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Doctoral Thesis

A Two-Factor Binomial Model for Pricing Hybrid Securities: A Simplified Approach

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Declaration of Authorship

I, Antonio De Simone, declare that this thesis and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
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Signed: Antonio De Simone

Date: 05 March 2013

"Everything should be made as simple as possible. But not simpler."

Albert Einstein

"We must not forget that when radium was discovered no one knew that it would prove useful in hospitals. The work was one of pure science. And this is a proof that scientific work must not be considered from the point of view of the direct usefulness of it. It must be done for itself, for the beauty of science, and then there is always the chance that a scientific discovery may become like the radium a benefit for humanity."

Marie Curie

"There are in fact two things, science and opinion; the former begets knowledge, the latter ignorance."

Hippocrates

"Men will gather knowledge no matter what the consequences. Science will go on whether we are pessimistic or optimistic, as I am. More interesting discoveries than we can imagine will be made, and I am awaiting them, full of curiosity and enthusiasm."

Linus Pauling

UNIVERSITY OF NAPOLI FEDERICO II

Abstract

Faculty of Economics
Department of Mathematics and Statistics

Ph.D. in Mathematics for Economics and Finance

A Two-Factor Binomial Model for Pricing Hybrid Securities: A Simplified Approach

by Antonio De Simone

This thesis develops a numerical procedure for pricing financial contracts whose contingent claims are exposed to two sources of risk: the stock price and the short interest rate. Particular emphasis here is placed on hybrid financial securities, i.e. on a group of financial contracts that combine the elements of the two broader groups of securities, debt and equity. Moreover, we focus on "American style" financial products, i.e. financial contracts (such as American options or convertible bonds) giving the owner a right to be exercised within a certain date. In particular, the proposed pricing framework assumes that the stock price dynamics is described by the Cox, Ross Rubinstein (CRR, 1979) binomial model under a stochastic risk free rate, whose dynamics evolves over time accordingly to the Black, Derman and Toy (BDT, 1990) one-factor model. We also show how to apply the numerical procedure to compute the price of three financial contracts with increasing complexity: a vanilla (European and American) call option, a callable convertible bond and a participating policy, i.e. an insurance contract where the benefit for the policyholder is partly fixed and partly variable, depending on the profit of the insurance company. We also discuss some issues related to the implementation and calibration of such a two factors numerical procedure and in particular how the dynamics of each risk factor can calibrated to the observed market prices. Finally, in order to assess the validity of the model, its advatages and drawbacks, we conduct an empirical analysis where, in particular, the role of the correlation between stock price and interest rate is emphasized. We study different possible ways for calibrating the correlation parameter, including implied correlation and multivariate GARCH forecast.

Keywords: Hybrid products, interest rate risk, rolling numeraire, binomial lattices, calibration, implied correlation

JEL code: C63, C65, G13

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Dedicated to my family, without which this work would not have been possible.

Chapter 1

Introduction

The aim of this thesis is to develop a general framework suitable for pricing contingent claims whose value depends on two sources of risk: the spot interest rate and the stock price. In particular, we focus on "American style" financial products, i.e. financial contracts (such as American options or convertible bonds) giving the owner a right to be exercised within a certain date.

The final result is a numerical procedure where the information from the binomial tree of the interest rate is combined with information about stock price within the well celebrated Cox, Ross and Rubinstein (CRR [1]) model. We choose binomial models mainly for their simplified approach in pricing and their flexibility, since many interest rate dynamics can be described by means of binomial trees. Moreover, binomial (and trinomial) lattice is one the most common pricing technologies adopted in the financial industry. The final result is a simplified pricing framework that can be of practical use for trading, hedging and risk management purposes. It is important to notice that in this context we assume the financial markets are complete, and we also partly verify what happened if it is not the case, even if we do not go in details of pricing in incomplete markets. In developing the pricing framework, the following issues have been addressed.

First of all, the dynamics of the interest rate and of the stock price need to be as consistent as possible with the observed market data of stocks, bonds, and their respective volatilities. This is one of the most desirable properties for developing a pricing framework to use for trading purposes because it ensures that the price assigned to the contract is such that arbitrage opportunities are not possible. This means that the prices provided by the model must be as close as possible to the arbitrage free prices of the securities, having the same contractual features, that are observed in the financial market. The concept of "closeness" of the prices should be considered both from a qualitative and a quantitative perspective. From a qualitative perspective, for each target security, the model price should be not biased or, in other terms, there should not exist systematic differences between market and model price, or at least such differences should be not significant. Moreover, bearing in mind that the object of the proposed pricing framework are hybrid financial products, we require that a reasonable pricing quality should be guaranteed with respect to a broad variety of traded securities (quantitative perspective). In fact, hybrid financial products can be defined as a group of financial contracts that combine the elements of the two broader groups of securities, debt and equity. This means that the replicating portfolio for a hybrid security is composed at least from the two kinds of financial contracts mentioned above. In this thesis however, we also discuss what happens to such a replicating portfolio if the other two major kinds of financial contracts (that are derivatives and insurance contracts) are included. The addition of derivatives to the replicating portfolio (no matters if they are included deliberately as a unique contract or if such an inclusion is consequence of the particular combination of equity and debt securities) imply that the pricing object is not simply a hybrid security but can become a “structured product”, i.e. a pre-packaged investment strategy based on derivatives, such as a single security, a basket of securities, options, indices, commodities, debt issuance and/or foreign currencies, and to a lesser extent, swaps.

During the last decade, the financial industry has increasingly often provided investors with financial contracts embedding life insurance policies. If such policies grant benefits that are based (at least partly) on the profit of the insurance company, we talk about participating policies, i.e. an insurance contract where the benefit for the policyholder is, in general, partly fixed and partly variable, depending on the profit of the insurance company. In most cases, the participation rate is applied to an intermediate level of the economic and actuarial result referring to a predefined dedicated quota of the investment or to a specific fund. From the policyholder perspective, such a contract can be thought of as a hybrid product embedding an option on the insurance company itself. However, since insurance policy have not an active secondary market, as it is for stocks and bonds, the emphasis here is rather on the managerial aspects of this contracts, such as those related to hedging and risk management purposes.

It can be argued that the existence of an enormous variety of derivatives listed in the financial markets would imply a too ambitious scope for this research. For the sake of feasibility, we dedicate attention only on two risk factors that are the stock price and the term structure of interest rates and, consequently, only on the equity-like and debt contracts and derivatives. We do not consider other risk factors such as the foreign exchange rate while credit risk is included in the pricing framework only as an exogenous variable.

It is worth noting that, the necessity for a dynamics that is consistent with the observed market data arises not only because the model should be able to "reproduce" the observed market prices of as many (traded) financial assets as possible, but also because it should be calibrated directly using the market observables. It is recommended to avoid complex estimation techniques for the unknown parameters describing the evolution of the risk factors. We think that this is a necessary (even though not sufficient) condition to develop a consistent framework that can be used for trading purposes because too often econometric techniques are used inappropriately, given the great deal of subjective choices necessary for the estimation. This is in the spirit of a financial industry that is intended to standardize information to facilitate its diffusion and the adoption of worldwide market standard and conventions. For the stock price we select the CRR model because, as widely known, the option price obtained by means of the binomial model is, in the limit, equal to the Black and Scholes ([**?**]) formula. The problem of calibration is however exacerbated if we look at the interest rate dynamics. The real problem here is to find a dynamics that can offer a satisfactory calibration to the observed market data of default free bonds. As for the stock price, the possibility to calibrate the model simply by means of the observed market volatility of the stock prices would be a desirable property (especially if the model is going to be used for trading purposes). However, selecting the interest rate process is much more complicated. This is the case because, by market conventions, the market volatility of the interest rates is computed (from caps, floors and swap options markets) according to the Black formula, where the underlying interest rate is the forward rate and its distribution at each time is lognormal. However, interest rates show a tendency to regress toward their long run average when the level of the rate is very high or very low, tendency that is not ensured by a lognormal process. We think that it does not exist a "perfect" dynamics for the interest rate, that there is not a dynamics that is superior to all the others. For this reason the pricing framework should be sufficiently flexible to allow for the use of different specification of the interest rate dynamics. As told before, the flexibility is another reason for using binomial lattices.

1.1 The background

Many pricing models with stochastic interest rates have been proposed in literature, starting from the seminal paper by Merton [2], where the Gaussian process was adopted to describe the continuous-time short rate dynamics. The adoption of a Gaussian process was very common in the '80s and in the early '90s before the advent of the lognormal term structure models. A discrete time dynamics for short rate process equivalent to those adopted by Merton was subsequently discussed by Ho and Lee ([3]), while other option pricing formulae under Gaussian interest rate were introduced by Rabinovitch ([4]) and Amin and Jarrow ([5]).

The success of the Gaussian models of the term structure relies on the mathematical tractability and thus on the possibility of obtaining closed formulas and solutions for the price of stock and bond options. In fact, the Gaussian process was for the first time adopted to derive the price of bond options by Vasicek ([6]). Furthermore, the calibration of the Gaussian models does not require particularly demanding computational effort.

Although the Gaussian models have been very successful for research purposes, some relevant inner drawbacks prevented their diffusion among the practitioners, as for example the possibility for the interest rate trajectories to assume negative values. In response, other equilibrium models for the interest rate term structure have been developed. One of this is the well-known Cox, Ingersoll and Ross ([7]) model (CIR), where the interest rate dynamics is described by a square root mean reversion process that, under the Feller ([8]) condition, does not allow the interest rate to become negative. CIR dynamics has subsequently been adopted also to describe the stochastic short rate framework for pricing stock option (Kunitomo and Kim, [9] and [10]) and for pricing endowment policies, with an asset value guarantee, where the benefit is linked to fixed income securities (Bacinello and Ortu, [11]).

However, as showed elsewhere, equilibrium models are in general not able to ensure an efficient calibration of the interest rate dynamics, because they are based on a limited number of parameters, in general unable to guarantee an acceptable fitting of the model prices to market prices. Moreover, a satisfactory calibration of the model is sometimes a hard task, because many equilibrium models rely on an instantaneous interest rate (spot or forward) that, in general, is not directly observable on the market. The relevance of this problem increased over time, especially after the diffusion of standard market practices of pricing derivatives within the Black and Scholes environment (Black and Scholes, [**?**] ; Black, [12]).

As mentioned, limitations of the equilibrium models may be overcome by market models, as for example the Black-Derman-Toy (BDT, [13]) model and the Libor Market Model (Brace et al. [14]). A particular characteristic of both models is the assumption of lognormal interest rate dynamics, even if this hypothesis applies only asymptotically for the BDT model. Such feature allows for a satisfactory calibration by adopting implied volatility measures according to the standard market practices. However, opposite to many equilibrium models, market models do not in general allow to obtain closed price formulas so that the price of interest rates derivatives has to be evaluated numerically. Between the two mentioned models, our attention is in particular devoted to the BDT because of its simplified approach in pricing interest rate derivatives. The BDT model allows to obtain a binomial tree for the dynamics of the Libor rate by adopting, as input data, the term structure of interest rates and of the corresponding volatilities, and to use it to price interest rate derivatives according to the local expectation hypothesis. An exhaustive explanation of the procedure adopted for the construction of the tree is reported in Neftci ([15]) and Ritchken ([16]). Figure 1.1 shows an hypothetical BDT tree for the 12-monts spot Libor rate $L(t, s)$, where s $t = 12$ months for each t, s.

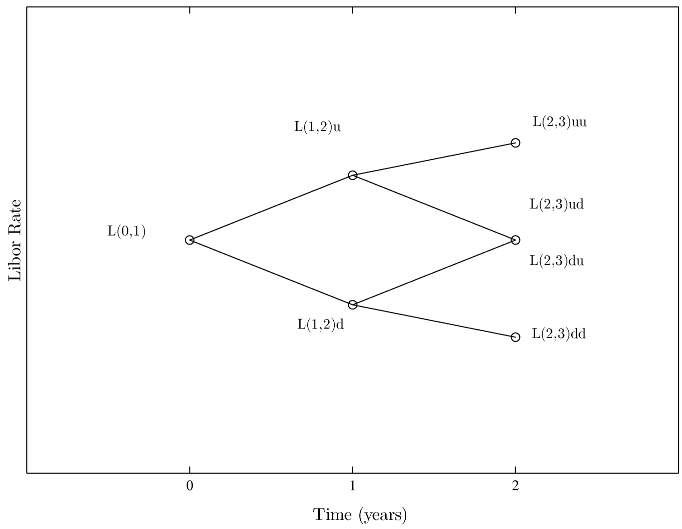


FIGURE 1.1: The BDT tree of the 12-months Libor rate $L(t, s)$.

At time $t = 0$ the 12-months spot Libor rate $L(0,1)$ is directly observable and therefore not stochastic. After one year, at time $t = 1$, the following 12-months Libor rate $L(1,2)$ can go up to the level $L(1,2)u$ or down to the level $L(1,2)d$. Similarly, at time $t = 2$ the Libor rate $L(1,2)j$, in the state of the world $j=u, d$, may go up or down with equal risk neutral probability. We finally notice that since $L(t, s)ud=L(t, s)du$ the BDT tree is recombining.

The goal of this research is to develop a framework where the information from the binomial tree of the interest rate is combined with information about stock price within the well celebrated Cox, Ross and Rubinstein (CRR 1979) model. As told before, we choose binomial models mainly for their simplified approach in pricing and their flexibility, since many interest rate dynamics can be described by means of binomial trees.

In the next sections we report a study that gives some other insights about the choice of the interest rate models. We also discuss some other issues related to the calibration of the BDT model and a comparison between interest rate models will be proposed.

1.2 The choice of the interest rate model

This section deals with the question of which interest rate model practitioners should choose to compute the fair value of Over The Counter (OTC) interest rate derivatives. In fact, as told before, it is possible to point out that an effective pricing process should allow to obtain the unknown price of a financial contract as consistent as possible with the observed prices of other instruments, so that arbitrage opportunities are secluded. For this reason, an effective pricing model should replicate the observed current prices of other financial securities, as far as it is possible.

As far as interest rate models are concerned, it is worth noting that in the financial markets, after the advent of the "market models", choosing the right methodology in pricing has become almost a trivial task, because these models offer an easy way to calibrate the future trajectories of interest rates, so that the current market prices can generally be replicated. Remark that not every interest rate model offers this possibility: for example, the Libor Market Model (LMM, Brace et al. [14]), which actually is one of the most popular, allows to obtain prices consistently with the standard market practice of pricing caps, floors and swap-options by using the Blacks formula (Black, [12]). However, if on the one hand the LMM seems to be a powerful tool in pricing interest rate derivatives, on the other hand, in some circumstances, its usage does not give satisfactory results, so that one could think to use some others interest rate models to get a "better" price.

In the remaining part of this chapter, we point out when LMM does not give appreciable results and show some empirical criteria on how to choose the right methodology, when practitioners face the task of evaluating interest rate derivatives. This effort is aimed to lead the choice of the interest rate model that is more suitable for pricing, especially when trading purposes are concerned.

We focus in particular on three interest rate models, which are the most famous and the most used in practice: the first one (model A) is the Cox, Ingersoll and Ross (CIR, [7]), and it is one of the first stochastic model of the term structure proposed in literature; the second one (model B) is the Black, Derman and Toy model (BDT, [13]), and it was one of the most popular before the advent of Libor Market Model; the third model (model C) is the still mentioned Libor Market Model, and more exactly the Lognormal Forward-Libor Model (LFM), in the version proposed for the first time by Brace, Gatarek and Musiela (Brace et al. [14]).

In performing this analysis, we follow an inductive approach by reporting empirical evidence, whose results can suggest some general rules. Starting by pricing a simple interest rate derivative (e.g. a cap) by means of the three mentioned models, some shortcomings arise, as well as some other interesting aspects involved in pricing derivatives. In fact, by pricing a target contract will be evident when the use of the LMM could not give appreciable results; moreover, by applying every model to the same target contract, it will be shown the quantitative differences between the prices generated by each one. At the end of the comparison interesting suggestions on which model practitioners should generally choose will be available.

Although plenty of papers on pricing interest rates derivatives have been written, the same does not hold for topics related to the comparison between interest rate models. In fact, it is remarkable that this paper originally provides a comparison of models whit heterogeneous features, because its aim is closely linked to the necessity of choosing the pricing methodology in pricing interest rate derivatives, from a professional point of view. On the other hand it is noticeable that the literature about comparisons of interest rate derivatives pricing models appear not to be very wide; moreover, the comparison is often made among models with homogeneous characteristics.

To begin with, it may be pointed out that a relevant work that try to compare interest rate models with heterogeneous characteristics can be found in Khan et al. ([17]), where the comparison involves the Hull-White and the Black-Karasinski model. However, the most popular market models are not considered in those work, also because its aim is linked to risk management rather than pricing issues.

It can be highlighted that important consideration on the drawbacks of the models, which are of fundamental importance to establish whether and to what extent an interest rate model can be successfully used in pricing derivatives, can be found in the works of the authors that for the first time developed the models themselves, and in particular some attention can be paid to the works of Cox, Ingersoll and Ross ([7]), Black, Derman and Toy ([13]), and Brace, Gatarek and Musiela ([14]), and their successive developments.

Some other works that deals with the comparison between models have been developed, both, by a theoretical and empirical point of view. A comparison of valuation model can be found in Jacobs ([18]), where one of the key issues faced by the author is to establish criteria for model quality; issue which is somewhat linked to this work. However it can be pointed out that the attention of Jacobs focuses on continuous-time stochastic interest rate and stochastic volatility models, such as CIR model and Heat, Jarrow and Morton (HJM, [19]) model, but it does not take into account the LMM or any other discrete-time stochastic interest rate model.

Another important work related to this, which is closely linked in particular to the market models, can be found in Plesser, de Jong and Driessen ([20]), where nevertheless, the attention is focused on the Libor Market model and on the Swap Market model only, and no comparison is made between continuous-time and discrete-time interest rate models; comparison that, on the other hand, is central in this work.

A broader, interesting analysis on interest rate derivatives pricing models is carried out by Barone ([21]), where almost each kind of model is studied, included continuous and discrete time models, equilibrium and arbitrage models, one factor and multifactor models as well. However the comparison between all this models are based on a theoretical point of view only, where aspects linked to the concrete application to pricing, hedging and risk management issues are not central in those work.

This work will in fact try to use an approach similar to those followed by Jacobs and Plesser, which is based on empirical analysis, without renounce to report some important considerations on the financial theories on which models are based; considerations which can moreover be used to carry out some important conclusions about the use of interest rate models themselves.

Finally, it is noticeable that consideration about the usage of the BDT model can be found in Bali et al. ([22]), where a comparison between two different approach in determining the volatility parameters is offered. Also if the approach in estimating the volatility is completely different, this issue is faced in this paper too.

1.3 Pricing interest rate derivatives

In this section target contract, necessary data and applied models are presented. To begin with, the target contract and the models will be presented (section 1.3.1 and 1.3.2 respectively); data will be shown (par 1.3.3) also after, because necessary data depend both, on the kind of contract and on the model considered. The models will be sketched since we want to understand how to use it in pricing, and to highlight qualities and drawbacks of each one for the comparison that will be done in the next section. The BDT model is presented in more detail, because it will be used in the next chapters as a building block of the two factors numerical procedure.

1.3.1 The target contract

To put in place the comparison between models, a simple contract is chosen also because in this way it will be possible to understand how large is the difference between the price provided by each model and the price provided by the Blacks formula. In this way it will also be possible to understand what are the reasons for such differences in prices, and some advice on how to minimize this difference could arise. This is a key consideration when a practitioner face the task of choosing the pricing model, also because the use of the Black and Scholes ([**?**]) approach is recommended by the central banks. So it can be important to understand if, and for what reasons, the model under observation produces a price considerably different form market standards.

For these reasons, the contract that will be priced in the next section is a one year plain vanilla cap (that can be easily priced by using the Black formula), written on the three-months Euribor and made of four paid-in-arrears caplet. This means that the pay off of each caplet, $C(T_j+\delta)$, with maturity date T_j, with $j = 1, 2, 3, 4$ and tenor $\delta = 0.25$, at the settlement date $T_j + \delta$, will be:

$$C(T_j + \delta) = [L(T_j, T_j + \delta) - K]^+ N\delta \tag{1.1}$$

where $T_1 = .25$, $T_{j+1} = T_j + \delta$, $L(T_j, T_j + \delta)$ is the three-months Euribor at the reset date T_j, K is the strike rate and N is the notional amount, and it equals to \$100,000. If this is the case, the value of the caplet at each maturity date, $C(T_j)$, will be the present value of $C(T_j + \delta)$:

$$C(T_j) = \frac{C(T_j + \delta)}{1 + L(T_j, T_j + \delta)\delta} \tag{1.2}$$

At the valuation date $t = 0$ the value of the cap will be given by the sum of each t-time caplet.

1.3.2 Interest rate derivatives pricing models

As told before, the models considered in this work are the CIR, the BDT and the LMM. The CIR model (model A) is a continuous-time equilibrium model where the instantaneous short rate dynamic under the risk-neutral probability measure is described by the following stochastic differential equation:

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t \tag{1.3}$$

where r_t is the t-time value of the instantaneous short rate, k, θ, and σ are positive constants representing respectively the mean reversion rate, the long period mean and the volatility of r_t in the CIR model; dW_t is a Wiener increment. The Feller condition

$2k\theta > \sigma$ ensures that the origin is inaccessible to the process (1.3), so that the short rate will never be negative.

One of the main problem of the CIR model is how to estimate the constants k, θ, and σ. In fact it is generally known that an estimate of these parameters ensuring a perfect fitting of the observed term structure is extreme difficult and not always satisfying. This drawback can be however removed by using some particular extension of the model, such as the CIR++ (Brigo and Mercurio, [23]), where a correction term is added to the short rate so that the bond prices provided by it are identical to those observed in the market. Although it is possible to improve the model, this extension will not be considered in this work.

However, to estimate the parameters of equation 1.3 a procedure based on current market data is put in place. The approach is similar to those used by Brown and Dybvig ([24]): the vector of the parameters $\beta = [\phi_1, \phi_2, \phi_3, r_t]$ is estimated by minimizing the squared differences between the observed bond prices $v(t, T_j)$ and the theoretical bond prices $v(t, T_j, \beta)$ provided by the model, where:

$$v(t, T_j, \beta) = A(t, T_j)e^{-B(t,T_j)r_t} \tag{1.4}$$

where $A(t, T_j)$ and $B(t, T_j)$ are respectively a state contingent cash flow and a temporal parameter generated on the base of equation 1.1, and are defined by the follows:

$$A(t, T_j) = \left(\frac{\phi_1 \exp\{\phi_1(T_j - t)\}}{\phi_2(\exp\{\phi_1(T_j - t)\} - 1) + \phi_1} \right)^{\phi_3} \tag{1.5}$$

$$B(t, T_j) = \frac{\exp\{\phi_1(T_j - t)\} - 1}{\phi_2(\exp\{\phi_1(T_j - t)\} - 1) + \phi_1} \tag{1.6}$$

In this way we can obtain the following parameters ϕ_1, ϕ_2, ϕ_3 where:

$$\phi_1 = k - \lambda,\ \phi_2 = k,\ \phi_3 = -\frac{\theta}{\lambda} \tag{1.7}$$

with λ constant representing the market price of risk, and where the volatility parameter of the process is given by:

$$\sigma = \sqrt{2(\phi_1\phi_2 - \phi_2^2)} \tag{1.8}$$

To obtain the vector β it will be assumed that:

$$Y = v(t, T_j, \beta) + \epsilon \tag{1.9}$$

where Y represents the vector of the observed market prices, $v(t, T_j, \beta)$ is the vector of the theoretical prices and ϵ is the vector of the errors. In this way, the vector β can be obtained by solving the following problem by means of Marquards algorithm:

$$\min_{\beta}[Y - v(t, T_j, \beta)]'[Y - v(t, T_j, \beta)]. \tag{1.10}$$

Once the parameters are estimated, the price of a paid-in-arrears caplet can be obtained firstly by calculating the price of a call bond option written on a coupon bond with strike price $X = 1/(1 + K\delta)$, by using the following formula:

$$\begin{aligned} ZBC(t, T_j, T_i, X) = v(t, T_j, \beta)\chi^2 \left(2\bar{r}[\rho + \psi + B(T_j, T_i)]; \frac{\sigma^2 k\theta}{\sigma^2}, \frac{2\rho^2 r \exp\{h(T_j - t)\}}{\rho + \psi + B(T_j, T_i)}\right) + \\ +Xv(t, T_j, \beta)\chi^2 \left(2\bar{r}[\rho + \psi]; \frac{\sigma^2 k\theta}{\sigma^2}, \frac{2\rho^2 r \exp\{h(T_j - t)\}}{\rho + \psi}\right), T_j < T_i \end{aligned} \tag{1.11}$$

where $\chi^2(x; a, b)$ is the noncentral chi-squared distribution function with a degrees of freedom and non-centrality parameter b, and where:

$$\rho = \frac{2h}{(\sigma^2 exp(T_j - t)h - 1)}, \psi = \frac{(k + h)}{\sigma^2}, \bar{r} = \frac{\ln(A(T_j, T_i)/X)}{B(T_j, T_i)}, h = \sqrt{(k^2 + 2\sigma 2)}.$$

Secondly, the price of the corresponding put bond option can be obtained by using the put-call parity (Black and Scholes, [**?**]):

$$ZBP(t, T_j, T_i, X) = ZBC(t, T_j, T_i, X) - v(t, T_i) + Xv(t, T_j) \tag{1.12}$$

Thirdly, the price of the caplet is obtained by the following relations:

$$C(t) = N(1 + X\delta)ZBP(t, T_j, T_i, X) \tag{1.13}$$

The BDT model (model B) is an arbitrage free discrete-time short rate model, which allows to obtain a binomial tree for the dynamic of the short rate. Once the tree is obtained, the fundamental theorem of the finance (Duffie, [25]) can be applied to calculate the price of a wide range of interest rates derivatives. Despite the CIR model, it cannot allow to obtain closed form formulas and the price of interest rates derivatives shall be evaluated numerically.

On the other hand, this model provides for an excellent calibration to the observed bond prices which, in every time, can be perfectly replicated from the model. Unfortunately, the same does not hold for the price of derivatives, which cannot be perfectly duplicated by the model, as it will be shown after.

