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PhD Thesis

Models and Algorithms for Facility Location Problems with Equity Considerations

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Introduction

A Facility Location Problem (FLP) consists in defining the position of a set of points (facilities) within a given location space on the basis of the distribution of demand points (users) to be allocated to the facilities. In the practical applications either in private or in public sector, these problems deal with strategic and long term decisions involving huge investment costs.

In general when a facility is positioned, it produces effects, positive or negative, on the users (actual or potential), whose intensity can be considered depending on the mutual distance. Of course if the effects are positive (desirable facilities) effective positions for the facilities are expected as close as possible to the demand points. It is the case of public utility sites such as schools, hospitals, shops, banks, metro stations and so on. On the contrary, in case of negative effects (undesirable or obnoxious facilities) users wish that facilities are as far as possible. Examples of this kind concern power or nuclear plants, rubbish dumps. However also in these cases, when facilities are too far from the demand points additional costs have to be paid in terms of logistic costs. For this reason it is necessary to find compromise solutions able to balance the different aspects.

In many practical applications of FLPs it is important to consider how the effects (positive or negative) are distributed among the users. In particular it should be proper that the effects are distributed in "equitable or equal" manner. However, although equity and/or equality represent critical issues, they are not straightforward concepts to be defined and then to be measured. As a consequence many efforts have been produced to find means of measuring them in many fields and from different point of views. In general measures proposed to formulate FLPs have been adapted from other contexts (political, economic, social) and concern calculations related to the distribution of distances between users and patronized facilities. For this reason different measures have been proposed and various models and methods have been provided to solve mathematical models representing FLPs with equitable and equality measures.

In this work we mainly focus on the following objectives: investigating on properties of available measures for FLPs in order to highlight differences and similarities and, then, to support decision makers in the choice of the most appropriate ones; formulating and solving an optimization problem occurring in logistics as a FLP with an equality measure as objective function.
The contents of the work are organized as follows.

In Chapter 1 we introduce the main FLPs describing the characterizing elements and showing the most popular mathematical formulations.

In Chapter 2 we illustrate the concepts of equity and equality, analysing how they are intended in different contexts and stressing their importance in decision making process. Then we show the most popular measures proposed in the literature.

In Chapter 3 we describe how the equality concept and their measures can be used in FLPs as objective function. In particular we provide the results of a literature analysis oriented to highlight theoretical contributions about the use of equality measures and proposals of models and methods to solve FLPs with equity and/or equality measures.

In Chapter 4 we propose some new properties to be associated to equality measures in order to describe characteristics which may be useful to drive optimization procedures in the search of optimal (or near-optimal) solutions. We also provide the results of an empirical analysis with the aim of underlining differences and similarities between pairs of equality measures.

In Chapter 5 we formulate a transportation problem with multiple sources and single destination in terms of FLP. In particular in order to reduce risks of congestion in the dynamic of flow arrivals at the common destination, an appropriate equality measure is introduced.

In Chapter 6 some heuristic methodologies to solve the defined transportation problem are described. Computational results on randomly generated test problems show opportunities and limits to efficiently solve the problem.

Chapter 1

Generalities of Location Theory

1.1 Introduction

In this chapter we introduce the subject of this work: the Facility Location Problems. We propose an overview about them explaining what is a Facility Location Problem and illustrating their basic elements. After the proposition of several classifications introduced in the literature we provide the formulation of the most used models. We describe the generalities about these problems as first stage for the analysis and the development, in the following of this work, of new facility location models.

1.2 The Five W's for Facility Location Problems

The answer to the question "**What** is a localization problem" can get different replies from people coming from different scientific areas (Eiselt and Marianov, 2011). A mathematician can say that he finds a number of additional points that optimize a function of the distance between new and existing points in a given metric space. A geographer looks for the position of a number of centers that serve market places or communities in a given region. A manager firm's decide on where to locate a single product firm to maximize an expected utility based on the demand of clients adding some constraints like transportation cost in a predefined time horizon.

Apart from the difference that arises from the different perspectives, a general definition including all the common aspects can be:

A Facility Location Problem (FLP) consists in positioning a set of structures (facilities) in a given space in order to satisfy the demand (actual or potential) expressed by a set of customers.

So the additional points of the mathematician are the firms that manager wants to locate or the center of the geographer, that we call facilities; the existing points are the market places or the demand of clients as well as the demand expressed by customers or demand points. The function of the distance will be the expected utility or the satisfaction of the communities, that is our satisfaction of the demand.

The birth date (**When**) of FLPs can be assumed in correspondence of the ideas formulated by Fermat (1601-1665) in the early seventeenth century; the mathematician proposed the problem of finding a fourth point when we have three points in a plane such that the sum of its distances to the three given points is as small as possible. A solution for this problem was successively found by Evangelista Torricelli (1608-1647). For the very first formulation of location problem we have to wait the XX century with the work of Weber, while the seminal paper is retained the ones of Hakimi [1964]; from that point to right now there is an continuous and constant progress in these scientific area with a lot of models proposed (for an historical prospective see Wesolowsky, 1993).

Resuming of **Where** the location models are applied it's a very hard work; the applications and sector of them are the most varied. Typically they are divided in private and public sector problems. The first are used for seeking the location of sites that maximize a profit. Some examples can be the decision about the location of an assembly plant within a region, or finding the most efficient position for a distribution center and/or a warehouse in the context of a supply chain. In contrast, public sector problems seek facility sites that optimize the population's access to those facilities or universality of the service. Some examples can be schools, hospitals, but, also stations of a system transportation. In addition FLPs can be used not only in the geographic context but also for supporting the decision about the layout of electronic components or industrial products.

Most of the applications involve decisions on the strategic level as they usually require huge investment costs, so **Who** use the FLPs is decision makers aim at defining the position of one or more facilities. As such, decisions tend to be long-term, which implies that much of the data used in the decision-making process, will be quite uncertain. For this reason decision makers should select sites not only on the basis of current performance indicators but also considering long-term system evolution including, for instance, environmental factors changes, population shifts and market trends.

Due to their strategic characteristics (**Why**), decisions about facility locations significantly affect the efficiency of several short-term operational aspects, including the performance of the production and logistics systems.

For the importance and the complexity of these problems a lot of scholars developed very different qualitative and quantitative approaches to solve them. In this chapter we put our attention to the use of mathematical models, a proved particularly method to tackle such problems (e.g. Daskin *et al.*, 1988; ReVelle and Eiselt, 2005; ReVelle *et al.*, 2008).

1.3 Elements of Facility Location Problems

Generally Location Problems are characterized by the following main elements (Eiselt and Laporte, 1995):

- Space where the facilities are to be located;
- Facilities (already existing and new) to be located;
- Customers expressing service demand;
- Interaction between customers and facilities;

- Metrics to measure distances between customers and facilities;
- Constraints to be satisfied.

The **space** usually corresponds to a geographical region representing the area where we can locate the facilities. In general, the facilities and the customers' positions occupy very small areas compared to the space where they are located. For this reason, they can be considered dimensionless points. In most of the applications, the study area can be described in a two-dimensional plane. The space can be continuous if any point is feasible for a new facility. In this case it is also possible to include the presence of "forbidden zones", where facilities cannot be located due to geographical obstacles or technical constraints. Otherwise, if locations must be chosen within a set of candidate points, the problem is discrete. In some applications where the problem is naturally described through a reference network, facilities can be positioned either on the links of the network or in correspondence of the nodes (network based models). However, it is possible to formulate problems also in a multi-dimensional space. It is the case, for example, of positioning a company in a market than can be described as a space in a set of economic variables.

The **facilities** are characterized by some attributes that are: number, capacity, service to supply and cost (Scaparra and Scutella, 2001). Very often the number of new facilities to be located is fixed and in the simplest one a single facility is to be located. It is also possible that the number of facilities is a decision variable. Some facility can have infinite capacity (uncapacitated problems), namely they can receive a infinite number of customers, while some others no (capacitated problems). In a location model we can fix facility of different typology like in a multi-level distribution system where at each level we have plants or warehouse (multi-echelon problems). The facilities, eventually can supply one or more different services (multi-services facilities). Finally facilities can have different configurations; in fact, they can be considered to occupy either points locations or area locations or even a specialized shape, like a graph or a tree. Generally when a facility is located at a candidate site, the decision maker incurs in fixed location costs including, for instance, property acquisition, facility construction and long-term management costs. They could depend on the site location (i.e. establishing a new retail outlet in an urban center could be more expensive than in a suburban area), on its size and characteristics.

The service demand of customers have to be satisfy from the facilities. Also the **customers** can be characterized by attributes like distribution and demand (Scaparra and Scutella, 2001). In fact the customers, like the facilities, can be distributed in a continuous space or located at specific points (called properly demand points). Typically at each demand point is associated a value of demand (a weight) that can be the same for all the customers or different; this last case is typical when it represents the demand of an area, but often is made the assumption that all demand will be supply by just one facility. The demand in some case can be also a random variable with some probability distribution (stochastic representation).

The customers **interact** with the facilities in the sense that they are allocated to one of them. In case of more facilities, the customers can choose to which facilities allocate. They may be free to patronize their own facilities (i.e. customers of supermarkets or trade centers) or they may be obliged to follow predetermined criteria. In a lot of FLPs there is a closest assignment constraint (Espejo *et al.*, 2012) that implies that customers are assigned to the most near facility. A quite common hypothesis is that at this interaction is associated a cost, called allocation cost that is variable and is that to be paid to serve the demand; they are strongly related to the proximity of the facility to the demand points. When facilities represent services of general interest (i.e. hospitals, schools, post offices, or private retail outlets), these costs are charged to users; however, the decision maker is interested in minimizing them as they represent a proxy measure of accessibility to the service. When a company needs to locate facilities (i.e. plants, warehouses, distribution systems) within a supply chain, allocation costs correspond to transportation and delivery costs and, therefore, the decision maker is interested in minimizing them as they affect the final profit. In both cases, the efficiency of the service strongly depends on the position of the facility with reference to the distribution of demand points; for this reason, facility location decisions represent a critical element in strategic planning. Sometimes, in addition to the interaction between customer and facilities, we can also have the interaction among facilities themselves, that compete with each other to capture available demand.

The cost of allocation is typically calculated as a distance between demand and facilities. Distance is a numerical description of how far customers and facilities are, at any given moment. Distance may refer to a physical length, a period of time, or it can be estimated on the basis of other criteria. A fundamental aspect is represented by the way of measuring distances, namely the selected **metric**. Distances in real dimension-space are most often derived from Minkowski distances, which are defined as a family of distances with a single parameter p. In particular, the d_{ij}^p distance between a point i with coordinates (a_i, b_i) in a two-dimensional space, and a point j with coordinates (a_j, b_j) is defined as:

$$d_{ij}^p = [|a_i - a_j|^p + |b_i - b_j|^p]^{1/p}$$

if p is equal to 1 we obtain the Manhattan distance, while with p equal to 2 the Euclidean distance, that is the most used.

Location problems can be characterized by the presence of many **constraints** that solutions must satisfy. Typical examples are topological constraints (i.e. minimum and/or maximum distances between facilities, zoning laws), capacity constraints (i.e. maximum demand that each facility can serve), technical and/or technological restrictions, economic and budget constraints

On the basis on how these elements listed are chosen and combined we can define a wide variety of location problems.

1.4 Objectives of Facility Location Problems

Objectives to be considered in decision making can be distinguished by many factors because the presence of facilities can produce different kinds of effects. If these effects are considered positive by customers, facilities are defined "desirable":

it is the case of many services (public or private) such as schools, public offices, shopping centers, metro stations. On the other hand if facilities are source of risks and/or damages, they are considered “undesirable or obnoxious”: examples of this kind are landfills, nuclear reactors, chemical plants, military installations.

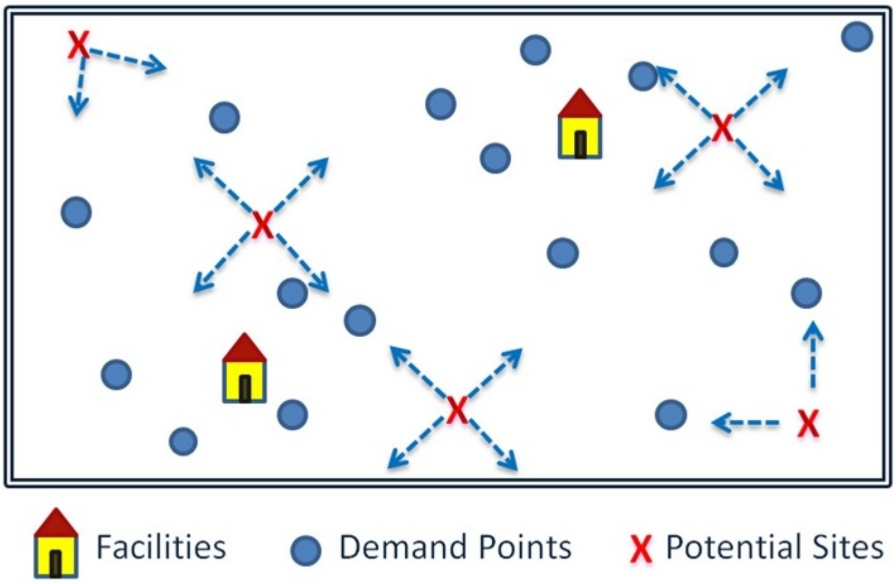

Figure 1.1: Locations Based on Pull Objectives

If facilities are desirable, the decision maker aims at positioning them as close as possible to the customers. In other words, customers attract ("pull") the facility to them (**pull objectives**). In order to obtain this kind of solution, it is possible to use efficiency measures such as the minimization of the distance between facilities and their assigned customers (MiniSum problems). Figure 1.1 shows an example of location based on pull objectives. There is a set of potential sites and a set of demand points in a given location space where the decision maker wants to locate two facilities. Each potential facility is pulled by customers through a sort of “attractive force” (represented by the arrows in the Figure). Consequently, the facilities will be located in the area with the largest number of demand points.

When facilities are considered "undesirable" or "obnoxious", customers aim at avoiding their presence and try to keep such facilities far away from them (**push objectives**). In Figure 1.2 customers push them away through "repulsive forces". Therefore, they will be located in the area with the smallest number of customers. A possible objective to describe this problem is the maximization of the distance between facilities and their assigned customers (MaxiSum problems).

However, the adoption of this measure would contribute to locate the facilities too far (in theory as far as possible) and the resulting solution would be not realistic in terms of efficiency. For this reason, solutions should be selected considering a trade-off between two conflicting objectives: for efficiency reasons facilities should be not too far from the area they serve but, at the same time, they should be far from the customers.

In addition, further location problems have been defined pursuing the objective of the "equality" of distances between demand points and the set of facilities

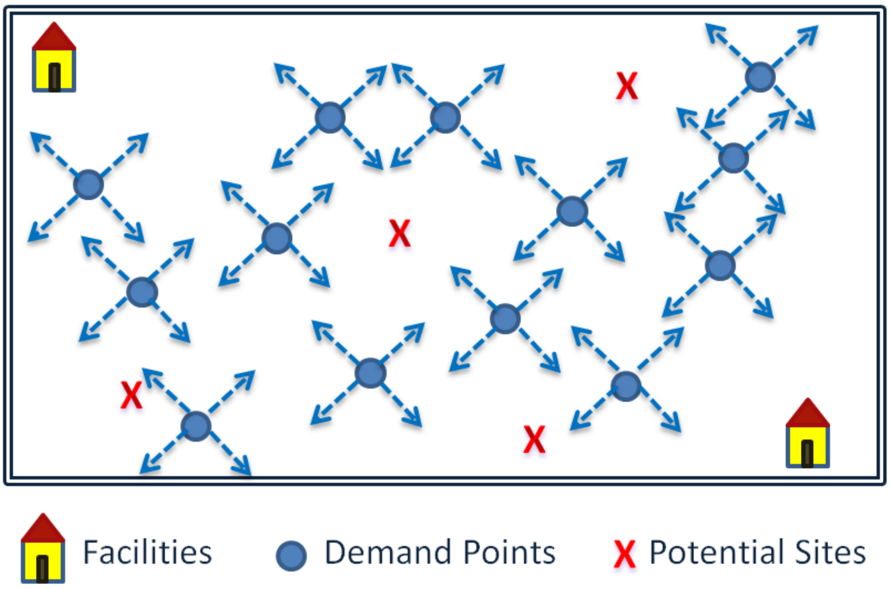

Figure 1.2: Locations Based on Push Objectives

(**balancing objectives**). This type of objectives is used, for example, when the location of public facilities must take into account not only the achievement of efficiency targets but also the more complex concept of "equity". In the Chapter 3 we give a description of these objectives.

1.5 Classifications of Facility Location Problems

A Facility Location Problem could take a variety of forms depending on how are combined the different elements previously introduced; each problems is useful in order to explore the different aspects of the problem and support facility location decisions in different contexts.

In the literature many schemes and taxonomies have been proposed in the last decades. One of them is based on the objectives (Eiselt and Laporte, 1995) as previously illustrated.

According to Hamacker and Nickel [1998] we can find those proposed by Handler and Mirchandani [1979] based on a multiple-position string in which each position is representative of a distinctive characteristic of the problem; in particular he suggested a 4-position scheme for network problems with center-type objective. Brandeau and Chiu [1989] gave a taxonomy to distinguish location problems with respect to three criteria (objective, decision variables, system parameters) without providing a formal classification scheme. Eiselt *et al.* [1993] used a 5-position scheme specialized on competitive location models.

Carrizosa *et al.* [1995] presented a 6-position scheme for classifying planar model where both demand rates and service times are given by a probability distribution. Hamacker and Nickel [1998] designed a 5-position classification scheme, to take into account every class of location problem in a single framework that represent, at the moment, the most detailed attempt to provide a universal classification of Location Problems. According with this proposal we specify the following aspects:

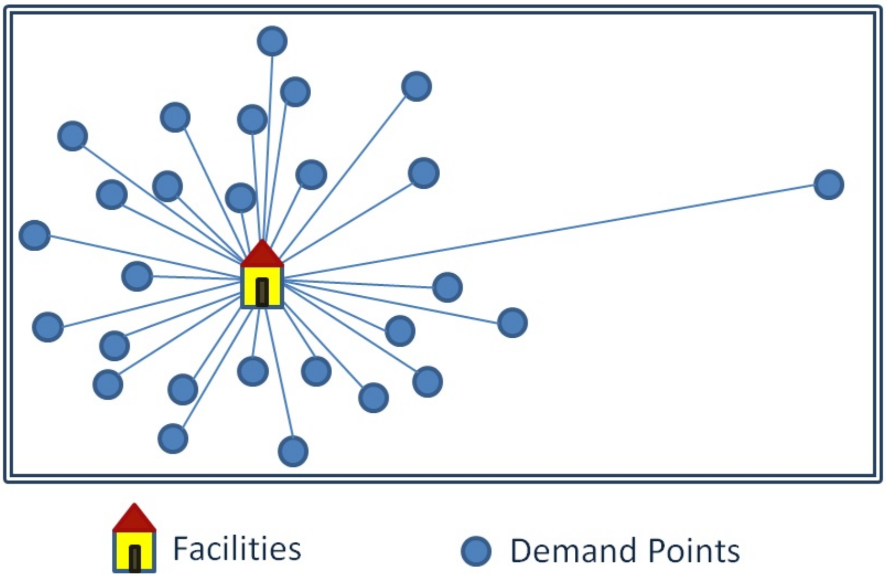

Figure 1.3: MiniSum Models

- Information about the number and type of new facilities;
- Characteristics of the location space (continuous, network and discrete models);
- Additional constraints such as information about the feasible solutions, capacity restrictions, etc;
- Relation between new and existing facilities expressed very often by a distance function or by assigned costs;
- Objective function.

ReVelle *et al.* [2008] instead indicated four possible categories for the models which can be adopted to represent a problem:

- Analytic models, where there are a lot of simplification assumption, like the uniform distribution of the demand or the same cost for fixing facility in every position.
- Continuous models, where facilities can be located in every point of a continuous space, while demand typically is concentrated in points.
- Network models, where facilities and customers are positioned on a network. Demand is typically associated to nodes, while cost to the links connecting demand points.
- Discrete models, where we have a discrete set of demand points, and a discrete set of candidate of potential locations. Such problems are often formulated as integer or mixed integer programming problems.

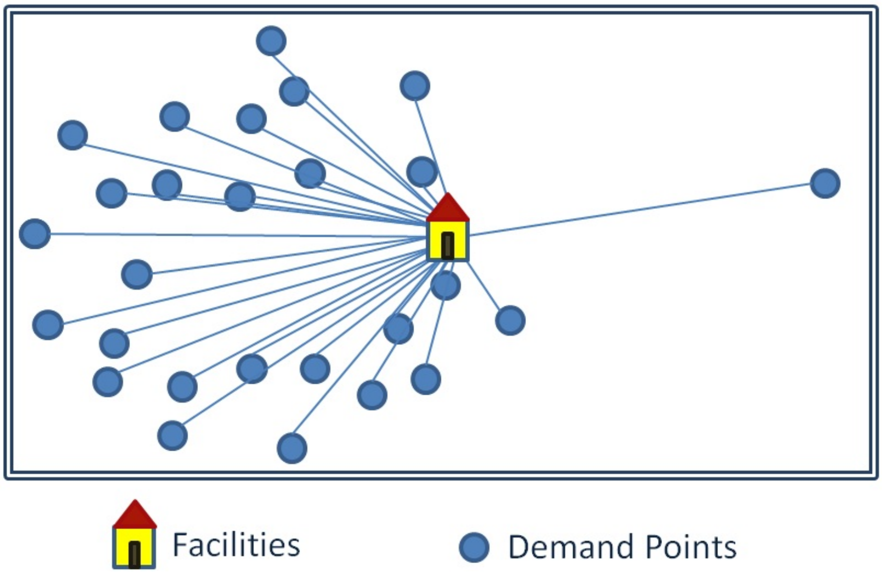

Figure 1.4: MiniMax Models

But, among the existing classification frameworks for which a summary is reported in Table 1.1, the most common one (Eiselt and Marianov, 2011) identifies three classes of problems: MiniSum, MiniMax and covering problems. All these problems deal with the location of desirable facilities but each of them focuses on different criteria. In the case of MiniSum problems the new facilities are located in order to maximize an aggregate indicator of the accessibility of the customers to the service. Typically, they minimize the average travel distance to reach the facilities and locate them as close as possible to the most customers. Due to the particular distribution of demand points showed in Figure 1.3, the optimal solution, which minimizes the average facility-customer distance, penalizes one customer which is at a distance significantly higher from the patronized facility.

In some cases, for example in the organization of an emergency service, this circumstance is not feasible as it is necessary to ensure a given level of accessibility to any customer. In order to take this aspect into account and achieve a major equality in evaluating the distances, other classes of problems can be identified. For example, the MiniMax problems aim at locating facilities so as to minimize the maximum distance between a customer and its assigned facility; in this way they seek to protect the customer in the worst condition. In Figure 1.4, it can be noticed that the facility has been moved towards the customer in the worst condition, thus reducing the maximum distance value. The solution presents a higher average distance value from demand points and, moreover, it results in lower efficiency

In contrast with the first two classes of problems, in which the distance between each customer and his closest facility is explicitly considered, the covering models do not explicitly take them into consideration. The concept of coverage implies that a customer can be adequately served (covered) when a facility is located within a given threshold distance or travel time from it (Figure 1.5).

In other words, a circle with radius centered in the facility location can be identified in order to distinguish covered demand points (within the circle) from uncovered demand points (outside the circle). Given this constraint, these models

Classification	**Reference**	**Applicability**	**Criteria**
4-position	Handler and Mirchandani [1979]	Center Problems	Type of new facilities Type of existing facilities Number of new facilities Type of network
Taxonomy	Brandeau and Chiu [1989]	All Models	Objective Decision variables System parameters
5-position	Eiselt *et al.* [1993]	Competitive Models	Location space Number of player Pricing policy Rules of the game Behavior of customers
6-position	Carrizosa *et al.* [1995]	Planar Models	Distribution existing facilities Distribution new facilities Number of new facilities Shape of existing facilities Metric
5-position	Hamacker and Nickel [1998]	All Models	Type of new facilities Location space Additional constraints Relation new /existing facilities Objective function
Model Category	ReVelle *et al.* [2008]	All Models	Analytic Continuous Network Discrete
Objective Function Category	Eiselt and Marianov [2011]	All Models	MiniSum MiniMax Equity

Table 1.1: Classifications for Location Problems

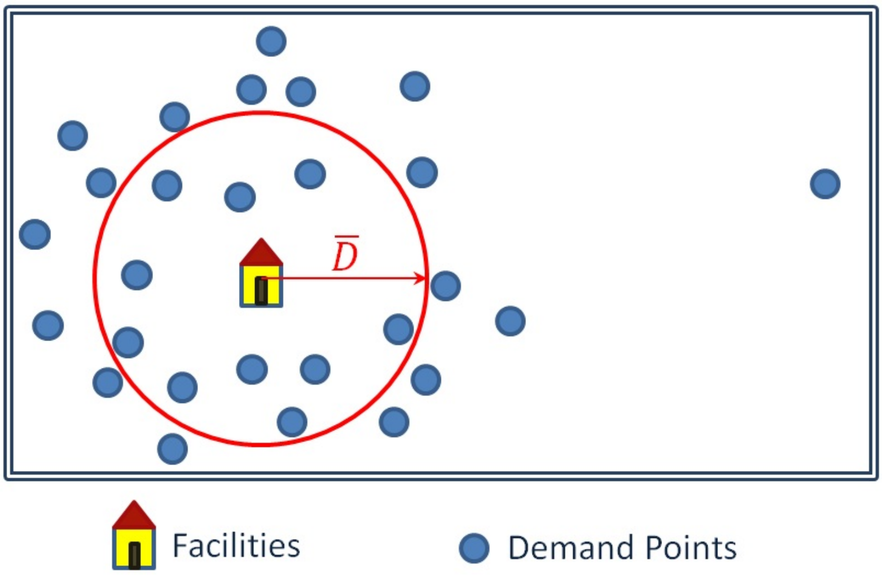

Figure 1.5: Covering Problems

seek to maximize the number of customer covered or to minimize the incurred costs to cover all the demand points In the following paragraph we introduce a discrete location model for each category introduced.

1.6 Basic Discrete Facility Location Problems

In this paragraph we provide the most used mathematical models for FLPs. The models are defined in a discrete location space, namely we have a set of demand points (customers) with an associated weight and a set of facility location that are the possible position for the new facility that we want to locate. Between customer i and potential facility j, we evaluate a distance, in some way among them indicated previously (e.g Euclidean distance).

1.6.1 Median Problems

The p-median problem aims at the minimization of the weighted sum of the distances between p facilities to be opened and a set of demand points. Several versions of the problem have been defined in the literature, and it has been used in many different applications varying from the location of industrial plants and warehouses or public facilities like school (see, for example, ReVelle and Eiselt [2005] for a list of applications) but also as a tool for data mining applications (Ng and Han, 1994). In the model there is the important assumption that we have to locate exactly p facilities, it is a situation considerable very near to the practice as when we know, for example, a budget constraint for the number of facilities to locate. The other important consideration is that the customers will be located to the most near facility.
So we introduce the follow notations, common for the three proposed models:

I set of demand points,

J set of possible locations for the facilities,
d_{ij} distance between customer i and potential facility j,
p total number of facilities to located,
h_i weight associated to each demand point (demand or number of customers).

