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Spectral Regularization and its Applications in Quantum Field Theory

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Dottorando
Maxim Kurkov

Tutore
Prof. Fedele Lizzi

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Abstract

The argument of this thesis is the ultraviolet Spectral Regularization of Quantum Field Theory (QFT). We describe its genesis, its definition and apply it to physically interesting models. One of the main applications of the Spectral Regularization is its application to the Bosonic Spectral Action (BSA), appearing in the noncommutative geometry approach to the Standard Model. Conformal anomaly, appearing in QFT of fermions, moving in a fixed bosonic background under Spectral Regularization is expressed in terms of the BSA. Generalizing this formalism to bosonic degrees of freedom, the phenomena of induced Sakharov gravity and trace anomaly induced inflation are described on an equal footing. The second part of the thesis is devoted to some models, naturally exhibiting the ultraviolet cutoff scale: we compute high momenta asymptotic of BSA, and find that it possesses a phase transition in the ultraviolet, and only at low momenta BSA reproduces the conventional QFT. Afterwards we consider the strong unification generalization of the Standard Model, based on a presence of the Universal Landau Pole for all gauge couplings at the Planck scale. Introducing the physical ultraviolet cutoff scale, such a model naturally resolves the instability problem of the Higgs potential.

Contents

Chapter 1

Introduction

In this thesis we discuss some applications of the ultraviolet spectral regularization in Quantum Field Theory (QFT) and some QFT's naturally possessing a physical ultraviolet cutoff.

In QFT in the *background field formalism* [1] one has to study a classical background field Φ surrounded by a quantized field ψ. The latter might be a field of quantum fluctuations of Φ itself, however that is not necessary. There are situations, when the field Φ is classical by its nature, like a gravitational field[1] $g_{\mu\nu}$, while all other fields (fermions, gauge fields, Higgs scalars) fluctuating around their vacuum expectation values are ψ [2]. What we want to study is a dynamics of Φ taking into account quantum effects.

Such systems are typically described by the classical action $S[\Phi, \psi]$ and neglecting by quantum effects i.e setting $\psi = 0$ the dynamics of Φ derives from the *classical* equations of motion:

$$\frac{\delta S\ [\Phi, 0]}{\delta \Phi} = 0$$

In order to take into account quantum effects i.e. take average over quantum fluctuations one should perform the functional integration over ψ:

$$\left\langle \frac{\delta S\ [\Phi, \psi]}{\delta \Phi} \right\rangle_{\psi} = Z^{-1} \int [d\psi] \left(\frac{\delta S\ [\Phi, \psi]}{\delta \Phi} \right) e^{-S[\Phi,\psi]}, \quad Z \equiv \int [d\psi] e^{-S[\Phi,\psi]}$$

Equations of motion for Φ that take into account quantum corrections read:

$$\frac{\delta W}{\delta \Phi} = 0, \quad W \equiv -\log Z.$$

[1]We emphasize, that no self consistent theory of quantum gravity is known.

where W is a *quantum effective action*.

Troubles appear when one tries to compute the quantum effective action, because in many physically relevant models, the path integral

$$\int [d\psi] e^{-S[\Phi,\psi]}$$

is not well defined, in particular frequently oscillating configurations of ψ give divergent contribution; that is called *ultraviolet* or UV divergence [1]. Therefore the functional integral must be regularized in order to get rid of such a problem.

A quantum theory may be either *effective* or *fundamental*. In the former case we are in the presence of the energy scale Λ which defines its region of applicability: the theory is valid at energies below Λ thereby exhibiting an intrinsic UV regulator. The UV cutoff scale Λ has a physical meaning of a transition scale to new physics thus coefficients of the corresponding low energy effective QFT should depend on this cutoff scale. In the latter case one may introduce a *formal* UV cutoff Λ and then subtract the UV divergent part considering the limit $\Lambda \to \infty$, or to use a regularization which does not exploit the UV cutoff at all e.g. dimensional [3, 4] or ζ functional ones [5, 6]. In any case the final result must be independent on any UV regulator. We emphasize, that in the present study we consider QFT in the first sense i.e. Λ is a physical parameter.

There are different techniques of regularization based on different assumptions and we will focus our attention on the special one, that introduces the UV cutoff scale Λ. Another important issue, one should care about is a *symmetry*. Standard truncation of momenta in loop diagrams is not a gauge invariant procedure [1]. It is more preferable to have such a regularization, that

* respects gauge invariance and general covariance (when one works in curved spacetime) and

* in the meantime introduces the ultraviolet cutoff scale Λ.

One regularization which is very interesting from mathematical and physical points of view and satisfies both requirements is the *Spectral Regularization*.

Spectral regularization was first introduced by Andrianov, Bonora and Gamboa-Saravi for gauge invariant description of fermionic determinants [7–9]. It was applied for chiral and scale anomalies in Quantum Chromo Dynamics (QCD) in order to derive low energy effective theory of mesons in [10–12], the spacetime was considered to be flat. Later this approach was generalized for curved spacetime in the context of induced gravity [13, 14].

In the present study, the result of [13, 14] is extended up to quadratic order in curvature in order to describe the relation with the bosonic spectral action [15–18]. The bosonic spectral action is interesting itself, because based on the *spectrum* of the Dirac operator, it recovers the bosonic Lagrangian of the Standard Model, coupled with gravity in the noncommutative geometry approach to particles physics [19–22]. In this thesis we show, that this object is nothing but the Weyl anomaly in a fermonic theory on bosonic background upon the spectral regularization. From another side the bosonic spectral action is a perfect example of a theory, leading to an effective QFT at low energies, however qualitatively different at high energy scale, thus exhibiting the *physical* cutoff scale Λ in the sense discussed above [23]. We also generalize the spectral regularization for all quantized fields, in order to study in a systematic way the influence of quantum vacuum fluctuations on the gravitational dynamics [24].