To obtain an interest rate tree, it is necessary to solve a system of n non linear equation in n unknown, where n depends on the length of the tree. To obtain a tree, arbitrage free prices of zero coupon bonds, as well as a term structure of the volatility, are necessary. For example, to obtain a tree steps tree for the one year Libor rate L_j, with $j = 0, 1, 2$ it is necessary to solve the following system:

$$\begin{cases} v(t,T_1) = \dfrac{1}{1+L_0} \\ v(t,T_2) = E_t^p\left[\dfrac{1}{(1+L_0)(1+L_1)}\right] \\ v(t,T_3) = E_t^p\left[\dfrac{1}{(1+L_0)(1+L_1)(1+L_2)}\right] \\ \sigma(t,T_1) = \frac{1}{2}\ln\left(\dfrac{L_1^u}{L_1^d}\right) \\ \sigma(t,T_2) = \frac{1}{2}\ln\left(\dfrac{L_2^{uu}}{L_2^{ud}}\right) \\ \dfrac{L_2^{uu}}{L_2^{ud}} = \dfrac{L_2^{du}}{L_2^{dd}} \\ L_2^{du} = L_2^{ud} \end{cases} \tag{1.14}$$

where $v(t, T_{j+1})$ is the arbitrage free price of a zero coupon bond with maturity T_{j+1}, L_j^m is the one-year Libor rate at the reset date T_j in the state of world m, $\sigma(t, T_{j+1})$ is the observed volatility, used by the BDT model, of the short rate for the maturity t_i and where E_t^p indicates the conditional expected value, at the information available at time t, under the risk-neutral probability p, so that we can obtain, for example:

$$v(t,T_2) = \frac{1}{1+L_0}\left(p\frac{1}{1+L_1^u} + (1-p)\frac{1}{1+L_1^d}\right). \tag{1.15}$$

Equation 1.15 is referred to as the “local expectation hypothesis” in many textbooks, and this hypothesis is verified if and only if there is no arbitrage opportunity. It also implies that no arbitrage opportunity is possible if $F(t, T_j, T_{j+1}) = E_t[v(T_j, T_{j+1})]$, where $F(t, T_j, T_{j+1})$ is the forward price of a pure discount bond at time t to be sell at time T_j, and expiring at time T_{j+1}. More generally, the local expectation hypothesis can be stated as follows:

$$v(t,T_{j+s}) = v(t,T_j)E_t[v(T_j,T_{j+s})] \;\; \forall s \in \mathbb{N}^+. \tag{1.16}$$

In this way the tree in figure 1.1 can be extracted from the market information. To sketch the mechanics of the model, consider the example of the BDT arbitrage free dynamics of the one year Libor rate and of a corresponding risk free bond illustrated in figure 1.2(a) and 1.2(b) respectively.

Notice that the spot Libor rate at time $t = 0$, $L(0, 1)$ is equal to 2% and it is not a random variable. Moreover, an important feature of the BDT model is that the risk neutral up probability is constant over time and is equal to 1/2. Therefore, at time $t = 1$ the one year Libor rate, $L(1, 2)$, is a random variable assuming the values of 2.9% or 5.2% with equal risk neutral probability.

At time $t = 2$ the Libor rate $L(2, 3)$ can rise from 5.2% to 12.9% or drop to 5% with equal probability, and so on. Since the tree is recombining, at time $t = 2$ the Libor rate can reach the value of 5% following two different paths (2% $\rightarrow$ 5.2% $\rightarrow$ 5% or 2% $\rightarrow$ 2.9% $\rightarrow$ 5%). As for the interest rate, the value of the risk free bond can be determined using an appropriate discount rate according to the interest rate dynamics.

As it can be easily insight from figure 1.2(b), the risk free bond is assumed to have a par value equal to 100 and the maturity is set at time $t = 3$. According to the interest rate tree, the bond price at time $t = 2$ will be equal to 98.11, 95.25 and 88.51 if the interest rate at time $t = 2$ is equal to 1.9%, 5% and 12.5% respectively. Proceeding backward, at each node, the bond price will be equal to the expected value of the bond at the successive node (children node), discounted with the appropriate interest rate. For instance, at time $t = 2$ the bond value can be equal to $\frac{98.11 \times 0.5 + 95.25 \times 0.5}{1.029} = 93.95$ or to $\frac{88.51 \times 0.5 + 95.25 \times 0.5}{1.052} = 87.34$ when the Libor rate is equal to 2.9% and 5.2% respectively. Similarly, the current observed market price of the bond is equal to $\frac{93.95 \times 0.5 + 87.34 \times 0.5}{1.02} =$ 88.86

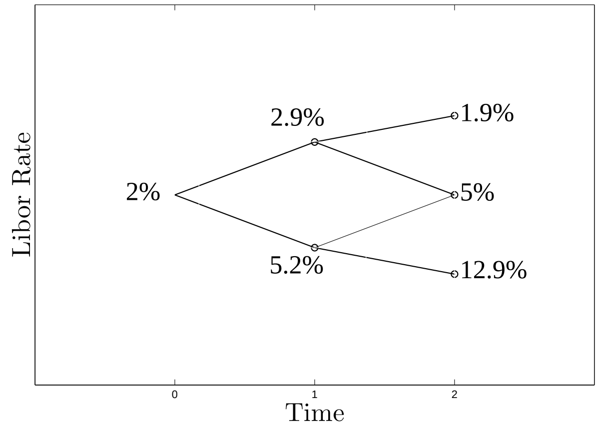
Once the tree is obtained, it can be used to get, for example, the t price of a paid-in-arrears caplet with maturity $T_2 = 2$ years, written on the one-year Libor rate (assuming $p = 1/2$ constant through the time):

$$C(t) = \left[\frac{C_{T_2}^{uu}}{(1+L_0)(1+L_1^u)(1+L_2^{uu})} + \frac{C_{T_2}^{ud}}{(1+L_0)(1+L_1^u)(1+L_2^{ud})} \right] \frac{1}{4} + \tag{1.17}$$

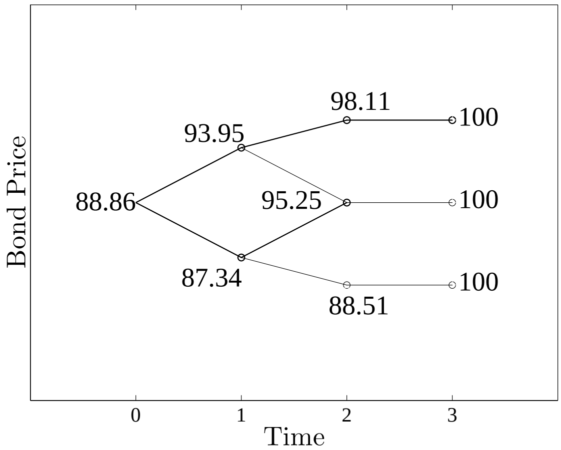
$$\left[\frac{C_{T_2}^{du}}{(1+L_0)(1+L_1^d)(1+L_2^{du})} + \frac{C_{T_2}^{dd}}{(1+L_0)(1+L_1^d)(1+L_2^{dd})} \right] \frac{1}{4}$$

where:

$$C_{T_2}^m = [L_2^m - K]^+ \tag{1.18}$$



(a) The BDT dynamics of the Libor rate



(b) The BDT dynamics of the straight bond

FIGURE 1.2: Example: the BDT arbitrage free dynamics of the Libor rate and of a straight bond

It is remarkably that the application of the BDT model requires to specify the values of $\sigma(t, T_{j+1})$. In practice the implied volatility is largely used to calibrate this model, because this measure of volatility is affected only by the information at the valuation date, and the past information cannot influence its value. However, from a theoretical point of view, the implied volatility obtained by using the Black formula, should represent the volatility of the forward rate dynamic, not of the spot rate, which is the lonely risk factor considered in the model. On the other hand, since no closed-form formula is attainable from the model, it is not possible to get an equivalent implied volatility using BDT.

The Libor Market model (model C) is an arbitrage free, multifactor continuous-time forward rate model which can allow, in every instant of time to reproduce both, the observed arbitrage free prices of bonds and of standard derivative such as caps and floors. This is the case because, as demonstrated by Brace, Gatarek and Musiela ([14]), if the forward Libor rate at the time t, $F_t \equiv F(t, T_j, T_{j+\delta})$, defined as:

$$F(t, T_j, T_{j+\delta}) = \left[\frac{v(t, T_j)}{v(t, T_{j+\delta})} - 1\right]\frac{1}{\delta} \tag{1.19}$$

follows, under the working $T_{j+\delta}$ forward measure probability, the process:

$$dF_t = F(t, T_j, T_{j+\delta})\xi(t, T_j)dW_t^{T_j+\delta} \tag{1.20}$$

where $\xi(t, T_j)$ is the volatility of F_t used in the LMM, the t-time price of a paid-in-arrears caplet with maturity T_j, and settlement date $T_j + \delta$ is given by the Black formula. Assuming the volatility to be constant $\xi(t, T_j) = \xi$, this means that the price of such caplet is:

$$C(t) = v(t, T_{j+\delta})[F(t, T_j, T_{j+\delta})\Phi(d_1) - K\Phi(d_2)]\delta N \tag{1.21}$$

where $\Phi(x)$ is the normal standard distribution function, with parameters:

$$d_1 = \frac{\ln{(F_t/K)} + \xi^2(T_j - t)\frac{1}{2}}{\xi\sqrt{(T_j - t)}}; \quad d_2 = d_1 - \xi\sqrt{(T_j - t)}$$

It is remarkably that, the equation 1.19 can be used to evaluate numerically the price of derivatives, written on the δ-Libor rate, for what a closed formula does not exist. The resulting price will be consistent not only with the observed term structure of interest rates, but also with the observed arbitrage free prices of caps and floors. This also mean that this model needs the implied Black and Scholes volatility for the calibration, and to use another measure of volatility does not ensure a perfect replication of the observed arbitrage free prices. As a consequence, if the implied volatility is not available, the use of other measures of volatility produces results that, in general, cannot be consistent with observed prices so that, in this case, all the remarks about the difficulties in the calibration for the CIR model, would hold for the Libor Market model too.

In general it is possible to assert that the Libor Market model can be considered as a powerful tool to create an association between observed prices and prices of not listed contract, where the link between them is represented by the implied volatility. If the observed prices are not efficient, included when the market is not liquid enough, the resulting price will be consistent with a price that is not considerable as “fair value”, and it should not be considered fair value as well.

1.3.3 Data

Different models require different input data to price a target contract. The model which requires less information is the CIR, because all the parameters are estimated using only the information from the term structure of interest rates observed at the valuation date. Because the risk driver of the contract is, in our application, the three-months Euribor, to estimate the parameters, all the maturities in the Euribor yield curve are used. On the other hand, for the Libor Market model and the BDT model also data about the volatilities of interest rates are necessary.

The comparison is made over a period of about six months and, more exactly, it is made by calculating the price, using all the three models, from 14/11/2008 to 15/5/2009, for a total amount of 121 observations. It is interesting to highlight that:

- the use of the implied volatility from the Black formula is generally recommended only if the market is efficient because, otherwise, the price of the target contract would not be efficient as well, and this holds independently from how the calibration is done;
- the efficient implied volatility should be used always to calibrate the Libor Market model, but the same does not hold for the BDT model where, in particular cases, the historical volatility allow to fit better the observed caps and floors prices.

To provide some evidence about the second statement, in the next section the BDT price of the target contract will be calculated twice for each date, using two different measures of volatility: the historical volatility and the implied Black formula volatility.

The historical daily volatility $\sigma^{hist}(day)$ (where the superscript stays for historical) will be estimated using the following estimator (that is known to be an unbiased estimator):

$$\sigma^{hist}(day) \equiv \sqrt{var[L(0;3)]} = \sqrt{\frac{\sum_{j=1}^{n}[L_j(0;3) - \bar{L}(0;3)]^2}{n-1}} \tag{1.22}$$

where $L_j(0;3)$ is the 3-month Euribor at the date T_j, with $j = 1, 2...n$, with $n = 252$, $\bar{L}(0;3)$ is the sample mean of $L_j(0;3)$. Since the first price is calculated on 14/11/2008, the time series of the 3-month Euribor have to begin on 14/11/2007. On the other hand, for all the models, only the observed rates are necessary; so the time series of the whole Euribor yield curve is necessary only over the period in which the comparison is made (from 14/11/2008 to 15/05/2009). The time series of the Euribor is available on www.euribor.org.

Once the daily volatility have been estimated, to obtain the volatility at 3, 6, 9 and 12 months, the square root rule is used:

$$\sigma^{hist}(T) = \sigma^{hist}(day)\sqrt{T} \tag{1.23}$$

where $\sigma^{hist}(T)$ is the historical volatility for the maturity T expressed in days. On the other hand, when the implied volatility is used to calibrate the BDT model, the reverse problem occurs: the implied volatility can be considered as one year volatility and thus the issue of determine the 3, 6 and 9 months volatility arise. To solve this problem, the volatility for the other maturities will be interpolated using a cubic spline interpolation method.

To calibrate the BDT model and Libor Market model, the implied volatility of caps written on the three-months Euribor is used. The volatility data are provided by Bloomberg. In particular the volatility used in pricing the target contract is a "flat volatility", i.e. the volatility which solve the Black formula with respect to the whole cap. For the comparison are also available the implied volatility for three strike prices: ATM, 2% and 6%, so it can be evaluated how the moneyness of the option can affect the resulting prices.

Finally is remarkable that, since the pay off of the last caplet in the target contract will be paid only after 15 month from the valuation date, the time series of the Euribor appear not sufficient. To complete the data, the Eurirs curve (the swap curve on the Euribor) will be used to interpolate the 15 months rate. In this case a linear interpolation method is used.

1.4 The comparison

First of all, the comparison is made through the period mentioned in the previous section, by fixing the strike price at the value of 2%. Afterward, the time will be fixed and the price for each contract will be calculated for three strike prices: ATM, 2% and 6%.

Figure 1.3 shows the price of the target contract, with a strike price of 2%, calculated by using all the three methodologies, from 14/11/2007 to 15/05/2007. It s noticeable that, by the end of the series, the price by the CIR model appears to be very near to the price by the Libor Market model (Black formula), and very far in the first part of the period considered. This can suggest that the CIR model can provide a results very near to the market standard when the market appears to be stable. In fact from 14/11/2008 to the 16/03/2009 the three months Euribor decreased from the 4.22% to 1.69%, while from the 16/03/2009 to the 15/05/2009 the same rate has decreased till 1.25%. This can suggests that during periods in which the interest rate is volatile, the CIR model does not produce appreciable results, because it can generate prices too far from the observed prices.

By looking at the BDT price, it is firstly noticed that the price obtained by using the implied volatility is always higher than the Blacks price, while the price obtained by using the historical volatility is always below it. It could be straightforward to remark that this is the case because the implied volatility is always higher than the historical volatility.

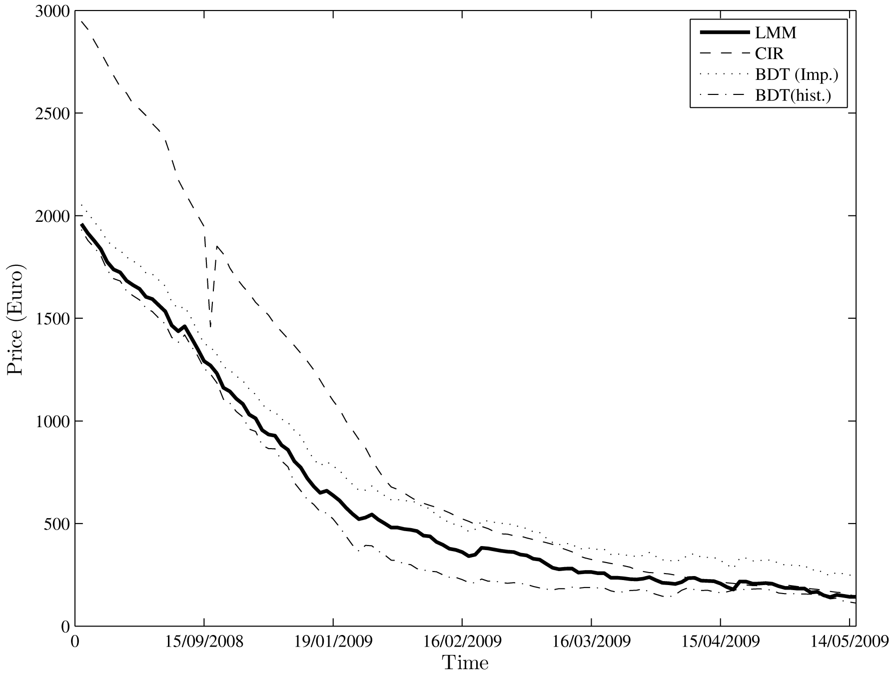


FIGURE 1.3: The price of the target contract through the time by different pricing models.

However it is also possible to notice that the market price of the target contract is almost always higher than the historical volatility price and always less than the implied volatility price, so that a mean between them may be very near to the market model price.

Another important issue is that, in most cases, the BDT model provide prices that are nearer to the market standard if the historical volatility is used rather than the implied. In fact, on 121 observations, only thirty times the use of the implied volatility produce a price nearer the Black price than the use of the historical volatility. This is consistent with the theory, which suggests that the implied Black volatility can be referred to the dynamic of the forward rate, and not of the short rate. In fact this last assumption would imply that the interest rates of the yield curve are perfectly correlated or, better, that the term structure is flat and that it moves only accordingly with additive shift.

A little evidence to support the previous statement can be found by analysing the figure 1.4. It shows the term structure of interest rates in two different dates, at the 12/02/2009 and at the 15/05/2009. This dates are not randomly selected. It an be noticed that, between these two dates, the European Central Bank (ECB) decided to cut its reference rate three times:

- on March 11, from 2% to 1.5%;
- on April 8, from 1.5% to 1.25%
- on May 13, from 1.25% to 1%

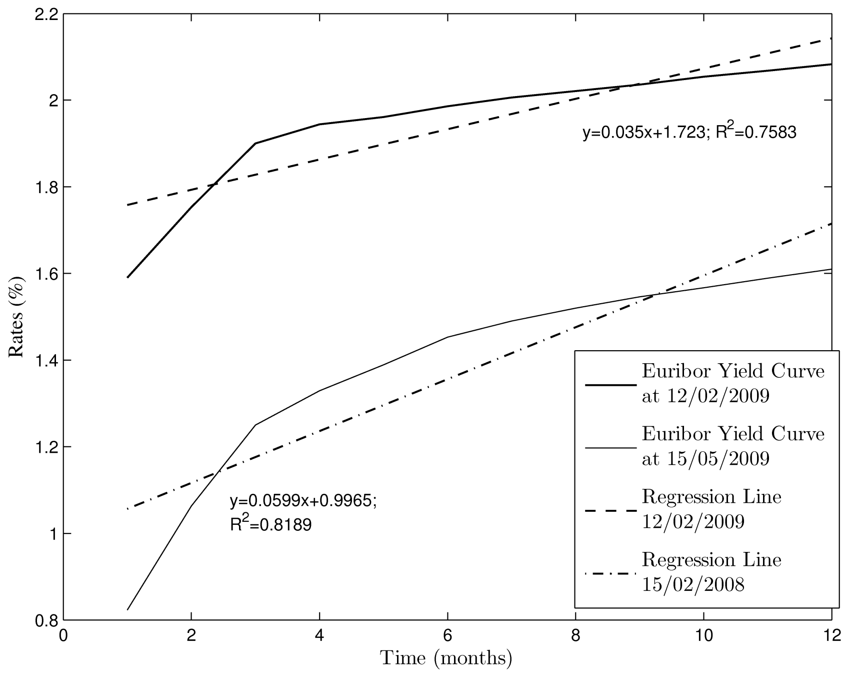


FIGURE 1.4: Comparison between the slope of the yield curve before and after the ECB reference rate cuts.

The regression line is also shown for both the yield curve, as well as the R-squared index. It can be easily inferred that the slope of the regression line at the 12/02/2009 is of 0.0350 with $R^2 = 0.75$, and is flatter than the slope in the other regression line that is of 0.0599 with $R^2 = 0.82$. We also calculate the average slope of the term structure observed 30 days before 12/02/2009 and 30 days after 15/05/2009. We notice that such averages are of 0.0282 and 0.0678 respectively and that such a difference is significant at the 1% level. Therefore, we can reasonably infer that the decision of the ECB have affected not only the intercept, but also the slope of the Euribor yield curve.

However, independently from the considerations on the slope of the yield curve, it can be noticed that, from the 22/01/2009 to the 04/03/2009, the use of the implied volatility produce a result more efficient than the use of the historical volatility, while the opposite occurs form the 04/03/2009 to the end of the series. This is probably due to the fact that the historical volatility, as obvious, takes into account for information that is not actual. As the theory suggests (Fama, [26]), if the financial markets are efficient, the current value of prices and rates keep all the information available, while the old prices are however affected by past information. For this reason when new information is available on the market, it is immediately reflected by the new value of prices and, thus, of the volatility. The same does not hold for the historical volatility.

If attention is paid to the comparison between prices when the moneyness of the option changes, some other issues can be noticed. In tables 1.1 and 1.2 the result of such comparison are shown. Table 1.1 shows how the prices for each model on the 15/05/2009 changes as the strike price changes, passing from 1.7% to 2% and to 6%, while table 1.2 shows the difference in percentage, for each strike price, between the price from Libor Market model and the price from each other model.

Strike Price	LFM	CIR	BDT(Imp.)	BDT (Hist.)
ATM(1.7%)	243,47	253.89	340.92	208.66
2%	143.20	145.36	249.06	112.44
6%	1.18	0.02	34.44	0.01

Table 1.1: Target contract price at the 15/05/2009 by strike price

Strike Price	CIR	BDT(Imp)	BDT(Hist)
ATM(1.7%)	-4.28%	-40.03%	14.30%
2%	-1.51%	-73.93%	21.48%
6%	98.69%	-2814.36%	99.15%

Table 1.2: Differences, in percentage, between the LMM and each other model

It is possible to show that the distance in percentage of the Black price from the prices by the other models increase as the moneyness decrease. In fact the higher the strike price, the higher the difference in prices. This effect is true almost always, except when the CIR price pass from the strike 1.7% to 2%. Only in this case the decreasing of the moneyness produce an improvement of the price. However it can be generally noticed that the price generated by the models gets further from the market price as the moneyness decreases. This effect is due to the fact that the CIR and the BDT models do not take into account the smile effect, in spite of the Libor Market model which is perfectly calibrated by using the value of the implied volatility related to the moneyness of the contract. This suggests that the less is the moneyness of the contract, the less efficient will be the price provided by CIR and BDT models. A easy way to improve the efficiency of the BDT price is to use always the ATM volatility which, being less than the other volatility (because of the smile effect), can allow to obtain a price nearer to the market standard price.

1.5 A good compromise between computational effort and calibration efficiency

The first remark concerns the procedure used in estimate the parameters in the CIR model because, by using the Marquards algorithm, the optimization problem can have an infinite number of solution, depending on the low and up bound used for the iterations, and by the start value of the parameters from which the iterations begin. It would be necessary an objective criteria on how to choose those values, so that the theoretical term structure provided by the CIR model could fit the observed term structure as well as possible.

Secondly it can be argued that, in the BDT model, implied volatility and historical volatility along the different maturities are determined by using different methodologies: the cubic spline for the implied and the square root for the historical. However this choice allows to obtain a value for the implied 3, 6 and 9 month volatility as lower as the value obtainable by using the square root rule and, so doing, to obtain a price nearer the market standard than otherwise.

Thirdly, it is remarkable that the comparison among strike prices is done just for one maturity; to keep stronger results it should be necessary to calculate the prices through the time. However, this remark can be neglected because the aim of the comparison is to highlight that the CIR and the BDT models was not thought to take into account problems linked to the volatility smile.

A final remark concerns the connection between the movements in the slope of the term structure and the possibility to use the implied volatility in the BDT model. In fact the evidence provided in the previous section is quite not strong, because the R-squared index, especially in the first case, is not high enough, so that to assert a difference in the slope, a higher order interpolation method should be necessary.

It is possible to conclude by illustrating that the evidence provided in this paper suggests that the Libor Market model can provide a price, for the target derivative, rigorously consistent with the prices observed in the market of caps and floors; however, it can be used if, and only if, data on the implied volatilities are available and are based on efficient prices. If the market is not arbitrage free, or if it is not liquid enough, the price will be consistent with a cap/floor price that cannot be considered a fair value.

Furthermore if implied volatility is not available or useful, could be necessary to use other pricing models because, in this case, all the three models considered in this work cannot ensure a satisfactory calibration with respect to the observed caps and floor prices. In fact there is no reason that can suggest to use other measures of volatility to calibrate the Libor Market model, so that its use does not produce appreciable advantages in respect to other models, especially when a closed form formula is available.

By observing the prices obtained, it is noticeable that if the implied volatility is not available, and the interest rates appear to be not volatile, the CIR model, as well as BDT model, can offer a price not far from the market standard. Moreover, despite the lack of data on the implied volatilities, it is still possible to adopt the BDT model and, on the other hand, it is remarkable that the use of such a kind of volatility measure is not always recommended, for the BDT, especially if the term structure appear not to be flat, so that the dynamic of the short rate can be logically considered different from the dynamic of the forward rate. This seems to be the case since the risk factor considered by the BDT model (the short rate) is somewhat different from the one used by the Blacks formula (the forward rate), and the coincidence of their dynamics is not always verified. In this cases the use of the historical volatility measure, notwithstanding its weaknesses, can still allow a good fit of the BDT price to the market price, if the market is not considerably volatile. However, as shown in the previous results, the BDT model appears to be a good compromise between computational effort and calibration efficiency. We think so because, if on the one hand the BDT model allows for reliable results both, using implied and historical volatility, on the other hand, the binomial lattices (where the length of the step cannot be arbitrarily set, for the reasons explained above) allow practitioners to adopt a really simplified approach in pricing, without renounce to pricing precision.

As far as further research is concerned, it would be very interesting to know if the CIR++ can provide a price nearer to the market standard in respect to the CIR model, so to understand if the use of a more complicated model can be justified from a higher precision. Generally, it would be interesting to extend this kind of analysis to other, more sophisticated pricing models and particularly to the models with a stochastic volatility.

Chapter 2

The two-factor numerical procedure

In this chapter, we explain how to determine the arbitrage free price of a European stock option by means of a numerical procedure that accounts for two sources of risk: the stock price and the interest rate. In doing this, an important aspect to take into account is that the risk free rate dynamics influences the price of an equity derivative in two ways. First, the path followed by the interest rate can influence the discount factor adopted to determine the present value of the payoff of the derivative. Secondly, under the risk neutral probability measure, the expected stock (and derivative) price depends on the risk free rate level. Therefore, as the interest rate changes, both the final payoff and the discount factor change too while, on the contrary, we assume that the stock price dynamics do not influence the interest rate values. For this reason, the stock price dynamics cannot be specified until the interest rate is known.

2.1 The assumptions of the model

As mentioned at the beginning of the previous chapter, we assume the financial markets are arbitrage free and complete. It is worth noting that the problem of incomplete markets can be addressed, from the point of view of a practitioner, by calibrating the model using historical measures of volatility. We see that, for the BDT model, empirical evidence suggests that historical measures of volatility can be reliable in many cases. However, we do not go in details of pricing in incomplete markets and remand this aspect to future research. Furthermore, we do not assume that the markets are in equilibrium, but we consider this as a natural consequence of the arbitrage free hypothesis.

To begin with, we show how to model the "marginal" dynamics of the two considered risk factors, even if the word marginal is here used improperly because, as anticipated before, we cannot set the dynamics of the stock price independently from the dynamics of the interest rate, unless we set the latter not stochastic. This is the reason why we begin from interest rate modelling and only after we illustrate the stock price dynamics. We rather attribute to the word marginal "marginal" a probabilistic meaning, since the joint probability function of a co-movements of stock price and interest rate is determined starting by the marginal probabilities of the two risk factors. We therefore state some hypotheses to the aim of combining together the two risk factors and obtain their joint dynamics. We begin by setting the assumption of zero correlation between short rate and stock price and after we release it.

In the majority of the cases, when there is a two risk factors pricing model we have to ascertain:

1. the dynamics according to which the two factors evolve;
2. the measure of the correlation between the two dynamics;
3. the estimate of relevant parameters under risk neutral environment.

The three tasks can be extremely difficult to perform properly. As a consequence, a certain level of approximation is often required. Even the simulation based on copula functions, not always available with reference to the distribution under observation, can be very questioning with reference to the choice of an appropriate correlation measure. In this order of ideas, we developed a numerical procedure to get the arbitrage free price of a European call option in a stochastic short rate framework. To begin with, we notice that the numerical procedure adopts the results from the BDT and from the CRR models, whose assumptions hold also for our model.

