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OBSERVABLES AND STATES, REDUCTION, QUANTUMNESS AND CLASSICALITY

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1

INTRODUCTION

The main objective of this dissertation is the study of the theory of Lie–Jordan Banach algebras, their role in the framework of classical and quantum mechanics, and their applications to different aspects of quantum systems.
The algebraic approach was started by W. Heisenberg and his positivistc attitude to use only observable quantities which he identified with transition frequencies in spectroscopy.
In the usual interpretation of quantum mechanics (the “Copenhagen interpretation”), the physical observables are represented by self-adjoint operators on a Hilbert space or Hermitian matrices. The basic operations on operators are multiplication by a complex scalar, addition, multiplication of operators, and forming the adjoint operator. But these underlying operations are not “observable”: the scalar multiple of a Hermitian matrix is not again Hermitian unless the scalar is real, the product is not Hermitian in general unless the factors happen to commute, and the adjoint is just the identity map on Hermitian matrices.

In 1933 the physicist Pascual Jordan proposed a program to discover a new

algebraic setting for quantum mechanics [Jor33], which would be free from the matrix structure but still enjoy all the same algebraic benefits as the Copenhagen model. He wished to study the intrinsic algebraic properties of Hermitian matrices and recast them in formal algebraic properties in order to see what other possible non-matrix systems satisfied these axioms.
Jordan decided that the fundamental observable operation was the symmetric product

$$a \circ b \equiv \frac{1}{2}(ab + ba), \tag{1.0.1}$$

now called *Jordan product* [HOS84],[McC04]. The key law governing this product, besides its obvious commutativity, is the weak associativity

$$a^2 \circ (b \circ a) = (a^2 \circ b) \circ a. \tag{1.0.2}$$

A real commutative algebra satisfying the property (1.0.2) is called *Jordan algebra*, and is called *special* if it can be realized as the Jordan algebra of an associative algebra as in Eq. (1.0.1), otherwise it is called *exceptional*.
Jordan's hopes were that by studying finite-dimensional algebras he could find families of simple exceptional algebras parameterized by natural numbers, so that in the infinite limit this would provide a suitable infinite-dimensional exceptional generalization for quantum mechanics.

In a fundamental paper in 1934 [JvNW34], Jordan, von Neumann and Wigner showed that there are only five basic types of simple finite-dimensional Jordan algebras: four types of Hermitian $n \times n$ matrix algebras $M_{n\times n}(\mathbb{K})$ where $\mathbb{K}$ can be the field of real numbers, complex numbers, quaternions and octonions (but for octonions only $n \leq 3$ is allowed), and the spin factors. The spin factors turn out to be realized as a subspace of Hermitian matrices, whereas the 27–dimensional Jordan algebra of 3×3 matrices with octonion entries, $M_{3\times 3}(\mathcal{O})$, is an exceptional Jordan algebra, now called *Albert algebra*.
This result was quite disappointing to physicists since the only exceptional algebra $M_{3\times 3}(\mathcal{O})$ was too tiny to provide an arena for quantum mechanics and the possible existence of infinite-dimensional exceptional algebras.
In 1979 the mathematician Zelmanov finally showed that even in infinite dimensions there are no simple exceptional Jordan algebras other than the Albert algebra

[Zel79]: it is an unavoidable fact of mathematical nature that simple algebraic systems obeying the basic Jordan identity (1.0.2) must (except in dimension 27) be derived from an associative structure.

However, the Jordan structure allows to recover most of the mathematical basis for the description of quantum systems, like the concept of compatible observables and the joint probability distribution for them [Emc84]. Eventually, the mathematical language becomes easier if one makes the technical assumption that the Jordan algebra $\mathcal{L}$ can be embedded in a complex extension $\mathcal{A} = \mathcal{L} \oplus i\mathcal{L}$ generated by complex linear combinations of elements of $\mathcal{L}$. This led Segal to the foundations of the theory of C^*–algebras [Seg47], which have had a profound influence on both the foundations and applications of quantum physics and quantum field theory [Haa96].
From the above discussion it is clear that special Jordan algebras admit such an extension. One of the main results of this thesis is the novel proof of a theorem which characterizes the Jordan (-Banach) algebras that are in a unique correspondence with C^*–algebras. As explained in Chapter 2, the novelty lies in the introduction of a Lie structure $[\cdot, \cdot]$ on the algebra, which is required to be compatible with the Jordan one [FFIM13c]. This means that the Leibniz identity is verified:

$$[a, b \circ c] = [a, b] \circ c + b \circ [a, c] \,, \tag{1.0.3}$$

and the associator of the Jordan product is related to the Lie associator by

$$(a \circ b) \circ c - a \circ (b \circ c) = \kappa \, [b, [c, a]] \,, \tag{1.0.4}$$

where κ is a positive real number. This leads us to the study of Lie–Jordan Banach algebras [Emc84], [Lan98].
This problem of when a given Jordan–Banach algebra is the real part of a C^*–algebra had already been faced in the past by A. Connes on one side [Con74] and Alfsen and Shultz on the other [AS98]. In particular the characterization obtained by Alfsen and Schultz in terms of the existence of a dynamical correspondence on the Jordan–Banach algebra amounts to state that the relevant structure to discuss the properties of the state space of a quantum system is exactly that of a Lie–Jordan Banach algebra. By making explicit this connection with the Lie structure also the physical interpretation becomes clear since it reflects the dual role

played by the observables: they are measurable quantities but also the generators of motions of the state space. For example, one and the same variable plays the role of an observable, called the energy, and of a generator, called the Hamiltonian. The fundamental significance of this pointwise identification of two sets of conceptually different objects manifests itself in the description of the measurement process. In fact, the observable-generator duality is at the root of the Bohr-Heisenberg principle of equivalence between definability and measurability in physics, a principle which has played a fundamental role in the discussions of the foundations of quantum mechanics.

One of the merits of Lie–Jordan algebras is that they also provide a neat algebraic framework common to classical and quantum mechanics. In the Hamiltonian picture of classical mechanics, one is naturally lead to the Lie algebraic structure of the Poisson brackets, which provide the equations of motion of the classical system. This algebraic unification could eventually help us to shed light on the intriguing problem of the classical limit of quantum mechanics and the quantization procedures.
There have been various ways in the literature of constructing quantum systems out of classical ones. All of them rely on a certain geometrical structure already present in the classical system, for example the Weyl quantization, the geometrical quantization and the deformation quantization. Following the previous ideas we would arrive to various descriptions of quantum systems, mainly of their algebra of observables, but the geometry that we used originally has faded out. However not all descriptions of quantum systems hide so thoroughly its geometrical structure. Because of a theorem by Kadison [Kad51] it is well-known that the C^*–algebra of observables of a given quantum system is isomorphic to the space of affine continuous functions on the convex space of states of the system. Thus, it would be convenient to identify the geometrical structures on the state space of a quantum system that will make Kadison correspondence more transparent. Such programme has been successfully developed along the last twenty years providing a consistent decription of the fundamental geometrical structures of quantum systems [CL84], [AS99], [CCGM07]. Moreover, a geometrical description of dynamical systems provides a natural setting to describe symmetries, and/or constraints. For instance, if the system carries a symplectic or Poisson

structure, several procedures were introduced along the years to cope with them, like Marsden–Weinstein reduction, symplectic reduction, Poisson reduction, reduction of contact structures, etc. However, it was soon realized that the algebraic approach to reduction provided a convenient setting to deal with the reduction of classical systems [GLMV94], [IdLM97].

Whenever constraints are imposed on a quantum system or symmetries are present, both dynamical or gauge, some reduction on the state space must be considered either because not all states are physical and/or because families of states are equivalent. In the standard approach to quantum mechanics, constraints are imposed on the system by selecting subspaces determined by the quantum operators corresponding to the constraints of the theory, called Dirac states, and equivalence of quantum states was dealt with by using the representation theory of the corresponding group of symmetries. However many difficulties emerge when implementing this analysis for arbitrary singular Lagrangian systems or other singularities arise (like singular level sets of momentum maps for instance) or quantum anomalies.

Taking as a departing point the algebraic approach to quantum mechanics and quantum field theory the problem of reduction of the quantum system becomes the problem of reducing the C^*–algebra of the system. Such programme was successfully developed for some gauge theories and was called T–reduction [GH85].

Another one of the main objectives of this thesis is to address the fundamental problem of reducing classical and quantum degenerate systems by using the theory of Lie–Jordan Banach algebras. It is the task of the physicist to extract the relevant physical (sub)system from such a degenerate one. Reduction means exactly the procedure aiming to identify this physical algebra. In Chapter 3 we develop the algebraic framework for reducing systems in classical and quantum mechanics. This is done by identifying the ideal generated by the constraints and quotienting its Lie normalizer with respect to it. This procedure turns out to be free from the problems of gauge anomalies which is often encountered in a heuristic approach to the quantization of constraints. We then prove that our reduction procedure is equivalent to the T–reduction developed by Grundling *et al.* for C^*–algebras.

One of the main outstanding problems of mathematical physics is to construct a C^*–algebra which describes a nonlinear field theory (higher than quadratic). An interesting feature of the reduced system in the classical realm is that it may turn out to be non-linear even if the starting one was linear. Motivated by this consideration, the quantum algebraic reduction theory could be very valuable in understanding how to provide mathematical descriptions of non-linear field theories.

In Chapter 4, we study the composition of physical systems from an algebraic point of view. By using LJB–algebras we obtain the general expression for the composition of both classical and quantum systems and in particular we derive the restrictions on the possible composable systems. This is, we find that two (classical or quantum) systems can be composed if and only if their defining constants κ, appearing in Eq. (1.0.4), are equal. Then, from the correspondence principle we know that κ is an homogeneous polynomial function of the Planck's constant $\hbar$ and hence we get that classical systems can only be composed with other classical systems, and quantum systems can only be composed if the Planck's constant is unique. This is a strong result since, from algebraic considerations only, we are able to prove the uniqueness of the Planck's constant. As shown further, the existence of multiple $\hbar$ would lead to violations of basic space-time conservation laws, which are not observed experimentally.

Finally, in Chapter 5, we give an algebraic characterization of commutative algebras, which is useful in order to find **quantumness** tests which are model-independent and refer to the most fundamental mathematical differences between classical and quantum theory. This problem, which goes back to the foundations of quantum mechanics, has become particularly relevant for the field of quantum information and quantum computation. A useful quantum computer should be a rather macroscopic machine which nevertheless preserves certain fundamental quantum properties. Moreover there are a number of tasks in computation and communication that can be performed only if quantum resources are available. By using the Jordan product we are able to provide a general definition of classical states and the associated quantumness witness which is suitable to infinite-dimensional systems.
Then we propose a measure of nonclassicality based on the incompatibility of

states relative to each other, rather than on correlations. Surprisingly, this new measure turns out to be experimentally suitable to a direct estimation by using a quantun circuit based on an interference experiment. In the last part we describe this experiment which is feasible with current technology.

The main message we tried to convey in this thesis work is that Lie–Jordan algebras provide not only an interesting mathematical framework for classical and quantum physics, but also open new horizons to groundbreaking works in the foundations and tests of the quantum theory.

2

STATES AND OBSERVABLES IN CLASSICAL AND QUANTUM SYSTEMS: C^*–ALGEBRAS AND LIE–JORDAN BANACH ALGEBRAS

2.1. The emergence of the algebraic approach in the quantum theory

The idea of an algebraic approach to quantum mechanics was already present in the matrix formulation developed by Heisenberg, Born, Jordan, Dirac and others. At this stage von Neumann formulated the quantum theory as an eigenvalue problem in a Hilbert space [vN96] and analyzed the concept of state from the point of view of the theory of probability. The reader is assumed to be familiar with the usual mathematical formulation of quantum mechanics which we will review in this section in the form of three postulates.

Postulate 2.1.1. *To each* **observable** A *on a given physical system there corresponds a linear* **self-adjoint** *operator* $\pi(A)$ *acting on a Hilbert space* $\mathcal{H}_\pi$ *and conversely.*

Notice that the converse part of the postulate is now known to be untenable due to the existence of superselection rules [SS78]. However we will not be concerned with this possibility in the following.
If we denote by $\mathcal{L}$ the set of all observables on the physical system considered, we can already observe that the first postulate equips $\mathcal{L}$ with the structure of a **real vector space**.
Notice also that if A and B are two arbitrary elements of $\mathcal{L}$, $\pi(A)\pi(B)$ in general does not belong to $\pi(\mathcal{L})$, whereas the combinations $\pi(A)\pi(B) + \pi(B)\pi(A)$ and $i(\pi(A)\pi(B) - \pi(B)\pi(A))$ do.[1]
The **symmetrized** product

$$a \circ b = \frac{1}{2}(ab + ba), \quad \forall\, a, b \in \mathcal{L}, \tag{2.1.1}$$

satisfies a number of interesting properties. It is commutative and bilinear and its introduction does not require the knowledge of the ordinary product of two noncompatible observables (i.e., two observables such that the corresponding operators do not commute in the ordinary sense). By defining $a^2 = a \circ a$, we have indeed

$$a \circ b = \frac{1}{2}((a + b)^2 - a^2 - b^2), \tag{2.1.2}$$

which involves only operations like (2.1.1). This symmetrized product is not associative in general, that is

$$(a \circ b) \circ c - a \circ (b \circ c) \neq 0 \tag{2.1.3}$$

for arbitrary $a, b, c \in \mathcal{L}$, as can be seen by simple inspection. The product (2.1.1) is called the **Jordan product** [Jor33],[JvNW34] and will be further examined in Section 2.2 where we will give the axiomatic algebraic formulation of quantum mechanics motivated by the previous considerations.

The **state** of a physical system is understood intuitively as a way to express the maximal simultaneous knowledge of the expectation values of all observables on

[1] In the following we will indicate, for brevity, the operators $\pi(A), \pi(B), \ldots$ with lower case letters $a, b, \ldots$, and commit the sin of denoting $\pi(\mathcal{L})$ with $\mathcal{L}$ itself.

the physical system considered. From the standard theory of quantum mechanics we know that to each state is associated a density matrix:

Postulate 2.1.2. *To each state ψ of the physical system considered corresponds a positive self-adjoint operator ρ of trace 1, acting on the Hilbert space $\mathcal{H}_\pi$ of Postulate 2.1.1, and such that the expectation values $\psi(a)$ of the observable a in the state ψ, are given by $\psi(a) = \mathrm{Tr}\,(\rho a)$.*

Let us stress some properties of states that follow immediately from Postulate 2.1.2. For any linear combination of elements $a_i \in \mathcal{L}$

$$\psi\Big(\sum_i c_i a_i\Big) = \mathrm{Tr}\,\Big(\rho \sum_i c_i a_i\Big) = \sum_i c_i \mathrm{Tr}\,(\rho a_i) = \sum_i c_i \psi(a), \qquad (2.1.4)$$

i.e. states act linearly on observables.

Finally concerning the dynamics we have the third postulate

Postulate 2.1.3. *The* **dynamical evolution** *of a closed quantum system described by a density state ρ is given by von Neumann's equation*

$$i\hbar \frac{\partial \rho}{\partial t} = [H, \rho], \qquad (2.1.5)$$

where H is the Hamiltonian (observable) operator of the system.

This is *Schrödinger's equation* in the space of density states. If we require that the states do not evolve in time, then we can equivalently describe the dynamics by letting the observables evolve and obtaining the *Heisenberg equation* of motion:

$$\frac{d}{dt} a(t) = \frac{i}{\hbar}[H, a(t)] + \frac{\partial a(t)}{\partial t}. \qquad (2.1.6)$$

Remark. Observe that the above dynamical equations are valid only for closed quantum systems, i.e. systems which do not interact with any external environment and do not have dissipative behaviour.

The commutator

$$[H, \rho] = H\rho - \rho H \tag{2.1.7}$$

which arises here because of Postulate 2.1.3, endows the physical observables with the role of **generators** of the motion on state space and satisfies a number of remarkable properties. Thus it is immediate to check that it is bilinear, antisymmetric and satisfies the Jacobi identity:

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0, \tag{2.1.8}$$

for all $a, b, c \in \mathcal{L}$. The antisymmetry of the bracket ensures that time-independent Hamiltonians are conserved [CS99]. The linearity guarantess that if $a(t)$ and $b(t)$ are two observables which only have dynamical evolution (i.e. without intrinsic time dependence), and λ_1, λ_2 two real constants, the observable $c_1(t) = \lambda_1 a(t) + \lambda_2 b(t)$ is also free of intrinsic time dependence. The Jacobi identity ensures that the observable $c_2(t) = [a(t), b(t)]$ also evolves dynamically:

$$\begin{aligned} \frac{dc_2}{dt} &= \left[\frac{da}{dt}, b\right] + \left[a, \frac{db}{dt}\right] && (2.1.9)\\ &= [[a, H], b] + [a, [b, H]] && (2.1.10)\\ &= [[a, b], H] = [c_2, H]. && (2.1.11) \end{aligned}$$

In particular this property also ensures the preservation of the canonical relations among canonical variables during time evolution.

2.1.1. Topological structure of the algebra of observables

With the wealth of information contained in their paper [JvNW34], Jordan, von Neumann, and Wigner demonstrated the power of a purely algebraic approach to quantum theories. However, there is a major weakness in their pioneering work, namely that they assumed that $\mathcal{L}$ has a finite linear basis. This had to be corrected by the introduction of an appropriate topological structure before the claim could be made that the theory provides a formalism general enough for the need of quantum problems. The aim would be to imitate the weak topology of operators acting on Hilbert spaces. Before proceeding to an axiomatic presentation of

LJB–algebras in Section 2.2 we show how to endow $\mathcal{L}$ with a natural topology in which the concept of state plays a significant role.

If we denote by $\mathcal{S}(\mathcal{L})$ the space of states associated to the quantum system, and define $\|a\| \equiv \sup_{\phi\in\mathcal{S}(\mathcal{L})} |\phi(a)|$, it follows immediately that for all $\lambda \in \mathbb{R}$, all a and b in $\mathcal{L}$, $\|\lambda a\| = |\lambda|\|a\|$, $\|a+b\| \leq \|a\| + \|b\|$ and that the vanishing of $\|a\|$ occurs only when $a = 0$. Therefore $\|\cdot\|$ is a norm for $\mathcal{L}$ and $\phi(a) \leq \|a\|$ for all $a \in \mathcal{L}$, $\phi \in \mathcal{S}(\mathcal{L})$.
As a result of these considerations, $\mathcal{L}$ is now equipped with the structure of a real Banach space relative to the natural norm introduced above, and the states ϕ in $\mathcal{S}$ are continuous (positive linear) functionals on $\mathcal{L}$ with respect to the topology induced by this norm.

From a phenomenological point of view we might remark at this point that one actually never deals in the laboratory with any observable a for which $\phi(a)$ is not finite; it is current practice nevertheless to consider in the theory “idealized observables” that are unbounded. There are probably novel approaches to get rid of this troubles by relaxing the Banach structure in favour of the more flexible structures like Frechet or Riesz structures.

2.1.2. The algebra of observables in classical mechanics

In classical mechanics one is usually introduced to the Newtonian formalism whose laws are generically shown to be equivalent to the more mathematically convenient Hamiltonian formalism. In Hamiltonian mechanics, we describe the state of a system by a point (q, p) in a symplectic manifold P, known as phase space. In all real physical systems, the position q and momentum p of the particle must remain bounded, but not uniformly. In general it will be identified with a cotangent bundle.
It is an experimental fact that we can never measure something with infinite precision. There are however quantities that we can, in principle, measure to an arbitrary precision. We call such quantities **classical observables**.
We would like to come up with a mathematically precise, physically motivated

way to characterize classical observables. A first natural requirement is that observables depend on the state of the system, that is, observables are functions of q and p. Moreover these functions must be real valued since we cannot measure complex quantities. Let us assume, as an experimental fact, that in the classical realm we can always measure q and p with arbitrary precision. Assume now that we want to measure the function f with error less than some $\epsilon > 0$. Since we can make the error in q and p arbitrarily small, there exist errors δ_q and δ_p such that $\forall\, q \in (q_0 - \delta_q, q_0 + \delta_q)$ and $p \in (p_0 - \delta_p, p_0 + \delta_p)$, the experimental value of $f(q, p)$ satisfies

$$f(q_0, p_0) - \epsilon < f(q, p) < f(q_0, p_0) + \epsilon.$$

But this is just the definition of a continuous function. We are then naturally lead to the characterization of observables in classical mechanics as the **continuous real-valued functions** on the phase space P.
We shall introduce in the algebra of observables one more operation, which is connected with the evolution of the mechanical system and we also require differentiability. For simplicity the discussion to follow is conducted using the example of a system with one degree of freedom.
The equations of motion are given by **Hamilton's equations** which have the form:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \quad H = H(q, p), \tag{2.1.12}$$

with solutions $q(t)$ and $p(t)$. The equations above generate a one-parameter group of transformations of the phase space into itself and in turn they generate a family of transformations of the algebra of observables:

$$f(q, p, t) = f(q(t), p(t)). \tag{2.1.13}$$

The function $f(q, p, t)$ satisfies the differential equation

$$\frac{df}{dt} = \frac{\partial H}{\partial p}\frac{\partial f}{\partial q} - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p} + \frac{\partial f}{\partial t} = \{H, f\} + \frac{\partial f}{\partial t}, \tag{2.1.14}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket, which makes the classical observables a **Poisson algebra**. That is, it satisfies

i) $\{f,g\} = -\{f,g\}$,

ii) $\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0$,

iii) $\{f,gh\} = g\{f,h\} + \{f,g\}h$,

for all $f,g,h \in C^\infty(P)$. It is interesting to point it out that the Poisson bracket is not defined on the full algebra of classical observables $C^0(P)$ but only on a dense subalgebra, its smooth part $C^\infty(P)$. Notice again that the smooth part is determined by the choice of a smooth structure on the phase space P.

2.2. Lie–Jordan Banach algebras

Motivated by the previous considerations, we are ready to define in this section the abstract algebraic properties describing classical and quantum observables.

Let $\mathcal{L}$ be a real vector space on which it is defined a symmetric bilinear distributive product $\circ$, called **Jordan product** which satisfies the generalized associative law:

$$(a^2 \circ b) \circ a = a^2 \circ (b \circ a), \quad \forall\, a,b \in \mathcal{L}, \tag{2.2.1}$$

which is the usual replacement for associativity for Jordan algebras; and an antisymmetric **Lie product** $[\cdot,\cdot]$ satisfying the Jacobi identity

$$[[a,b]\,,c] + [[c,a]\,,b] + [[b,c]\,,a] = 0, \quad \forall\, a,b,c \in \mathcal{L}. \tag{2.2.2}$$

We require these two operations to be compatible in the sense that Leibniz identity is verified:

$$[a, b \circ c] = [a,b] \circ c + b \circ [a,c]\,, \tag{2.2.3}$$

or, in other words, the linear map $D_a(\cdot) \equiv [a,\cdot]$ is a derivation of the Jordan product $\circ$.

By abstracting the previous properties, one says in general that a vector space with a symmetric operation $\circ$ and an antisymmetric one $[\cdot,\cdot]$ satisfying the properties (2.2.1),(2.2.2),(2.2.3), is called **"unlocked" Lie–Jordan algebra**. The complete definition of a ("locked") **Lie–Jordan algebra** requires that the associator of the structure product is related to the Lie product by:

$$(a \circ b) \circ c - a \circ (b \circ c) = \kappa \, [b, [c, a]] \, , \tag{2.2.4}$$

κ being a positive real number. Then we will call $(\mathcal{L}, \circ, [\cdot, \cdot])$ satisfying (2.2.4) a Lie–Jordan algebra with constant κ. The rational behind axiom (2.2.4) comes from the example discussed in the previous section on physical observables as self-adjoint operators on a Hilbert space and that we will discuss again from this perspective. Thus if we consider for instance the real vector space of bounded self-adjoint operators on a Hilbert space $\mathcal{H}$, and the Jordan product $\circ$ defined by

$$a \circ b = \frac{1}{2}(ab + ba), \tag{2.2.5}$$

and the Lie product given by the scaled commutator

$$[a, b] = i\lambda(ab - ba), \tag{2.2.6}$$

with $\lambda \in \mathbb{R}$, we obtain that (2.2.4) requires:

$$\kappa\lambda^2 = \frac{1}{4}, \tag{2.2.7}$$

if $\kappa \neq 0$. We have introduced an additional factor $i\lambda$ in (2.2.6) with respect to the familiar definition of commutator (2.1.7). In this way the space of observables actually acquires the structure of a Lie algebra and also takes into account the freedom of a scale factor λ. Moreover we insert the constant $\hbar$ for dimensional reasons and we actually see that the Lie–Jordan algebras thus defined depend on the physical constant $\hbar$.

Notice that if the Jordan product is associative and $\kappa \neq 0$ then, as it is proved in the next Theorem, the Lie structure becomes commutative, i.e. $[a, b] = 0 \; \forall \, a, b \in \mathcal{L}$.

Theorem 2.2.1. *A Lie–Jordan algebra $\mathcal{L}$ with constant $\kappa \neq 0$ is commutative if and only if the Jordan product is associative.*

Proof. Assume first that $\mathcal{L}$ is commutative. Then, trivially, from the associator identity (2.2.4) it follows that the Jordan algebra $\mathcal{L}$ is associative. Conversely, if the Jordan product is associative, then any triple commutator vanishes, so that $\forall\, a, b \in \mathcal{L}$

$$\begin{aligned} 0 &= [a, [b^2, a]] \\ &= [a, 2b \circ [b, a]] \\ &= 2b \circ [a, [b, a]] + [a, 2b] \circ [b, a] \\ &= 2b \circ [a, [b, a]] - 2[a, b]^2 \\ &= -2[a, b]^2, \end{aligned}$$

where we used the Leibnitz identity in the second and the third equality. In conclusion $[a, b] = 0,\ \forall\, a, b \in \mathcal{L}$. □

If we consider a classical carrier space, for instance a Poisson manifold, the algebra of smooth functions on the manifold becomes a Lie–Jordan algebra with constant $\kappa = 0$ when equipped with the (associative) pointwise product $f \circ g\,(x) = f(x)g(x)$, and Lie bracket $[f, g] = \{f, g\}$, with $\{\cdot, \cdot\}$ being the Poisson bracket defined on the manifold. Thus it follows that from an algebraic point of view it is quite appropriate to consider a Poisson algebra as a Lie–Jordan algebra with $\kappa = 0$. From this perspective we may consider the parameter κ a sort of deformation parameter between the classical and the quantum picture. With this intuition in mind we may call Lie–Jordan algebras with $\kappa = 0$ classical. Notice that as we mentioned already there is no Lie–Jordan algebra structure on $C^0(P)$, however it could be a good idea to call them unbounded Lie–Jordan algebras.
In order to accomodate infinite dimensional systems in this formalism, we need to define a topological structure on the algebra.

Definition 2.2.2. A Lie–Jordan Banach algebra (or LJB–algebra for short) is Lie-Jordan algebra $(\mathcal{L}, \circ, [\cdot, \cdot])$ such that it carries a complete norm $\|\cdot\|$ verifying:

i) $\|a \circ b\| \leq \|a\|\,\|b\|$,

ii) $\|a^2\| = \|a\|^2$,

iii) $\|a^2\| \leq \|a^2 + b^2\|$,

$\forall\, a, b \in \mathcal{L}$.