We define the, allocation decisions, namely which facility j satisfy the demand expressed by a customer i, through the following x-variables:

$$x_{ij} = \begin{cases} 1 & \text{if demand point } i \text{ is allocated to facility } j \\ 0 & \text{otherwise} \end{cases} \qquad \forall i, j \in J,$$

and location decisions, are represented with:

$$y_j = \begin{cases} 1 & \text{if a facility is located at point } j \\ 0 & \text{otherwise} \end{cases} \qquad \forall j \in J.$$

The formulation proposed by Hakimi [1964] and ReVelle and Swaim [1970], retained the "classical" is

$$\min \sum_{i \in I, j \in J} h_i d_{ij} x_{ij}$$

$$\text{s.t.} \sum_{j \in J} x_{ij} = 1 \qquad \forall i \in I, \tag{1.1}$$

$$x_{ij} \leq y_j \qquad \forall i \in I, \forall j \in J, i \neq j, \tag{1.2}$$

$$\sum_{j \in J} y_j = p \tag{1.3}$$

$$x_{ij}, y_j \in \{0, 1\} \qquad \forall i \in I, \forall j \in J. \tag{1.4}$$

Constraints (1.1) ensure that all the demand points are allocated. Constraints (1.2) guarantee that a point receives allocation only if it is a plant. Constraint (1.3) fixes the number of plants to p. Constraints (1.4) states that all variables are binary.

It is classified as NP-hard (Kariv and Hakimi, 1969) so for solving it we find in literature a very huge number of exact method and metaheuristic approaches that look for a good solution (sometimes the optimal solution) when the problem is characterized by a big number of demand points and facilities.

A very comprehensive survey about heuristic approaches for median problems is provided by Mladenovic *et al.* [2007] that resumes both classical heuristic methods than metaheuristic approaches, giving also important indication of the instances used in the literature for testing them. For what concerning the exact method we can indicated, among others, the landmark study of Beasley [1969] and Galvao and Raggi [1969].

There also a lot of extension to the problem like the capacitated version studied for example by Mulvey and Beck [1984] and recently by Lorena and Senne [2004], or the generalized version with more than one type of facility called Multi-Weber problem (Cooper, 1963; Cooper, 1964).

The MiniSum objectives is also used in the other well known model called Simple Plant Location Problem (Erlenkotter, 1978) where the number of facilities to fix is a variable of the problem.

1.6.2 Center Problems

An important class of problem owned to MiniMax problem are the p-center problems. The p-center problem seeks to minimize the maximum distance between any demand and its nearest facility. At difference of the previous class we want that the maximum distance between a demand point and its closest facility is as small as possible instead to minimize the total distance between demand points and facility. Indicating with D the maximum distance between a demand node and the nearest facility, and using the same variables of the previous case the problem can be formalized as follows (Hakimi, 1964):

$$
\begin{aligned}
& \min D \\
\text{s.t.} \quad & \sum_{j \in J} x_{ij} = 1 && \forall i \in I, && (1.5) \\
& x_{ij} \leq y_j && \forall i \in I, \forall j \in J, i \neq j, && (1.6) \\
& \sum_{j \in J} y_j = p && && (1.7) \\
& D \geq \sum_{j \in J} d_{ij} x_{ij} && \forall i \in I, && (1.8) \\
& x_{ij}, y_j \in \{0, 1\} && \forall i \in I, \forall j \in J. && (1.9)
\end{aligned}
$$

The objective function is to minimize the maximum distance between any demand node and its nearest facility. Constraints (1.5) to (1.7) are identical to (1.1) to (1.3) of the p-median problem. Constraint (1.8) defines the maximum distance between any demand node i and the nearest facility j. Finally, constraints (1.9) are binary constraints for the decision variables. If the number of facilities to located is equal to 1 we call the problem Absolute center Problems (Hakimi, 1964). In some cases at each demand point is also associated a weight (Daskin, 1995) and the objective function become:

$$D \geq h_i \sum_{j \in J} d_{ij} x_{ij} \forall i \in I$$

If facility locations are restricted to the nodes of the network, the problem is a vertex center problem (Daskin, 1995). For example Burkard and Dollani [2007] formalized a 1-center problem on a network with positive and negative vertex weights with the objective to minimize a linear combination of the maximum weighted distances of the center to the vertices with positive weights and to the vertices with negative weights. Ozsoy and Pinar [2006] introduced the capacity restrictions on the facilities. Moreover a lot of applications can be described with center models. Biazaran and SeyediNezhad [2009] summarized the possibilities in the location of emergency services, like hospitals and fire stations and computer network services like location of the data files; but also in the distribution system, or for military purpose and public facilities like parks, post boxes and bus stops.

1.6.3 Covering Problems

One of the classical objectives in location modeling is “coverage” which seeks to ensure that each customer is "covered" namely serve by a certain facility if the distance between them is lower than a certain threshold, or required distance. The first model of this type was proposed by Church and ReVelle [1974] and it is called the p-Maximal Covering Location Problem; it consists in locating p facilities that can cover the maximum amount of demand. Using the following z-variables (covering variables):

$$z_i = \begin{cases} 1 & \text{if customer } i \text{ is covered by some facility} \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in I.$$

A possible formulation for the the p-Maximal Covering Location Problem is:

$$\max \sum_{i \in I} h_i z_i$$

$$\sum_{j \in J} y_j \geq z_i \qquad \forall i \in I, \tag{1.10}$$

$$\sum_{j \in J} y_j = p \tag{1.11}$$

$$y_j, z_i \in \{0,1\} \qquad \forall i \in I, \forall j \in J. \tag{1.12}$$

Constraints (1.10) guarantee that a point can covered only by facility opened. Constraints (1.11) fixes the number of plants to p. Constraints (1.12) states that all variables are binary.

Referring to Berman *et al.* [2010] covering problems born in order to locate emergency services that is necessary that assure (cover) the maximum number possible of users; e.g. an ambulance have to be located at a distance such that the travel distance from it to the maximum number of customers will be included in a certain threshold. Other applications have been considered the location of retail facilities or those for the signal transmission. The maximal covering problem is also NP-hard (Megiddo *et al.*, 1983), and for this reason many scholars proposed heuristics (see e.g. Daskin, 1995 and Current *et al.*, 2009) . There are a lot of modifications of this model for example considering also negative weights (Berman *et al.*, 2009) or with capacity constraints for the facilities (Chung *et al.*, 1983)

In the literature there are many others problem concerning the cover concepts as recently surveyed by Berman *et al.* [2010]. The most important is the Set Covering Location Problem, introduced by Hakimi [1964] and formulated as integer programming by Toregas *et al.* [1971] that consists in founding the minimum number or the minimum cost set of facilities such that every demand points is covered by some facilities.

In addition there are several covering problems born from modulating in different way the covering concept. We can have the back-up coverage, in which demand points are required to be covered by more than one open facility. Storbeck [1982] in his model maximized, in addition to the demand covered by the facilities, also the demand covered by at least two facilities. Moreover Daskin *et al.* [1988] formulated a problem in which has to be maximized only the back-up coverage.

Church and Roberts [1984] and successively Karasakal and Karasakal [2004] introduced the concept of gradual or partial coverage where for each facility we have two covering radius: a minimum covering radius and a maximum covering radius; demand points within the minimum radius are considered to be totally covered, while the ones falling in the area between the circles described by the two radii are considered to be partially covered.

1.6.4 Other Location Models

In addition to the above described problems, further ones have received considerable attention of researchers. We will briefly introduce some of the most significant ones.

The above-mentioned problems deal with the location of desirable facilities that customers wish to have as close as possible. However, the scenario changes when dealing with undesirable or obnoxious facilities (undesirable facility location problems) where the customers want to stay as far as possible from the facilities that are retained danger for the customers near to the facility, while they provide some services from the rest of the society and for this reason are necessary (Erkut and Neuman, 1989). For instance, a production plant provide goods and it is important that it stays as close as possible to its client and server, but it can produce polluting so nobody wants it near; sport facilities which should be well accessible may generate quite annoying effects like noise, congestion or even vandalism.

Sometimes also the closeness among the facilities can be considered undesirable, like in the case of franchises or facilities that should be dispersed to the greatest possible distance in order to minimize the damage to others caused by an accident at one of the them. These problems are called dispersion problems and the typical example is the p-dispersion problem that consists of locating p facilities so that a function of the distances among the open facilities is optimized (see Kuby, 1987 and Erkut *et al.*, 1994).

Often the model proposed in the literature are the opposite of those resumed in the previous paragraph. One of this called *maxian* or *anti-median* problem and proposed by Church and Garfinkel [1978], it is identical to the median problem except that the objective function is maximizing instead to minimizing the sum of the weighted distances between facilities and the others demand points.

Berman and Huang [2008] introduced the Minimum Covering Location Problem that consists to locate a fixed number of facilities with the objective to minimize the number of covered customers (where, as stated above, a customer is considered covered if its distance to the closest facility is less than a pre-determined radius) by respecting a constraint on the minimum distance among facilities themselves. Again, they referred to locate facilities that may pose a serious danger to the individuals living nearby so the aim is covered fewer people as possible.

Whereas the usual location models locate facilities based on the wishes and objectives of a single decision maker, competitive location problems consider the presence of multiple decision makers which compete with each other in accordance with coinciding or overlapping objectives (i.e. the maximization of the expected profit). In the definition of models able to effectively describe this situation, crucial aspects are represented by the number of decision makers, the pricing policies,

References	Topic
Owen and Daskin [1998]	General Survey
Scaparra and Scutella [2001]	General Survey
Hale and Moberg [2003]	General Survey
Klose and Drexl [2005]	General Survey
Drezner and Hamacher [2001]	General Survey
ReVelle and Eiselt [2005]	General Survey
Nickel and Puerto [2005]	Book
ReVelle *et al.* [2008]	General Survey
Current *et al.* [2009]	General Survey
Melo *et al.* [2009]	Main Applications
Farahani and Hekmatfar [2009]	Book
Berman *et al.* [2010]	Covering
Eiselt and Marianov [2011]	Book

Table 1.2: Main References for Facility Location Problems

the presence of restrictions to the possible choices, the customers' behaviors in patronizing their facility, the availability of information about the competitors' decisions (for a review see Eiselt *et al.*, 1993).

When location analysis includes aspects related to the impact of various types of uncertainty, it is necessary to develop probabilistic or stochastic location models. Typical sources of uncertainty are future demand, customer-facility travel times, facility breakdowns, future trends for management costs (Snyder, 1987).

1.7 Conclusion

In this chapter we proposed an overview of Location Problems a relevant and very used class of optimization problems. We provided the basic elements of these models, indicating different classifications and also the formulation for some of them.

In the Table 1.2 we report the most important references in the field of location theory considering books and survey more recently in the different topics indicated in the chapter. As indicated some of them concern generalities on location theory, often proposing a review of the main formulation models and the most important resolution procedures presented in the literature. Others, instead focus on single aspect or a single typology of the problem.

This introduction is necessary as stage for the development, in the following of this work, of other Location Models.

Chapter 2

Equity Concept: definitions and measures in a generic context

2.1 Introduction

In this chapter we deal with the general definition of equity and equality. We illustrate the meaning of the concepts and in which contexts were defined and we point out about the differences between equity and equality. We introduce also the issue related to how measuring equality in a generic context. This chapter is necessary for explaining the numerous facets that this concept can assume and why is important taking it into account in a very critical strategic decision like those concerning facility location problems.

2.2 Equity: a Philosophical Principle for Strategic Decision

"Aequitas est quasi superior regula humanorum actuum", this statement of San Tommaso (1225-1274) can be a sort of justification for using equity in any kind of decision.

The Latin word *aequitas* is amenable to the Greek word *epikeia*. Aristotle (384-324 a.C.) was the first that used this word defining *epikeia* as something that is a corrective of the justice. In fact in the *Nicomachean Ethics* he wrote "What creates the problem is that the equitable is just, but not the legally just but a correction of legal justice"; in other words, the equity can serve when the universal nature of the law can not include the totality of the possible cases and so it is necessary adapt it to single cases. Aristotle also described, in the *Rhetoric*, a "material equity", intended as the disposition of people who tend to take less than his portion though he has the law on his side.

Many other philosophers tried to give a definition of the concept. For Hobbes (1588-1679) the equity is a human quality characterizing everyone, or only some individuals, and at the same time it is a law of nature that humans are obliged to follow. In the *Levantian* he said: "For though the action be against the law of nature as being contrary to equity".

Also Kant (1724-1804) in *The Metaphysical Elements of Justice* referred to equity as when "one he is basing his claim on his right rather than ethical duties of others; however, in the case of a right of equity conditions for determining how much and what kind of remedy should be allowed are absent". Kant provided a very interesting example: suppose that a commercial partner company, in which profits are equally shared with other partners, contributes more than other members and then accidentally looses more than the others; if it would ask an additional request it would not have the right because, according to the contract, the income has to be equally distributed . However, for equity reasons he should receive more than the other parts.

Rousseau (1712-1778) affirmed that, although by nature men may be unequal, by force or intelligence, thanks to conventions and legal rights, it is established some form of equity that is moral and legitimate.

The philosophical theories are ideal expressions of the search of equity as principle that leads the action of the individuals. For this reason many authors have tried to apply this general principle in their specific field of studies.

So, if the equity concept born in the jurisdictional-philosophical field, probably the first theoretic formulation and systematization is provided in the sociological field. Homans [1958], Blau [1964] and Adams [1965] defined inequity as the equivalence of the outcome/input ratios; this happens when an individual is in a direct exchange relationship of goods with another individual, or when they are both in a exchange relationship with a third one. Consider, for example, two individuals A and B that mutually transfer resource each other; namely the output of A is the input of B and vice versa. If the ratio between inputs of A to his outcomes (input of B) is not equivalent to that of B then there is inequity.

In a resource distribution system Deutsch [1975] suggested that an allocation norm that guarantees equity can lead goods to those individuals that had been able to effectively use them in the past; who have previously used in a better way the resources would receive an high number of them in a successive redistribution.

More generally, Walzer [1983] highlighted that inequity occurs when who is in the best condition infuses his power, money or influence to get in a better condition.

Perelman [1991] said that equity requires that part of the same category are equally treated. Dasgupta [1993] defined equity as a measure of the relative similarity among individuals or groups when they enjoy material resources, technologies, health, education or socio-political rights. Equity is achieved when each group receives its fair share.

We have provided this very concise review of some definitions proposed for the equity concept in order to underline how this issue is still object of current debates. In this context the concept of equity is often treated in relation to the equality concept. In the next section we focus on this aspect highlighting the differences to be considered.

2.3 Equity vs Equality

Equality and equity concepts are very often confused. The notions are presented in many debates on social and public policy, and also in many others contexts but it seems that there is not a real agreement on what they mean. For example Bronfenbrenner [1973] said that equity is something of subjective while equality is objective. In particular Espinoza [2007] sustained that while equality involves only a quantitative assessment, equity involves both a quantitative assessment and a subjective moral or ethical judgement. Equity assessments are more problematic because people have different idea about the concepts of fairness and justice.

Dalton [1920] was the first to introduce the problem on how defining and evaluating equality. First of all he clarified that equality depends on the context in which the definition is applied. For example in an economic context a situation of perfect equality, indicated as the maximum economic welfare, is reached when the total income is distributed among a given number of persons in equal parts. As a consequence inequality can be defined in relation this condition of equal distribution. However in order to evaluate equality or inequality it is necessary to specify these concepts should be measures.

Schutz [1951] sustained that the equality of income distribution is found when every 'income-receiving unit' receives its proportional share of the total income. Of course either the concept of unit or that of income can have several meanings for different purposes.

For instance Atkinson [1970] specified that income should be distinguished between post-tax and pre-tax.

Bronfenbrenner [1973] described exactly the difference between equity and equality. In fact he said that, despite their phonetic similarity and philological connections, the two notions are quite distinct. The equity is non-mechanical in principle and is largely, if not completely, a subjective matter. For achieving equity the distribution of wealth will be done in accordance with principles of justice. Instead equality is mainly a mechanical or statistical matter. In fact the equality is related to a measure that can be equal, like income or wealth per unit.

Allison [1978] concentrated his efforts on understanding when a distribution of goods or resources can be more or less equal than another one. To this aim he defined and analyzed useful criteria to select and check what are the main and most important measures to be used.

Dworkin [1981] defined two types of equality concepts. The first, called equality of welfare, is obtained when resources are distributed or transferred until no further transfer can realized without causing difference in "success". However the concept of success is subjective as it depends on different preferences, goals and ambitions; so equality can be achieved only all people share the same the success concept. The second, equality of resources, concerns the equal distribution of resources.

For Frankfurt [1987] economic equality is reached when everyone have enough. If everyone had enough, it would be of no moral consequence whether some had more than others.

2.4 Measuring Equality

If equity is a principle that could inspire our decisions, equality can be considered the basis to evaluate equity as it focuses on how resources or goods are equally distributed. In other words equality requires the individuation and the definition of methodology that permits to understand if two distributions are less or more equal. With this aim in the literature a lot of equality measures has been introduced.

In the following we introduce some of the most popular equality measures. We assume that a given amount of resources S (goods or income) has to be distribute among a set of n members (or groups). We indicate with s_i the share assigned to each member and with $\bar{s} = \frac{S}{n}$ the average value. We assure that each member is characterized by the value of an attribute a_i whose average value is $\bar{a} = \sum_i \frac{a_i}{n}$. The attribute represents a factor that could involve the distribution of resources. For instance if the distribution concerns group of individuals this attribute can represent the dimension of each group.

In order to illustrate the list of measures we refer to the classification proposed by Marsh and Schilling [1994] that collect a very big number of them defining also a framework for the calssification. In particular they analyzed three categories. The first called, reference distribution, is related to which element the shares of the individuals are compared. Typically the reference is the mean effect but it is possible to compare also the with an attribute of some type or with the effect on another group. The second is the metric, namely how is constructed this comparison like a difference of sum or the maximum difference. The third is the scaling factor. Often the measures are divided for the mean in order to compare distribution that are different in the size.

We indicate first all measures that are not normalized and after the ones normalized from which the values are included between 0 and 1. After we show the measures that compare the share of the groups with some attributes.

2.4.1 Not Normalized Measures

Worst Condition

Rawls [1971] defined that equality improve when is improved the condition of who is worst-off. The measure, called Worst Condition (CW) is:

$$CW = \min_i s_i$$

Range

Brill *et al.* [1976] sustained that we can verify if there is equality analyzing the differences between the best and the worst shares in a set of individuals. If this difference is small means that the difference in condition of individuals are similar. The measure, called Range (RG) is:

$$RG = \max_i s_i - \min_i s_i$$

There is also an alternative way to calculate the Range as:

$$ARG = \max_{i,j} |s_i - s_j|$$

Mean Absolute Deviation

This index is used in order to capture the dispersion for the values in a distribution; if the value is small means that there is small distribution of the values and, thinking in equality way, individuals have a share of goods as much similar as possible. The Mean Absolute Deviation (MAD) is a very simple way to capture the dispersion of variate values; it represents the averaged sum of the absolute deviation from the mean of the distribution. It can be indicated as:

$$MAD = \sum_{i=1}^{n} |s_i - \bar{s}|$$

Maximum Deviation

It is also possible to evaluate the Maximum Deviation (MD) from the average value to the others values indicated as:

$$MD = \max_i |s_i - \bar{s}|$$

Variance

The Variance (VAR) is frequently used as indicator of equality in different contexts, thanks to the possibility to take into account the variation of a distribution (i.e. Gastwirth, 1973). It presents two peculiarities; the first is that differences between variate values and the mean value are squared and so the differences that are relatively large are accentuated. The second is mean-dependent, in fact two distributions could display the same relative variation but the distribution with the smaller mean would have a lower variance. The expression is:

$$VAR = \sum_{i=1}^{n} (s_i - \bar{s})^2$$

Variance of logs

Exist also the variance of the logs (LOGVAR) that combine logarithms with variance; the effects of using logs is that tends to focus on which individual have lower level of shares. It is given by:

$$LOGVAR = \frac{1}{n} \sum_{i} (\log s_i - \log \bar{s})^2$$

Sum of absolute deviation

Often as equality measure is used the sum of absolute deviation (AD), that take into account all the differences of all possible pairs of variate values (Keeney, 1980). The expression is:

$$AD = \sum_{i=1}^{n} \sum_{j=1}^{n} |s_i - s_j|$$

In the Table 2.1 we report a resume of all not normalized equality measures introduced.

2.4.2 Normalized Measures

Coefficient of Variation

Code	**Measure**	**Formulation**	**References**
CW	Worst Condition	$\min_i s_i$	Rawls [1971]
RG	Range	$\max_i s_i - \min_i s_i$	Brill *et al.* [1976]
ARG	Alternative Range	$\max_{i,j} \lvert s_i - s_j \rvert$	Brill *et al.* [1976]
MAD	Mean Absolute Deviation	$\sum_{i=1}^{n} \lvert s_i - \bar{s} \rvert$	Marsh and Schilling [1994]
MD	Maximum Absolute Deviation	$\max_i \lvert s_i - \bar{s} \rvert$	Marsh and Schilling [1994]
VAR	Variance	$\sum_{i=1}^{n} (s_i - \bar{s})^2$	Gastwirth [1971]
LOG-VAR	Variance of logs	$1/n \sum_i (\log s_i - \log \bar{s})^2$	Theil [1967]
AD	Absolute Difference	$\sum_{i=1}^{n} \sum_{j=1}^{n} \lvert s_i - s_j \rvert$	Keeney [1980]

Table 2.1: Not Normalized Equality Measures Resume

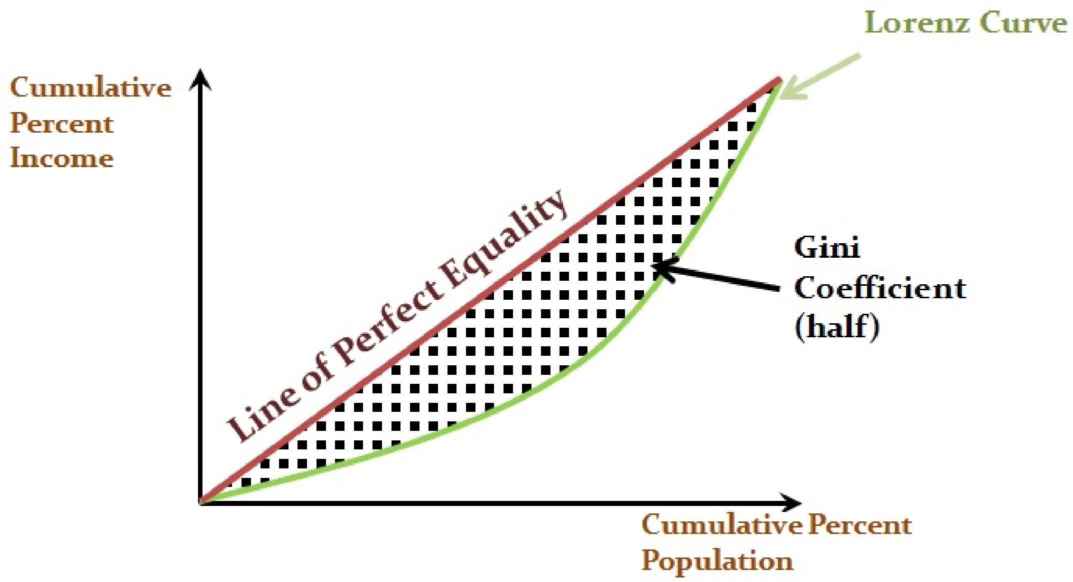

Figure 2.1: Gini Coefficient and Lorenz Curve

The mean-dependent of the Variance, can be avoided by using, instead of the variance the coefficient of variation (VC),that normalize the variance, dividing it for the average value; so it becomes:

$$VC = \frac{\sum_{i=1}^{n}(s_i - \bar{s})^2}{\bar{s}}$$

Gini Coefficient and Lorenz Curve

In order to measure equitable (or inequitable) distribution of a good across a population, a graphical representation was proposed by Lorenz (Lorenz curve). The representation is done, traditionally, on a $X - Y$ axis where the abscissa measures the cumulative percentage of the population (member of the group) while the ordinate the cumulative percentage of the good, such as income or wealth. The Lorenz curve has extremes at points (0,0) and (1,1) because the 0 % of the population holds 0% of the income, and 100 % of the population hold 100 % of the income; the most equitable distribution where a certain percentage k of the population has k percentage of the good, is representable with the straight 45 degree line that connects the two extremes. When a k percentage of the population holds less that k percentage of the good the Lorenz curve drops below the straight equality line proportionally to the disparity. The area, between the Lorenz curve and the straight 45 degree line is a measure of inequality in the distribution (see Figure 2.1).

The Gini Index was developed by Gini [1912] and it is strictly linked to the Lorenz Curve because it measures the ratio of the area between the Lorenz Curve and the line of perfect equality to the entire area below the Lorenz curve and it is included between 0 and 1; in Figure 2.1 we can visualize it as the half value of the area depicted in black. If it is equal to 0, then there is no inequality, and all members have the same share of the good. That is, the Lorenz curve is the straight equality line and the area is zero. If it is equal to 1, one member has all the good, with the area of inequality being equal to the entire area under the straight equality line. Therefore, smaller is the Gini coefficient more equitable is the distribution.

An expression of the Gini coefficient (GC) , using the formalism previously introduced can be:

$$GC = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}|s_i - s_j|}{2n^2\bar{s}}$$

Theil's Index

The Theil's Index (TI) (Theil, 1967) has origin in information theory and involves a logarithmic transformation of certain variable fractions. The weakness point is the difficult to compute. A possible expression is:

$$TI = \frac{\sum_i |s_i \log s_i - \bar{s}\log\bar{s}|}{\bar{s}}$$

The Schutz's Index

The Schutz's Index (SI) proposed by Schutz [1951], is the normalized version of the Mean Absolute Deviation and is given by:

$$SI = \frac{1}{2n\bar{s}}\sum_i |s_i - \bar{s}|$$

The Atkinson's Index

The Atkinson's Index (ATK) derived from the theory enveloped by Atkinson [1970]. A possible expression is:

$$ATK = 1 - n^{\frac{1}{\delta-1}}[\sum(\frac{s_i}{\bar{s}})^{\frac{1}{\delta-1}}]$$

with δ a parameter included between 0 and 1.

In the Table 2.2 we report all the normalized measures introduced indicating the references in which they are defined.