2.1.1 The marginal dynamics of the short rate

We assume that the interest rate considered as risk factor is the spot interbank offered rate, such as the m-month Libor or Euribor rate, and that its dynamics is described according to the BDT binomial model. Despite the Libor rate refers to banks AA or AA- rated, we consider it as a reasonable good proxy of the risk free rate especially during the periods of financial turmoil. We think this because stressed market conditions, where typically the risk aversion of the investors increases, put pressure on AAA rated securities increasing their price excessively. As a consequence, the corresponding interest rate is too low and, as occurred during the 2012, it can even become negative. In section 2.1.5 we study how to release this assumption and a possible solution to this issue is proposed.

More specifically, we assume that the m-month Libor rate at time T_j, with $j = 1...k$, in the state of the world $m = u, d$, is denoted by $L(T_j, T_{j+1})^m$, where u is for up state and d is for down state and $\delta = T_{j+1} - T_j$ representing the tenor of the Libor rate expressed as fraction of year. The Libor rate at the successive time step can go up to $L(T_{j+1}, T_{j+2})^{mu}$ or down to $L(T_{j+1}, T_{j+2})^{md}$ with equal risk neutral (marginal) probability. Let σ_L denotes the yearly volatility of the Libor rate that, for simplicity, is set constant over time. Section 1.3.2, and more specifically equation 1.14, explains how to get the values of the interest rate in the different "states of the world" (nodes). This implies that the interest rate is piecewise constant over time so that, once the Libor rate is known, it can be adopted as risk free rate to determine the probability of the up movement of the stock price through the successive time step and the corresponding discount factor. Notice also that, at time T_0 (current time) the Libor rate $L(T_0, T_1)$ is not a random variable and is equal to the observed fixing. For this reason, we drop the superscript m. Finally, recall that, since $L(T_j, T_{j+1})^{ud} = L(T_j, T_{j+1})^{du}\ \forall j > 0$ the tree is recombining. This is another advantage of the Ho-Lee/BDT model with respect to other (probably more sophisticated) interest models, such as the Heat, Jarrow and Morton (HJM, [19]) model where the non-recombining tree imposes a major computational burden.

The advantages in using this model are that the price of a straight bond is exogenously determined so that the observed term structure of interest rate can perfectly be reproduced. Moreover, the model can be satisfactorily calibrated by using the implied volatilities from caps, floors and swap option markets.

2.1.2 The marginal dynamics of the stock price

The second risk factor is the stock price and we assume that it evolves over time according to the CRR model. We choose this model because of its simplified approach in pricing, and also because the discrete time dynamics in the CRR framework approximates the Black and Scholes dynamics of the stock price, according to the following stochastic differential equation (SDE):

$$\Delta S_t = \mu S_t \Delta + \sigma_S S_t \sqrt{\Delta} \epsilon_t \tag{2.1}$$

where S_t is the stock price at time t, μ is the drift of the process Δ is the time distance between two observations, σ_S is the instantaneous volatility (or better the pseudo-instantaneous, since it does not refer to a continuous time model) coefficient and ϵ_t can be interpreted as the outcome of a binomial random variable under the natural probability. Notice that as $\Delta \to 0$, ϵ_t tends in distribution to a standard normal random variable so that $\sqrt{\Delta}\epsilon_t$ can be interpreted as a Wiener increment. As a result, the final stock price tends in distribution to a lognormal random variable. This property entails that the adoption of implied volatility measures for pricing equity linked contingent claims ensures a satisfactory calibration of the model to the market data. Of course, as the drift changes, the expected value of the stock price changes too because of the subsequently modification of the probability space. On the contrary, the levels of the future stock prices in each state of the world only depend on the current stock price and on the volatility parameter.

According to the stated environment, the arbitrage free dynamics of the stock price is described by equation 2.1. Therefore the risk neutral dynamics of the stock price is defined as follows:

$$\Delta S_t = r(t) S_t \Delta + \sigma_S S_t \sqrt{\Delta} \epsilon_t^P \tag{2.2}$$

where $r(t)$ is the instantaneous risk free interest rate at time t under the corresponding appropriate risk neutral probability P. The crucial point in our approach is that the instantaneous risk free rate is piecewise constant and evolves over time according to the BDT dynamics, since the spot rate $r(t)$ is not a random variable unless the tenor δ of the rate matures.

In other words, once we choose a term structure (in our examples the Libor rate term structure), the variability of the rate can be observed in practice according to the "nodes" of the term structure. Therefore, if the rate is, as in our case, the Libor, the variability time interval goes from 1 week up to 12 months and coincides with the tenor of the chosen rate. Therefore, once we adopted the 12 months rate, the dynamics over time on a discrete time interval is equal to 12 months. This accounts for a piecewise constant dynamics whose advantage is the reduced computational effort especially for very long term contracts, while the main disadvantage is that longer time steps may have an impact on the early exercise, because for example the dividend date may not necessarily coincide with a time step. For this reason, we also studied a possible way to make the number of steps of the stock tree independent on the number of steps of the interest rate tree. The length of the time interval of the process describing the rate dynamics is therefore set according to the relevant node of the term structure. Theoretically, any node could be used, but if we decide to go for a BDT application we need also volatility data. Since not all the nodes have the same liquidity, the significance of the corresponding implied volatility is not the same across the term structure. We are therefore forced towards those nodes showing the maximum liquidity, since this guarantees the more efficient measure of implied volatility. If we assume (or better observe) that the most liquid node is 12 months, we will adopt a BDT model on a year basis. As a consequence on a certain time horizon (longer than one year in the case under estimation), we will have an array of one year rates defining a corresponding set of probability spaces which can be used for evaluating the stock price dynamics. It follows that the evaluating numeraire is piecewise constant and in a sense "rolls over time" according to the term structure tenor. Therefore what can be regarded as a random variable at the evaluation date is the δ-rate. The risk free rate $r(t)$ and the δ-Libor rate $L(T_j, T_{j+1})$, with $\delta = T_{j+1}T_j$ and $t = T_j$, are connected as follows:

$$\exp\{-r(T_j)n\Delta\} = [1 + L(T_j, T_{j+1})\delta]^{-1} \tag{2.3}$$

where n is the number of steps and is such that $n\Delta = \delta$. As the Libor rate changes, the drift and the up probability in equation 2.2 also change. We notice that the pseudo-instantaneous risk free rate, $r(t)$ is constant from T_j to T_{j+1}, so that $r(l) = r(l + \Delta)$ for each $l = t, ...T_{j+1}\Delta$. Afterwards, the current Libor rate changes and the instantaneous risk free rate will change too.

Assuming for simplicity that the time interval between two observations of the stock price S (and of the interest rate) is equal to the tenor of the Libor rate δ, the risk neutral marginal probability $P^m_{T_j}$ of an up movement of the stock price between time T_j and time T_{j+1}, given that the Libor rate prevailing at time T_j is $L(T_j, T_{j+1})^m$, is equal to:

$$P^m_{T_j} = \frac{B^m_{T_j} - D_S}{U_S - D_S} \tag{2.4}$$

where $B^m_{T_j} = 1 + L(T_j, T_{j+1})^m\delta$, $U_S = \exp\{\sigma_S\delta\}$, $D_S = \exp\{-\sigma_S\delta\}$ and σ_S is the yearly stock price volatility and, for simplicity, it is assumed to be constant over time. This implies that, at each time step, the up probability changes according to the changes of the Libor rate. Moreover, as for the interest rate tree, also the stock tree is recombining because $U_S D_S = 1$.

2.1.3 The joint dynamics of stock price and interest rate under the hypothesis of independence. The one-step case.

In this section we show how to obtain the joint dynamics of the stock price and of the interest rate in the case of independence between them. As generally known, independence is a concept referred exclusively to the probability and not also to the values of the random variables. This means that we can obtain, at each time step, the joint probability density function of the two random variables simply by multiplying their marginal probabilities.

As a consequence of the independence, at each time, the correlation between the future stock price and the future spot rate, conditional on the current value of the spot rate, is zero. This assumption is clearly not realistic, but it is presented in this section as a showcase, while in the next sections we show how to release it. We begin with considering only the first step of the trees while in the next section the two-step case is presented. The example involving only the first step may easily explain the implications of the assumption of independence. As far as the one step case is concerned, let us simplify slightly the notation. Suppose that t is the current time while T is the future time, and suppose for simplicity (and without loss of generality) that $T - t = \delta$, the tenor of the Libor rate. Recalling that the current Libor rate, $L(t,T)$, is not a random variable because it is directly observed, we drop the subscript and superscript from the marginal up probability of the stock price, that is now denoted by p. Moreover, since the up probability of the Libor rate is equal to $1/2$ and is constant over time, it is denoted by q.

According to the BDT model, the future stock price at time T may assume the value of $L(T,T+\delta)^u$ with (risk neutral) probability $q = 1/2$ or the value of $L(T,T+\delta)^d$ with probability $1-q = 1/2$. At the same time, according to the CRR model, the future stock price may assume the value of $S_T^u = S_t U_S$ with (risk neutral) probability p or of $S_T^d = S_t D_S$ with probability $1-p$. In other word, we refer to p and q as the marginal up probability respectively of the stock price and of the Libor rate. If we set the hypothesis of null correlation between $L(T,T+\delta)^m$ and S_T^m, with $m = u, d$, conditional on the current value of the short rate and of the stock price, the joint conditional probability mass function of stock price and short rate can easily obtained by multiplying the two probability measures. A graphic representation of the four states of the world resulting as all possible combinations of the states of the world associated to each risk factor is shown in figure 2.1.

2.1.4 The joint dynamics of stock price and interest rate under the hypothesis of independence. The two-step case.

To show how the procedure can be effectively applied, we consider a two-step scheme. The length of each period is equal to the tenor δ of the Libor rate that is, for simplicity, set equal to 1. The number of steps n of the stock tree is thus equal to 2 and, consequently, the step size Δ is equal to 1.

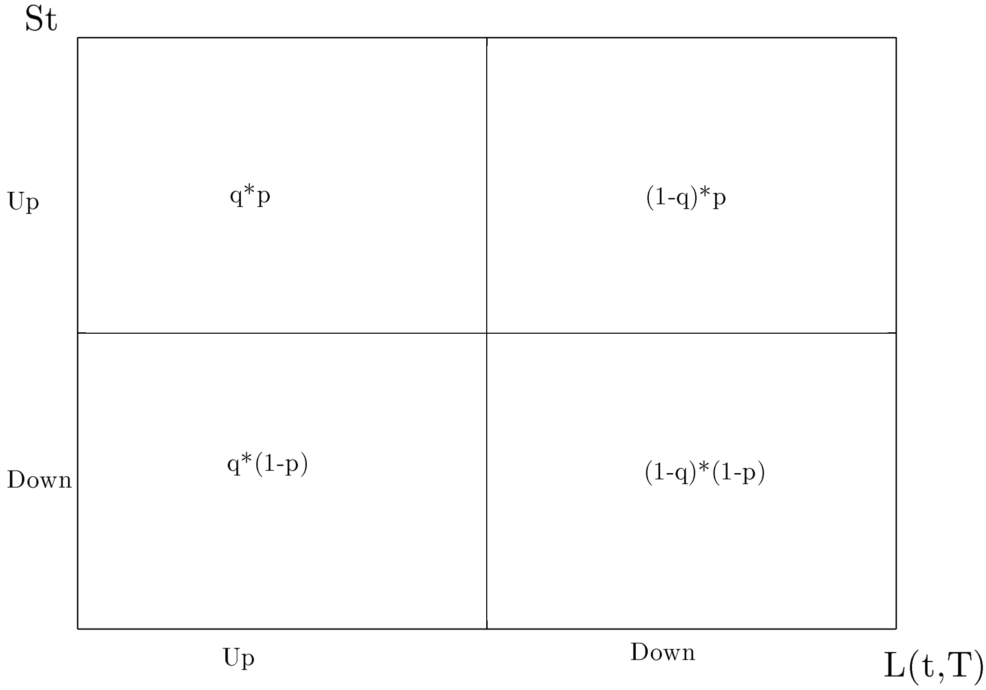


FIGURE 2.1: The joint probabilities of the short rate and of the stock price in the case of independence

Let $L(0,1)$ be the current 12 month Libor rate, whose value may increase to $L(1,2)^u$ or decrease to $L(1,2)^d$ with equal (marginal) probability ($q = 1/2$), according to the BDT model. Analogously, let S_0 be the current stock price whose value, according to the implied stock tree model, may increase to $S_1^u = S_0 U_S$ with risk neutral marginal probability $p_0 = \frac{B_0 - D_S}{U_S - D_S}$ or decrease to $S_1^d = S_0 D_S$ with probability $1 - p_0$, where $B_0 = 1 + L(0,1)$. We can therefore identify four "states of the world" whose probabilities are reported in figure 2.2.

Now, let us consider the node in which the short rate has increased to $L(1,2)^u$ and the stock price has increased to S_1^u. Under the assumption that the two events are independent, the probability that they occur contemporaneously is:

$$Pr\left(S_1^u \cap L(1,2)^u | S_0, L(0,1)\right) = Pr\left(S_1^u | S_0, L(0,1)\right) Pr\left(L(1,2)^u | S_0, L(0,1)\right) = p_1 \frac{1}{2} \tag{2.5}$$

This procedure may be repeated for each possible couple of the short rate and stock price at time $T = 1$. Now, let us consider the state of the world where stock price is S_1^u and the Libor rate is $L(1,2)^u$. At the successive time step, $t = 2$, the stock price may again increase to S_2^{uu} with risk neutral marginal probability p_1^u or decrease to S_2^{ud} with

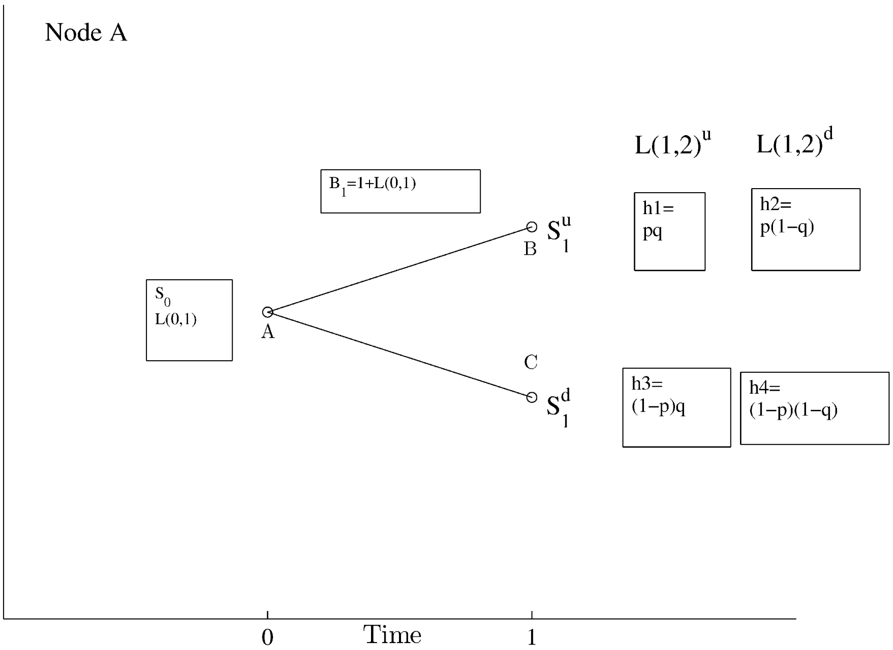


FIGURE 2.2: The first step of the tree

probability $1 - p_1^u$, with $p_1^u = \frac{B_1^u - D_S}{U_S - D_S}$ and $B_1^u = 1 + L(1,2)^u$. At the same time, the Libor rate may increase to $L(2,3)^{uu}$ or decrease to $L(2,3)^{ud}$ with equal probability. The probability that both the stock price and the Libor rate increase for two consecutive times is therefore:

$$Pr\left(S_2^{uu} \cap L(2,3)^{uu} | S_1^u, L(1,2)^u\right) = p_0 \times p_1^u \times \frac{1}{2} \times \frac{1}{2} \tag{2.6}$$

We repeat this procedure until, at the end of the second time interval, all the 16 final states of the world and their respective probabilities are available, as described in table 2.1.

The current stock price S_0 and the future stock price S_2 are thus linked as follows:

$$S_0 = \mathbb{E}_0^h \left[\frac{S_2}{1 + L(1,2)}\right] \frac{1}{1 + L(0,1)} \tag{2.7}$$

where h_t^i is the joint probability measure at time t (in this case $t = 2$) associated to the future values of the stock price and of the Libor rate in the state of the world $i = 1...16$ and $\mathbb{E}_0^h$ denotes the expected value under the probability measure h conditional on the information available at the valuation date $t = 0$. Equation 2.7 states that the current stock price S_0 is the expected value of the future stock price S_2 discounted using an appropriate stochastic discount factor. We notice that the discount factor is only partly

State of the world	Libor rate	Stock price	Probability
1	$L(2,3)^{uu}$	S_2^{uu}	h_2^1
2	$L(2,3)^{ud}$	S_2^{uu}	h_2^2
3	$L(2,3)^{du}$	S_2^{uu}	h_2^3
4	$L(2,3)^{dd}$	S_2^{uu}	h_2^4
5	$L(2,3)^{uu}$	S_2^{ud}	h_2^5
6	$L(2,3)^{ud}$	S_2^{ud}	h_2^6
7	$L(2,3)^{du}$	S_2^{ud}	h_2^7
8	$L(2,3)^{dd}$	S_2^{ud}	h_2^8
9	$L(2,3)^{uu}$	S_2^{du}	h_2^9
10	$L(2,3)^{ud}$	S_2^{du}	h_2^{10}
11	$L(2,3)^{du}$	S_2^{du}	h_2^{11}
12	$L(2,3)^{dd}$	S_2^{du}	h_2^{12}
13	$L(2,3)^{uu}$	S_2^{dd}	h_2^{13}
14	$L(2,3)^{ud}$	S_2^{dd}	h_2^{14}
15	$L(2,3)^{du}$	S_2^{dd}	h_2^{15}
16	$L(2,3)^{dd}$	S_2^{dd}	h_2^{16}

TABLE 2.1: An example of the discrete joint probability density function of the stock price and of the Libor rate

included in the expectation operator (i.e. it is only "partly" stochastic) because, at time $t = 0$, the current Libor rate $L(0,1)$ is known while the Libor rate that will run during the period from 1 to 2 is a random variable. Since the discount factor and the stock price are independent random variables, it can be shown that:

$$S_0 = \mathbb{E}_0^p [S_2] \, \mathbb{E}_0^q \left[\frac{1}{1 + L(1,2)} \right] \frac{1}{1 + L(0,1)} = \frac{\mathbb{E}_0^p [S_2]}{1 + L(0,2)} \tag{2.8}$$

where the last equality holds because of the local expectation hypothesis.

2.1.5 Releasing the independence assumption

In the previous sections we see how to compute the probability associated to each couple of possible levels of stock price and interest rate or, in other words, we proposed a specific joint probability mass function of the two series of random variables. More specifically, we adopted a "naïve" way to do this, very common especially in the practice of pricing convertible bonds (see for instance Hung and Wang, [27]), assuming that stock return and interest rate are independent, so that the joint probability is equal to the product of the marginal probabilities. However, beyond the common belief that interest rates and stock prices are somewhat connected, there exist a significant evidence of negative correlation between the returns on stocks and bonds (see among others Flannery and James, [28]), so that the assumption of zero correlation cannot in general be satisfactory.

However, the restrictive assumption of independence can be released by rearranging in a different way the joint probabilities calculated in the case of independence among the possible states of the world. For instance, if we want to set a perfectly negative correlation, it is sufficient to equally distribute the probabilities, calculated in the case of independence, of contemporaneous up and down movements of both stock price and interest rate to the other two states of the world, according to figure 2.3.

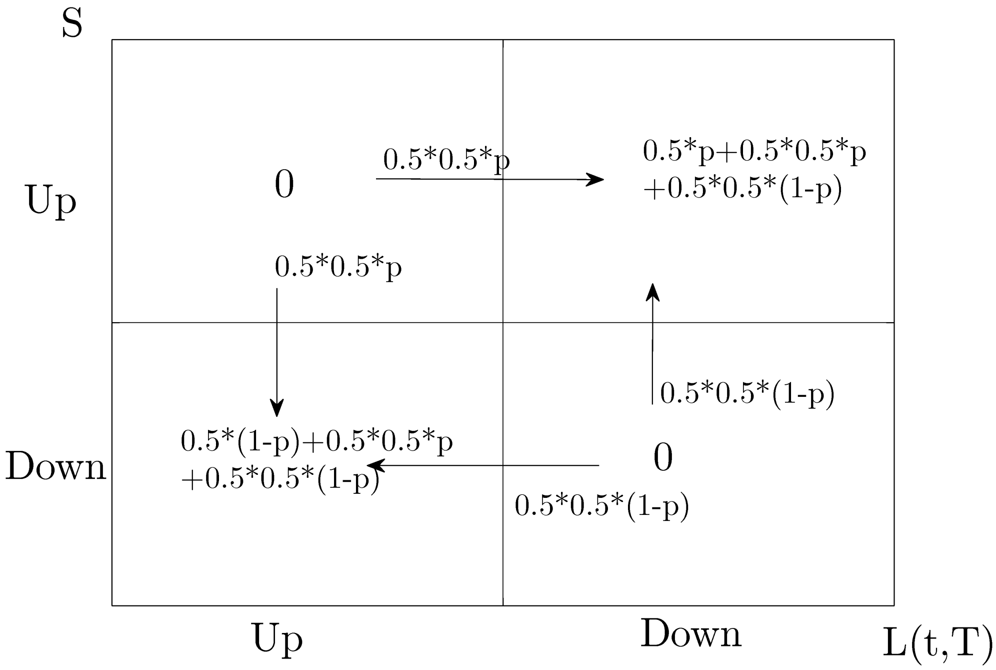


FIGURE 2.3: Perfectly negative correlation between stock price (S_t) and Libor rate ($L(t,T)$)

In general, at each time t, to set a correlation equal to ρ_t, with $\rho_t < 0$, it will be sufficient to equally redistribute to the other two states of the world a percentage of the probabilities of contemporaneous up and down movements of interest rate and stock price. On the contrary, to set a correlation equal to $\rho_t < 0$, it will be necessary to equally redistribute to the other two states of the world a percentage of the probability of opposite up and down movements (i.e. rate goes up/stock goes down and vice versa), of the two risk factors. Table 2.2 reports the joint probability distribution function at time t for the three cases of positive, null and negative correlation.

2.2 The price of a European call option

To show how the procedure can be exploited for pricing derivatives we begin by pricing a European plain vanilla call option, while in the next section the issue of the early exercise will be discussed for pricing American style options.

State of the world			Correlation		
Stock price	**Libor rate**	**Prob.**	$\rho_t < 0$	$\rho_t = 0$	$\rho_t > 0$
up	up	h_t^1	$p_t 0.5(1+\rho_t)$	$p_t 0.5$	$0.5(p_t + 0.5\rho_t)$
down	down	h_t^2	$(1-p_t)0.5(1+\rho_t)$	$(1-p_t)0.5$	$0.5(1-p_t+0.5\rho_t)$
up	down	h_t^3	$0.5(p_t - 0.5\rho_t)$	$p_t 0.5$	$p_t 0.5(1-\rho_t)$
down	up	h_t^4	$0.5(1-p_t-0.5\rho_t)$	$(1-p_t)0.5$	$(1-p_t)0.5(1-\rho_t)$

TABLE 2.2: Joint probability mass function at each node

Equation 2.8 implies a very important property: the probability measure h, defined as the joint probability associated to the future values of the stock price and of the Libor rate, is such that the stock price is a martingale. Once the joint PDF of the final stock price and of the Libor rate is obtained, it is quite a simple task to determine the price ξ_0 of a European call option at time $t = 0$, with strike price X:

$$\xi_0 = \mathbb{E}_0^h \left[\frac{[S_2 - X]^+}{1 + L(1,2)} \right] \frac{1}{1 + L(0,1)} \tag{2.9}$$

Equation 2.9 implies that the price of a call option is approximately equal to the price of the same option calculated by means of the CRR model. In other words, the model here developed can be regarded as an extension of the implied stock tree model on a roll-over basis. However, this could not be the case if we release the hypothesis of zero correlation between interest rate and stock price, because in the original CRR model the interest rate is not stochastic and therefore there is no reason for any relationship between the rate and the stock price. On the contrary, a change of the term structure of interest rates produces a difference in the price of the call option, since that change can be regarded as the adoption of a different parameter. Therefore, there is a certain interest in evaluating pricing differences emerging from the implementation of one model against the other. To this aim it is necessary to get the joint PDF of the stock price and of the interest rate.

Let us consider the simple framework of the previous section where $n = 2$, $\delta = 1$. We set the stock price equal to 100 and its implied annual volatility to 20%. Finally, we assume that the spot Libor rate is 1% for the first year and 2% for the second year and 3% for the third year, while the term structure of the interest rate volatility is equal to 20% and 30% for the second and the third year respectively. All the parameters are reported in table 2.3.

Two remarks are necessary. Firstly, we notice that the Libor rate for maturities over 12 months is not directly observable on the market. Those values can however be obtained from the swap curve on the Libor rate by means of bootstrap technique (details in Hull, [29]). Secondly, we notice that to the aim of pricing a plain vanilla call option it

Parameters	Time (years)		
	0	1	2
S_0	100	-	-
$L(0,T)$	1%	2%	3%
σ_S	-	20%	30%
σ_S	-	20%	30%

TABLE 2.3: Parameters adopted for the calibration of the model

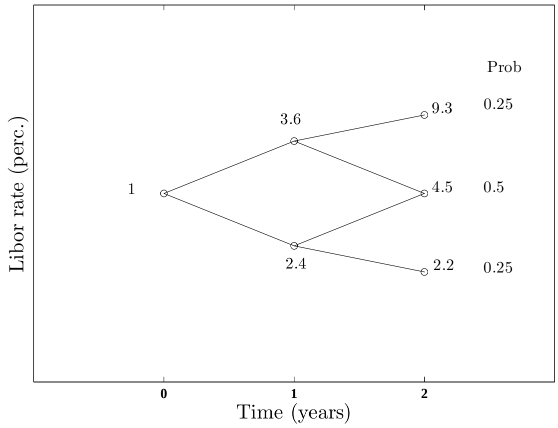
is not necessary to specify the 3-year interest rate. We decide however to consider it to show the final joint PDF of the interest rate and stock price. Such PDF may be adopted to compute the price of financial products whose value depends contemporaneously on the level of stock price and interest rate at the maturity, as for example convertible bonds.

Figures 2.4(a) and 2.4(b) show respectively the BDT interest rate tree and the binomial stock tree (with marginal probabilities) calculated adopting the parameters shown in table 2.3, while figure 2.5 shows an example of a three periods lattice representing the joint dynamics of the Libor rate and of the stock price.

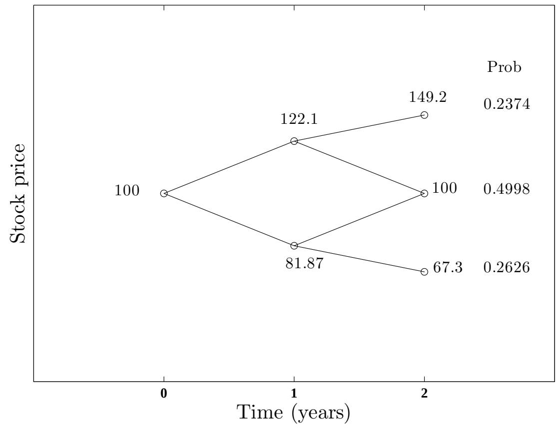
Given the values in the figures 2.4(a) and 2.4(b), it is a simple task to determine the joint PDF of the interest rate and of the stock tree. For example, the probability that the stock price will be 100 at the end of the 2nd year jointly with an interest rate level of 9.3% is 0.25×0.4998.