In particular a LJB–algebra is a Jordan–Banach algebra (or JB–algebra) when considered with the Jordan product alone. On the other hand, if we are given a LJB–algebra $\mathcal{L}$, by taking combinations of the two products we can define an associative product on the complexification $\mathcal{L}^{\mathbb{C}} = \mathcal{L} \oplus i\mathcal{L}$. Specifically, we define:

$$ab = a \circ b - i\sqrt{\kappa}\,[a, b]\,, \quad \forall\, a, b \in \mathcal{L},$$

and extend it by linearity to $\mathcal{L}^{\mathbb{C}}$. Then $\mathcal{L}^{\mathbb{C}}$ becomes an associative *–algebra, where $(a + ib)^* = a - ib$. Such associative algebra equipped with the norm $\|x\| = \|x^* x\|^{1/2}$ where $x = a + ib$, is the unique C^*–algebra whose real part is precisely $\mathcal{L}$ (see Section 2.3).

Notice that if the LJB–algebra $\mathcal{L}$ is classical, i.e. $\kappa = 0$, its associated C^*–algebra is isomorphic to the space of continuous functions on a compact topological space with the supremum norm, hence if such space carries a differentiable structure the Lie bracket will define a family of unbounded derivations on the dense subspace of smooth functions, otherwise trivial. In other words we will need weaker topologies to accomodate classical LJB–algebras in the same picture. Then, from now on, we will just consider non–classical LJB–algebras, i.e., $\kappa \neq 0$. We must point out here that the study of unbounded LJB–algebras has never been started.

2.2.1. Spectrum and states of Lie–Jordan Banach algebras

The concept of spectrum of an observable is very important since it provides the possible outcomes of a measurement of the observable on the physical system. In this subsection we explore the definition of spectrum and states in algebraic terms.

Definition 2.2.3. Let $\mathcal{L}$ be a unital LJB–algebra. The spectrum $\sigma(a)$ of $a \in \mathcal{L}$ is defined as the set of those $\lambda \in \mathbb{R}$ for which $a - \lambda\mathbb{1}$ has no inverse in $\mathcal{L}$.

Note that a LJB–algebra $\mathcal{L}$ is a complete order unit space with respect to the positive cone [HOS84]:

$$\mathcal{L}^+ = \{\, a^2 \mid a \in \mathcal{L}\,\} \tag{2.2.8}$$

or equivalently an element is positive if and only if its spectrum is positive.[2]
We shall in this section prove some useful properties of the spectrum and then the Cauchy–Schwarz like inequalities.

Lemma 2.2.4.

$$\sigma(a_1^2 + a_2^2 + \mu[a_1, a_2]) \cup \{0\} = \sigma(a_1^2 + a_2^2 - \mu[a_1, a_2]) \cup \{0\} \qquad (2.2.9)$$

$\forall\, a_1, a_2 \in \mathcal{L}$ *and* $\forall\, \mu \in \mathbb{R}$.

Proof. For $\lambda \neq 0$ the invertibility of $a_1^2 + a_2^2 + \mu[a_1, a_2] - \lambda\mathbb{1}$ implies the invertibility of $a_1^2 + a_2^2 - \mu[a_1, a_2] - \lambda\mathbb{1}$. Namely, one computes that

$$\begin{aligned}(a_1^2 + a_2^2 + \mu[a_1, a_2] - \lambda\mathbb{1})^{-1} \;=\; & \lambda^{-1}\{2a_1 \circ (b \circ a_1) - a_1^2 \circ b + 2a_2 \circ (b \circ a_2) + \\ & -a_2^2 \circ b + 2[a_1, b \circ a_2] + 2a_1 \circ [b, a_2] - \mathbb{1}\}\end{aligned}$$

with $b = \{a_1^2 + a_2^2 + \mu[a_1, a_2] - \lambda\mathbb{1}\}^{-1}$. □

Lemma 2.2.5.

$$\sigma(a_1^2 + a_2^2 + \mu[a_1, a_2]) \subset \mathbb{R}^- \Rightarrow a_1^2 + a_2^2 + \mu[a_1, a_2] = 0 \qquad (2.2.10)$$

$\forall\, a_1, a_2 \in \mathcal{L}$ *and* $\forall\, \mu \in \mathbb{R}$.

Proof. Note that $a_1^2 + a_2^2 - \mu[a_1, a_2] = 2a_1^2 + 2a_2^2 - (a_1^2 + a_2^2 + \mu[a_1, a_2])$ and then under the assumptions of the lemma $\sigma(a_1^2 + a_2^2 - \mu[a_1, a_2]) \subset \mathbb{R}^+$. This implies, by the previous lemma, that $\sigma(a_1^2 + a_2^2 - \mu[a_1, a_2]) = \{0\}$. □

Theorem 2.2.6.

$$X = a_1^2 + a_2^2 - \mu[a_1, a_2] \in \mathcal{L}^+ \qquad (2.2.11)$$

$\forall\, a_1, a_2 \in \mathcal{L}$ *and* $\forall\, \mu \in \mathbb{R}$.

[2] An alternative way to express this would be: an element $a \in \mathcal{L}$ is positive if $\sigma(a) \subset \mathbb{R}^+$. It is possible to show that the cone of positive elements is given by $\mathcal{L}^+ = \{\, a^2 \mid a \in \mathcal{L} \,\}$. Moreover it is a consequence of the completeness of a LJB–algebra to show that it is a complete order unit space.

Proof. Every $X \in \mathcal{L}$ has the decomposition [Lan98] $X = X_+ + X_-$, where $X_+, X_- \in \mathcal{L}^+$ and $X_+ \circ X_- = [X_+, X_-] = 0$. It follows that $X_-^3 = -(b_1^2 + b_2^2 - \mu[b_1, b_2]) \geq 0$ with $b_1 = a_1 \circ X_- + \mu[a_2, X_-]$ and $b_2 = \mu[a_1, X_-] + a_2 \circ X_-$. But $X_-^3 = -2b_1^2 - 2b_2^2 + (b_1^2 + b_2^2 + \mu[b_1, b_2])$ which is a negative quantity and then in turn implies that $X_- = 0$ and then $X = X_+ \geq 0$. □

Motivated by the considerations of Section 2.1, we can define the space of **states** $\mathcal{S}(\mathcal{L})$ of a LJB–algebra as the set of all **real normalized positive linear functionals** on $\mathcal{L}$, i.e.

$$\rho\colon \mathcal{L} \to \mathbb{R} \tag{2.2.12}$$

such that $\rho(\mathbb{1}) = 1$ and $\rho(a^2) \geq 0$, $\forall\, a \in \mathcal{L}$. The state space is convex and compact with respect to the w*–topology.

We shall now prove the Lie–Jordan algebra version of the Cauchy–Schwarz inequalities. These are a very important "ingredient" for many subsequent proofs.

Theorem 2.2.7. *Let $\mathcal{L}$ be a unital LJB–algebra with constant $\hbar^2$ and ρ a state on $\mathcal{L}$. Then if $a, b \in \mathcal{L}$ we have*

$$\rho(a \circ b)^2 \leq \rho(a^2)\rho(b^2), \tag{2.2.13}$$

and

$$\rho([a, b])^2 \leq \frac{1}{\hbar^2}\rho(a^2)\rho(b^2). \tag{2.2.14}$$

Proof. Let $\lambda \in \mathbb{R}$, then we have

$$0 \leq \rho(\lambda a + b)^2) = \lambda^2\rho(a^2) + 2\lambda\rho(a \circ b) + \rho(b^2). \tag{2.2.15}$$

If $\rho(a^2) = 0$ then $\rho(a \circ b) = 0$ since λ is arbitrary. If $\rho(a^2) \neq 0$, let $\lambda = -\rho(a \circ b)\rho(a^2)^{-1}$, and the first proof is immediate.
The second inequality is proved similarly by using the positivity of $a_1^2 + a_2^2 + 2\hbar[a_1, a_2]$ as stated in Thm. (2.2.6). □

Example 2.2.8. As we have discussed before the self-adjoint subalgebra $\mathcal{B}_{sa} = \mathcal{L}(\mathcal{H})$ of the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ with the operator norm is a Lie–Jordan Banach algebra and the states are the positive linear functional on $\mathcal{B}_{sa}$. Let ρ be a continuous state with respect to the ultrastrong topology on $\mathcal{B}(\mathcal{H})$ [vN36], i.e. the topology on $\mathcal{B}(\mathcal{H})$ given by the open neighbourhood base

$$N(a; (x_i)_1^\infty, \epsilon) = \{ b \in \mathcal{B}(\mathcal{H})\colon \sum_{i=1}^{\infty} \|(a-b)x_i\|^2 < \epsilon \}, \tag{2.2.16}$$

for $a \in \mathcal{B}(\mathcal{H})$, $\epsilon > 0$ and any sequence $(x_i) \in \mathcal{H}$ satisfying $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$. Then there is a positive linear trace class operator $\widetilde{\rho} \in \mathcal{B}_{sa}$ such that

$$\rho(a) = \mathrm{Tr}(\widetilde{\rho} a) \tag{2.2.17}$$

for all $a \in \mathcal{B}_{sa}$.
Conversely, if ρ is a positive trace class operator, then the functional $a \mapsto \mathrm{Tr}(\widetilde{\rho} a)$ defines an ultrastrongly continuous positive linear functional on $\mathcal{B}_{sa}$.

We have shown that the algebra of observables of a quantum mechanical system is a LJB–algebra and described the quantum states as positive linear functionals on the algebra. As it is evident from the previous example not all the states on the algebra can be realized as density matrices. Those states realized as density matrices are called normal. It is nevertheless recognized the important of non-normal states in the mathematical approaches to quantum statistical mechanics [BR03].
A natural question may now arise. That is, can the algebraic framework accomodate something more general than the standard quantum theory we have seen? Is it possible to provide realizations of a LJB–algebra (or equivalently a C^*–algebra) different from the usual quantum mechanics? The answer to this question gives actually a solid background to the algebraic theory since it can be proved that LJB–algebras and C^*–algebras can always be represented as algebras of operators on a Hilbert space.

Theorem 2.2.9 (Gelfand, Naimark, Segal)**.** *Let $\mathcal{L}$ be a unital LJB–algebra. A* **representation** *of $\mathcal{L}$ on a complex Hilbert space $\mathcal{H}$, is a strongly continuous Lie–Jordan homomorphism π of $\mathcal{L}$ into the self-adjoint bounded operators on $\mathcal{H}$, i.e.*

$\forall\, a, b \in \mathcal{L}$

$$\pi(a \circ b) = \pi(a) \circ \pi(b) \tag{2.2.18}$$

$$[\pi(a), \pi(b)] = \pi([a, b]). \tag{2.2.19}$$

Moreover given a state ω of $\mathcal{L}$ there exists a Hilbert space $\mathcal{H}_\omega$ and a representation $\pi_\omega\colon \mathcal{L} \to \mathcal{B}_{sa}(\mathcal{H}_\omega)$ and a unit vector $|0\rangle \in \mathcal{H}_\omega$ such that for all $a \in \mathcal{L}$, $\omega(a) = \langle 0|\pi_\omega(a)|0\rangle$.

We will say that the representation π is *nondegenerate* if $\mathrm{span}\{\pi(a)|\phi\rangle \ \mid a \in L,\ |\phi\rangle \in \mathcal{H}\}$ is dense in $\mathcal{H}$.
Given two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of $\mathcal{L}$, we say that they are equivalent if there exists a unitary map $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that $U \circ \pi_1(a) = \pi_2(a) \circ U$, $\forall\, a \in \mathcal{L}$.

Remark. If we replace the LJB–algebra by a Jordan–Banach algebra, as it was originally proposed by P. Jordan, then the above theorem is not true. It is in fact known (as discussed in the Introduction) that there is an “exceptional” Jordan algebras (i.e. a Jordan algebra which do not arise from an associative product) which cannot represented as an algebra of operators on a Hilbert space. This is the so called *Albert algebra* of 3×3 matrices with values in the Octonions.

2.3. C^*–algebras and dynamical correspondence

In this section we study the identification of observables and generators from an algebraic point of view. Each of the sets of observables and generators is an algebra, and the observable-generator duality manifests itself as a map from the space of observables to the space of generators. This map interraltes the two algebras, imposing restrictions on their structures. Our purpose is to investigate these restrictions. The background for our approach is to consider a quantum system as described by a C^*–algebra $\mathcal{A}$ whose real part are the observables of the system, and its quantum states ω are normalized positive complex functionals on it. However the state space $\mathcal{S}$ of the quantum system does not determine univocally the C^*–algebra structure of the system but only its Jordan–Banach real algebra part [JvNW34]–[Seg47]. In fact as Kadison's theorem shows [Kad51], the real (or self-adjoint) part of a C^*–algebra $\mathcal{A}$, is isometrically isomorphic to the space of all w*– continuous affine functions on the state space of $\mathcal{A}$. A. Connes on one side [Con74] and Alfsen and Shultz on the other [AS98], solved the problem of when a given Jordan–Banach algebra is the real part of a C^*–algebra. The characterization obtained by Alfsen and Schultz in terms of the existence of a dynamical correspondence on a Jordan–Banach algebra amounts to state that the relevant structure to discuss the properties of the state space of a quantum system is that of a LJB–algebra [Emc84]–[Lan98]. In fact the topological properties of the state space are completely captured by the Jordan–Banach algebra structure and the Lie algebra structure allows to construct the C^*–algebra setting for them, their GNS representations, etc.

We are now going to prove one of the main results in the theory of LJB–algebras, which we already anticipated in the previous sections. Namely, the equivalence between the category of C^*–algebras and the category of LJB–algebras. We will prove that a C^*–algebra is always the complexification of a LJB–algebra. In order to do this, we briefly give few definitions on C^*–algebras and derivations of LJB–algebras.

Definition 2.3.1. A C^*–algebra $\mathcal{A}$ is a Banach algebra over the field of complex numbers, together with an antilinear map $^*\colon \mathcal{A} \to \mathcal{A}$ called **involution**, which satisfies $(x^*)^* = x$ and

$$\|x^* x\| = \|x\|\|x^*\|, \quad \forall\, x \in \mathcal{A}. \tag{2.3.1}$$

Following [AS98] we will define a derivation of a JB–algebra $\mathcal{L}$ by focusing first only on the order structure with respect to the positive cone $\mathcal{L}^*$ defined before, ignoring for the moment the algebraic multiplicative aspect. All the proofs contained in [AS98] will be omitted.

Definition 2.3.2. A bounded linear operator δ on a JB–algebra $\mathcal{L}$ is called an **order derivation if** $\mathrm{e}^{t\delta}(\mathcal{L}^+) \subset \mathcal{L}^+$, $\forall\, t \in \mathbb{R}$**.**

We denote the Jordan multiplier determined by an element $b \in \mathcal{L}$ by δ_b. Thus for all $a \in \mathcal{L}$

$$\delta_b(a) = b \circ a.$$

Notice that $\mathrm{e}^{t\delta_b}$ is the multiplier associated to $\mathrm{e}^{tb} = (\mathrm{e}^{\frac{tb}{2}})^2 \in \mathcal{L}^+$. Then Jordan multipliers δ_b are order derivations $\forall\, b \in \mathcal{L}$.

Definition 2.3.3. An **order derivation** δ on a unital JB–algebra $\mathcal{L}$ is self-adjoint if there exists $a \in \mathcal{L}$ such that $\delta = \delta_a$ and is skew-adjoint if $\delta(\mathbb{1}) = 0$.

Again, it can be shown that if δ is an order derivation, then δ is skew if and only if δ is a **Jordan derivation**, i.e., it is a derivation with respect to the Jordan product:

$$\delta(a \circ b) = \delta a \circ b + a \circ \delta b, \quad \forall\, a, b \in \mathcal{L}. \tag{2.3.2}$$

We will establish now the main notion in [AS98].

Definition 2.3.4. A **dynamical correspondence** [AS98] on a unital JB–algebra $\mathcal{L}$ is a linear map

$$\psi\colon a \to \psi_a \tag{2.3.3}$$

from $\mathcal{L}$ into the set of skew order derivations $\psi_a\colon \mathcal{L} \to \mathcal{L}$ which satisfies:

i) there exists $\kappa \in \mathbb{R}$ such that $\kappa\, [\psi_a, \psi_b] = -\, [\delta_a, \delta_b]\,, \quad \forall\, a, b \in \mathcal{L}$, and[3]

ii) $\psi_a a = 0, \quad \forall\, a \in \mathcal{L}$.

It follows immediately from the definitions that:

$$\psi_a b = -\psi_b a, \quad \forall\, a, b \in \mathcal{L}. \tag{2.3.4}$$

The dynamical correspondence then assigns a “skew order derivation” ψ_a to each element a of the given algebra $\mathcal{L}$. The skew order derivations are generators of one-parameter groups of unital order automorphisms of $\mathcal{L}$ [AS98], and by duality also of one-parameter groups of motions on the state space of $\mathcal{L}$. Thus a dynamical correspondence gives the elements of $\mathcal{L}$ a double identity, which reflects the dual role of physical variables as observables and as generators of a one-parameter group of motions of the state space.

[3]The notations $[\psi_a, \psi_b]$ and $[\delta_a, \delta_b]$ are not related to any Lie bracket and stand for the commutator of the operators in the arguments, i.e. $[\psi_a, \psi_b] = \psi_a\psi_b - \psi_b\psi_a$, $[\delta_a, \delta_b] = \delta_a\delta_b - \delta_b\delta_a$.

Definition 2.3.5. Let $\mathcal{L}$ be a unital JB–algebra. A C^*–product compatible with $\mathcal{L}$ is an associative product on the complex linear space $\mathcal{L} \oplus i\mathcal{L}$ which induces the given Jordan product on $\mathcal{L}$ and makes $\mathcal{L} \oplus i\mathcal{L}$ into a C^*–algebra with involution $(a+ib)^* = a - ib$ and norm $\|x\| = \|x^* x\|^{1/2}$ where $x = a + ib$.

Note that if a JB–algebra $\mathcal{L}$ is the self-adjoint part of a C^*–algebra $\mathcal{A}$, then there are a natural product and a norm induced in $\mathcal{L} \oplus i\mathcal{L}$ by using the representation $A = a + ib$ with $A \in \mathcal{A}$ and $a, b \in \mathcal{L}$. Such product and norm organize $\mathcal{L} \oplus i\mathcal{L}$ into a C^*–algebra. It follows that a JB–algebra is the self-adjoint part of a C^*–algebra if and only if there exists a C^*–product compatible with $\mathcal{L}$ on $\mathcal{L} \oplus i\mathcal{L}$. Yhe main result in [AS98] provides an explicit relation between JB–algebras and C^*–algebras provided that the former are equipped with a dynamical correspondence.

Theorem 2.3.6 ([AS98])**.** *A unital JB–algebra $\mathcal{L}$ is Jordan isomorphic to the self-adjoint part of a C^*–algebra if and only if there exists a dynamical correspondence on $\mathcal{L}$. Each dynamical correspondence ψ on $\mathcal{L}$ determines a unique associative C^*–product compatible with $\mathcal{L}$ defined as*

$$ab = a \circ b - i\sqrt{\kappa}\, \psi_a b \tag{2.3.5}$$

and each C^*–product compatible with $\mathcal{L}$ arises in this way from a unique dynamical correspondence ψ on $\mathcal{L}$.

We will now show that the existence of a dynamical correspondence on $\mathcal{L}$ is equivalent to the existence of a Lie product organizing $\mathcal{L}$ into a LJB–algebra. First we need the following lemmas:

Lemma 2.3.7. *Let $(\mathcal{L}, [\cdot,\cdot]_{\mathcal{L}}, \circ)$ be a LJB–algebra. Then there exists an associative bilinear product on $\mathcal{L} \times \mathcal{L}$ defined as*

$$a \cdot b = a \circ b - i\sqrt{\kappa}\ [a,b]_{\mathcal{L}}, \quad \forall\, a, b \in \mathcal{L}, \tag{2.3.6}$$

and extended linearly to $\mathcal{L} \oplus i\mathcal{L}$.

Proof. Bilinearity of the product follows directly from the bilinearity of the Jordan and Lie products. We have to prove the associativity, i.e.:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c, \quad \forall\, a, b, c \in \mathcal{L}. \tag{2.3.7}$$

The l.h.s. of the previous equation leads to:

$$a\cdot(b\cdot c) = a\circ(b\circ c) - i\sqrt{\kappa}\, a\circ[b,c]_{\mathcal{L}} - i\sqrt{\kappa}\, [a,b]_{\mathcal{L}}\circ c - i\sqrt{\kappa}\, b\circ[a,c]_{\mathcal{L}} - \kappa\, [a,[b,c]_{\mathcal{L}}]_{\mathcal{L}}\,,$$

and the r.h.s.:

$$(a\cdot b)\cdot c = (a\circ b)\circ c - i\sqrt{\kappa}\, b\circ[a,c]_{\mathcal{L}} - i\sqrt{\kappa}\, a\circ[b,c]_{\mathcal{L}} - i\sqrt{\kappa}\, [a,b]_{\mathcal{L}}\circ c - \kappa\, [\,[\,a,b\,]_{\mathcal{L}}\,,c\,]_{\mathcal{L}}\,,$$

Then

$$\begin{aligned} a\cdot(b\cdot c) - (a\cdot b)\cdot c &= a\circ(b\circ c) - (a\circ b)\circ c - \kappa\, [a,[b,c]_{\mathcal{L}}]_{\mathcal{L}} - \kappa\, [c,[a,b]_{\mathcal{L}}]_{\mathcal{L}} \\ &= \kappa\left([b,[c,a]_{\mathcal{L}}]_{\mathcal{L}} + [a,[b,c]_{\mathcal{L}}]_{\mathcal{L}} + [c,[a,b]_{\mathcal{L}}]_{\mathcal{L}}\right) \\ &= 0. \end{aligned}$$

where we have used (2.2.2),(2.2.3) and (2.2.4). □

Note that the Jordan and Lie products can be obviously expressed in terms of the associative product as:

$$a\circ b = \frac{1}{2}(a\cdot b + b\cdot a), \tag{2.3.8}$$

$$[a,b]_{\mathcal{L}} = \frac{i}{2\sqrt{\kappa}}(a\cdot b - b\cdot a). \tag{2.3.9}$$

Lemma 2.3.8. *Let* $(\mathcal{L},[\cdot,\cdot]_{\mathcal{L}}\,,\circ)$ *be a LJB–algebra. Then* $\mathrm{e}^{[a,\cdot]_{\mathcal{L}}}$ *is a Jordan automorphism* $\forall\, a\in\mathcal{L}$.

Proof. We have to prove that

$$\mathrm{e}^{[a,\cdot]_{\mathcal{L}}}\,(b\circ c) = (\mathrm{e}^{[a,\cdot]_{\mathcal{L}}}\,b)\circ(\mathrm{e}^{[a,\cdot]_{\mathcal{L}}}\,c). \tag{2.3.10}$$

By Hadamard's formula [Ser65], the l.h.s. of the previous equation is:

$$\mathrm{e}^{[a,\cdot]_{\mathcal{L}}}\,(b\circ c) = \mathrm{e}^{a}\cdot(b\circ c)\cdot\mathrm{e}^{-a}\,.$$

By using formula (2.3.8), the r.h.s. of (2.3.10) becomes:

$$\begin{aligned}(\mathrm{e}^{[a,\cdot]_{\mathcal{L}}}\ b)\circ(\mathrm{e}^{[a,\cdot]_{\mathcal{L}}}\ c) &= (\mathrm{e}^{a}\cdot b\cdot \mathrm{e}^{-a})\circ(\mathrm{e}^{a}\cdot c\cdot \mathrm{e}^{-a}) \\ &= \frac{1}{2}\ \mathrm{e}^{a}\cdot(b\cdot c)\cdot \mathrm{e}^{-a}+\frac{1}{2}\ \mathrm{e}^{a}\cdot(c\cdot b)\cdot \mathrm{e}^{-a} \\ &= \mathrm{e}^{a}\cdot(b\circ c)\cdot \mathrm{e}^{-a}\,.\end{aligned}$$

□

Lemma 2.3.9. *Let $(\mathcal{L},[\cdot,\cdot]_{\mathcal{L}},\circ)$ be a LJB–algebra. Then $[a,\cdot]_{\mathcal{L}}$ is an order derivation on $\mathcal{L}\ \forall\, a\in\mathcal{L}$.*

Proof. From Definition 2.3.2, we have to prove that $\mathrm{e}^{t[a,\cdot]_{\mathcal{L}}}\ (\mathcal{L}^+)\subset\mathcal{L}^+,\ \forall\, a\in\mathcal{L}$ and $\forall\, t\in\mathbb{R}$. Since $\mathrm{e}^{t[a,\cdot]_{\mathcal{L}}}$ is a Jordan automorphism (Lemma 2.3.8), we have:

$$\mathrm{e}^{t[a,\cdot]_{\mathcal{L}}}(b\circ b)=(\mathrm{e}^{t[a,\cdot]_{\mathcal{L}}}\,b)\circ(\mathrm{e}^{t[a,\cdot]_{\mathcal{L}}}\,b),$$

$\forall\, a,b\in\mathcal{L}$ and $\forall\, t\in\mathbb{R}$, i.e. $\mathrm{e}^{t[a,\cdot]_{\mathcal{L}}}$ preserves the positive cone (2.2.8) $\mathcal{L}^+$. □

Then we can finally conclude:

Theorem 2.3.10 ([FFIM13c]). *Let $\mathcal{L}$ be a unital JB–algebra. There exists a dynamical correspondence ψ on $\mathcal{L}$ if and only if $\mathcal{L}$ is a LJB–algebra with Lie product $[\cdot,\cdot]_{\mathcal{L}}$ such that*

$$[a,b]_{\mathcal{L}}=\psi_a b \tag{2.3.11}$$

Proof. First assume that $\mathcal{L}$ is a LJB–algebra. From Definition 2.3.4 we have to check that $\forall\, a,b\in\mathcal{L}$

$$\kappa\,[\psi_a,\psi_b]=-\,[\delta_a,\delta_b]$$

that is

$$\kappa([a,[b,c]_{\mathcal{L}}]_{\mathcal{L}}-[b,[a,c]_{\mathcal{L}}]_{\mathcal{L}})=b\circ(a\circ c)-a\circ(b\circ c)$$

which is an easy computation once the Jordan and Lie products are expressed as in (2.3.9) and (2.3.8). From the antisymmetry of the Lie product it is also true that $\psi_a a=[a,a]_{\mathcal{L}}=0\ \forall\, a\in\mathcal{L}$. Hence the linear map $a\to[a,\cdot]_{\mathcal{L}}$ from the LJB–algebra $\mathcal{L}$ to the skew-order derivations on $\mathcal{L}$ is a dynamical correspondence.