2.4.3 Measures with Attribute

The Hoover's Index

The Hoover's Index (HI) (Hoover, 1941), is the half sum of the absolute differences between the ratio of each share with the average value and the ratio between the corresponding attribute and the average value. It can be expressed by:

$$HI = \frac{1}{n}\sum_i |\frac{s_i}{\bar{s}} - \frac{a_i}{\bar{a}}|$$

The Coulter's Index

Coulter [1981] introduced this equality measure in a police service settings, concerning the difference between the amount of service delivered under the requirement of a specified equality standard. The measure takes the square root of the sum of squared deviations between the fraction of service delivered to an area and that area's proportion of the total population. Coulter notes that the squaring levies, proportionately, a greater penalty on greater inequality than on

Code	Measure	Formulation	References
VC	Coefficient of Variation	$\frac{\sum_{i=1}^{n}(s_i-\bar{s})^2}{\bar{s}}$	Gastwirth [1971]
GC	Gini Coefficient	$\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\lvert s_i-s_j\rvert}{2n^2\bar{s}}$	Gini [1912]
TI	Theil's Index	$\frac{\sum_i \lvert s_i \log s_i - \bar{s}\log\bar{s}\rvert}{\bar{s}}$	Theil [1967]
SI	Schutz's Index	$\frac{1}{2n\bar{s}}\sum_i \lvert s_i-\bar{s}\rvert$	Schutz [1951]
ATK	Atkinson's Index	$1-n^{\frac{1}{\delta-1}}[\sum(\frac{s_i}{\bar{s}})^{\frac{1}{\delta-1}}]$	Atkinson [1970]

Table 2.2: Normalized Equality Measures Resume

lesser inequality. This mathematically reflects his philosophy that a small amount of inequality is expected and is politically stainable while a substantial amount is intolerable. The formalization, considering the amount of service as attribute, is:

$$CO = \sqrt{\frac{1}{n}\sum_i(\frac{s_i}{\bar{s}} - \frac{a_i}{\bar{a}})^2}$$

Variance with Attribute

Mayhew and Leonardi [1982] minimized the difference between the ratio of predicted service availability, the share, to expected needs, the attributes, and the ratio of system wide resources to system wide needs, respectively the average value of the shares and the average value of the attributres; it can be expressed as:

$$VARA = \sum_i(\frac{s_i}{a_i} - \frac{\bar{s}}{\bar{a}})^2$$

The Adam's Index

The measure derives from his Equity theory (Adams, 1965), and compares the relative outcome to some parameters such as input considered as attribute. The equality holds if the ratio between these two quantities is the same for all the participants. AD can be formulated as:

$$ADI = \sum_i\sum_h\lvert\frac{s_i}{a_i} - \frac{s_h}{a_h}\rvert$$

The modified Adam's Index

Walster *et al.* [1973] proposed a variation of the Adams' formula for circumstance in which negative values are possible. We called this MADI and is expressed as:

$$MADI = \sum_i\lvert\frac{s_i - s_i}{a_i}\rvert$$

Code	**Measure**	**Formulation**	**References**
HI	Hoover's Index	$\frac{1}{n}\sum_i \lvert\frac{s_i}{\bar{s}} - \frac{a_i}{\bar{a}}\rvert$	Hoover [1941]
CO	Coulter's Index	$\sqrt{\frac{1}{n}\sum_i (\frac{s_i}{\bar{s}} - \frac{a_i}{\bar{a}})^2}$	Coulter [1981]
VARA	Variance with attribute	$\sum_i (\frac{s_i}{a_i} - \frac{\bar{s}}{\bar{a}})^2$	Mayhew and Leonardi [1982]
ADI	Adam's Index	$\sum_i \sum_h \lvert\frac{s_i}{a_i} - \frac{s_h}{a_h}\rvert$	Adams [1965]
MADI	Modified Adam's Index	$\sum_i \lvert\frac{s_i - a_i}{a_i}\rvert$	Walster *et al.* [1973]
MADA	Mean Absolute Deviation with attributes	$\sum_i \lvert s_i - a_i\rvert$	Heiner *et al.* [1981]
SCSI	Schutz's Sociospatial Index	$\sum_i (\frac{s_i a_i}{\sum_k s_k a_k} - \frac{a_i}{n\bar{a}})$	Schutz [1951]

Table 2.3: Equality Measures with Attribute Resume

Mean Absolute Deviation with attribute

Heiner *et al.* [1981] formulated this measure in their examination of the allocation of services to the mentally retarded. The individuals in the study are differentiated by the level of disability. They compared the differences between the goal to be met by each group and a level of achievement resulting (our shares), with the amount of service allocated (our attributes). Looking for minimum values of the measure means that we obtain most equality distribution of the resource. The Mean Absolute Deviation with Attribute is given by:

$$MADA = \sum_i \lvert s_i - a_i \rvert$$

Schutz's Sociospatial Index

Schutz [1951] proposed in his studies also another measure that take into account also attributes that characterized the distribution. It is given by:

$$SCSI = \sum_i \left(\frac{s_i a_i}{\sum_k s_k a_k} - \frac{a_i}{n\bar{a}}\right)$$

In the Table 2.3 we report the measures with attribute indicating the references in which they are defined.

	Element 1	Element 2	Element 3	Element 4
Distribution A	10	20	30	40
Distribution B	5	25	25	45
Distribution C	5	10	30	55
Attribute	50	65	35	100

Table 2.4: Shares of Distributions

2.5 Example of Evaluation for Equality Measures

In the following we provide a simple example of evaluation of all equality measures introduced. We suppose the presence of 4 elements and a total number of available goods equal to 100. We have 3 different distributions of goods on the 4 elements as indicated in Table 2.4.

We suppose that each element is characterized by an attribute whose values are indicate also in Table 2.4.

There is no equal distribution, i.e. distribution with equal values for each element. For this reason we aim at evaluating degree of inequality calculated according various described measures. We determine for all these distributions the absolute values of the equality measures, and the related values in comparison with the minimum value of each one. In Tables 2.5 and 2.6 we provide the values obtained.

We can note that the distribution A is the most equitable for all the measures except for the Atkinson's Index where C is indicated with more equitable; it is important to highlight that more small is value of the measures, more equitable is retained the distribution. It is possible, indeed verify that the distribution B is more equitable that distribution C, again this not happens for the Atkinson's and Theil's Index. The normalized values indicated are very different; in particular for LOGVAR and MD there is a very high variability, while is low with WC. Instead, for the others measure, is more or less the same.

The evaluation of the measures with attribute show a different behavior. In fact for two measures the more equitable is the C distribution, and in one case the B, while in the others distribution A. So we can not identify a more equitable distribution, and this appears also from the normalized values that are all high. The results are reported in Table 2.7.

2.6 Choosing a Measure

The example above illustrated shows the possibility of representing the degree of inequality of a given distribution using different measures with different intensity.

	Absolute Values			**Normalized Values**		
Measure	**A**	**B**	**C**	**A**	**B**	**C**
Worst Condition	40,00	45,00	55,00	1 ,00	0,89	0,73
Range	30,00	40,00	50,00	1,00	0,75	0,60
Mean Absolute Deviation	40,00	40,00	70,00	1,00	1,00	0,57
Maximum Ab. Deviation	15,00	20,00	30,00	1,00	0,75	0,50
Variance	500,00	800,00	1550,00	1,00	0,62	0,32
Variance of logs	0,053	0,14	0,19	1,00	0,38	0,26
Absolute Difference	200,00	240,00	340,00	1,00	0,83	0,59

Table 2.5: Evaluation of Not Normalized Equality Measures

	Absolute Values			**Normalized Values**		
Measure	**A**	**B**	**C**	**A**	**B**	**C**
Coefficient of Variation	20,00	32,00	62,00	1,00	0,63	0,32
Gini Coefficient	0,25	0,30	0,43	1,00	0,83	0,58
Theil's Index	542,95	579,68	495,22	0,91	0,85	1,00
Schutz's Index	0,20	0,20	0,35	1,00	1,00	0,57
Atkinson's Index	0,62	0,80	0,35	0,58	0,44	1,00

Table 2.6: Evaluation of Normalized Equality Measures

	Absolute Values			Normalized Values		
Measure	**A**	**B**	**C**	**A**	**B**	**C**
Hoover's Index	0,32	0,32	0,60	1,00	1,00	0,53
Coulter's Index	0,40	0,39	0,67	0,98	1,00	0,58
Variance with Attribute	0,26	0,19	0,41	0,73	1,00	0,46
Adam's Index	4,13	3,82	2,90	0,70	0,76	1,00
Modified Adam's Index	2,06	2,35	3,45	1,00	0,85	0,58
Mean Ab. Deviation Attribute	150	150	150	1,00	1,00	1,00
Schutz's Sociospatial Index	0,39	0,44	0,77	1,00	0,89	0,51

Table 2.7: Evaluation of Equality Measures with Attribute

For this reason we should ask how it is possible to choose a measure in order to appropriately point ut the correct degree of inequality. It should be underline that the choice of a specific measure is related to the meaning of inequality namely how we intend inequality also from a theoretical point of view. Allison [1978] proposed to take the decision for choosing among the numerous measures of inequality, based their choice on convenience, familiarity, or on vague, methodological grounds, and in practice many authors adopt this approach.

Moreover some authors have defined a set of criteria that a measure have to satisfy in order to be a good measure. Champernowne [1974] has collected seven criteria for selecting good index of inequality:

1. Simpleness to evaluate or estimate in a easy and understandable form.

2. Impartiality among subjects involved in the measure, namely if we are evaluating the equality of income distribution among a group of individuals the measure depends just by the share possessed by the single member and not by the ranking of the individuals nor by other factors like race, wealth, power, political advantage.

3. Invariance with respect to the number of elements among the measure is evaluated. If we evaluate for the distribution of the income the measure with a number of person larger or smaller if the income are distribute equally the value of the measure have to be the same.

4. Scale Invariance with respect to uniform increase (or decrease) of the size of shares for each element of the group. More precisely the index should be unaffected if each income is altered by the same proportion.

5. The Pigou-Dalton criterion, also called principle of transfer, requires that if a distribution is modified by altering two incomes only so as to leave their total unaltered, then the index concerned must be increased, unchanged or decreased, according as the absolute difference between the two incomes is increased, unchanged or decreased. There is also a sensitivity on transfer that reveals if the "quantity" of how the measure takes into account this change.

6. Lower and upper bound on the value of the measure are desirable and especially the measure shall we included between 0 to 1; in particular the index must take the value 0 for all distributions in which every share is equal and 1 when the distribution is totally in favor of a single individual.

7. Suitability as a specialist measure of one particular aspect of inequality in distinction from the others.

2.7 Conclusion

In this chapter we gave an overview of what is intended with equity and equality. After the introduction of both concepts through the analysis of the definitions proposed in the literature we listed the most famous measure defined and we calculated for a very small example of distribution. The criteria with which we choose the measures can be different and we have listed some of them introduced in the literature. All these aspects are present and will be faced also in the equity analysis in the location theory as we will see in the next chapter.

Chapter 3

Equality in Location Problems

3.1 Introduction

In this chapter we analyze how the equity principle can be used in locations problems. Starting from the literature first of all we describe in which context it is useful considering equality in choosing locations and how equity is generally intended in this field. Through a literature review of the papers concerning this topic we individuate some gaps that will be analyzed in the following chapters.

3.2 Considering Equality in Locating Facility

When we locate a facility, as indicated in Chapter 1, we choose its position on the basis of an objective. In addition to pull and push objectives we can adopt balancing objectives, i.e. objectives that balance the effects of the facility on the users in order to guarantee equality in the distribution of costs and/or benefits among users.

These types of objectives are particularly suitable to describe problems in the public sector, where users should have considered in the same way (Marianov and Serra, 2002). Examples can be the location of hospitals, schools or government agency (desirable facilities). In other cases (undesirable facilities), despite the users want to stay as far as possible from the facility, the balancing objectives aim at distributing damages among users in an equal way.

More precisely, the level of equality in decision of choosing a new site for a facility can be evaluated comparing the effects, both positive and negative, resulting for the groups that benefit or suffer it (Marsh and Schilling, 1994). If each group receives an equal share then equality is guaranteed.

The groups can be defined according to different dimensions. The spatial dimension, also determined by jurisdictional boundaries, divide the users in groups on the basis of the spatial distribution. So people in a state or a in a province can be considered a group; but also those that are included in an area of a certain number of square kilometers. Following the demographic dimension group can be constituted by people that are characterized by the same level of some characteristics like the age or their income. An interesting dimension is the temporal one in which groups are constituted by users that share a same characteristic in the time. The dimension

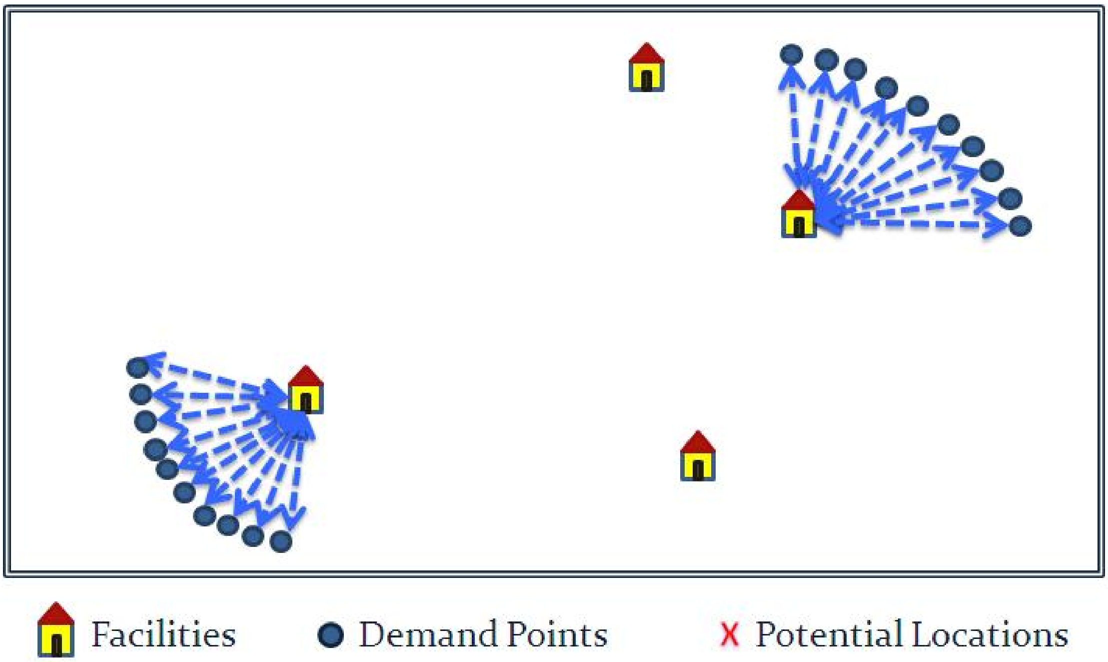

Figure 3.1: Equality Location Problems: Ideal Situation

has to be choose on the basis of the specific application in the proximity of the facility. For example in the case of garbage dump sites groups can be identified by residents in areas (spatial dimension). If the new site is a school users can be identified thanks to their age (demographic dimension). In our discussion we use demand points, that can indicate a single user or a group depending on the analyzed case.

The level of the effect is generally dependent from the distance between the facility and the elements of the group. Assuming that effects on the users are proportional to distances from facilities, guaranteing equality means obtaining the maximum possible level of equality of the distances between facilities and users. In a ideal situation all the users have to be positioned at the same distance from facilities, as illustrated in Figure 3.1. However, this happens rarely. So the purpose is looking for solutions in which distances are in some way similar. To this aim equality measures, derived from other contexts, can be adapted in order to measure a degree of equality in the distribution of distances.

Also in the location theory there is confusion between terms equity and equality. According to Mulligan [1991] and Eiselt and Laporte [1995] the evaluation of equality distribution is one of the concept of equity; so equality indicates the way in which we evaluate if the configuration of the facilities guarantee an equal distributions of the effects deriving from the facility. For this reason we use equality measures, and not equity measures as instead used by others authors (i.e. Marsh and Schilling, 1994 and Mesa *et al.*, 2003).

3.3 Equality Measures for Location Problems

Many proposals have been provided in which equality measures, as defined in other contexts, have been adapted to the location context.

In three different surveys (Erkut, 1993; Marsh and Schilling, 1994; Eiselt and Laporte, 1995) are shown a list of equality measures used in location problems. In Table 3.1 we indicate the list of measures cited by each survey. Each symbol $\odot$ marks that measure have been indicated in that paper.
For formulating the measures the following notations are used:

$I = \{1, \ldots, n\}$ the set of the n demand points;

d_i the distance between the demand point i and the facility;

$\bar{d}$ the average distance between the demand points and the facility defined as $\sum_i \frac{d_i}{n}$;

As we can see some measures take into account the evaluation of the spread of deviation, more precisely Center (CEN) considers the user more disadvantaged, namely at the biggest distance, while Range (RG) reports the differences among the user at the smallest distance and the one at the largest one. Others evaluate the deviation from a central point; in particular while Mean Absolute Deviation (MAD) and Maximum Absolute Deviation (MD) compare respectively the sum and the maximum deviation, the Variance (VAR) square the sum of the deviations of all distances from the average distance. Some of them are used for minimizing differences in the distances between all pairs of facilities; while Absolute Differences sum all the differences, the others combine maximization and sum of differences (Sum Maximum Absolute Differences, Maximum Maximum Absolute Differences, Maximum Sum Absolute Differences).

Moreover others are normalized and so their values will be included between 0 and 1. While Theil's Index, and Variance of logs include the evaluation of logarithm, Atkinson Index is a very different measure also involving a new parameter δ that can be chosen between 0 and 1. Instead Gini Coefficient, Schutz's Index and Coefficient of Variation are respectively the normalized version (divided for the average of the distances) of Absolute Difference, Mean Absolute Deviation and Variance.

In the Table 3.2, instead the distances are related to some others attribute that will depend from the type of facility that we are locating. We used the following additional notations:

a_i the attribute associated to each demand point i;

$\bar{a}$ the average value of attribute $\sum_i \frac{a_i}{n}$.

For better understanding the meaning of equality measure for location problems we explain how works one of the most popular measures, the Gini Coefficient, that is defined and obtained as explained in the previous chapter. In the Figure 3.2 we report a graphical representation of the value of the measure on a $X - Y$ axis. On the X axis there is the cumulative percent of users while on the Y axis there is the cumulative percent of distances. So the bisector line indicates that the same percent of users have the same portion of distances, namely all the users are at the same distance from the facility. In this case the Gini Coefficient is equal to zero.

Code	**Measure**	**Formulation**	Marsh and Schilling [1994]	Eiselt and Laporte [1995]	Erkut [1993]
CEN	Center	$\max_i d_i$	⊙	⊙	⊙
RG	Range	$\max_{i,j} \|d_i - d_j\|$	⊙	⊙	
ARG	Alternative Range	$\max_i d_i - \min_j d_j$	⊙		
MAD	Mean Absolute Deviation	$\sum_{i=1}^{n} \|d_i - \bar{d}\|$	⊙	⊙	⊙
MD	Maximum Absolute Deviation	$\max_{i=1} \|d_i - \bar{d}\|$	⊙	⊙	⊙
VAR	Variance	$\sum_{i=1}^{n} (d_i - \bar{d})^2$	⊙	⊙	⊙
AD	Absolute Differences	$\sum_{i=1}^{n} \sum_{j=1}^{n} \|d_i - d_j\|$	⊙	⊙	⊙
SMAD	Sum Maximum Absolute Differences	$\sum_{i=1}^{n} \max_{j \in I} \|d_i - d_j\|$			⊙
MMAD	Maximum Maximum Absolute Differences	$\max_{i \in I} \max_{j \in I} \|d_i - d_j\|$			⊙
MSAD	Maximum Sum Absolute Differences	$\max_{i \in I} \sum_{j=1}^{n} \|d_i - d_j\|$			⊙
LOG-VAR	Variance of logs	$1/n \sum_i (\log d_i - \log \bar{d})^2$	⊙	⊙	
GC	Gini Coefficient	$\frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \|d_i - d_j\|}{2n^2 \bar{d}}$	⊙	⊙	⊙
TI	Theil's Index	$\frac{\sum_i \|d_i \log d_i - \bar{d} \log \bar{d}\|}{\bar{d}}$	⊙	⊙	
SI	Schutz's Index	$\frac{1}{2n\bar{d}} \sum_i \|d_i - \bar{d}\|$	⊙	⊙	
VC	Coefficient of Variation	$\frac{\sum_{i=1}^{n} (d_i - \bar{d})^2}{\bar{d}}$	⊙	⊙	
ATK	Atkinson's Index	$1 - n^{\frac{1}{\delta-1}} [\sum (\frac{d_i}{\bar{d}})^{\frac{1}{\delta-1}}]$	⊙	⊙	

Table 3.1: Equality Measures for Location Problems

Code	**Measure**	**Formulation**	Marsh and Schilling [1994]	Eiselt and Laporte [1995]	Erkut [1993]
HI	Hoover's Index	$\frac{1}{n}\sum_i \lvert \frac{d_i}{\bar{d}} - \frac{a_i}{\bar{a}} \rvert$	⊙	⊙	
CO	Coulter's Index	$\sqrt{\frac{1}{n}\sum_i (\frac{d_i}{\bar{d}} - \frac{a_i}{\bar{a}})^2}$	⊙	⊙	
VARA	Variance with attribute	$\sum_i (\frac{d_i}{a_i} - \frac{\bar{d}}{\bar{a}})^2$	⊙	⊙	
ADI	Adam's Index	$\sum_i \sum_h \lvert \frac{d_i}{a_i} - \frac{d_h}{a_h} \rvert$	⊙	⊙	
MADI	Modified Adam's Index	$\sum_i \lvert \frac{d_i - a_i}{a_i} \rvert$	⊙	⊙	
MADA	Mean Absolute Deviation with attributes	$\sum_i \lvert d_i - a_i \rvert$	⊙	⊙	

Table 3.2: Equality Measures with Attribute for Location Problems

When the users are not at the same distance, reporting for each cumulative portions of users the correspondent value of cumulative proportion of distances, we obtain points below the bisector line; the line that connects these points, called Lorenz Curve, is below the bisector to an extent that will be grater when the distances are more different. So, the area between the line of perfect equality and the Lorenz's curve is representative of half value of the Gini coefficient.

3.4 A literature Review: Statistics

To the aim to verify the interest in using the equality concept in the location context we performed an extensive State-of-the-Art survey. We used the web-based tool Google Scholar (including the most widespread academic search engines), for searching papers in all international referred journals in the time interval from when we retain the beginning of using equity in location problems, 70's years, until now days. We look for the words equity, equality, equitable and balancing objectives in the title, keywords and abstract of the papers for which the main topic is the choice of a new location for a facility. In this way we selected only the papers in which the authors want to pursuit some form of equality. At the same time we excluded papers in which the equality is not a purpose of the problem but it is obtained just like a consequence of using others types of objectives.

This way 36 papers have been retrieved most of them published in the last decade as highlighted in the Figure 3.3.

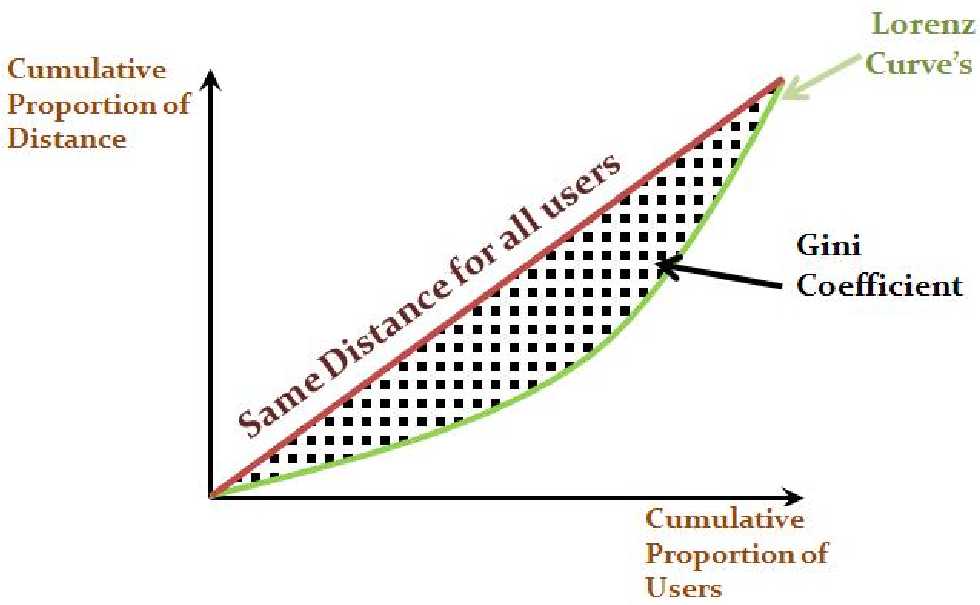

Figure 3.2: Gini Coefficient for Location Problems

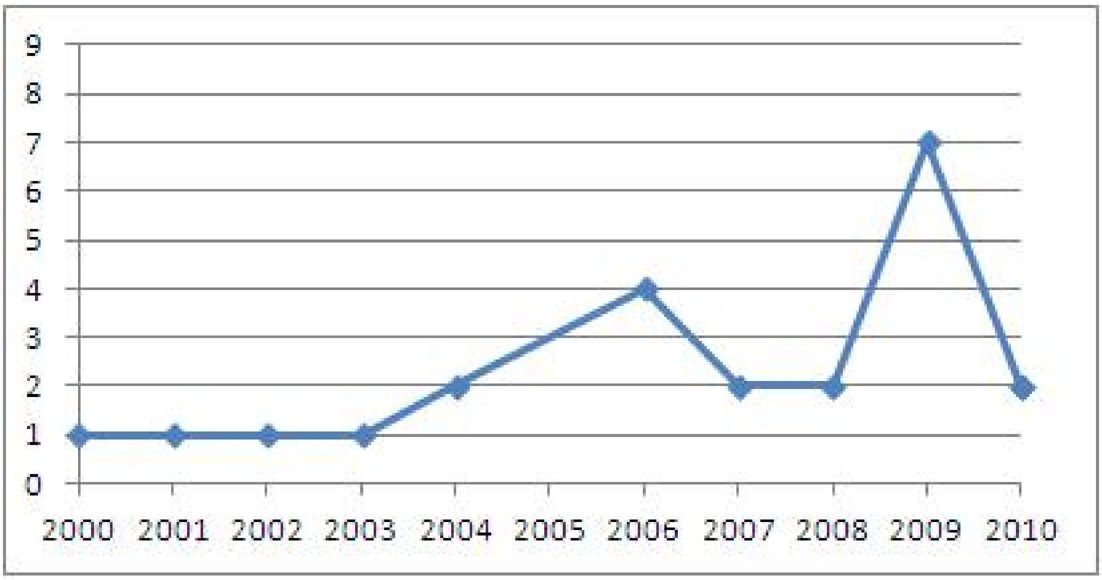

Figure 3.3: Number of Papers in the Last Decade

Journal	**Frequency**
European Journal of Operational Research	9
Annals of Operations Research	4
Location Science	4
Geographical Analysis	3
Computers & Operations Research	3
Transportation Science	2
Discrete Applied Mathematics	2

Table 3.3: Journals and Frequencies

Table 3.3 reports journals which hosted at least two papers. They account for 26 total papers out of 36 (72.22% of the total number of papers) with *European Journal of Operational Research* as top contributor.

Table 3.4 reports keywords retrieved at least two times in the surveyed papers, and the number of occurrences for each keyword. It emerges that the words *Location*, *Equity*, *Facility Location* are the most cited. There are also words like *Equality*, *Inequality Measure* that indicate how the objective or the principle of equity is formalized.

Keywords like *multiple criteria*, *multiobjective* and *efficiency* also are cited, putting in evidence that very often the equity criteria are used together with an efficiency criteria in a multiobjective context.

It is interesting to note that also the word *public sector* is among the most used as problems with equity considerations are often used to solve problems in the public sector.

A further analysis was performed about the used approaches to solve the problems (Figure 3.4). Proposals concern the definition of both exact and heuristic methods. In general exact methods are used to solve problems of limited sizes. In particular with general solver we indicate the use of optimization solvers (i.e. Cplex or Xpress) or a simple evaluation made by very simple softwares (i.e Excel). Instead heuristics are defined when formulations are characterized by the presence of non linearity in the objective function; general solver in this case means a default procedure defined in a optimization software.

As highlighted in Figure 3.5 the most part of proposed models is tested on appropriately generated instances. In some cases papers focus on real applications, especially in the public sector, like emergency recover or hospital management

Keywords	**Frequency**
Location	10
Equity	7
Facility location	5
Multiple criteria	4
Efficiency	4
MultiObjective model	3
Network	3
Fairness	3
Network	3
Heuristics	2
Public sector	2
Equality measures	2
Algorithm	2
Inequality measures	2

Table 3.4: Keywords and Frequency

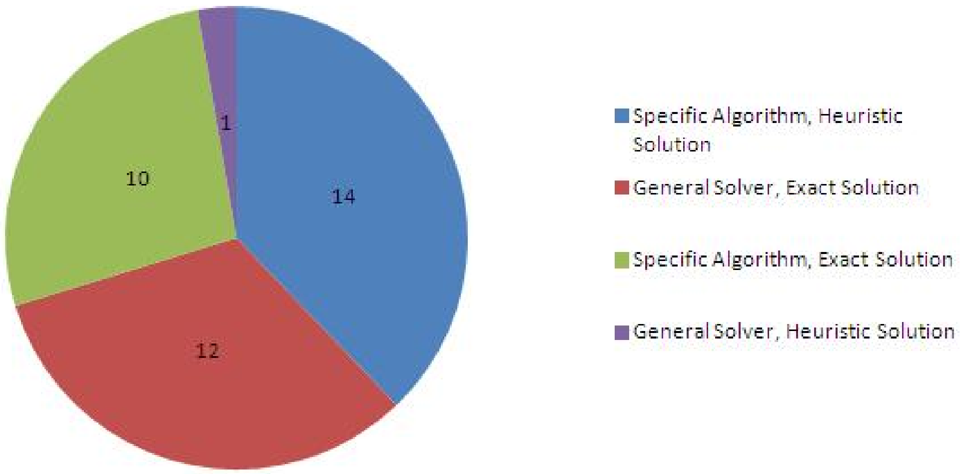

Figure 3.4: Solution Approaches

location problems.