To the aim of pricing a stock option, it is necessary to calculate the terminal payoff of the option and then roll back the tree represented in figure 2.5 till time $t = 0$. The terminal nodes are the simpler. Take node E, I and J as an example. The value of the stock option at node E, ξ_E, is therefore the present value, calculated using the rate $L(2,3)^{uu}$, of the weighted average of the payoff of the call at nodes I and J using as weights the probabilities P_2^{uu} and $1 - P_2^{uu}$ respectively. Notice that computing the present value of the average is possible only at the terminal nodes. In all the other cases, it is necessary to compute the average only after the present value is calculated.

It is worth noting that at nodes I and K the stock price is the same, being equal to $S_3^{uuu} = S_0 U_S^3$, and thus, the payoff of the call option is $\xi_I \equiv \xi_K = (S_3^{uuu} - X)^+$, where X is the strike price. The difference between the two cases is in the discount rate that, at node I is equal to $L(2,3)^{uu}$ while at node K is equal to $L(2,3)^{ud}$. Once the payoff of the call at node E, F, G and H are calculated, we can use the joint probability mass function, computed by means of table 2.2, to determine the price of the call at node A, ξ_A. Such value is therefore equal to



(a) Example of the BDT tree



(b) Example of implied binomial stock tree

FIGURE 2.4: Example: binomial lattices of Libor rate and stock price

$$\xi_A = \frac{\xi_E h_2^1}{1 + L(1,2)^u} + \frac{\xi_F h_2^3}{1 + L(1,2)^d} + \frac{\xi_G h_2^4}{1 + L(1,2)^u} + \frac{\xi_H h_2^2}{1 + L(1,2)^d} \tag{2.10}$$

Similarly, it is possible to determine the time $t = 0$ value of the stock option by rolling back the tree.

2.2.1 A numerical example

We remark that, if we set the hypothesis of zero correlation, the price of the call option calculated by means of the procedure exposed in sections 2.1.3 and 2.1.4 and coincide with the price of the same option calculated by means of the CRR model in the case where the term structure of the interest rate is not flat. More precisely, the risk free rate for the first year coincides with the corresponding spot rate while the risk free rate

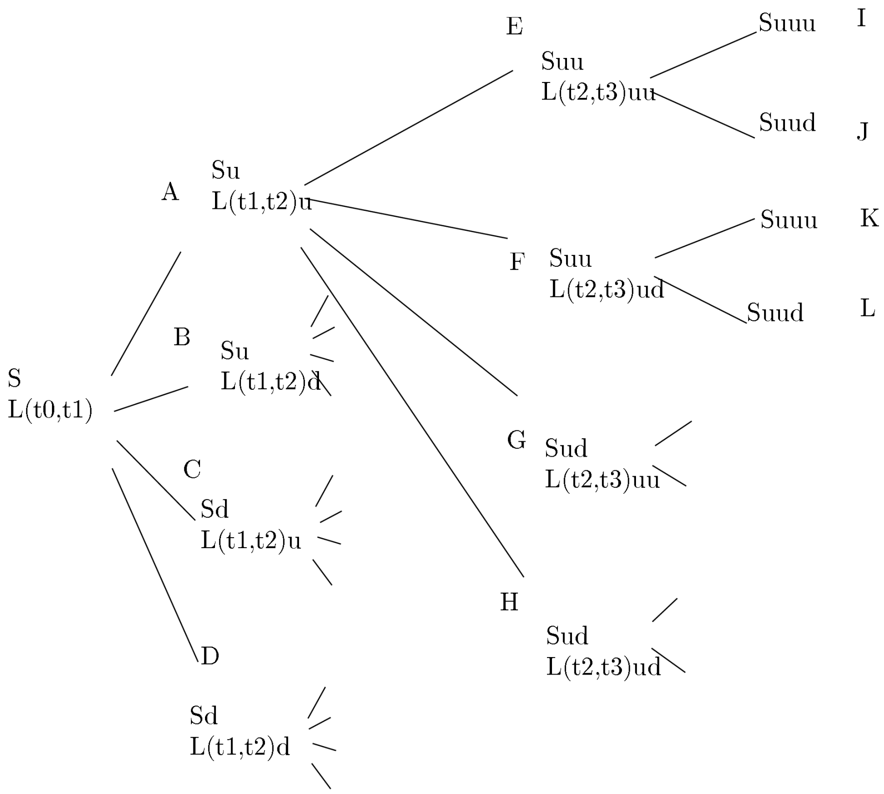


FIGURE 2.5: Example of the joint dynamics of the stock price (S) and Libor rate ($L(t,T)$)

for the second year coincides with the corresponding forward rate. In this way we are able to incorporate market expectations (at the valuation date) on the future interest rates in the pricing of option.

To show how interest rate expectations affect the stock option pricing, we compare the price of a call option, derived in our framework, to the Black and Scholes price. We decide to adopt the Black and Scholes model for the comparison firstly, because the CRR price tends to the Black and Scholes price as $\delta \to 0$, and secondly, because the Black and Scholes price is not able to capture the expectations on the future interest rates. Table 2.4 reports the differences (in percentage) between the price of a two year ATM vanilla call option calculated by means of our procedure (ξ) and the price of the same option calculated by means of the Black and Scholes (1973) model (Bls), according to the following formula:

$$\frac{\xi - Bls}{Bls} \tag{2.11}$$

Such differences are calculated for different term structures of the interest rates. The other parameters adopted to evaluate the price differences are those reported in table

2.3 but this time we consider, for each year, a higher number of steps for the stock tree. More precisely, since the stock exchange is open about 254 days per year, we thus set $n = 508$ ($\Delta = 1/254$).

We thus notice that if the term structure is flat (see the numbers on the diagonal of table 2.4), the percentage difference with the Black and Scholes formula is quite negligible, from .03% to .05%. However, as expected, such differences tend to increase as the difference between the interest rates for the two maturities increases. If, on the contrary, $L(0,1) < L(0,2)$ ($L(0,1) > L(0,2)$) the term structure is upward (downward) sloping and the price differences are positive (negative).

		$L(0,2)$				
		1%	**2%**	**3%**	**4%**	**5%**
	1%	.05	6.96	13.10	18.58	23.48
	2%	-7.41	.04	6.68	12.60	17.90
$L(0,1)$	**3%**	-15.12	-7.10	.04	6.41	12.11
	4%	-23.06	-14.45	-6.80	.04	6.15
	5%	-31.22	-22.01	-13.82	-6.51	.03

TABLE 2.4: Price differences with respect to the Black and Scholes ([**?**]) formula for different interest rate term structures.

2.3 The replicating portfolio

To emphasize some advantages of the procedure, in this section we show a different approach for pricing a European call option based on the replicating portfolio and on partial differential equations (PDEs), where the interest rate is stochastic. This approach is the same used by Brennan and Schwartz ([30]) for pricing convertible bonds, but we use it here for a simpler example. In the next chapters, we will use the approach based on PDEs for pricing convertible bonds to the aim of performing a comparison between the proposed numerical procedure and other convertible bond pricing models.

2.3.1 Binomial model and replicating portfolio

Suppose that, at time t, we are interested in composing a replicating portfolio by using Y stocks with price equal to S_t and an investment of Z Euro in the bank account B_t^m, where m denotes the particular state of the world in which the Libor rate can be observed.

To begin with, consider the terminal nodes of the lattice represented in figure 2.4. Take nodes I, J and E as an example. As a matter of fact, the composition of the replicating portfolio here is not different from that illustrated by Cox et al. ([1]) within their traditional binomial model. The issue is to set the values of Y and Z such that the replicating portfolio assumes the value of $\xi_I = (S_3^{uuu} - X)^+$ if the stock price from node E goes up to node I, and the value of $\xi_J = \left(S_3^{uud} - X\right)^+$ if the price goes down to node J. Therefore, the values of Y and Z are the two unknowns of the following system of two linear equations:

$$\begin{cases} Y S_3^{uuu} + Z B_2^{uu} = \xi_I \\ Y S_3^{uud} + Z B_2^{uu} = \xi_J \end{cases} \tag{2.12}$$

whose solutions are:

$$Y = \frac{\xi_I - \xi_J}{S_3^{uuu} - S_3^{uud}} = \frac{\xi_I - \xi_J}{(U_S - D_S) S_2^{uu}} \tag{2.13}$$

$$Z = \frac{U_S \xi_J - D_S \xi_I}{(U_S - D_S) B_2^{uu}} \tag{2.14}$$

By the law of one price, the call value at node E must equal the value of the replicating portfolio of stocks and bank account using the weights illustrated by equations 2.13 and 2.14. The value of the call option is therefore:

$$\xi_E = \frac{1}{B_2^{uu}} \left(\frac{B_2^{uu} - D_S}{U_S - D_S} \xi_I + \frac{U_S - B_2^{uu}}{U_S - D_S} \xi_J \right) \tag{2.15}$$

It is easy to recognize that, as far as terminal states are concerned, the risk neutral probabilities are the same as in the traditional binomial model. On the contrary, differently from the traditional binomial model, notice that the values of Y and Z at each time say $t+1$ depend not only on the specific value of the stock price, but also on the value of the interest rate prevailing at time t and expiring at $t+1$.

The other steps of the tree ("non-terminal steps or nodes") are more interesting even if the technique of the replicating portfolio is exactly the same. However, in this case we have 4 possible states of the world (take for instance E, F, G, H) and thus a system of 4 linear equations in 4 unknowns. Intuitively, this unknowns are the values of Y and Z conditional on the prevailing Libor rate or, in other words: Y^u, Z^u, Y^d, Z^d, where the superscripts denote the particular state of the Libor rate (up or down). This is the case because the weights of the replicating portfolio will now depend not only on S, but also on the Libor rate. The system can be stated as follows:

$$\begin{cases} Y^u S_2^{uu} & + Z^u B_1^u & = \xi_E \\ & Y^u S_2^{ud} + Z^u B_1^u & = \xi_G \\ Y^d S_2^{uu} & & + Z^d B_1^d = \xi_F \\ & Y^d S_2^{ud} & + Z^d B_1^d = \xi_H \end{cases} \tag{2.16}$$

The system can be solved in different ways, using the Cramer's rule or by substitution. Such coefficients are zero because the states of the world are mutually exclusive. If we also add the hypothesis (strong but very common when pricing models in complete markets are concerned) that the states are also collectively exhaustive, we can state that the system has a solution if there is no arbitrage opportunity and that, if the market is complete, the solution is unique. This situation is consistent with the Arrow and Debreu ([31]) state price model. However, to solve this system it is worth noting that it can be split into the two following sub-systems, depending on the prevailing Libor rate:

$$\begin{cases} Y^u S_2^{uu} + Z^u B_1^u = \xi_E \\ Y^u S_2^{ud} + Z^u B_1^u = \xi_G \end{cases} \quad \begin{cases} Y^d S_2^{uu} + Z^d B_1^d = \xi_F \\ Y^d S_2^{ud} + Z^d B_1^d = \xi_H \end{cases} \tag{2.17}$$

The solutions of these systems are exactly the same as before. However in this case, the price of the call depends on the Libor rate. Suppose for example that the investor will solve the left system in 2.17 with probability q and the right system with probability $1-q$. Therefore, the price of the call will be:

$$\xi_A^u = \frac{1}{B_1^u}\left(\frac{B_1^u - D_S}{U_S - D_S}\xi_E + \frac{U_S - B_1^u}{U_S - D_S}\xi_G\right) \tag{2.18}$$

with probability q, if the rate increases and

$$\xi_A^d = \frac{1}{B_1^d}\left(\frac{B_1^d - D_S}{U_S - D_S}\xi_F + \frac{U_S - B_1^d}{U_S - D_S}\xi_H\right) \tag{2.19}$$

with probability $1-q$ if the rate decreases. If the stock price and the Libor rate are supposed to be independent, the call price at node A is therefore the average of ξ_A^u and ξ_A^d calculated using as weights the up and down probabilities of the interest rate respectively. If on the contrary the two risk factors are not independent, an assumption is necessary in order to determine the call price at node A, as the one stated in section 2.1.5. Appendix A shows that the risk neutral probabilities associated to each of the four possible states of the world are those illustrated in table 2.2. .

2.3.2 The partial differential equation approach

In this section we propose a replicating portfolio where the risk factors are represented by continuous stochastic processes. Suppose that at time t the change in the spot interest rate is assumed to be given by the following stochastic differential equation:

$$dr(t) = \mu(r(t), t)dt + \sigma(r(t), t)dW_r(t) \tag{2.20}$$

where $\mu(r(t), t)$ and $\sigma(r(t), t)$ are the drift and the diffusion coefficient of the process, $dW_r(t)$ is a Wiener increment. The first issue to face within a continuous time model of the interest rate is to define the rate itself. As a matter of fact, the instantaneous rate is not directly observable on the financial markets. Therefore, to reconcile the model with the market conventions we need to define the short rate as the semielasticity of a pure discount bond $v(t, T)$ with respect to time, similarly to equation 2.3:

$$v(t, T) = [1 + L(t, T)\delta]^{-1} = \exp\{-\int_t^T r(t)du\} \tag{2.21}$$

where the spot rate is therefore defined as follows:

$$r(t) = -\frac{\partial \log v(t, T)}{\partial t} = -\frac{1}{v(t, T)}\frac{\partial v(t, T)}{\partial t} \tag{2.22}$$

Assume also that the stock price follows a Ito's process such as that described by equation 2.20:

$$dS(t) = \alpha(S(t), t)dt + \beta(S(t), t)dW_S(t) \tag{2.23}$$

where $\alpha(S(t), t)$ and $\beta(S(t), t)$ are the drift and diffusion coefficient of the process and where:

$$dW_S(t)dW_r(t) = \rho(t)dt \tag{2.24}$$

where $\rho(t)$ is the instantaneous correlation between $r(t)$ and $S(t)$.

Suppose that we are interested in evaluating a derivative whose value depends on the stock price and on the interest rate. Let $Q(r(t), S(t), t)$ denotes the time t value of such derivative. Given the above assumptions, it is shown in the Appendix B that the value of the derivative satisfies the following partial differential equation:

$$\frac{\partial Q}{\partial S}\alpha + \frac{\partial Q}{\partial r}\left[\mu - \lambda r\sigma\right] + \frac{\partial Q}{\partial t} + \frac{1}{2}\frac{\partial^2 Q}{\partial S^2}S^2\beta^2 + \frac{1}{2}\frac{\partial^2 Q}{\partial r^2}r^2\sigma^2 + \frac{\partial^2 Q}{\partial S \partial r}\beta\sigma\rho - rQ = 0 \quad (2.25)$$

where the parentheses are omitted for convenience of the reader, λ is the market price of interest rate risk. It can be thought as the reward to variability ratio of a portfolio whose rate of return is perfectly correlated with changes in the interest rate (e.g., any long-term bond) and, as shown in Appendix B, it is equal to:

$$\lambda = \frac{\mu_{v_i} - r}{\sigma_{v_i}} \quad (2.26)$$

for each bond $i, v_i(t, T_i, r)$, with $i = 1, 2...k$. μ_{v_i} and σ_{v_i} are the drift and the diffusion coefficient of the process describing the change in the value of the bond. If the pure expectations theory of the term structure holds then $\lambda = 0$. As emphasized at the beginning of this section, a particular issue of this approach is that the interest rate is not directly observable and as a consequence, we need to specify the model by estimating its parameters each time. As illustrated in Chapter 1, the calibration of interest rate model such as CIR or Vasicek is not a simple task. On the contrary, the numerical procedure proposed above admits only input parameters directly observable on the financial markets.

Once the parameters are estimated, we need to solve the partial differential equation in 2.25 under the boundary conditions describing the final value of the derivative. In the case of a plain vanilla stock option with maturity T and strike price X, the condition is:

$$Q(r(T), S(T), T) = [S(T) - X]^+ \quad (2.27)$$

2.3.3 A comparison between PDE approach and binomial model

In this section we perform a comparison between the PDE approach and the numerical procedure that is the object of this thesis. To this aim, we must choose a stochastic process for the interest rate. To perform a significant comparison, it would be necessary to choose a stochastic model that is as consistent as possible with the dynamics of the Libor rate adopted by the BDT model. To the best of our knowledge, it does not exist a perfectly comparable model or dynamics. Even if we chose a lognormal dynamics and used the implied volatility as diffusion coefficient, the problem of calibrating the drift component would remain. A possible solution to this problem is the adoption of the HJM ([19]) dynamics of the interest rate, allowing to obtain the Hoo and Lee ([3])

implied tree, with a constant volatility structure. However, also this solution is not satisfactory at least for three reasons:

- the risk factor considered by the HJM model is the instantaneous forward rate, and not the spot rate;
- the volatility structure cannot fit the observed term structure of the interest rate volatility, unless binomial lattices are concerned;
- the trajectories of the interest rate can assume negative values or may not be mean reverting.

To understand the first point, consider the HJM dynamics of the instantaneous forward rate $f(t,T)$ under the assumption of constant volatility. Such dynamics can be written as follows:

$$df(t,T) = \sigma(T-t)dt + \sigma dW_f(t) \tag{2.28}$$

To obtain the dynamics of the spot rate, it is necessary to consider the limit of equation 2.28 for $T \to t$. This would yield the following dynamics:

$$df(t,t) \equiv dr(t) = \sigma dW_f(t) \tag{2.29}$$

that is clearly too simple for describing the dynamics of the term structure of interest rate. First of all, it is not ensured the reversion of the interest rate level towards the long period average. We remark that also in the BDT model the mean reversion is not ensured. In general, if mean reverting features are not directly embedded in the model (such as Vaiscek or CIR model), an analogous result can be obtained imposing a particular "shape" to the term structure of the volatility of the interest rate. More precisely, a decreasing or a "humped" volatility structure can ensure a reasonable reversion of the interest rate towards the long run average. A slightly more complex model can be obtained if the volatility structure would be allowed to change over time, at least deterministically, to account for non-flat term structure of volatility. But this lead to the second point: this is quite not a simple task if binomial trees or Monte Carlo simulations are not involved.

For this reason, we choose a different process of the short rate. More precisely, we adopt the CIR model, as described in 1.3.2. The parameters are estimated on the base of only two prices, calculated by using equation 2.21. The inputs of the model are those described in table 2.3. We also set a correlation equal to zero, because the aim of this section is only to compare two methods instead of the numerical results.

Another issue si that, although some authors suggest (see for example Wilmott, [32]) the use of PDEs as the single most useful tool in derivatives pricing, others suggest that they are quite vulnerable to numerical difficulties and, while acknowledging the role of finite difference methods, they suggest the use of lattice-based methods whenever possible (see, e.g., Luenberger [33]).

Finally, as far as American options are concerned, another issue raises: for each time before expiration, there is a critical value for the price of the underlying asset at which it is optimal to exercise the option. Depending on the option type (call or put), it will also be optimal to exercise the option for prices above and below the critical price. So with American options we should cope with a free boundary, i.e., a boundary within the domain, which separates the exercise and no-exercise region.

On the contrary, for the stock price, we use the dynamics represented in equation 2.3 in a risk neutral environment, i.e. where $\alpha(S(t),t) = r(t)S(t)dt$ and $\beta(S(t),t) = \beta S(t)$, and β is the market volatility of the stock, calculated by means of the Black and Scholes formula.

Since the aim of this thesis is far from a complete explanation of the numerical methods used to solve the PDE in equation 2.25, it is sufficient here to recall that the methods selected is the forward finite difference approximation, using discretization steps equal to the number of steps adopted in section 2.2.1. Further details about the procedure can be found in Brandimarte [34].

In the table below, we compare the price of a call option, derived in the framework developed in this thesis, to the price of the same option calculating using the PDE approach. Table 2.4 reports the differences (in percentage) between the price of a two year ATM vanilla call option calculated by means of our procedure (ξ) and the price of the same option calculated by means of the PDE approach as described above (PDE), according to the following formula:

$$\frac{\xi - PDE}{PDE} \tag{2.30}$$

As in 2.2.1, the differences are calculated for different term structures of the interest rates. The other parameters adopted to evaluate the price differences are those reported in table 2.3. We consider, for each year, 254 steps and thus set $n = 508$ ($\Delta = 1/254$).

$L(0,2)$						
		1%	**2%**	**3%**	**4%**	**5%**
	1%	-3.03	7.26	10.34	-8.04	-9.72
	2%	8.88	2.94	6.68	7.22	-10.33
$L(0,1)$	**3%**	-15.12	-10.89	-1.02	-6.67	-4.89
	4%	-9.44	-7.54	-6.80	-.30	1.24
	5%	14.38	11.01	-7.69	-4.38	-1.64

Table 2.5: Price differences with respect to the partial differential equation approach for different interest rate term structures.

As it can be noticed by comparing table 2.4 and 2.5, in general, the differences in the second table are, in absolute value narrower than it is in the first table. Moreover, also in this case, the smallest differences are on the diagonal. We think that this evidence is consistent with the findings of chapter 1 that, during periods in which the interest rate is volatile (at it is the case when the term structure is very steep), the CIR model does not produce appreciable results, because it can generate prices too far from those observed on the financial market.

Finally, the differences are very heterogeneous, and no pattern can be recognized. Also this result is consistent with the fact that the estimation of the parameters time by time (depending on the observed term structure) may result in a poor dynamic behaviour, as pointed out also by Ritchken ([16]). Dynamic accuracy is important, especially if hedges are to be constructed and replicating strategies implemented. For the sake of clarity, it is important to emphasize that the term structure constrained models (such as Hull and White model [35]) have been criticized because of their *ad hoc* nature. This is the case because at a certain point in time, say at the beginning of the trading day, the initial conditions are determined under the hypothesis that the parameters does not change. Unfortunately, it could be possible that, after few hours (not to say minutes) the entire model has changed. For this reason, it is necessary to compute the price as soon as the market data is available.

Chapter 3

The price of American options

As generally known, options may be distinguished between European and American options. The former can be exercised only on the expiration date while the latter can be exercised at any time on or before the expiration date. The pricing of European options has been presented in the previous chapter. After a brief review of the American option pricing theory, in this chapter we explain how to apply the two-factor numerical procedure for pricing American call and put options. We focus particularly on put-call parity, the early exercise of American options and the way our numerical procedure deals with such issues. We illustrate the theory by distinguishing whether the underlying stock pays dividends or not.

3.1 Basic properties of American options. The case of non-dividend paying stock

Let $\Xi(S;t;T;X)$, $\xi(S;t;T;X)$, $\Pi(S;t;T;X)$ and $\pi(S;t;T;X)$ denote the values at time t of American and European calls and American and European puts respectively. All the options are written on a stock with current price of S, have expiration date T and strike price $X \geq 0$. As always occurs in traditional option pricing theory (see Merton, [2]), we assume that both the stock and the option are traded in arbitrage free markets and there are not taxes or transactions costs.

For a European call option, at the maturity date T the holder has to decide between paying X dollars for buying one share of common stock whose value is S_T or decide to not exercise the option receiving nothing. Clearly, the call will be exercised only if $S_T > X$.

For an American call option the problem is more complex because, in addition to exercise the option at the maturity, the same right is also grant any time prior to maturity. Also in this case, the option will be left unexpired if $S_t < X$. However, this does not mean that the investor will always exercise the option when $S_t > X$. This is the case because the time value may suggest that selling the option on the market may be worth more.

For European puts similar reasoning applies. We assume in this section that the stock has "limited liability", i.e. the holder of the stock is never obliged to pay for cash injections and he or she pays only for the market price when the stock is sold. This implies (among other things) that $S_t = 0$ if and only if $S_T \equiv 0$ for all $T > t$. Moreover, since the holder can choose whether or not to exercise, the options have limited liability to him or her (long party) but not to the writer (short party).

The following propositions are very basic statements in option pricing theory, and we recall that for the convenience of the reader. In particular, this propositions depend only on the very basic assumption that investors utility function is increasing in the final wealth and there are not dominated securities. The intuition of each proposition is given below in this section while the proofs (when necessary) are shown in Appendix C.

- **Proposition 1**:

$$\Xi(\cdot) \geq 0; \qquad \xi(\cdot) \geq 0; \qquad \Pi(\cdot) \geq 0; \qquad \pi(\cdot) \geq 0.$$

- **Proposition 2**:

$$\Xi(S;T;T;X) = \xi(S;T;T;X) = (S - X)^+;$$
$$\Pi(S;T;T;X) = \pi(S;T;T;X) = (X - S)^+.$$

- **Proposition 3**:

$$\Xi(S;t;T;X) \geq S - X; \qquad \Pi(S;t;T;X) \geq X - S:$$

- **Proposition 4**: *for* $T_2 > T_1$:

$$\Xi(\cdot;T_2) \geq \Xi(\cdot;T_1) \quad \textit{and} \quad \Pi(\cdot;T_2) \geq \Pi(\cdot;T_1).$$

- **Proposition 5**:

$$\Xi(\cdot) \geq \xi(\cdot) \quad \textit{and} \quad \Pi(\cdot) \geq \pi(\cdot).$$

- **Proposition 6**: *for* $X_1 < X_2$:

$$\Xi(\cdot; X1) \leq \Xi(\cdot; X2) \quad \textit{and} \quad \xi(\cdot; X1) \leq \xi(\cdot; X2);$$
$$\Pi(\cdot; X1) \geq \Pi(\cdot; X2) \quad \textit{and} \quad \pi(\cdot; X1) \geq \pi(\cdot; X2).$$

- **Proposition 7**:

$$S \equiv \Xi(S; t; \infty; 0) \geq \Xi(S; t; T; X) \geq \xi(S; t; T; X).$$

- **Proposition 8**:

$$\xi(0; \cdot) = \Xi(0; \cdot) = 0.$$

Proposition 1 restates the limited liability of option contracts: the holder of the options have the exercise right, and he will be never asked for a cash injection into its positions but the initial payment. Since the option imply only additional rights, its value cannot be negative.

Proposition 2 states that at the expiration date the option will be exercised if it has a positive "intrinsic value" (i.e. the value if exercised, defined as $S - X$ for calls and $X - S$ for puts); otherwise it will be left unexercised.

Proposition 3 states a very basic no arbitrage condition: American options must sell for at least their intrinsic value or an immediate arbitrage profit could be made simply by purchasing the option and exercising. This may not be the case for European options since early exercise is not permitted so that this arbitrage is not available.

Propositions 4 and 5 are the same concept of proposition 1 generalized in a dominance argument: additional rights cannot have negative value. In proposition 4 the dominance argument refers to two different time to mautity $(T_1; T_2)$ while in Proposition 5 it refers to the possibility to exercise the option also before the expiration date.

Proposition 6 states that calls (puts) are nonincreasing (nondecreasing) functions of the exercise price. It can be proved with a very simple dominance argument for both American and European options. To well understand this point, it can be worth thinking that the first call is exercised for a profit $(S - X_1)$ that is lower than that of the second call $(S - X_2)$ in every future possible state of the world.

Proposition 7 follows directly from Propositions 4 and 6 (first inequality) and Proposition 5 (second inequality). Proposition 8 follows from Proposition 7 and from the assumption of limited liability for long option positions. From a different point of view, Propositions 7 and 8 depend on the assumptions that $X \geq 0$ and the stock has limited liability.

Notice that the time path followed by the stock price is not relevant for European options and does not affect their value. On the contrary, American options are "path-dependent" financial products and interim values can affect the decision whether or not to exercise before the maturity.