Conversely, assume $\mathcal{L}$ is a JB–algebra with a dynamical correspondence ψ. Then from (2.3.4) $\psi_a b = [a, b]_{\mathcal{L}}$ is antisymmetric. The Jacobi property (2.2.2) follows from the defining property i) of the dynamical correspondence (Definition 2.3.4), the Leibniz identity (2.2.3) follows from (2.3.2) and also the compatibility condition (2.2.4) is easy to check with a simple computation using the properties of the dynamical correspondence (Definition 2.3.4). Hence a JB–algebra with a dynamical correspondence is a LJB–algebra. □

Corollary 2.3.11. *A unital JB–algebra $\mathcal{L}$ is Jordan isomorphic to the self-adjoint part of a C^*–algebra if and only if it is a LJB–algebra.*

Proof. This is an obvious consequence of Theorems 2.3.6 and 2.3.10. □

This finally proves the equivalence between the category of C^*–algebras and that of LJB–algebras. We conclude with

Corollary 2.3.12. *Let $(\mathcal{L}, \circ, [\cdot,\cdot]_{\mathcal{L}})$ be a LJB–algebra and $\mathcal{A} = \mathcal{L}^{\mathbb{C}}$ the natural C^*–algebra defined by the complexification of $\mathcal{L}$. Then there is a natural identification between the states $\mathcal{S}(\mathcal{L})$ of $\mathcal{L}$ and the states $\mathcal{S}(\mathcal{A})$ of the C^*–algebra $\mathcal{A}$.*

Proof. Given a state ω of $\mathcal{L}$, we define a linear functional $\tilde{\omega}$ of $\mathcal{A}$ by extending it linearly. The linear functional $\tilde{\omega}$ is positive and normalized because ω is positive and normalized. Notice that if $x = a + ib \in \mathcal{L}^{\mathbb{C}}$, then $x^*x = a^2 + b^2$, then if $\widetilde{\omega}$ is a functional extending ω (chosen continuous by the Hahn–Banach theorem), then

$$\widetilde{\omega}(x^*x) = \widetilde{\omega}(a^2 + b^2) = \omega(a^2 + b^2) \geq 0, \tag{2.3.12}$$

because ω is positive. The converse is trivial. □

Notice in addition that if $\alpha \colon \mathcal{A} \to \mathcal{B}$ is a morphism of C^*–algebras, then $\alpha(a^*) = \alpha(a)^*$, thus α restricts to a morphism $\alpha_{sa} \colon \mathcal{A}_{sa} \to \mathcal{B}_{sa}$. Now let $\sigma \colon \mathcal{L}_1 \to \mathcal{L}_2$ be a morphism of Lie–Jordan algebras, i.e. $\forall\, a, b \in \mathcal{L}_1$

$$\sigma(a \circ b) = \sigma(a) \circ \sigma(b) \tag{2.3.13}$$

$$\sigma([a, b]) = [\sigma(a), \sigma(b)] \tag{2.3.14}$$

then we can define $\widetilde{\sigma}\colon \mathcal{L}_1^{\mathbb{C}} \to \mathcal{L}_2^{\mathbb{C}}$ as

$$\widetilde{\sigma}(a+ib) = \sigma(a) + i\sigma(b). \tag{2.3.15}$$

Then we have for all $a, b \in \mathcal{L}_1$

$$\begin{aligned}
\widetilde{\sigma}(a\cdot b) &= \widetilde{\sigma}(a\circ b - i\sqrt{\kappa}[a,b]) = \sigma(a\circ b) - i\sqrt{\kappa}\sigma([a,b]) && (2.3.16)\\
&= \sigma(a)\circ\sigma(b) - i\sqrt{\kappa}[\sigma(a),\sigma(b)] && (2.3.17)\\
&= \widetilde{\sigma}(a)\cdot\widetilde{\sigma}(b). && (2.3.18)
\end{aligned}$$

Therefore we have proved

Theorem 2.3.13. *Given a morphism $\sigma\colon \mathcal{L}_1 \to \mathcal{L}_2$ of LJB–algebras, there is a unique extension $\widetilde{\sigma}$ of σ to a morphism of the corresponding C^*–algebras $\widetilde{\sigma} = \mathcal{A}_1\colon \mathcal{L}_1^{\mathbb{C}} \to \mathcal{A}_2 = \mathcal{L}_2^{\mathbb{C}}$ and $\widetilde{\mathbb{1}_{\mathcal{L}}} = \mathbb{1}_{\mathcal{L}^{\mathbb{C}}}$. Moreover the functors from the category $\mathcal{LJB}$ of Lie–Jordan Banach algebras in the category $\mathcal{C}^*\mathcal{A}$ of C^*–algebras is an isomorphism of categories.*

3

Reduction of Lie–Jordan Banach algebras

By a degenerate system we mean a physical system that when described mathematically possesses extra, nonphysical degrees of freedom. Then a complete description of it is usually attained by adding supplementary conditions, by the action of a gauge group on it or by any other mean that allows to identify the true degrees of freedom of the theory. The task of the physicist is to extract the relevant physical subsystem from such a degenerate one. Indeed, physical information such as boundary conditions or constraints is often injected into a theory through the use of supplementary constraints. The treament of degenerate systems in classical mechanics was developed by Dirac [Dir01] who provided an algorithmic procedure and has now reached a high degree of mathematical maturity. It was formalized first by P. Bergmann [Ber61] and later on M. Gotay *et al.* set its geometrical foundation, being known as the presymplectic constraints algorithm [GNH78], [GN79], [GN80]. Later on Marmo, Mendella and Tulczyjew established its simplest geometrical structure by considering it as a consistency condition for implicit differential equations [MMT95]. As for the quantum setting, these systems still remain within heuristic formulations without earning much from their classical rigor, due to the dubious nature of quantization.

The aim of this chapter is to review first the classical treatment of constraints in the language of differential geometry and translate it in the more abstract algebraic framework, which turns out to be appropriate for both classical and quantum constraints. We then provide a quantum mechanical procedure for eliminating the degeneracy in a mathematically consistent way, by focusing on the algebra of observables and comparing this approach with the T-procedure in the C^*–algebra framework, developed along the years by Grundling and collaborators [GH85], [GH88], [GL00].

3.1. Symplectic reduction and Dirac's theory of constraints

In this section we rephrase Dirac's theory of constraints in the modern language of symplectic geometry. We start discussing the symplectic reduction with respect to a coisotropic submanifold of a symplectic submanifold. Then the parallel with Dirac's theory of constraints is naturally elucidated.

3.1.1. Symplectic reduction

Let (M, Ω) be a symplectic manifold, that is, Ω is a closed non-degenerate 2-form. Fix a point $p \in M$ and consider the vector space T_pM of tangent vectors to M at p. The symplectic form determines a non-degenerate antisymmetric form on T_pM, making it into a **symplectic vector space**. In a symplectic vector space V, we can define four kind of subspaces. Let W be a subspace of V, and denote by $W^\perp$ its **symplectic complement** relative to the symplectic form Ω:

$$W^\perp = \{X \in V \mid \Omega(X, Y) = 0 \quad \forall Y \in W\}. \tag{3.1.1}$$

Subspaces W obeying $W \subseteq W^\perp$ are called **isotropic** and they necessarily obey $\mathrm{dim}W \leq \frac{1}{2}\mathrm{dim}V$. On the other hand, if $W \supseteq W^\perp$, W is called **coisotropic** and it must obey $\mathrm{dim}W \geq \frac{1}{2}\mathrm{dim}V$. If W is both isotropic and coisotropic, then it is its own symplectic complement, it obeys $\mathrm{dim}W = \frac{1}{2}\mathrm{dim}V$ and it is called a **lagrangian** subspace. Finally, if $W \cap W^\perp = 0$, W is called **symplectic**.

Notice that if W is lagrangian, the restriction of Ω to W is identically zero; whereas if W is symplectic, Ω restricts to a symplectic form. In particular, symplectic subspaces are even dimensional. If W is coisotropic, Ω restricts to a nonzero antisymmetric bilinear form on W which, nevertheless, is degenerate since any vector in $W^\perp \subseteq W$ is symplectically orthogonal to all of W. But it then follows that the quotient $W/W^\perp$ inherits a well defined symplectic form and hence becomes a symplectic vector space. The passage from V to $W/W^\perp$ (which is a subquotient) is known as the **symplectic reduction** of V relative to the coisotropic subspace W. In the following we will make this procedure global by generalizing it to symplectic manifolds.

We similarly define a submanifold M_0 to be **isotropic**, **coisotropic**, **lagrangian**, or **symplectic** according to whetever at *all* points $p \in M_0$, the tangent spaces T_pM_0 are isotropic, coisotropic, lagrangian or symplectic subspaces of T_pM respectively.

Suppose now that a submanifold M_0 is a **coisotropic submanifold** of M, let $\iota\colon M_0 \hookrightarrow M$ denote the immersion and $\Omega_0 = i^*\Omega$ the pull back of the symplectic form of M onto M_0. This defines a distribution which we denote by $TM_0^\perp$, as follows. For $p \in M_0$ we let $(TM_0^\perp)_p := (T_pM_0)^\perp$. This distribution is involutive: let $X, Y \in TM_0^\perp$, for all vector fields Z tangent to M_0, we have that

$$
\begin{aligned}
0 &= d\Omega_0(X,Y,Z) && (3.1.2)\\
&= X\Omega_0(Y,Z) - Y\Omega_0(X,Z) + Z\Omega_0(X,Y) && (3.1.3)\\
&\quad -\Omega_0([X,Y],Z) + \Omega_0([X,Z],Y) - \Omega_0([Y,Z],X). && (3.1.4)
\end{aligned}
$$

But all terms except the fourth are automatically zero since they involve Ω_0 contractions between TM_0 and $TM_0^\perp$. Therefore the fourth term is also zero and this implies $[X,Y] \in TM_0^\perp \;\forall X, Y \in TM_0^\perp$. Therefore, by Frobenius' theorem, $TM_0^\perp$ are the tangent space to the leaves of a foliation and we denote by $\pi\colon M_0 \to \widetilde{M}$ the natural surjection mapping the points of M_0 to the unique connected leaf they belong to. Then if $\widetilde{M}$ is a smooth manifold, whose tangent space at a leaf would be isomorphic to $T_pM_0/T_pM_0^\perp$ for any point p lying in that leaf. We can therefore give $\widetilde{M}$ a symplectic structure $\widetilde{\Omega}$ by demanding that $\pi^*\widetilde{\Omega} = \Omega_0$.

In other words, let $\widetilde{X}$, $\widetilde{Y}$ be vectors tangent to $\widetilde{M}$ at a leaf. To compute $\widetilde{\Omega}(\widetilde{X},\widetilde{Y})$ we merely lift $\widetilde{X}$ and $\widetilde{Y}$ to vectors X_0 and Y_0 tangent to M_0 at a point p in the leaf and then compute $\Omega_0(X_0, Y_0)$. The result is clearly independent of the particular lift since the difference of any two lifts belongs to $TM_0^\perp$ and is independent of the particular chosen point p of the leaf since, if Z is a tangent vector to the leaf, the Lie derivative of Ω_0 along Z

$$L_Z\Omega_0 = di_Z\Omega_0 + i_Z d\Omega_0 = 0. \tag{3.1.5}$$

Therefore $(\widetilde{M},\widetilde{\Omega})$ becomes a symplectic manifold and it is called the **symplectic reduction** of (M,Ω) relative to the coisotropic submanifold (M_0,Ω_0).

Suppose now that M_0 is a **symplectic submanifold** of M and let $i\colon M_0 \hookrightarrow M$ denote its inclusion. We can give M_0 a symplectic structure merely by pulling back Ω to M_0. Hence (M_0,Ω_0), $\Omega_0 = i^*\Omega$, becomes a symplectic manifold, called the **symplectic restriction** of M onto M_0. In this case we can obtain explicitly the Poisson bracket of M_0 in terms of the Poisson bracket of M, as we are going to show in the following.

Let f and g be smooth functions on M_0, and let us extend them to smooth functions on M, and we will use the abuse of notation of still calling them f and g. Let X_f and X_g be their respective hamiltonian vector fields on M (see Appendix A). Since M_0 is symplectic, the tangent space of M at every point $p \in M_0$ can be decomposed in the following direct sum

$$T_pM = T_pM_0 \oplus (T_pM_0)^\perp, \tag{3.1.6}$$

according to which a vector field X can be decomposed as the sum of two vectors: X_T, tangent to M_0, and $X^\perp$ symplectically perpendicular to M_0. The Poisson bracket of the two functions f and g on M_0 is simply given by

$$\{f,g\}_0 = \Omega(X_f - X_f^\perp, X_g - X_g^\perp). \tag{3.1.7}$$

Now suppose that $\{Z_\alpha\}$ is a local basis for $TM_0^\perp$. Then the normal part $X^\perp$ of any vector X can be written

$$X^\perp = \sum_\alpha \lambda_\alpha Z_\alpha. \tag{3.1.8}$$

Then we notice that

$$\Omega(X, Z_\alpha) = \Omega(X^\perp, Z_\alpha) = \sum_\beta \lambda_\beta \Omega(Z_\beta, Z_\alpha) \tag{3.1.9}$$

and define the square matrix M whose entries are $M_{\alpha\beta} = \Omega(Z_\alpha, Z_\beta)$, which is invertible since M_0 is a symplectic submanifold. Hence we call the inverse $M^{\alpha\beta}$ which satisfies

$$\sum_\beta M_{\alpha\beta} M^{\beta\gamma} = \delta^\gamma_\alpha. \tag{3.1.10}$$

It follows that the coefficients λ_α are given by

$$\lambda_\beta = \sum_\alpha \Omega(X, Z_\alpha) M^{\alpha\beta}. \tag{3.1.11}$$

Then by putting Eq. (3.1.11) into Eqs. (3.1.8) and (3.1.7) we obtain

$$\{f, g\}_0 = \{f, g\} - \sum_{\alpha,\beta} \Omega(X_f, Z_\alpha) M^{\alpha\beta} \Omega(Z_\beta, X_g). \tag{3.1.12}$$

If we further assume that the vector fields Z_α are hamiltonian vector fields associated (via Ω) to the funcions χ_α, then

$$\{f, g\}_0 = \{f, g\} - \sum_{\alpha,\beta} \{f, \chi_\alpha\} M^{\alpha\beta} \{\chi_\beta, g\}. \tag{3.1.13}$$

Therefore $\{\cdot, \cdot\}_0$ is nothing but the **Dirac bracket** associated to the constraints χ_α.

3.1.2. First and second class constraints

In this subsection we will show that the submanifold defined by a set of first/second class constraints is respectively coisotropic/symplectic.
Let (M, Ω) be a symplectic manifold on which it is defined a set of smooth functions $\{\psi_a\}$ which are called **constraints**. This means that the allowed "phase space" of the relevant dynamical system is the zero locus of the constraints

$$\{p \in M \mid \psi_a(p) = 0 \quad \forall\, a\}. \tag{3.1.14}$$

Any other set of functions with the same zero locus gives an equivalent description of the physics. This fact will be crucial in the algebraic description of constraints of the subsequent sections.

Following Dirac [Dir01] let us denote by Ψ the linear subspace generated by the $\{\psi_a\}$, and by $\mathcal{J}$ the ideal of $C^\infty(M)$ they generate, i.e. linear combinations of the $\{\psi_a\}$ whose coefficients are arbitrary smooth functions. Then let F be a maximal subspace of Ψ with the property that

$$\{F, \Psi\} \subset \mathcal{J}. \tag{3.1.15}$$

Let $\{\varphi_i\}$ be a basis of F: Dirac defines this functions as **first class constraints**. Let now define the subspace $S \subset \Psi$ complementary to F to be spanned by the functions $\{\chi_\alpha\}$: Dirac calls these functions **second class constraints**.
Dirac proves that the matrix of functions $\{\chi_\alpha, \chi_\beta\}$ is nowhere degenerate, which is equivalent to the statment that the submanifold defined by the second class constraints is symplectic. In fact, let us define the function $\chi\colon M \to \mathbb{R}^k$ whose components are the second class constraints, i.e.

$$\chi(m) = (\chi_1(m), \ldots, \chi_k(m)) \tag{3.1.16}$$

and assume that the submanifold $N = \chi^{-1}(0)$ is a closed imbedded submanifold of M. Then the vectors tangent to N are precisely those vectors which are perpendicular to the gradients of the constraints. That is, X is a tangent vector to N if and only if $d\chi_\alpha(X) = 0$ for all $\alpha = 1, \ldots k$. By definition of hamiltonian vector fields Z_α associated to the constraints, the above condition is

$$X \in TN \Longleftrightarrow \Omega(X, Z_\alpha) = 0 \quad \forall\, \alpha. \tag{3.1.17}$$

It follows that the Z_α span the symplectic complement of TN. Therefore we can restrict ourselves to the symplectic manifold N with the Poisson bracket given by (3.1.13).

We now restrict the first class constraints $\{\varphi_i\}$ to N where they are still first class constraints and we will denote them again by $\{\varphi_i\}$, with a little abuse of notation. We again put them together by defining the function $\varphi\colon N \to \mathbb{R}^l$ and

assume that $N_0 \equiv \varphi^{-1}(0)$ is a closed imbedded submanifold. We will now show that N_0 is a coisotropic submanifold of N.
The tangent space to N_0 is again characterized by those vectors which are annihilated by the gradients of the constraints

$$X \in TN_0 \Longleftrightarrow d\varphi_i(X) = 0 \, \forall \, i \tag{3.1.18}$$

which, by using the definition of the hamiltonian vector fields X_i associated to the constraints $\{\varphi_i\}$, it translates into

$$TN_0 = \langle X_i \rangle^{\perp}, \tag{3.1.19}$$

where $\langle X_i \rangle$ is the linear span of the X_is. Since the constraints now are first class, it follows

$$d\varphi_i(X_j) = \{\varphi_i, \varphi_j\} = c_{ij}^k \varphi_k, \tag{3.1.20}$$

which is zero on N_0. Therefore the X_i are tangent to N_0. This is equivalent to

$$TN_0^{\perp} \subset TN_0 \tag{3.1.21}$$

and hence N_0 is a coisotropic submanifold of N.

There is a slightly more geometrical version of the previous picture. It can be done by combining Gotay's presymplectic embedding theorem [GS81] with the previous discussion. This is, according to the presymplectic embedding theorem, given any presymplectic manifold (C, ω) where exists an essentially unique, symplectic manifold $(S, \widetilde{\Omega})$ such that C is embedded in S, $\iota \colon C \hookrightarrow S$ and $\omega = \iota^* \widetilde{\Omega}$. Then given a symplectic manifold (M, Ω) and a submanifold $C \subset M$, provided that the restriction ω of Ω to C is presymplectic (here we assume constant value of ω), then there exists a symplectic manifold $(S, \widetilde{\Omega})$ and a symplectic map $j \colon S \hookrightarrow M$ such that $j^* \Omega = \widetilde{\Omega}$ and $\iota^* \widetilde{\Omega} = \omega$. Notice that S is defined as a submanifold of M by the ideal $\mathcal{J}_S$ of functions vanishing at S. Because S is symplectic the ideal is generated by second class constraints. Moreover C is defined inside S by another ideal $\mathcal{J}_C$ and because C is coisotropic in S, this ideal consists of first class constraints.

3.2. Reduction of Poisson algebras

The power of the algebraic formalism is that it continues to make sense in situations where the geometry might be singular. The aim of this section is to show how it is possible to recast the symplectic and coisotropic reduction purely in the category of Poisson algebras.

Dual to a manifold M we have the commutative algebra $C^\infty(M)$ of its smooth functions which characterize it completely. To every point $p \in M$ there corresponds a closed maximal ideal $I(p)$ of $C^\infty(M)$ consisting of those functions vanishing at p. It turns out that these are all the maximal closed ideals. So that as a set, the manifold M is the set of maximal closed ideals of $C^\infty(M)$.

Similarly, if $\iota\colon M_0 \hookrightarrow M$ is a submanifold, it can be described by an ideal $I(M_0)$ consisting of the smooth functions vanishing on M_0. Clearly $I(M_0) = \bigcap_{p\in M_0} I(p)$. For the submanifolds described by the regular zero locus of a set of smooth functions, the ideal $I(M_0)$ is generated by the constraints on the manifold. We have the following isomorphism:

$$C^\infty(M_0) \cong C^\infty(M)/I(M_0). \tag{3.2.1}$$

If (M, Ω) is a symplectic manifold and $M_0 \hookrightarrow M$ is a symplectic submanifold then $I(M_0)$ is generated by the second class constraints.
We now provide an algebraic description of the case in which M_0 is coisotropic. Recall that vector fields are derivations of the algebra of functions: $\mathfrak{X}(M) = \operatorname{Der} C^\infty(M)$. From the above isomorphism, a derivation of $C^\infty(M)$ gives ries to a derivation of $C^\infty(M_0)$ if and only if it preserves the ideal $I(M_0)$:

$$\operatorname{Der} C^\infty(M_0) = \{X \in \operatorname{Der} C^\infty(M) \mid X(I(M_0)) \subset I(M_0)\}. \tag{3.2.2}$$

As we have seen in the previous subsection, the vector fields in $TM_0^\perp$ are precisely the hamiltonian vector fields which arise from functions if $I(M_0)$, whence the coisotropy condition $TM_0^\perp \subset TM_0$ becomes the condition that the vanishing ideal is closed under the Poisson bracket: $\{\mathcal{J}, \mathcal{J}\} \subset \mathcal{J}$.

If we denote by Ω_0 the restriction of Ω to M_0, then $\ker \Omega_0$ is an integrable distribution (we assume that the rank of Ω_0 is constant). Then the quotient space $\widetilde{M}$ of M_0 with respect to the connected leaves of $\ker \Omega_0$, inherits a symplectic structure provided it is a manifold. Finally the functions on $\widetilde{M}$ are those functions on M_0 which are constant on the leaves of the foliation defined by $\ker \Omega_0$. Since the tangent vectors to the leaves are the hamiltonian vector fields of functions in $I(M_0)$, we have an isomorphism

$$C^\infty(\widetilde{M}) = \{f \in C^\infty(M_0) \mid \{f, I(M_0)\} = 0\}, \tag{3.2.3}$$

where $\{f, I(M_0)\} = 0$ on M_0. Extending f to a function on M, the isomorphism becomes

$$C^\infty(\widetilde{M}) = \{f \in C^\infty(M) \mid \{f, I(M_0)\} \subset I(M_0)\}/I(M_0). \tag{3.2.4}$$

By generalizing these constructions, if we have a Poisson algebra (i.e. an associative Lie–Jordan algebra) $(\mathcal{L}, \{\cdot,\cdot\})$ and an ideal $\mathcal{J}$ with respect to the Jordan product, we can work out the algebraic reduction by taking the normalizer $\mathcal{N}_\mathcal{J}$ with respect to the ideal $\mathcal{J}$

$$\mathcal{N}_\mathcal{J} = \{x \in \mathcal{L} \mid \{x, \mathcal{J}\} \subset \mathcal{J}\} \tag{3.2.5}$$

which is a Poisson subalgebra, and a straightforward computation shows $\mathcal{N}_\mathcal{J} \cap \mathcal{J}$ is its Poisson ideal. Therefore $\widetilde{\mathcal{L}} = \mathcal{N}_\mathcal{J}/(\mathcal{N}_\mathcal{J} \cap \mathcal{J})$ inherits the structure of a Poisson algebra.

The example in which (M, Ω) is a symplectic manifold, $\mathcal{L} = C^\infty(M)$ and $\mathcal{J}$ is the ideal

$$\mathcal{J} = \{f \in C^\infty(M) \mid f|_{M_0} = 0\} \tag{3.2.6}$$

shows the connection with the discussion before. By using the second isomorphism theorem for vector spaces

$$\mathcal{N}/(\mathcal{N} \cap \mathcal{J}) \simeq (\mathcal{N} + \mathcal{J})/\mathcal{J} \tag{3.2.7}$$

and taking into account that the quotient by $\mathcal{J}$ can be identified with the restriction to the submanifold M_0, the right hand side can be described as the restriction to M_0 of the functions in $\mathcal{N}_\mathcal{J} + \mathcal{J}$, but $\mathcal{N}_\mathcal{J} + \mathcal{J} = C^\infty(M)$ for second class constraints.

3.2.1. Reduction by symmetries

Suppose that we have a Lie group G acting on a symplectic manifold M and we want to restrict our Poisson algebra to functions that are invariant under the action of the group.

The infinitesimal action of the group induces a family of vector fields $E \subset \mathfrak{X}(M)$ that are an integrable distribution and actually the action of G on M induces a map $\hat{\rho}\colon \mathfrak{g} \to \operatorname{Der} C^\infty(M) = \mathfrak{X}(M)$ which is a Lie algebra homomorphism. Then $E = \hat{\rho}(\mathfrak{g})$. If the action of G on M is faithful then $E \cong \hat{\rho}(\mathfrak{g})$. With these geometric data we introduce the subspace

$$\mathcal{E} = \{f \in C^\infty(M) \mid Xf = 0,\ \forall\, X \in E\} \tag{3.2.8}$$

that is a Jordan subalgebra ($\mathcal{E} \circ \mathcal{E} \subset \mathcal{E}$), but not necessarily a Lie subalgebra. When this is the case, i.e. if

$$\{\mathcal{E}, \mathcal{E}\} \subset \mathcal{E}, \tag{3.2.9}$$

the restrictions of the operations to $\mathcal{E}$ endows it with the structure of a Poisson subalgebra.

From the algebraic point of view the action of vector fields on functions is a derivation of the Jordan algebra product $\circ$:

$$X(f \circ g) = Xf \circ g + f \circ Xg, \tag{3.2.10}$$

and if this derivation is also a Lie derivation:

$$X\{f, g\} = \{Xf, g\} + \{f, Xg\}, \tag{3.2.11}$$

then one easily sees that $\mathcal{E}$ is a Lie subalgebra.

An example of the previous situation happens when E is a family of Hamiltonian vector fields, i.e. there exists a Lie subalgebra $\mathcal{G} \subset \mathcal{C}^\infty(\mathcal{M})$ such that $X \in E$ if and only if there is a $g \in \mathcal{G}$ with $Xf = \{g, f\}$ for any $f \in C^\infty(M)$. This kind of derivations, defined through the Lie product, are called inner derivations, they are always Lie derivations and therefore they define a Lie-Jordan subalgebra with the procedure described above.

Then if $J\colon M \to \mathfrak{g}^*$ denotes the momentum map of the action, for any $\xi \in \mathfrak{g}$,

$X_\xi \in E = \hat{\rho}(\mathfrak{g})$ is a Hamiltonian vector field with Hamiltonian $J_\xi = \langle J, \xi\rangle$. Thus we have $X_\xi(f) = \{J_\xi, f\}$ and in this case

$$\begin{aligned} X_\xi\{f,g\} &= \{J_\xi,\{f,g\}\} = \{\{J_\xi,f\},g\} + \{f,\{J_\xi,g\}\} \\ &= \{X_\xi f, g\} + \{f, X_\xi g\}. \end{aligned} \tag{3.2.12}$$

The submanifold $J^{-1}(0)$ (provided that 0 is a regular value of J) is coisotropic and the corresponding reduction is called Marsden–Weinstein reduction [MW74]. Actually Marsden–Weinstein reduction corresponds to reduce with respect to the manifold $J^{-1}(\mu)$, $\mu \in \mathfrak{g}^*$ which now is not coisotropic in general. As a particular instance of this situation consider a symplectic manifold (M, Ω) with a strongly Hamiltonian action of the connected Lie group G.

3.3. More general Poisson reductions

One attempt to combine the previous reductions (by constraints and by symmetries) to define a more general one is contained in [MR86]. We shall rephrase here in algebraic terms the original construction that was presented in geometric language.

The data are an embedded submanifold $\iota : N \to M$ of a Poisson manifold and a subbundle $B \subset T_N M := \iota^*(TM)$. With these data we define the Jordan ideal $\mathcal{I} = \{f \in C^\infty(M) \mid f|_N = 0\}$ as before, and the Jordan subalgebra $\mathcal{B} = \{f \in C^\infty(M) \mid Xf = 0\, \forall\, X \in \Gamma(B)\}$. The goal is to define an associative Lie-Jordan structure in $\mathcal{B}/(\mathcal{B} \cap \mathcal{I})$.

Following [MR86] we assume that $\mathcal{B}$ is also a Lie subalgebra, then if $\mathcal{B} \cap \mathcal{I}$ is a Lie ideal of $\mathcal{B}$ the sought reduction is possible.

However, the condition that $\mathcal{B}$ is a subalgebra is a rather strong one [FZ08] and, consequently, the reduction procedure is much less general than initially expected. Actually, as we will show, it consists on a succesive application of the reductions introduced in the previous section. One can prove the following result.