Table 3.5 indicates for each measure how many times it is used in problems' formulations is used.

We can notice that the most used measures are the Maximum Deviation, the Mean Absolute Deviation and the Variance. While the first two are linear and so can be used in a linear programming model, the use of the Variance makes the problem more complex.

In addition most of the proposed models deal with discrete problems even if many proposals are also focused on network and continuous problems as shown in Figure 3.6.

3.5 A Literature Review: Contents

In the literature Mesa *et al.* [2003] proposed to subdivide papers about equality measures in those concern the proposal of models and algorithms and papers that analyze the characteristics of the measures. Within this general scheme, we propose a more detailed classification also considering the development of the literature in this field. In particular we distinguish among:

- **First Formulations**, i.e. papers in which first formulations are included and that introduce the fundamental ingredients of these problems; we included papers until the first years of 90's.
- **Survey and Properties Analysis**, where are resumed equality measures adapted in location theory and theoretical properties are analyzed.
- **New Forms of Equality**, papers in which alternatives proposals of equality measures are indicated.
- **Location Models with Equality Measures**, i.e. papers describing models and methods in which equality measures are introduced as objective function

Generated Instances	27
Case Study	8
OR Library	1

Sector Application	
Hazardous Material Transhipments	**1**
Casualty Collection Points	**2**
Healt Service	**3**
Subsized house	**1**
Construction of Infrastructure	**1**

Figure 3.5: Testing Models and Applications

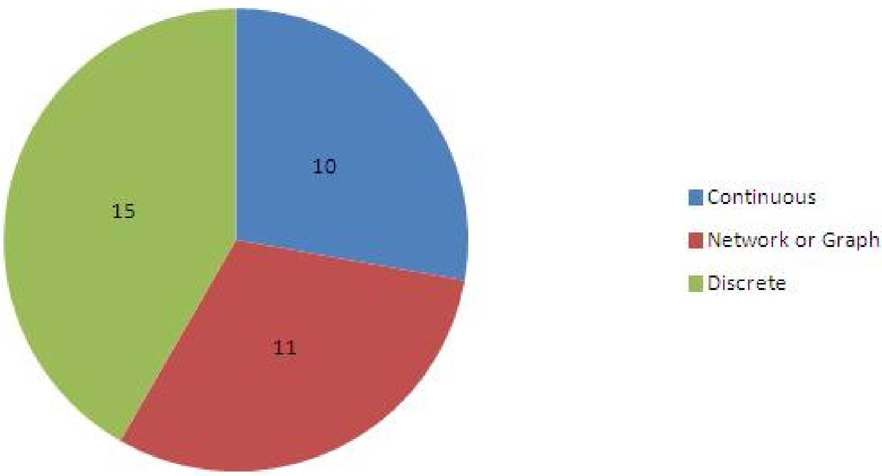

Figure 3.6: Space in which are Defined Models

Code	**Measure**	**Number of Times**
CEN	Center	10
RG	Range	5
ARG	Alternative Range	1
MAD	Mean Absolute Deviation	10
VAR	Variance	11
MD	Maximum Absolute Deviation	12
VAR	Variance	11
AD	Absolute Difference	8
SMAD	Sum Maximum Absolute Difference	1
MMAD	Maximum Maximum Absolute Difference	1
MSAD	Maximum Sum Absolute Difference	1
TI	Theil's Index	1
LOG VAR	Variance of logs	1
SI	Schutz's Index	1
VC	Coefficient of Variation	1
GC	Gini Coefficient	6
ATK	Atkinson's Index	1

Table 3.5: Number of Times in which each Equality Measure is Used in Problems Formulations

and/or constraints.

- **Applications**, that describe practical and real applications in which equality is considered.

In the following we provide a brief description of the main contributions of each of the indicate classes.

3.5.1 First Formulations

A first location decision based on equality aspects can be considered the one proposed by Mumphrey and Wolpert [1973] for choosing the position of a bridge evaluating positive and negative effects on the population; they proposed a mechanism of compensation by the government for the group that are considered disadvantaged.

The first model, using equality measure can be considered that of McAllister [1973] which focused on the system of public service centers. He used the variance of the distances among facilities and populations for evaluating the level of equality in the distribution of the effects deriving from the facility, jointly with an efficiency criterion.

Afterwards, Lindner-Dutton *et al.* [1991] considered a problem in which route for hazardous material shipments had to be defined in order to assure the equitable distribution of risk among the zones of the community. In their integer programming model, the minimization of the sum of the maximum differences of risk that exist between any pairs of zones is optimized. Also Current and Ratick [1995] proposed for this typology of problem a multiobjective model in which risks and equity, determined spatially, are minimized for the users in the worst condition.

Instead, Berman and Kaplan [1990] treated equality question using taxes and side payments to redress benefit inequities. A very simple tax or side payment scheme was proposed in order to equalize benefits for all customers in the system. They also defined a problem on a three-node network in order to minimize the sum of the absolute differences and the maximum deviation of the distance, showing that solutions were comparable to those obtained by the first approach.

3.5.2 Survey and Properties Analysis

A significative set of papers have been devoted to the study of the measures adopted in the location context and by the analysis of theoretical properties which can be considered useful to characterize equality measures.

Morrill and Symons [1977] evaluated the value of different measures when we optimize an efficiency criterion for choosing the location of a facility.

Maimon [1986] investigated properties of the variance for tree networks.

Mulligan [1991] showed a comparative analysis of equality curves trend on some measures (Gini coefficient, Mean deviation, Hoover's Index, Variance and Theil's index).

Erkut [1993] provided the first review on the field resuming all the equality measures indicated in the literature and proposing others. He tested them on a very small example highlighting some properties for each one.

Marsh and Schilling [1994], as we put in evidence in the rest of the chapter, proposed a very comprehensive survey. They listed the possible measures and also proposed a framework for the classification of them. In addition they indicated a set of properties to be satisfied by equality measures.

Hay [1995] discussed the concepts of equity, fairness and justice for location theory.

Lopez-De-Los-Mozos and Mesa [2001] analyzed properties of the maximum absolute deviation.

Furthermore Drezner and Drezner [2009] indicated some new properties for the Gini coefficient and the Absolute Difference for continuous location problems.

3.5.3 New forms of Equality

Some authors have proposed models introducing alternative formulations of equality measures different from the classical ones.

Baron *et al.* [2006] considered the problem of locating a given number of facilities on a continuous space so as to minimize the maximum demand faced by each facility subject to closest assignments and coverage constraints. This way they minimized the condition of users in the worst condition.

Berman and Huang [2008] found a position for a given number of facilities in order to minimize the maximum total weight attracted by each facilities on a network.

Moreover, Marín [2011] proposed a discrete facility location problem where the difference between the maximum and minimum number of customers allocated to every plant has to be balanced formulating it as an integer programming model solved with a branch and cut procedure.

A new form of equality criterion was defined by Espejo *et al.* [2009] that approach a discrete facility location problem in which demand points have strict preference order on the sites where the plants can be located. The goal is to minimize the total envy felt by the entire set of demand points. The new total envy criterion is defined as the absolute difference felt by the users and several integer linear programming formulations are provided.

Prokopyev *et al.* [2009] formulated some measures for the so called dispersion problem in which facilities have to be allocated at the most possible distance among them. They proposed the equitable dispersion problem that minimizes range and mean absolute deviation of the distribution of the distances among pairs of facilities.

3.5.4 Location Models with Equality Measures

Ghosh [1996] defined a problem of locating a number of facilities along a line; in order to minimize the maximum distance between two adjacent facilities they also considered a variation where the objective is not only to minimize the maximum distance, but also to hierarchically minimize the second maximum distance and so on. In the model it was assumed that there was a cost for siting a facility at a given point, and considered bi-criteria extensions where the objective was to simultaneously achieve efficiency and equity.

A new concept for evaluating properties of equality measures is introduced in Ogryczak [2000] that formulated a bi-criteria optimization model in which minimize the mean distance and the mean absolute deviation measure. The solutions of the model satisfy the new concept of equitable efficiency. These results are further generalized and is improved in Ogryczak and Zawadzki [2002], Kostreva *et al.* [2004] and Ogryczak and Zawadzki [2009] including also more measures.

Mesa *et al.* [2003] showed algorithms for single facility location problems on networks with several equality measures: the variance, the sum of weighted absolute deviations, the maximum weighted absolute deviation, the sum of absolute weighted differences, the range, and the Gini Coefficient measures.

Drezner *et al.* [2006] proposed a minimax regret multi-objective formulation that follows the idea of the minimax regret conception decision analysis. The model aimed at minimizing the maximum percentage deviation of individual as objective function; in doing so they implement a descent heuristic and a tabu search procedure.

Drezner and Drezner [2006] considered as objective function the variance of total demand attracted from each facility. In the model the gravity rule is used for the allocation of demand among facilities rather than assuming that each customer selects the closest facility. They proposed heuristic solution procedures for the problem in the plane.

Ohsawa *et al.* [2006] chose the location of a facility within a given region taking into account two criteria of equity and efficiency. Equality is sought by minimizing the sum of the absolute differences between all pairs of squared Euclidean distances from users to own facility; efficiency is measured through optimizing the sum of squared users-facility distances, either to be minimized or maximized for a desirable or obnoxious facility respectively. Afterwards Ohsawa *et al.* [2008] extended their model considering a bicriteria model with different measures to locate a semi-obnoxious facility within a convex polygon.

Drezner and Drezner [2007] investigated planar location models with two equality objectives: the minimization of the variance, and the minimization of the range of the distances. The problems were solved using a global optimization technique.

Lopez-De-Los-Mozos *et al.* [2008] exploited the concept of a particular formulation, called the ordered weighted averaging formulation, for defining a model which unifies and generalizes several inequality measures on several kinds of networks. They developed a polynomial-time algorithms to solve them

Drezner and Drezner [2009] proposed a location model minimizing the Gini coefficient based on service distances. They provide an algorithm that finds the optimal location of one facility in a bounded area in the plane when demand is generated at a set of demand points.

Puerto *et al.* [2009] dealt with the problem of locating path-shaped facilities of unrestricted length on networks. They defined the following problems: locating a path which minimizes the range, that is, the difference between the maximum and the minimum distance from the vertices to a facility, and locating a path which minimizes a convex combination of the maximum and the minimum distance from the vertices of the network to a facility, also known as the Hurwicz criterion.

3.5.5 Applications

In some cases papers are mainly oriented to solve real problems in which the introduction of measures able to take into account equality aspect is useful.

Johnson and Hurter [1998] presented an optimization model for evaluating different locations for the rent-subsidized housing in a large metropolitan area, taking into account effects for different groups like including residents of subsidized housing, owners of nearby single-family housing, employers and society at large. They looked for a balance of the number of peoples in the houses considering the size of the entire population and the number of residents in the subsized houses.

Drezner [2004] found the best location of casualty collection points that are expected to become operational in case of a human-made or natural disaster with mass casualties, such as a high-magnitude earthquake. He suggested and analyzed five objective functions including the Variance and the Gini Coefficient. In addition a multi-objective model has been proposed also applied to a scenario based on a large earthquake hitting Orange County.

Galvao *et al.* [2006] formulated a bi-criterion model in which perinatal facilities in the municipality of Rio de Janeiro have to be located minimizing the distances between the new facilities and the users and also for guaranteeing the balance of the loading of the facility, minimizing the maximum deviation of the loads of each facility.

Medaglia *et al.* [2009] proposed a bi-objective obnoxious facility location model for the disposal of hospital waste generated in the Department of Boyaca (Colombia). The objective deals with the tradeoff between a low-cost operating network and the balancing of negative effects on the population living near the waste management facilities.

Kim and Kim [2010] focused on the problem of determining locations for long-term care facilities with the objective of balancing the numbers of patients assigned to the facilities.

3.6 Conclusion

In this chapter we analyzed the concept of equality in the context of location theory. We put in evidence that there are a lot of equality measures and also models involving them.

The proposals of models with different equality measures highlighted that they seem to be equivalent in capturing complex concepts such as equality, equity fairness.

Literature has also underlined the presence of papers oriented to define properties that measures should be satisfy in order to be considered appropriate to represent equality aspects. These properties are mainly introduced in axiomatic way and they do not generally provide useful information from the computational point of view, i.e. in supporting the optimization process.

In the next chapter we try to deepen these aspects by introducing new properties and by investigating possible correlations among measures.

Chapter 4

Equality Measures: Properties

4.1 Introduction

We have illustrated the most used equality measures, typically derived from the economic field and adapted to the location context. In this chapter we describe some properties that have been introduced in order to characterize equality measures. In addition we propose further properties able to put in evidence similarity among groups of measures and what happens when we optimize a measure rather than another.

4.2 Classification of Properties for Equality Measures

In the literature several authors have mentioned one or more criteria that should be considered when an equality measure is selected.

In their survey Marsh and Schilling [1994] indicated seven properties that should be defined as axioms.

Anyway these properties highlight only if the measures are well defined or not. Instead it is necessary to identify new properties describing the behavior of equality measures in an optimization context, as has been provided for one of them by Drezner and Drezner [2009]. We define a set of significative ones in order to understand what happens when we adopt one of the different measures respect to the others. In this way we list a second category of properties that we call *optimization properties*.

In addition we classify the properties, for each one of the categories identified, in binary and computable. With binary we intend properties that can be satisfied or not by an equality measure; while with computable we indicate the properties that can be satisfied with a different degree of satisfaction.

4.3 Properties Proposed in the Literature

In this section we illustrate the properties defined in the literature for location problems. In the category of binary properties we insert: Principle of Transfer,

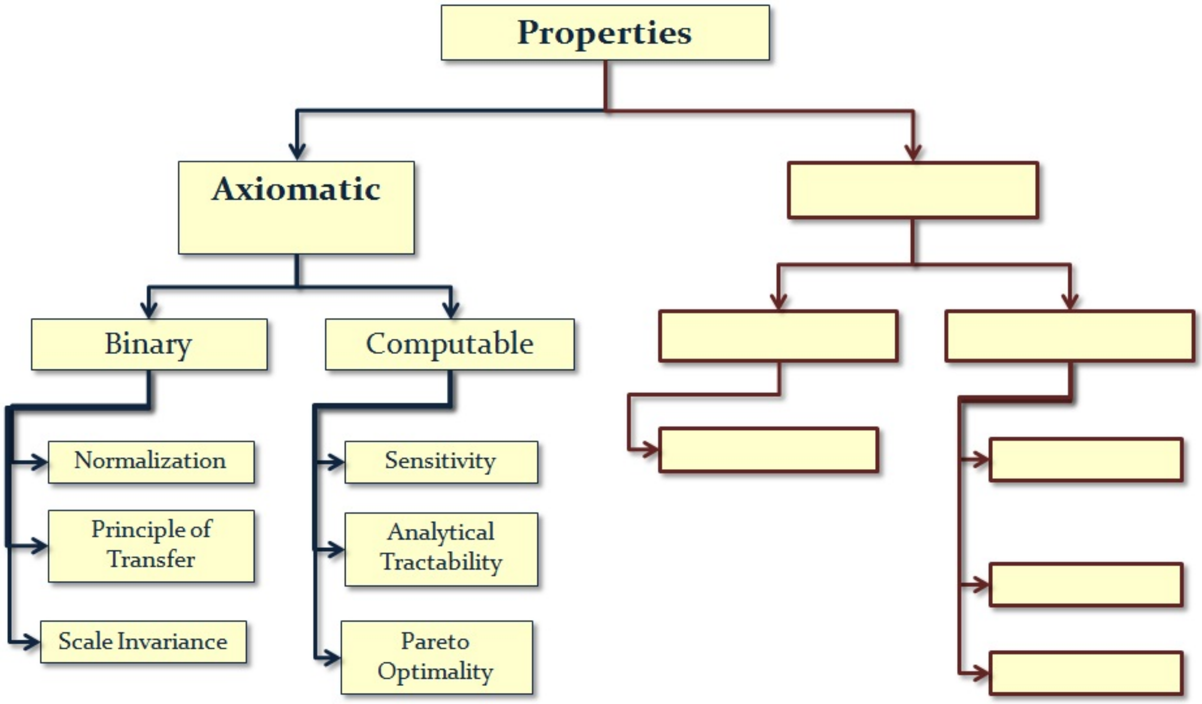

Figure 4.1: Properties for Equality Measures in Location Context

Scale Invariance, Normalization, Impartiality. Afterwards we describe the Analytic Tractability, Sensitivity and Pareto Optimality that we have included in the computable properties category (Figure 4.1).

4.3.1 Axiomatic Properties

The *Principle of Transfer* known as the Pigou-Dalton condition (from the name of its inventors), provides that a distribution of income should become less unequal if a monetary unit is transferred from a person, who is in a better economic situation than the average, to a person in a worse situation. In a location context, as defined by Erkut [1993], the distribution of distances should become less inequal if a farther user becomes closer to its patronized facility at the expense of someone else who was closer and move away, keeping constant all the other distances. Namely, given two distributions of distances sorted in increasing way, $S_1 = \{d_1, d_2, ..., d_i, d_j, \ldots, d_n\}$ and $S_2 = \{d_1, d_2, ..., d_i + \epsilon, d_j - \epsilon, \ldots, d_n\}$, S_2 is more equitable than S_1, if the absolute difference between $(d_i + \epsilon)$ and $(d_j - \epsilon)$ is less than the absolute difference between d_i and d_j. Ohsawa *et al.* [2008] demonstrated the property for the Absolute Difference measure. In similar manner it is possible to prove it for all measures.

The *Scale Invariance* principle (Erkut, 1993) is satisfied if the degree of equity does not change varying the type of scale used to assess the measure itself. In a location problem this means that a measure should not vary if the distances are calculated according to a different scale.

The *Normalization* (Marsh and Schilling, 1994) occurs when measures are somehow scaling or compared to a statistical measure. This way it is also possible to compare distributions in presence of different number of elements and of a different average distance. The normalization is in relation to the scalar invariance principle because if measures are normalized are also invariant.

The *Impartiality* property highlights that equity should only depend on the social factors and data and not from other aspects like race, color, age or political. In the location context this property is automatically satisfied because users are not distinguished according these aspects.

4.3.2 Computable Properties

The *Analytic Tractability* property (Marsh and Schilling, 1994) concerns the computational complexity of a measure. In this sense it can be calculated as the number of operations needed to evaluate a given measure. However it should be defined considering the contribute to the complexity of a given problem. For instance it is expected that a non-linear measure makes a problem more complex instead of a linear measure.

Sensitivity (Marsh and Schilling, 1994) is the feature that defines how the solution varies with the variations of any parameter of the problem. In the context of location problems the property takes into account variations of the measure in dependence on changes of positions of demand points.

The *Pareto Optimality* solution will be considered better than another if at least one user has a shorter distance from the facilities. Often this condition is not considered necessary because this is a measure of efficiency and not of equality (Campbell, 1990).

Finally, we can have the property of *Appropriateness*. Mulligan [1991] summarized this concept arguing that some measures are not intuitively satisfactory and most of the time, the use of an inappropriate measure in a decision-making process leads to a certain failure; moreover, used measure should be easily understood in order to be able to choose between different alternatives.

4.4 New Optimization Oriented Properties

Apart the Transformation Invariance property recently proposed by Drezner and Drezner [2009] the existing defined properties are mainly axiomatic.

Then they do not provide indications about the behavior of a measure from the optimization point of view, indications that could be useful to support the design of effective optimization methods (exact and heuristic). For this reason we introduce some "optimization oriented" properties (Figure 4.2) that we define considering a standard regular location space. In particular we consider, as in Drezner and Drezner [2009] a uniformly distributed demand in a disk of unit radius. Then we consider the following properties in relation to the presence of this demand distribution.

The *Transformation Invariance* was introduced in the context of location by Drezner and Drezner [2009]. This property is satisfied if we transformed the space of location and the position of the facility in the same way, we should obtain the same result for the calculated measure. Drezner and Drezner [2009] analyzed three

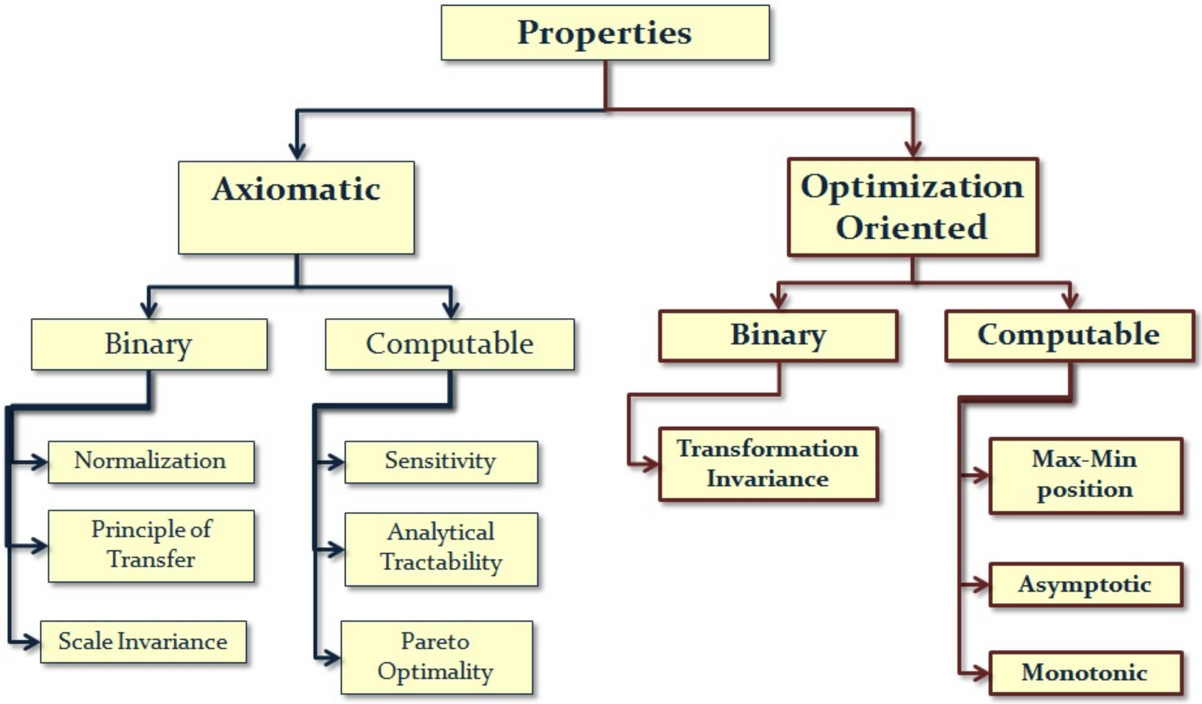

Figure 4.2: Optimization Oriented Properties for Equality Measures in Location Context

different kinds of transformation (Translation, Rotation and Expansion) and they verified the property for the Gini Coefficient.

The *Max-Min Position* reveals the expected position of the maximum and the minimum value inside and outside the circle.

The *Monotonic* property analyzes the trend of a measure over the distance from the center of the demand distribution (Figure 4.3).

The *Asymptotic* property evaluates what is the value of a measure far from the center of the demand distribution. A measure has an asymptotic behavior when, moving the facility from the center of distribution, further than a certain distance, the measure tends to assume the same value. Possible values for the asymptote can be finite, infinite or zero (Figure 4.4).

We performed a study oriented to highlight the behavior of each measure in

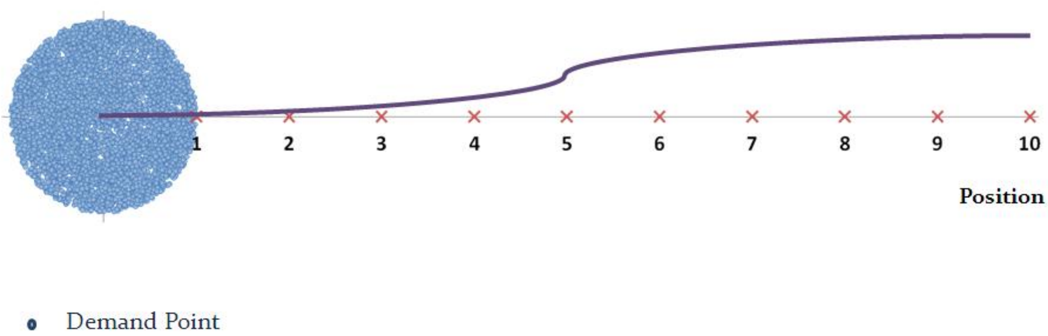

Figure 4.3: Trend of an Equality Measure

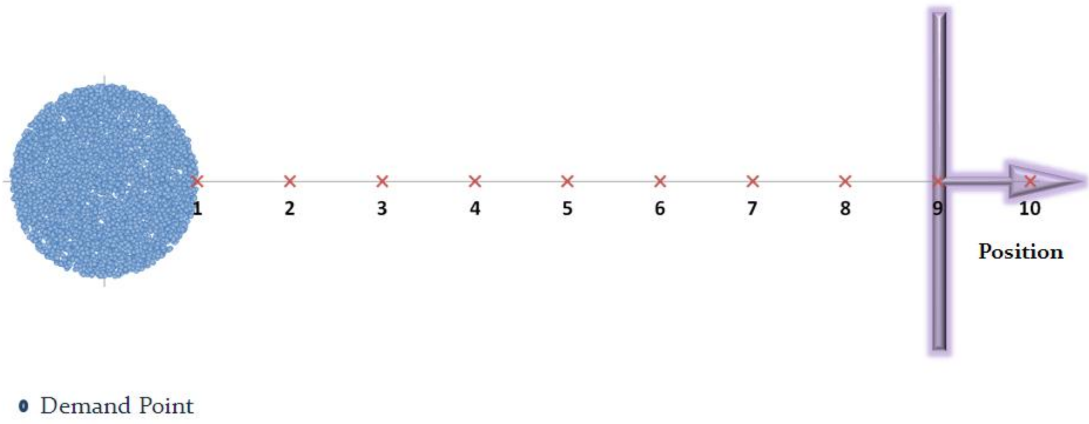

Figure 4.4: Evaluation of an Equality Measure at Big Distance

terms of optimization oriented properties. To this aim we realized an empirical analysis considering a single facility location problem using as objective one by one the equality measures indicated in Table 4.1. In order to better illustrate the results we recall the notation introduced in Chapter 3:

$I = \{1, \ldots, n\}$ the set of the n demand points;

d_i the distance between the demand point i and the facility;

$\bar{d}$ the average distance between the demand points and the facility defined as $\sum_i \frac{d_i}{n}$;

In the Table 4.1 9 of 12 measures of the list performed by Eiselt and Laporte [1995] and one measure indicated by Erkut [1993] are included. Measures 1-7 are not normalized, while the remaining ones (8-9) are normalized. Furthermore, measures 3-5 and 8-9 represent deviations from the mean distance distribution.