Independently on the context of BSA there are indications that at very high energy, of the order of Planck mass $M_{\rm Pl} = 10^{19}$ GeV, the behavior of particles is profoundly altered by the onset of gravitational effects. The first to notice this has been Bronstein [25] in 1936 and since then there have been several attempts to describe the quantum field theory at high energy or small distances. Also in string theory the very high energy behavior in the scattering of particles [26, 27] shows the existence of some sort of generalized uncertainty, whose Hilbert space representation [28] leads to a position operator which has self-adjoint extensions defined on a set of continuous lattices, so that nearby points cannot be described by the same operator. In loop quantum gravity it is the area operator which is quantized [29], while an operatorial analysis of spacetime non commutativity in quantum field theory is in [30].

Since for all this project, the bosonic spectral action is of special importance, the second chapter is devoted to a brief introduction to the Spectral action principle.

In third chapter we show how the bosonic spectral action emerges from the Weyl anomaly in a theory of fermions, moving in a fixed gauge and gravity background. The Weyl anomaly generating functional is obtained in terms of slightly modified bosonic spectral action, coupled to the dilaton. Then the full Higgs-Dilaton action, describing Weyl anomaly is computed.

In the fourth chapter, generalizing the spectral regularization also on bosonic degrees of freedom, we compute the Weyl anomaly and express the anomaly generating functional through a collective scalar degree of freedom of all quantum vacuum fluctuations. Such a formulation allows us to describe induced gravity on an equal footing with the anomaly-induced effective action, in a self-consistent way. We then show that requiring stability of the cosmological constant under loop quantum corrections, Sakharov's induced gravity and Starobinsky's anomaly-induced inflation are either both present or both absent, depending on the particle content of the theory.

The fifth and sixth chapters are devoted to models, naturally possessing the ultraviolet cutoff. In fifth chapter we discuss the propagation of bosons (scalars, gauge fields and gravitons) at high momenta in the context of the bosonic spectral action. Using heat kernel techniques, we find that in the high-momentum limit the quadratic part of the action does not contain positive powers of the derivatives. We interpret this as the fact that the two point Green functions vanish for nearby points, where the proximity scale is given by the inverse of the cutoff.

The sixth chapter describes some natural generalization of the Standard Model of elementary particles, also exhibiting the cutoff in ultraviolet. Indeed, our understanding of quantum gravity suggests that at the Planck scale the usual geometry loses its meaning. If so, the quest for grand unification in a large non-abelian group naturally endowed with the property of asymptotic freedom may also lose its motivation. Instead we propose an unification of all fundamental interactions at the Planck scale in the form of a *Universal Landau Pole* (ULP), at which all gauge couplings diverge [31,32]. The Higgs quartic coupling also diverges while the Yukawa couplings vanish. The unification is achieved with the addition of fermions with vector gauge couplings coming in multiplets and with hypercharges identical to those of the Standard Model. The presence of these particles also prevents the Higgs quartic coupling from becoming negative, thus avoiding the instability (or metastability) of the Standard Model vacuum.

Chapter 2

The Spectral Action Principle

In this chapter we give an introduction to the relevant aspects of the spectral action principle. The reader conversant with the topic may skip this part. A more thorough introduction can be found in [33].

2.1 Fields, Hilbert Spaces, Dirac Operators and the (Non)commutative Geometry of Spacetime

The main idea of the whole programme of Connes' noncommutative geometry [34] is to describe ordinary mathematics, and physics, in terms of the spectral properties of operators.

Let us introduce a (Euclidean) space-time and thereby implicitly the algebra $\mathcal{A}$ of complex valued continuous functions of this space-time. There is in fact a one-to one correspondence between (topological Hausdorff) spaces and commutative C^*-algebras, i.e. associative normed algebras with an involution and a norm satisfying certain properties. This is the content of the Gelfand-Naimark theorem [35, 36], which describes the topology of space in terms of the algebras. In physicists terms we may say the the properties of a space are encoded in the continuous fields defined on them. This concept, and its generalization to noncommutative algebras is one of the starting points of Connes' noncommutative geometry programme [34]. The programme aims at the transcription of the usual concepts of differential geometry in algebraic terms and a key role of this programme is played by a *spectral triple*, which is composed by an algebra $\mathcal{A}$ acting as operators on a Hilbert space $\mathcal{H}$ and a (generalized) Dirac operator $D\!\!\!\!/$.

The spectral triple contains the information on the geometry of space-time. The algebra as we said is dual to the topology, and the Dirac operator enables the translation of the metric and differential structure of spaces in an algebraic form. There is no room in this chapter to describe whole this programme, and we refer to the literature for details [34, 36–38].

Within this general programme a key role is played by the approach to the Standard Model. This is the attempt to understand which kind of (noncommutative) geometry gives rise to the standard model of elementary particles coupled with gravity. The roots of this approach is to have the Higgs appear naturally as the "vector" boson of the internal noncommutative degrees of freedom [39–41]. The most complete formulation of this approach is given by the *spectral action*, which is presented in [20].

To obtain the standard model take as algebra the product of the algebra of functions on spacetime times a finite dimensional matrix algebra

$$\mathcal{A} = C(\mathbb{R}^4) \otimes \mathcal{A}_F \tag{2.1}$$

Likewise the Hilbert space is the product of fermions times a finite dimensional space which contains all matter degrees of freedom, and also the Dirac operator contains a continuous part and a discrete one

$$\mathcal{H} = \mathrm{Sp}(\mathbb{R}^4) \otimes \mathcal{H}_F \tag{2.2}$$

In the NCG approach to the Standard Model we have to consider instead of the the algebra of continuous complex valued function, matrix valued functions. The underlying space in this case is still the ordinary spacetime, technically the algebra is "Morita equivalent" to the commutative algebra, but the formalism is built in a general way so to be easily generalizable to the truly noncommutative case, when the underlying space may not be an ordinary geometry.