Theorem 3.3.1 ([FZ08])**.** *With the previous definitions, if $\mathcal{B}$ is the proper subalgebra of $C^\infty(M)$ then the following statements hold:*

a) $\mathcal{B} \subset \mathcal{N} := \{g \in C^\infty(M) \mid \{\mathcal{I}, g\} \subset \mathcal{I}\}$.

b) $\mathcal{B} \cap \mathcal{I}$ *is Poisson ideal of* $\mathcal{B}$.

c) $\mathcal{B}/(\mathcal{B} \cap \mathcal{I})$ *always inherits a Poisson bracket.*

d) Take another $0 \neq B' \subset T_N(M)$ *and define* $\mathcal{B}'$ *accordingly. If* $B \cap TN = B' \cap TN \Leftrightarrow \mathcal{B} + \mathcal{I} = \mathcal{B}' + \mathcal{I}$ *by the second isomorphism theorem we have*

$$\mathcal{B}/(\mathcal{B} \cap \mathcal{I}) \simeq (\mathcal{B} + \mathcal{I})/\mathcal{I} \simeq \mathcal{B}'/(\mathcal{B}' \cap \mathcal{I}) \tag{3.3.1}$$

and the two Poisson brackets induced on $(\mathcal{B} + \mathcal{I})/\mathcal{I}$ *coincide.*

Proof. We prove a) by contradiction. Assume that $\mathcal{B} \not\subset \mathcal{N}$ then there exist functions $f \in \mathcal{B}$, $g \in \mathcal{I}$ and an open set $U \subset N$, such that

$$\{g, f\}(p) \neq 0, \quad \text{for any } p \in U. \tag{3.3.2}$$

But certainly $g^2 \in \mathcal{B}$ as a simple consequence of the Leibniz rule for the action of vector fields. Therefore, using that $\mathcal{B}$ is a Lie subalgebra we have

$$\{g^2, f\} = 2g\{g, f\} \in \mathcal{B} \tag{3.3.3}$$

and due to the fact that $g \in \mathcal{I}$ and $\{g, f\}(p) \neq 0$ this implies $g \in \mathcal{B}_U$, where $\mathcal{B}_U$ is the set of functions whose restriction to U coincide with the restriction of someone in $\mathcal{B}$.

So far we know that $g \in \mathcal{B}_U \cap \mathcal{I}$ and therefore $hg \in \mathcal{B}_U \cap \mathcal{I}$ for any $h \in C^\infty(M)$. But using that $\mathcal{B}_U$ is a Lie subalgebra as it is $\mathcal{B}$ (due to the local character of the Poisson bracket) we have

$$\{hg, f\} = h\{g, f\} + g\{h, f\} \in \mathcal{B}_U \Rightarrow h\{g, f\} \in \mathcal{B}_U \Rightarrow h \in \mathcal{B}_U. \tag{3.3.4}$$

But h is any function, then $\mathcal{B}_U = C^\infty(M)$ and $B|_U = 0$ which implies $B = 0$ as we assumed that it is a subbundle. This contradicts the hypothesis of the theorem and a) is proved.

b) follows immediately from a). Actually if $\mathcal{B} \subset \mathcal{N}$ we have $\{\mathcal{I}, \mathcal{B}\} \subset \mathcal{I}$ and moreover $\{\mathcal{B}, \mathcal{B}\} \subset \mathcal{B}$. Then $\{\mathcal{I} \cap \mathcal{B}, \mathcal{B}\} \subset \mathcal{I} \cap \mathcal{B}$.

c) is a simple consequence of the fact that $\mathcal{B}$ is a Lie-Jordan subalgebra and $\mathcal{B} \cap \mathcal{I}$ its Lie-Jordan ideal.

To prove d) take $f_i \in \mathcal{B}$ and $f'_i \in \mathcal{B}'$, $i = 1, 2$, such that $f_i + \mathcal{I} = f'_i + \mathcal{I}$. The Poisson bracket in $(\mathcal{B} + \mathcal{I})/\mathcal{I}$ is given by

$$\{f_1 + \mathcal{I}, f_2 + \mathcal{I}\} = \{f_1, f_2\} + \mathcal{I} \in (\mathcal{B} + \mathcal{I})/\mathcal{I}, \tag{3.3.5}$$

where for simplicity we use the same notation for the Poisson bracket in the different spaces, which should not lead to confusion. We compute now the alternative expression $\{f'_i + \mathcal{I}, f'_2 + \mathcal{I}\} = \{f'_1, +f'_2\} + \mathcal{I}$. We assumed $f'_i = f_i + g_i$ with $g_i \in \mathcal{I} \cap (\mathcal{B} + \mathcal{B}')$ and therefore, as a consequence of a), we have $\{f_1, g_2\}, \{g_1, f_2\}, \{g_1, g_2\} \in \mathcal{I}$, which implies

$$\{f'_1, +f'_2\} + \mathcal{I} = \{f_1, f_2\} + \mathcal{I} \tag{3.3.6}$$

and the proof is completed. □

Last property implies that the reduction process does not depend effectively on B but only on $B \cap TN$. Actually one can show that this procedure is simply a successive application of the two previous reductions presented before: first we reduce the Poisson bracket by constraints to N and then by symmetries with $E = B \cap TN$.

For completeness we would like to comment on the situation when $B = 0$. In this case $\mathcal{B} = C^\infty(M)$ and, of course, it is always a Lie subalgebra. Under these premises the reduction is not possible unless $\mathcal{I}$ is a Lie ideal which is not the case in general. Anyhow, if the conditions to perform the reduction are met and we consider some $B' \neq 0$ such that $B' \cap TN = 0$ and $\mathcal{B}'$ is a Lie subalgebra, then we obtain again property d) of the theorem: the Poisson brackets induced by $B = 0$ and B' on $\mathcal{B}/\mathcal{I}$ are the same.

The question that arises here actually is, is this reduction the most general one that can be performed using $\mathcal{N}$ and $\mathcal{B}$? Or, in other words, if we are given $\mathcal{N}$ and $\mathcal{B}$ does there exists a more general way to obtain the desired associative Lie-Jordan structure in $\mathcal{B}/(\mathcal{B} \cap \mathcal{I})$ where $\mathcal{B}$ and $\mathcal{I}$ are defined as before?

To answer this question we will rephrase the problem in purely algebraic terms. We shall assume that together with an associative Lie-Jordan algebra we are given a Jordan ideal $\mathcal{I}$ and a Jordan subalgebra $\mathcal{B}$. Of course, a particular example of this is the geometric scenario discussed before. Under these premises $\mathcal{B}\cap\mathcal{I}$ is a Jordan ideal of $\mathcal{B}$ and $\mathcal{B}+\mathcal{I}$ is a Jordan subalgebra, then it is immediate to define Jordan structures on $\mathcal{B}/(\mathcal{B}\cap\mathcal{I})$ and on $(\mathcal{B}+\mathcal{I})/\mathcal{I}$ such that the corresponding projections π_B and π are Jordan homomorphisms. Moreover, the natural isomorphism between both spaces is also a Jordan isomorphism. The problem is whether or not we can also induce a Poisson bracket in the quotient spaces compatible with the Jordan product. One first step to carry out this program is contained in the following theorem.

Theorem 3.3.2 ([FFIM13a]). *Given an associative Lie-Jordan algebra,* $(\mathcal{L},\circ,\{\ ,\ \})$, *a Jordan ideal* $\mathcal{I}$ *and a Jordan subalgebra* $\mathcal{B}$*, assume*

$$\text{a) } \{\mathcal{B},\mathcal{B}\}\subset\mathcal{B}+\mathcal{I}, \qquad \text{b) } \{\mathcal{B},\mathcal{B}\cap\mathcal{I}\}\subset\mathcal{I}, \tag{3.3.7}$$

then the following commutative diagram

$$\begin{array}{ccccc} \mathcal{B}\times\mathcal{B} & & \xrightarrow{\{\ ,\ \}} & & \mathcal{B}+\mathcal{I} \\ \big\downarrow{\scriptstyle \pi_B\times\pi_B} & & & & {\scriptstyle \pi}\big\downarrow \\ \mathcal{B}/(\mathcal{B}\cap\mathcal{I})\times\mathcal{B}/(\mathcal{B}\cap\mathcal{I}) & \longrightarrow & \mathcal{B}/(\mathcal{B}\cap\mathcal{I}) & \overset{\simeq}{\longleftrightarrow} & (\mathcal{B}+\mathcal{I})/\mathcal{I} \end{array} \tag{3.3.8}$$

defines a unique bilinear, antisymmetric operation in $\mathcal{B}/(\mathcal{B}\cap\mathcal{I})$ *that satisfies the Leibniz rule.*

Proof. In order to show that we define uniquely an operation we have to check that π_B is onto and that $\ker(\pi_B)\times\mathcal{B}$ and $\mathcal{B}\times\ker(\pi_B)$ are mapped into $\ker(\pi)=\mathcal{I}$. But first property holds because π_B is a projection and the second one is a consequence of (3.3.7,b). The bilinearity of the induced operation follows form the linearity or bilinearity of all the maps involved in the diagram and its antisymmetry derives form that of $\{\ ,\ \}$. Finally Leibniz rule is a consequence of the same property for the original Poisson bracket and the fact that π and π_B are Jordan homomorphisms. □

The problem with this construction is that, in general, the bilinear operation does not satisfy the Jacobi identity as shown in the following example.

Example 3.3.3. Consider $M = \mathbb{R}^3 \times \mathbb{R}^3$, with coordinates $(\mathbf{x}, \mathbf{y})$ and Poisson bracket given by the bivector $\Pi = \sum_{i=1}^{3} \frac{\partial}{\partial_{x_i}} \wedge \frac{\partial}{\partial_{y_i}}$. Take $N = \{(0, 0, x_3, \mathbf{y})\}$ and for a given $\lambda \in C^\infty(N)$ define $B = \operatorname{span}\{\partial_{x_1}, \partial_{x_2} - \lambda\partial_{y_1}\} \subset T_N M$ and

$$\mathcal{B} = \{f \in C^\infty(M),\ |\ Xf|_N = 0, \forall\, X \in \Gamma(B)\}. \tag{3.3.9}$$

Notice that $T_N M$ is a direct sum of B and TN, therefore we immediately get

$$\{\mathcal{B}, \mathcal{B}\} \subset \mathcal{B} + \mathcal{I} = C^\infty(M) \quad \text{and} \quad \{\mathcal{B}, \mathcal{B} \cap \mathcal{I}\} \subset \mathcal{I}, \tag{3.3.10}$$

and we meet all the requirements to define a bilinear, antisymmetric operation on $\mathcal{B}/(\mathcal{B} \cap \mathcal{I}) \simeq C^\infty(N)$.

Using coordinates $(x^3, \mathbf{y})$ for N the bivector field is

$$\Pi_N = \frac{\partial}{\partial_{x_3}} \wedge \frac{\partial}{\partial_{y_3}} + \lambda \frac{\partial}{\partial_{y_1}} \wedge \frac{\partial}{\partial_{y_2}} \tag{3.3.11}$$

that does not satisfy the Jacobi identity unless $\partial_{x_3}\lambda = \partial_{y_3}\lambda = 0$.

Now the problem is to supplement (3.3.7) with more conditions to guarantee that the induced operation satisfies all the requirements for a Poisson bracket. We do not know a simple description of the minimal necessary assumption but a rather general scenario is the following proposition:

Proposition 3.3.4 ([FFIM13a]). *Suppose that in addition to the conditions of theorem 3.3.2 we have two Jordan subalgebras $\mathcal{B}_+$, $\mathcal{B}_-$*

$$\mathcal{B}_- \subset \mathcal{B} \subset \mathcal{B}_+ \quad \textit{and} \quad \mathcal{B}_\pm + \mathcal{I} = \mathcal{B} + \mathcal{I}, \tag{3.3.12}$$

such that

$$\text{a) } \{\mathcal{B}_-, \mathcal{B}_-\} \subset \mathcal{B}_+, \quad \text{b) } \{\mathcal{B}_-, \mathcal{B}_+ \cap \mathcal{I}\} \subset \mathcal{I}. \tag{3.3.13}$$

Then the antisymmetric, bilinear operation induced by (3.3.8) is a Poisson bracket, i.e. it fulfils the Jacobi identity.

Proof. To prove this statement consider any two functions $f_1, f_2 \in \mathcal{B}$ and, for $i = 1, 2$, denote by $f_{i,-}$ a function in $\mathcal{B}_-$ such that $f_i + \mathcal{I} = f_{i,-} + \mathcal{I} \subset \mathcal{B} + \mathcal{I}$. Due to (3.3.7) we know that

$$\{f_{1,-}, f_{2,-}\} + \mathcal{I} = \{f_1, f_2\} + \mathcal{I}, \tag{3.3.14}$$

but if (3.3.13,a) also holds,

$$\{f_{1,-}, f_{2,-}\} \in \mathcal{B}_+, \tag{3.3.15}$$

in addition we have that

$$\{f_{1,-}, f_{2,-}\}_- - \{f_{1,-}, f_{2,-}\} \in \mathcal{B}_+ \cap \mathcal{I}, \tag{3.3.16}$$

and using (3.3.13,b)

$$\{\{f_{1,-}, f_{2,-}\}_-, f_{3,-}\} + \mathcal{I} = \{\{f_{1,-}, f_{2,-}\}, f_{3,-}\} + \mathcal{I}. \tag{3.3.17}$$

Therefore the Jacobi identity for the reduced antisymmetric product derives from that of the original Poisson bracket. □

Notice that the whole construction has been made in algebraic terms and therefore it will have an immediate translation to the quantum realm. But before going to that scenario we reexamine the example before, Ex. 3.3.3, to show how it fits into the general result.

Example 3.3.5. We take definitions and notations from example 3.3.3. Now let $\tilde{\lambda}$ be an arbitrary smooth extension of λ to M, i.e. $\tilde{\lambda} \in C^\infty(M)$ such that $\tilde{\lambda}|_N = \lambda$, we define $E = \text{span}\{\partial_{x_1}, \partial_{x_2} - \tilde{\lambda}\partial_{y_1}\} \subset TM$ and $\mathcal{B}_- = \{f \in C^\infty(M) \mid Xf = 0,\ \forall X \in \Gamma(E)\}$.

If we define $\mathcal{B}_+ = \mathcal{B}$, it is clear that $\mathcal{B}_- \subset \mathcal{B} \subset \mathcal{B}_+$, $\mathcal{B}_\pm + \mathcal{I} = \mathcal{B} + \mathcal{I}$ and $\{\mathcal{B}_-, \mathcal{B}_+ \cap \mathcal{I}\} \subset \mathcal{I}$. But $\{\mathcal{B}_-, \mathcal{B}_-\} \subset \mathcal{B}_+$ if and only if $\partial_{x_3}\lambda = \partial_{y_3}\lambda = 0$.

Therefore, in our construction we can accommodate the most general situation in which the example provides a Poisson bracket. We believe that this is not always the case, but we do not have any further counterexamples.

We want to end this section with a comment on the possible application of the reduction described in this section to quantum systems. In this case the Lie-Jordan algebra is non-associative and due to the associator identity there is a deeper connection between the Jordan and Lie products. As a result the different treatment between the Jordan and the Lie part, that we considered in the case of associative algebras, is not useful any more and the natural thing to do is to consider a more *symmetric* prescription.

3.3.1. Generalized reduction of Lie–Jordan algebras

We propose in this subsection a generalisation of the standard reduction procedure (the quotient of subalgebras by ideals) for Lie–Jordan algebras along similar lines to those followed in the associative case.

The statement of the problem is the following: given a Lie-Jordan algebra $\mathcal{L}$ and two subspaces $\mathcal{B}, \mathcal{S}$ the goal is to induce a Lie-Jordan structure in the quotient space $\mathcal{B}/(\mathcal{B} \cap \mathcal{S})$.

If we assume the following conditions:

$$\mathcal{B} \circ \mathcal{B} \subset \mathcal{B} + \mathcal{S}, \qquad [\mathcal{B}, \mathcal{B}] \subset \mathcal{B} + \mathcal{S}, \tag{3.3.18a}$$

$$\mathcal{B} \circ (\mathcal{B} \cap \mathcal{S}) \subset \mathcal{S}, \qquad [\mathcal{B}, \mathcal{B} \cap \mathcal{S}] \subset \mathcal{S}, \tag{3.3.18b}$$

then a diagram similar to the one in Theorem 3.3.2 allows to induce commutative and antisymmetric bilinear operations in the quotient. Now, in order to fulfil the ternary properties (Jacobi, Leibniz and associator identity) we need more conditions. We can show that, again, it is enough to have two more subspaces $\mathcal{B}_- \subset \mathcal{B} \subset \mathcal{B}_+$ such that $\mathcal{B}_\pm + \mathcal{S} = \mathcal{B} + \mathcal{S}$ and moreover we get the conditions substituting (3.3.18a) and (3.3.18b):

$$\mathcal{B}_- \circ \mathcal{B}_- \subset \mathcal{B}_+, \qquad [\mathcal{B}_-, \mathcal{B}_-] \subset \mathcal{B}_+, \tag{3.3.19a}$$

$$\mathcal{B}_- \circ (\mathcal{B}_+ \cap \mathcal{S}) \subset \mathcal{S}, \qquad [\mathcal{B}_-, (\mathcal{B}_+ \cap \mathcal{S})] \subset \mathcal{S}. \tag{3.3.19b}$$

Then, under these conditions, one can correctly induce a Lie-Jordan structure in the quotient. Conditions (3.3.19a) constitute a weaker version of the notion of Lie–Jordan subalgebra. Actually if $\mathcal{B}_- = \mathcal{B} = \mathcal{B}_+$, then we are just claiming that $\mathcal{B}_-$ is a Lie–Jordan subalgebra. On the other hand conditions (3.3.19b) constitute a weaker version of the notion of ideal. If $\mathcal{B}_- = \mathcal{B} = \mathcal{B}_+$ then (3.3.19b) just implies that S is an ideal of $\mathcal{B}$. Because of this we will say that the pair $\mathcal{B}_-$, $\mathcal{B}_+$ is a **weak Lie–Jordan subalgebra**, and that S is a **weak Lie–Jordan ideal** of $(\mathcal{B}_-, \mathcal{B}_+)$. Then we have proved:

Theorem 3.3.6. *Let $\mathcal{L}$ be a Lie–Jordan algebra and $\mathcal{B}_- \subset \mathcal{B} \subset \mathcal{B}_+$ a weak Lie–Jordan subalgebra and S a weak Lie–Jordan ideal of $(\mathcal{B}_-, \mathcal{B}_+)$. Then $\mathcal{B}/\mathcal{B} \cap S$ inherits a canonical Lie–Jordan structure*

There are at least two aspects of this construction that need more work. The first one is to find examples in which this reduction procedure is relevant, similarly to what we did for the classical case in the previous section. The second problem is of topological nature: given a Banach space structure in the big algebra $\mathcal{L}$, compatible with its operations, we can correctly induce a norm in the quotient provided $\mathcal{B}$ and $\mathcal{S}$ are closed subspaces. However, the induced operations need not to be continuous in general; though they are, if $\mathcal{B}$ is a subalgebra and $\mathcal{S}$ an ideal. The study of more general conditions for continuity and compatibility of the norm will be the subject of further research.

3.4. Quantum constraints and reduction of Lie–Jordan Banach algebras

In this section we show how to deal with **quantum constraints** in the setting of LJB–algebras [FFIM13c]. We assume for simplicity that there are no second-class constraints, otherwise one could introduce the Dirac bracket and treat the problem with first-class constraints only, as described in Sec. 3.1. In the Dirac quantization method, one assumes that it is possible to quantize the classical constraints c_i, by associating them the operators $\hat{c}_i$ acting on some Hilbert space $\mathcal{H}$ and require that every physical state should remain unchanged if one performs a transformation generated by the constratins. Hence the constraints form the set $\{\mathcal{C},\ \hat{c}_i \in \mathcal{C}\}$ and the selection condition

$$\hat{c}_i|\psi\rangle = 0, \quad \forall\, \hat{c}_i \in \mathcal{C}, \tag{3.4.1}$$

identifies the subspace of physical states:

$$\mathcal{H}^C \equiv \{|\psi\rangle \ \mid \ \hat{c}_i|\psi\rangle = 0 \quad \forall\, \hat{c}_i \in \mathcal{C}\}. \tag{3.4.2}$$

The first-class classical constraints c_i satisfy:

$$[c_i, c_j] = f_{ij}^k c_k. \tag{3.4.3}$$

This relation is not always preserved at the quantum level, and it may be spoiled by extra terms of quantum mechanical origin,

$$[\hat{c}_i, \hat{c}_j] = \hat{f}_{ij}^k \hat{c}_k + \hat{D}_{ab}. \tag{3.4.4}$$

If this is the case, the physical states $|\psi\rangle$ should also obey the extra conditions

$$\hat{D}_{ab}|\psi\rangle = 0. \tag{3.4.5}$$

This last condition has no classical analog and may restrict the physical subspace too much. When $\hat{D}_{ab} \neq 0$ one says that **gauge invariance** is **broken** at the quantum level, and the operator $\hat{D}_{ab}$ is called a **gauge anomaly** [HT92]. But if gauge invariance is broken by quantum effects, it is meaningless to search for gauge-invariant states, i.e. to impose the conditions (3.4.1). Thus, one sees that the Dirac method is not directly applicable when there is a gauge anomaly. By using the algebraic reduction procedure described in this section, we will be able to identify the algebraic restrictions such that the physical subspace will not suffer the gauge anomaly problem.

In the LJB–algebra framework, one starts with an algebra $\mathcal{L}$ containing all physical observables, and assume that the constraints should appear in $\mathcal{L}$ as a subset $\mathcal{C}$. This assumption is justified by the fact that $\ker \hat{c}_i = \ker\, \hat{c}_i^* \hat{c}_i$, hence we can assume the constraint algebra to be self-adjoint, this is it belongs to the LJB–algebra $\mathcal{L}$.

Remark. Note that in physical relevant situations the operators $\hat{c}_i$ are not bounded. Then, by following the seminal works of Grundling *et al.* [GH85], [GL00] some possibilities may arise which we can handle within the algebraic framework:

1. if the $\hat{c}_i$ are unbounded but essentially selfadjoint, we can take the unitaries $\mathcal{U} \equiv \{\exp(it\hat{c}_i) \mid t \in \mathbb{R}\}$ and identify the constraint set with $\mathcal{C} = \{U - \mathbb{1} \mid U \in \mathcal{U}\}_{sa}$, i.e. taking the selfadjoint part defined by those unitaries.

2. If the $\hat{c}_i$ are unbounded and normal, we can identify $\mathcal{C}$ with $\{f(\hat{c}_i) \mid j \in I\}$, where f is a bounded real valued Borel function with $f^{-1}(0) = \{0\}$.

3. If the $\hat{c}_i$ are unbounded, closable and not normal, we can replace each $\hat{c}_i$ with the essentially selfadjoint operator $\hat{c}_i^* \hat{c}_i$, which is justified by the fact that as mentioned above $\ker \hat{c}_i = \ker\, \hat{c}_i^* \hat{c}_i$, reducing then to the case of essentially selfadjoint constraints.

The constraint set $\mathcal{C}$ select the physical state space, also called **Dirac states**

$$\mathcal{S}_D = \{\, \omega \in \mathcal{S}(\mathcal{L}) \mid \omega(c^2) = 0, \quad \forall\, c \in \mathcal{C}\,\}$$

where $\mathcal{S}(\mathcal{L})$ is the state space of $\mathcal{L}$. This is analogous to selecting the constraint submanifold in the classical reduction and to the quantum condition (3.4.1). Now, in our algebraic setting, we are able to define a "generalized" constraint subalgebra, which is a constraint subalgebra of $\mathcal{L}$ which gives rise to the same set of Dirac states. Hence we define the **vanishing subalgebra** $\mathcal{V}$ as:

$$\mathcal{V} = \{\, a \in \mathcal{L} \mid \omega(a^2) = 0, \quad \forall\, \omega \in \mathcal{S}_D\,\}.$$

Proposition 3.4.1. *$\mathcal{V}$ is a non-unital LJB–subalgebra.*

Proof. Let $a, b \in \mathcal{V}$. From (2.2.4) it follows:

$$(a \circ b)^2 = \kappa\, [b, [a \circ b, a]\,] + a \circ (b \circ (a \circ b)\,). \tag{3.4.6}$$

If we introduce $c = [a \circ b, a]$ and $d = b \circ (a \circ b)$, Eq. (3.4.6) becomes:

$$(a \circ b)^2 = \kappa\, [b, c\,] + (a \circ d\,). \tag{3.4.7}$$

From the inequalities (2.2.13)(2.2.14) it is easy to show that if $\omega(a^2) = 0$ then

$$\omega(a \circ b) = 0 = \omega([a, b]) \quad \forall\, b \in \mathcal{L}. \tag{3.4.8}$$

Then if we apply the state ω to the expression (3.4.7), from (3.4.8) it follows:

$$\omega(\,(a \circ b)^2\,) = \kappa\, \omega([b, c\,]) + \omega(a \circ d\,) = 0. \tag{3.4.9}$$

By definition of $\mathcal{V}$, this means that $\forall\, a, b \in \mathcal{V}$, $a \circ b \in \mathcal{V}$.
By applying the state ω to the relation

$$(a \circ b)^2 - \kappa\, [a, b]^2 = a \circ (b \circ (a \circ b)) - \kappa\, a \circ [b, [a, b]\,],$$

we obtain $\omega(\,[a, b]^2\,) = \omega((a \circ b)^2) = 0$, that is $\forall\, a, b \in \mathcal{V}$, $[a, b] \in \mathcal{V}$. Hence $\mathcal{V}$ is a Lie–Jordan subalgebra.
$\mathcal{V}$ also inherits the Banach structure since it is defined as the intersection of closed subspaces. □

The Prop. 3.4.1 tells us that the right requirement for overcoming the problems with gauge anomalies is to use the vanishing subalgebra $\mathcal{V}$ as constraint algebra. Since it is a Lie-subalgebra, if $c_i, c_j \in \mathcal{V}$, then also $[c_i, c_j] \in \mathcal{V}$.
We can also use the vanishing subalgebra to provide an alternative description of the Dirac states that will be useful later.

Proposition 3.4.2. *With the previous definitions we have*

$$\mathcal{S}_D = \{\omega \in \mathcal{S}(\mathcal{L}) \mid \omega(a) = 0, \quad \forall a \in \mathcal{V}\}$$

Proof. As $\mathcal{V}$ is a subalgebra and it contains $\mathcal{C}$ it is clear that the right hand side is included into $\mathcal{S}_D$.

To see the other inclusion it is enough to consider that for any state $\omega(a)^2 \leq \omega(a^2)$, therefore any Dirac state should vanish on $\mathcal{V}$. □

Define now the Lie normalizer as

$$\mathcal{N}_{\mathcal{V}} = \{\, a \in \mathcal{L} \mid [a, \mathcal{V}] \subset \mathcal{V} \,\} \tag{3.4.10}$$

which corresponds roughly to Dirac's concept of "first class variables" [Dir01], i.e. the set of physical observables which preserve the constraints.