In order to simulate a demand space with continuous uniformly distributed demand we consider a space consisting in a unit circle in which we generated 5000 demand points according to a uniform distribution. To this purpose we followed the procedure indicated in Weisstein [2011]. The procedure is characterized by the following steps to define the position of a single demand point:

- generating two random numbers ρ and θ with uniform distribution such that:

$$\rho : 0 \leq \rho \leq 1$$

$$\theta : 0 \leq \theta \leq 2\pi$$

- calculating the coordinates of each demand point as:

$$X = \sqrt{\rho} \cos \theta$$

$$Y = \sqrt{\rho} \sin \theta$$

An example of demand space generated according to this procedure is illustrated in Figure 4.5. In presence of a circular uniformly distributed demand, in order to verify the above mentioned properties. We introduce an X axis whose origin corresponds to the center of the demand space and whose direction coincide with

Code	**Measure**	**Formulation**
CEN	Center	$\max_{i \in I} d_i$
RG	Range	$\max_{i \in I} d_i$- $\min_{i \in I} d_i$
MAD	Mean Absolute Deviation	$\sum_{i \in I} \lvert d_i - \bar{d} \rvert$
VAR	Variance	$\sum_{i \in I} (d_i - \bar{d})^2$
MD	Maximum Deviation	$\max_{i \in I} \lvert d_i - \bar{d} \rvert$
AD	Absolute Difference	$\sum_{c \in I, d \in I} \lvert d_c - d_d \rvert$
SMDA	SumMaxDiffAbs	$\sum_{c \in I} \max_{d \in I} \lvert d_c - d_d \rvert$
SI	Schutz's Index	$\frac{1}{2N\bar{d}} \sum_{i \in I} \lvert d_i - \bar{d} \rvert$
VC	Coefficient of Variation	$\frac{\sum_{i \in I} (d_i - \bar{d})^2}{\bar{d}}$
GC	Gini Coefficient	$\frac{\sum_{c \in I, d \in I} \lvert d_c - d_d \rvert}{2n^2\bar{d}}$

Table 4.1: Selected Equality Measures for the Single Facility Location Problem

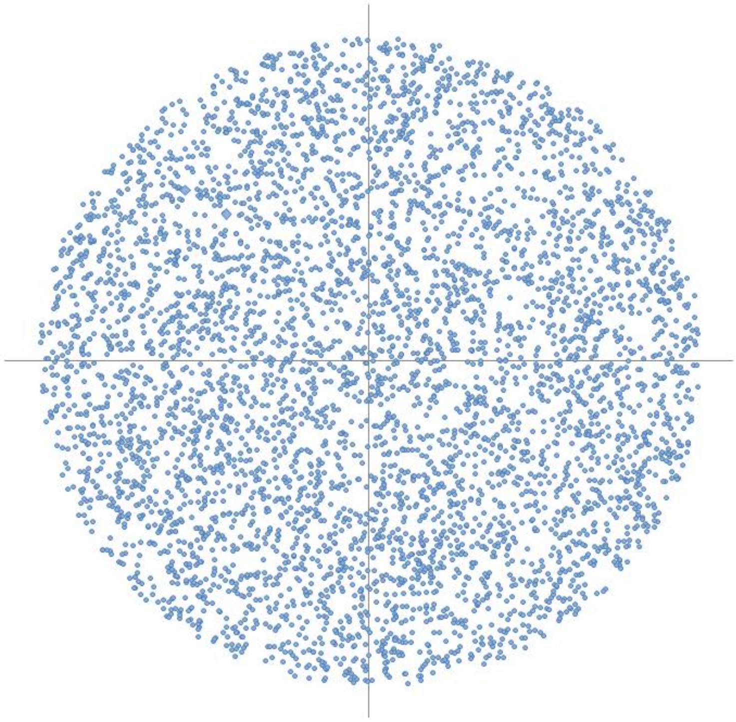

Figure 4.5: Example of Test Problem

the direction of a radius. Then we calculated each measure inside and outside the circular demand space. In particular inside the circle, we considered every point on the X-axis for $X = 0$ to $X = 1$ assuming a step equal to 0,05 (Figure 4.6). Outside the circle we considered points from $X = 1$ to $X = 10$ with a step equal to 1 (Figure 4.7).

4.5 Analysis of the Optimization Oriented Properties

The calculation of the value of each measure along the X axis let one analyzes the above defined properties. To this aim we generated 10 different circular demand spaces. We focus on the set of optimization oriented properties. In particular we considered the transformation invariance as defined by Drezner and Drezner [2009] an the new ones introduced:

- **Max-Min Position**
- **Monotonicity**
- **Asymptoticity**

In the following we describe the performed analysis and the conclusions drawn about the behavior of each measure.

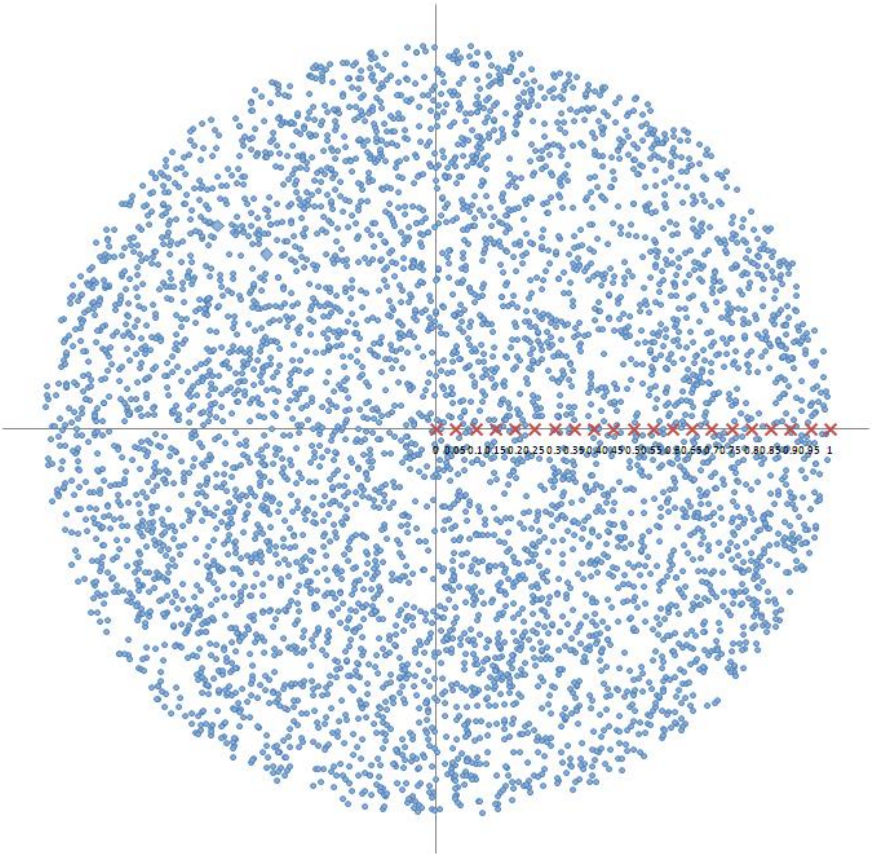

Figure 4.6: The Considered Facility Inside the Space

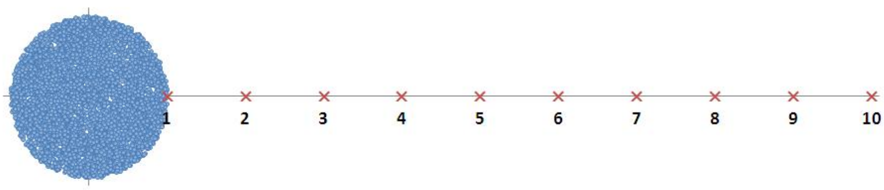

Figure 4.7: The Considered Facility Outside the Space

	Translation Invariance	Rotation Invariance	Expansion Invariance
CEN	1	1	1
RG	1	1	K
MAD	1	1	K
VAR	1	1	$> K$
MD	1	1	K
AD	1	1	K
SMDA	1	1	K
SI	1	1	1
VC	1	1	1
GC	1	1	1

Table 4.2: Transformation Invariance Properties

Transformation Invariance

In order to test the transformation invariance property we apply three different transformations to the location space. In particular for verifying the *Translation invariance* for each of the 10 generated instances we provided a "translated" version of each instance by changing the position of each demand point incrementing the value of the coordinates of the same quantity; i.e. indicating with K this increment, a demand point with coordinates (X_1, Y_1) after the transformation its coordinates become $(X_1 + K, Y_1 + K)$. We also incremented, of the same quantity, the coordinates of the position occupied by the facility.

For the *Rotation invariance* the methodology adopted is the same but, we produced the new instances rotating all the coordinates of the points and of the facility of the same angle α. Again for the *Expansion invariance* we "expanded" both the coordinates of a certain value, i.e a demand point (X_1, Y_1) become $(X_1 \times K, Y_1 \times K)$.

In order to evaluate the properties we calculated the ratio between the measure in the original space and the value obtained in the modified space at the center of the distribution (Table 4.2). Thanks to the experimentation we can establish that the rotation invariance and the translation invariance are satisfied for all equality measures analyzed, because the value of the measures in the original and in the generated instance is the same for all of them and so the ratio is equal to 1. For the Expansion Invariance it can be observed that while the normalized measures (SI, VC, GC) satisfy the property, among the others apart the center all the measures present an increment of the measure equal to K. We can note that all normalized measures are invariant scalar, indeed the ratio is equal to 1; instead the not normalized measures have all the same ratio corresponding to the coefficient of multiplication K; only the variance (VAR) presents a ratio plus than K.

Max-Min position, Monotonicity, Asymptoticity

In order to describe the behavior of each measure in terms of min and max position, monotonicity and asymptoticity we calculated for each test problem the values of each measure along the X axis, inside and outside the circular space according to the steps above described.

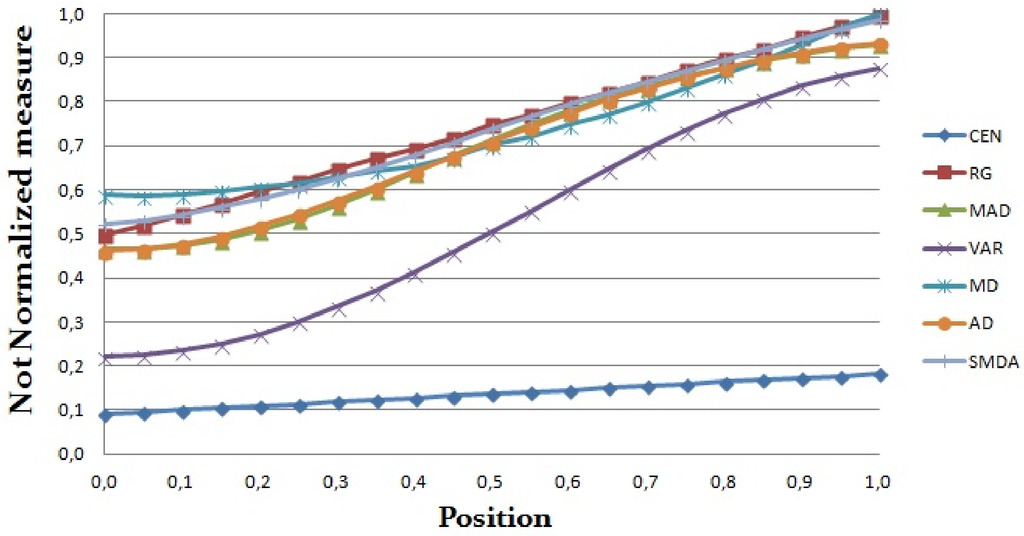

Figure 4.8: Not Normalized Measures: Monotonic Property

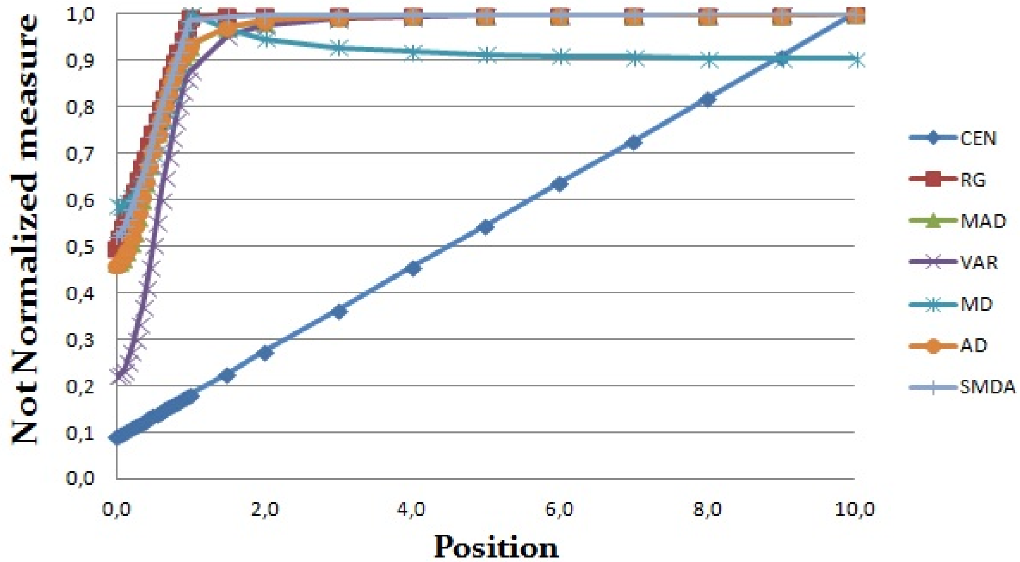

Figure 4.9: Not Normalized Measures: Asymptotic Property

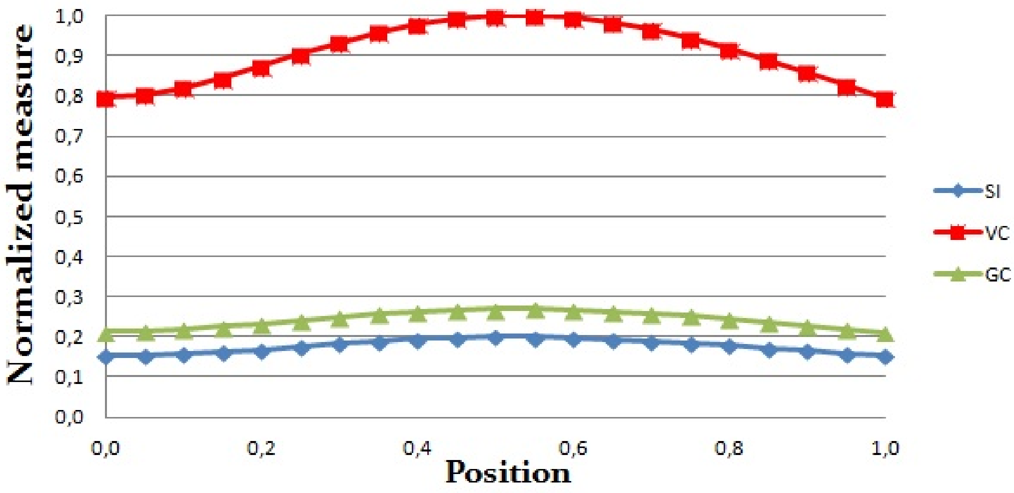

Figure 4.10: Normalized Measures: Monotonic Property

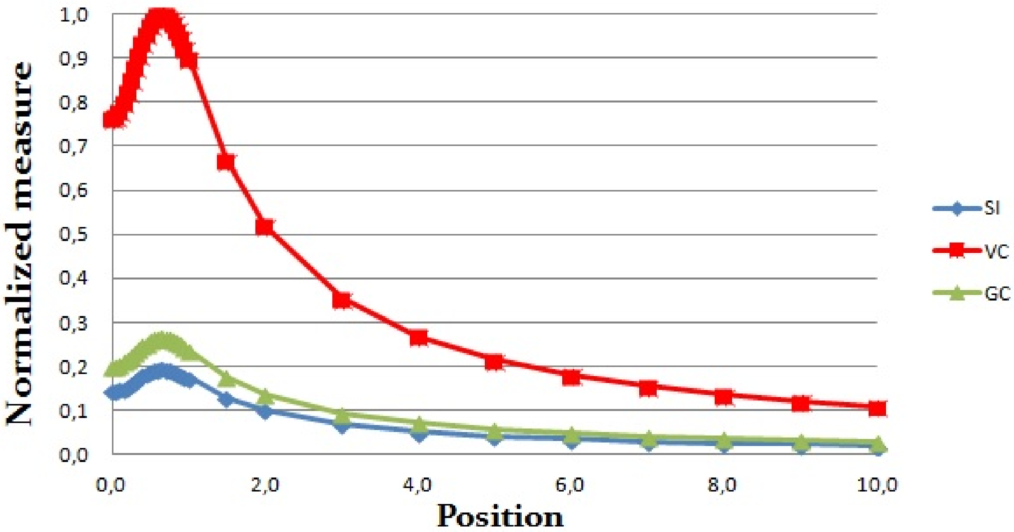

Figure 4.11: Normalized Measures: Asymptotic Property

	Facility Position		
	$X = 10$	$X = 100$	$X = 1000$
CEN	10.99669	100.99661	1000.99661
RG	1.99139	1.99156	1.99157
MAD	2122.08092	2123.7162	2123.74822
VAR	1251.63765	1252.83111	1252.83446
MD	1.00692	1.00098	1.00121
AD	14409275.14	14416984.28	14417021.75
MSDA	7101.52532	7103.26279	7103.35337
SI	0.02119	0.00212	0.00021
VC	3.53352	0.35395	0.0354
GC	0.02878	0.00288	0.00029

Table 4.3: Values of Measures at Points Outside the Circular Space

In addition to verify the asimptoticity behavior we also evaluated measures at point $X = 10, 100, 1000$, i.e. at a distance from the center respectively equal to 10, 100 and 1000 times. In Figures 4.8, 4.9, 4.10 and 4.11, we plot the average values of the measures on the 10 generated instances in function of the distance of the considered facility from the center. Moreover, since the measures have different order of magnitude, for not normalized measures we divided all values for the maximum value obtained in a given instance for each measure; in this way all values will be included between 0 and 1.

In particular we show in Figures 4.8 and 4.9 the behavior of the not normalized measures inside and outside the circular space while in Figures 4.10 and 4.11 we indicate the same for the normalized measures. In Table 4.3 we report the average values on the 10 instances obtained at points outside the circular space.

On the basis of the obtained results we point out that all the not normalized measures present an increasing similar behavior inside the circle, from the center to the extreme point (Figure 4.8).

The Center measure (CEN) has a constant increase as it represents the distance from the farthest demand point; in presence of a very dense demand space if the facility is positioned at point $X = 10$ its value is well approximated by the value in the center plus the distance from the center. Moreover the Center measure (CEN) is the only that has an asymptote at infinite value, while the others not normalized measures have an asymptote at finite value (Figure 4.9).

The RG has the same behaviour of CEN inside the circle, while at larger distances the difference between maximum and minimum distance tends to diminish.

The MAD, AD, MSDA and VAR have a same trend with an inflection point immediately after the origin of the axis and another at the end of the distribution of the points. The Measure MD has a trend more fluctuating inside the circle (Figure 4.8).

The asymptote for these measure is equal to the respective value in correspondence of the facility positioned in point $X = 1$ as put in evidence from the value in Table 4.3.

Code	Min Position	Frequency	Max Position	Frequency
CEN	Center	100%	Farthest	100%
RG	Center	100%	Farthest	100%
MAD	Center	100%	Farthest	100%
VAR	Center	100%	Farthest	100%
MD	Center	40%	Farthest	100%
AD	Center	100%	Farthest	100%
SMDA	Center	100%	Farthest	100%
SI	Farthest	100%	Inside (65% of radius)	100%
VC	Farthest	100%	Inside (85% of radius)	100%
GC	Farthest	100%	Inside (65% of radius)	100%

Table 4.4: The Max/Min Position and Frequency on Ten Instances

Instead the normalized measures (GC, SI, VC) have an unimodal trend inside the circle in fact they first increase and after decrease (Figure 4.10). On the contrary outside the circle they assume a decreasing trend that tends asymptotically to zero at a very high distance (Figure 4.11). At the position $X = 1000$, as indicated in Table 4.3, the value is almost equal to 0.

In Table 4.4 we indicate the position of the facility we found the maximum and the minimum values for all measures and also the frequency intended as the number of times on ten for which the same correspondent position is obtained. We point out that the minimum values for not normalized measures is obtained always in correspondence to the centre except for the measure MD but, the position is very close to the centre. On the contrary the maximum value is obtained for the not normalized measures inside the circle always in correspondence of the position on the circumference. For the normalized measures the minimum is always in the farthest position analyzed. For the Gini Coefficient (GC), in according with the analysis conducted by Drezner and Drezner [2009], the maximum value is about 65% of the radius of the circle. Still the maximum values of SI is about 65%, while for VC is 85% of the radius of the circle.

Finally in Table 4.5 we provide a resume on how the optimization oriented properties are satisfied; for the binary properties we indicate if they are satisfied (*Yes*) or not (*No*), while for the computable ones the respective degree of computation. In particular, with *center* we intend the center of the circle (the location space), while with *farthest* a position outside the circle and a very big distance from it; *inside*, instead, indicates a position in the circle different from the center. *Increasing* and *unimodal* refer, as indicated before, to the monotonic property.

4.6 Analysis of the Correlation between Pairs of Measures

Another objective of our analysis is oriented to determine correlations between each pair of measures, i.e what happens to a measure when we optimize another measure. To this aim we formulate the mathematical model of a general problem

Code	Transformation Invariance	Asymptotic	Monotonic	Min Position	Max Position
CEN	Yes	Infinite	Increasing	Center	Farthest
RG	No	Finite	Increasing	Center	Farthest
MAD	No	Finite	Increasing	Center	Farthest
VAR	No	Finite	Increasing	Center	Farthest
MD	No	Finite	Increasing	Center	Farthest
AD	No	Finite	Increasing	Center	Farthest
SMDA	No	Finite	Increasing	Center	Farthest
SI	Yes	Zero	Unimodal	Farthest	Inside
VC	Yes	Zero	Unimodal	Farthest	Inside
GC	Yes	Zero	Unimodal	Farthest	Inside

Table 4.5: Summary Proposed Properties

characterized by the location of p facilities with the objective of minimizing an equality measure.

We use the following notation:

$I = \{1, \dots, n\}$ the set of the n demand points;
$J = \{1, \dots, n\}$ the set of potential locations for the facilities;
d_{ij} the distance between the demand point i and the facility in j;
$\bar{d}$ the average distance between the demand points and the facilities defined as $\sum_{i,j} \frac{d_{ij}}{n}$;
Then, allocation decisions are represented through the following x-variables:

$$x_{ij} = \begin{cases} 1 & \text{if demand point } i \text{ is allocated to facility } j \\ 0 & \text{otherwise,} \end{cases} \qquad \forall i \in I, j \in J,$$

and location decisions are represented by

$$y_j = \begin{cases} 1 & \text{if a facility is located at point } j \\ 0 & \text{otherwise,} \end{cases} \qquad \forall j \in J.$$

The proposed formulation is

$$\min \ f_k(d_{ij}, x_{ij})$$

$$\text{s.t.} \ \sum_{j \in J} x_{ij} = 1 \qquad \forall i \in I, \tag{4.1}$$

$$x_{ij} \leq y_j \qquad \forall i \in I, j \in J, i \neq j, \tag{4.2}$$

$$\sum_{j \in J} y_j = p \tag{4.3}$$

$$\sum_{j} d_{ij} x_{ij} + (M - d_{ij}) y_j \leq M \qquad \forall i \in I, j \in J, \tag{4.4}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i \in I, j \in J, \tag{4.5}$$

$$y_j \in \{0, 1\} \qquad \forall j \in J. \tag{4.6}$$

where $f_k(d_{ij}, x_{ij})$ is one of equality measures chosen among those indicated in Table 4.6 and M is a very large number ($M \geq \max_{i,j} d_{ij}$).

Constraints (4.1) ensure that all the demand points are allocated. Constraints (4.2) ensure that a point may receive allocation only if it is active. Constraint (4.3) fixes the number of facilities to p. Constraints (4.4), as pointed out by Espejo *et al.* [2012] permit to allocate each demand point to the closest facility. Constraints (4.5) and (4.6) indicate that variables are binary.

The characteristic of this model depends on the chosen measure. If $f_k(d_{ij}, x_{ij})$ is not linear a linearization process can be applied.

In particular we adopt the approach used by Chang [2001]. We show how this works in the case of the Gini Coefficient.

Introducing the following new variables:

t a real variable equal to $t = \frac{1}{\sum_{i \in I, i \in J} d_{ij} xij}$;
f_{ij} a real variable substituting the product $x_{ij} t$ in the objective function;
z_{ij} a real variable substituting the product $x_{ij} t$ in the constraints;

Code	**Measure**	**Formulation**
CEN	Center	$\max_{i\in I,j\in J} d_{ij}x_{ij}$
RG	Range	$\max_{i\in I,j\in J} d_{ij}x_{ij}$- $\min_{i\in I,j\in J} d_{ij}x_{ij}$
MAD	Mean Absolute Deviation	$\sum_{i\in I} \mid \sum_{j\in J} d_{ij}x_{ij} - \bar{d}\mid$
VAR	Variance	$\sum_{i\in I}(\sum_{j\in J} d_{ij}x_{ij} - \bar{d})^2$
MD	Maximum Deviation	$\max_{i\in I} \sum_{j\in J} \mid d_{ij}x_{ij} - \bar{d}\mid$
AD	Absolute Difference	$\sum_{c\in I,d\in I} \mid \sum_{j\in J} d_{cj}x_{cj} - \sum_{j\in J} d_{dj}x_{dj}\mid$
SMDA	SumMaxDiffAbs	$\sum_{c\in I} \max_{d\in I} \mid \sum_{j\in J} d_{cj}x_{cj} - \sum_{j\in J} d_{dj}x_{dj}\mid$
SI	Schutz's Index	$\frac{1}{2N\bar{d}} \sum_{i\in I,j\in J} \mid d_{ij}x_{ij} - \bar{d}\mid$
VC	Coefficient of Variation	$\frac{\sum_{i\in I,j\in J}(d_{ij}x_{ij}-\bar{d})^2}{\bar{d}}$
GC	Gini Coefficient	$\frac{\sum_{c\in I,d\in I} \mid \sum_{j\in J} d_{cj}x_{cj}-\sum_{j\in J} d_{dj}x_{dj}\mid}{2n^2\bar{d}}$

Table 4.6: Objective Functions Based on Equality Measures

m_{ij} a real variable for the linearization of the absolute value.

The model becomes:

$$\min \frac{1}{2n} \sum_{i \in I, j \in J} m_{ij}$$

$$\text{s.t. } f_{ij} \geq M(x_{ij} - 1) + t \quad \forall i \in I, \forall j \in J \tag{4.7}$$

$$f_{ij} \geq 0 \quad \forall i \in I, \forall j \in J, \tag{4.8}$$

$$f_{ij} \leq x_{ij} \quad \forall i \in I, \forall j \in J, \tag{4.9}$$

$$f_{ij} \leq t \quad \forall i \in I, \forall j \in J, \tag{4.10}$$

$$\sum_{i \in I, j \in J} x_{ij} d_{ij} = 1 \tag{4.11}$$

$$z_{ij} \geq M(x_{ij} - 1) + t \quad \forall i \in I, \forall j \in J, \tag{4.12}$$

$$z_{ij} \leq M(1 - x_{ij}) + t \quad \forall i \in I, \forall j \in J, \tag{4.13}$$

$$z_{ij} \leq M(x_{ij}) \quad \forall i \in I, \forall j \in J, \tag{4.14}$$

$$\sum_{j \in J} f_{cj} d_{cj} - \sum_{j \in J} f_{dj} d_{dj} \leq m_{cd} \quad \forall c, d \in I, \tag{4.15}$$

$$\sum_{j \in J} f_{cj} d_{cj} - \sum_{j \in J} f_{dj} d_{dj} \geq -m_{cd} \quad \forall c, d \in I, \tag{4.16}$$

$$\sum_{j \in J} x_{ij} = 1 \quad \forall i \in I, \tag{4.17}$$

$$x_{ij} \leq y_j \quad \forall i \in I, j \in J, i \neq j, \tag{4.18}$$

$$\sum_{j \in J} y_j = p \tag{4.19}$$

$$\sum_{j} d_{ij} x_{ij} + (M - d_{ij}) y_j \leq M \quad \forall i \in I, j \in J, \tag{4.20}$$

$$x_{ij} \in \{0, 1\} \quad \forall i, j \in J, \tag{4.21}$$

$$y_j \in \{0, 1\} \quad \forall j \in J. \tag{4.22}$$

The constraints (4.7, 4.8, 4.9,4.10) assure that the new variable f_{ij} is equal to $x_{ij}t$ as suggested in Chang [2001]. The constraints 4.7 assure that the variable t is equal to $\frac{1}{\sum_{i \in I, i \in J} d_{ij} xij}$. The constraints (4.12, 4.13,4.14) assure that the new variable z_{ij} are equal to $x_{ij}t$ in the formulation of the constraints, again as suggested Chang [2001]. Constraints (4.15, 4.16) are the classical ones for linearizing the absolute value in the objective function with the introduction of the new variables m_{ij}. The other constraints are the same expressed in the p equality model.