- **Proposition 9**: *for a stock paying no dividends,* $\Xi(S;t;T;X) \geq S - Xv(t,T)$. *In this case exercise of the American call will never occur prior to maturity, and* $\Xi(\cdot) \equiv \xi(\cdot)$.

Proof. See Appendix C.

It is worth noting that the same does not hold for American put options and in general early exercise has positive value. To illustrate this, suppose that the stock price has dropped to a point where $S_t < X - Xv(t,T)$. In this case the value of the option if exercised, $X - S$ is greater than the present value of the maximum payoff from exercise at maturity, $Xv(t,T)$. Therefore, waiting until expiration is not optimal.

For European puts and calls with the same striking prices, the optimal exercise strategies are perfect complements. At maturity one and only one of the two will be exercised and if the options are at-the-money then both may be exercised at a zero value. For American options the optimal exercise strategies for puts and calls are not perfect complements. This means that the put-call parity relation that holds for American options is different from the case of European options.

- **Proposition 10**: *put-call relation for American options.*

$$\Pi(S;t;T;X) \geq \Xi(S;t;T;X) - S + Xv(t,T) : \tag{3.1}$$

Proof. See Appendix C.

One proposition which has not been taken into account here is that the value of a call (put) option is an increasing (decreasing), convex function of the underlying stock price. These properties probably appear to be eminently reasonable, perhaps in the first case even "obvious". Nevertheless, it do not necessarily apply in all cases, since it depends on the stochastic process of the underlying asset.

To illustrate this, consider the following economy where the price of the underlying asset is 50 at current time, and it will change to either 100 or 0 after one period.

Alternatively, the current price of such asset may be 60, and after one period the price will change to either 70 or 50. In both cases possible values of the probabilities supporting this economy are each equal to $\frac{1}{2}$. In other words, this is a valid economy since there exists a probability measure, say p such that $50 = p \times 100 + (1-p) \times 0$ and $60 = p \times 70 + (1-p) \times 50$. This is the case by setting $p = .5$. Moreover, if the market is complete, such probability measure is unique.

Now consider a call option with an exercise price of 60. If the stock price is 50, then the call value is:

$$\xi(50; 60) = .5 \times (100 - 60) + .5 \times 0 = 20. \tag{3.2}$$

If the stock price is 60 he call value is:

$$\xi(60; 60) = .5 \times (70 - 60) + .5 \times 0 = 5. \tag{3.3}$$

It is clear that, for the stochastic process just described, the call value is not an increasing function of the stock price. In this case we say that the option price is not homogeneous in S and X. Recall that the concept of homogeneity of degree n applies if the considered variables (S and X in this case) are to the power of n. In this case, we are interested in verifying a very natural properties of option prices: they must be homogeneous of degree 1 in S and X, implying for example, for a call option, it must be that:

$$\xi(\theta S; t; T; \theta X) = \theta \xi(S; t; T; X) \tag{3.4}$$

for all $\theta \geq 0$.

In addition, notice that since $\xi(0; 60) = 0$, the function is not convex either. Using the put-call parity (whose details are shown in Appendix C), we can also see that the put with an exercise price of 60 would not be decreasing or convex in the stock price.

Even if the stochastic process is assumed to be proportional to the underlying asset is not sufficient to guarantee this result, as it has been claimed. In fact, recall that a stochastic process is said to be proportional if the distribution of percentage changes in the stock price does not depend on the level of the stock price. An example of proportional stochastic process is the lognormal distribution with constant mean and variance.

As a counterexample to this second statement, consider the situation where the current stock price is 40, and the following four states of the world are possible in the next period:$\{56; 44; 36; 24\}$. Alternatively, consider a situation where the current stock price is 50, and the following four states of the world are possible in the next period: $\{70; 55; 45; 30\}$. Suppose the probability of each outcome is $\frac{1}{4}$. Notice that in each case the stock price can increase or decrease by 40% or 10%, so the stochastic process is proportional.

Suppose in the first case that the supporting state prices (i.e. the amount to pay for receiving one dollar if a certain state of the world occurs) are $\{.4; .1; .1; .4\}$ and in the second case that they are $\{.02; .48; .48; .02\}$. Then the values of calls with an exercise price of 55 are respectively the following:

$$\xi(40; 55) = .4 \times (56 - 55) = .4; \tag{3.5}$$

$$\xi(50; 55) = .02 \times (70 - 55) = .3. \tag{3.6}$$

Even in this case, we see that not necessarily the call price is an increasing function of the underlying stock.

To be sure that the pricing function is increasing in S and convex it is sufficient to assume that the state price associated to a particular percentage return is independent of the current stock price. In other words, if $\nu(S_T|S_t)$ is the state price for the "metastate" in which the outcome is S_T conditional on the current state of S_t, then for all $\theta > 0$:

$$\nu(\theta S_T|\theta S_t) = \nu(S_T|S_t) = \nu\left(\frac{S_T}{S_t}\right) \tag{3.7}$$

where $\nu\left(\frac{S_T}{S_t}\right)$ is the state price related to the return and can be interpreted as a "normalized" state price.

Since The assumption that the risk-neutral probabilities are proportional to the state prices $p(S_T/S_t) \equiv \frac{\nu(S_T/S_t)}{v(t,T)}$) is equivalent to saying that the risk-neutral stochastic process for the stock is proportional, and the interest rate is the same for every initial state. In the next section we show in which way the proposed two-factor numerical procedure can generalized this result by using a different pricing kernel (i.e. a different state price per unit probability). This last assumption guarantees that the normalizing constant in the denominator (sometimes referred to as the "numeraire asset") $v(t, T) = \sum \nu(S_T/S_t)$

is the same. In other terms, this is a different way to state that in a risk neutral world, the expected return on a risky asset is equal to the risk free rate.

As generally known, put and call option pricing functions are monotone and convex in S. We prove this by means of the following more general proposition.

Proposition 11: *if the risk-neutral stochastic process for the stock price is proportional, then put and call option prices are homogeneous of degree one in the stock price and the exercise price, and they are monotone convex functions of the former.*

Proof. See Appendix C

3.2 Assumptions concerning the risk-neutral stochastic process

If other assumptions about the risk-neutral stochastic process for the stock price are stated, then further restrictions on option prices can be derived. In this section we discuss some of these restrictions.

Propositions 1 and 7, combined with the consideration that if the stock pays no dividends between t and T then $\xi(S;t;T;X) \geq S - Xv(t,T)$ (see Appendix C for a formal proof),imply the price of a call option is bounded as follows:

$$(S - Xv(t,T)) \leq F(S;t) \leq S \tag{3.8}$$

However, it is worth noting that the range of these bounds is quite wide and they can be narrowed by imposing additional assumptions about the nature of the risk-neutral dynamics of the stock price. In this respect, the assumption that the ratio of the risk-neutral to the real world (or physical) probabilities is a decreasing function of the stock price, may be very useful. This means that:

$$\frac{p_m}{\hat{p}_m} \leq \frac{p_l}{\hat{p}_l} \tag{3.9}$$

where $\frac{p}{\hat{p}}$ is the ratio between risk-neutral and true probabilities, and where $m > l$. This assumption ensures (among other things) that the beta of the stock is positive. This ratio is $\frac{p_m}{\hat{p}_m} = \frac{\Lambda_m}{v(t,T)}$, where Λ_m is the martingale pricing measure for the different states of the world denoted by the stock prices. It can be noted that such a martingale measure is, in each of these metastates, also proportional to the average marginal utility of each investor. If we assume that it is a decreasing function it follows that on average every investor is better off in states when the stock price is higher.

- **Proposition 12**: *If the state price per unit probability is a monotone decreasing function of the stock price, then a call options value is bounded by*:

$$\begin{aligned} v(t,T)\{\mathbb{E}\left[(S_T - X)^+ - S_T\right] + S_t\} &\leq \Xi(S_t;t) && (3.10)\\ &\leq \frac{\mathbb{E}\left[(S_T - X)^+\right]}{\mathbb{E}\left[S_T\right]/S_t} && (3.11)\end{aligned}$$

Proof. See Appendix C

Notice that these bounds are considerably narrower than that provided by equation 3.8.

Moreover, the property expressed by proposition 12 holds also for the Black-Scholes model. To understand this point, consider first the following discrete-time economy, with risk-neutral investors, where higher expected payoff at maturity always imply a higher asset price. This is the case simply because the discount rate (i.e. the numeraire or proportionality constant) is the same for all asset($v(t,T)$). We will see in next sections that what we propose in our numerical procedure is to allow for the numeraire asset to change through the time. Nevertheless, such numeraire continue to be the same for all assets.

Furthermore, stocks with higher variance (and therefore more price dispersion) must have in the risk-neutral economy, a higher price. This is the case because stocks with a higher variance parameter also have higher expected payoffs, while the discount rate is the same for all the stocks. As a consequence, its options must be more valuable as well. We can therefore derive a valid general proposition relating the riskiness of the stock to the price of the option.

- **Proposition 13**. *If $S1 = S2$ at time t, but the latter is riskier at time T when measured using the risk-neutral probabilities p_m (using information available at time t), then $\xi_1(S_1;t;T;X) \leq \xi_2(S2;t;T;X)$. The same result is valid for European puts and, if there are no dividends, American calls.*

3.3 The two-factor numerical procedure and the basic properties of American options

In this section, we give some intuition allowing to asses whether our numerical procedure complies with the Propositions 1-13. First of all, notice that Propositions 1 and 2 depends on the contract itself, and from the general properties of the probability function $p(\cdot)$ that it sums to one and $0 \geq p(\cdot) \geq 1$. These properties are ensured, for risk neutral probability function, by the fact that

$$U_S > 1 + L(t,T)^m > D_S \tag{3.12}$$

In general, the right hand side of equation 3.12 is always satisfied by the specific features of the BDT model ensuring arbitrage free prices, but this is not true for the left hand side. In fact, notice that the BDT price of a pure discount bond with unit redemption value is never higher then 1 because there is no chance that the interest rate trajectory becomes negative. On the contrary, there is actually no reason for preventing the interest rate to become higher than $U_S - 1$. This situation may occur for instance during the periods of high inflation, high economic growth or in presence of market bubbles, when the levels of interest rates and their volatilities are typically very high with respect to the stock price volatility. In these cases, the pricing problem becomes more complex and the valuation of financial products very uncertain so that more tools are necessary to obtain reliable prices.

However, notice that a humped term structure of interest rate volatility can sensibly mitigate this problem, since the volatility itself tends to zero as $t \to \infty$. Moreover, higher uncertainty about interest rate levels typically affects also the stock market and may result in higher stock volatilities. However, we can report no evidence supporting this intuition and therefore, we advice to check each time whether the value of the up probability is greater than U_D, and if not, we think that the best solution is to choose a "re-distribution rule" for the probabilities that is different from that illustrated in table 2.2. This solution is very simple, but clearly not fully satisfactory. To the best of our knowledge, there is not a best way for doing this, and we hope that further research may shed a light about this issue.

Proposition 3, 5 and 9 depend on the fact that the owner of the option holds the right of early exercise. This problematic is illustrated in the next section. We anticipate here that the technique to account for early exercise is borrowed by Cox et al. ([1]).

Proposition 4 is ensured by the fact that the total variability of the stock price increases as time passes and more steps of the tree are concerned while proposition 6 is ensured by the fact that the terminal payoff is in the second case less then in the first case (or at least equal), for each state of the world.

As stated in the previous section, Proposition 7 follows directly from Propositions 4 and 6 (first inequality) and Proposition 5 (second inequality), while Proposition 8 follows from Proposition 7 and from the assumption of limited liability for long option positions. Therefore, such conditions are automatically satisfied.

Proposition 10 is the put-call relation for American options, and may be very useful for computing the prices of put options starting from the price of the call. However, one of the assumptions stated for the proof of Proposition 10 is that the interest rate is constant while in our model is stochastic. In general, as far as the interest rate is allowed to change, the arbitrage argument used here to prove the put-call relation may no longer be valid, and therefore we recommend to not use it unless the role of interest rate changes is assessed.

Proposition 11 is satisfied because what we use here is the CRR model that in the limit coincides with the Black and Scholes model. And we already pointed out that the latter is a proportional stochastic process.

Proposition 12 states that if the pricing kernel is a monotonic decreasing function of the stock price, than narrower bounds for the stock option value can be recorded. The pricing kernel (or Radon-Nykodim derivative) is defined as the state price per unit probability. It is the major tool to switch from the real world probability to a risk neutral probability measure. This is done by expressing the value of all the securities using a different unit measure, that we call "numeraire asset".

In finance theory, the price of an asset at time t, denoted by Q_t, can be expressed as the present value of the expected payoff maturing at time T, y_T, discounted using an appropriate interest rate:

$$Q_t = \mathbb{E}\left[y_T \tilde{v}(t,T)\right] = \sum_m y_T^m \nu_m \tag{3.13}$$

where $\tilde{v}(t,T)$ is a stochastic discount factor accounting for the riskiness of the asset Q and ν_m is the state price associated to the metastate m, i.e. the price to pay today to receive 1 dollar in the future if and only if the state m occurs. As far as $\tilde{v}(t,T)$ is a random variable, it cannot be extracted from the expected value operator $\mathbb{E}_t$ that is calculated by using the real world probability p^* and the information available at time t. To switch from the real to the risk-neutralized world, we need to "adjust" the probabilities. In doing this, also the discount factor must change in order to preserve the left hand side equality in 3.13. The pricing kernel Λ, defined as the state price per unit probability, is therefore:

$$\Lambda_m = \nu_m/\hat{p}_m \tag{3.14}$$

where $\hat{p}_m$ is the risk neutral probability associated to the state of the world m.

In our model we denoted by h^m what here is denoted by $\hat{p}_m$. In this case, it is really simple to asses that the pricing kernel increases as the stock price increases. Now, to check whether the pricing kernel in our model is a monotonic decreasing function of the stock price, recall that $S_T^u = S_t U_S$ and $S_T^d = S_t D_S$. Rearranging we have that $\frac{S_T^u}{S_t} = U_S$ and $\frac{S_T^d}{S_t} = D_S$. Plugging these two equations in 2.4 (dropping the time indices from P for convenience of notation) and rearranging we have:

$$P^u = \frac{B_t S_t - S_T^d}{S_T^u - S_T^d} \tag{3.15}$$

Now, consider that, no matter if the correlation is positive null or negative, the probability h^m is obviously a monotonic increasing function in S_t and therefore, the pricing kernel is a monotonic decreasing function of the same variable. In our model, the risk neutral up probability of the stock price P is always multiplied for a scale factor depending on the up probability of the interest rate and on the correlation parameter. As a consequence, h_m is an increasing function of S_t and therefore, the pricing kernel is a monotonic decreasing function of S_t. Before concluding, we illustrate the pricing kernel representation of our model. If we express equation 3.13 in terms of the pricing kernel, we have:

$$Q_t = \mathbb{E}_t\left[y_T \tilde{v}(t,T)\right] = \sum_m y_T^m \nu_m = \sum_m y_T^m \Lambda_m \hat{p}_m = \mathbb{E}_t^{\hat{p}}\left[y_T \Lambda\right] \tag{3.16}$$

where it is clear that the pricing kernel Λ is our stochastic discount factor. To better understand this point, simply compare 3.16 with 2.10 as follows:

$$\sum_m y_T^m \Lambda_m \hat{p}_m = \sum_m \xi_m h^m v(t,T)^m \tag{3.17}$$

If we set $y_T^m = \xi_m$, recalling that $\hat{p}_m$ is in our model denoted by h^m, it must be that $\Lambda_m = v(t,T)^m$.

3.4 Dividend paying stock options and early exercise

When the underlying stock pays dividends, pricing options is more complicated, despite the fact that the option is American or European. For the former in particular, the "rationality" condition that suggests disadvantage from early exercise may no longer be valid. Moreover, the size and in particular cases also the date of the dividend may be stochastic and the consequent uncertainty must be considered in the pricing problem, for example in the Ito's expansion. In this section we deal with the first issue. For the second issue, we only advice to treat the dividend as an information from the financial market, like the stock volatility or the correlation, that is susceptible to change as far as new information is available. We think this because increasing the number of stochastic processes in the pricing problem only rarely can considerably improve the model without increasing the complexity. This is not in the spirit of the two-factor model that we proposed where the aim is to simplify the pricing approach. For this reason, we concentrate our attention only on the first aspect, concerning the early exercise in American options when the stock pays dividends.

First of all, the two-factor numerical procedure proposed in the previous section is based on the Cox, Ross and Rubinstein model. As suggested by these authors, to deal with the early exercise it is sufficient, at each node and each date, to compare the price of the European counterpart with its value if exercised, that is, with its intrinsic value.

To well understand this point, consider again the example proposed in figure 2.5 and take nodes E, F, G, H as an example. We see in chapter two how to compute the call value at these nodes by means of a replicating portfolio and using an arbitrage argument. Suppose that a (discrete) dividend $\zeta(S, T_j)$ is paid on the stock at time T_j with $T_0 < T_j \leq T_n$ and where T_0 and T_n denote respectively the valuation and the expiration date of the option.

Suppose that the dividend is proportional to the current stock price (i.e. stock price at time T_0). Suppose for example that the dividend is a percentage γ of S^- (so that γ can be interpreted as the dividend yield), the stock price the day before the dividend date. So we can write:

$$\zeta(S, T_j) = \gamma S^- \tag{3.18}$$

with $0 < j \leq n$. Let S^+ denotes the ex-dividend stock price. Assuming that the Miller-Modigliani ([36]) proposition holds, it must be that:

$$S^{+} = S^{-} - \zeta(S, T_j) = (1-\gamma)S^{-} \tag{3.19}$$

Therefore, the stock price may go up to $S^{+u} = S^{-}(1-\gamma)^{\mathbb{1}}U_S$ or down to $S^{+d} = S^{-}(1-\gamma)^{\mathbb{1}}D_S$ where $\mathbb{1}$ is an indicator function assuming the value of 1 if the end period is an ex-dividend date, and zero otherwise. The assumption that the dividend is certain imply that γ and $\mathbb{1}$ are not stochastic. They are input parameters that may change with time, as it is the stock price volatility.

It can be noticed that the only change with respect to chapter 2 is that $S^{-}(1-\gamma)$ substitute S_t^m in equation 2.16, as illustrated as follows:

$$\begin{cases} Y^u S_1^{-u}(1-\gamma)U_S + Z^u B_1^u = \xi_E \\ Y^u S_1^{-u}(1-\gamma)D_S + Z^u B_1^u = \xi_F \end{cases} \quad \begin{cases} Y^d S_1^{-u}(1-\gamma)U_S + Z^d B_1^d = \xi_G \\ Y^d S_1^{-u}(1-\gamma)D_S + Z^d B_1^d = \xi_H \end{cases} \tag{3.20}$$

as before, the left system is solved with probability q and the right one with probability $(1-q)$.

It can be easily noticed that early exercise may be optimal, if for instance $\mathbb{1} = 1$ and $S^{-}(1-\gamma)^{\mathbb{1}}D_S > X$. In fact, since $U_S > D_S$, then also, $S^{-}(1-\gamma)^{\mathbb{1}}U_S > X$. In this case, we have that $\Xi_E = S^{-}(1-\gamma)U_S$ and $\Xi_F = S^{-}(1-\gamma)D_S$, and so on and so forth. Therefore, the solving the system 3.20 for Y and B and plugging the solution into the equations 2.18 and 2.19 to obtain the value of the call price at node A. If this is the case, notice that the early exercise does not depend on the probabilities (both, physical or risk neutral) and from equation 3.20 also notice that the presence of stochastic interest rate does not affect the call value at a specific node. In fact, the value of the stock price in our model does not depend on the value of the interest rate. Only the risk neutral probability is affected from the prevailing level of the interest rate.

Therefore, if the stock price is sufficiently high, the value of the call option (Ξ_A, for instance) can obviously be less than $S - X$. As a consequence, in such circumstances, investors would not be willing to hold the call for one more period waiting for the expiration simply because the maximum present value form expiration will be less than the value that can be obtained if the call is exercised.

In other term, to avoid arbitrage opportunities, there must exist a critical value S^{*} such that if $S > S^{*}$ the call is immediately exercised. Such value is such that

$$\xi_A = \frac{\xi_E h_2^1}{1+L(1,2)^u} + \frac{\xi_F h_2^3}{1+L(1,2)^d} + \frac{\xi_G h_2^4}{1+L(1,2)^u} + \frac{\xi_H h_2^2}{1+L(1,2)^d} = S - X \tag{3.21}$$

for non-terminal nodes, and to

$$\left[p\xi^u, +(1-p)\xi^d\right] v(t,T) = S - K \tag{3.22}$$

for terminal nodes.

In fact, S^* can be interpreted as the lowest stock price at which the value of the hedging portfolio exactly equals $S - X$ and, other things equal, the higher the dividend yield, the lower the interest rate, and the lower the striking price, the lower will be S^*

We can extend the analysis to an arbitrary number of periods in the same way as before, just by rolling back the tree. There is only one additional difference, a minor modification in the hedging operation. Now the funds in the hedging portfolio will be increased by any dividends received, or decreased by the restitution required for dividends paid while the stock is held short.

Similar reasoning apply for put options, but in this case we should pay attention to the fact that the value of the European counterpart should be compared with the intrinsic value for a put $(X - S)$. Nevertheless, the procedure is very similar. As anticipated before, attention must be paid to the possibility of early exercise for the holder of an American put even if dividends are not paid. To simplify the pricing problem, if no dividends are paid, the put value can be obtained via put-call parity, as illustrated by Proposition 10.

Chapter 4

Valuation of convertible bonds

In the previous sections we show the hypotheses and the mechanics for developing and implementing our two-factor numerical procedure. As explained above, this model is thought to model contemporaneously two sources of risk: the stock price and the interest rate. To facilitate the exposition of the model, we show how it can be applied for pricing both European and American (plain vanilla) options. Although the value of a broad variety of financial products can potentially be computed, it is inconvenient to use our numerical procedure for pricing vanilla securities. This is the case because in general the higher pricing efficiency (if any) attainable from the inclusion of more sources of risk, cannot justify the increased computational burden and complexity required for the implementation of the procedure, especially because the pricing efficiency of simpler models (developed in the literature and in practice) is already acceptable. As far as simpler pricing models are available, we recommend to not use this numerical procedure for pricing vanilla products.

However, the pricing of more complex securities, such as hybrid financial products, may require very sophisticated models where the inclusion of additional risk factors does not necessarily imply a dramatic increase of the computational burden. A hybrid financial security can be defined as a financial contracts that combine the elements of the two broader groups of securities, debt and equity. For these kind of securities, it is therefore natural that the pricing problems focuses on both the above mentioned sources of risk. Moreover, very often such financial products show path dependency, requiring that more sophisticated tools must be involved in the pricing problem. This is the main reason why in the case of hybrid securities it may be worth to increase the complexity of the model for a major pricing efficiency, as it is the case for convertible bonds. By applying our model for pricing convertible bonds, it will be clear the advantage of our numerical procedure coming from the simplicity if its calibration.

4.1 Convertible bonds

Convertible bonds are hybrid products very common in the financial industry. A fundamental role played by such securities is to provide funding for firms at lower costs with respect to other financial contracts, such as ordinary bonds, mainly because they bear a lower interest coupon payment. On the other hand, such advantages are offset by the right, granted to the bondholder, to modify his "status", switching from creditor to shareholder, if he or she found it convenient. Moreover, it can be noticed that convertibles are commonly adopted by the management to enhance the capital structure of the firm. This is the case because the exercise of the conversion privilege allows contemporaneously to reduce the debt exposure and to increase the common equity of the issuer. This last can doubtless be a desirable feature especially during the periods characterized by increasing interest rates and by the necessity for deleveraging.

If on the one hand convertible bonds may be considered as an important tool for the capital structure management, on the other hand, it can be noticed that these securities are considerably more complex with respect to other financial instruments, such as ordinary bonds, options and warrants, and their pricing imply to account for many sources of risk and to deal with complex evaluation frameworks and models.

Many contributions in literature may be recognized beginning from the seminal papers by Ingersoll ([37]) and by Brennan and Scwhartz ([38], [30]) where, for the first time, a replicating strategy and numerical approximation methods for convertible bond pricing are proposed. In particular Brennan and Scwhartz ([30]) develop a pricing framework for convertible bonds that accounts for two sources of risk that are the firm value and the interest rate. Furthermore, the approach adopted by Brennan and Scwhartz in credit risk modeling is somewhat similar to the one adopted by Merton ([39]), sometimes defined as "structural approach", where the default event is endogenously defined by the case of an asset value lower than the liability value. The Brennan and Schwartz model suffers the main drawback, generally related to the structural approach, that the firm value and volatility are not directly observable. For this reasons a different approach, sometimes defined as "reduced-form approach", have been successively developed first by Jarrow and Turnbell ([40]) and then by of Duffie and Singleton ([41]). The main idea of this new approach is that the default risk is exogenously characterized by a jump diffusion process and that the probability of default is inferred by financial market data.

Finally, very relevant are the contribution by Tsiveriotis and Fernandes ([42]), where partial differential equations are concerned within a reduced-form approach, and by Hung and Wang ([27]) where the dynamics of the risk factors are described by means of binomial trees.

4.2 Review of the convertible bonds pricing theory

A convertible bond (CB) is a hybrid security allowing the bondholder to convert the nature of his or her investment from bond to equity, at particular conditions and maturities defined by the contract. Very often the faculty to opt for an equivalent sum of money instead of receiving the shares is granted to the bondholder. Anyway, such instruments combine typical features of ordinary bonds with the upside potential of an investment in common equity shares. The conversion privilege can be seen as an American (or sometimes Bermudan) option that the bondholder can exercise at his discretion. However, such discretion is commonly limited by the fact that the issuer retains the right to call the convertible for redemption even if, very often, such right cannot be exercised before a certain period of time has expired. When the call provision is exercised, the bondholder can decide to convert the bond in common equity shares, at a predefined conversion ratio, or to redeem it at a predefined call price. It is worth to notice that the conversion strategy set by the bondholder will depend on the call strategy set by the issuer and vice versa, and that the equilibrium value of the convertible is the value that offers no arbitrage opportunity to both the parties. For this reason, the value of the convertible cannot be determined until the two strategies are simultaneously set.

To begin with, it is worth noting that to determine the arbitrage free value of the convertible it is necessary to set the optimal conversion strategy followed by the bondholder and the optimal call strategy followed by the issuer. In doing this, it is important to consider that issuer and bondholder have conflicting interests because, the market value of the firm being the same, the strategy of maximizing the value of the common equity can be pursued only by minimizing the value of the convertibles and vice versa. This situation imply that the issue of convertibles do not influence the market value of the firm. The assumption is a consequence of the Miller and Modigliani ([36]) theorem and appears to be quite restrictive, especially if we consider the role played by convertibles in optimizing the capital structure to the aim of value creation.

4.2.1 Optimal conversion strategy

It is noticeable that, in general, the bondholder will always find it optimal to exercise the conversion privilege if the price of the convertible falls below the conversion value, that is the value of the shares the bondholder receives as result of the conversion. This way he or she renounces to the market value of the convertible to obtain its conversion value. On the contrary, if the price of the convertible is above the conversion value, the bondholder does never find it optimal to convert since this would imply to renounce to a value (the price of the convertible) that the market considers higher than the conversion value. Therefore, the optimal conversion strategy imposes, the bondholder to convert as soon as the price of the convertible equals the conversion value.