Proposition 3.4.3. *$\mathcal{N}_{\mathcal{V}}$ is a unital LJB–algebra and $\mathcal{V}$ is a Lie–Jordan ideal of $\mathcal{N}_{\mathcal{V}}$.*

Proof. Let $a, b \in \mathcal{N}_{\mathcal{V}}$ and $v \in \mathcal{V}$. Then by definition of normalizer it immediately follows:

$$[[a, b], v] = [[a, v], b] + [[v, b], a] \in \mathcal{V}. \tag{3.4.11}$$

Let us now prove that $\forall\, v \in \mathcal{V},\ v \circ a \in \mathcal{V}$, this is $\mathcal{V}$ is a Jordan ideal of $\mathcal{N}_{\mathcal{V}}$:

$$\omega((v \circ a)^2) = \kappa\, \omega([a, [v \circ a, v]]) + \omega(v \circ (a \circ (a \circ v))) \tag{3.4.12}$$

which gives zero by repeated use of properties (2.2.13) and (2.2.14).
Then it becomes easy to prove that $\mathcal{N}_{\mathcal{V}}$ is a Jordan subalgebra:

$$[a \circ b, v] = [a, v] \circ b + a \circ [b, v] \in \mathcal{V}. \tag{3.4.13}$$

Finally, since the Lie bracket is continuous with respect to the Banach structure, it also follows that $\mathcal{N}_{\mathcal{V}}$ inherits the Banach structure by completeness. □

In the spirit of Dirac, the physical algebra of observables in the presence of the constraint set $\mathcal{C}$ is represented by the LJB–algebra $\mathcal{N}_{\mathcal{V}}$ which can be reduced by the closed Lie–Jordan ideal $\mathcal{V}$ which induces a canonical Lie–Jordan algebra structure in the quotient:

$$\widetilde{\mathcal{L}} = \mathcal{N}_{\mathcal{V}}/\mathcal{V}. \tag{3.4.14}$$

We will denote in the following the elements of $\widetilde{\mathcal{L}}$ by $\widetilde{a}$.
The quotient Lie-Jordan algebra $\widetilde{\mathcal{L}}$ carries the quotient norm,

$$\|\widetilde{a}\| = \| \, [a] \, \| = \inf_{b\in\mathcal{V}} \|a+b\|,$$

where $a \in \mathcal{N}_{\mathcal{V}}$ is an element of the equivalence class $[a]$ of $\mathcal{N}_{\mathcal{V}}$ with respect to the ideal $\mathcal{V}$. The quotient norm provides a LJB–algebra structure to $\widetilde{\mathcal{L}}$.

Hence the reduction of the Lie–Jordan algebra $\mathcal{L}$ with respect to the constraint set $\mathcal{C}$ is given by the short exact sequence

$$0 \to \mathcal{V} \to \mathcal{N}_{\mathcal{V}} \to \widetilde{\mathcal{L}} \to 0. \tag{3.4.15}$$

In the Subsection 3.4.2 we will prove that the states on the reduced LJB–algebra $\widetilde{\mathcal{L}}$ are exactly the Dirac states restricted to the physical algebra of observables $\mathcal{N}_{\mathcal{V}}$.

3.4.1. Reduction of Lie–Jordan Banach algebras and constraints in C^*–algebras

Following [GH85], [GL00] we briefly recall how to deal with quantum constraints in a C^*–algebra setting. The aim of this section is to prove that the reduction procedure of C^*–algebras used to analyze quantum constraints, also called T–reduction, can be equivalently described by using the theory of reduction of LJB–algebras discussed above.

A quantum system with constraints is a pair $(\mathcal{F}, \mathcal{C})$ where now the field algebra $\mathcal{F}$ is a unital C^*–algebra containing the self–adjoint constraint set $\mathcal{C}$, i.e. $C = C^* \; \forall \, C \in \mathcal{C}$. The constraints select the Dirac states

$$\mathcal{S}_D \equiv \{ \, \omega \in \mathcal{S}(\mathcal{F}) \mid \omega(C^2) = 0, \quad \forall \, C \in \mathcal{C} \, \}$$

where $\mathcal{S}(\mathcal{F})$ is the state space of $\mathcal{F}$.

Define $\mathcal{D} = [\mathcal{F}\mathcal{C}] \cap [\mathcal{C}\mathcal{F}]$ where the notation $[\cdot]$ denotes the closed linear space generated by its argument and satisfies the following

Theorem 3.4.4. *$\mathcal{D}$ is the largest non-unital C^*–algebra in* $\bigcap_{\omega \in \mathcal{S}_D} \ker \omega$.

For any set $\Omega \subset \mathcal{F}$, define as before its normalizer or "weak commutant" as

$$\Omega_W = \{ F \in \mathcal{F} \mid [F, H] \subset \Omega, \quad \forall H \in \Omega \}. \tag{3.4.16}$$

Consider now the multiplier algebra of Ω as

$$\mathcal{M}(\Omega) = \{ F \in \mathcal{F} \mid FH \in \Omega \text{ and } HF \in \Omega, \quad \forall H \in \Omega \} \tag{3.4.17}$$

i.e. the largest set for which Ω is a bilateral ideal. $\mathcal{M}(\Omega)$ is clearly a unital C^*–algebra and we have the following

Theorem 3.4.5. $\mathcal{O} \equiv \mathcal{D}_W = \mathcal{M}(\mathcal{D})$.

That is, the weak commutant of $\mathcal{D}$ is also the largest set for which $\mathcal{D}$ is a bilateral ideal and it will be denoted by $\mathcal{O}$. It follows that the maximal (and unital) C^*–algebra of physical observables determined by the constraints $\mathcal{C}$ is given by:

$$\widetilde{\mathcal{F}} = \mathcal{O}/\mathcal{D}. \tag{3.4.18}$$

To show that this procedure is equivalent to the reduction of the corresponding LJB–algebra (as discussed in Section 3.4), we need to prove some simple statements.

Lemma 3.4.6. *Let $\mathcal{Z}$ and $\mathcal{I}$ be two Lie-Jordan subalgebras of a LJB–algebra $\mathcal{L}$. Then $\mathcal{Z}^{\mathbb{C}} = \mathcal{Z} \oplus i\mathcal{Z}$ is the weak commutant (or Lie normalizer) of $\mathcal{I}^{\mathbb{C}} = \mathcal{I} \oplus i\mathcal{I}$ if and only if $\mathcal{Z}$ is the Lie normalizer of $\mathcal{I}$, i.e. $\mathcal{Z} = \mathcal{N}_{\mathcal{I}}$.*

Proof. Assume first $\mathcal{Z}^{\mathbb{C}}$ is the weak commutant of $\mathcal{I}^{\mathbb{C}}$ and let $a + ib \in \mathcal{Z}^{\mathbb{C}}$ with $a, b \in \mathcal{Z}$. By definition

$$[a + ib, \mathcal{I} \oplus i\mathcal{I}] \subset \mathcal{I} \oplus i\mathcal{I}$$

that is

$$[a+b,\mathcal{I}]\subset\mathcal{I}\quad\text{and}\quad[a-b,\mathcal{I}]\subset\mathcal{I}.$$

Since the normalizer is a vector space, this implies

$$[a,\mathcal{I}]\subset\mathcal{I}\quad\text{and}\quad[b,\mathcal{I}]\subset\mathcal{I},\quad\forall\, a,b\in\mathcal{Z},$$

that is $\mathcal{Z}$ is the Lie normalizer of $\mathcal{I}$. Conversely assume $\mathcal{Z}$ is the Lie normalizer of $\mathcal{I}$:

$$[a,\mathcal{I}]\subset\mathcal{I},\quad\forall\, a\in\mathcal{Z},$$

then it follows

$$[a+ib,x+iy]\in\mathcal{I},\quad\forall\, a,b\in\mathcal{Z}\quad\text{and}\quad\forall\, x,y\in\mathcal{I},$$

that is $\mathcal{Z}^{\mathcal{C}}$ is the weak commutant (or Lie normalizer) of $\mathcal{I}^{\mathcal{C}}$. □

Lemma 3.4.7. *Let $\mathcal{Z}$ and $\mathcal{I}$ be two Lie-Jordan subalgebras of $\mathcal{L}$. Then $\mathcal{I}$ is a Lie–Jordan ideal of $\mathcal{Z}$ if and only if $\mathcal{I}^{\mathcal{C}}=\mathcal{I}\oplus i\mathcal{I}$ is an associative bilateral ideal of $\mathcal{Z}^{\mathcal{C}}=\mathcal{Z}\oplus i\mathcal{Z}$.*

Proof. Using the expressions provided by eqs. (2.3.8) and (2.3.9), the statement becomes an easy computation. □

Let us define $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ such that $\mathcal{F}=\mathcal{L}\oplus i\mathcal{L}$ and $\widetilde{\mathcal{F}}=\widetilde{\mathcal{L}}\oplus i\widetilde{\mathcal{L}}$, i.e. they are the self-adjoint part of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ respectively. From Corollary 2.3.11 it follows that $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are unital LJB–algebras. Similarly define the LJB–algebras $\mathcal{N}_{\mathcal{J}}$ and $\mathcal{J}$ as the self-adjoint parts of $\mathcal{O}$ and $\mathcal{D}$ respectively, i.e. $\mathcal{O}=\mathcal{N}_{\mathcal{J}}\oplus i\mathcal{N}_{\mathcal{J}}$, $\mathcal{D}=\mathcal{J}\oplus i\mathcal{J}$.

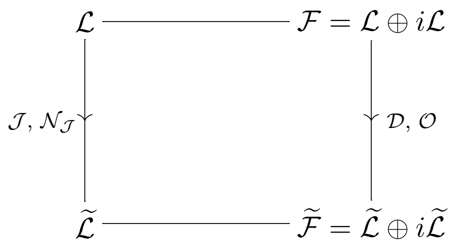
Theorem 3.4.8. *With the notations above, let $\mathcal{F}=\mathcal{L}\oplus i\mathcal{L}$ be the field algebra of the quantum system and $\mathcal{C}$ a real constraint set. Let $\mathcal{D}=[\mathcal{F}\mathcal{C}]\cap[\mathcal{C}\mathcal{F}]$, $\mathcal{O}=\mathcal{D}_W$ be as in Thm. 3.4.5, and $\widetilde{\mathcal{F}}=\mathcal{O}/\mathcal{D}=\widetilde{\mathcal{L}}\oplus i\widetilde{\mathcal{L}}$ be the reduced field algebra. Then:*

$$\widetilde{\mathcal{L}}=\mathcal{N}_{\mathcal{V}}/\mathcal{V},$$

with $\mathcal{V}$ and $\mathcal{N}_{\mathcal{V}}$ being the vanishing subalgebra of $\mathcal{L}$ and its Lie normalizer respectively.

Proof. Observe that the space of states on $\mathcal{F}$ is the space of states on $\mathcal{L}$ extended linearly by complexification and conversely $\mathcal{S}(\mathcal{L}) = \mathcal{S}(\mathcal{F})|_{\mathcal{L}}$. Then from Thm. 3.4.4 it follows that $\mathcal{D}$ is exactly the vanishing subalgebra for $\mathcal{S}_D$, that is $\mathcal{D} = \mathcal{V} \oplus i\mathcal{V}$. Then from the Lemmas 3.4.6 and 3.4.7 everything goes straightforward and the two procedures are clearly equivalent. □

The equivalence of the two approaches can be illustrated pictorially by the following "functorial" diagramme:



3.4.2. The space of states of the reduced LJB–algebra

The purpose of the remaining two sections is to discuss the structure of the space of states and the GNS construction of reduced states for reduced LJB–algebras with respect to the space of states of the unreduced one.

As it was discussed in the previous section, let $\mathcal{A}$ be a C^*–algebra, $\mathcal{L} = \mathcal{A}_{\rm sa}$ its real part and $\mathcal{V}$ the vanishing subalgebra of $\mathcal{L}$ with respect to a constraint set $\mathcal{C}$ and let $\mathcal{N}_{\mathcal{V}}$ be the Lie normalizer of $\mathcal{V}$. Then we will denote as before by $\widetilde{\mathcal{L}}$ the reduced Lie–Jordan Banach algebra $\mathcal{N}_{\mathcal{V}}/\mathcal{V}$ and its elements by $\widetilde{a}$.

Let $\widetilde{\mathcal{S}} = \mathcal{S}(\widetilde{\mathcal{L}})$ be the state space of the reduced LJB–algebra $\widetilde{\mathcal{L}}$, i.e. $\widetilde{\omega} \in \widetilde{\mathcal{S}}$ means that $\widetilde{\omega}(\widetilde{a}^2) \geq 0\ \forall \widetilde{a} \in \widetilde{\mathcal{L}}$, and $\widetilde{\omega}$ is normalized. Notice that if $\mathcal{L}$ is unital, then $\mathbb{1} \in \mathcal{N}_{\mathcal{V}}$ and $\mathbb{1} + \mathcal{V}$ is the unit element of $\widetilde{\mathcal{L}}$. We will denote it by $\widetilde{\mathbb{1}}$.
We have the following:

Lemma 3.4.9. *There is a one-to-one correspondence between normalized positive linear functionals on $\widetilde{\mathcal{L}}$ and normalized positive linear functionals on $\mathcal{N}_{\mathcal{V}}$ vanishing on $\mathcal{V}$.*

Proof. Let $\omega' \colon \mathcal{N}_{\mathcal{V}} \to \mathbb{R}$ be positive. The positive cone on $\widetilde{\mathcal{L}}$ consists of elements of the form $\widetilde{a}^2 = (a + \mathcal{V})^2 = a^2 + \mathcal{V}$, i.e.

$$\mathcal{K}^+_{\widetilde{\mathcal{L}}} = \{\, a^2 + \mathcal{V} \mid a \in \mathcal{N}_{\mathcal{V}} \,\} = \mathcal{K}^+_{\mathcal{N}_{\mathcal{V}}} + \mathcal{V}.$$

Thus if ω' is positive on $\mathcal{N}_{\mathcal{V}}$, $\omega'(a^2) \geq 0$, hence:

$$\omega'(a^2 + \mathcal{V}) = \omega'(a^2) + \omega'(\mathcal{V})$$

and if ω' vanishes on the closed ideal $\mathcal{V}$, then ω' induces a positive linear functional on $\widetilde{\mathcal{L}}$. Clearly ω' is normalized then the induced functional is normalized too because $\widetilde{\mathbb{1}} = \mathbb{1} + \mathcal{V}$.

Conversely, if $\widetilde{\omega} \colon \widetilde{\mathcal{L}} \to \mathbb{R}$ is positive and we define

$$\omega'(a) = \widetilde{\omega}(a + \mathcal{V})$$

then ω' is well-defined, positive, normalized and $\omega'|_{\mathcal{V}} = 0$. □

Notice also that given a positive linear functional on $\mathcal{N}_{\mathcal{V}}$ there exists an extension of it to $\mathcal{L}$ which is positive too.

Lemma 3.4.10. *Given a closed Jordan subalgebra $\mathcal{Z}$ of a LJB–algebra $\mathcal{L}$ such that $\mathbb{1} \in \mathcal{Z}$ and ω' is a normalized positive linear functional on $\mathcal{Z}$, then there exists $\omega \colon \mathcal{L} \to \mathbb{R}$ such that $\omega(a) = \omega'(a)$, $\forall\, a \in \mathcal{Z}$ and $\omega \geq 0$.*

Proof. Since $\mathcal{L}$ is a JB–algebra, it is also a Banach space. Due to the Hahn–Banach extension theorem, there exists a continuous extension ω of ω', i.e. $\omega(a) = \omega'(a)$, $\forall\, a \in \mathcal{Z}$, and moreover $\|\omega\| = \|\omega'\|$.

From the equality of norms and the fact that ω' is positive we have $\|\omega\| = \omega'(\mathbb{1})$, but ω is an extension of ω' then $\|\omega\| = \omega(\mathbb{1})$, which implies that ω is a positive functional and satisfies all the requirements stated in the lemma. □

We can now prove the following:

Theorem 3.4.11. *The set $\mathcal{S}_D(\mathcal{N}_{\mathcal{V}})$ of Dirac states on $\mathcal{L}$ restricted to $\mathcal{N}_{\mathcal{V}}$ is in one-to-one correspondence with the space of states of the reduced LJB–algebra $\widetilde{\mathcal{L}}$.*

Proof. In Prop. 3.4.2 we characterised the Dirac states as those that vanish on $\mathcal{V}$. Combining this result with that of Lemma 3.4.9 the proof follows. □

3.4.3. The GNS representation of reduced states

Finally, we will describe the GNS representation of a reduced state in terms of data from the unreduced LJB–algebra. Let $\widetilde{\mathcal{L}}$ be, as before, the reduced LJB–algebra of $\mathcal{L}$ with respect to the constraint set $\mathcal{C}$. Denote by $\widetilde{\mathcal{A}} = \widetilde{\mathcal{L}} \oplus i\widetilde{\mathcal{L}}$ the corresponding C^*–algebra and by $\widetilde{\mathcal{S}}$ its state space. Let $\widetilde{\omega} \in \widetilde{\mathcal{S}}$ be a normalized state on $\widetilde{\mathcal{A}}$. The GNS representation of $\widetilde{\mathcal{A}}$ associated to the state $\widetilde{\omega}$, denoted by

$$\pi_{\widetilde{\omega}} \colon \widetilde{\mathcal{A}} \to \mathcal{B}(\mathbf{H}_{\widetilde{\omega}}),$$

is defined as

$$\pi_{\widetilde{\omega}}(\widetilde{A})(\widetilde{B} + \mathcal{J}_{\widetilde{\omega}}) = \widetilde{A}\widetilde{B} + \mathcal{J}_{\widetilde{\omega}}, \quad \forall\, \widetilde{A}, \widetilde{B} \in \widetilde{\mathcal{A}},$$

where the Hilbert space $\mathbf{H}_{\widetilde{\omega}}$ is the completion of the pre–Hilbert space defined on $\widetilde{\mathcal{A}}/\mathcal{J}_{\widetilde{\omega}}$ by the inner product

$$\langle \widetilde{A} + \mathcal{J}_{\widetilde{\omega}}, \widetilde{B} + \mathcal{J}_{\widetilde{\omega}} \rangle \equiv \widetilde{\omega}(\widetilde{A}^* \widetilde{B})$$

and $\mathcal{J}_{\widetilde{\omega}} = \{\widetilde{A} \in \widetilde{\mathcal{A}} \mid \widetilde{\omega}(\widetilde{A}^* \widetilde{A}) = 0\}$ is the Gelfand left-ideal of $\widetilde{\omega}$. Let ω be a state on $\mathcal{A} = \mathcal{L} \oplus i\mathcal{L}$ that extends the state ω' on $\mathcal{N}_{\mathcal{V}}^{\mathbb{C}}$ induced by $\widetilde{\omega}$ according to Lemmas 3.4.9 and 3.4.10. Notice that ω vanishes on $\mathcal{V}$, thus the Gelfand ideal $\mathcal{J}_\omega$ of ω contains $\mathcal{V}$. We will have then:

Theorem 3.4.12. *There is a unitary equivalence between $\mathbf{H}_{\widetilde{\omega}}$ and the completion of the pre–Hilbert space:*

$$\mathbf{H}' = \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \Big/ \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \cap \mathcal{J}_\omega$$

with the inner product defined by

$$\langle A + \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \cap \mathcal{J}_\omega, B + \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \cap \mathcal{J}_\omega \rangle' \equiv \omega(A^* B), \quad \forall\, A, B \in \mathcal{N}_{\mathcal{V}}^{\mathbb{C}}.$$

Proof. Notice first that $\langle \cdot, \cdot \rangle'$ is well defined because of the properties of the Gelfand ideal $\mathcal{J}_\omega$. Morover we have that

$$\mathbf{H}_{\widetilde{\omega}} = \widetilde{\mathcal{A}} / \mathcal{J}_{\widetilde{\omega}}$$

and from Thm. 3.4.8, $\widetilde{\mathcal{A}} = \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} / \mathcal{V}^{\mathcal{C}}$ and $\mathcal{J}_{\widetilde{\omega}} = \mathcal{J}_{\omega'} / \left(\mathcal{J}_{\omega'} \cap \mathcal{V}^{\mathcal{C}} \right)$.
Hence because $\mathcal{J}_{\omega'} = \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \cap \mathcal{J}_\omega$ and $\mathcal{V}^{\mathcal{C}} \subset \mathcal{J}_{\omega'}$, we have:

$$\begin{aligned} \mathbf{H}_{\widetilde{\omega}} &= \widetilde{\mathcal{A}} / \mathcal{J}_{\widetilde{\omega}} = \left(\mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \big/ \mathcal{V}^{\mathcal{C}} \right) \Big/ \left(\mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \cap \mathcal{J}_\omega \big/ \mathcal{V}^{\mathcal{C}} \right) \\ &\cong \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \Big/ \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \cap \mathcal{J}_\omega . \end{aligned}$$

□

Notice that

$$\mathbf{H}' = \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \Big/ \mathcal{N}_{\mathcal{V}}^{\mathbb{C}} \cap \mathcal{J}_\omega \cong \left(\mathcal{N}_{\mathcal{J}}^{\mathcal{C}} + \mathcal{J}_\omega \right) \Big/ \mathcal{J}_\omega .$$

Thus the reduced GNS construction corresponding to the state $\widetilde{\omega}$ is the GNS construction of any extension ω of $\widetilde{\omega}$ restricted to $\mathcal{N}_{\mathcal{V}}^{\mathbb{C}} + \mathcal{J}_\omega$. Notice that $\widetilde{\omega}$ will be a pure state if and only if $\pi_{\widetilde{\omega}}$ is irreducible, i.e. if the representation of π_ω of ω restricted to $\mathcal{N}_{\mathcal{V}}^{\mathbb{C}} + \mathcal{J}_\omega$ is irreducible. Then if $\mathcal{N}_{\mathcal{V}}^{\mathbb{C}} + \mathcal{J}_\omega = \mathcal{A}$, π_ω will be irreducible if ω is a pure state. If $\mathcal{N}_{\mathcal{V}}^{\mathbb{C}} + \mathcal{J}_\omega \subsetneq \mathcal{A}$, then the state ω extending $\widetilde{\omega}$ might be non pure.

4

ALGEBRAIC IMPLICATIONS OF COMPOSABILITY OF PHYSICAL SYSTEMS

4.1. Composition of classical and quantum systems: state space

The state spaces of systems in classical mechanics compose according to the Cartesian product. Namely, let X, Y be two such systems, each with the respective state spaces given by the manifolds M, N, then the state space of the composite system X and Y is the Cartesian product $M \times N$. Here one can note that composing two classical systems need not necessarily mean that the two systems shall interact, or that the states of the two systems may enter into a correlation. And if they do not, then the state space will be the whole $M \times N$. However, if they interact, and certain states of the component systems become correlated, then the state space of "$X \; and \, Y$" can be a strict subset of $E \times F$.

If X,Y are two quantum systems, with the respective state space associated with the Hilbert spaces $\mathcal{H}_X$ and $\mathcal{H}_Y$, then the state space of the composite quantum system will be associated with the *tensor product* $\mathcal{H}_X \otimes \mathcal{H}_Y$. The novelty in the quantum way of composing systems is that the resulting quantum composite state space is significantly larger than the classical composite state space. For istance, if $\mathcal{H}_X$ and $\mathcal{H}_Y$ are finite dimensional Hilbert spaces, with respective dimension n and m, then the dimension of $\mathcal{H}_X \otimes \mathcal{H}_Y$ is nm, whereas the dimension of $\mathcal{H}_X \times \mathcal{H}_Y$ is only $n + m$. Furthermore, one has the injective mapping

$$\begin{aligned} \mathcal{H}_X \times \mathcal{H}_Y &\to \mathcal{H}_X \otimes \mathcal{H}_Y && (4.1.1)\\ (x, y) &\mapsto x \otimes y. && (4.1.2) \end{aligned}$$

The composite states in

$$(\mathcal{H}_X \otimes \mathcal{H}_Y) \setminus (\mathcal{H}_X \times \mathcal{H}_Y) \tag{4.1.3}$$

do not correspond to any pair of component states $x \in \mathcal{H}_X$, $y \in \mathcal{H}_Y$, and thus provide a quantum feature that does not exist in classicl mechanics. The states $z \in (\mathcal{H}_X \otimes \mathcal{H}_Y) \setminus (\mathcal{H}_X \times \mathcal{H}_Y)$ are called *entangled* (see Sec. 4.4 for an algebraic treatment of entangled states).

4.2. Composition of classical and quantum systems: algebra of observables

From an algebraic point of view the composition of classical and quantum systems admits a unified picture because in both cases the relevant algebras of observables are composed via tensor product. In fact, in the classical case, if M and N are two manifolds, then the space of smooth functions (with compact support) $C^\infty(M \times N) \cong C^\infty(M) \bar{\otimes}\, C^\infty(N)$ [Trè06].

In the following we will denote a generic algebraic physical system as $\mathcal{U} = (\mathcal{L}, \circ, [,])$ where $\mathcal{L}$ is the associated Lie–Jordan Banach algebra with products $\circ$ and $[,]$. Inspired by Grgin and Petersen [GP76] we introduce the composability category $\mathfrak{U}$ composed by the algebraic systems $\mathcal{U}$, equipped with a product $\bigcirc$ satisfying the following properties:

(i) if $\mathcal{U}_1, \mathcal{U}_2 \in \mathfrak{U}$ and $\mathcal{U}_{12} = \mathcal{U}_1 \bigcirc \mathcal{U}_2$, then $\mathcal{L}_{12} = \mathcal{L}_1 \otimes \mathcal{L}_2$,
i.e. the underlying algebra is the tensor product algebra of the component algebras.

(ii) Associativity:

$$(\mathcal{U}_1 \bigcirc \mathcal{U}_2) \bigcirc \mathcal{U}_3 = \mathcal{U}_1 \bigcirc (\mathcal{U}_2 \bigcirc \mathcal{U}_3). \quad (4.2.1)$$

(iii) Existence of a unit.
The underlying field of real numbers $\mathbb{R}$ is a unit for the composition product:

$$\mathbb{R} \bigcirc \mathcal{U} = \mathcal{U} = \mathcal{U} \bigcirc \mathbb{R}, \quad (4.2.2)$$

for all $\mathcal{U} \in \mathfrak{U}$.

With these conditions on the composition product $\bigcirc$, the composition category $\mathfrak{U}$ is a **semigroup** with a unit.

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two algebras belonging to the category $\mathfrak{U}$, with Lie–Jordan algebras $(\mathcal{L}_1, \circ, [\cdot,\cdot], \kappa_1)$ and $(\mathcal{L}_1, \circ, [\cdot,\cdot], \kappa_2)$. The most general way to endow the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$ with a Lie–Jordan structure with constant κ is by defining the products:

$$(a_1 \otimes b_1) \circ (a_2 \otimes b_2) := (a_1 \circ a_2) \otimes (b_1 \circ b_2) + \sqrt{\kappa_1 \kappa_2}\, [a_1, a_2] \otimes [b_1, b_2], \quad (4.2.3)$$

$$[a_1 \otimes b_1, a_2 \otimes b_2] = \sqrt{\frac{\kappa_2}{\kappa}}\, (a_1 \circ a_2) \otimes [b_1, b_2] + \sqrt{\frac{\kappa_1}{\kappa}}\, [a_1, a_2] \otimes (b_1 \circ b_2), \quad (4.2.4)$$

where $a_1, a_2 \in \mathcal{L}_1$ and $b_1, b_2 \in \mathcal{L}_2$. It can be checked by simple inspection that these products 4.2.3 and 4.2.4 satisfy all the axioms 2.2.1-2.2.4. Regarding the Banach structure, we know from the theory of tensor products of C^*–algebras (see the Appendix B) that is always possible to define a compatible Banach structure on the tensor product algebra.