In order to evaluate the degree of similarity between pairs of measures we solve the model considering the equality measure v as objective function and we calculate the value of another measures u, f_v^u. For instance f_{MAD}^{AD} represents the value of measure Mean Absolute Deviation (MAD) when the model is solved by using the measure Absolute Difference (AD) as objective function. This way considering a set of problems, for each pair (u, v), two data sets can be obtained: the first representing all the optimal solution values f_u^* calculating using measure u as objective; the

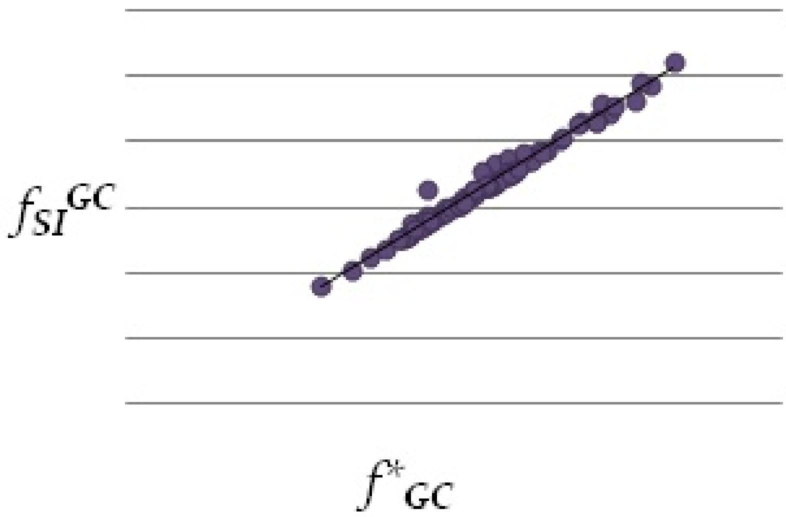

Figure 4.12: An example of Correlation Coefficient

second representing the values of the measure u, f_v^u calculated using as objective function v. For instance, Figure 4.12 reports the results of the experiments assuming u corresponding to the Gini Coefficient (GC) and v corresponding to the Schutz's Index (SI).

If we calculate the correlation coefficient $r_{v,u}$ it can be viewed as a measure or the degree of similarity of the measures (u, v) when a problem is solved. In fact when the points are collected on a straight line means that the two measures are perfectly correlated $r_{v,u} = 1$; this means that if we use u or v as objective function we systematically find the same solution. On the contrary, if points are more distributed in the plane, the measures are less correlated. It has to be noticed that, given its particular definition, generally the correlation coefficient $r_{v,u}$ is not symmetrical. This correlation measure indicates how much a measure can be considered a good "proxy" for another measure. In practice, high values of $r_{v,u}$ indicate that by optimizing measure v one can obtain values very close to the optimum ones for the measure u.

4.7 Empirical Analysis

We proposed an empirical analysis oriented to quantify potential correlations between pair of measures. To this aim, we solved the general p-equality formulation with the different measures indicated in Table 4.6.

In our analysis we considered two sets of test problems. The first one $(10X10)$ assuming $|I| = |J| = 10$. The second one $(20X20)$ with $|I| = |J| = 20$ (an example is shown in Figure 4.13). The experiments were conducted on 100 randomly generated instances assuming with $p = 2, 3, 4, 5$ for the $10X10$ set and $p = 2, 4, 6, 8, 10$ for $20X20$ set. Throughout all the testing, we used a Pentium IV with 2.40 GigaHertz and 4.00 GigaBytes of RAM running. The solver was Cplex v 12.00. For each pair

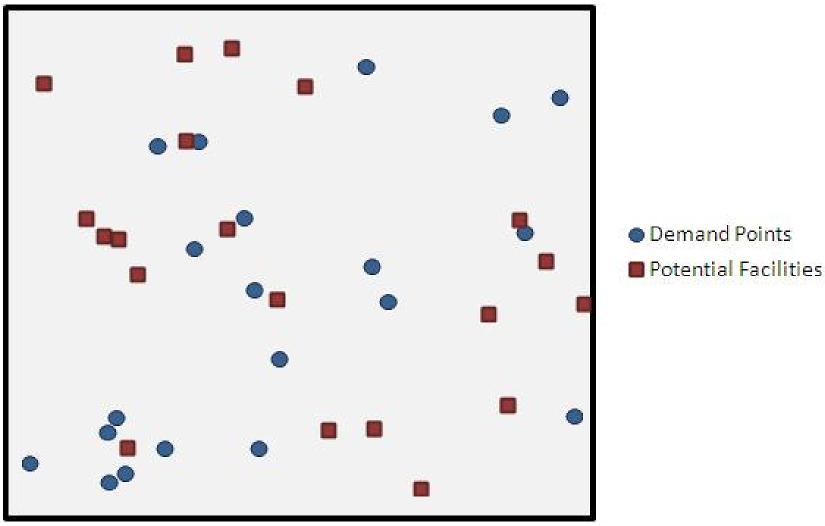

Figure 4.13: An Example of Randomly Generated Instance with $|I| = |J| = 20$

of selected measures we calculated:

- the number of instances, out of 100, where two measures obtained the same optimal solution;
- the correlation coefficient $r_{v,u}$.

	CEN			RG			MAD			VAR			MD			AD			SMDA			SI			VC			GC		
	max	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*
CEN	100	100	100	55	52	49	56	44	37	59	48	42	48	46	44	56	46	40	57	50	45	28	20	12	59	48	42	28	20	15
RG				100	100	100	48	35	26	58	45	37	65	52	40	57	43	31	71	57	49	40	36	28	63	60	56	45	42	36
MAD							100	100	100	75	68	63	39	30	23	78	72	66	63	52	45	52	47	41	63	56	47	48	42	37
VAR										100	100	100	50	39	32	87	79	72	77	68	61	45	42	38	72	67	60	43	41	38
MD													100	100	100	45	33	26	62	52	43	25	21	16	43	36	31	29	24	20
AD																100	100	100	71	58	49	47	43	38	67	63	52	45	43	39
SMDA																			100	100	100	38	36	34	67	61	58	41	39	37
SI																						100	100	100	63	57	54	75	72	71
VC																									100	100	100	63	62	61
GC																												100	100	100

Table 4.7: Number of Instances where Two Measures Obtained the Same Optimal Solution on $10X10$ Randomly Generated Instances

	CEN			RG			MAD			VAR			MD			AD			SMDA			SI			VC			GC		
	max	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*
CEN	100	100	100	65	50	36	59	33	14	73	44	22	67	46	30	71	42	20	76	52	33	28	13	4	73	44	22	33	15	4
RG				100	100	100	20	7	2	34	16	4	50	23	3	47	36	30	1	69	65	11	5	3	32	15	6	18	8	4
MAD							100	100	100	57	45	39	18	6	1	66	54	46	35	23	13	32	27	23	42	32	22	31	24	19
VAR										100	100	100	32	13	2	80	71	60	57	41	29	24	20	16	63	51	40	33	28	22
MD													100	100	100	24	10	2	52	24	6	11	4	1	26	11	2	17	5	1
AD																100	100	100	47	34	24	31	23	19	59	46	35	40	30	23
SMDA																			100	100	100	19	14	8	45	36	30	21	17	9
SI																						100	100	100	39	33	26	63	53	46
VC																									100	100	100	55	53	50
GC																												100	100	100

Table 4.8: Number of Instances where Two Measures Obtained the Same Optimal Solution on $20X20$ Randomly Generated Instances

We have in Tables 4.7 and 4.8 the minimum (*min*), the average (*avr*) and the maximum (*max*) number of coinciding solutions among those derived from all the instances solved in the two different location spaces. We can note that a very significant high number is obtained in the case of some particular pairs (RG-AD; MAD-AD; VAR-AD; VAR-VC; GC-GC; GC-SI). On the other hand, for some pairs of measures, for instance SI-MD, it is obtained the same optimal solutions a much lower number of times. The most significant relationships noticed in the $10X10$ case still persist also in the $20X20$ case.

	CEN			RG			MAD			VAR			MD			AD			SMDA			SI			VC			GC		
	max	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*
CEN	1	1	1	0,87	0,74	0,61	0,84	0,75	0,69	0,88	0,74	0,59	0,87	0,78	0,70	0,87	0,77	0,71	0,90	0,79	0,72	0,60	0,33	0,17	0,70	0,58	0,34	0,55	0,31	0,10
RG	0,87	0,74	0,61	1	1	1	0,95	0,92	0,88	0,98	0,97	0,97	0,99	0,97	0,92	0,96	0,95	0,94	1	1	0,99	0,80	0,62	0,49	0,95	0,90	0,87	0,81	0,66	0,47
MAD	0,72	0,67	0,55	0,93	0,90	0,86	1	1	1	0,99	0,99	0,99	0,88	0,85	0,81	1,00	1,00	0,99	0,93	0,92	0,89	0,83	0,74	0,68	0,96	0,93	0,88	0,83	0,71	0,62
VAR	0,78	0,62	0,40	0,98	0,96	0,94	0,99	0,99	0,99	1	1	1	0,95	0,91	0,87	1,00	1,00	0,99	0,96	0,95	0,94	0,78	0,62	0,47	0,97	0,92	0,84	0,78	0,59	0,40
MD	0,81	0,71	0,62	0,97	0,96	0,93	0,89	0,85	0,82	0,93	0,91	0,89	1	1	1	0,90	0,87	0,84	0,98	0,96	0,93	0,76	0,59	0,5	0,88	0,84	0,80	0,77	0,59	0,45
AD	0,78	0,68	0,56	0,97	0,95	0,92	1,00	1,00	0,99	1,00	1,00	1,00	0,94	0,90	0,86	1	1	1	0,99	0,98	0,95	0,84	0,72	0,64	0,97	0,94	0,88	0,84	0,71	0,58
SMDA	0,86	0,73	0,59	0,99	0,99	0,99	0,97	0,96	0,94	0,99	0,99	0,99	0,99	0,98	0,94	0,97	0,97	0,96	1	1	1	0,82	0,66	0,55	0,96	0,93	0,90	0,82	0,66	0,49
SI	0,48	0,40	0,27	0,77	0,70	0,62	0,85	0,78	0,70	0,85	0,76	0,68	0,73	0,67	0,62	0,84	0,78	0,70	0,80	0,73	0,63	1	1	1	0,92	0,91	0,86	0,99	0,99	0,98
VC	0,72	0,58	0,45	0,96	0,93	0,90	0,96	0,95	0,95	0,98	0,97	0,96	0,92	0,90	0,88	0,97	0,97	0,96	0,97	0,95	0,93	0,96	0,92	0,89	1	1	1	0,97	0,92	0,86
GC	0,53	0,42	0,26	0,82	0,77	0,68	0,85	0,8	0,73	0,85	0,78	0,69	0,79	0,75	0,66	0,85	0,80	0,71	0,82	0,76	0,66	0,99	0,99	0,98	0,94	0,93	0,91	1	1	1

Table 4.9: Correlation Coefficient Among Measures in the $10X10$ Instances

	CEN			RG			MAD			VAR			MD			AD			SMDA			SI			VC			GC		
	max	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*	*max*	*avr*	*min*
CEN	1	1	1	0,96	0,53	0,24	0,90	0,50	0,26	0,96	0,53	0,19	0,97	0,58	0,24	0,95	0,56	0,32	0,97	0,59	0,29	0,62	0,26	0,12	0,81	0,43	0,21	0,61	0,29	0,13
RG	0,88	0,55	0,31	1	1	1	0,92	0,81	0,63	0,97	0,93	0,87	0,99	0,97	0,92	0,96	0,88	0,78	1,00	0,99	0,96	0,67	0,55	0,47	0,94	0,88	0,80	0,82	0,63	0,45
MAD	0,77	0,50	0,34	0,85	0,79	0,66	1	1	1	0,99	0,98	0,95	0,86	0,78	0,64	1,00	0,99	0,98	0,92	0,88	0,76	0,87	0,72	0,57	0,95	0,91	0,82	0,85	0,70	0,55
VAR	0,85	0,42	0,20	0,92	0,88	0,84	0,99	0,98	0,96	1	1	1	0,93	0,89	0,82	1,00	1,00	0,99	0,99	0,97	0,92	0,85	0,60	0,40	0,96	0,93	0,89	0,84	0,65	0,45
MD	0,92	0,54	0,27	0,97	0,96	0,94	0,92	0,79	0,60	0,95	0,88	0,78	1	1	1	0,95	0,83	0,66	0,99	0,97	0,95	0,81	0,58	0,44	0,92	0,84	0,76	0,83	0,62	0,48
AD	0,82	0,52	0,37	0,89	0,86	0,82	1,00	0,99	0,99	1,00	1,00	0,99	0,89	0,84	0,78	1	1	1	0,96	0,94	0,89	0,86	0,68	0,50	0,98	0,95	0,88	0,84	0,69	0,53
SMDA	0,91	0,55	0,31	0,99	0,98	0,96	0,96	0,89	0,76	0,99	0,97	0,94	0,99	0,98	0,96	0,98	0,93	0,86	1	1	1	0,84	0,63	0,49	0,96	0,93	0,88	0,84	0,67	0,51
SI	0,51	0,38	0,28	0,73	0,63	0,52	0,84	0,79	0,74	0,79	0,72	0,62	0,64	0,58	0,51	0,79	0,74	0,65	0,72	0,63	0,58	1	1	1	0,97	0,92	0,86	0,99	0,99	0,98
VC	0,77	0,52	0,36	0,91	0,86	0,82	0,96	0,95	0,94	0,97	0,96	0,95	0,89	0,85	0,81	0,97	0,96	0,94	0,96	0,92	0,88	0,96	0,88	0,81	1	1	1	0,97	0,92	0,88
GC	0,55	0,40	0,26	0,76	0,68	0,58	0,82	0,79	0,75	0,81	0,76	0,66	0,68	0,63	0,58	0,81	0,77	0,68	0,75	0,69	0,65	1,00	0,99	0,99	0,98	0,95	0,91	1	1	1

Table 4.10: Correlation Coefficient among Measures in the $20X20$ Instances

In Tables 4.9 and 4.10 we report the minimum (*min*), the average (*avr*) and the maximum (*max*) values of the correlation coefficient, for each pair of measures, obtained on all instances solved for the two location spaces (respectively $10X10$ and $20X20$) imposing a different number of opened facilities. We can note very high values, in general, within homogeneous groups (namely, among pairs of normalized measures and pairs of not-normalized ones). Among the selected measures, the Coefficient of Variation (VC) and the Gini Coefficient (GC) show a very high correlation with all the other measures. It is interesting to point out that Schutz's Index (SI) does not appear to be highly correlated to Mean Absolute Deviation (MAD), nevertheless SI represents the normalized version of MAD.

4.8 Conclusion

In this chapter we described a classification of properties that can be associated to equality measures. In addition to properties indicated in the literature, we provided mew ones oriented to describe the behavior of properties in presence of a "regular" distribution of demand points. Then we proposed a measure of similarities between pairs of measures and through an empirical analysis we pointed out how much pairs of measures can be considered correlated.

Chapter 5

The Balancing Two Stage Location Problem

5.1 Introduction

In this chapter we analyze how the equality measures can be used for formulating problems apparently different from classical location problem. In particular we tackle a transportation problem related to flows of material from some origins to a depot whose performance can be formulated introducing an appropriate measure of equality. In this chapter we propose two formulations for the problem, while in the following chapter we show some developed solving procedures.

5.2 The problem

In its classical version, the transportation problem consists in to finding the way of transporting homogeneous product from a set of origins to a set of destinations so that the total cost can be minimized Hitchcock [1999].

For some real-world applications, the transportation problem is often extended to satisfy several other additional constraints or it is performed in two or more stages.

The transportation problem in two-stages was formulated, for the first time by Geoffrion and Graves [1974] and afterwards by many others (i.e. Gen *et al.*, 2006; Hindi and Basta, 1999; Klose, 2000; Marín and Pelegrín, 1997, 1999).

The general idea underlying this problem is the following. In a first stage, the customers' demands are transported from the production plants to certain distribution centers. Plants and distribution centers can be regarded as fixed or their location may form part of the decision problem. Their capacity can be finite or not limited. In a second stage, the demand is transported from the distribution centers to the final customers. The problem to be solved involves designing an optimal distribution structure which takes into account the installation cost of the different plants, distribution centers and depots and the transportation costs associated with both the stages. Any variation such as additional constraints or costs give rise to different versions of the problem.

We propose a new two-stages version of the transportation problem that presents

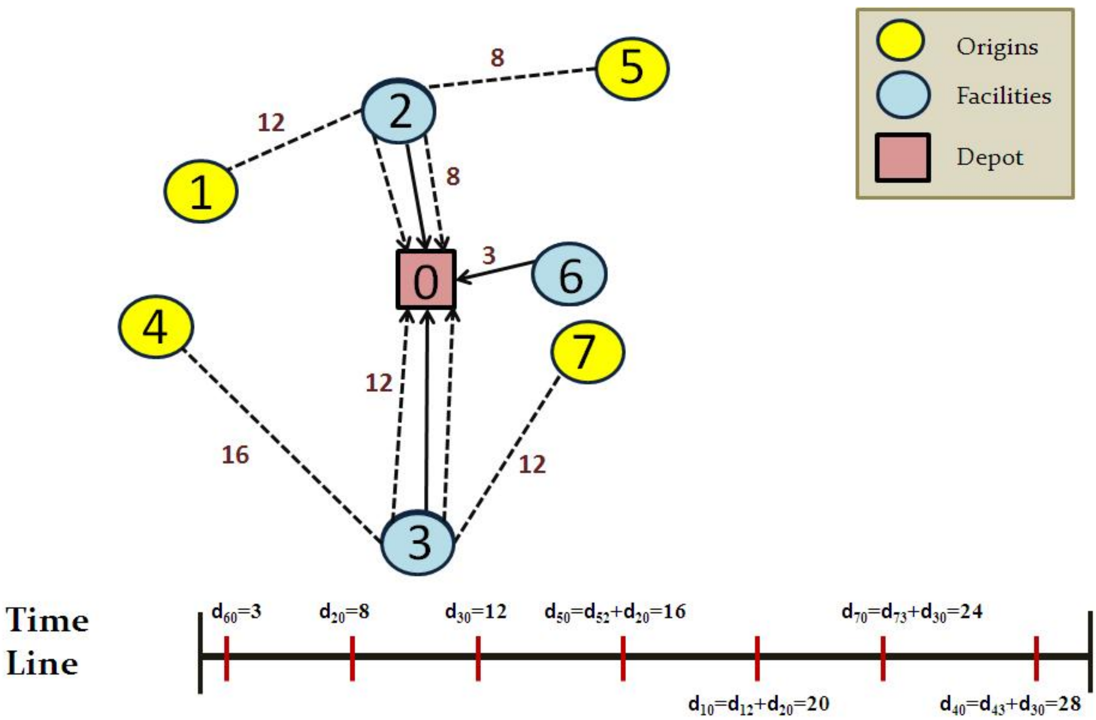

Figure 5.1: Example: Situation A

various particularities. In the first stage we suppose that material goes from a set of origins to some intermediate Distribution Centers (DCs) while in the second stage it has moved from DCs to a depot. While the position of the origins and of the depot is fixed, those of DCs have to be chosen within the set of the positions of origins.

We suppose that the depot has a limited operational capacity so the arrivals of material have to be separated in time as much as possible. So, while in the classical version of a transportation model the objective is the minimization of the transportation and installation cost, in our case the objective is balancing the arrival times of material to the depot, without considering explicitly capacity constraints, but guaranteing that the operational capacity is respected thanks to the maximization of the time between consecutive arrivals.

An example of practical applications is a distribution system of petrol. For instance, assume that petrol has to be sent from several refineries (origins) to pump stations (DCs) where it is pumped and after to a a storage tank (depot), that receive the petrol from all the plants. If the storage tank can not receive all the flows of material for its limited capacity, an appropriate schedule of the arrivals in the time can avoid situation in which the depot can not receive more petrol.

In analogy with the location theory we see the origins a demand points and the DCs as the facilities. So, while the position of the demand points and of the depot is fixed, we have to choose the facilities (plants) to be opened. Once positions of facilities have been determined, we have to allocate each demand point to one of the available facility.

We introduce an example for better showing the problem. In Figure 5.1 (Situation A) we have seven demands points numbered by 1 to 7, that are also potential

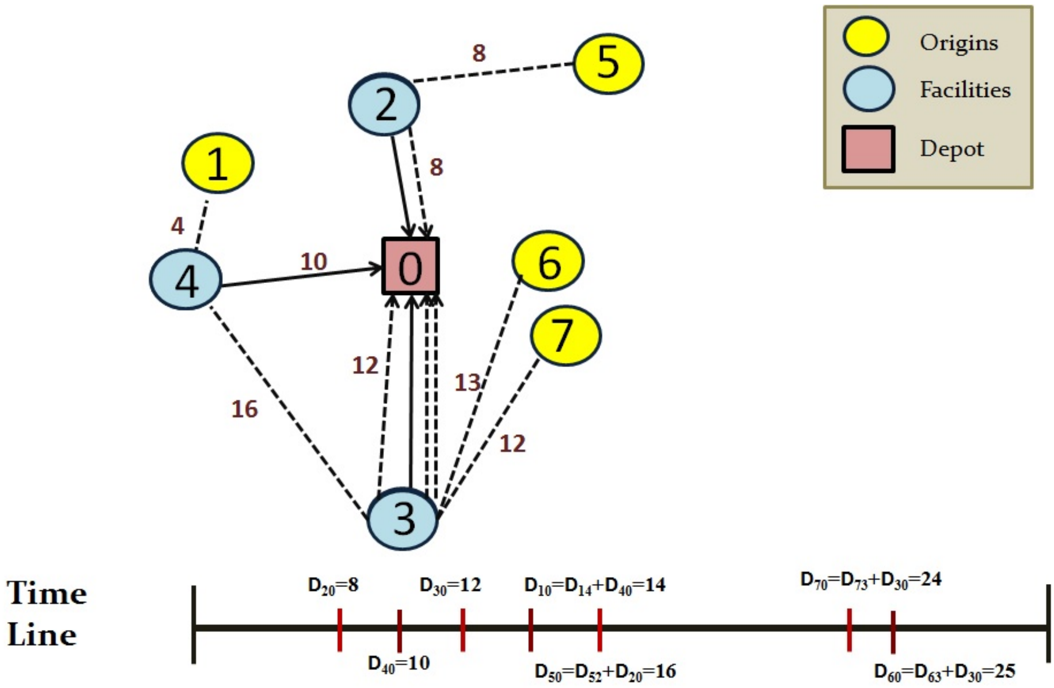

Figure 5.2: Example: Situation B

facilities; among these, the three plants have been represented with blue filled circles (2, 6, 3). The depot has been depicted with a square. The flows represented in the Figure 5.1 show that demand points 2, 6 and 3 have to be covered only the distance among themselves and the depot. This because we suppose that if in a demand point there is a facility, the correspondent demand point will be allocated to that facility. Instead, the demand points 1 and 5 are allocated to the facility 2, while the demand points 4 and 7 to 3. The demand points in which a facility is not located are free to patronize each one of the opened facility.

We suppose that the arrival times are proportional to the distance to cover. So, the total distance to cover for each demand point, representative of the complete arrival time from the origin to the depot, is the sum of the distance from the demand point to the facility at which its is allocated plus the distance from the facility to the depot; in the Figure 5.1 these are indicated, for each demand point, with a segmented arrow at which is associated a value.

We report also on a time line the arrival times of the different demand points. From this we highlight the distribution in the time of the arrivals; if the "space" between consecutive arrivals is more or less equal then the solution respect the limited operational capacity of the depot. The "space" between consecutive time arrivals can be identified as the differences between two consecutive time arrivals.

Think again to the distribution system of petrol. If the storage tank can not adsorb all the flows of petrol at the same time, if the consecutive arrival times are away in the time the flow, once time arrived, can be adsorb from the tank that in this way can be ready to accept other flows of petrol.

While in the situation A the differences among arrival times are more or less the same, instead in situation B, represented in Figure 5.2, they are very different. In

	Situation A	**Situation B**
Origins	*Arrival Times*	*Arrival Times*
1	20	14
2	8	8
3	12	12
4	28	10
5	16	16
6	3	25
7	24	24

Table 5.1: Arrival Times

Situation A		**Situation B**	
Ordered Arrival Times	**Differences**	**Ordered Arrival Times**	**Differences**
3	-	8	-
8	5	10	2
12	4	12	2
16	4	14	2
20	4	16	2
24	4	24	8
28	4	25	1

Table 5.2: Consecutive Arrival Times and Differences

this case we have different positions for the plants and also different flows defined, namely at which facility are allocated the demand points. We can see that, in this case, the arrival times are not equally dispersed in the time as in the first case but, the arrival times in some cases are very near to each others.

Given the aim to do not cause operational stop for the depot we would like to find solutions for the problem more similar to the first one.

For avoiding that arrival times are too near we can try to maintain the differences among consecutive arrival times more bigger as possible. In the Table 5.1 we report the arrival times of the two situations described before.

We can see in Table 5.2 that in the situation A the differences between two consecutive arrival times, calculated as the biggest one minus the smallest, are very similar (equal to 4 except in one case); instead, in the situation B there are very small differences and in particular the last one is equal to one. So the aim of our model can be considered to maximize all the differences among the arrival times, or that is the same, the maximization of the minimum differences among two consecutive arrival times.

We model this two stage-transportation problem as a facility location model in which we choose the facilities to be opened maximizing the minimum difference among consecutive arrival times, constraining just the number of plants to be opened. We call this Balancing Two-Stage Location Problem (BTLP) for putting in evidence the new type of objective function introduced and the two stage characteristics. In the next section we formalize the qualitative elements described until now.

5.3 Elements of the Model

The model is defined with the following elements: (i) a set $J = \{1, ..., M\}$ of demand points which also represent potential locations for the facilities, (ii) a depot sited at point 0, (iii) a fixed number p of facilities (plants) to be located, and (iv) an $M \times (M+1)$ distances matrix $d = (d_{ij})$ that represent either the distance (cost, travel time) between a demand point situated in i and the facility in j (if $i, j \in J$) or the distance from the facility in i and the depot (when $j = 0$). Here we assume $d_{ii} = 0\ \forall i \in J$ and $d_{ij} > 0\ \forall i \in J, j \in J \cup \{0\} :\ i \neq j$.