In its most recent form due to Chamseddine, Connes and Marcolli [20] a crucial role is played by the mathematical requirements that the noncommutative algebra satisfies the conditions to be a manifold. Then under some physical requirements the internal algebra is almost uniquely derived to be

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \tag{2.3}$$

that corresponds to the gauge group $SU(3) \times SU(2) \times U(1)$. In other words from a purely algebraic scheme the gauge group of the Standard Model is singled out.

The Hilbert space $\mathcal{H}$ is assumed to be "chiral", i.e. split into a left and a right spaces:

$$\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \tag{2.4}$$

A generic matter field will therefore be a spinor

$$\Psi = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} \tag{2.5}$$

and in this representation the chirality operator, which we call γ, is a two by two block diagonal matrix with plus and minus one eigenvalues. The two components are spinors themselves and we are not indicating the gauge indices, nor the flavor indices.

We note, that the Bosonic Spectral action is defined for a Riemannian manifold with Euclidean signature of metric. In contrast to the bosonic case, the "Euclidisation" of fermions is not just analytical continuation but is a more delicate issue. One way of the Euclidisation, being the most suitable for the noncommutative geometry (see [33, 42, 43] for discussions), is based on the doubling of fermionic degrees of freedom. The idea is the following: each two component chiral spinor of the SM must be replaced by the four component Dirac fermion, and left and right fermions are treated as independent degrees of freedom, in particular

$$\psi_{\mathrm{L}}^{\mathrm{Eucl}} \neq \frac{1}{2}(1-\gamma_5)\,\psi_{\mathrm{L}}^{\mathrm{Eucl}}, \quad \psi_{\mathrm{R}}^{\mathrm{Eucl}} \neq \frac{1}{2}(1+\gamma_5)\,\psi_{\mathrm{R}}^{\mathrm{Eucl}}. \tag{2.6}$$

We stress, that *both* $\psi_{\mathrm{L}}^{\mathrm{Eucl}}$ and $\psi_{\mathrm{R}}^{\mathrm{Eucl}}$ have four independent components each, i.e. 8 independent components totally. It is important, that when one computes the partition function Z or conformal anomaly, RG equations etc. one must put *by hand* a factor of $1/2$, where needed, e.g. $Z^{\mathrm{Mink}} = \left(Z^{\mathrm{Eucl}}\right)^{\frac{1}{2}}$. *Only* when one comes back to Minkowski signature one reduces number of fermions, imposing the projection

$$\psi_{\mathrm{L}}^{\mathrm{Mink}} = \frac{1}{2}(1-\gamma_5)\,\psi^{\mathrm{Mink}}, \quad \psi_{\mathrm{R}}^{\mathrm{Mink}} = \frac{1}{2}(1+\gamma_5)\,\psi^{\mathrm{Mink}}. \tag{2.7}$$

The (generalized) Dirac operator $D\!\!\!\!/$ [19] is given by a 2×2 matrix acting on spinors of the kind (2.5)

$$D\!\!\!\!/ = \begin{pmatrix} D\!\!\!\!/_G & \gamma_5 \otimes S \\ \gamma_5 \otimes S^\dagger & D\!\!\!\!/_G \end{pmatrix} \tag{2.8}$$

where $D\!\!\!\!/_G$ is a "geometric" part of the Dirac operator,[1]

$$D\!\!\!\!/_G = i e_k^\mu \gamma^k \left(\partial_\mu - \frac{i}{2}\omega_\mu^{mn}\sigma_{mn} - iA_\mu^a T^a\right), \tag{2.9}$$

[1]Following a well established tradition, we use greek indexes to label coordinates, latin letters k, l, m, n for Lorentz indexes and a, b, c for gauge indexes.

that contains the spin connection ω_μ^{mn} and gauge fields A_μ^a, while S contains the information about Higgs field, Yukawa couplings, mixings i.e. all terms which couple the left and right spinors; T^a and σ^{mn} stand for generators of gauge and Lorentz groups in spinor representation correspondingly. The gravitational background is in general nontrivial, and the metric is encoded in the anticommutator of the γ's: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

The generalized Dirac operator describes all metric aspects of the theory, and the behavior of the fundamental matter fields, represented by vectors of the Hilbert space, and it also contains all boson fields, including the mediators of the forces (intermediate vector bosons), and the Higgs field. The dynamics of the fermions is given by coupling them to a gauge and gravitational background. This coupling is performed by a classical action, which is given by the scalar product:

$$\begin{aligned} S_F &= \langle \Psi | \not{D} | \Psi \rangle \\ &= \int d^4x \sqrt{|g|} \Psi(x)^\dagger \not{D} \Psi(x). \end{aligned} \tag{2.10}$$

2.2 The Spectral Action and the Standard Model coupled to Gravity

As we have seen, the fermionic action of the Standard Model naturally appears in the NCG formalism based on the spectral properties of the Dirac operator $\not{D}$. The bosonic action, however is still missing. The spirit of NCG prescribes to *define* the bosonic action, based on the spectrum of the Dirac operator $\not{D}$ as well. The bosonic part of the spectral action reads [19]

$$S_B = \mathrm{Tr}\chi\left(\frac{\not{D}}{\Lambda}\right), \tag{2.11}$$

where χ is an even cutoff function, which can be e.g. the sharp cutoff,

$$\chi(x) = \begin{cases} 0 & x < -1 \\ 1 & x \in [-1, 1] \\ 0 & x > 1 \end{cases} \tag{2.12}$$

and in that case it counts eigenvalues smaller than the cutoff scale Λ.

$$(S_B)_{\text{sharp cutoff}} = \text{number of eigenvalues of } \not{D}^2 \text{ smaller than } \Lambda^2 \tag{2.13}$$

Sometimes it is more convenient to use C^∞ cutoff function and for simplicity one can use the exponential cutoff $\chi(z) = \exp\left(-z^2\right)$: in this case (2.11) is nothing but a heat kernel trace. The bosonic spectral action so introduced is always finite by its nature, it is purely spectral and it depends on the cutoff Λ.