Remark. Notice that the above definitions satisfy the associativity condition (ii) of the compsability category, i.e.:

$$\{(a_1 \otimes b_1) \otimes c_1\} \circ \{(a_2 \otimes b_2) \otimes c_2\} - \{a_1 \otimes (b_1 \otimes c_1)\} \circ \{a_2 \otimes (b_2 \otimes c_2)\} = 0 \quad (4.2.5)$$

and

$$[(a_1 \otimes b_1) \otimes c_1, (a_2 \otimes b_2) \otimes c_2] - [a_1 \otimes (b_1 \otimes c_1), a_2 \otimes (b_2 \otimes c_2)] = 0. \quad (4.2.6)$$

Remark. Notice that from our Theorem 2.3.6 we know we can obtain the associative $*$–product on $\mathcal{A}_1 \otimes \mathcal{A}_2 = (\mathcal{L}_1 \oplus i\mathcal{L}_1) \otimes (\mathcal{L}_2 \otimes \mathcal{L}_2)$ by

$$(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = (x_1 \otimes y_1) \circ (x_2 \otimes y_2) - i\sqrt{\kappa}\,[x_1 \otimes y_1, x_2 \otimes y_2], \quad (4.2.7)$$

where $x_1, x_2 \in \mathcal{A}_1$ and $y_1, y_2 \in \mathcal{A}_2$. As one would expect, it follows

$$(x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = (x_1 \cdot x_2) \otimes (y_1 \cdot y_2), \quad (4.2.8)$$

where $(x_1 \otimes y_1) = x_1 \circ y_1 - i\sqrt{\kappa_1}\,[x_1, y_1]$ according to Eq. (2.3.6) and the same holds for the second subsystem. We then choose an appropriate norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$ as to make it a C^*–algebra, as explained in the Appendix B.

4.3. Algebraic proof of the uniqueness of the Planck's constant $\hbar$

The concept of a universal quantization constant has been central to modern physics since its introduction by Planck in 1900. As accustomed as we have become to thinking of Planck's constant $\hbar = \frac{h}{2\pi}$, in terms of which both angular momentum and action are measured, it cannot be logically excluded that different realms of physics are in fact described by distinct quantization constants. In order to understand that such a possibility is meaningful phenomenologically, it is useful to recall the classical experiment of Beth [Bet36] on the measure of the angular momentum of the photon. By passing circularly polarized light of known intensity through a quartz retardation plate suspended from a torsion fiber, Beth was able to determine the angular momentum transmitted to the plate by a single photon. His result was consistent with the theoretical expectation $J = \hbar$. The Beth experiment can, in principle, be adapted to measure the intrinsic angular momentum of any particle citefischbach1991new.

The existence of a multiplicity of quantization constants has important implications when one goes beyond the single-particle quantum mechanics. This follows from the realization that the introduction of multiple Planck's constants in a system with two or more particles leads to apparent violations of space-time conservation laws. To see this, consider two particles having a common mass m, whose coordinates q_1, q_2 and momenta p_1, p_2 satisfy[1]

$$[q_1, p_1] = i\hbar_1, \tag{4.3.1}$$
$$[q_2, p_2] = i\hbar_2, \tag{4.3.2}$$
$$[q_1, q_2] = [p_1, p_2] = 0. \tag{4.3.3}$$

We assume that the Hamiltonian H of the system is given by

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(q_1 - q_2), \tag{4.3.4}$$

where the interaction potential V depends only on the separation of the two particles, and hence is translationally invariant. If we introduce the relative and center-of-mass coordinates in the usual way,

$$u = q_1 - q_2, \quad k = \frac{1}{2}(p_1 - p_2), \tag{4.3.5}$$
$$R = \frac{1}{2}(q_1 + q_2), \quad P = p_1 + p_2, \tag{4.3.6}$$

then H can be written in the conventional form

$$H = \frac{P^2}{2M} + \frac{k^2}{2\mu} + V(u), \tag{4.3.7}$$

where $M = 2m = 4\mu$. The new coordinates satisfy:

$$[u, R] = [k, P] = 0, \tag{4.3.8}$$
$$[u, k] = [R, P] = \tfrac{1}{2} i(\hbar_1 + \hbar_2), \tag{4.3.9}$$
$$[u, P] = 4[R, k] = i(\hbar_1 - \hbar_2). \tag{4.3.10}$$

From Eq. (4.3.10) it follows that

$$[H, P] = [V(u), P] \neq 0. \tag{4.3.11}$$

Hence in this very simple model the center-of-mass momentum P is not a constant of the motion. Due to these implications, it is then significant to understand the

[1] Here we stick to the usual definition of commutator $[a, b] = ab - ba$.

possibility for the existence of multiple Planck's constants. Using the theory of Lie–Jordan algebras and the results of the previous section on the composition of two such algebras, we will be able to prove, from algebraic considerations only, that the Planck's constant is **unique**.

Remember that a Lie–Jordan algebra is defined with a "free" parameter, κ, which links the Jordan with the Lie associator, Eq. (2.2.4):

$$(a \circ b) \circ c - a \circ (b \circ c) = \kappa \, [b, [c, a]] \, , \tag{4.3.12}$$

and it is in turn linked to the extension to the associative *–product, Eq. (2.3.6):

$$ab = a \circ b - i\sqrt{\kappa} \, [a, b]. \tag{4.3.13}$$

This freedom reflects also in the Lie product, which can be expressed in terms of the associative product as in Eq. (2.2.6), i.e. it is a scaled commutator:

$$[a, b] = i\lambda(ab - ba), \tag{4.3.14}$$

where

$$\kappa\lambda^2 = \frac{1}{4}. \tag{4.3.15}$$

κ can take any positive value, but for the correspondence principle, we would like to require that in the limit $\hbar \to 0$, the Jordan product becomes associative, that is $\kappa \to 0$. Hence, the simplest expression of κ must be a linear homogeneous function of $\hbar$ or at most a quadratic homogeneous function in $\hbar$. This means that if we are able to prove the uniqueness of the parameter κ, we have proved the uniqueness of $\hbar$.

In the previous section, given two Lie–Jordan algebras $\mathcal{L}_1$ and $\mathcal{L}_2$, we have obtained the explicit expression for the definition of a Lie–Jordan structure on the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$, given by Eqs. (4.2.3), (4.2.4). We will now require another natural property on $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2$, i.e. that the restriction of the products on $\mathcal{L}$ to the two subalgebras $\mathcal{L}_1$ and $\mathcal{L}_2$ must be compatible. From the subalgebra immersions

$$\mathcal{L}_1 \hookrightarrow \mathcal{L}_1 \otimes \mathbb{1}, \quad \mathcal{L}_2 \hookrightarrow \mathbb{1} \otimes \mathcal{L}_2, \tag{4.3.16}$$

the restriction requirement means that

$$(a_1 \otimes \mathbb{1}) \circ (a_2 \otimes \mathbb{1}) = (a_1 \circ a_2) \otimes \mathbb{1} \tag{4.3.17}$$

and

$$[a_1 \otimes \mathbb{1}, a_2 \otimes \mathbb{1}] = [a_1, a_2] \otimes \mathbb{1}, \tag{4.3.18}$$

for all $a_1, a_2 \in \mathcal{L}_1$ and the same for $b_1, b_2 \in \mathcal{L}_2$. In particular the Eqs. 4.3.18 and 4.2.4 imply that

$$\kappa_1 = \kappa_2 = \kappa, \tag{4.3.19}$$

that is all the Lie–Jordan algebras must have the same defining constant κ in order to be composed. Since κ is an homogeneous polynomial function of $\hbar$, this implies the **uniqueness** of the Planck's constant $\hbar$.

Remark. Note that Eq. (4.3.19) also implies that we cannot compose a classical with a quantum system, since for the latter $\kappa \neq 0$ and for the former $\kappa = 0$. This is in agreement with the results of [CS99], which essentially mean that we cannot compose a classical with a quantum algebra without relaxing some of the axioms 2.2.1-2.2.4 defining Lie–Jordan algebras.

4.4. Algebraic aspects of quantum entanglement

To describe subsystems of the quantum system, we choose some subalgebras of the total algebra $\mathcal{L}$. In the context of separability of quantum states, we consider a pair $(\mathcal{L}_A,\ \mathcal{L}_B)$ of isomorphic subalgebras of $\mathcal{L}$ with the following properties:

(i) the subalgebras $\mathcal{L}_A$ and $\mathcal{L}_B$ are statistically independent, in the sense that for all $a \in \mathcal{L}_A$ and $b \in \mathcal{L}_B,\ [a, b] = 0$,

(ii) the subalgebras $\mathcal{L}_A$ and $\mathcal{L}_B$ generate the total algebra i.e. $\mathcal{L} = \mathcal{L}_A \vee \mathcal{L}_B$.

Any pair of subalgebras satisfying the above conditions will be called *a Bell pair* of subalgebras of the total algebra $\mathcal{L}$.

Definition 4.4.1. Let $(\mathcal{L}_A,\, \mathcal{L}_B)$ be a Bell pair of subalgebras of $\mathcal{L}$. The pure state ω on $\mathcal{L}$ is $(\mathcal{L}_A,\, \mathcal{L}_B)$–separable if

$$\omega(ab) = \omega(a)\omega(b), \quad a \in \mathcal{L}_A,\ b \in \mathcal{L}_B. \tag{4.4.1}$$

A mixed state is $(\mathcal{L}_A,\, \mathcal{L}_B)$–separable if it can be expressed as a convex combination of pure $(\mathcal{L}_A,\, \mathcal{L}_B)$–separable states.

The state ω is $(\mathcal{L}_A,\, \mathcal{L}_B)$–*correlated* or *non-separable* if it is not $(\mathcal{L}_A,\, \mathcal{L}_B)$–separable. To indicate how much a given pure state ω differs from the separable one for a fixed choice of Bell pair of subalgebras, we may introduce a quantity which measures the *total correlations* in the state ω. It is defined as

$$C_\omega(\mathcal{L}_A,\, \mathcal{L}_B) = \sup_{a,b} |\omega(ab) - \omega(a)\omega(b)|, \tag{4.4.2}$$

where the supremum is taken over all normalized elements $a \in \mathcal{L}_A$ and $b \in \mathcal{L}_B$. It follows that

$$0 \le C_\omega(\mathcal{L}_A,\, \mathcal{L}_B) \le 1. \tag{4.4.3}$$

In the next subsection we apply the idea of algebraic non-separability to the case of two qubits.

4.4.1. Two qubits

Consider the four-level quantum system given by the Hilbert space $\mathcal{H} = \mathbb{C}^\triangle$ with the canonical basis $\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3$ and $\mathbf{e}_4$. The total algebra $\mathcal{L}$ can be considered as generated by the identity $\mathbb{1}$ and the elements $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{15}$, where

$$\boldsymbol{\lambda}_i = \mathbb{1} \otimes \sigma_i, \quad \boldsymbol{\lambda}_{3+i} = \sigma_i \otimes \mathbb{1}, \quad i = 1, 2, 3 \tag{4.4.4}$$

and $\boldsymbol{\lambda}_j,\ j = 7, \ldots, 15$ are given by the tensor products of the Pauli matrices σ_i taken in the lexicographical order. In the following we will write

$$\mathcal{L} = [\, \mathbb{1}, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_{15} \,]. \tag{4.4.5}$$

An arbitrary element $a \in \mathcal{L}$ has the form

$$a = c_0 \mathbb{1} + \sum_{j=1}^{15} c_j \boldsymbol{\lambda}_j, \quad c_0,\, c_j \in \mathbb{C}, \tag{4.4.6}$$

so every state is defined by formula

$$\omega(a) = c_0 + \sum_{j=1}^{15} c_j w_j, \tag{4.4.7}$$

where

$$w_j = \omega(\boldsymbol{\lambda}_j), \quad j = 1, \ldots, 15 \tag{4.4.8}$$

are the real numbers.

In the case of two qubits, it is convenient to take subalgebras $\mathcal{L}_A$ and $\mathcal{L}_B$ defined in the following way. Let $a_1,\, a_2,\, a_3$ and $b_1,\, b_2,\, b_3$ be the linearly independent hermitian elements of $\mathcal{L}$, which satisfy

$$a_i^2 = b_i^2 = \mathbb{1}, \quad [a_i, b_j] = 0, \quad i, j = 1, 2, 3 \tag{4.4.9}$$

and

$$a_i \circ a_j = b_i \circ b_j = 0, \quad i \neq j,\, i, j, = 1, 2, 3 \tag{4.4.10}$$

We put

$$\mathcal{L}_a = [\, \mathbb{1},\, a_1,\, a_2,\, a_3 \,], \quad \mathcal{L}_b = [\, \mathbb{1},\, b_1,\, b_2,\, b_3 \,]. \tag{4.4.11}$$

Consider the elements $a \in \mathcal{L}_A$, $b \in \mathcal{L}_B$ defined as

$$a = \mu_1 a_1 + \mu_2 a_2 + \mu_3 a_3, \quad b = \nu_1 b_1 + \nu_2 b_2 + \nu_3 b_3. \tag{4.4.12}$$

where $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$, $\vec{\nu} = (\nu_1, \nu_2, \nu_3)$ are the real vectors. From the Eqs. (4.4.9) and (4.4.10)

$$a^2 = \|\vec{\mu}\|^2\, \mathbb{1}, \quad b^2 = \|\vec{\nu}\|^2\, \mathbb{1}, \tag{4.4.13}$$

so if the vectors $\vec{\mu}$ and $\vec{\nu}$ are normalized, $a^2 = \mathbb{1}$ and $b^2 = \mathbb{1}$. From now on we will always assume that $\|\vec{\mu}\| = \|\vec{\nu}\| = 1$. Let ω be an arbitrary pure state on $\mathcal{L}$. Notice that for a and b defined by (4.4.12)

$$\omega(ab) - \omega(a)\omega(b) = \langle\, \vec{\mu},\, Q\vec{\nu}\,\rangle, \tag{4.4.14}$$

where *the correlation matrix* $Q = (q_{ij})$ has the matrix elements

$$q_{ij} = \omega(a_i b_j) - \omega(a_i)\omega(b_j). \tag{4.4.15}$$

Thus

$$C_\omega(\mathcal{L}_A,\, \mathcal{L}_B) = \sup_{\vec{\mu},\vec{\nu}} |\langle\, \vec{a},\, Q\vec{b}\,\rangle| = \|Q\|, \tag{4.4.16}$$

where the supremum is taken over all normalized vectors $\vec{\mu},\, \vec{\nu} \in \mathbb{R}^3$. Thus in the case of two qubits, the total correlation in the pure state ω can be computed by finding the matrix norm of the corresponding correlation matrix Q i.e. the largest singular value of Q.

4.4.2. Bell pairs and entanglement

The most natural choice of Bell pair is obtained by considering the subalgebras

$$\mathcal{L}_{A_0} = [\, \mathbb{1},\, \boldsymbol{\lambda}_1,\, \boldsymbol{\lambda}_2,\, \boldsymbol{\lambda}_3\,], \quad \mathcal{L}_{B_0} = [\, \mathbb{1},\, \boldsymbol{\lambda}_4,\, \boldsymbol{\lambda}_5,\, \boldsymbol{\lambda}_6\,]. \tag{4.4.17}$$

All conditions defining a Bell pair are trivially satisfied. Notice also that $(\mathcal{L}_{A_0},\, \mathcal{L}_{B_0})$–correlated states can be identified with standard entangled states with respect to the partition $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$, therefore the Bell pair given by (4.4.17) will be called *canonical Bell pair* of subalgebras. An interesting link between the algebraic theory of non-separability and standard theory of entanglement is given by the following result [VPC04], [DGJ12]:

Theorem 4.4.2. *Let $(\mathcal{L}_{A_0},\, \mathcal{L}_{B_0})$ be the canonical Bell pair of subalgebras of the total algebra of two-qubit system. For an arbitrary pure state ω*

$$C_\omega(\mathcal{L}_{A_0},\, \mathcal{L}_{B_0}) = C(\omega), \tag{4.4.18}$$

where $C(\omega)$ is the concurrence of ω.

We are going to show by considering an explicit example that the notion of entanglement is highly non-unique. Non-separability of a state is always relative to the measurement setup, which fixes the specific choice of observables, forming statistically independent subalgebras $\mathcal{L} : A$ and $\mathcal{L}_B$.

We start the discussion of this point considering the states which are obviously separable with respect to the canonical subalgebras $\mathcal{L}_{A_0}$ and $\mathcal{L}_{B_0}$ 4.4.17. Take the vector states defined by the basic vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$, but consider observables belonging to different subalgebras of $\mathcal{L}$. In this case, let $\mathcal{L}_A$ and $\mathcal{L}_B$ be defined as follows [DGJ12]

$$\begin{aligned}
\mathcal{L}_A &= \left[\mathbb{1},\, \frac{1}{\sqrt{2}}(\boldsymbol{\lambda}_4 + \boldsymbol{\lambda}_{11}),\, \frac{1}{\sqrt{2}}(\boldsymbol{\lambda}_{10} - \boldsymbol{\lambda}_{12}),\, -\frac{1}{2}(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_3 - \boldsymbol{\lambda}_{13} + \boldsymbol{\lambda}_{15})\right], \\
\mathcal{L}_B &= \left[\mathbb{1},\, \frac{1}{\sqrt{2}}(\boldsymbol{\lambda}_7 + \boldsymbol{\lambda}_9),\, -\frac{1}{\sqrt{2}}(\boldsymbol{\lambda}_5 + \boldsymbol{\lambda}_8),\, \frac{1}{2}(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_3 - \boldsymbol{\lambda}_{13} - \boldsymbol{\lambda}_{15})\right].
\end{aligned}$$

Using the relations between the generators $\boldsymbol{\lambda}_j$, one can easily check that $(\mathcal{L}_A,\, \mathcal{L}_B)$ is a Bell pair. The states $\mathbf{e}_1$ and $\mathbf{e}_2$ have correlation matrices with all zero elements, but the correlation matrices corresponding to $\mathbf{e}_3$ and $\mathbf{e}_4$ are given by

$$Q_{\mathbf{e}_3} = \operatorname{diag}(1, 1, -1), \quad Q_{\mathbf{e}_4} = \operatorname{diag}(-1, -1, -1),$$

and

$$\|Q_{\mathbf{e}_3}\| = \|Q_{\mathbf{e}_4}\| = 1.$$

Thus the states $\mathbf{e}_1$ and $\mathbf{e}_2$ are $(\mathcal{L}_A,\, \mathcal{L}_B)$–separable, whereas the states $\mathbf{e}_3$ and $\mathbf{e}_4$ are maximally $(\mathcal{L}_A,\, \mathcal{L}_B)$–entangled.

Consider now the family of states which are maximally entangled with respect to canonical subalgebras $\mathcal{L}_{A_0}$ and $\mathcal{L}_{B_0}$. As it is known [BJO01], such property have the states $\omega_{\delta,\varphi,\vartheta}$, defined by vectors

$$\Psi_{\delta,\varphi,\vartheta} = \alpha\, \mathbf{e}_1 + \beta\, e^{i\varphi}\, \mathbf{e}_2 + \beta\, e^{i\vartheta} \mathbf{e}_3 - \alpha\, e^{i(\varphi+\vartheta)}\, \mathbf{e}_4, \tag{4.4.19}$$

where

$$\alpha = \frac{\delta}{\sqrt{2}}, \quad \beta = \sqrt{\frac{1-\delta^2}{2}},$$

and $\delta \in [0,1]$, $\varphi,\ \vartheta \in [0, 2\pi]$. This time we ask about entanglement properties of $\omega_{\delta,\varphi,\vartheta}$ but with respect to the experimental setup given by the pair $(\mathcal{L}'_A,\ \mathcal{L}'_B)$ defined below

$$\begin{aligned}
\mathcal{L}'_A &= \left[\mathbb{1},\ -\frac{1}{2}(\boldsymbol{\lambda}_3 - \boldsymbol{\lambda}_6 - \boldsymbol{\lambda}_7 + \boldsymbol{\lambda}_{11}),\ -\boldsymbol{\lambda}_{10},\ -\frac{1}{2}(\boldsymbol{\lambda}_3 + \boldsymbol{\lambda}_6 - \boldsymbol{\lambda}_7 - \boldsymbol{\lambda}_{11})\right], \\
\mathcal{L}'_B &= \left[\mathbb{1},\ \frac{1}{\sqrt{2}}(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_9), -\frac{1}{\sqrt{2}}(\boldsymbol{\lambda}_5 - \boldsymbol{\lambda}_{14}),\ \boldsymbol{\lambda}_{15}\right].
\end{aligned}$$

The corresponding correlation matrix is

$$C_{\omega_{\delta,\varphi,\vartheta}} = \|Q\| = \sqrt{\delta^2(1-\delta^2)\,(2+\cos 2\,\varphi - \cos 2\,\vartheta)}, \tag{4.4.20}$$

and we see that all states $\omega_{\delta,\varphi,\vartheta}$ with $\delta = 0$ or $\delta = 1$ and $\varphi,\ \vartheta$ arbitrary, are $(\mathcal{L}'_A, \mathcal{L}'_B)$–separable. The same property have the states with $\varphi = \pi/2,\ \vartheta = 0$ and any $\delta \in (0,1)$. On the other hand, the state defined by the vector

$$\Psi_{\frac{1}{\sqrt{2}},0,\frac{\pi}{2}} = \frac{1}{2}\,(\mathbf{e}_1 + \mathbf{e}_2 + i\,\mathbf{e}_3 - i\,\mathbf{e}_4\,)$$

gives the maximal value of the norm (4.4.20), so it is not only maximally correlated with respect to the canonical Bell pair $(\mathcal{L}_{A_0},\ \mathcal{L}_{B_0})$ but also with respect to the pair $(\mathcal{L}'_A,\ \mathcal{L}'_B)$. So we see that depending on the experimental setup, separable states can be maximally entangled and vice versa: maximally entangled states can be separable.

To avoid ambiguities in deciding if a given state is separable or entangled, we should always specify which statistically independent subalgebras of the total algebra of observables are considered. In the Sec. 5.7, we propose a measure of entanglement which overcomes these problems by focusing on the noncommutativity of the algebra.

5

QUANTUMNESS AND CLASSICALITY

Classical mechanics is the crystallization of our everyday experience of matter and motion. However, we have found that to deal with matter in the microscopic world we must use quantum mechanics. A fundamental aspect of modern physics is to characterize the crossover between the quantum world and the classical world. There are many different approaches to answer this question, yet no single approach captures the whole breath of physics. It is clear that for some systems classical physics arises from quantum physics in the large number limit, as formulated in the Bohr *correspondence principle*. Yet it is not fully understood how the many-particle limit gives rise to classical physics, and how much of quantum physics still remains. The validity of this principle and the range of its application still remains open. It seems however that with the improved experimental techniques the actual border between quantum and classical worlds has been moved into larger and larger systems [GJK$^+$03]. The most daring challenge to the correspondence principle remains the idea of macroscopic quantum systems. For example, there exists quite convincing evidence that certain macroscopic systems, like Josephson junctions, Bose-Einstein condensates or Rydberg atoms, preserve fundamental quantum properties. On the other hand, these arguments are strongly model-dependent and do not completely exclude the existence of an approximate classical description. There are many operational tasks that give sufficient conditions for the **quantumness** of a system. For instance, quantumness is revealed in the form of nonlocality by the violation of Bell's inequality. Therefore it is desirable to have a broad notion of quantumness that encompasses all (or many) other notions of quantumness. This problem, which goes back to the foundations of quantum mechanics, has become particularly relevant for the field of quantum information and quantum computation. A useful quantum computer should be a rather macroscopic machine which nevertheless preserves certain fundamental quantum properties. Moreover there are a number of tasks in computation and communication that can be performed only if quantum resources are available. This is the case, for example, when nonclassical states of light are employed in communication or metrology [Gla63], or when mathematical models can be simplified beyond classical limits by tailored quantum systems [GWRV12]. It is also believed that some examples of macroscopic quantum systems can provide promising implementations of quantum information processing.

5.1. Commutative algebras

Quantum mechanics has many points of similarity with classical mechanics and these aided in developing quantum mechanics; but there are also many essential points of difference. The most important is that not all dynamical variables can be measured at the same time. The dynamical variables constitute a noncommuting algebra from which a commuting subalgebra is selected by any possible measurement. In other words, some properties of quantum systems cannot be specified simultaneously. Well-known examples are the position and the momentum of any quantum particle, or the spin components of a spin particle.
Our aim is to find the quantumness tests which are model-independent but still operational and refer to the most fundamental mathematical differences between classical and quantum theory. In light of this, we present in this section a theorem characterizing commutative (i.e. classical) algebras. We will use a Lie–Jordan algebra $\mathcal{L}$ or its associated C^*–algebra $\mathcal{A} = \mathcal{L} \oplus i\mathcal{L}$ without any ambiguity when

it will be more useful.

Theorem 5.1.1. *Given a LJB–algebra* $(\mathcal{L}, \circ, [\cdot, \cdot])$ *with positive cone* $\mathcal{L}^+$*, the following statements are equivalent:*

(i) $(\mathcal{L}, [,])$ *is commutative;*

(ii) $(\mathcal{L}, \circ)$ *is associative;*

(iii) $\mathcal{L}$ *is isomorphic to* $C(X, \mathbb{R})$*, for some locally compact Hausdorff space* X*; if* $\mathcal{L}$ *has a unit,* X *is compact;*

(iv) if $a, b \in \mathcal{L}^+$ *such that* $a - b \in \mathcal{L}^+$*, then* $a^2 - b^2 \in \mathcal{L}^+$*;*

(v) if $a, b \in \mathcal{L}^+$*, then* $a \circ b \in \mathcal{L}^+$*.*

Proof. The equivalence between (i) and (ii) is the content of Thm. (2.2.1). The equivalence of (i) and (iii) is essentially a rephrase of the famous Gelfand-Naimark theorem for commutative algebras [BR03]. The equivalence between (i) and (iv) has been proved in [Oga55] and generalized in [Ped79]. Finally the proof of the equivalence between (iv) and (v) can be found in [APVR08]. □

5.2. Classical states and Quantumness Witnesses

As a consequence of Thm. 5.1.1, for a quantum system one can always find pairs of observables $a, b \in \mathcal{L}^+$ or $x, y \in \mathcal{L}^+$, $x - y \in \mathcal{L}^+$ such that the observables

$$q_{AVR} = a \circ b \tag{5.2.1}$$

or

$$q_V = x^2 - y^2 \tag{5.2.2}$$

are *not* positive semidefinite. Thus $q_{AVR}, q_V \in \mathcal{L}$ are **witnessing** the quantumness (i.e. noncommutativity) of the algebra $\mathcal{L}$ [APVR08], [FPVY12]. In order to provide a rigorous definition of quantumness witness we first make rigorous the intuitive definition of **classical states** [FPVY12].

Definition 5.2.1 (Classical states)**.** We say that a state $\rho \in \mathcal{S}(\mathcal{L})$ is **classical** if

$$\rho([a,b]) = 0, \quad \text{for any pair } a, b \in \mathcal{L}. \tag{5.2.3}$$

A state that is not classical is **quantum**.

Notice that we can have classical states even when the algebra is noncommutative (namely, even when there exist a and a such that $[a,b] \neq 0$). In words, classical states are trasparent to commutators, they do not "perceive" nonvanishing commutators. Moreover, the Def. 5.2.1 of classical state is weaker than the notion of classicality that emerges from Thm. 5.1.1. Indeed, $\mathcal{L}$ is commutative (or associative) if and only if *every* state $\rho \in \mathcal{S}(\mathcal{L})$ is classical.

Remark. Let us notice that, in general, mixtures are not classical states. For example, a qubit state $\rho = p|0\rangle\langle 0| + q|1\rangle\langle 1|$ is not classical, since it possesses coherence, e.g. $\langle -|\rho|+\rangle = c_0 c_1 (p-q)$ for $|+\rangle = c_0|0\rangle + c_1|1\rangle$ and $|-\rangle = c_1^*|0\rangle - c_0^*|1\rangle$, which is nonvanishing provided $p \neq q$ and $c_0, c_1 \neq 0$. On the other hand, the completely mixed state $\rho = \mathbb{1}/2$ is classical, in that it does not possess any coherence, $\langle -|\rho|+\rangle = 0$ for any c_0 and c_1.

Let us now define **Quantumness Witnesses** (QW).