The aim is to locate p plants among the M candidates and to allocate the remaining $M - p$ demand points to a plant (which is not necessarily the closest one). Let a_i be the plant to which demand point i has been allocated (assuming $a_i = i$ if i is a plant itself). Then, M distances from demand points to the depot will be obtained as $\delta_i := d_{ia_i} + d_{a_i0}\ \forall i \in M$. We call these *travel distances*. The goal is to maximize the minimum difference between two consecutive values in the vector $(\delta_1, \ldots, \delta_M)$.

Note that, when d represents times, the model can be easily extended by considering an additional processing time in the plants. Also note that the minimum distance between two demand points allocated to the same plant will be greater than or equal to the optimal value of the problem, that is to say, the model naturally spaces out the arrivals to the plants. A possible alternative is to force closest allocation of demand points to facilities. To this end, closest allocation constraints (CAC) have to be added to the formulations introduced in the next sections, drastically worsening the solutions of the problem. CAC in discrete location have been deeply studied in Espejo *et al.* [2012], where a complete classification of all possibilities previously considered in the literature was carried out.

The value of the objective function can be determined comparing the travel distance of each demand point with those of the others. Among all these differences the minimum will be the value that the objective function maximizes.

In order to formulate the problem as an Integer Programming model it is important to note that maximizing the minimum difference between consecutive travel distances is equivalent to maximizing the minimum difference between any pair of travel distances (associated with different demand points). The drawback when formulating this problem is to identify which travel distance is greater than or equal to the other, that is to say, absolute values of travel distances differences have to be considered.

From the distances matrix (d_{ij}) we define the matrix (D_{ij}) which measures the distances from point i to the depot through plant j, that is to say,

$$D_{ij} := d_{ij} + d_{j0} \qquad \forall i, j \in J.$$

5.4 Classical Style Formulation for BTLP

What we mean as "classical style formulation" is a model formulated through variables generally used in classical discrete location problems, as shown in the Chapter 1. Then, allocation decisions are represented through the following x-

variables:

$$x_{ij} = \begin{cases} 1 & \text{if demand point } i \text{ is allocated to facility } j \\ 0 & \text{otherwise,} \end{cases} \qquad \forall i, j \in J:\ i \neq j,$$

and location decisions are represented with

$$x_{jj} = \begin{cases} 1 & \text{if a facility is located at point } j \\ 0 & \text{otherwise,} \end{cases} \qquad \forall j \in J.$$

Variable z will represent the minimum difference between travel distances, i.e., the objective function to be maximized. The proposed formulation is

$$
\begin{aligned}
\text{(CBTLP)}\ \max\ & z \\
\text{s.t.}\ & \sum_{j \in J} x_{ij} = 1 && \forall i \in J, && (5.1)\\
& x_{ij} \leq x_{jj} && \forall i, j \in J, i \neq j, && (5.2)\\
& \sum_{j \in J} x_{jj} = p && && (5.3)\\
& z \leq \sum_{\ell \in J} |D_{ia} - D_{j\ell}| x_{j\ell} + && && (5.4)\\
& + (z_{UB} - \min_{\ell \in J} |D_{ia} - D_{j\ell}|)(1 - x_{ia}) && \forall i, j, a \in J:\ i \neq j, && (5.5)\\
& x_{ij} \in \{0, 1\} && \forall i, j \in J. && (5.6)
\end{aligned}
$$

Constraints (5.1) ensure that all the demand points are allocated. Constraints (5.2) ensure that a point receives allocation only if it is a plant. Constraint (5.3) fixes the number of plants to p. Constraints (5.5) are used to obtain the value of the objective function. In particular, due to constraints (5.1), for any $j \in J$ the first addend in the right hand side of (5.5) will take the value $|d_{ia} - d_{j\ell}|$ for that plant l to which j is allocated. If, additionally, site i is allocated to plant a, z will be upperly bounded by $|d_{ia} - d_{j\ell}|$, as wished. Otherwise z will be bounded by z_{UB}, a known upper bound on the optimal value of the problem, plus a non negative amount.

Since an upper bound is necessary for the formulation, we get a trivial one in the following way:

$$z_{UB} := (\max_{i,j \in J}\{D_{ij}\} - \min_{i \in J}\{D_{ii}\})/(M - 1).$$

This formulation has M^2 binary variables and $M^3 + 1$ constraints (excluding binarity constraints). In many other discrete location problems, it suffices with forcing the binarity of x_{jj} $\forall j \in J$, reducing in this way the complexity of some resolution methods. This is not the case with this formulation, as we show in the following example.

Consider an instance with $n = 4$, four points in the plane located at $(2, 2)$, $(1, 1)$, $(1, 4)$ and $(5, 0)$, respectively. The depot is in $(2, 3)$. For ease of computation, we use the Manhattan distance $d((i_1, i_2), (j_1, j_2)) := |i_1 - j_1| + |i_2 - j_2|$. The optimal solution to the instance with $p = 2$ is to locate facilities in points 1 and 4, and to allocate 2 and 3 to 4. The corresponding travel distances are 1, 11, 14 and 6, giving an optimal value of 3. Relaxing in (CBTLP) the integrity of x_{ij} with $i \neq j$, the optimal solution is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

i.e., exactly the same solution except that each half of point 3 is assigned to a different facility. This fractional solution gives an objective value of 5.

5.5 Ordered Formulation for BTLP

We introduced an alternative formulation for the BTLP, using the so-called ordered median formulation, well explained in Nickel and Puerto [2005].

In order to build the ordered formulation, some preprocessing is needed. First, all different potential travel distances $D_{ij} = d_{ij} + d_{j0}$, $i, j \in J$, have to be sorted in (strictly) increasing sequence:

$$D_{(0)} := 0 < D_{(1)} < D_{(2)} < \cdots < D_{(g)} := \max_{i,j \in J}\{D_{ij}\}.$$

We call $G := \{0, \ldots, g\}$ the set of corresponding indexes.

Similarly, all different travel distances from each point i are sorted in increasing order:

$$D^i_{(0)} := 0 < D^i_{(1)} < \cdots < D^i_{(g_i)} := \max_{j \in J}\{D_{ij}\}.$$

The corresponding sets of indexes are named $G_i := \{0, \ldots, g_i\}$, $i \in J$.

Then, for this second formulation we introduce new binary variables y_{ik} and w_{ik}. Allocation decisions are represented through y-variables as follows ($i \in J, k \in G_i$):

$$y_{ik} = \begin{cases} 1 & \text{if the travel distance for point } i \text{ is } D^i_{(k)}, \\ 0 & \text{otherwise,} \end{cases} \quad .$$

The objective function is got from the y-variables through the w-variables, defined as follows ($j \in J, k \in G$):

$$w_{jk} = \begin{cases} 1 & \text{if the } (n-j+1)\text{-th travel distance is less than or equal to } D_{(k)} \\ & \text{and the } (n-j)\text{-th travel distance is strictly greater than } D_{(k)}, \\ 0 & \text{otherwise,} \end{cases} \quad .$$

Then the ordered formulation is

$$
\begin{aligned}
\text{(OBTLP)}\quad & \max z && && \\
\text{s.t.}\quad & \sum_{k\in G_i} y_{ik} = 1 && \forall i\in J, && (5.7)\\
& \sum_{i\in J} y_{i1} = p && && (5.8)\\
& y_{ik} \le \sum_{\substack{j\in J\\ D_{ij}=D^i_{(k)}}} y_{j1} && \forall i\in J, k\in G_i, && (5.9)\\
& \sum_{j\in J} j\cdot w_{jk} = \sum_{i\in J}\sum_{\ell\ge \ell^i_k} y_{i\ell} && \forall k\in G, && (5.10)\\
& \sum_{j\in J} w_{jk} \le 1 && \forall k\in G, && (5.11)\\
& z \le \sum_{k=1}^{g-1} D_{(k)}(w_{jk}-w_{j,k+1}) && \forall j\in J, && (5.12)\\
& y_{ik}\in\{0,1\} && \forall i\in J, k\in G_i, && (5.13)\\
& w_{jk}\in\{0,1\} && \forall j\in J, k\in G, && (5.14)
\end{aligned}
$$

where

$$
\ell^i_k := \begin{cases} \min\{s \;:\; D^i_{(s)} \ge D_{(k)}\} & \text{if } D_{(k)} \le D_{(g_i)},\\ g_i+1 & \text{otherwise.}\end{cases}
$$

Constraints (5.7) ensure that all demand points have to be allocated at some given distance $D^i_{(k)}$. Constraint (5.8) ensures that exactly p plants are opened (note that y_{i1} is equal to 1 if and only if the demand point i is allocated with a distance $D^i_{(1)}$, i.e., allocated to itself). Constraints (5.9) make sure that $y_{ik} = 0$ if no plant is opened at a distance equal to $D^i_{(k)}$. The w_{ik} variables are introduced in order to calculate for each demand point the difference between the travel distance of demand point i and the nearest travel distance of the other points. We need to link these variables with the allocation variables y_{ik}. This can be done forcing variables w_{ik}, for each demand point i, to be equal to one for all k for which $D_{(k)}$ is less than or equal to its travel distance, and greater than the travel distance of another demand point (namely when variable y_{ik} is equal to one for a different demand point). Constraints (5.10) together with (5.11) assure this relationship. Constraints (5.12) force the objective function to assume the correct value. Using telescopic sum for the variables w_{ik} we assure for each demand point the evaluation of the minimum difference among the travel distance from the other demand points. Constraints (5.13) and (5.14) ensure the binarity of the variables.

The number of variables and constraints in (OBTLP) depends on the number of different potential travel distances. In the worst case (all potential travel distances different), this formulation has $M^3 + M^2$ variables and $3M^2 + 2M + 1$ constraints (excluding binarity constraints). Binarity of the y-variables can be relaxed, since they are linked to w-variables only through constraints (5.10) and not present in the objective function. On the contrary, even under the binarity of y-variables, binarity of w-variables cannot be relaxed.

Consider again the data of the previous example, for CBTLP. Relaxing the binarity of w-variables leads to the solution

$$y = \begin{pmatrix} 1 & 0 & 0 & - \\ 0 & 0 & 1 & - \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & - \end{pmatrix} \qquad w = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 1 & 0.25 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0.25 & 0 & 0 \end{pmatrix}$$

with an objective value of 4.

5.6 An Illustrative Example for the OBTLP Formulation

Let $J = \{1, \ldots, 4\}$ and assume $p = 2$ plants to located. Let the potential travel distances matrix D_{ij} be as follows:

$$(D_{ij}) = \begin{pmatrix} 4 & 11 & 13 & 12 \\ 13 & 6 & 10 & 8 \\ 12 & 7 & 3 & 10 \\ 16 & 9 & 8 & 7 \end{pmatrix}.$$

Sorting all different values in this matrix we obtain

$$(D_{(1)}, \ldots, D_{(11)}) = (3, 4, 6, 7, 8, 9, 10, 11, 12, 13, 16)$$

with $G = \{1, \ldots, g = 11\}$, while sorting each row we get $G_i = \{1, \ldots, g_i = 4\}$, $i = 1, \ldots, 4$, $(D^1_{(k)}) = (4, 11, 12, 13)$, $(D^2_{(k)}) = (6, 8, 10, 13)$, $(D^3_{(k)}) = (3, 7, 10, 12)$, $(D^4_{(k)}) = (7, 8, 9, 16)$. In the optimal solution the values of the y-variables are:

$$(y_{ik}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The location of facilities is determined by the variables in the first column taking value 1 (y_{11} and y_{41}). Points 1 and 4 are allocated to themselves, so the travel distance from 1 is $D^1_{(1)} = 4$ while for 4 is $D^4_{(1)} = 7$. Instead, demand points 2 and 3 are allocated, respectively, to 1 with $D^2_{(4)} = 13$ and 4 with $D^3_{(3)} = 10$, corresponding to variables y_{24} and y_{33}.

Since y-variables are defined only in $k \in G_i$, in (5.10) we use the index ℓ^i_k. For example, for $i = 3$ we have

k	1	2	3	4	5	6	7	8	9	10	11
$D_{(k)}$	**3**	4	6	**7**	8	9	**10**	11	**12**	13	16
ℓ^3_k	1	2	2	2	3	3	3	4	4	5	5
$y_{3\ell^3_k}$	0	0	0	0	1	1	1	0	0	0	0

In this way, constraint (5.10) with $k = 7$ is

$$w_{17} + 2w_{27} + 3w_{37} + 4w_{47} = y_{12} + y_{13} + y_{14} + y_{23} + y_{24} + y_{33} + y_{34} + y_{44} = 2.$$

From (5.11), it follows $w_{27} = 1$.

	$D_{(1)}$ 3	$D_{(2)}$ **4**	$D_{(3)}$ 6	$D_{(4)}$ **7**	$D_{(5)}$ 8	$D_{(6)}$ 9	$D_{(7)}$ **10**	$D_{(8)}$ 11	$D_{(9)}$ 12	$D_{(10)}$ **13**	$D_{(11)}$ 16
y_{ik}	·	1	·	·	·	·	·	0	0	0	·
	·	·	0	·	0	·	0	·	·	1	·
	0	·	·	0	·	·	1	·	0	·	·
	·	·	·	1	0	0	·	·	·	·	0
$\sum_{\ell=\ell_k^i}^{g_i} y_{i\ell}$	1	1	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	1	1	0
	1	1	1	1	1	1	1	0	0	0	0
	1	1	1	1	0	0	0	0	0	0	0
w_{jk}	0	0	0	0	0	0	0	1	1	**1**	0
	0	0	0	0	1	1	**1**	0	0	0	0
	0	0	1	**1**	0	0	0	0	0	0	0
	1	**1**	0	0	0	0	0	0	0	0	0

Figure 5.3: Relationship between y- and w-variables

The Table provided in Figure 5.3 shows the values of the variables y_{ik}, $\sum_{\ell=\ell_k^i}^{g_i} y_{i\ell}$ and w_{jk} and how they are linked. We can visualize, for example, that the second row of sums of y-variables is linked to the first row of w-variables. In this way, the rows of w-variables will represent the travel distances sorted in decreasing order.

Moreover, for each pair of demand points the differences between the two $D_{(k)}$-values corresponding to the last $w_{ik} = 1$ (depicted bold) are the differences between travel distances. With the telescopic sum in (5.12) we evaluate for each demand point only the minimum difference. For example, for $j = 3$

$$z \leq D_{(1)}(w_{31} - w_{32}) + D_{(2)}(w_{32} - w_{33}) + \ldots + D_{(10)}(w_{3,10} - w_{3,11}) + D_{(11)}w_{3,11} =$$

$$3{\cdot}(0-0)+4{\cdot}(0-1)+6{\cdot}(1-1)+7{\cdot}(1-0)+8{\cdot}(0-0)+9{\cdot}(0-0)+\ldots+16{\cdot}0 = 7-4 = 3,$$

the difference between the second and third travel distances.

5.7 Valid Inequalities for OBTLP

We introduce fixing variable and valid inequalities for improving the OBTLP formulation. We also define a separation procedure for selecting only some of them. For each family we also propose a small example in order to show how they works.

First of all, on the basis of the variable meaning and structure we can trivially fix some w-variables. First, $w_{M1} = 1$. Similarly, the w-variables in the right bottom corner and in the left upper corner can be fixed to 0. Equivalently, we can add the following constraints to the formulation:

$$\sum_{j=M-k+1}^{M} w_{jk} = 1 \qquad \forall k = 1, \ldots, M-1, \tag{5.15}$$

BTLP has some common aspects with other ordered discrete location problems approached in the literature (Nickel and Puerto, 2005), for which several valid inequalities have been developed and tested. Less obvious, there are also common aspects with several optimization problems in the field of Bioinformatics, like protein sequence alignment. We consider here several families of valid inequalities for (OBTLP). The first family, given by

$$\sum_{\ell \in J} \min\{a, \ell\} w_{\ell k} \geq \sum_{i \in A} \sum_{\ell=\ell_k^i}^{g_i} y_{i\ell} \qquad \forall k \in G,\ \forall A \subseteq J:\ |A| = a, \tag{5.16}$$

is based on similar constraints used in Marín *et al.* [2009] for the Discrete Ordered Median Problem.

Although there are $O(M^2 2^M)$ inequalities in family (5.16), an efficient separation procedure can be implemented. Starting with an optimal solution (y^*, w^*) to the linear relaxation of (OBTLP), for each $k \in G$ we sort $(\sum_{\ell=\ell_k^1}^{g_1} y_{1\ell}^*, \ldots, \sum_{\ell=\ell_k^M}^{g_M} y_{M\ell}^*)$ in decreasing order, thus getting $Y_1 \geq \ldots \geq Y_M$. Then, for all $a \in J$, if $\sum_{\ell \in J} \min\{a, \ell\} w_{\ell k}^* < \sum_{i=1}^{a} Y_i$, we add to the formulation the inequality in (5.16) corresponding with the current value of k and the a maximum values of Y_i.

Consider an instance with $M = 4$, $p = 2$, distances

$$\begin{pmatrix} 4 & 10 & 8 & 6 \\ 11 & 3 & 7 & 9 \\ 8 & 6 & 4 & 6 \\ 7 & 9 & 7 & 3 \end{pmatrix}$$

where $g = 8$ and $D_{(k)} = (3, 4, 6, 7, 8, 9, 10, 11)$. The optimal solution to the linear relaxation of (OBTLP) (including constraints (5.15)), with optimal value 2.2, has y- and w-values

$$\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & - \\ 1 & 0 & 0 & - \end{array} \qquad \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.66 & 0.53 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

The separation algorithm only identifies the first violated inequality for each column k. There are two columns in this case. For $k = 4$ (respectively $k = 5$) the inequality corresponds with $|A| = 1$ (resp. $|A| = 1$), namely

$$w_{14} + w_{24} + w_{34} + w_{44} \geq y_{23} + y_{24},$$

$$w_{15} + w_{25} + w_{35} + w_{45} \geq y_{23} + y_{24}.$$

After re-optimizing, the linear relaxation gives optimal value 2, with the same y-values as before and w-values

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0.5 & 0.5 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0.5 & 0.5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0. \end{array}$$

Two more violated inequalities, corresponding with $k = 4$, $|A| = 2$ and $k = 5$, $|A| = 2$ are detected. The re-optimization of the extended formulation gives the optimal integer solution to the instance, with optimal value 1, the same y-values as before and w-values

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0. \end{array}$$

The second family of valid inequalities is

$$\sum_{j=1}^{M} w_{jk_j} \leq 1 \qquad \forall k_1 \leq k_2 \leq \ldots \leq k_M. \tag{5.17}$$

Since the w-variables, taking value 1 in any feasible solution, has a ascending stair shape when organized as in Figure 5.3, any two w-variables cannot take value 1 simultaneously if one of them is on the right and above (or at the same level) with respect to the other. That is to say, all the variables in a descending stair (like in (5.17)) are jointly bounded by 1. A similar situation happens in the comparison of two proteins whose amino acids are represented by points in two parallel lines. Here, points in both lines must be linked by a segment without crossing lines. Variables which are similar to w are used to determine which pairs of points are linked, and they arrange in an (strictly) ascending stair. This problem has been studied in Lancia [2004].

There are $(g+M-1)!/(M!(g-1)!)$ constraints if family (5.17), so we have devised a separation procedure also for this family. Starting with constraint

$$\sum_{j=1}^{M} w_{j1} \leq 1,$$

from a fractional solution w^* we successively get valid inequalities in (5.17) with larger left hand sides by comparing $\sum_{j=1}^{M} w^*_{jk_j}$ with

$$\sum_{j=1}^{M} w^*_{j,k_j+a^\ell_j}, \qquad \forall \ell = 1, \ldots, M$$

where $a^\ell_j = 1$ for all $j \geq \ell$ such that $k_j = k_\ell$, $a^\ell_j = 0$ otherwise. The process stops after one step without improvement. If the left hand side of the last inequality is greater than 1, the inequality is added to the formulation in the corresponding node of the branching tree. We then repeat the process starting with $\sum_{j=1}^{M} w_{j2} \leq 1$ and so on. After re-optimizing, separation is carried out again using the new optimal fractional solution.

Consider an instance with $M = 4$, $p = 2$, distances

$$\begin{pmatrix} 2 & 6 & 4 & 6 \\ 5 & 3 & 7 & 5 \\ 4 & 8 & 2 & 8 \\ 5 & 5 & 7 & 3 \end{pmatrix}$$

where $g = 7$ and $D_{(k)} = (2, 3, 4, 5, 6, 7, 8)$. The optimal solution to the linear relaxation of (OBTLP) (including constraints (5.15)) has w-values

0	0	0	0	1	1	1
0	0	1	0.4	0	0	0
0	1	0	0.4	0	0	0
1	0	0	0	0	0	0

and optimal value 1.4. The first violated inequality found by the procedure is

$$w_{13} + w_{23} + w_{37} + w_{47} \leq 1.$$

Adding this constraint, the optimal value 1 is found, corresponding with the w-values

0	0	0	0	1	1	1
0	0	1	1	0	0	0
0	1	0	0	0	0	0
1	0	0	0	0	0	0.

The third family includes a small number of inequalities which can be added to the formulation without separation:

$$\sum_{j \in J} w_{jk} \geq \sum_{j \in J} w_{j,k+1} \qquad \forall k = 1, \ldots, g-1.$$

The fourth family is, in some sense, complementary with respect to the first one:

$$\sum_{\ell=M+1-a}^{M} (\ell + a - M) w_{\ell k} \leq \sum_{i \in A} \sum_{\ell=\ell_k^i}^{g_i} y_{i\ell} \qquad \forall k \in G, \ \forall A \subseteq J : \ |A| = a, \tag{5.18}$$

Again the cardinality of this family is $O(M^2 2^M)$. The separation procedure is similar to that of the first family. Starting with an optimal solution (y^*, w^*) to the linear relaxation of (OBTLP), for each $k \in G$ we sort $(\sum_{\ell=\ell_k^1}^{g_1} y^*_{1\ell}, \ldots, \sum_{\ell=\ell_k^M}^{g_M} y^*_{M\ell})$ in increasing order, thus getting $Y'_1 \leq \ldots \leq Y'_M$. Then, for all $a \in J$, if $\sum_{\ell=M+1-a}^{M} (\ell + a - M) w^*_{\ell k} > \sum_{i=1}^{a} Y'_i$, we add to the formulation the inequality in (5.18) corresponding with the current value of k and the a minimum values of Y'_i.

Consider an instance with $M = 4$, $p = 2$, distances

$$\begin{pmatrix} 3 & 9 & 3 & 7 \\ 8 & 4 & 4 & 4 \\ 6 & 8 & 0 & 4 \\ 8 & 6 & 2 & 2 \end{pmatrix}$$

where $g = 8$ and $D_{(k)} = (0, 2, 3, 4, 6, 7, 8, 9)$. The optimal solution to the linear relaxation of (OBTLP) (including constraints (5.15)) has y- and w-values

0	0	1	–
1	0	–	–
0	0	0	1
1	0	0	–

0	0	0	0	0	0	0	1
0	0	0	0	0.222	1	1	0
0	0	1	0.407	0.519	0	0	0
1	1	0	0.444	0	0	0	0

and optimal value 2.44. The first violated inequalities found by the procedure are

$$w_{44} \leq y_{42} + y_{43},$$

$$w_{35} + 2w_{45} \leq y_{22} + y_{42} + y_{43}.$$

Adding these constraints, a better optimal value of 2.18 is found, corresponding with the w-values

0	0	0	0	0	0.666	0.333	1
0	0	0.185	0	0.666	0	0.666	0
0	0.333	0.629	0.555	0	0.333	0	0
1	0.666	0.185	0.333	0.166	0	0	0.

The next iteration produces inequalities

$$w_{34} + 2w_{44} \leq y_{42} + y_{43} + y_{32} + y_{33} + y_{34},$$

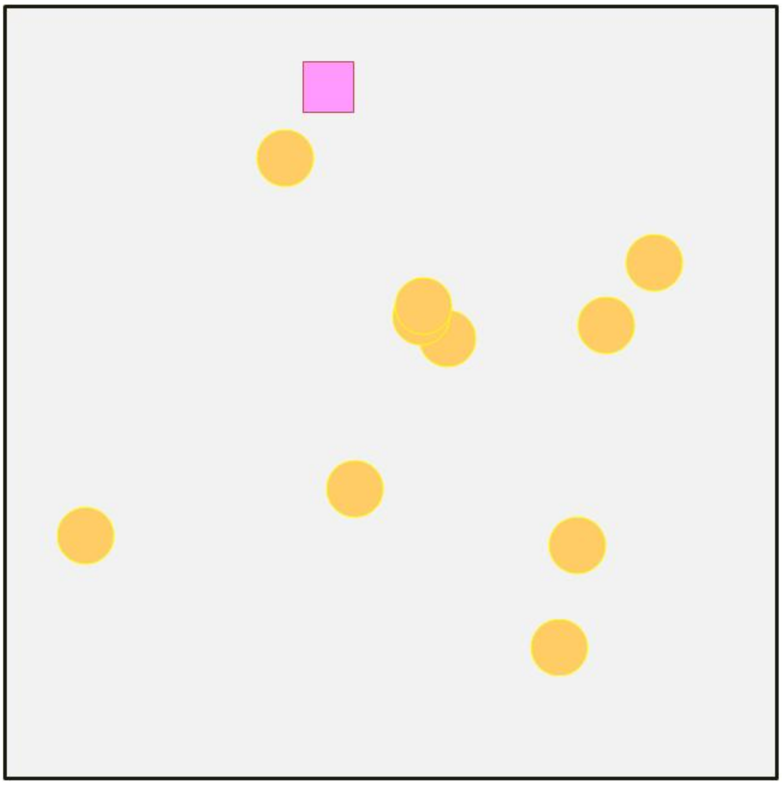

Figure 5.4: Instance: M=10, Depot in Random Position

$$w_{45} \leq y_{22},$$
$$w_{36} + 2w_{46} \leq y_{22} + y_{43},$$

and the optimal value 2.

The last family is also given by a small number of inequalities which can be added to the formulation without separation:

$$\sum_{i \in J} \sum_{\ell = \ell_k^i}^{g_i} y_{i\ell} \leq 1 + \sum_{j=2}^{M} (j-1) w_{jk} \qquad \forall k \in G.$$

5.8 Computational Study

Since, to the best of our knowledge, this is the first definition of BTLP we have to define new instances in order to test our models.

We decide to generate them in this way. In a location space delimited in a square of a fixed dimension we generate randomly demand points and we fix the depot in three different position: in the center of the square, in the right-low corner of the square and in a random position of the space.

In Figure 5.4 is reported one of the instances defined, with the depot fixed in a random position and M =10.

The testbed is composed of nine instances three instances for each position occupied by the depot. We tested different combinations of M in {10, 15, 20, 25, 30} with different values of p depending on the case.

Instance		Xpress					
		Time (s)			Nodes		
M	p	Max	Avr	Min	Max	Avr	Min
10	2	0,81	0,70	0,60	139	100	58
10	4	0,72	0,58	0,48	145	82	35
10	6	0,53	0,39	0,25	117	53	15
15	2	8,63	5,93	4,71	769	527	276
15	4	10,60	8,02	4,90	2421	1487	511
15	6	6,74	4,90	3,45	1455	807	293
20	2	47,93	37,18	22,20	2253	1698	547
20	4	598,22	333,80	161,76	66150	37512	14674
20	6	464,37	176,52	78,98	56169	25094	11801
20	10	75,08	35,45	18,53	10568	3777	1320
25	2	294,70	201,49	98,94	8563	4860	1800
25	4	> 2h	> 2h	> 2h	> 2h	> 2h	> 2h
25	6	> 2h	> 2h	> 2h	> 2h	> 2h	> 2h
25	12	717,07	235,84	35,37	45111	12941	53
30	2	1145,45	746,97	475,85	17391	9571	6055
30	4	> 2h	> 2h	> 2h	> 2h	> 2h	> 2h
30	6	> 2h	> 2h	> 2h	> 2h	> 2h	> 2h
30	15	4947,73	1635,16	457,56	161071	48814	8695

Table 5.3: Computational Results for CBTLP with Xpress (no cut strategy applied)

We considered among the demand points and the facilities and the depot Euclidean distances. For instance, given the coordinates of a a demand point (a_i, b_i) that is allocated to a facility in a point j with coordinates (a_j, b_j) the travel distance within the depot in the center of coordinates (a_c, b_c) is:

$$D_{ij} := [|a_i - a_j|^2 + |b_i - b_j|^2]^{1/2} + [|a_j - a_c|^2 + |b_j - b_c|^2]^{1/2}$$

.