Moreover, it can be proved that the bondholder does never find it optimal to convert an uncalled convertible bond except immediately prior the dividend date or the date at which a change in the term condition occurs or, finally, at the expiration date. The reason for this is similar to the reason why the holder of an American call option, written on a non paying dividend asset, will never find it optimal to exercise the option before the expiration, as illustrated in section 3.1. Intuitively, this is the case because only in the above mentioned cases the price of the convertible may be equal or less the conversion value. Otherwise, the bondholder prefers to sell the bond on the market instead of converting it. This imply the following lower bound condition:

$$C \geq CV \tag{4.1}$$

where C is the market value of the convertible and CV is the conversion value and it is equal to the value of the shares the bondholder receives if he or she exercises the conversion privilege.

Since it is never optimal to convert an uncalled CB (except in the above mentioned cases) a call provision can therefore be included in the contract in order to force the conversion before the expiration date.

4.2.2 The optimal call strategy

The call provision implies that, if the bond is called for redemption by the issuer, the bondholder has the option to choose if to exercise the conversion privilege and receive the conversion value, or if to redeem it and receive the call price (CP).

Bearing in mind that the issuers objective is to minimize the value of the convertible, an optimal call strategy implies to call the CB as soon as the value if called (VIF) is equal to the value if not called. This is the case because, if the CB were left uncalled at a market price higher than the value if called, the issuer would maximize the stock value by calling it. On the contrary, if the CB were called at a market price lower than the value if called, an extra-profit would be offered to bondholders so that the issuer would minimize the value of the convertible by leaving it uncalled. The first result is therefore that:

$$C = VIC \tag{4.2}$$

Moreover, the value of the convertible if called is:

$$VIC = [CV, CP]^{+} \tag{4.3}$$

and it is clear that VIC reaches the minimum value of CP when $CV = CP$. The logical consequence is that the issuer will call the bond as soon as the conversion value is at least equal to the call price. Moreover, after the convertible becomes callable, if the conversion value were higher than the call price, the issuer will decide to call the bond because, by condition 1, it would be sold at a price higher (or at least equal) than the conversion value that, in this case, is the value if called.

For this reason, an optimal call strategy is to call the convertible as soon as and the following upper bound condition can be recognized:

$$C \leq CP \tag{4.4}$$

4.3 A basic model for the pricing of convertible bonds

In evaluating the arbitrage free value of a CB the need for a cash flow mapping arises. To this aim, at least four elements should be considered: the stock price, the equity component of the convertible (that is the expectation about the conversion), the debt component (the expectation in case of no conversion) and, finally, the total value of the CB. This last is therefore given by the sum of the two components, if lower than the call price. Otherwise, the issuer find it optimal to call the CB and its value is equal to the conversion value. Many pricing models describe the dynamics of such components by means of binomial trees. Following this approach, it is first necessary to determine the payoff of the convertible at maturity, and then to roll back the tree structure till the evaluation date, checking at each time step if the optimal conversion and call strategies are respected (that is to check if conditions 1 and 4 are respected).

At maturity date T, if the conversion value (that in this case is the equity component) is higher than the face value of the CB, the bondholder decides to convert. In this case the debt component of the CB is equal to zero, while the equity component is equal to the number of shares offered in conversion for each CB, times the stock price at the expiration date. On the contrary, if the face value of the CB is higher than the conversion value, the bondholder does not decide to convert, and the payoff is equal to the face value of the CB. Rolling back the tree, it should be necessary to check if the sum of the equity component and of the debt component is greater than the call price, if the convertible is already callable. A numerical example can simplify the explanation. In this framework, it is assumed that the interest rate term structure is flat and not stochastic. Moreover, the event of default is for the moment not explicitly considered, even if the discount rate applied to the debt part of the CB is a risky rate.

Figure 4.1 shows a binomial tree of the four component of the CB. It is assumed that the risk free rate (r) is constant over time and is equal to 3%, while the risky yield is equal to 6% (that is to fix a spread s=3%). The risk free yield is adopted to construct the implied binomial stock tree, while the risky yield is adopted to discount the debt part of the convertible bond.

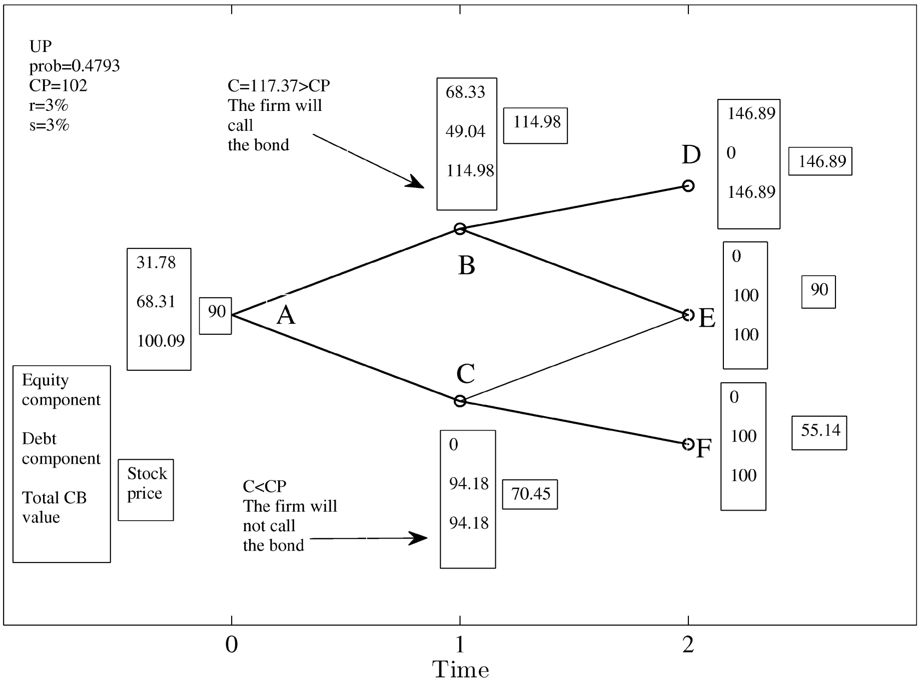


FIGURE 4.1: The convertible bond dynamics without interest rate and credit risk

Note that, since the risk free rate is constant over time, the up probability of the stock price is constant too, and in this case is equal to 0.4793. For simplicity, it is assumed that each CB can be converted into one share (i.e. the conversion ratio is 1), whose price at $t = 0$ (the evaluation date) is equal to 90. For simplicity, it is assumed that the CB pays no coupons, has a two years maturity and can be immediately called by the firm at a price of 102. Finally, the stock pays no dividend during the following two years. From figure 4.1 it can be noticed that, at node D, the stock price is equal to 146.89. This is the equity component at time $t = 2$ and the bondholder decides to convert. Since the conversion occurs, the debt component of the convertible falls to zero. On the contrary, at node E and F the stock price is equal to 90 and 55.14 respectively, and is in both cases is lower than the face value of the convertible. The bondholder decides to not convert it, so that the equity component falls to zero, and the CB value is given from the debt component only.

Proceeding backward, at node B there is a probability of 47.93% that the equity component will rise to 146.89 in the next period, and a probability of 52.17% that it will fall to zero. The equity component is therefore equal to $(0.4793 \times 146.98 + 0.5217 \times 0)\exp\{-0.03\} = 68.33$. Moreover, at the same node, there is a probability of 52.17% that the debt part will rise to 100 and a 47.93% of probability that it falls to zero. The debt component is therefore equal to $(0.5217 \times 100)\exp\{-0.06\} = 49.04$. At node B we notice that the CB value, that is equal to the sum of the two components (117.37), is greater than the call price (102). Condition 4.4 is violated so that the firm decides to call the bond, forcing the conversion. In turn, the bondholder decides to convert the bond because this way he or she receives a stock whose value (114.98) is higher than the call price. On the contrary, at node C it can be noticed that the equity component is equal to zero because there is no chance to convert the bond at maturity. The debt part is equal to $100\exp\{-0.06\} = 94.18$. Since the value if not called is lower than the value if called, the firm will decide to not call the bond. Therefore, at node C, the value of the convertible is 94.18. Rolling back the tree, it is possible to determine the value of the convertible at $t = 0$ just by computing the sum of the equity and of the debt part, and checking for condition 4.1 and 4.4.

4.4 Default risk

From the previous section, it is possible to notice that the interest rate adopted to discount the equity part is the risk free yield, while for the debt part the risky yield is concerned. However, the event of default has not properly been considered. It is possible to account for the default risk by inferring the default probability from the market prices of risky bonds having an equivalent credit rating of the CB or of the issuer. Let $v(t, T_j)$ and $g(t, T_j)$ be the market prices, observed at time t and maturing at time T_j, with $j = 1, 2$ of a risk free bond and of a risky bond respectively, having face value equal to 1, and let RR be the recovery rate in case of default. The probability that the default occurs between time t and T_1, φ_1, can be estimated as follows:

$$g(t, T_1) = (1 - \varphi_1)v(t, T_1) + \varphi_1 RR v(t, T_1) \tag{4.5}$$

$$\varphi_1 = \frac{g(t, T_1)v(t, T_1)^{-1} - 1}{RR - 1} \tag{4.6}$$

$S(0)$	90	CP	102	p	0.52
$L(0,1)$	2%	$v(0,1)$	0.98	q	0.5
$L(0,2)$	3%	$v(0,2)$	0.94	h_1	0.23
$s(0,1)$	3%	$g(0,1)$	0.95	h_2	0.29
$s(0,2)$	3%	$g(0,2)$	0.89	h_3	0.22
$\sigma(S)$	30%	$\rho(S,L)$	0.1	h_4	0.26
$\sigma(L(1,2))$	20%	V	100	·	·

TABLE 4.1: Parameters adopted for pricing the convertible bond

Once φ_1 has been estimated and assuming that the dynamics of the risk free bond is described according to the BDT model, it is possible to infer the probability that the default occurs between time T_1 and time T_2, φ_2, as follows:

$$\begin{aligned} g(t,T_2) &= \\ & v(t,T_1)\{q(1-\varphi_1)v(T_1,T_2)^u\left[(1-\varphi_2)+\varphi_2 RR\right]+ \\ & +(1-q)(1-\varphi_1)v(T_1,T_2)^d\left[(1-\varphi_2)+\varphi_2 RR\right]+ \\ & +q\varphi_1 RRv(T_1,T_2)^u+(1-q)\varphi_1 RRv(T_1,T_2)^d\} \\ &= v(t,T_2)\{\varphi_1 RR-(1-\varphi_1)\left[\varphi_2 RR+(1-\varphi_2)\right]\} \qquad (4.7) \\ \varphi_2 &= \left(\frac{g(t,T_2)v(t,T_2)^{-1}-\varphi_1 RR}{1-\varphi_1}-1\right)/(RR-1) \qquad (4.8) \end{aligned}$$

4.5 Pricing convertible bonds subject to interest rate risk and default risk.

In this section, a numerical example is reported to describe how the arbitrage free price of a convertible bond can be determined. Assume that the convertible pays no coupon, and that the underlying stock pays no dividend during the entire period. Moreover, the convertible has a maturity of two years and a par value (V) of 100, it is immediately callable at the price of 102, and the conversion ratio is 1 (meaning that each CB can be converted in one share of common stock). The arbitrage free dynamics of the Libor rate is described by figure 1.2(a) while the spread is of 3% and is constant over time. Table 1 reports the parameters employed to determine the price of the convertible. It is worth noting that the up probability from $t = 1$ to $t = 2$ depends on the value of the Libor rate that, at $t = 1$, that is stochastic. Therefore, 4 states of the world are considered, each of probability h^m, with $m = 1, \ldots, 4$.

More specifically, h_1 and h_3 are, respectively, the probabilities of a contemporaneous up and down movement of stock price and Libor rate, while h_2 and h_4 are the probabilities of opposite movements of the two risk factors (the stock price increases the Libor rate decreases and vice versa) respectively. Since at time $t = 0$ the Libor rate is not a random variable, only two states of the world are concerned, so that p represents the probability of an up movement of the stock price from $t = 0$ to $t = 1$.

To begin with, as in the basic model explained in section 4.3, it is necessary to start by constructing the stock tree and by setting the payoff of the convertible at time $t = 2$. In doing this, it is necessary to pay attention to the default event. For this reason, we have two basic scenarios for each state of the world, one where the default occurs between $t = 1$ and $t = 2$, and the other where the default does not occur. The CB tree is shown in figure 4.2.

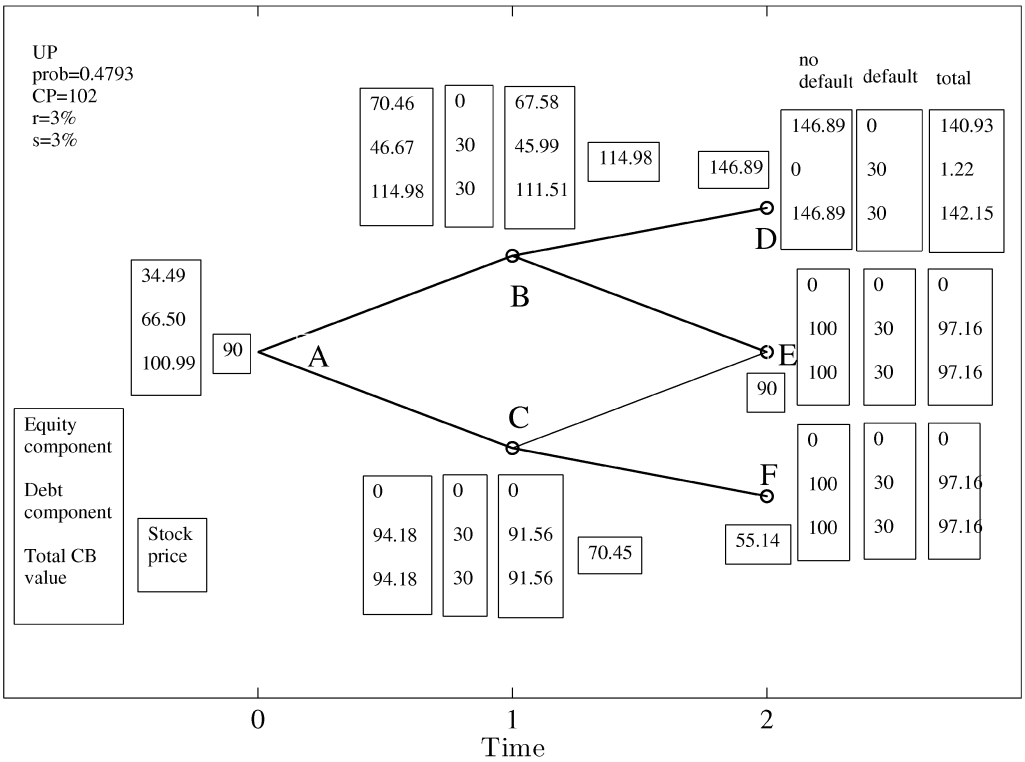


FIGURE 4.2: The convertible bond dynamics with default and interest rate risk

Since the event of default is included in the model, we have tree columns at each node: the first column is the pay off in the case the default does not occur; the second is the payoff in case of default; the third is the "certain equivalent" of the two states. Take node D, E and B as an example. At node D the stock price is 146.98 and, if default does not occur, the bondholder will decide to convert. The CB value is thus formed only by the equity part, that is equal to 146.98.

On the contrary, if the default occurs, the bondholder will receive only the face value times the recovery rate, the CB value is formed only by the equity part and is equal to 30. The certain equivalent is the expected value of this two cases, calculated by means of the default probability φ_2. More precisely, the certain equivalent of the equity part and of the bond part are $0\varphi_2 + 146.98(1-\varphi_2) = 140.93$ and $30\varphi_2 + 0(1-\varphi_2) = 1.22$ respectively. At the same time, the payoff at node E can be calculated. In this case, the equity part is zero irrespective of the default event, while the debt part will be 100 or 30 if the default occurs.

Once the certain equivalent have been calculated for the equity and the debt part and for the CB value, it is necessary to discount it using as (stochastic) discount rate, the Libor rate for both the equity and the debt part. In particular, notice that node E corresponds to an up movement of the stock price. Thus, the discount rate is 5.2 and 2.9 with probability equal to h_1 and h_2 respectively. On the contrary, node D corresponds to a down movement of the stock price, and for this reason, the discount rate is 5.2 and 2.9 with probability h_3 and h_4 respectively. At node B the equity part is therefore equal to $\frac{140.93}{1.052}h_1 + \frac{140.93}{1.029}h_2 + \frac{0}{1.052}h_3 + \frac{0}{1.029}h_4 = 70.46$, while the debt part is $\frac{1.22}{1.052}h_1 + \frac{1.22}{1.029}h_2 + \frac{97.16}{1.052}h_3 + \frac{97.16}{1.029}h_4 = 45.42$. The CB value at node B is therefore equal to the sum of this two parts (115.88) and, if default does not occur between $t = 0$ and $t = 1$, the issuer will decide to call the CB, whose total value is therefore given by the stock price. This is the value of the CB at node B (if default does not occur) because the bondholder decides to convert, being the conversion value higher than the call price. Rolling back the tree, it is possible to get the $t = 0$ value of the convertible. Notice that, calculating the present value of the certain equivalent from $t = 1$ to $t = 0$ is similar to the basic case because the interest rate between these two dates is not a random variable.

Chapter 5

Participating policies: A risk management application of the two-factor numerical procedure

The previous chapters illustrates how the two-factor numerical procedure can be applied for hedging and trading purposes. We see the case of European and American options and also the case of convertible bonds. In this chapter we discuss a risk management application for this model, involving the valuation of participating policies.

A participating (or with-profits) policy is an insurance contract that participates in the profits of a life insurance company, according to a rate (participation rate) equal to a predefined level alpha (generally < 1). In most cases, the participation rate is applied to an intermediate level of the economic and actuarial result referring to a predefined dedicated quota of the investment or to a specific fund.

In particular, we see how to apply the numerical procedure for the computation of risk-adjusted performance indicators that can serve for appreciating the optimality of the issue with reference to both relevant product design variables and investment decisions.

The complex risk system involving life insurance business, mainly life and pension annuities, requires a structured equipment of measurement and management tools, suitable to evaluate the policies' performance within an ex-post perspective as well as within a forecasting context, consistently with the recent regulatory and accounting rules set up by the international supervisory associations and committees. This chapter focuses on the impact of the financial risk sources on annuity contracts by working out on an ex-ante year by year analysis. The study specifically refers to the case of participating policies, which strongly suffer from the financial risk components, by developing a scenario analysis by means of our numerical procedure. The analysis addresses a twofold goal: on the one hand it provides performance indicators based on the operating income, on the other hand it obtains managerial guidelines, within a risk-adjusted performance framework, apt to evaluate the participation rate.

5.1 The participating policy

A participating (or with-profits) policy is an insurance contract that participates in the profits of a life insurance company, according to a rate (participation rate) equal to a predefined level alpha (generally < 1). In most cases, the participation rate is applied to an intermediate level of the economic and actuarial result referring to a predefined dedicated quota of the investment or to a specific fund. As far as the payoff scheme is concerned, these policies embed an option which can be described as follows: if the result $R(t+1)$ of the t-year net of the annual quota of the administrative expenses is positive, the insurer will recognize an additional inflow equal to a percentage α of $R(t+1) - \gamma$

This additional value will be paid immediately after having recognized it; otherwise nothing will be added to the current benefit. Let us consider an annual cost due to administrative expenses equal to γ. Hence, the insurer exhibits a result which is:

$$R(t+1) = V(t) \exp\left\{ \int_t^{t+1} r(u)du \right\} - N^x(t+1) - N^x(t+1) \sum_{j=1}^{n-t} {}_jp_{x+t} \exp\left\{ - \int_t^{t+j} \hat{r}(u)du \right\} \tag{5.1}$$

where $r(u)$ is the return on assets, $N^x(t+1)$ is the unitary installment paid to the actual number of survivors at age $x+t+1$, denoted $B(t+1)$ being the portfolio contractual benefit at time $t+1$, and $\hat{r}(u)$ is the discount rate applied in the valuation of the mathematical provision. If the result is positive ($R(t+1) > 0$),the insurer will recognize to policyholders an additional benefit:

$$\tilde{B}(t+1) = \alpha\left(R(t+1) - \gamma\right) \tag{5.2}$$

On a portfolio basis, it is a call option sold by the insurer. Therefore from the insurer perspective the inclusion of the option in the policy scheme will restyle its result as

$$\tilde{R}(t+1) = R(t+1) - \tilde{B}(t+1) \tag{5.3}$$

Considering also the capital invested by the insurer we have

$$\tilde{U}(t+1) = \tilde{R}(t+1) + K(t)\left[\exp\left\{\int_{t}^{t+1} r(u)du\right\} - 1\right] \tag{5.4}$$

As a consequence the total value of the insurer capital at the end of the period is

$$K(t+1) = K(t) + \tilde{U}(t+1) \tag{5.5}$$

The computational point is to evaluate the option payoff on the result which is actually a function, in a financial management perspective, of two risk factors: the return on the investment $(r(u))$ and the discount factor $(\hat{r}(u))$, of the provision.

In a management perspective, the decision on the participation rate has to evaluate the sustainability of the issue. Within this context, a primary insight can be gained through the analysis of the expected profit and of the linked expected return on equity from the position the intermediary is implementing. These indicators provide the management with an immediate picture of future outcomes and differentials of portfolio alternatives, although they do not weight the information with the risk associated to the participation rate choice. As a consequence, the return has to be compared with the risk undertaken: to this aim, value-at-risk (VaR) measures are considered. In this case, being the VaR an estimate of the maximum likely loss with a certain confidence level over an identified time horizon, we have to evaluate the distribution of the final result with reference to the whole asset and liability portfolio. Therefore, as far as the result is concerned, the focus is on the difference between the initial allocated capital (K_0) and the final value of the portfolio (K_n) over a holding period (n) matching the duration of the policy.

More specifically, in the risk evaluation of the entire issue, the VaR can be interpreted as the VaR of a portfolio whose initial value is the allocated capital and whose flow dynamic is given by the initial asset allocation and the subsequent intermediate dynamic. The VaR of the entire value can be interpreted as the VaR of a portfolio exposed to those risk factors that are the changing parameters of the net value itself. It can be measured by modeling the risk factors and by using risk filters to proportionate the effect on the net value.

To this aim risk adjusted performance metrics (RAPMs) have become popular in the finance industry. As known, business evolution can be described by means of both the intermediation portfolio (economic value approach) and the income flows (current earning approach). The first approach accounts for the difference between asset and liability at any time of the portfolio cycle, while the second explains the difference between the profit components periodically accrued. In order to compute RAPMs, we used both and concentrated on an average yearly measure, since we are concerned with the global profitability of the issue.

Many acronyms and definitions for RAPMs can be found in the literature. We use the following: the expected return on equity ($\mathbb{E}[RoE]$) defined as the ratio of the expected profit to initial capital, whose annual intensity is set as

$$\mathbb{E}[RoE] = \frac{\mathbb{E}[\tilde{U}_n]}{nK_0} \tag{5.6}$$

the expected risk-adjusted return on capital (E[RARoC]) defined as the ratio of the expected profit to the VaR of the portfolio, whose corresponding average intensity value is given by

$$\mathbb{E}[RoE] = \frac{\mathbb{E}[\tilde{U}_n]}{n|VaR|} \tag{5.7}$$

5.2 The stochastic context

Assuming as preliminary remark that the demographic forecast is consistent with the best prediction of the survival trend, we focus our attention on the financial risk drivers. Within this context we model a stochastic framework involving the evolution in time for both the financial components, consisting, respectively, of the random movements of the return rates as well as the interest rates involved in the estimate of the provisions' forecasted cash flows (within a forward perspective). In this order of ideas the stochastic scenario evolves in time on the basis of the information flows summarized by a filtered probability space $(\Omega; \mathscr{F}; \mathbb{P})$, where $\mathscr{F}$ contains the information flows about the financial history of both the accumulation process of the financial resources and the discounting process pertaining to the reserve calculation. We point out that the financial filtration $\mathscr{F}$ and the demographic filtration $\mathscr{F}'$ are independent.

Focusing on the contract afore described, we assume that the total amount of $V(t)$ is calculated on the basis of a stochastic process describing the return rates, say $\{r(s), s \geq 0\}$, whilst the provisions' calculation is opportunely made on the basis of a stochastic discount process $\{v(t,s), 0 \leq t \leq s\}$, coherently with a current estimate calculation.

Moreover, considering a life annuity, denoted by T the deferment period and by τ the premium payment period,

$$V(t) = \sum_{i=t+1}^{\infty} (b_i \mathbb{1}_{\{T \leq i \leq K(x)\}} - P_i \mathbb{1}_{\{i \leq \tau\}}) v(t,i) = \sum_{i=t+1}^{\infty} z_i v(t,i) \tag{5.8}$$

where the indicator function $\mathbb{1}_{\{T \leq i \leq K(x)\}}$ takes the value 1 if $T \leq i \leq K(x)$, and 0 otherwise, whilst the indicator function $\mathbb{1}_{\{i \leq \tau\}}$ takes the value 1 if $i \leq \tau$ and 0 otherwise.

So, considering the t-th accounting period, the insurer's mean result in $t+1$ is given by

$$\begin{aligned} \mathbb{E}[\tilde{R}(t+1)] &= \mathbb{E}[R(t+1) - \alpha(R(t+1) - \gamma)^+] = \\ &= \mathbb{E}[min(R(t+1), (1-\alpha)(R(t+1) + \alpha\gamma))] \end{aligned} \tag{5.9}$$

In formula 5.9 $\mathbb{E}$ represents the expectation under a risk neutral probability measure structured according to the suggestions of the guidelines of the International Accounting Board (IAIS, FASB 2004).

5.3 The computational procedure

The portfolio we are modeling is therefore exposed to two main risk factors: the dynamic of interest rates used for the valuation of the provision and the dynamic of the return of the reference asset. Computation of the final result has been derived by numerical methods, using binomial trees opportunely combined to a consistent framework under the same risk neutral probability measure. As far as the interest rate is concerned, the Black Derman and Toy model was selected for its outstanding ratio of computational efforts to calibration efficiency.

Under BDT, using a binomial lattice (figure 5.1) we calibrated the model parameters to fit both the current term structure of interest rates (yield curve), and the volatility structure for interest rate caps. Using the calibrated lattice we then valued the mathematical provision year by year along the duration of the policy.