Definition 5.2.2. We say that an observable $Q \in \mathcal{L}$ is a QW if

(i) for any classical state $\rho \in \mathcal{S}(\mathcal{L})$ one gets $\rho(Q) \geq 0$,

(ii) there exists a (quantum) state $\sigma \in \mathcal{S}(\mathcal{L})$ such that $\sigma(Q) < 0$.

The fact that the particular observables q_{AVR} in (5.2.1) are QWs follows from the following lemma proved in [FPVY12]. In the following two lemmas we confine ourselves to finite-dimensional systems and use the ciclicity property of states inherited by its representation as density matrices, according to Eq. (2.2.17). We will discuss the infinite dimensional case in the next section.

Lemma 5.2.3. *For any classical state $\rho \in \mathcal{S}(\mathcal{L})$ and for any pair $a, b \in \mathcal{L}$ with $a, b \in \mathcal{L}^+$ it happens that*

$$\rho(a \circ b) \geq 0. \tag{5.2.4}$$

We now prove that also the observables q_V defined in (5.2.2) are QWs.

Lemma 5.2.4. *For any classical state $\rho \in \mathcal{S}(\mathcal{L})$ and for any pair $x, y \in \mathcal{L}$ with $x, y \in \mathcal{L}^+$ and $y \geq x$, it happens that*

$$\rho(y^2 - x^2) \geq 0. \tag{5.2.5}$$

Proof. Consider the quantity

$$(y + x)(y - x) = y^2 - x^2 - 2i[y, x]. \tag{5.2.6}$$

Since both $(y + x)$ and $(y - x)$ are positive operators by assumption, there exists $a, b \in \mathcal{L}$ such that

$$(y + x)(y - x) = a^*ab^*b. \tag{5.2.7}$$

By evaluating the classical state we get

$$\rho((y + x)(y - x)) = \rho(a^*ab^*b) = \rho(ba^*ab^*) = \rho(c^*c) \geq 0, \tag{5.2.8}$$

where $c = ab^* \in \mathcal{L}$. From Eq. (5.2.6) we conclude that for any classical state ρ

$$\rho(c^*c) = \rho(y^2 - x^2) \geq 0. \tag{5.2.9}$$

□

Remark. In words, classical states do not even perceive the possible negativity of the operators $a \circ b$ and $y^2 - x^2$.

5.3. More on classical states and QWs in infinite dimensional systems

As emphasized in the previous section, classical states are defined to be "transparent" to all commutators in the algebra so that they do not "detect" the non-commutativity of the operators, which is the main algebraic property distinguishing quantum from classical systems. Observe however that the Def. 5.2.1 would not be suitable in the infinite-dimensional case. Think for example to the situation

in which we impose the canonical commutation relations (CCR) $[\hat{x}, \hat{p}] = i\hbar\mathbb{1}$[1], which would imply $\rho([\hat{x}, \hat{p}]) = i\hbar$, i.e. the non-existence of classical states.
In this section we provide a more general definition of classical states by an intelligent use of the Jordan product which coincides with the previous one in finite dimensions and avoids the problems related to the CCRs and is applicable to infinite-dimensional systems [FFMP14].

Definition 5.3.1. We say that a state $\rho \in \mathcal{S}(\mathcal{L})$ is *classical* if

$$\rho((a \circ b) \circ c - a \circ (b \circ c)) = 0, \qquad \forall\, a, b, c \in \mathcal{L} \tag{5.3.1}$$

In other words, classical states do not "detect" the lack of associativity of the algebra and therefore (nonvanishing) triple commutators.

Remark. Observe that the above definition of classical states is applicable also to algebras of unbounded operators.

As a consequence of Thm. 2.2.1 for a quantum (i.e. non-commutative) system, it is always possible to find a triple of observables a, b, c such that the observable

$$q = (a \circ b) \circ c - a \circ (b \circ c) \tag{5.3.2}$$

is non-vanishing. Moreover, classical states (5.3.1) vanish on q-observables. Thus $q \in \mathcal{L}$ is a candidate "witness" for the quantum nature of the algebra of observables. Notice that, unlike the quantumness witnesses introduced in the previous section, q detects quantumness as soon as $q \neq 0$. However, if one wants to consider only positive witnesses, one can always use q^2 instead of q.

Let us now prove that the Def. 5.3.1 is equivalent to Def. 5.2.1 for normal states, i.e. states that can be realized as density matrices as defined in Eq. (2.2.17). In this case we will commit the sin of not distinguishing between states ρ and density matrices $\widetilde{\rho}$.

Definition 5.3.2. Given a Lie algebra $\mathfrak{g}$, the derived algebra is $[\mathfrak{g}, \mathfrak{g}]$, i.e. the subalgebra generated by taking all possible Lie commutators.

[1] We use here the standard definition of commutator $[\hat{x}, \hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x}$, different from (2.2.6).

Remark. In many relevant cases the derived algebra is the whole algebra, and is called "perfect algebra". A more stringent result also holds true, that is all semisimple Lie algebras can be generated by repeated commutators of only two elements [Kur51]. This is true, e.g. for the algebras $\mathfrak{su}(\mathrm{n})$.

Lemma 5.3.3. *A normal state $\rho \in \mathcal{S}(\mathcal{L})$ is classical if and only if its density matrix $\tilde{\rho}$ belongs to the center of the derived algebra $[\mathcal{L}, \mathcal{L}]$.*

Proof. By using the the compatibility condition (2.2.4), the Leibniz identity (2.2.3) and the properties of the trace we have:

$$\begin{aligned} \rho((a \circ b) \circ c - a \circ (b \circ c)) &= \mathrm{Tr}(\widetilde{\rho} \circ [b, [c, a]]) \\ &= \mathrm{Tr}([\widetilde{\rho} \circ b, [c, a]]) - \mathrm{Tr}(b \circ [\widetilde{\rho}, [c, a]]) \\ &= \mathrm{Tr}(b \circ [\widetilde{\rho}, [a, c]]). \end{aligned} \tag{5.3.3}$$

Hence $\rho((a \circ b) \circ c - a \circ (b \circ c)) = 0$ for all $a, b, c \in \mathcal{L}$ implies $[\widetilde{\rho}, [a, c]] = 0$ for all $a, c \in \mathcal{L}$, i.e. $\widetilde{\rho}$ is in the center of $[\mathcal{L}, \mathcal{L}]$. The converse is obviously true from Eq. (5.3.3). □

Lemma 5.3.4. *A density matrix $\widetilde{\rho}$ is in the center of the algebra $\mathcal{L}$ if and only if the corresponding state ρ satisfies*

$$\rho([a, b]) = 0, \tag{5.3.4}$$

for all $a, b \in \mathcal{L}$.

Proof. If $\widetilde{\rho}$ is in the center $Z(\mathcal{L})$ then

$$\mathrm{Tr}(\widetilde{\rho} \circ [a, b]) = \mathrm{Tr}([\widetilde{\rho} \circ a, b]) = 0, \tag{5.3.5}$$

for all $a, b \in \mathcal{L}$.
Conversely, if $\rho([a, b]) = 0 \; \forall \, a, b \in \mathcal{L}$ then

$$\mathrm{Tr}(\widetilde{\rho} \circ [a, b]) = \mathrm{Tr}([\widetilde{\rho} \circ a, b]) - \mathrm{Tr}(a \circ [\widetilde{\rho}, b]) = -\mathrm{Tr}(a \circ [\widetilde{\rho}, b]) = 0, \tag{5.3.6}$$

implies $[\widetilde{\rho}, b] = 0 \; \forall \, b \in \mathcal{L}$, i.e. $\widetilde{\rho} \in Z(\mathcal{L})$. □

From these lemmas it immediately follows the following

Theorem 5.3.5. *A normal state ρ on a LJB–algebra of observables $\mathcal{L}$ is classical if and only if*

$$\rho([a,b]) = 0, \tag{5.3.7}$$

for all $a, b \in \mathcal{L}$.

5.3.1. Generating the algebra of observables with Jordan products

The strategy adopted in this section, hinging upon the identity (2.2.4), makes use of commutators and anticommutators that involve *three* operators. If one aims at an operational approach, towards experiments, one must make careful use of resources. For example, (traces of) anticommutators involving n operators are related to nth-order interference experiments and increasingly complicated quantum circuits [FMP$^+$13]. See the Sec. 5.1 for an experimental measure of the quantumness.
In light of this observation it is interesting to understand how one can generate the whole algebra by making use of a small set of generators via the Jordan product [FFMP14]. We are going to prove in Thm. 5.3.9 that under suitable hypotheses, three generators are enough.

Definition 5.3.6. A set of elements $a_1, \ldots, a_k$ is said to *generate* an algebra $\mathcal{L}$ if every element of $\mathcal{L}$ is linearly dependent on products of $a_1, \ldots, a_k$; the elements $a_1, \ldots, a_k$ are then called *generators* of $\mathcal{L}$.

Kuranishi proved a sufficient condition for Lie algebras to be generated by two elements [Kur51]:

Theorem 5.3.7. *Let $\mathfrak{g}$ be a semisimple Lie algebra over the real or complex numbers. Then there exist two elements a and b which generate $\mathfrak{g}$.*

Definition 5.3.8. A non-unital Lie–Jordan algebra $(\mathcal{L}, \circ, [\cdot,\cdot])$ is called semisimple if the Lie algebra $(\mathcal{L}, [\cdot,\cdot])$ (i.e. the full algebra considered with the Lie product alone) is semisimple.

Remark. Observe that the unit cannot be generated by Lie products, and hence it will be "added by hand" whenever necessary.

The analogous for Lie–Jordan algebras of Kuranishi's theorem is the following

Theorem 5.3.9. *Let* $(\mathcal{L}, \circ, [\cdot, \cdot])$ *be a semisimple Lie–Jordan algebra. Then the Jordan algebra* $(\mathcal{L}, \circ)$ *(i.e. the full algebra considered with the Jordan product alone) is generated by the Jordan products of three elements, plus the identity. In particular, one can use two generators* a*,* b *of the Lie algebra* $(\mathcal{L}, [\cdot, \cdot])$*, and their commutator* $c = [a, b]$.

Proof. For simplicity assume, without loss of generality, that the algebra is non-unital. Since the Lie algebra $(\mathcal{L}, [\cdot, \cdot])$ is semisimple it can be generated by repeated Lie products of two elements a and b. Then starting from a and b we generate with a first Lie bracket:

$$c = [a, b] \tag{5.3.8}$$

then

$$[a, c] = [a, [a, b]], \quad [b, c] = [b, [a, b]] \tag{5.3.9}$$

and repeating

$$[a, [a, c]], [a, [b, c]], [b, [a, c]], [b, [a, c]], [b, [b, c]], [c, [a, c]], [c, [b, c]] \tag{5.3.10}$$

and so on. We see that all the elements generated by a, b and c are of the form of a triple commutator. Recalling the associator identity (2.2.4)

$$(a \circ b) \circ c - a \circ (b \circ c) = [b, [c, a]]\,, \tag{5.3.11}$$

it follows that every element can be expressed as a linear combination of triple Jordan products, that is generated by Jordan products of a, b and $c = [a, b]$.

□

For finite-dimensional quantum systems, the space of quantum states can be immersed into the semisimple Lie–Jordan algebra $\mathfrak{u}(\mathrm{N})$, which can be Jordan-generated by three elements. This means that one could witness the properties of a system by repeated measures of anticommutators of appropriate elements of the algebra, which is in principle experimentally feasible. Further investigation is required to check if the elements generating the algebra correspond to realization of states as projectors in the algebra.

5.4. QWs and Entanglement witnesses

Let our system be made up of two *subsystems*, that will conventionally be sent to Alice and Bob, whose observations are independent. The notion of independence is reflected in the fact that the total algebra of observables is assumed to factorize in two subalgebras

$$\mathcal{C} = \mathcal{A} \otimes \mathcal{B}. \tag{5.4.1}$$

Namely, the two subalgebras commute with each other, but each subalgebra can be noncommutative.

Definition 5.4.1. A state $\rho \in \mathcal{S}(\mathcal{C})$ is said to be *separable* (with respect to the given bipartition $\mathcal{A} \otimes \mathcal{B}$) if it can be written as a convex combination of product states, namely,

$$\rho = \sum_k p_k \rho_k \otimes \sigma_k, \qquad p_k > 0, \qquad \sum_k p_k = 1, \tag{5.4.2}$$

where $\rho_k \in \mathcal{S}(\mathcal{A})$ and $\sigma_k \in \mathcal{S}(\mathcal{B})$ are states of $\mathcal{A}$ and $\mathcal{B}$, respectively. A state that is not separable is said to be *entangled* (with respect to the given bipartition).

Remark. The definition of separability depends on the algebra $\mathcal{C}$ of the composed system, that in general can be reducible, i.e. the matrices $C \in \mathcal{C}$ are block-diagonal, $C = \bigoplus_k C_k$. If states are identified with density matrices belonging to the algebra, then they inherit the block-diagonal form of the latter.

Definition 5.4.2 ([HHHH09])**.** We say that an observable $E \in \mathcal{C}$ is an *entanglement witness* (EW) if

(i) for any separable state $\rho \in \mathcal{S}(\mathcal{C})$ one gets $\rho(E) \geq 0$,

(ii) there exists a (entangled) state $\sigma \in \mathcal{S}(\mathcal{C})$ such that $\sigma(E) < 0$.

By following [FPVY12] we now show that every EW is also a QW. We first consider a preliminary lemma.

Lemma 5.4.3. *Any classical state is separable.*

Proof. Notice first that if the algebra $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$ is the full algebra of operators

$$\mathcal{C} = B(\mathbb{C}^n) \otimes B(\mathbb{C}^m), \tag{5.4.3}$$

then the only classical state is the totally mixed state,

$$\rho = \mathbb{1}_{nm}/nm = \mathbb{1}_n/n \otimes \mathbb{1}_m/m, \tag{5.4.4}$$

which is obviously separable. In general, however, the (sub)algebras $\mathcal{A}$ and $\mathcal{B}$ are reducible (i.e. they are proper subalgebras of the full matrix algebra) and one has

$$\mathcal{C} = \Big(\bigoplus_k B(\mathbb{C}^{n_k})\Big) \otimes \Big(\bigoplus_l B(\mathbb{C}^{m_l})\Big) = \bigoplus_{k,l} B(\mathbb{C}^{n_k}) \otimes B(\mathbb{C}^{m_l}) =: \bigoplus_{k,l} \mathcal{C}_{kl}, \tag{5.4.5}$$

where each $\mathcal{C}_{kl}$ is an irreducible algebra of dimension $n_k m_l$. All observables are block-diagonal and the classical states have the form

$$\rho = \bigoplus_{k,l} p_{kl} \mathbb{1}_{n_k}/n_k \otimes \mathbb{1}_{m_l}/m_l, \tag{5.4.6}$$

with $p_{kl} \geq 0$ and $\sum_{kl} p_{kl} = 1$, i.e. they are separable. □

Remark. Notice that if the two subalgebras are reducible, states inherit their block-diagonal structure. See Remark after Eq. (5.4.2).

Our main theorem is now an easy consequence of the lemma just proved.

Proposition 5.4.4. *Any EW is a QW.*

Proof. Consider an EW $E \in \mathcal{C}$. By definition $\rho(E) \geq 0$ for any separable $\rho \in \mathcal{S}$. But by the previous lemma all classical states are separable. It follows that $\rho(E) \geq 0$ for any classical state ρ. Moreover, by definition, $\sigma(E) < 0$ for some entangled state σ, which by the previous lemma must be a quantum state. Thus, E is a QW. □

Remark. The converse is, of course, not true. If the algebra $\mathcal{A}$ is noncommutative, and $Q \in \mathcal{A}$ is a QW of the quantum state $\sigma \in \mathcal{S}(\mathcal{A})$, then

$$\tilde{Q} = Q \otimes \mathbb{1} \in \mathcal{C} \tag{5.4.7}$$

is also a QW (of the total algebra), but it is *not* an EW. Indeed, it is negative on separable states of the form $\sigma \otimes \omega$ [for any $\omega \in \mathcal{S}(\mathcal{B})$], namely,

$$(\sigma \otimes \omega)(\tilde{Q}) < 0. \tag{5.4.8}$$

5.5. QWs and Bell inequality

In the previous section we have shown that an EW is always a QW. In view of that, among all QWs, those of the simple form (5.2.1), that we shall call *anti-commutator QWs*, are quite interesting for possible applications, for example for efficiently generating EWs out of anticommutators. Therefore, here we shall investigate whether an EW E can be written in the *particular* form (5.2.1), namely, whether there exists a pair of positive operators a and b such that $E = a \circ b$ [FPVY12].

An example of EW for a $d \times d$ system is the swap operator [HHHH09]

$$S = \sum_{i,j=0}^{d-1} |i\rangle\langle j| \otimes |j\rangle\langle i|. \tag{5.5.1}$$

Here the algebra is the full algebra of matrices $B(\mathbb{C}^d) \otimes B(\mathbb{C}^d)$, and $\{|j\rangle\}_j$ is a chosen orthonormal basis of $\mathbb{C}^d$ (computational basis). S is nonnegative, $\rho(S) \geq 0$, for all separable states ρ, but it possesses an eigenvalue equal to -1.

Another interesting example of EW is the Bell-CHSH observable

$$E_{\text{Bell}} = 2 \pm (A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2), \tag{5.5.2}$$

where $A_{1,2} \in \mathcal{A}$ and $B_{1,2} \in \mathcal{B}$ are dichotomic observables (with eigenvalues ± 1) of Alice and Bob, respectively, and $A_{1,2}^2 = \mathbb{1}$, $B_{1,2}^2 = \mathbb{1}$. If $\rho(E_{\text{Bell}}) < 0$, E_{Bell} witnesses the violation of the Bell-CHSH inequality in the entangled state ρ.

For instance, if we take

$$\begin{aligned} A_1 = \sigma_x, \qquad B_1 = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y), \\ A_2 = \sigma_y, \qquad B_2 = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y), \end{aligned} \tag{5.5.3}$$

where $\sigma_{x,y,z}$ are Pauli operators

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|, \;\; \sigma_y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|, \;\; \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|, \tag{5.5.4}$$

then

$$E_{\text{Bell}} = 2 \pm \sqrt{2}(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y). \tag{5.5.5}$$

Observe now that the swap operator (5.5.1) and the Bell-CHSH observable (5.5.5) are related by

$$S = P_{00} + P_{11} \pm \frac{1}{2\sqrt{2}}(E_{\text{Bell}} - 2), \tag{5.5.6}$$

where

$$P_{ij} = |i\rangle\langle i| \otimes |j\rangle\langle j| \qquad (i, j = 0, 1) \tag{5.5.7}$$

are projections.

Due to the negative shift -2 in (5.5.6), S is more efficient at witnessing entanglement than E_{Bell}: S can actually detect entangled states that do not violate the Bell inequality. For instance, let $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, then the entanglement of the vector state

$$|\chi\rangle = a|+\rangle \otimes |-\rangle + b|-\rangle \otimes |+\rangle \tag{5.5.8}$$

is witnessed by S if $\text{Re}(a^* b) < 0$, while E_{Bell} in Eq. (5.5.5) (with the $+$ sign) is negative only for $\text{Re}(a^* b) < -(\sqrt{2} - 1)/2$.

By following [FPVY12], let us define

$$\begin{aligned} a &= 2 \pm (A_1 \otimes B_1 - A_2 \otimes B_2) \geq 0, \\ b &= 2 \pm (A_1 \otimes B_2 + A_2 \otimes B_1) \geq 0, \end{aligned} \tag{5.5.9}$$

with dichotomic observables $A_{1,2} \in \mathcal{A}$, $B_{1,2} \in \mathcal{B}$. The operators a and b are symmetric under the exchange $A \leftrightarrow B$. Then,

$$\begin{aligned} ab &= 2E_{\text{Bell}} + [A_1, A_2] \otimes \mathbb{1} + \mathbb{1} \otimes [B_1, B_2], \\ ba &= 2E_{\text{Bell}} - [A_1, A_2] \otimes \mathbb{1} - \mathbb{1} \otimes [B_1, B_2], \end{aligned} \tag{5.5.10}$$

so that

$$q_{\text{AVR}} = a \circ b = 4E_{\text{Bell}}. \tag{5.5.11}$$

This shows that the EW E_{Bell} is also an anticommutator QW: if the Bell-CHSH inequality is violated by an entangled state ρ, then $\rho(Q_{\text{AVR}}) < 0$.

An interesting remark is the following one: assume you have two particles, on which Alice and Bob measure dichotomic observables. They put together their results and find that a state ρ exists such that $\rho(E_{\text{Bell}}) < 0$. Then they can conclude that their local observables do not commute.[2] In this sense, one can say that the Bell inequality is testing quantumness, and not simply entanglement: by looking only at the correlations of the two subsystems, one can check whether the two local (sub)algebras are noncommutative.

5.6. Direct estimation of linear and nonlinear functionals of quantum states

Certain properties of a quantum state ρ, such as its purity, degree of entanglement, or its spectrum, are of significant importance in quantum information science. They can be quantified in terms of linear or non-linear functionals of ρ. Linear functionals, such as average values of observables $\{A\}$, given by $\mathrm{Tr}\, A\rho$, are quite common as they correspond to directly measurable quantities. Non-linear functionals of state, such as the von Neumann entropy $-\mathrm{Tr}\, \rho \ln \rho$, eigenvalues, or a measure of purity $\mathrm{Tr}\, \rho^2$, are usually extracted from ρ by classical means i.e. ρ is first estimated and once a sufficiently precise classical description of ρ is available, classical evaluations of the required functionals can be made. However,

[2] In this case *both* algebras $\mathcal{A}$ and $\mathcal{B}$ are noncommutative. Indeed, it is easy to prove that if one of the two algebras were classical then any state ρ of the composed system would necessarily be separable. See e.g. Prop. 2.5 in [Key02].

if only a limited supply of physical objects in state ρ is available, then a direct estimation of a specific quantity may be both more efficient and more desirable. For example, the estimation of purity of ρ does not require knowledge of all matrix elements of ρ, thus any prior state estimation procedure followed by classical calculations is, in this case, inefficient. However, in order to bypass tomography and to estimate non-linear functionals of ρ more directly, we need quantum networks performing quantum computations that supersede classical evaluations.

In this section, we describe a simple quantum network which can be used as a basic building block for direct quantum estimations of both linear and non-linear functionals of any ρ [EAO$^+$02]. The network can be realized as multiparticle interferometry. While conventional quantum measurements allow the estimation of $\mathrm{Tr}\, A\rho$ for some observable A, this quantum circuit can also provide a direct estimation of the overlap of any two unknown quantum states ρ_a and ρ_b, i.e. $\mathrm{Tr}\, \rho_a \rho_b$. Here, and in the following, we use terminology developed in quantum information science. For a comprehensive overview of this terminology, including quantum logic gates and quantum networks see, for example, [NC10].

Consider a typical interferometric setup: Hadamard gate which serves as beam splitter, phase shift φ, mirrors, Hadamard gate, followed by a measurement in the computatinal basis. The beam pair spans a two dimensional Hilbert space $\mathcal{H} = \{|0\rangle, |1\rangle\}$. The state vectors $|0\rangle$, $|1\rangle$ can be taken as wave packets that move in two given directions defined by the geometry of the interferometer. In this basis, we may represent mirrors, beam-splitters (Hadamard gate) and relative $U(1)$ phase shifts by the unitary operators

$$\begin{aligned} U_M &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\; U_H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\ U_\varphi &= \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{5.6.1}$$

respectively. An input pure state $\rho_{\text{in}} = |0\rangle\langle 0|$ of the interferometer transforms into the output state

$$\rho_{\text{out}} = U_H U_M U_\varphi U_H \rho_{\text{in}} U_B^\dagger U_\varphi^\dagger U_M^\dagger U_H^\dagger \tag{5.6.2}$$

$$= \frac{1}{2}\begin{pmatrix} 1-\cos\varphi & -i\sin\varphi \\ i\sin\chi & 1+\cos\varphi \end{pmatrix} \tag{5.6.3}$$

that yields the intensity along $|1\rangle$ as

$$I \propto 1 + \cos\varphi. \tag{5.6.4}$$

Thus the relative $U(1)$ phase φ could be observed in the output signal of the interferometer [SPE+00].

Now assume that the particles carry additional internal degrees of freedom, e.g., spin. This internal spin space $\mathcal{H}_\text{i} \cong \mathbb{C}^N$ is spanned by the vectors $|k\rangle$, $k = 1, 2, \ldots N$, chosen so that the associated density operator is initially diagonal

$$\rho = \sum_k w_k |k\rangle\langle k| \tag{5.6.5}$$

with w_k the classical probability to find a member of the ensemble in the pure state $|k\rangle$. The density operator could be made to change inside the interferometer

$$\rho \longrightarrow U_\text{i} \rho U_\text{i}^\dagger \tag{5.6.6}$$

with U_i a unitary transformation acting only on the internal degrees of freedom. Mirrors and beam-splitters are assumed to leave the internal state unchanged so that we may replace U_M and U_H by $\widetilde{U}_M = U_M \otimes \mathbb{1}_i$ and $\widetilde{U}_B = U_B \otimes \mathbb{1}_i$, respectively, $\mathbb{1}_i$ being the internal unit operator. Furthermore, we introduce the unitary transformation

$$\widetilde{U} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes U_\text{i} + \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1}_i. \tag{5.6.7}$$

The operators $\widetilde{U}_M, \widetilde{U}_B$, and $\widetilde{U}$ act on the full Hilbert space $\mathcal{H} \otimes \mathcal{H}_\text{i}$. $\widetilde{U}$ correponds to the application of U_i along the $|1\rangle$ path and the $U(1)$ phase φ similarly along $|0\rangle$.

Let an incoming state given by the density matrix $\widetilde{\rho}_{\text{in}} = \rho_{\text{in}} \otimes \rho = |\tilde{0}\rangle\langle\tilde{0}| \otimes \rho$ be split coherently by a beam-splitter and recombine at a second beam-splitter after being reflected by two mirrors. Suppose that $\widetilde{U}$ is applied between the first beam-splitter and the mirror pair. The incoming state transforms into the output state

$$\widetilde{\rho}_{\text{out}} = \widetilde{U}_H \widetilde{U}_M \widetilde{U} \widetilde{U}_H \widetilde{\rho}_{\text{in}} \widetilde{U}_H^\dagger \widetilde{U}^\dagger \widetilde{U}_M^\dagger \widetilde{U}_H^\dagger, \tag{5.6.8}$$

as shown in Fig. 5.1:

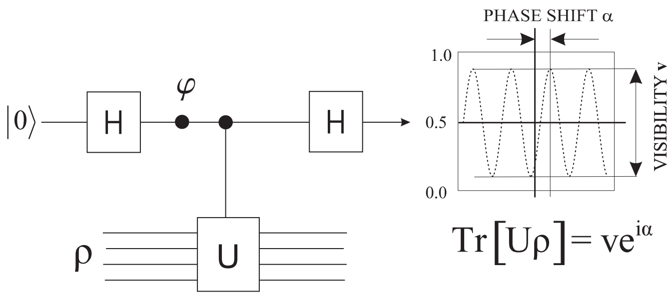


Figure 5.1: Both the visibility and the shift of the interference patterns of a single qubit (top line) are affected by the controlled-U operation on a general state ρ.