The formulations were implemented in the commercial solver Xpress IVE running on a Pentium IV with 2.40 GHz and 4 GB of RAM memory. In Table 5.3 we report the maximum, the average and the minimum computational times in seconds of the overall solution process and the maximum, the average and the minimum number of nodes of the branching tree obtained with the first formulation. The time limit was fixed to two hours of CPU; in the Table "> 2h" indicates that this time has been exceeded, at least, in solving one of the instances.

The results have been obtained disabling the cut generation option of Xpress in order to show the performance of the formulation cleanly. We can see that the time needed to solve the problem for small instance is very low, but it increases a lot increasing the number of demand points. The time also increases when we increment the number of facilities to be opened but decreases when this become larger; in fact for the instances with 25 and 30 demand points we are not able to solve the problem with p equal to 4 and 6 while we obtain the optimal solutions with a value of p equal to $M/2$.

Instance		Xpress + Cut					
		Time (s)			Nodes		
M	p	Max	Avr	Min	Max	Avr	Min
10	2	1,55	0,96	0,51	41	12	1
10	4	1,94	0,74	0,18	11	3	1
10	6	0,23	0,18	0,16	1	1	1
15	2	7,92	6,51	5,01	210	102	7
15	4	10,21	7,92	6,02	849	430	210
15	6	8,23	4,74	1,43	177	81	1
20	2	22,92	17,91	12,25	386	279	165
20	4	126,63	85,50	53,79	16724	7582	2815
20	6	109,57	61,06	27,38	20628	6648	1639
20	10	21,67	7,70	3,18	217	25	1
25	2	62,17	54,31	40,61	675	564	470
25	4	2403,36	1244,356	598,85	150271	68646	33993
25	6	> 2h	> 2h	> 2h	> 2h	> 2h	> 2h
25	12	283,78	91,26	9,49	22380	5490	1
30	2	184,97	158,75	145,14	1374	1035	802
30	4	> 2h	> 2h	> 2h	> 2h	> 2h	> 2h
30	6	> 2h	> 2h	> 2h	> 2h	> 2h	> 2h
30	15	59,86	46,52	31,00	769	95	1

Table 5.4: Computational Results for CBTLP with Xpress Default

Instance		Xpress						Branch and Cut					
		Time (s)			Nodes			Time (s)			Nodes		
M	p	Max	Avr	Min	Max	Avr	Min	Max	Avr	Min	Max	Avr	Min
10	2	2898	1037	41	231163	91168	25429	2303	819	59	262434	128246	30273
10	4	1695	628	166	163100	113936	64477	1853	690	96	272273	136895	40427
10	6	2698	390	2	92560	36276	701	931	168	17	262434	99370	2395

Table 5.5: Computational Results for OBTLP

In Table 5.4 we provide results using default cut strategy of software Xpress that improves computational times and the number of nodes analyzed that decrease for all instances. It also solves instances with 25 demand point and $p = 4$.

The second formulation is less efficient. We are able to find only solutions for M=10 and not for all the other instances, as we can see in Table 5.5. In addition for all the instances there is at least one in which the software does not find the optimal solution in the time limit of two hours. Using the formulation with the branch and cut illustrated in Section 5.7 we improve these results in terms of computational times and number of nodes. Only in one instance for p equal to 6 the problem was unsolved in the time limit imposed.

5.9 Conclusion

In this chapter we have defined a new location model for representing a two-stage transportation problem. We approach this model in a innovative way using an equality measure as objective function in order to reach the balancing among the arrival times of the material to the depot. We have proposed two formulations, one of classical type and one based on so-called ordered formulation, putting in evidence the computational complexity and also showing how they work. We also have formulated valid inequalities for the second formulation. In the next chapter we propose heuristic approaches for finding solutions for the problem.

Chapter 6

Procedures for the Balancing Two Stage Location Problem

6.1 Introduction

In this chapter we introduce two different heuristics for finding solutions for the Balancing Two Stage Location Problem. We first explain the implementation of the genetic algorithm. Afterwards we provide the steps of a greedy procedure. We also describe how we can use them in combination giving the solutions derived from the greedy algorithm as initial solutions for the genetic procedure. We show the results obtained by the empirical analysis.

6.2 Genetic Algorithm

Genetic Algorithm (GA) are a family of methodologies inspired by evolution (Whitley, 1994). These algorithms encode a potential solution to a specific problem on a simple chromosome-like data structure and apply recombination operators to these structures so as to preserve critical information. Genetic algorithms are often viewed as function optimizers, although GA have been applied to many problems. An implementation of a GA begins with a population of chromosomes. One then evaluates these structures and allocates reproductive opportunities in such a way that those chromosomes which represent a better solution to the target problem are given more chances to reproduce than those chromosomes which are poorer solutions. The goodness of a solution is typically defined with respect to the current population.

So, the first step in the implementation of any genetic algorithm is to generate an initial population. In GA each member of this population will be a string of length L which corresponds to the problem encoding. Each string is referred to a "chromosome". In most cases the initial population is generated randomly. After creating an initial population, each string is then evaluated and assigned a fitness value.

The notion of evaluation and fitness are sometimes used interchangeably (Whitley, 1994). However, it is useful to distinguish between the evaluation function and the fitness function used by a GA. The fitness function transforms that measure of performance into an allocation of reproductive opportunities. The evaluation of a string representing a set of parameters is independent of the evaluation of any other string. The fitness of that string, however, is always defined with respect to other members of the current population.

In GA fitness is defined by $\frac{f_i}{\bar{f}}$ where f_i is the evaluation associated with string i and $\bar{f}$ is the average evaluation of all the strings in the population. Fitness in other ways for example with a tournament selection.

The execution of GA can be divided in steps. It starts with the current population. After selection is applied to the current population to create an intermediate population. Then recombination and mutation are applied to the intermediate the population to create the next population. The process of going from the current population to the next population is a generation in the execution of GA.

In the first generation the current population is also the initial population. After calculating $\frac{f_i}{\bar{f}}$ for all the strings in the current population, selection is carried out. In GA the probability that strings in the current population are copied and placed in the intermediate generation is proportion to their fitness. There are a number of ways to do selection. We might view the population as mapping onto a roulette wheel, where each individual is represented by a space that proportionally corresponds to its fitness. By repeatedly spinning the roulette wheel, individuals are chosen using stochastic sampling with replacement to fill the intermediate population.

After selection has been carried out the construction of the intermediate population is complete and recombination can occur. This can be viewed as creating the next population from the intermediate population. Crossover is applied to randomly paired chromosomes with a given probability. After recombination we can apply a mutation operator. Each element of the chromosome in the population, is mutated with a given low probability (mutation probability). Typically the mutation rate is applied with less than 1 % probability.

After the process of selection, recombination and mutation is concluded the next population can be evaluated. The process of evaluation, selection, recombination and mutation forms one generation in the execution of a genetic algorithm.

In the following we define the characterizing elements of GA: encoding, crossover and mutation operators.

6.2.1 Encoding

In order to explain the encoding scheme, let us consider the solution shown in Figure 6.1, with a total number of 8 nodes with 3 opened facilities (1, 5, 8). Each demand point is allocated to one of the opened facilities as indicated by the arrows in the Figure. In particular nodes 1, 2, 4, 6 are allocated to node 1 while nodes 3 and 5 are allocated to node 5 and nodes 7 and 8 are allocated to 8. This allocation scheme can be represented through a string of 8 elements in which the generic element j indicates the node patronized by demand point j. With reference to the mentioned solution the corresponding string is:

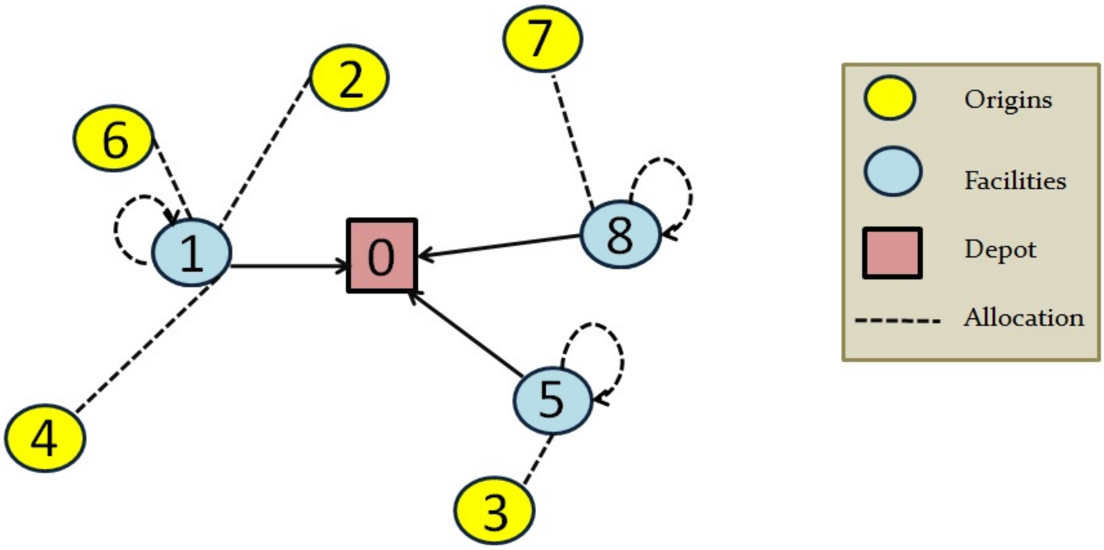

Figure 6.1: A solution for BTLP

Allocation	1	1	5	1	5	1	8	8
Demand Points	1	2	3	4	5	6	7	8

In order to effectively encode either the opened facilities and the allocation of each demand point, we consider a string of M genes divided in two substrings. The first substring (of length p) is representative of the p opened facilities. In our case this substring is given by

Substring	8	1	5
Index	1	2	3

It is useful to stress that facilities can appear in this substring in any order.

The second substring (of length $M - p$) should indicate the allocation of the demand points different from those corresponding to the opened facilities. In our example the demand points are 2, 3, 4, 6, 7. In order to do that each generic element k of this substring can assure a value between 1 and p which represents the index value of the first substring. In practice in our example, the final encode would result

	8	1	5
Index	1	2	3

	2	3	2	2	1
Demand Point	2	3	4	6	7

For instance, the value 2 associated to the demand point 4 is allocated to the second element of the first substring, corresponding to node 1.

6.2.2 Operators

Another peculiarity in the implementation of a genetic algorithm is given by the genetic operator; in our proposal we consider the two classical operators (crossover and mutation).

Crossover Operator

We implemented two different schemes for the two considered substrings. For the first we distinguish between "fixed" gens and "probable" genes. Fixed genes are those (if existing) comparing in both parents. For each children each of these genes is assigned to an element randomly chosen. The probable genes are used to fill the remaining part of the substring in random way.

For instance let us consider the two parents substrings:

Parent 1	8	1	5
Parent 2	1	2	4

The fixed genes are represented by node 1 while the set of probable genes are given by nodes 2, 4, 5, 8. Then node 1 randomly assigned to some elements of child 1 as for instance the following.

Child	1		

then nodes 2, 4, 5, 8 are randomly chosen to fill the empty parts of the substrings, providing, for instance, the substring

Child	1	2	5

The same procedure is applied to generate the child 2.

For the second substring we used the classical crossover implementation by randomly generating a cross point and combining the first part of the Parent 1 and the second part of the Parents1 to obtain the child 1 and viceversa for the child 2. As an example considering the two substrings in Figure 6.2.

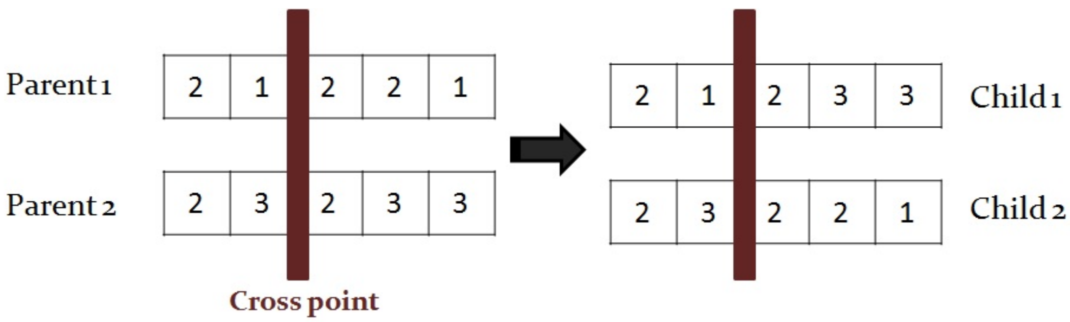

Figure 6.2: Crossover Second Substring

Mutation Operator

We apply a mutation operator in different ways in the two substrings. In the first substring we randomly select one of the opened facility and we replace it with one not opened. Then the demand points that were allocated to the closed plant will be randomly allocated to one of opened plants . In the second part we randomly change the allocation of some demand points allocating.

6.2.3 Setting Parameters

For the implementation of the algorithm we need to set some parameters as

- Number of iterations;
- Population size;
- Crossover Probability;
- Mutation Probability.

In particular for the mutation probability we propose to adapt it to the diversification of the population. In other words, when the population is composed by elements that are all similar, at least with the same fitness value, we increase the mutation probability in order to guarantee the diversification of the population. We also regenerate some or all elements of the population after a specified number of iterations for which the average fitness of all chromosomes it is the same.

6.3 The Greedy Heuristic

The algorithm developed is classifiable among the greedy heuristic approaches in which a solution is constructed adding each element at a time. The procedure proposed is divided in two steps. First we choose the set of p facilities to be opened and after we decide the allocation the other demand points.

For the first step we initialize the algorithm imposing that all the facilities are opened; so in the initial solution the travel distances from each demand point correspond to the direct distance from each point to the depot (this list is indicated with TL). After having sorted these distances in increasing way, obtaining TL^0, we evaluated the differences between each couple of consecutive elements. In order to have the minimum difference between travel distances as large as possible we want that the minimum difference individuated in TL^0 will not be in the final solution. For this reason, we eliminate from TL^0 one of the two travel distances that determine this minimum difference, choosing one of them at random. In this way, we impose that at each eliminated distance we can not put a facility. We iterate this procedure until the number of element in TL^0 is equal to M (Figure 6.3).

In the second step of the heuristic we have to allocate the demand points to the opened facilities.

Suppose that we have opened facilities 1,2,3. So we have considered In the following we show an example of how the second step of the heuristic works.

Let $J = \{1, \ldots, 8\}$ and assume $p = 3$ plants to located. Let the travel distances

Step 1: Opening Facilities

For $i \in 1, \ldots, M$
  Set $TL(i) = D_{ii}$
end-for
Set TL^0 TL sorted in increasing way
repeat Set $i^* = \min_{1,\ldots,|TL^0|-1}(TL^0(i+1) - TL^0(i))$
  Choose at random between i^* and i^*+1
  Eliminate the correspondent element from TL^0
until $|TL^0| = p$

Figure 6.3: Greedy Algorithm for Opening Facilities

matrix D_{ij} be as follows:

$$(D_{ij}) = \begin{pmatrix} 20 & 57 & 42 & 134 & 64 & 75 & 93 & 102 \\ 88 & 46 & 93 & 152 & 98 & 132 & 153 & 76 \\ 34 & 76 & 35 & 176 & 75 & 91 & 81 & 93 \\ 201 & 55 & 114 & 98 & 112 & 142 & 98 & 78 \\ 185 & 88 & 146 & 108 & 28 & 167 & 142 & 53 \\ 120 & 63 & 184 & 159 & 32 & 56 & 118 & 42 \\ 77 & 72 & 165 & 167 & 31 & 79 & 77 & 82 \\ 149 & 67 & 175 & 104 & 87 & 87 & 170 & 18 \end{pmatrix}.$$

Suppose we have opened facilities 1, 3, 7. So we have to consider the distances of the remaining demand pints to these facilities. In this way we have only these travel distances indicated in the following matrix

$$(D_{ij}) = \begin{pmatrix} \mathbf{20} & & \\ 88 & 93 & 153 \\ & \mathbf{35} & \\ 201 & 114 & 98 \\ 185 & 146 & 142 \\ 120 & 184 & 118 \\ & & \mathbf{77} \\ 149 & 175 & 170 \end{pmatrix}.$$

In practice as each opened facility is allocated to itself we have to fix the value for each row (2, 4, 5, 6, 8) in such a way that the minimum of the mutual distances will be as large as possible. In order to do that, we sort all the elements in a vector in an increasing way. For each element we also indicate the indexed corresponding to the position in the matrix D_{ij}. For instance the element of value 120 is the element of row 6 and column 1 in the matrix D_{ij}. In this vector we individuate the pair of consecutive elements which provides the minimum difference. In our example we individuated the pair of values 184 and 185.

D_{ij}	*Row*	*Column*
20	1	1
35	3	3
77	7	7
93	2	3
98	2	1
114	4	3
118	6	7
120	6	1
142	5	7
146	5	3
149	8	1
153	2	7
170	8	7
175	8	3
184	**6**	**3**
185	**5**	**1**
201	4	1

So in order to avoid the contemporary presence of these two elements in the final solution we randomly chose one of these two elements to be deleted. Eliminating, for instance, the element 184 we obtained a reduced vector. This procedure is iterated until we obtain a final vector of M elements in which there is only one element for each row. Applying this procedure to our example we obtain the final solution corresponding to the element of matrix D_{ij}.

D_{ij}	*Row*	*Column*
20	1	1
35	3	3
77	7	7
93	2	3
114	4	3
142	5	7
149	8	1
184	6	3

For the randomness of these procedure, determined by the choice of the eliminated distances, we can have different solutions every time that we repeat the procedure. We define two possible ways of implementing the heuristic:

1. **RSP** Generation random of opened facilities and allocation of demand points with step 2;

2. **FPSP** Generation of opened facilities for a fixed number of solutions with step 1 and allocation of demand points with step 2;

6.4 Computational Experience

We test the methodologies described on the same instances defined in the previous chapter for which we are able to find the optimal solution.

For the genetic algorithm we fix the number of iterations equal to 200. The number of elements of the population is chosen equal to 100, 200 and 300. The crossover probability is equal to 0,7 and mutation probability is equal to 0,3. We report the results with this combination of parameters because produced the best solutions among the others tested. We use as starting solutions both random solutions that the ones obtained with the greedy heuristic, and in particular we verify that the best solutions are obtained with genetic algorithms in combination with heuristic RSP; this happens because the solutions produced by FPSP are more similar and so not useful for GA that converges too early.

The results are shown in Table 6.1. We have the number of elements (starting solutions) for the initial population and the average times on the nine instances employed by the procedure for each combination of demand points and opened facilities. We also indicate a gap from the optimal value obtained with the exact methodology. The gaps are evaluated for each instance as:

$$Gap = \frac{OptimalValue - HeuristicValue}{OptimalValue}\%;$$

we again report the average value obtained on the nine instances generated for each combination of demand points and opened facilities.

The quality of the solutions improves when we increase the number of elements of the population used, with a limited increment of the computational times. It is also evident that the results are better when we adopt as starting solutions the ones obtained with the heuristic procedure, with a limited increment of time for the small instances. For instances with a small number of demand points we obtain the optimal solutions in many cases and for some combinations of M and p, where gap is equal to 0%, for all. With a bigger number of demand points the solutions are not optimal, but are near to the optimal value. Moreover when also the number of facilities opened is big, as for the case $M = 30$ and $p = 15$, the gaps are small.

We report the obtained results with FPSP; in this case we have to decide the number of solutions to create. Given the random choice of the elements to add in the construction of the solution, finding good solutions depends also by randomness so increasing the number of solutions created we have a bigger probability of finding better solutions. In particular we generate respectively 100, 1000 5000 and 50000 for each combination of M and p (50000 are generated only when we do not find optimal solutions for all instances with a smaller number of solutions generated). We show, in Table 6.2 the results concern the application of FPSP.

We can put in evidence how the gaps are small, often equal to zero that means that we find the optimal solution in all the tested instances; with small M (10, 15) we obtain optimal solutions in computational times small, but that depend on the number of solutions generated. The instances complicated seem to be the ones with M equal to 20 and p equal to 6 for which the gap is quite high.

We derive that for the Balancing Two Stage Location Problem there are instances simple to solve that are the ones with small number of demand points (10, 15).

Instance		Genetic Algorithm			Genetic Algorithm + RSP		
M	p	N. Elements	Time(s)	Gap	N. Elements	Time(s)	Gap
10	2	100	20,38	0%	100	21,26	0%
	2	200	41,09	1%	200	42,45	0%
	2	300	60,84	0%	300	65,22	0%
10	4	100	21,68	2%	100	23,74	0%
	4	200	44,48	1%	200	48,23	0%
	4	300	66,00	1%	300	72,54	0%
10	6	100	23,28	1%	100	25,36	0%
	6	200	45,84	0%	200	52,54	0%
	6	300	69,26	0%	300	79,91	0%
15	2	100	21,93	12%	100	23,82	2%
	2	200	43,78	7%	200	47,46	0%
	2	300	67,30	6%	300	71,76	0%
15	4	100	23,40	14%	100	27,50	16%
	4	200	48,69	16%	200	56,92	11%
	4	300	75,29	10%	300	86,12	9%
15	6	100	24,76	15%	100	30,91	6%
	6	200	50,04	11%	200	61,43	2%
	6	300	76,67	6%	300	92,59	1%
20	2	100	24,92	16%	100	27,19	5%
	2	200	47,44	8%	200	55,13	1%
	2	300	74,97	5%	300	82,72	0%
20	4	100	26,74	24%	100	33,21	19%
	4	200	51,55	20%	200	67,08	16%
	4	300	82,07	16%	300	100,41	14%
20	6	100	26,88	31%	100	38,48	25%
	6	200	54,29	22%	200	76,37	20%
	6	300	83,24	20%	300	115,66	17%
20	10	100	33,63	6%	100	46,95	0%
	10	200	59,75	6%	200	93,97	0%
	10	300	89,70	0%	300	143,64	0%
25	2	100	27,73	19%	100	31,03	16%
	2	200	54,40	15%	200	62,57	9%
	2	300	82,34	9%	300	93,50	7%
25	12	100	37,54	15%	100	67,60	8%
	12	200	77,17	7%	200	146,57	6%
	12	300	115,99	7%	300	242,61	6%
30	2	100	29,45	15%	100	34,46	0%
	2	200	59,19	12%	200	69,00	0%
	2	300	91,65	8%	300	104,20	0%
30	15	100	37,62	13%	100	103,59	7%
	15	200	76,24	8%	200	208,35	8%
	15	300	118,57	7%	300	311,26	3%

Table 6.1: Computational Results for Genetic Algorithm

Instance		Greedy Heuristic (FPSP)		
M	p	N. Solutions	Time(s)	Gap
10	2	100	0,23	2%
	2	1000	2,21	1%
	2	5000	11,29	1%
10	4	100	0,43	4%
	4	1000	4,20	1%
	4	5000	21,18	0%
10	6	100	0,47	0%
	6	1000	4,56	0%
	6	5000	23,13	0%
15	2	100	0,40	9%
	2	1000	4,12	0%
	2	5000	19,93	0%
15	4	100	0,97	20%
	4	1000	9,81	14%
	4	5000	47,74	7%
	4	50000	475,02	4%
15	6	100	1,37	11%
	6	1000	13,89	5%
	6	5000	72,35	3%
	6	50000	702,05	2%
20	2	100	0,61	11%
	2	1000	6,03	4%
	2	5000	30,24	1%
	2	50000	297,27	1%
20	4	100	1,77	29%
	4	1000	17,57	17%
	4	5000	86,62	12%
	4	50000	838,32	7%
20	6	100	2,83	28%
	6	1000	25,62	23%
	6	5000	139,66	17%
	6	50000	1392,03	11%
20	10	100	3,98	6%
	10	1000	39,56	0%
	10	5000	196,28	0%
25	2	100	0,88	21%
	2	1000	9,11	8%
	2	5000	44,25	4%
	2	50000	446,72	3%
	12	100	9,37	10%
	12	1000	93,64	3%
	12	5000	453,04	3%
	12	50000	4583,86	1%
30	2	100	1,25	14%
	2	1000	12,38	11%
	2	5000	59,81	8%
	2	50000	601,68	2%
	15	100	18,60	1%
	15	1000	181,70	3%
	15	5000	907,32	1%
	15	50000	8850,26	0%

Table 6.2: Computational Results for Greedy Heuristic

Moreover the instances with p equal to 2 and p equal to $M/2$ are simpler to solve respect to the ones with p equal to 4 and 6 and this is evident using both the heuristics. The genetic algorithm works individuating solutions of worst quality in terms of value but, for large instances the computational times are smaller. In this regard we test GA also for big instances until 1000 demand points and the computational times are about 300 seconds. Instead the greedy heuristic finds better solutions with values more near to the optimal ones but the computational times depend on the number of solutions generated. Given the criterion of choice of elements, that presents some random characteristics, increasing the number of solutions, is more likely finding good solutions. Anyway the best approach is to use in combination the two heuristics that produces the best results.

6.5 Conclusion

In this chapter we illustrated two heuristics for finding solutions for BTLP. The genetic algorithm and the greedy heuristic provided solutions that are often equal or very near to the optimal solution for the same instances that we have solved with exact methodologies. The computational analysis have shown the differences in terms of computational times and quality of solutions. Anyway BTLP seems to be very complicated and for this reason in the future we will define new heuristic approaches (i.e. Lagrangian Relaxation) also in combination with exact methodologies.

Conclusion

In this work we have analyzed a specific class of Facility Location Problems (FLPs) in which equality measures are used as objective function. The analysis of the literature about these problems shows a relevant interest of the scientific community mainly oriented toward two different research streams: the theoretical analysis about properties and characteristics that an equality measure should present in order to effectively describe a FLP in which equality aspects in the definition of the final solution are crucial; the proposal of mathematical models and solution methods including equity and equality aspects either as objective function or as constraints.

These two streams does not appear linked each other. In fact it would seem that theoretical analysis does not provide a significant help in deciding, in a given (mathematical and/or practical) context which measure would better represent the problem.

For this reason the first objective of this work have been the definition of new properties able to support the choice of decision makers of the more appropriate measures to be adopted to effectively represent a FLP in a given context. To this aim some empirical analyses have been performed in order to understand the typical behavior of each measure in presence of uniform distributed demand in a regular circular location space. In addition we have proposed a correlation coefficient able to capture similarities between pairs of equality measures in solving discrete FLPs. The evaluation of this coefficient has shown that some measures can be considered more representative as they present higher correlation with the others. This first significant result should be confirmed by additional experiments considering different kinds of location spaces.

We also defined and formulated a transportation problem with multiple sources and single destination in terms of FLP. In particular to reduce risks of congestion in the dynamic of flow arrivals at the common destination, an appropriate equality measure is introduced. In order to solve the formulated problem, some exact and heuristic methodologies have been developed and implemented. Computational results performed on randomly generated test problems show opportunities and limits to efficiently solve the problem. The actual proposed procedures are able to effectively solve problems of limited size. Further developments of research should focus on the refinement of the proposed methodologies with the aim of increase the size of problems solvable in optimal (or near optimal) way.

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