As far as the investment is concerned, we assumed that the insurer selected a stock market investment and we modeled the investment return by means of Cox Ross and Rubinstein binomial pricing model. This is done by means of another binomial lattice

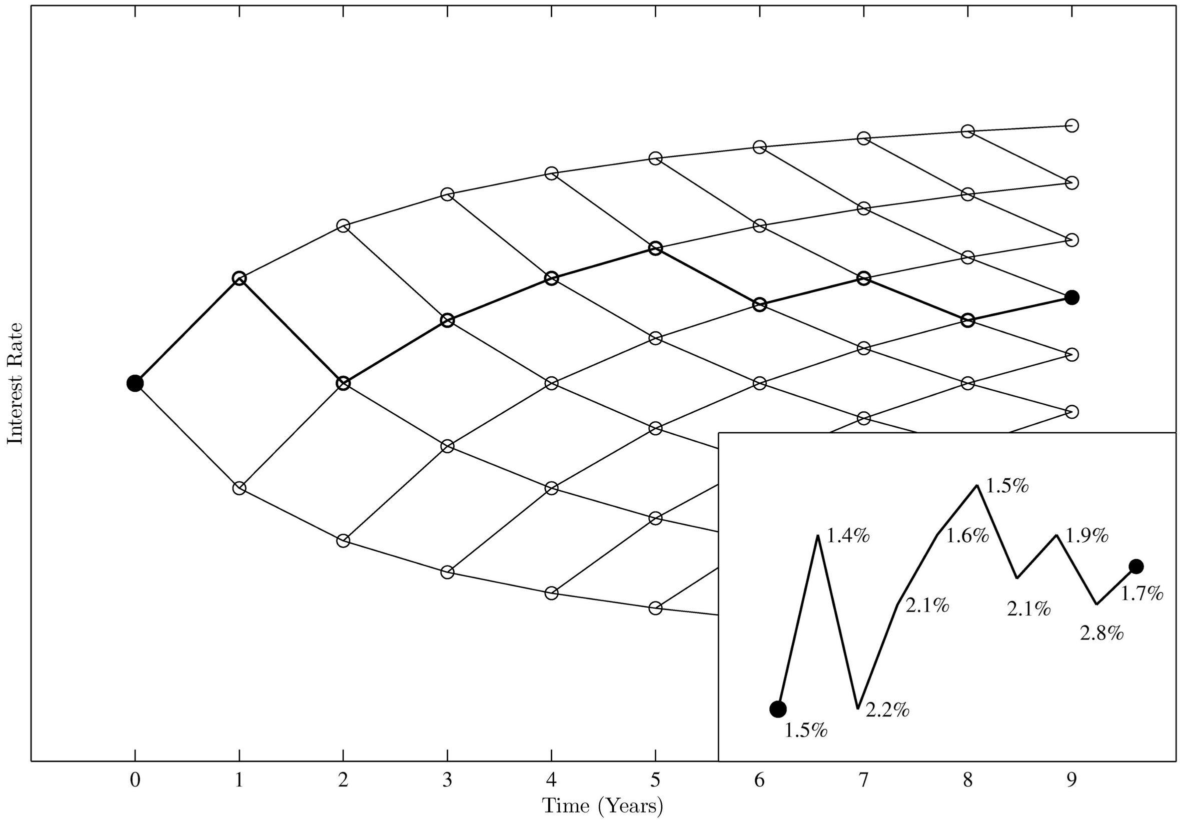


FIGURE 5.1: The BDT tree and the dynamics of $\hat{r}(u)$

(figure 5.2), for a number of time steps between the valuation and critical dates. Each node in the lattice represents a possible price of the underlying at a given point in time. Therefore, the CRR tree depends on the BDT rate for the risk-free rate and evolves in time and space according to the stock market volatility.

Valuation of the results and of all the relevant figures are performed iteratively, after having defined the two dimensional binomial tree under the information set available at issue date. As far as the starting dataset is concerned, a portfolio of 1,000 homogeneous temporary annuities has been selected. Italian IPS55 mortality table was selected as the best estimated for the mortality rates. Calibration has been set on December, 20, 2010 on Euribor dataset (Bloomberg). As fa as the stock market parameters are concerned the FTSE MIB Index was selected as the benchmark stock market index for the Borsa Italiana, the Italian National Stock Exchange, consisting of the 40 most-traded stock classes on the exchange. A full description of the data is reported in the following table.

Our goal is the computation of the expected profit at the end of the business together with the connected value at risk (95%) in order to appraise the sustainability of the participation rate.

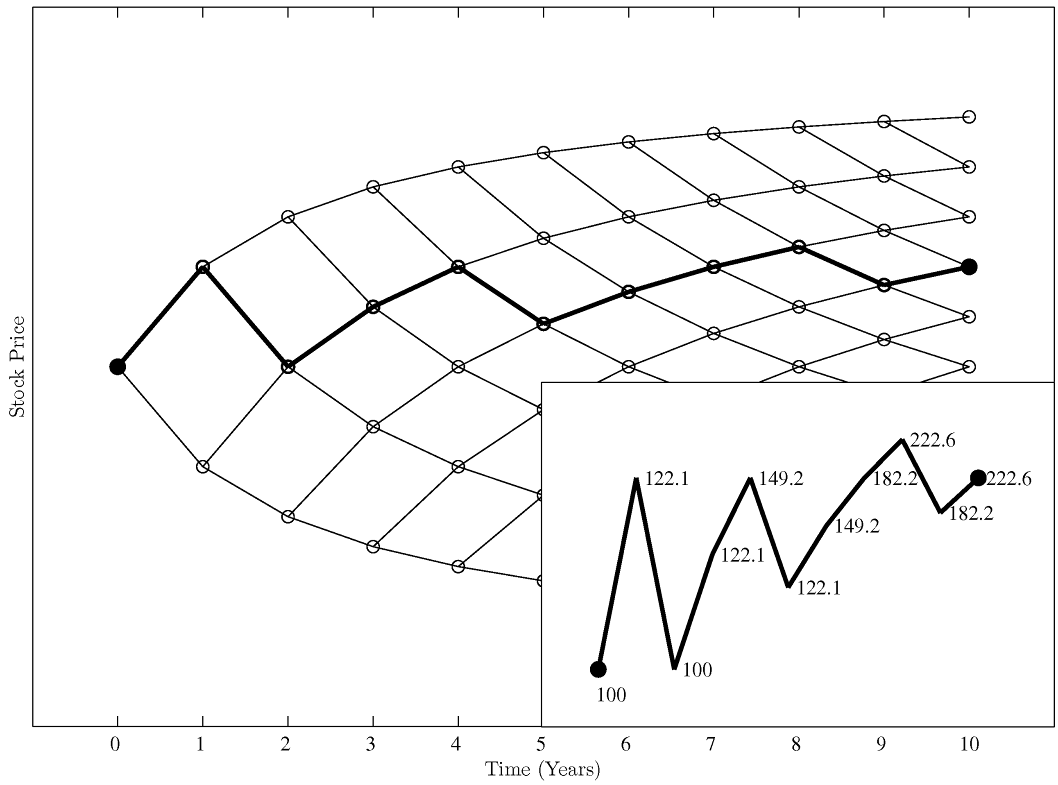


FIGURE 5.2: The implied stock tree and the dynamics of $r(u)$

Policy Duration(n)	10 years
Mortality table	IPS55M (Italy, Male)
Insured age (x)	40 years
$B(t)$	unitary
Technical rate (Pure Premium)	2% (annual rate)
Loading rate (Office Premium)	7%
Pure Premium	8.94
Office Premium	9.57
Annual Expenses Portfolio basis(γ)	62.60
Number of sold policies	1,000
Initial Capital (K_0)	1913.70
Interest rate curve	Zero Rates from Euroswap Curve
Interest rate volatility surface	Cap Volatilities from Euroswap Curve
Stock market volatility	FTSE MIB Implied Volatility ATM 12 month
Calibration date	20 December 2010
Data Source	Bloomberg Dataset

TABLE 5.1: Participating policies and risk adjusted performance measures: Data description

α	$\mathbb{E}[Roe]$	$\mathbb{E}[Rarorac]$	$\mathbb{E}[Roe]/\mathbb{E}[Raroc]$
100,00%	-4,95%	n.a.	n.a.
90,00%	-3,54%	13,11%	-27,00%
80,00%	-2,13%	12,64%	-16,86%
70,00%	-0,72%	12,29%	-5,88%
60,00%	0,68%	12,45%	5,50%
50,00%	2,09%	12,34%	16,96%
40,00%	3,50%	12,33%	28,39%

Table 5.2: Risk adjusted performance measures per different levels of alpha

5.4 Results

Table 5.2 shows the risk adjusted measures per different levels of α.

The computation of risk-adjusted performance indicators can serve for appreciating the optimality of the issue with reference to both relevant product design variables and investment decisions. Within this respect, we observe that product design decision can be usefully evaluated by indicators we adopted. The main result is the possibility to optimize the issue by means of the adopted indicators, as reference target in the management of discretionary issue parameters (mainly the participation rate but also the loading rates). The optimization problem, therefore, can be addressed as maximizing the RoE and/or Raroc target under the constraint of sustainability by changing the participation rate and eventually other discretionary parameters. Of course, the evaluation process can be enriched by including within the optimization problem even the randomness of the mortality rate and by delimiting the capital at risk.

Chapter 6

The calibration of the correlation parameter

In this chapter we discuss some issues related to the implementation and calibration of the two-factor numerical procedure whose details have been illustrated in the previous chapters. In particular, we discuss how it can be possible to calibrate the dynamics of each risk factor to the observed market prices. We also offer different possible ways for calibrating the correlation parameters and analyse the pricing performance of each one. An empirical analysis is conducted in order to assess whether, as for the volatility of each risk factor, it is possible to set the correlation parameter so that the model can efficiently reproduce the observed market prices of the securities adopted as "benchmark" for the calibration. We then discuss how it is possible to determine an "implied correlation" coefficient from the observed market prices of stock options, showing whether and to what extent, such correlation measure can predict the future realized correlation.

6.1 Calibrating the model

To begin with, it is worth noting that the adoption of the BDT and of the CRR models respectively for modelling the dynamics of the interest rate and of the stock price facilitates the calibration of the tree, being most of the required input data directly observable on the financial markets. The only parameter that needs to be estimated is the correlation between Libor rate and stock price. To this aim, four different measures of correlation are concerned: i) historical correlation; ii) exponentially-weighted moving average (EWMA) correlation; iii) correlation forecasted by means of a bivariate GARCH(1,1) and iv) implied correlation.

For convenience of the reader, let us change slightly the notation with respect to previous chapters. The first measure of correlation is the most simple and intuitive. The n-days correlation between Libor rate and stock return at time t,$\rho_{t,n}^{hist}$, can be easily calculated, using the past observations of the interbank offered rate and of the stock price, by means of the following formula:

$$\rho_{t,n}^{hist} = \frac{\sum_{i=0}^{n}(R_{\delta L,t-i} - \mu_{\delta L})(R_{S,t-i} - \mu_S)}{\sqrt{\sum_{i=0}^{n}(R_{\delta L,t-i} - \mu_{\delta L})^2}\sqrt{\sum_{i=0}^{n}(R_{S,t-i} - \mu_S)^2}} \tag{6.1}$$

where $R_{S,t-i}$ is the log return of the stock at time t and μ_S is its simple average over the period from t to $t-n$; $R_{\delta L,t-i}$ is the log difference between the fixing at time t and that at time $t-1$ of the interbank offered rate with tenor δ and is the simple average of $\mu_{\delta L}$ over the period from t to $t-n$. Notice that this measure of correlation attributes an equal weight to each observation from $t-n$ to t. On the contrary the EWMA correlation, whose diffusion amongst practitioners is mainly due to its application by J.P. Morgans RiskMetrics$^{\text{TM}}$, gives more weight to recent data with respect to older data and, for this reason, it reacts faster to sudden changes of the risk factors. The n-days EWMA correlation between Libor rate and stock return at time t, $\rho_{t,n}^{EW}$, is this calculated as follows:

$$\rho_{t,n}^{EW} = \frac{\sum_{i=0}^{n}\tau^i(R_{\delta L,t-i} - \mu_{\delta L})(R_{S,t-i} - \mu_S)}{\sqrt{\sum_{i=0}^{n}\tau^i(R_{\delta L,t-i} - \mu_{\delta L})^2}\sqrt{\sum_{i=0}^{n}\tau^i(R_{S,t-i} - \mu_S)^2}} \tag{6.2}$$

where $\tau \in (0,1)$ is the decay factor and, in this thesis it is set equal to .94.

Following Lopez and Walter ([43]), the third measure of correlation considered here is the forecast obtained by means of a bivariate GARCH(1,1), in the diagonal VECH specification proposed by Bollerslev et al. ([44]). The equation for the variance-covariance matrix between the stock return and interest rate at each time t, Σ_t, is the following:

$$vech(\Sigma_t) = \Gamma + Avech(\varepsilon_t\varepsilon_t^{'}) + Bvech(\Sigma_{t-1}) \tag{6.3}$$

where *vech* is an operator that converts an N by N matrix into an N(N+1)/2 by 1 vector, Γ is a 1 by 3 vector of parameters to be estimated, A and B are 3 by 3 diagonal matrix of parameters to be estimated, and $\varepsilon_t = [R_{\delta L,t}R_{S,t}]^{'} = [R_{1,t}R_{2,t}]^{'}$. Notice that we set

$R_{\delta L,t} \equiv R_{1,t}$ and $R_{S,t} \equiv R_{2,t}$ to simplify the notation. Therefore, equation 6.3 can be rewritten as follows:

$$\begin{bmatrix} \sigma^2_{1,t} \\ \sigma_{12,t} \\ \sigma^2_{2,t} \end{bmatrix} = \begin{bmatrix} \gamma_{11} \\ \gamma_{12} \\ \gamma_{22} \end{bmatrix} + \begin{bmatrix} \alpha_1 1 & 0 & 0 \\ 0 & \alpha_1 2 & 0 \\ 0 & 0 & \alpha_2 2 \end{bmatrix} \begin{bmatrix} r^2_{1,t} \\ r_{1,t} r_{2,t} \\ r^2_{2,t} \end{bmatrix} + \begin{bmatrix} \beta_1 1 & 0 & 0 \\ 0 & \beta_1 2 & 0 \\ 0 & 0 & \beta_2 2 \end{bmatrix} \begin{bmatrix} \sigma^2_{1,t-1} \\ \sigma_{12,t-1} \\ \sigma^2_{2,t-1} \end{bmatrix} \tag{6.4}$$

where $\sigma^2_{1,t}$, $\sigma^2_{2,t}$ and $\sigma_{12,t}$ are respectively the daily variance of the Libor rate, the daily variance of the stock price and the daily covariance between them at time t. To compute the forecast of the n-days variance/covariance, $\mathbb{E}_t[\sigma_{ij,t,n}]$, with $i, j = 1, 2$, we exploit the properties that the estimates are not serially correlated, so that:

$$\mathbb{E}_t[\sigma_{ij,t,n}] = \sum_{k=1}^{n} \mathbb{E}_t[\sigma_{ij,t+k}] \tag{6.5}$$

where $\mathbb{E}_t[\sigma_{ij,t}]$ is the forecast of the daily variance/covariance at time t. In doing this for the whole sample period, we use a “rolling” time horizon. This means that the forecasts of the daily variance/covariance, $\mathbb{E}_t[\sigma_{ij,t+k}]$, for $k = 1 \ldots n$, are obtained by means of the GARCH parameters estimated using 256 observations prior of the day t. Finally, the n-days correlation forecast at time t, $\rho^G_{t,n}$ is equal to:

$$\rho^G_{t,n} = \frac{\mathbb{E}_t[\sigma_{12,t,n}]}{\sqrt{\mathbb{E}_t[\sigma^2_{1,t,n}]}\sqrt{\mathbb{E}_t[\sigma^2_{2,t,n}]}} \tag{6.6}$$

The major drawbacks related to the measures of correlation proposed above are that: i) they are “exogenous” to the model, meaning that the correlation is estimated regardless the mechanics of the model itself and thus, their use do not assure that the model can reproduce the observed prices of financial securities (e.g. of the options); ii) the correlation is always estimated using past data that, if the markets are efficient enough (see Fama [26]), cannot significantly improve the information set contained in the current prices. For this reason, we propose a different way to calibrate the correlation, based only on the current market prices and volatilities of the risk factors. As a matter of fact, the BDT model can be efficiently calibrated by using the implied volatilities from caps, floors and swap options market obtained by means of the Black ([12]) formula (see De Simone [45]). At the same time, the CRR model can efficiently be calibrated by means of the implied volatility from stock option markets, calculated by mean of the Black and Sholes formula. Therefore, the price at time t of a stock option (e.g. an

American call), calculated using the numerical procedure explained in Chapter 2, can be represented as a function of the current values of the stock price, of the Libor rate, of the corresponding volatilities and of the correlation parameter. Assuming for simplicity that the correlation and the volatility of both the interest rate and the stock price are constant over time, we can set the market price of a call option at time t, ξ_t^M, equal to the price ξ_t^{NP} of the same call calculated by means of the numerical procedure depicted in the previous chapters:

$$\xi_t^M = \xi_t^{NP}\left(S_t, L(t,T_j), \sigma_{S,t,n}^{bls}, \sigma_{\delta L,t,n}^{blk}, \rho_{t,n}^I | X, N, y_t\right) \tag{6.7}$$

where $N = T_j - t$ is the time to maturity of the option; $\sigma_{S,t,n}^{bls}$ and $\sigma_{\delta L,t,n}^{blk}$ are respectively the yearly volatilities of the stock price and Libor rate at time t, calculated by means of the Black and Scholes and of the Black formulas with reference to options having a time to maturity equal to N; y_t is the dividend yield of the stock at time t and $\rho_{t,n}^I$ is the n-days implied correlation measure that can be computed by using an appropriate algorithm of calculus. We therefore define as implied correlation that value of $\rho_{t,n}^I$ that equalizes (at least approximately) the observed market price of a stock option (i.e. an American call) to its theoretical price as represented by the right-hand side of equation 6.7.

Figure 6.1 reports the absolute value of differences, over the period 03/01/2011 30/12/2011, between the daily price of a 1 year constant maturity ATM American call option calculated by means of the numerical procedure (NP price) using each of the above mentioned correlation measures, and the price of the same option calculated by using the Barone Adesi and Whaley ([46]) analytical approximation (BAW price). The difference is expressed as percentage of the BAW price. As underlying, we select a security listed on the Italian stock market while as risk free rate we use the 12 month Euribor rate. The time series of prices, dividend yield, rates and their volatilities are provided by Bloomberg™.

We notice that the best performance is due to the implied correlation, since only occasionally the price difference is higher than 0.05%. Moreover, we notice that also the other correlation measures allow for an appreciable fitting of the NP price to the BAW price and thus to the observed market prices, being the difference between them never higher than 7.7%. If we consider that the price of the option ranges from 0.2 to 0.3 Euro, it means that the maximum absolute difference between NP price and BAW price ranges from 1.54 cents to 2.31 cents.

There are two main drawbacks related to the estimation of the proposed measure of implied correlation:

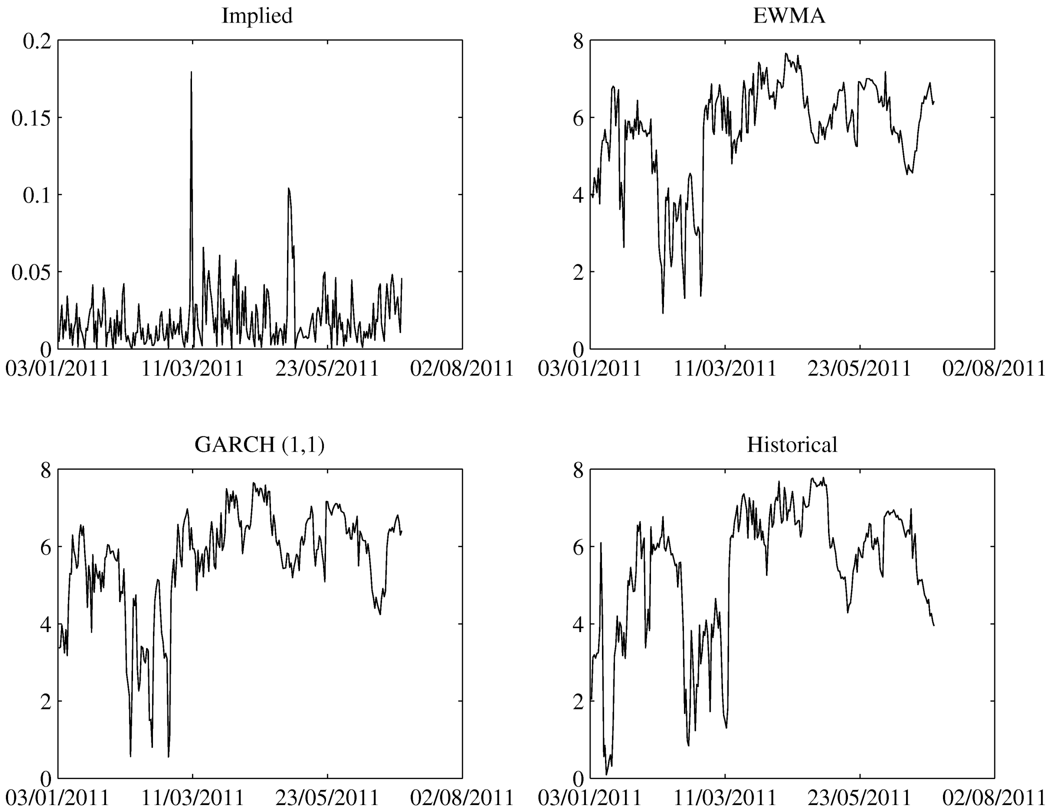


FIGURE 6.1: Differences between NP and BAW price in percentage of the BAW price

- the absence of a closed formula imply the necessity to state an algorithm of calculus to find the value of $\rho^I_{t,n}$ that allow equation 6.7 to hold. The higher the precision of the calculus algorithm the higher the computational burden of computing the correlation parameter;
- as for the volatility of the interest rate, it should be necessary to specify a term structure also for the implied correlation. To simplify the calibration procedure, we set the hypothesis that the correlation is constant over time even if it can be remarked that this is not exactly the case. As a matter of fact, the longer the observation period, the less variable is the correlation and it cannot be excluded that the sign of the correlation may differ as the observation period changes.

Moreover, the issue that we are interesting to address is whether and to what extent, the implied correlation estimated by using equation 6.7 can help to predict the future values of the correlation between interest rate and risk

6.2 Testing the predictive accuracy of the measures of correlation

We compute the 60-days correlation between the stock price of an Italian listed bank and the 3-month Euribor rate using the four measures of correlation depicted in the previous section (historical, EWMA with τ=.94, GARCH based correlation and implied correlation) from 01/01/2011 to 31/12/2011, for a total of 257 observations. All data are provided by BloombergTM database. The implied correlation is estimated with reference to an hypothetical constant maturity ATM 3 month call option, written on a security listed on the Italian stock market, calculated by means of the Barone Adesi and Whaley ([46]) formula. To this aim, we use the 3 month implied volatility provided by BloombergTM and, as risk free rate, the 3 month Euribor rate.

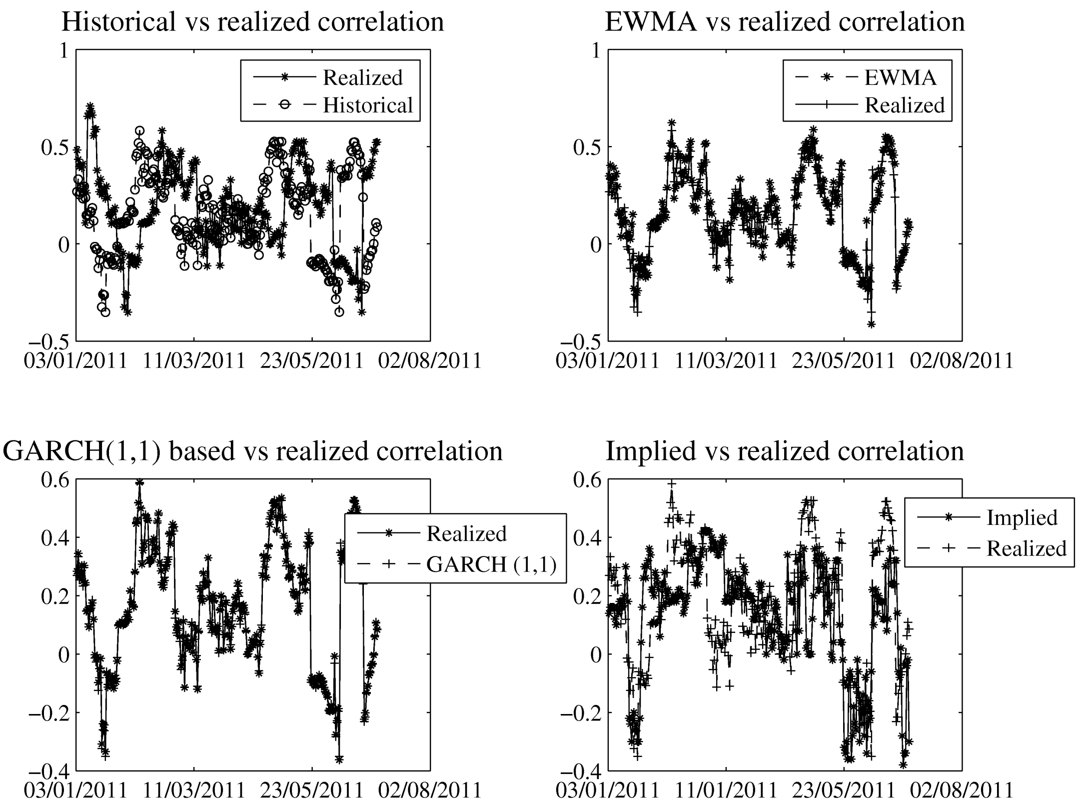


FIGURE 6.2: Comparison between four measures of correlation and the realized correlation

Figure 6.2 shows the comparison of each correlation forecast with the realized correlation. The evidence of positive correlation, for most of the observation period, between the 3 month Euribor and the stock price is consistent with the findings of Flannery and James ([28]), given the inverse relation between bond prices and interest rates. Moreover, we notice that, the implied correlation is much more volatile with respect to the other measures, meaning that such correlation reacts faster to sudden changes of the quotations. However, if we look at the predictive accuracy, the most affordable measure seems to be the one estimated via multivariate GARCH(1,1).

To test the predictive accuracy of the four types of correlation, we follow Lopez and Walter ([43]) and use three different methods:

i) analysis of the forecast error;

ii) partial optimality regression;

iii) encompassing regression.

The first method consists in analysing the correlation error ($\eta^j_{t,n}$), defined as the difference between the particular measure of n-days correlation forecast $\rho^j_{t,n}$, and the realized n-days correlation $\rho^{real}_{t,n}$: $\eta^j_{t,n} = \rho^j_{t,n} - \rho^{real}_{t,n}$ where:

$$\rho^{real}_{t,n} = \frac{\sum_{i=0}^{n}(R_{\delta L,t+i} - \mu_{\delta L})(R_{S,t+i} - \mu_S)}{\sqrt{\sum_{i=0}^{n}(R_{\delta L,t+i} - \mu_{\delta L})^2}\sqrt{\sum_{i=0}^{n}(R_{S,t+i} - \mu_S)^2}} \tag{6.8}$$

After $\eta^j_{t,n}$ is calculated, we regress it on a constant. If the estimated constant is significantly different from zero, the correlation j is said to be a biased forecast of the realized correlation. The second method, partial optimality regression consists in estimating, for each measure of correlation, the following equation:

$$\rho^{real}_{t,n} = a_0 + a_1\rho^j_{t,n} + e_t \tag{6.9}$$

If the coefficients a_0 and a_1 are not significantly different from 0 and 1 respectively, the correlation measure j is said to be a partial optimal forecast of the realized correlation. Finally, the third method, encompassing regression, consists in estimating, for each measure of correlation, the following equation:

$$\rho^{real}_{t,n} = b_0 + \sum_{j=1}^{s}\rho^j_{t,n} + e_t \tag{6.10}$$

Take $s = 2$ as an example. If b_0, b_1 and b_2 are not significantly different from 0, 1 and 0 respectively, than the correlation measure $j = 2$ "encompasses" the correlation measure $j = 1$, meaning that the information set included in the estimation of the former encompasses that included in the estimation of the latter.

6.3 Results

Table 6.1 reports the regression coefficients for the tree type of tests. In performing the regressions, we use the Newey and West ([47]) standard errors to account for potential heteroskedasticity and autocorrelation.