Inserting Eqs. (5.6.1) and (5.6.7) into Eq. (5.6.8) yields

$$\begin{aligned} \widetilde{\rho}_{\text{out}} &= \frac{1}{4}\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes U_{\text{i}}\rho U_{\text{i}}^\dagger + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes \rho \right. \\ &\quad + e^{i\varphi}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \otimes \rho U_{\text{i}}^\dagger \\ &\quad \left. + e^{-i\varphi}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \otimes U_{\text{i}}\rho \right]. \end{aligned} \tag{5.6.9}$$

The output intensity along $|1\rangle$ is

$$\begin{aligned} I &\propto \text{Tr}\left(U_{\text{i}}\rho U_{\text{i}}^\dagger + \rho + e^{-i\varphi}U_{\text{i}}\rho + e^{i\varphi}\rho U_{\text{i}}^\dagger\right) \\ &\propto 1 + |\text{Tr}\,(U_{\text{i}}\rho)\,|\cos\left[\varphi - \arg \text{Tr}\,(U_{\text{i}}\rho)\right], \end{aligned} \tag{5.6.10}$$

where we have used $\mathrm{Tr}(\rho U_{\mathrm{i}}^{\dagger}) = [\mathrm{Tr}\,(U_{\mathrm{i}}\rho)]^{*}$.

The important observation from Eq. (5.6.10) is that the interference oscillations produced by the variable $U(1)$ phase φ *is shifted by* $\phi = \arg Tr\,(U_i\rho)$ *for any internal input state* ρ, be it mixed or pure. Moreover the visibility of the interference pattern is $\nu = |\mathrm{Tr}\,(U_{\mathrm{i}}\rho)\,| \geq 0$, which reduces to the expected $\nu = |\langle\psi|U_{\mathrm{i}}|\psi\rangle|$ for pure states $\rho = |\psi\rangle\langle\psi|$.

The output intensity in Eq. (5.6.10) may be understood as an incoherent weighted average of pure state interference profiles as follows. The state k gives rise to the interference profile

$$I_k \propto 1 + \nu_k \cos\left[\varphi - \phi_k\right], \tag{5.6.11}$$

where $\nu_k = |\langle k|U_{\mathrm{i}}|k\rangle|$ and $\phi_k = \arg\langle k|U_{\mathrm{i}}|k\rangle$. This yields the total output intensity

$$I = \sum_k w_k I_k \propto 1 + \sum_k w_k \nu_k \cos\left[\varphi - \phi_k\right], \tag{5.6.12}$$

which is the incoherent classical average of the above single-state interference profiles weighted by the corresponding probabilities w_k. Eq. (5.6.12) may be written in the desired form $1 + \tilde{\nu}\cos(\varphi - \tilde{\phi})$ by making the identifications

$$\begin{aligned} \tilde{\phi} &= \arg\left(\sum_k w_k \nu_k e^{i\phi_k}\right) = \arg\mathrm{Tr}\,(U_{\mathrm{i}}\rho) = \phi, \\ \tilde{\nu} &= \left|\sum_k w_k \nu_k e^{i\phi_k}\right| = |\mathrm{Tr}\,(U_{\mathrm{i}}\rho)\,| = \nu. \end{aligned} \tag{5.6.13}$$

Let us now consider a quantum state ρ of two separable subsystems, such that $\rho = \rho_a \otimes \rho_b$. We choose our controlled-U to be the controlled-V, where V is the swap operator, defined as, $V|\alpha\rangle_A|\beta\rangle_B = |\beta\rangle_A|\alpha\rangle_B$, for any pure states $|\alpha\rangle_A$ and $|\beta\rangle_B$. In this case, the modification of the interference pattern given by Eq. (5.6.13) can be written as,

$$\nu = \mathrm{Tr}\,V\,(\rho_a \otimes \rho_b) = \mathrm{Tr}\,\rho_a\rho_b. \tag{5.6.14}$$

which is easily proved using the spectral decomposition of ρ_a and ρ_b [EAO+02]. Since $\mathrm{Tr}\,\rho_a\rho_b$ is real, we can fix $\varphi = 0$ and the probability of finding the qubit in state $|0\rangle$ at the output, P_0, is related to the visibility by $\nu = 2\,P_0 - 1$, as explained before. This construction, shown in Fig. (5.2), provides a direct way to measure $\mathrm{Tr}\,\rho_a\rho_b$:

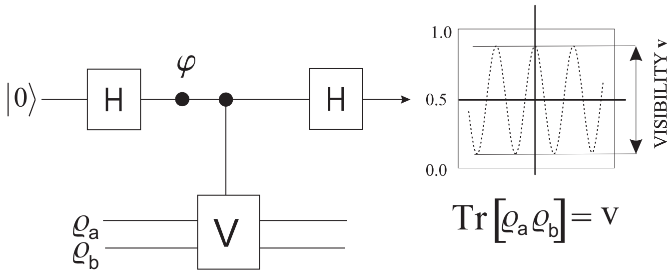


Figure 5.2: A quantum network for direct estimations of both linear and non-linear functions of state. The probability of finding the control (top line) qubit in state $|0\rangle$ at the output depends on the overlap of the two target states (two bottom lines). Thus estimation of this probability leads directly to an estimation of $\mathrm{Tr}\,\rho_a\rho_b = \nu = 2\,P_0 - 1$.

There are many possible ways of utilizing this result. For pure states $\rho_a = |\alpha\rangle\langle\alpha|$ and $\rho_b = |\beta\rangle\langle\beta|$ the formula above gives $\mathrm{Tr}\,\rho_a\rho_b = |\,\langle\,\alpha\,|\,\beta\,\rangle\,|^2$ i.e. a direct measure of orthogonality of $|\alpha\rangle$ and $|\beta\rangle$. If we put $\rho_a = \rho_b = \rho$ then we obtain an estimation of $\mathrm{Tr}\,\rho^2$. In the single qubit case, this measurement allows us to estimate the length of the Bloch vector, leaving its direction completely undetermined. For qubits $\mathrm{Tr}\,\rho^2$ gives the sum of squares of the two eigenvalues which allows to estimate the spectrum of ρ.

In general, in order to evaluate the spectrum of any $d \times d$ density matrix ρ we need to estimate $d-1$ parameters $\mathrm{Tr}\rho^2$, $\mathrm{Tr}\rho^3$,... $\mathrm{Tr}\rho^d$. For this we need the controlled-shift operation, which is a generalization of the controlled-swap gate. Given k systems of dimension d we define the shift $V^{(k)}$ as

$$V^{(k)}|\phi_1\rangle|\phi_2\rangle...|\phi_k\rangle = |\phi_k\rangle|\phi_1\rangle...|\phi_{k-1}\rangle, \tag{5.6.15}$$

for any pure states $|\phi\rangle$. Such an operation can be easily constructed by cascading $k-1$ swaps V. If we extend the network and prepare $\rho = \rho^{\otimes k}$ at the input then the interference will be modified by the visibility factor,

$$\nu = \mathrm{Tr}\, V^{(k)} \rho^{\otimes k} = \mathrm{Tr}\, \rho^k = \sum_{i=1}^{m} {\lambda_i}^k. \tag{5.6.16}$$

Thus measuring the average values of $V^{(k)}$ for $k = 2, 3...d$ allows us to evaluate the spectrum of ρ. Although we have not eliminated classical evaluations, we have reduced them by a significant amount. The average values of $V^{(k)}$ for $k = 2, 3...d$ provide just enough information to evaluate the spectrum of ρ but certainly not enough to estimate the whole density matrix.

5.7. Measure of quantumness via commutators

The macroworld is essentially classical because the accessible states all commute with each other, being positional eigenstates. The main idea of this chapter has been to extend this approach further to any system and define it to be classical precisely if all the admissable states are mutually compatible. What is advantageous in this approach is that references to the dynamics and correlations are removed. Correlations are always defined relatively to a particular set of experimental capabilities. As an illustration, maximal entangled states may be represented as product states and vice versa, simply by a suitable choice of the degrees of freedom, as the physical bipartition of the composite system considered (see [ZLL04] and Sec. 4.4).
As one way to overcome this problem, we propose here a measure of nonclassicality based on the incompatibility of states relative to each other, rather than on correlations. This approach can be helpful in studying the quantumness of complex processes, such as those encountered in photosynthetic systems, where it may be computationally unfeasible to compute measures of nonclassicality based on correlations. The fact that the experiments in photosynthesis show quantum effects [NBT11], while rivaling classical explanations for exciton transport also exist, suggests that our approach could find fruitful applications here.

The Hilbert-Schmidt norm of a bounded operator A is given by

$$\|a\|_{HS}^2 = \mathrm{Tr}\,(a^* a). \tag{5.7.1}$$

Given two states ρ_1 and ρ_2 (regarded as density matrices), we propose the measure of their mutual incompatibility to be twice the Hilbert-Schmidt norm of their commutator:

$$Q(\rho_1, \rho_2) = 2\|[\rho_1, \rho_2]\|_{HS}^2, \tag{5.7.2}$$

where the numerical pre-factor is put for normalization. Q so defined is a convenient measure of state incompatibility. It is symmetric in both arguments, and its interpretation as such is conceptually transparent.

Lemma 5.7.1. $Q(\rho_1, \rho_2) \geq 0$ *and the equality is satisfied if and only if* $[\rho_1, \rho_2] = 0$.

Proof. The theorem is actually slightly more general, i.e. it is true also when ρ_1 and ρ_2 are observables. We know that in the Lie–Jordan algebra of observables there are no nilpotent elements, that is $a^2 = 0 \Rightarrow a = 0$.
In finite dimension this is trivial, in infinite dimensions one just uses the Banach property

$$\|a^2\| = \|a\|. \tag{5.7.3}$$

The nilpotent elements arise only in the associated C^*–algebra obtained by complexification. Hence, given two observables a and b, if the commutator $[a, b] \neq 0$ then the square $[a, b] \circ [a, b] = [a, b]^2 \neq 0$. Moreover $[a, b]^2$ is a positive operator, hence it follows that $\mathrm{Tr}([a, b]^2) > 0$ if and only if $[a, b] \neq 0$. In other words, the operators a and b commute if and only if $\mathrm{Tr}([a, b]^2) = 0$.

□

The following has been proved in [ICS13]:

Theorem 5.7.2. $0 \leq Q(\rho_1, \rho_2) \leq 1$.

As shown in [FMP$^+$13], the noncommutativity of two states can be witnessed by using the Jordan product (i.e. the anticommutator) of the states, and the process can be realized experimentally by using an interferometer (as the one described in the previous section), but unfortunately only an indefinite iterative procedure is available, that is for some states it requires an indefinite number of copies in order to bring to light the quantumness. As we will show in the next section, the measure of Q is experimentally feasible with a precise quantum circuit which is independent on the input states [FFF$^+$14].

5.8. Direct estimation of the quantumness measure

In the previous section we have introduced the measure of quantumness

$$Q(\rho_1, \rho_2) = 2\|[\rho_1, \rho_2]\|^2_{HS}. \tag{5.8.1}$$

By simple algebraic computation we have in terms of the standard associative product:

$$Q(\rho_1, \rho_2) \equiv \frac{1}{2}\mathrm{Tr}([\rho_1, \rho_2]^2) = \mathrm{Tr}(\rho_1^2\rho_2^2 - \rho_1\rho_2\rho_1\rho_2). \tag{5.8.2}$$

Given the states ρ_1 and ρ_2, from (5.8.2) is evident that we can measure $Q(\rho_1, \rho_2)$ by using the interferometric setup displayed in Figure 5.3, where we control the unitary operation U on the quantum system, as described in Sec. 5.6.

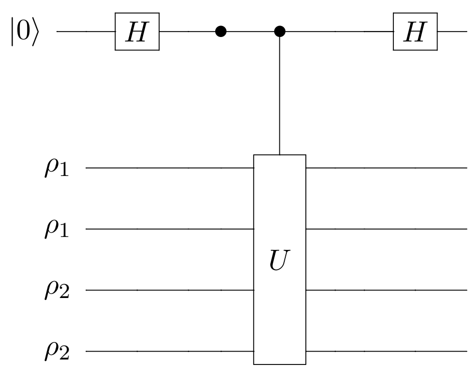


Figure 5.3: Controlled-U operation

The action of the controlled-U on the system modifies the interference pattern by the factor

$$\mathrm{Tr}\,(\rho U) = v\,\mathrm{e}^{i\alpha}, \tag{5.8.3}$$

where v is the new visibility and α is the shift of the interference fringes. The observed modification of the visibility gives an estimate of $\mathrm{Tr}\,(U\rho)$, i.e. the expectation value of the unitary operator U on the state ρ. In order to perform our experiment, we need to measure the quantities

$$\mathrm{Tr}\,(\rho_1\rho_2\rho_1\rho_2) \tag{5.8.4}$$

and

$$\mathrm{Tr}\,(\rho_1^2\rho_2^2) \tag{5.8.5}$$

in two separate experiments, or in the same interferometer by using two control qubits. Such operations can be easily constructed by cascading different swaps operators. We define the unitary operator S_{ij} as

$$S_{ij}|\phi_1\rangle|\phi_2\rangle\cdots|\phi_i\rangle\cdots|\phi_j\rangle\cdots = |\phi_1\rangle|\phi_2\rangle\cdots|\phi_j\rangle\cdots|\phi_i\rangle\cdots \tag{5.8.6}$$

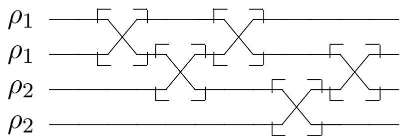
i.e. it exchanges the ket i with the ket j. If we denote by A, B, C, D the four parties of our state $\rho = \rho_1^A \otimes \rho_2^B \otimes \rho_3^C \otimes \rho_4^D$ then we have

$$\mathrm{Tr}\,(\rho_1\rho_2\rho_1\rho_2) = \mathrm{Tr}\,(S_{BC}S_{CD}S_{AB}S_{BC}S_{AB}\;\rho_1\otimes\rho_1\otimes\rho_2\otimes\rho_2) \tag{5.8.7}$$

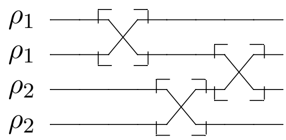
and

$$\begin{aligned}\mathrm{Tr}\,(\rho_1^2\rho_2^2) &= \mathrm{Tr}\,(S_{BC}S_{CD}S_{AB}\;\rho_1\otimes\rho_1\otimes\rho_2\otimes\rho_2) &\quad (5.8.8a)\\ &= \mathrm{Tr}\,(S_{AB}S_{BC}S_{CD}\;\rho_1\otimes\rho_1\otimes\rho_2\otimes\rho_2). &\quad (5.8.8b)\end{aligned}$$

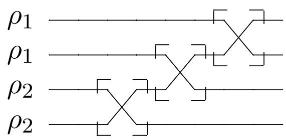
Eq. (5.8.7) is depicted in Figure 5.4(a), and Eqs. (5.8.8a) and (5.8.8b) in Figures 5.4(b) and 5.4(c) respectively.



(a) $\mathrm{Tr}(\rho_1\rho_2\rho_1\rho_2)$



(b) $\mathrm{Tr}(\rho_1^2\rho_2^2)$



(c) $\mathrm{Tr}(\rho_1^2\rho_2^2)$

Figure 5.4: Quantum circuits measuring $Q(\rho_1, \rho_2)$

A

DIFFERENTIABLE AND SYMPLECTIC GROUP ACTIONS

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Suppose G acts smoothly on a differentiable manifold M, i.e. there is a smooth map $\rho\colon\ G\times M\to M$ such that $\rho(g,m)\equiv g\cdot m$, and $g\cdot(h\cdot m)=(gh)\cdot m$, $\forall\, g,h\in G$, $m\in M$ and $e\cdot m=m\,\forall\, m\in M$, with e being the identity element of the group.

Let $\mathfrak{X}(M)$ denote the Lie algebra of vector fields on M. Given the action ρ we have a map

$$\begin{aligned}\hat{\rho}\colon \mathfrak{g} &\to \mathfrak{X}(M) \\ \xi &\mapsto X_\xi\end{aligned} \tag{A.0.1}$$

associating to each $\xi\in\mathfrak{g}$ a vector field X_ξ on M, called **fundamental vector field** defined by its action on the functions $f\in C^\infty(M)$:

$$X_\xi f(m)=\frac{d}{dt}f(\mathrm{e}^{-t\xi}\,m)\,|_{t=0}\ . \tag{A.0.2}$$

The map (A.0.1) is a Lie algebra homomorphism

$$X_{[\xi,\zeta]} = [X_\xi, X_\zeta], \tag{A.0.3}$$

where in the rhs we have the Lie bracket of vector fields.
If $G = \mathbb{R}$, then an action of $\mathbb{R}$ on M provides a one-parameter family of diffeomorphisms $\phi_S \colon M \to M$, which we call an autonomous dynamical system. Such terminology can be extended to a general action of G on M, however this is only done in the setting of C^*–algebras.
If $Y \in \mathfrak{X}(M)$, then $\mathfrak{g}$ acts on it via the Lie bracket $[X_\xi, Y]$.
Similarly, if $\theta \in \Omega^1(M)$ is a one-form, then for all $\xi \in \mathfrak{g}$ we have the action of ξ on θ given by:

$$L_{X_\xi}\theta(Y) = X_\xi\theta(Y) - \theta([X_\xi, Y]). \tag{A.0.4}$$

In general if $\omega \in \Omega^p(M)$ is a p-form, we define:

$$L_{X_\xi}\omega \equiv (di_{X_\xi} + i_{X_\xi}d)\omega, \tag{A.0.5}$$

where d is the exterior derivative and i_{X_ξ} is the contraction operator defined by

$$(i_{X_\xi}\omega)(Y_1, \ldots, Y_{p-1}) = \omega(X_\xi, Y_1, \ldots, Y_{p-1}). \tag{A.0.6}$$

By simple inspection, notice that its action agrees on functions and on one-forms (recall that the operator $L_X = di_X + i_X d$ is called the Lie derivative).

Now let (M, Ω) be a symplectic manifold, that is, Ω is a closed non-degenerate 2-form. In other words, $d\Omega = 0$ and the natural map $\hat{\Omega} \colon TM \to T^*M$ is an isomorphism. Thus on a symplectic manifold there is a natural map between vector fields and one-forms:

$$\beta \colon \mathfrak{X} \;\to\; \Omega^1(M) \tag{A.0.7}$$

$$X \;\mapsto\; i_X\Omega = \hat{\Omega}(X), \tag{A.0.8}$$

which is an isomorphism with inverse $\beta^{-1} \colon \Omega^1(M) \to \mathfrak{X}(M)$. In local coordinates,

$$\Omega = \frac{1}{2}\Omega_{ij}dx^i \wedge dx^j, \tag{A.0.9}$$

and nondegeneracy of Ω implies that $\det(\Omega_{ij}) \neq 0$.

We now take a connected Lie group G acting on M via *symplectomorphisms*, i.e. diffeomorphisms which preserve Ω. Infinitesimally, this means that if $\xi \in \mathfrak{g}$ then

$$\begin{aligned} 0 &= L_X \Omega && \text{(A.0.10)} \\ &= d i_X \Omega + i_X d\Omega && \text{(A.0.11)} \\ &= d i_X \Omega, && \text{(A.0.12)} \end{aligned}$$

where X is the vector field associated to ξ. The one-form $i_X \Omega$ is closed. A vector field X such that $i_X \Omega$ is closed is said to be a **symplectic vector field**. It is clear that the symplectic vector fields are the image of closed forms under β^{-1}.
If $\beta(X)$ is exact, we say that X is a **Hamiltonian vector field**. This means that there exists $f_X \in C^\infty(M)$ such that

$$\beta(X) + df_X = 0. \tag{A.0.13}$$

This function is not unique because we can add to it a locally-constant function and still satisfy the above equation. We have that the Hamiltonian vector fields are the images of exact form under β^{-1}. A G-action on M said to be **Hamiltonian** if

to every $\xi \in \mathfrak{g}$ we can assign an Hamiltonian vector field X_ξ.
In a symplectic manifold, the functions define a **Poisson algebra**. If $f, g \in C^\infty(M)$, we define the Poisson bracket by

$$\{f, g\} \equiv \Omega(X_f, X_g), \tag{A.0.14}$$

where X_f is the Hamiltonian vector field such that $\beta(X_f) + df = 0$. The Poisson bracket is clearly skew-symmetric and obeys the Jacobi identity (since $d\Omega = 0$), and is a derivation on functions. Hence it gives $C^\infty(M)$ the structure of a Lie algebra. A Hamiltonian action is said to be Poisson or **strongly Hamiltonian** if there is a Lie algebra homomorphism $\mathfrak{g} \to C^\infty(M)$ sending X to f_X in such a way that $\beta(X) + df_X = 0$ and that $f_{[X,Y]} = \{f_X, f_Y\}$. In such case we can define the **momentum map** $J\colon M \to \mathfrak{g}^*$ of the action by:

$$\langle J, \xi \rangle = f_{X_\xi}. \tag{A.0.15}$$

The map

$$\begin{aligned} a\colon \mathfrak{g} &\to C^\infty(M) \\ \xi &\mapsto f_\xi \end{aligned} \tag{A.0.16}$$

is sometimes called the **comomentum map**.

B

TENSOR PRODUCTS OF C^*–ALGEBRAS

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be C^*–algebras. We can define their *–algebra tensor product as the standard algebraic tensor product of algebras $A_1 \otimes \mathcal{A}_2$ with product $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ and involution $(a \otimes b)^* = a^* \otimes b^*$.
The norm of a C^*–algebra is unique in the sense that on a given $*$–algebra $\mathcal{A}$ there is at most one norm which makes $\mathcal{A}$ into a C^*–algebra. Still, on a $*$–algebra $\mathcal{A}$ there may exist different norms satisfying the C^*–property. The completion with respect to any of such norms results in a C^*–algebra which contains $\mathcal{A}$ as a dense subalgebra. This is precisely what happens when the tensor product of C^*–algebras is considered: in the general case there are many different norms on the algebraic tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ (which is a $*$–algebra) with the C^*–property.

For example we may define

$$\left\| \sum a_i \otimes b_i \right\|_\wedge = \sum \|a_i\| \|b_i\|. \tag{B.0.1}$$

This seminorm becomes a norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$ on an appropriate subspace, and its completion is denoted $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$ and called the projective tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$. We also have

$$\begin{aligned} \left\| \left(\sum a_i \otimes b_i \right)^* \right\|_\wedge &= \left\| \sum a_i^* \otimes b_i^* \right\|_\wedge = \sum \|a_i^*\| \|b_i^*\| \\ &= \sum \|a_i\| \|b_i\| = \left\| \sum a_i \otimes b_i \right\|, \end{aligned} \tag{B.0.2}$$

so $\mathcal{A}_1 \hat{\otimes} \mathcal{A}_2$ is a Banach *–algebra. But it fails to satisfy the C^*–axiom ($\|x^*x\| = \|x\|^2$):

$$\begin{array}{rcl} \|(\sum a_i \otimes b_i)^* (\sum a_i \otimes b_i)\| &=& \|(\sum a_i^* \otimes b_i^*) (\sum a_i \otimes b_i)\| \\ &=& \|\sum a_i^* a_j \otimes b_i^* b_j\| \\ &=& \sum \|a_i^* a_j\| \|b_i^* b_j\| \\ &\leq& \sum \|a_i\| \|a_j\| \|b_i\| \|b_j\| \\ &=& (\sum \|a_i\| \|b_i\|)^2 \\ &=& \|\sum a_i \otimes b_i\|^2 \end{array} . \tag{B.0.3}$$

It turns out that representations on $\mathcal{A}_1$ and $\mathcal{A}_2$ allow us to define norms on $\mathcal{A}_1 \otimes \mathcal{A}_2$ that make it a C^*–algebra.

Definition B.0.1. Let $\rho_{\mathcal{A}_1} : \mathcal{A}_1 \to \mathcal{B}(\mathcal{H}_1)$ and $\rho_{\mathcal{A}_2} : \mathcal{A}_2 \to \mathcal{B}(\mathcal{H}_2)$ be representations on $\mathcal{A}_2$ and $\mathcal{A}_2$. We define the **product representation** $\rho = \rho_{\mathcal{A}_1} \otimes \rho_{\mathcal{A}_2}$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ as

$$\rho(a \otimes b) = \rho_{\mathcal{A}_1}(a) \otimes \rho_{\mathcal{A}_2}(b) \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2). \tag{B.0.4}$$

Since we always have the trivial representations, the set of representationsappendixTensor of $\mathcal{A}_1$ on $\mathcal{H}_1$ and $\mathcal{A}_2$ on $\mathcal{H}_2$ are never empty.

Definition B.0.2. We define the **minimal** C^*–norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$ by

$$\begin{aligned} \left\| \sum a_i \otimes b_i \right\|_{\text{min}} &= \sup_{\rho_A, \rho_B} \left\| \rho \left(\sum a_i \otimes b_i \right) \right\| \\ &= \sup_{\rho_A, \rho_B} \left\| \sum \rho_A(a_i) \otimes \rho_B(b_i) \right\| \end{aligned} \tag{B.0.5}$$

where the two norms on the right are operator norms.

This is clearly finite (hence a norm) and satisfies the C^*–axiom. The completion of $\mathcal{A}_1 \otimes \mathcal{A}_2$ with this norm is a C^*–algebra called the **minimal** (or **spatial**) tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$ and will be denoted by $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$.

Definition B.0.3. appendixTensor
Let $\rho_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a representation and $N \subseteq \mathcal{H}$ be the largest subspace of $\mathcal{H}$ such that $\rho(a)(x) = 0$ for all $a \in \mathcal{A}$ and $x \in N$. Then $N^{\perp}$ is called the *essential subspace* of $\mathcal{H}$, and we will denote it $E(\mathcal{H})$. If $E(\mathcal{H}) = \mathcal{H}$, then ρ_A is said to be **nondegenerate**.

Proposition B.0.4. *If $\rho_{\mathcal{A}_1} : \mathcal{A}_1 \to \mathcal{B}(\mathcal{H})$ is a nondegenerate representation, then there are unique nondegenerate representations $\rho_{\mathcal{A}_1} : \mathcal{A}_1 \to \mathcal{B}(\mathcal{H})$ and $\rho_{\mathcal{A}_2} : \mathcal{A}_2 \to \mathcal{B}(\mathcal{H})$ such that $\rho(a \otimes b) = \rho_{\mathcal{A}_1}(a)\rho_{\mathcal{A}_2}(b) = \rho_{\mathcal{A}_2}(b)\rho_{\mathcal{A}_1}(a)$.*

But arbitrary representations of the tensor product of algebras cannot be broken into pieces. This gives us the following.

Definition B.0.5. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}_1$,$\mathcal{A}_2$ be C^*–algebra. We define the **maximal** C^*–norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$ as

$$\left\|\sum a_i \otimes b_i\right\|_{\max} = \sup_{\rho} \left\|\rho\left(\sum a_i \otimes b_i\right)\right\| \tag{B.0.6}$$

where $\rho : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{B}(\mathcal{H})$. This is also a C^*–norm, and the completion of $\mathcal{A}_1 \otimes \mathcal{A}_2$ under this norm is a C^*–algebra called the **maximal** tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$ and will be denoted by $\mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$.

An important result [Bla06] is that

$$\| \cdot \|_{\min} \leq \| \cdot \|_* \leq \| \cdot \|_{\max} \leq \| \cdot \|_{\wedge} \tag{B.0.7}$$

where $\| \cdot \|_*$ is any C^*–norm. It follows that $\|(a \otimes b)\|_* = \|a\|\|b\|$. Then clearly the natural map $\mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 \to \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ is continuous.

We conclude by defining *nuclear* C^*–algebras.

Definition B.0.6. A C^*–algebra $\mathcal{A}_1$ is **nuclear** if for every C^*–algebra $\mathcal{A}_2$, there is a unique C^*–norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$, i.e. $\mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 = \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$.

For instance if G is discrete, $C^*(G)$ is nuclear if and only if G is amenable (however this is not true if G is not discrete). Examples of non-nuclear algebras for discrete groups are given for instance by $C_r^*(F_2)$, the reduced C^*–algebra of the free group generated by two elements [Tak64].

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