Then the bosonic spectral action can be evaluated in low energy limit using standard heat kernel techniques [44] and the final result gives the full action of the standard model coupled with gravity. We restrain from writing it since it takes more than one page, see [20].

Technically the canonical bosonic spectral action is a sum of residues, and can be expanded in a power series in terms of Λ^{-1} as

$$S_B(\Lambda) = \sum_n f_n\, a_n(D\!\!\!\!/^2/\Lambda^2) \tag{2.14}$$

where the f_n are the momenta of χ

$$\begin{aligned} f_0 &= \int_0^\infty \mathrm{d}x\, x^3 \chi(x) \\ f_2 &= \int_0^\infty \mathrm{d}x\, x \chi(x) \\ f_{2n+4} &= (-1)^n \frac{n!}{(2n)!} \partial_x^{2n} \chi(x)\bigg|_{x=0} \quad n \geq 0 \end{aligned} \tag{2.15}$$

the a_n are the Seeley-de Witt coefficients which vanish for n odd. For $D\!\!\!\!/^2$ of the form

$$D\!\!\!\!/^2 = -(g^{\mu\nu}\partial_\mu\partial_\nu \mathbf{1} + \alpha^\mu\partial_\mu + \beta) \tag{2.16}$$

defining

$$\begin{aligned} \omega_\mu &= \frac{1}{2} g_{\mu\nu}\left(\alpha^\nu + g^{\sigma\rho}\Gamma^\nu_{\sigma\rho}\mathbf{1}\right) \\ \Omega_{\mu\nu} &= \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu] \\ E &= \beta - g^{\mu\nu}\left(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho\right) \end{aligned} \tag{2.17}$$

then

$$\begin{aligned} a_0 &= \frac{\Lambda^4}{16\pi^2}\int \mathrm{d}x^4 \sqrt{g}\,\mathrm{tr}\mathbf{1}_F \\ a_2 &= \frac{\Lambda^2}{16\pi^2}\int \mathrm{d}x^4 \sqrt{g}\,\mathrm{tr}\left(-\frac{R}{6} + E\right) \\ a_4 &= \frac{1}{16\pi^2}\frac{1}{360}\int \mathrm{d}x^4 \sqrt{g}\,\mathrm{tr}(-12\nabla^\mu\nabla_\mu R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} \\ &\quad +2R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60\nabla^\mu\nabla_\mu E + 30\Omega_{\mu\nu}\Omega^{\mu\nu}) \end{aligned} \tag{2.18}$$

tr is the trace over the inner indices of the finite algebra $\mathcal{A}_F$ and in Ω and E contain the gauge degrees of freedom including the gauge stress energy tensors and the Higgs scalar.

Performing the heat kernel expansion for the Dirac operator $D\!\!\!/$, of the form given by (2.8) one finds [19]

$$\begin{aligned} S_B(\Lambda) \;\simeq\; & \int d^4x\sqrt{g}\bigg(\frac{45\Lambda^4}{8\pi^2}+\frac{15\Lambda^2}{16\pi^2}\left(R-2y^2H^2\right) \\ & +\frac{1}{4\pi^2}\bigg(3y^2\left(D_\mu HD^\mu H-\frac{1}{6}RH^2\right)+3z^2H^4 \\ & +G^i_{\mu\nu}G^{\mu\nu i}+W^\alpha_{\mu\nu}W^{\mu\nu\alpha}+\frac{5}{3}B_{\mu\nu}B^{\mu\nu}-\frac{9}{16}C_{\mu\nu\rho\lambda}C^{\mu\nu\rho\lambda}\bigg)\bigg), \end{aligned} \tag{2.19}$$

where y^2 and z^2 stand for correspondingly quadratic and quartic combinations of the Yukawa couplings, whose precise definition can be found for example in [19, Eq. 3.17]. Since the Yukawa couplings are strongly dominated by the one of the top quark Y_t, one can safely consider, that $y^2 \simeq Y_t^2, z^2 \simeq Y_t^4$.

The bosonic spectral action (2.19) contains Einstein-Hilbert gravitational action, action for the Higgs field, conformally coupled with gravity, kinetic terms for gauge bosons. The action also contains nonstandard gravitational terms (quadratic in the curvature), which are currently being investigated for their cosmological consequences [45–52]. It is remarkable, that all coefficients in (2.19) are not arbitrary numbers, but come out from the Dirac operator $D\!\!\!/$, that contains only parameters, related with the fermions, and thus this construction has predictive power, in particular it predicts the Higgs mass. Although the first version of the Spectral Action predicts a wrong value of the Higgs mass ($\sim$ 170 GeV instead of $\sim$ 125 GeV), ”nonminimal” generalizations of this idea, lead to the correct value of the Higgs mass [53–56].