	Method 1	**Method 2**				**Method 3**
Intercept	-	0.0087 (0.0126)	0.0041 (0.0048)	-0.0012 (0.0008)	-0.2163^{**} (0.0549)	0.0001 (0.0001)
IMPLIED	-0.0022 (0.0227)	$0.5336^{\wedge\wedge}$ (0.0519)				-0.0006 (0.0006)
EWMA	0.0043 (0.0040)		0.9490^{++} (0.0162)			-0.1910 (0.2243)
GARCH	0.0007 (0.0006)			1.0033^{++} (0.0030)		$-1.1903^{\wedge\wedge}$ (0.0023)
HIST	0.0263 (0.0209)				$0.2016^{\wedge\wedge}$ (0.0175)	-0.0015 (0.0004)
#.of obs.	257	257	257	257	257	257

** the intercept is significantly different from zero at the 1% level

$^{\wedge\wedge}a_0$ is significantly different from 1 at the 1% level

++ the Wald test for the joint hypotheses that $b_0 = 0$ and $b_1 = 1$ cannot be rejected at the 1%

TABLE 6.1: Regression results

The table shows the regression coefficients, estimated by using the Newey and West ([47]) standard errors, from the analysis of the forecast error (Method 1), the partial optimality regression (Method 2) and the encompass regression (Method 3). Standard deviation are in parentheses. Under method 1 are reported, for each correlation measure, the results of the regression of on a constant. Under method 2 are reported, for each correlation measure, $\eta^j_{t,n}$ the estimates of the coefficient of equation 9, whilst under method 3 the coefficients of equation 10 are reported.

The results seems to confirm the intuition behind the analysis of figure 6.2. If we look at the first column of table 6.1, (Method 1), we notice that none of the correlation measures is said to be a biased forecast, even if only the EWMA and the GARCH correlation are partially optimal. In fact, the coefficients a_0 and a1 associated to both these measures of correlation are not significantly different from 0 and 1 respectively, while the coefficient a1 associated to the implied and to the historical volatility measures is significantly different from 1 at 1% level. Moreover, the Wald test for the joint hypotheses that the coefficients a0 and a1 are equal to 0 and 1 respectively, can be rejected at the 1% level only for the GARCH and for the EWMA correlations. Finally, from the last column of table 3 we notice that the coefficients associated to the correlation measures are not significantly different from 0, except for that associated to the GARCH correlation that is greater than 1. This evidence suggests that the GARCH measure of correlation "encompasses" all the other measures.

Chapter 7

Conclusions

This thesis develops a numerical procedure for pricing financial contracts whose contingent claims are exposed to two sources of risk: the stock price and the short interest rate. Particular emphasis here is placed on hybrid financial securities, i.e. on a group of financial contracts that combine the elements of the two broader groups of securities, debt and equity. Moreover, we focus on "American style" financial products, i.e. financial contracts (such as American options or convertible bonds) giving the owner a right to be exercised within a certain date. As generally known, since the price of these contracts depends on the interim values assumed by the risk drivers (path dependency), their valuation needs more sophisticated pricing tools and give rise to more interesting considerations.

In particular, the proposed pricing framework assumes that the stock price dynamics is described by the Cox, Ross Rubinstein binomial model under a stochastic risk free rate, whose dynamics evolves over time accordingly to the Black, Derman and Toy one-factor model. Such procedure represents a good compromise between the computational burden that in general characterizes the pricing of complex hybrid products (for which no closed formula is available), and calibration efficiency. This the case because the procedure allows to combine the main features of the two model mentioned above, so that an efficiently calibration to the observed market data can be performed. Moreover, the correlation between interest rate and stock price, that very often is neglected when binomial trees are concerned, is also taken into account, even if in this case the issue of computing such parameter arises.

It is worth noting that, the necessity for a dynamics that is consistent with the observed market data arises not only because the model should be able to "reproduce" the observed market prices of as many (traded) financial assets as possible, but also because it should be calibrated directly using the market observables. One major goal of the numerical procedure is that its adoption allows practitioners to avoid complex estimation techniques for computing the value of the unknown parameters describing the evolution of the risk factors. We think that this is a necessary (even though not sufficient) condition to develop a consistent framework that can be useful for trading purposes because too often econometric techniques are used inappropriately, given the great deal of subjective choices necessary for the estimation. This is in the spirit of a financial industry that is intended to standardize information to facilitate its diffusion and the adoption of worldwide market standards and conventions.

We also show how to apply the numerical procedure to compute the price of three financial contracts with increasing complexity: a vanilla (European and American) call option, a callable convertible bond and a participating policy, i.e. an insurance contract where the benefit for the policyholder is partly fixed and partly variable, depending on the profit of the insurance company.

The procedure suffers two major drawbacks. The first consists in the fact that the number of branches of the tree may considerably increase as the time to expiration of the product increases. However, we do not think that this is really a problem because, especially for very long run financial contracts, few time steps are necessary for the probability space to reach a certain desired “density”. In fact, keeping the number of desired final replicas of the risk factors constant, binomial trees require in general a higher number of time steps with respect to the procedure described above. In fact, the number of final states of the world in a standard binomial tree is 2^n while in our numerical procedure is 2×4^n, so that less number of steps are necessary for getting a certain pricing accuracy.

The second drawback is more serious, because it concerns the possibility that the up (down) return of the stock price is higher than the return on the bank account. This may result in an arbitrage opportunity. In this thesis we suggest to verify this possibility at each time step and also presents some ideas for preventing the risk neutral probabilities to be higher then 1 or less then zero. We also discuss the cases where it can be more likely for this situation to happen, as it is the case during periods of financial stress, market bubbles and inflation. We hope that further research may shed more light about this issue and try to find more valuable solutions.

Although the value of a broad variety of financial products can potentially be computed, it is inconvenient to use our numerical procedure for pricing vanilla securities. This is the case because in general the higher pricing efficiency (if any) attainable from the inclusion of more sources of risk, cannot justify the increased computational burden and complexity required for the implementation of the procedure, especially because the pricing efficiency of simpler models (developed in the literature and in practice) is already acceptable. As far as simpler pricing models are available, we recommend to not use this numerical procedure for pricing vanilla products.

In pricing convertible bonds, it is worth noting that our numerical procedure takes into account the default of the issuer, not simply discounting the bond part at a risky rate, but considering the event of default as a possible state of the world. On the other hand, one of the main drawbacks of the procedure proposed is that it does not take into account the possible correlation between the two risk factors and the default probability. In fact, it can be noticed that as the stock price decreases, the default is more likely to occur, while the opposite can be told about the relationship between interest rate and stock price. This point is on the contrary captured by pricing models within a structural approach, where the default is characterized by a market value of the firm lower than the market value of its liabilities.

Finally, it is important to remark that, we also discuss some issues related to the implementation and calibration of such a two factors numerical procedure and in particular how the dynamics of each risk factor can calibrated to the observed market prices. In particular, in order to assess the validity of the model, its advatages and drawbacks, we conduct an empirical analysis where, in particular, the role of the correlation between stock price and interest rate is emphasized. We study different possible ways for calibrating the correlation parameter, including implied correlation and multivariate GARCH forecast. In fact, by means of an empirical analysis, we try to assess the relative contribution of the correlation component. To this aim, four correlation measures are concerned, namely, historical, exponentially weighted moving average correlation, a bivariate GARCH forecasted correlation and implied correlation obtained from the stock option prices. We show that the best pricing performances are associated to the implied correlation measure, even if also the adoption of the other correlation measures allows to obtain prices that are reasonably close to the observed market prices. However, the main drawback in computing the implied correlation from the bi-factorial procedure is related to its low predictive accuracy. Compared to other correlation measures (especially GARCH correlation), implied correlation cannot be considered an affordable forecast of the future realized correlation as it occurs for volatility. We hope that further research may shed more light about the possibility to extract implied correlation measures from bi-factorial pricing models, also involving risk factors of different nature. In fact, the procedure is very general, and may be applied for obtaining the joint lattice of two risk factors whose dynamics are expressed by means of binomial trees. We support the idea that this procedure may be very valuable for hybrid products, but we do not investigate if it can also be helpful for describing the dynamics of other risk factors, such as exchange rate or credit spread.

Appendix A

Risk neutral probabilities in a two factor binomial lattice

We prove that the risk neutral probabilities in non-terminal nodes of the binomial lattice with stochastic interest rate are those illustrated in table 2.2, that we report here for the convenience of the reader.

State of the world			**Correlation**		
Stock price	**Libor rate**	**Prob.**	$\rho_t < 0$	$\rho_t = 0$	$\rho_t > 0$
up	up	h_t^1	$p_t 0.5(1+\rho_t)$	$p_t 0.5$	$0.5(p_t + 0.5\rho_t)$
down	down	h_t^2	$(1-p_t)0.5(1+\rho_t)$	$(1-p_t)0.5$	$0.5(1-p_t+0.5\rho_t)$
up	down	h_t^3	$0.5(p_t - 0.5\rho_t)$	$p_t 0.5$	$p_t 0.5(1-\rho_t)$
down	up	h_t^4	$0.5(1-p_t-0.5\rho_t)$	$(1-p_t)0.5$	$(1-p_t)0.5(1-\rho_t)$

Table A.1: Joint probability mass function at each node

To begin with, recall the value of the call option at node A illustrated by equations 2.18 and 2.19:

$$\xi_A^u = \frac{1}{B_1^u}\left(\frac{B_1^u - D_S}{U_S - D_S}\xi_E + \frac{U_S - B_1^u}{U_S - D_S}\xi_G\right)$$

$$\xi_A^d = \frac{1}{B_1^d}\left(\frac{B_1^d - D_S}{U_S - D_S}\xi_F + \frac{U_S - B_1^d}{U_S - D_S}\xi_H\right)$$

Substituting 2.4 into 2.18 and 2.19, and recalling that $B_{T_j}^m = 1 + L(T_j, T_{j+1})\delta$ with $T_{j+1} - T_j = \delta, m = u, d$ and $j \in \mathbb{N}^+$ yields:

$$\xi_A^u = \frac{[P_1^u \xi_E + (1 - P_1^u)\xi_G]}{1 + L(1,2)^u}$$

$$\xi_A^d = \frac{\left[P_1^d \xi_F + (1 - P_1^d)\xi_H\right]}{1 + L(1,2)^d}$$

In a risk neutral world, according to the BDT Libor rate tree, the up probability of the Libor rate is equal to $1/2$ and it is constant. Moreover, if the stock price and the Libor rate are independent, the probability associated to each couple of the two risk factors is simpy the product of their marginal probabilities. It follows that the call price at node A, ξ_A is equal to:

$$\begin{aligned}
\xi_A &= \frac{[P_1^u \xi_E + (1 - P_1^u)\xi_G]}{1 + L(1,2)^u}\frac{1}{2} + \frac{\left[P_1^d \xi_F + (1 - P_1^d)\xi_H\right]}{1 + L(1,2)^d}\frac{1}{2} \\
&= \frac{[P_1^u 0.5\xi_E + (1 - P_1^u)0.5\xi_G]}{1 + L(1,2)^u} + \frac{\left[P_1^d 0.5\xi_F + (1 - P_1^d)0.5\xi_H\right]}{1 + L(1,2)^d} \\
&= \frac{\xi_E h_2^1}{1 + L(1,2)^u} + \frac{\xi_F h_2^3}{1 + L(1,2)^d} + \frac{\xi_G h_2^4}{1 + L(1,2)^u} + \frac{\xi_H h_2^2}{1 + L(1,2)^d} \qquad \text{(A.1)}
\end{aligned}$$

where the probabilities h_2^i are the same of table 2.2 for the case of zero correlation.

If stock price and Libor rate are supposed to be not independent, a further assumption is necessary in order to compute the risk neutral probabilities. In doing this, recall that the states of the world are mutually exclusive and collettively exhaustive. Suppose now that the correlation between the two risk factors is positive ($\rho_t = +1$). This means that the when the stock price goes up (down) also the Libor rate goes up (down), and no other states of the world are possible. But if this is the case, it is impossible for the call price to reach the nodes F and G and the issue of redistributing the probabilities h among the other states of the world arises. If we redistribute the probabilities in the way illustrated in section 2.1.5 to the last line of equation A.1 and rearrange the proof follows immediately. Q.E.D.

Appendix B

The partial differential equation for two-factor derivatives

We show how to derive the partial differential equation 2.25 for a derivative Q subject to two sources of uncertainty: the uncertainty inherent in the interest rate and in the stock price.

First of all, recall the stochastic differential equation 2.20, whose compact version is illustrated below for the convenience of the reader.

$$dr = \mu dt + \sigma dW_r \tag{B.1}$$

By Ito's lemma, the instantaneous change in the price of a bond $i, v_i(t, T_i, r)$ with $i = 1, 2...k$, can be written as follows:

$$\frac{dv_i}{v_i} = \mu_{v_i} dt + \sigma_{v_i} dW_r \tag{B.2}$$

where

$$\begin{aligned}
\mu_{v_i} &= \left(\frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial r}\mu + \frac{1}{2}\frac{\partial^2 v_i}{\partial r^2} r^2 \sigma^2 \right) / v_i \\
\sigma_{v_i} &= \left(\frac{\partial v_i}{\partial r} r\sigma \right) / v_i
\end{aligned}$$

Consider a zero investment portfolio P_v formed by investing the amount $w_i, i = 1, 2$ in two bonds v_1 and v_2 and borrowing the amount $(w_1 + w_2)$ at the risk free rate r. The instantaneous change in the value of such portfolio is:

$$dP_v = [w_1(\mu_{v_1} - r) + w_2(\mu_{v_2} - r)]\, dt + (w_1\sigma_{v_1} + w_2\sigma_{v_2})\, dW_r \tag{B.3}$$

Choose w_i such that

$$(w_1\sigma_{v_1} + w_2\sigma_{v_2}) = 0. \tag{B.4}$$

The portfolio P_v is instantaneously non-stochastic and, if no arbitrage is possible, its return must be zero. This imply that:

$$w_1(\mu_{v_1} - r) = -w_2(\mu_{v_2} - r) \tag{B.5}$$

Solving B.4 for w_1 and plugging the result in B.5 yields:

$$\frac{(\mu_{v_1} - r)}{\sigma_{v_1}} = \frac{(\mu_{v_2} - r)}{\sigma_{v_2}} = \lambda(r(t), t) \tag{B.6}$$

The result of B.6 is general and holds for each couple of bonds. It implies that the market price of interest rate risk, $\lambda(r(t), t) \equiv \lambda$ is the same for each bond in the market.

Now, recall the compact form of equation 2.23:

$$dS = \alpha dt + \beta dW_S \tag{B.7}$$

The instantaneous rate of change in the value of the derivative $Q(r(t), S(t), t) \equiv Q$, using the Ito's lemma, is:

$$\frac{dQ}{Q} = \gamma dt + \frac{\partial Q}{\partial S}\frac{S\beta}{Q} dW_S + \frac{\partial Q}{\partial r}\frac{r\sigma}{Q} dW_r \tag{B.8}$$

where $\gamma(r(t), S(t), t) \equiv \gamma$ is given by the following:

$$\gamma = \left[\frac{\partial Q}{\partial S}\alpha + \frac{\partial Q}{\partial r}\mu + \frac{\partial Q}{\partial t} + \frac{1}{2}\frac{\partial^2 Q}{\partial S^2}S^2\beta^2 + \frac{1}{2}\frac{\partial^2 Q}{\partial r^2}r^2\sigma^2 + \frac{\partial^2 Q}{\partial S \partial r}\beta\sigma\rho\right] / Q \tag{B.9}$$

where ρ is the instantaneous correlation between r and S.

Consider to compose a zero net investment portfolio, denoted by Pf by investing amounts w_Q, w_v and w_S in the derivative, a default-free pure discount bond and the stock respectively, and borrowing the amount $(w_Q + w_v + w_S)$ at the instantaneous interest rate r. Using equation B.1, B.2 and B.7, the instantaneous change in the value of such portfolio is the following:

$$\begin{aligned} dPf &= [w_Q(\gamma - r) + w_S(\alpha - r) + w_r(\mu - r)]\, dt \\ &+ \left[w_Q \frac{\partial Q}{\partial S}\frac{S\beta}{Q} + w_S\beta \right] dW_S \\ &+ \left[w_Q \frac{\partial Q}{\partial r}\frac{r\sigma}{Q} + w_v\sigma_v \right] dW_r \end{aligned} \tag{B.10}$$

If we choose w_Q, w_v and w_S such that

$$\begin{aligned} \left[w_Q \frac{\partial Q}{\partial S}\frac{S\beta}{Q} + w_S\beta \right] &= 0 \\ \left[w_Q \frac{\partial Q}{\partial r}\frac{r\sigma}{Q} + w_v\sigma_v \right] &= 0 \end{aligned} \tag{B.11}$$

the portfolio Pf is instantaneously non-stochastic and, if no arbitrage is possible, its return must be zero. This imply that:

$$[w_Q(\gamma - r) + w_S(\alpha - r) + w_r(\mu - r)] = 0 \tag{B.12}$$

Eliminating w_Q, w_v and w_S between B.12 and B.12 and recalling equations B.9 and B.6 we obtain equation 2.25:

$$\frac{\partial Q}{\partial S}\alpha + \frac{\partial Q}{\partial r}[\mu - \lambda r\sigma] + \frac{\partial Q}{\partial t} + \frac{1}{2}\frac{\partial^2 Q}{\partial S^2}S^2\beta^2 + \frac{1}{2}\frac{\partial^2 Q}{\partial r^2}r^2\sigma^2 + \frac{\partial^2 Q}{\partial S \partial r}\beta\sigma\rho - rQ = 0$$

Q.E.D.

Appendix C

American option pricing theory: basic proofs

This appendix reports the proofs of some propositions illustrated in chapter 3. Further details about this proofs can be found in Ingersoll's book ([48]).

Proposition 9: *for a stock paying no dividends $\Xi(S;t;T;X) \geq S - Xv(t,T)$. In this case exercise of the American call will never occur prior to maturity, and $\Xi(\cdot) \equiv \xi(\cdot)$.*

Proof. First of all, to prove the statement, it is necessary to prove that if the stock pays no dividend between t and T, then

$$\xi(S;t;T;X) > S - Xv(t,T). \tag{C.1}$$

Let ν_m be the current value of a pure state security which pays one dollar in the state when $S_T = m$. We also refer to this security as "state price", since it can be interpreted as the amount to pay for receiving one dollar if the state m occurs. If we recall that $v(t,T)$ can be defined as the present value of one dollar for sure at time T and assuming that the sure rate of interest is always positive, so that $v(t,T) < 1$ for $t < T$, we have:

$$v(t,T) = \textstyle\sum_m \nu_m \tag{C.2}$$

$$\xi(S_t;t;T;X) = \textstyle\sum_m \nu_m (m-X)^+ = \displaystyle\sum_{m \geq X} \nu_m (m-X)^+ \tag{C.3}$$

$$S = \textstyle\sum_m \nu_m m \tag{C.4}$$

Notice that, from C.2 we have:

$$\xi(S_t;t;T;X) = \sum_m \nu_m (m-X)^+ \geq \sum_m \nu_m (m-X) = \sum_m \nu_m m - X \sum_m \nu_m m \quad \text{(C.5)}$$

To prove that if the stock pays no dividend between t and T, then $\xi(S;t;T;X) > S - Xv(t,T)$ simply plug C.2 and C.3 into C.5.

From Propositions 5 and from C.1, $\Xi > \xi > S - Xv(t,T)$. Furthermore, since $\Pi(\cdot) < 1$ for $t < T, \Xi > S - X$, which is the options value when exercised. Since the option is always worth more "alive" than when exercised, exercise will never occur prior to maturity. Finally, since the extra right of the American call is never used, the American call is worth no more than its European counterpart. Q.E.D.

Proposition 10: *put-call relation for American options.*

$$\Pi(S;t;T;X) \geq \Xi(S;t;T;X) - S + Xv(t,T) : \quad \text{(C.6)}$$

Proof. If the stock pays no dividends, then $\Pi(\cdot) \geq \pi(\cdot)$ and $Xi(\cdot) = \xi(\cdot)$ by Propositions 5 and 10. The relation given then follows from the put-call parity relation for European options:

$$\pi(S;t;T;X) = \xi(S;t;T;X) - S + Xv(t,T) \quad \text{(C.7)}$$

It can be worth to show a simplified proof of the put-call parity for European options of equation C.7 before illustrating the proof for American ones. To this aim, notice that the payoff on the portfolio which is long one call and the lending and short one share of the stock is $(S-X)^+ - S + X = (X-S)^+$. This is also the payoff on a put, so the two must have the same current value.

As far as American options are concerned, if the stock does pay dividends, it may pay to exercise the call option prior to its maturity. If it is exercised at time τ such that $t < \tau \leq T$, the portfolio is worth $S - X - S + Xv(\tau,T) < 0$, while the put must have a positive value. If the call is not exercised prior to its maturity, then the portfolio and the put will have the same values as in the proof of equation C.7 . Q.E.D.

Proposition 11: *if the risk-neutral stochastic process for the stock price is proportional, then put and call option prices are homogeneous of degree one in the stock price and the exercise price, and they are monotone convex functions of the former.*

Proof. To prove the statement, we first need to prove that a call option price is a convex function of the exercise price. That is, for $X1 > X2 > X3$:

$$\xi(\cdot; X_2) \leq \theta\xi(\cdot; X_1) + (1-\theta)\xi(\cdot; X_3); \tag{C.8}$$

where $\theta \equiv (X_3 - X_2)/(X_3 - X_1)$. To see this, notice that, since $(m - X)^+$ is convex in X, so is the positive linear sum $\sum \nu_m(m - X)^+$. Specifically, substituting C.2 into C.8 and rearranging yields:

$$\begin{aligned} &\theta\xi(\cdot; X_1) + (1-\theta)\xi(\cdot; X_3) - \xi(\cdot; X_2) \\ &\quad = \textstyle\sum_m \nu_m \left[\theta(m - X_1)^+(1-\theta)(m - X_3)^+ - (m - X_2)^+\right] \\ &\quad = \sum_{X_1 \leq m \leq X_2} \nu_m\theta(m - X_1) + \sum_{X_2 \leq m \leq X_3} \nu_m\left(\theta m - \theta X_1 - m - X_2\right) \geq 0 \end{aligned} \tag{C.9}$$

The first sum clearly has only positive terms. The second sum also has only positive terms; since $\theta < 1$, the summand is decreasing in m, so the smallest term is zero when $s = X_3$. This implies that the call option is a convex function of the underlying asset. Now, we will show that, if the stochastic process is proportional:

- option price is homogeneous of degree one in S and X;
- option price is a monotone convex function in S.

For a proportional risk-neutral stochastic process, the supporting price for a particular level of return does not depend on the current stock price, so:

$$\begin{aligned} \xi(\theta S_t; \theta X) &= \sum_m \nu(\theta S_T | \theta S_t)(\theta S_T - \theta X) \\ &= \sum_m \nu(\frac{S_T}{S_t})(\theta S_T - \theta X) \\ &= \theta\sum_m \nu(\frac{S_T}{S_t})(S_T - X) = \theta\xi(S_t; X). \end{aligned} \tag{C.10}$$

where the second equality follows from equation 3.7 and the last one follows form C.3. The remainder of the proof follows immediately from the properties of homogeneous functions. For $S_2 > S_1$:

$$\begin{aligned} \xi(S_2; X) - \xi(S_1; X) &= \xi(S_2; X) - \frac{S_1}{S_2}\xi\left(S_2; \frac{S_2}{S_1}X\right) \\ = \xi(S_2; X) - \xi\left(S_2; \frac{S_2}{S_1}X\right) &+ \left(1 - \frac{S_1}{S_2}\right)\xi\left(S_2; \frac{S_2}{S_1}X\right) \end{aligned} \tag{C.11}$$

where the first equality holds by multiplying and dividing the term $\xi(S_1; X)$ by $S_1 S_2$ and using equation C.10, while the second equality holds by adding and subtracting $\xi\left(S_2; \frac{S_2}{S_1} X\right)$ and rearranging. The last term is non-negative because, by assumption, $S_1 < S_2$ and $\xi(\cdot) > 0$. The difference of the first two terms is non-negative by Proposition 6.

To prove convexity, start with the relation in C.8. For $X_2 = \theta X_1 + (1-\theta) X_3$ and $\theta > 0$,

$$\xi(1; X_2) \leq \theta \xi(1; X_1) + (1-\theta)\xi(1; X_3); \tag{C.12}$$

Now define $\gamma = \theta X2/X1$ (note that $1 - \gamma = (1-\theta)X2/X3$) and $S_i = 1/X_i$. Using the homogeneity of the option function gives:

$$\frac{1}{X_2}\xi(S_2; 1) \leq \frac{1}{X_1}\theta\xi(S_1; 1) + (1-\theta)\frac{1}{X_3}\xi(S_3; 1) \tag{C.13}$$

that is equivalent to:

$$\xi(S_2; X_2) \leq \gamma\xi(S_2; X_1) + (1-\theta)\xi(S_2; X_3). \tag{C.14}$$

The proof for put options can be done in a similar manner, or it follows from the put-call parity theorem. Q.E.D.

Proposition 12: *If the state price per unit probability is a monotone decreasing function of the stock price, then a call options value is bounded by*:

$$v(t,T)\{\mathbb{E}\left[(S_T - X)^+ - S_T\right] + S_t\} \quad \leq \quad \Xi(S_t; t) \tag{C.15}$$

$$\leq \quad \frac{\mathbb{E}\left[(S_T - X)^+\right]}{\mathbb{E}\left[S_T\right]/S_t} \tag{C.16}$$

Proof. Consider a portfolio with $S_t - \Xi(St; t)$ dollars invested in bonds maturing at time T and one call option worth $\Xi(S_t; t)$. The payoff on this portfolio at time T is $(S_T - X) + (S_t - \Xi)/v(t,T) \equiv z(S_T)$. This payoff function is a positive constant for $S_T < X$. Above this level it increases dollar for dollar with the terminal stock price; thus, there is either one or no values S_T^* for which $z(S_T^*) = S_T^*$. Since the portfolio is currently worth as much as one share of stock, its payoff cannot always exceed that on the stock or it would dominate the stock. Thus, there must be exactly one value S_T^* of equality. Since the portfolio and the stock have the same current value,

$$
\begin{aligned}
0 &\geq \sum_{m<m^*} p_m^* \Lambda_{m^*}[z(m)-m] + \sum_{m \geq m^*} p_m^* \Lambda_{m^*}[z(m)-m] = \Lambda_{m^*} \sum_{m} p_m^*[z(m)-m] \\
& \textit{or} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(C.17)} \\
0 &\geq \mathbb{E}\left[z(S_T) - S_T\right] = \mathbb{E}\left[(S_T - X)^+ - S_T\right] + \frac{S_t - \Xi(\cdot)}{v(t,T)}
\end{aligned}
$$

The last inequality follows since $\Lambda_m > 0$ for all m. Solving this last inequality for $\Xi(\cdot)$ gives the lower bound.

To verify the upper bound, we compare a portfolio of $S_t/\Xi(S_t;t)$ options to a share of stock. The payoff on this portfolio is $St(S_T - X)^+/\Xi(S_t;t) \equiv Z(S_T)$. Again, this function has a single crossing point where $Z(S_T^*) = S_T^*$. This time the portfolios value exceeds the price of a share of stock when $S_T > S_T^*$. Since the portfolio is currently worth the same as one share of stock,

$$\mathbb{E}\left[\Lambda(Z(S_T) - S_T)\right] = 0 \tag{C.18}$$

Separating the sum as before gives:

$$0 \leq \mathbb{E}\left[(Z(S_T) - S_T)\right] = \mathbb{E}\left[\frac{S_t(S_T - X)^+}{\Xi(S_t,t)} - S_T\right] \tag{C.19}$$

Solving this last inequality for $\Xi(\cdot)$ gives the upper bound. Q.E.D.

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