star = 0$ and therefore $V_+ = 0$ in Ω.
By previous Step, equalities in (4.10) and (4.11) become:

$$\int_\Omega V_-(x)\ u^p = \int_\Omega (V_-)^\star(x)\ u^p = \int_0^{|\Omega|} (V_-)^*(s)\ (u^*(s))^p\ \ ds$$
$$\int_\Omega m(x)\ u^p = \int_\Omega m^\star(x)\ u^p = \int_0^{|\Omega|} m^*(s)\ (u^*(s))^p\ \ ds\ .$$

Since m and u are both nonnegative and since u is strictly radially decreasing, the equality condition in Hardy–Littlewood inequality with nonnegative functions Theorem 1.7 yields $m = m^\star$ (modulo translations). Analogously we get and $V_- = (V_-)^\star$ (modulo translations) and adding V_+ we finally find $V = -(V_-)^\star$ (modulo translations).

$\square$

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