In this chapter we briefly discussed the spectral action principle, i.e. we have shown, how the classical bosonic action of the Standard Model can be extracted from the fermionic Dirac operator in a presence of the cutoff scale Λ. In the next chapter we will discuss quantized fermions, moving in a fixed bosonic background under the spectral regularization, i.e. cutoff scale appears again, but this time upon a quantization. We will establish the relation between the (generalised) Weyl anomaly and the bosonic spectral action and in particular we will find, that the bosonic spectral Lagrangian (i.e. integrand in (2.19)) is nothing but the infinitesimal Weyl anomaly.

Chapter 3

Spectral regularization: bosonic spectral action and generalised Weyl anomaly.

In this chapter we will show the intimate relationships between Weyl anomaly and the bosonic spectral action in the framework of spectral regularization following refs. [16–18].

We start with a generic action for a chiral theory of fermions coupled to gauge fields, Higgs and gravity. The considerations here apply to the standard model, but we will not need the details of the particular theory under consideration. It is known (see the previous chapter), and this is the essence of the noncommutative geometry approach to the standard model, that the theory is described by a fermionic action and a bosonic action, both of which can be expressed in terms of the spectrum of the Dirac operator.

In what follows we will introduce the spectral regularization, then we will discuss the (generalized) Weyl invariance of the classical fermionic action and will compute the (generalized) Weyl anomaly. Afterwards we present Weyl anomaly generating functional introducing the auxilary field, that as we will see can be interpreted as a collective mode of all fermions dual to (generalised) conformal anomaly, therefore we will call this field ”collective dilaton”. Corresponding classical Higgs-dilaton action appears to be bosonic spectral action coupled with the dilaton in a special way.

3.1 Spectral regularization: definition

We start from the fermionic partition function

$$Z(D\!\!\!/) = \int [\mathrm{d}\Psi][\mathrm{d}\bar{\Psi}]e^{-S_F} = \det\left(\frac{D\!\!\!/}{\mu}\right) \tag{3.1}$$

where we needed to introduce a normalization scale μ for dimensional reasons, and the last equality is formal because the expression is divergent and needs regularizing.

In order to regularize the expression Eq. (3.1) we need to introduce a *cutoff* scale, which we call Λ. This cutoff scale may have the physical meaning of an energy in which the theory (seen as effective) has a phase transition, or at any rate a point in which the symmetries of the theory are fundamentally different (unification scale). Some QFT models, naturally exhibiting the ultraviolet cutoff scale will be considered in Chapters 5 and 6.

We will regularize the theory in the ultraviolet using a procedure introduced by Andrianov, Bonora and Gamboa-Saravi in [7–9] but leaving room for the normalization scale μ. Although this procedure predates the spectral action, it is very much in the spirit of spectral geometry, since it uses only the spectral data of the Dirac operator. The energy cutoff is enforced by considering only the eigenvalues of $D\!\!\!/$ smaller than the scale Λ. Consider the projector

$$P_N = \sum_{n=1}^{N} |\lambda_n\rangle\langle\lambda_n| ; \quad N = \max n \text{ such that } \lambda_n \leq \Lambda \tag{3.2}$$

where λ_n are the eigenvalues of $D\!\!\!/$ arranged in increasing order of their absolute value (repeated according to possible multiplicities), $|\lambda_n\rangle$ a corresponding orthonormal basis, and the integer N is a maximal number of eigenvalue that is smaller than Λ. This means that we are effectively using the N^{th} eigenvalue as cutoff. This number and the corresponding spectral density depends on coefficient functions of the Dirac operator, $N = N(D\!\!\!/)$. We emphasize, that everything is well defined and finite.

Instead of this sharp cutoff, which considers totally all eigenvalues up to a certain energy, and ignores all the rest of the spectrum, it is also possible to consider a smooth cutoff enforced by a smooth function. Choosing a function χ which is smoothened version of the characteristic function of the interval $[0, 1]$ one can consider the operator

$$P_\chi = \chi\left(\frac{D\!\!\!/}{\Lambda}\right) = \sum_n \chi\left(\frac{\lambda_n}{\Lambda}\right) |\lambda_n\rangle\langle\lambda_n| . \tag{3.3}$$

This operator is not a projector anymore, and it coincides with P_N for $\chi = \Theta$, where Θ is the Heaviside step function. The use of a smooth χ can be preferable in an expansion, such as the heat kernel expansion, nevertheless for the scopes of the present chapter a sharp cutoff is adequate.

In the framework of noncommutative geometry this is the most natural cutoff procedure, although as we said it was introduced before the introduction of the standard model in noncommutative geometry. It makes no reference in principle to the underlying structure of spacetime, and it is based purely on spectral data, thus is perfectly adequate to Connes' programme. This form of regularization could be also used for field theory which cannot be described on an ordinary space-time, as long as there is a Dirac operator, or generically a wave operator[1].

We define the regularized partition function $Z(D\!\!\!\!/, \mu)$ as follows

$$Z(D\!\!\!\!/, \mu) \equiv \prod_{n=1}^{N} \frac{\lambda_n}{\mu}. \tag{3.4}$$

If one choses $\mu = \Lambda$, the regularized partition function $Z(D\!\!\!\!/, \Lambda)$ has a transparent meaning. Let us express Ψ and $\bar{\Psi}$ as

$$\Psi = \sum_{n=1}^{\infty} a_n |\lambda_n\rangle \,; \qquad \bar{\Psi} = \sum_{n=1}^{\infty} b_n |\lambda_n\rangle \tag{3.5}$$

with a_n and b_n anticommuting (Grassman) quantities. Then $Z(D\!\!\!\!/, \Lambda)$ becomes (performing the integration over Grassman variables for the last step)

$$Z(D\!\!\!\!/, \Lambda) = \int \prod_{n=1}^{N} \frac{\mathrm{d}a_n \mathrm{d}b_n}{\Lambda} \mathrm{e}^{-\sum_{n=1}^{N} b_n \lambda_n a_n} = \det\left(D\!\!\!\!/_N\right) \tag{3.6}$$

where we defined[2]

$$D\!\!\!\!/_N = 1 - P_N + P_N \frac{D\!\!\!\!/}{\Lambda} P_N. \tag{3.7}$$

In the basis in which $D\!\!\!\!/ / \Lambda$ is diagonal it corresponds to set to Λ all eigenvalues of $D\!\!\!\!/$ larger than Λ. Note that $D\!\!\!\!/_N$ is dimensionless and depends on Λ both explicitly and intrinsically via the dependence of N and P_N.

[1]The spectrum must not be necessary descrete, see [57]

[2]Although P_N commutes with $D\!\!\!\!/$ we prefer to use a more symmetric notation.

Now we see what happens if one chooses $\mu \neq \Lambda$. In this case the partition function reads:

$$\begin{aligned} Z(\not{D}, \mu) &= \prod_{n=1}^{N} \frac{\lambda_n}{\mu} = \det\left(\mathbf{1} - P_N + P_N \frac{\not{D}}{\mu} P_N\right) \\ &= \det\left(\mathbf{1} - P_N + P_N \frac{\not{D}}{\Lambda} P_N\right) \det\left(\mathbf{1} - P_N + \frac{\Lambda}{\mu} P_N\right) \\ &= Z(\not{D}, \Lambda) \det\left(\mathbf{1} - P_N + \frac{\Lambda}{\mu} P_N\right). \end{aligned} \tag{3.8}$$

The latter factor in Eq. (3.8) can be rewritten as follows

$$\det\left(\mathbf{1} - P_N + \frac{\Lambda}{\mu} P_N\right) = \prod_{n=1}^{N} \frac{\Lambda}{\mu} = e^{-\log\left(\frac{\mu}{\Lambda}\right) \cdot N}$$

$$= \exp\left\{-\log\left(\frac{\mu}{\Lambda}\right) \cdot \ \mathrm{Tr}\chi\left(\frac{\not{D}}{\Lambda}\right)\right\}, \quad \chi(z) \equiv \Theta\left(1 - z^2\right), \tag{3.9}$$

thus we conclude that the ambiguity in choice of μ corresponds to the ambiguity of an addition of the bosonic spectral action, defined by Eq. (2.11), to the fermionic action S_F. Or equivalently in the spectral action principle it is not necessary to put the BSA by hand, one can consider from the very beginning *quantized* fermionic theory, regularized following our natural prescription and the BSA appears automatically!

3.2 Weyl invariance and the Fermionic Action

Now we demonstrate, that the fermionic action given by Eq. (2.10) is invariant under the *generalized* Weyl transformation

$$g_{\mu\nu} \to \mathrm{e}^{2\phi} g_{\mu\nu}, \quad \Psi \to \mathrm{e}^{-\frac{3}{2}\phi}\Psi, \quad H \to \mathrm{e}^{-\phi} H. \tag{3.10}$$

Note that the rescaling involves also the Higgs field. In this sense we differ form the usual usage of Weyl (or conformal) invariance which is only valid for massless fields. In what follows we will skip the word "generalized" for brevity.

It is sufficient to show, that under the Weyl transformation Eq. (3.10) of $g_{\mu\nu}$ and H, the Dirac operator $\not{D}$ given by Eq. (2.8) transforms in a homogeneous way:

$$\not{D} \to e^{-\frac{5}{2}\phi(x)} \not{D} e^{\frac{3}{2}\phi(x)} \tag{3.11}$$

The law of transformation of the Higgs field H is in agreement with Eq. (3.11). To prove the equation of Eq. (3.11) finally, we notice, that, under[3] Eq. (3.10), the geometric part of the Dirac operator $\not{D}_G$ given by Eq. (2.9) transforms as follows:

$$\not{D}_G \to \mathrm{e}^{-\frac{5\phi(x)}{2}} \not{D}_G \mathrm{e}^{\frac{3\phi(x)}{2}}. \tag{3.12}$$

The mentioned result is present in [58], however here we present a more detailed proof. The Weyl transformation of the metric tensor in terms of vierbeins is given by:

$$e_{\mu k} \to \mathrm{e}^{\phi(x)} e_{\mu k}, \quad \mathrm{e}^{\mu k} \to \mathrm{e}^{-\phi(x)} e^{\mu k}. \tag{3.13}$$

The spin connection ω_μ^{mn} has the following expression via vierbeins (see [58]):

$$\omega_\mu^{mn} = \frac{1}{2} e^{m\lambda} e^{n\rho} \left(C_{\lambda\rho\mu} - C_{\rho\lambda\mu} - C_{\mu\lambda\rho} \right), \tag{3.14}$$

where

$$C_{\lambda\rho\mu} = e_\lambda^k (\partial_\rho e_{k\mu} - \partial_\mu e_{k\rho}). \tag{3.15}$$

Substituting the transformation Eq. (3.13) in Eq. (3.15) and Eq. (3.14) we find the spin connection transformation law under Eq. (3.13):

$$\omega_\mu^{mn} \to \omega_\mu^{mn} + e_\mu^m e^{n\rho} \partial_\rho \phi - e_\mu^n e^{m\lambda} \partial_\lambda \phi. \tag{3.16}$$

The generators of the representation of Lorentz group σ_{mn} for spin $1/2$ have the following form in terms of the Dirac matrixes:

$$\sigma_{mn} = \frac{i}{4} \left[\gamma_m, \gamma_n \right]. \tag{3.17}$$

Therefore the transformation of the combination $\frac{i}{2} \omega_\mu^{mn} \sigma_{mn}$ reads: (this formula is presented in [58][4]):

$$\frac{i}{2} \omega_\mu^{mn} \sigma_{mn} \to \frac{i}{2} \omega_\mu^{mn} \sigma_{mn} - \frac{1}{2} \gamma_\mu \gamma^\alpha \partial_\alpha \phi + \frac{1}{2} \partial_\mu \phi. \tag{3.18}$$

So we have:

$$-\gamma^\mu \frac{i}{2} \omega_\mu^{mn} \sigma_{mn} \to -\gamma^{mu} \frac{i}{2} \omega_\mu^{mn} \sigma_{mn} + \frac{3}{2} \gamma^\mu \partial_\mu \phi. \tag{3.19}$$

[3]More carefully one should write the corresponding transformation of vierbeins instead of the transformation of a metric tensor.

[4]Comparing our and [58, Eq. (B.24)] formulas: note that Fujikawa and Suzuki use $\alpha(x)$ = -$\phi(x)$

Finally, using Eq. (3.19) and Eq. (3.13) we obtain:

$$\begin{aligned}
\not{D}_G &\equiv ie^\mu_k\gamma^k\left(\partial_\mu - \frac{i}{2}\omega^{mn}_\mu\sigma_{mn} - iA^a_\mu T^a\right) \to \\
&\to ie^\mu_k\gamma^k e^{-\phi}\left(\partial_\mu - \frac{i}{2}\omega^{mn}_\mu\sigma_{mn} - iA^a_\mu T^a + \frac{3}{2}\partial_\mu\phi\right) = \\
&= ie^\mu_k\gamma^k e^{-\frac{5\phi}{2}}\left(\partial_\mu - \frac{i}{2}\omega^{mn}_\mu\sigma_{mn} - iA^a_\mu T^a\right)e^{+\frac{3\phi}{2}} = e^{-\frac{5}{2}\phi(x)}\not{D}_G e^{+\frac{3}{2}\phi(x)}. \quad (3.20)
\end{aligned}$$

We also remark that we *do not* transform the gauge fields A^a_μ: they appear in $\not{D}_G$ multiplied by e^μ_k, so the correct transformation of the "gauge term" of $\not{D}_G$ is automatically provided by the transformation of the vierbeins.

We now proceed to quantize the theory. It can be proven [59] that although the classical theory is invariant, the measure in the quantum path integral is not. We have an anomaly: in contrast to a classical case, the quantum theory is not invariant against this symmetry transformation anymore. A textbook introduction to anomalies can be found in [58].

3.3 Generalized Weyl anomaly

Although, as we said before, the classical fermionic action is (generalized) Weyl invariant, the quantum effective action $W \equiv -\log Z$ is not. Now we would like to compute the difference between the initial quantum effective action $W \equiv W[g_{\mu\nu}, H]$ and the transformed one $W_\phi \equiv W[g_{\mu\nu}e^{2\phi}, He^{-\phi}]$.

$$W - W_\phi = \log\left(\frac{Z_\phi}{Z}\right) = \int_0^1 dt\, \partial_t \log\left(Z_{\phi\cdot t}\right) \quad (3.21)$$

The nontrivial step is to compute $\partial_t \log\left(Z_{\phi\cdot t}\right)$ In what follows we will use the notations

$$\not{D} \to \not{D}_\phi \equiv e^{-\frac{5\phi}{2}}\not{D}e^{\frac{3\phi}{2}}, \quad P_\Lambda\left[\not{D}\right] = \Theta\left(\Lambda^2 - \not{D}^2\right), \quad (3.22)$$

Using the standard relation "log det = Tr log" one has:

$$\begin{aligned}
\partial_t Z_{\phi\cdot t} &= \partial_t \exp\ \mathrm{Tr}\ \log\left(\frac{\not{D}_{\phi\cdot t}}{\mu}P_\Lambda\left[\not{D}_{\phi\cdot t}\right]\right) \\
&= Z_{\phi\cdot t}\cdot\partial_t\ \mathrm{Tr}\ \log\left(\frac{\not{D}_{\phi\cdot t}}{\mu}P_\Lambda\left[\not{D}_{\phi\cdot t}\right]\right). \quad (3.23)
\end{aligned}$$

Since the operator P_Λ is a projector i.e. $P_\Lambda^2 = 1$ one can write it outside of the sign of log:

$$\begin{aligned}
& Z_{\phi\cdot t} \cdot \partial_t \,\mathrm{Tr}\, \log\left(\frac{\not{D}_{\phi\cdot t}}{\mu} P_\Lambda\left[\not{D}_{\phi\cdot t}\right]\right) = Z_{\phi\cdot t} \cdot \partial_t \,\mathrm{Tr}\, \left\{\log\left(\frac{\not{D}_{\phi\cdot t}}{\mu}\right) P_\Lambda\left[\not{D}_{\phi\cdot t}\right]\right\} \\
= \;& Z_{\phi\cdot t} \cdot \partial_t \,\mathrm{Tr}\, \left\{\log\left(\frac{\not{D}_{\phi\cdot t}}{\Lambda}\right) P_\Lambda\left[\not{D}_{\phi\cdot t}\right] + \log\left(\frac{\Lambda}{\mu}\right) \cdot P_\Lambda\left[\not{D}_{\phi\cdot t}\right]\right\}. \qquad (3.24)
\end{aligned}$$

Let us compute the first term in Eq. (3.24):

$$\begin{aligned}
& \partial_t \,\mathrm{Tr}\, \left\{\log\left(\frac{\not{D}_{\phi\cdot t}}{\Lambda}\right) P_\Lambda\left[\not{D}_{\phi\cdot t}\right]\right\} \\
= \;& \mathrm{Tr}\, \left\{\partial_t\left(\log\left(\frac{\not{D}_{\phi\cdot t}}{\Lambda}\right)\right) P_\Lambda\left[\not{D}_{\phi\cdot t}\right] + \log\left(\frac{\not{D}_{\phi\cdot t}}{\Lambda}\right) \partial_t P_\Lambda\left[\not{D}_{\phi\cdot t}\right]\right\} \\
= \;& \mathrm{Tr}\, \left\{\left(\not{D}_{\phi\cdot t}\right)^{-1} \partial_t\left(\not{D}_{\phi\cdot t}\right) \cdot P_\Lambda\left[\not{D}_{\phi\cdot t}\right] - \frac{1}{2}\log\left(\frac{\not{D}^2_{\phi\cdot t}}{\Lambda^2}\right) \delta\left(\Lambda^2 - \not{D}^2_{\phi\cdot t}\right) \partial_t\left(\not{D}^2_{\phi\cdot t}\right)\right\} \\
= \;& \mathrm{Tr}\, \left\{\left(\not{D}_{\phi\cdot t}\right)^{-1} \left(-\frac{5\phi}{2}\not{D}_{\phi\cdot t} + \not{D}_{\phi\cdot t}\frac{3\phi}{2}\right) \cdot P_\Lambda\left[\not{D}_{\phi\cdot t}\right]\right. \\
& \left. -\frac{1}{2}\underbrace{\log\left(\frac{\Lambda^2}{\Lambda^2}\right)}_{0} \delta\left(\Lambda^2 - \not{D}^2_{\phi\cdot t}\right) \partial_t\left(\not{D}^2_{\phi\cdot t}\right)\right\} \\
= \;& -\mathrm{Tr}\, \left\{\phi \cdot P_\Lambda\left[\not{D}_{\phi\cdot t}\right]\right\}, \qquad (3.25)
\end{aligned}$$

where we used the definition Eq. (3.22) of projector P_Λ and performed a cyclic permutation of terms under the sign of trace where it was needed.

Now we work on the second term in Eq. (3.24).

$$\begin{aligned}
& \partial_t \,\mathrm{Tr}\, \left\{P_\Lambda\left[\not{D}_{\phi\cdot t}\right]\right\} = -\,\mathrm{Tr}\, \left\{\delta\left(\Lambda^2 - \not{D}^2_{\phi\cdot t}\right) \partial_t\left(\not{D}^2_{\phi\cdot t}\right)\right\} \\
= \;& -2\,\mathrm{Tr}\, \left\{\delta\left(\Lambda^2 - \not{D}^2_{\phi\cdot t}\right)\left(\not{D}_{\phi\cdot t}\right) \cdot \left(-\frac{5\phi}{2}\not{D}_{\phi\cdot t} + \not{D}_{\phi\cdot t}\frac{3\phi}{2}\right)\right\} \\
= \;& 2\,\mathrm{Tr}\, \left\{\phi \cdot \not{D}^2_{\phi\cdot t} \cdot \delta\left(\Lambda^2 - \not{D}^2_{\phi\cdot t}\right)\right\} = 2\,\mathrm{Tr}\, \left\{\phi \cdot \Lambda^2 \cdot \delta\left(\Lambda^2 - \not{D}^2_{\phi\cdot t}\right)\right\} \\
= \;& 2\,\mathrm{Tr}\, \left\{\phi \cdot \Lambda^2 \cdot \partial_{\Lambda^2}\Theta\left(\Lambda^2 - \not{D}^2_{\phi\cdot t}\right)\right\} = 2\Lambda^2\partial_{\Lambda^2}\,\mathrm{Tr}\, \left\{\phi \cdot P_\Lambda\left[\not{D}_{\phi\cdot t}\right]\right\}. \qquad (3.26)
\end{aligned}$$

Using the formulas Eq. (3.25) and Eq. (3.26) we obtain:

$$\partial_t \log Z_{\phi\cdot t} = -\left(1 - \Lambda^2 \log\frac{\Lambda^2}{\mu^2}\partial_{\Lambda^2}\right) \mathrm{Tr}\, \left\{\phi \cdot P_\Lambda\left[\not{D}_{\phi\cdot t}\right]\right\}. \qquad (3.27)$$

Substituting the result Eq. (3.27) in Eq. (3.21) we arrive to the answer:

$$W - W_\phi = -\left(1 - \Lambda^2 \log \frac{\Lambda^2}{\mu^2} \partial_{\Lambda^2}\right) \int_0^1 dt \,\mathrm{Tr}\left\{\phi\, \chi\left(\frac{\not{D}_{\phi\cdot t}}{\Lambda}\right)\right\}, \qquad (3.28)$$

where $\chi(z) \equiv \Theta\left(1 - z^2\right)$

As one can easily see, the structure

$$\int_0^1 dt \,\mathrm{Tr}\left\{\phi\, \chi\left(\frac{\not{D}_{\phi\cdot t}}{\Lambda}\right)\right\}$$

is very similar to the bosonic spectral action with the sharp cutoff, c.f. Eq. (2.11). It is possible to give an explicit functional expression to the projector in terms of the cutoff:

$$P_\Lambda\left[\not{D}\right] = \Theta\left(1 - \frac{\not{D}^2}{\Lambda^2}\right) = \lim_{\epsilon\to+0} \int\limits_{-\infty}^{\infty} \mathrm{d}\alpha \,\frac{1}{2\pi\mathrm{i}(\alpha - \mathrm{i}\epsilon)} \mathrm{e}^{\mathrm{i}\alpha\left(1-\frac{\not{D}^2}{\Lambda^2}\right)} \qquad (3.29)$$

This integral is well defined for a compactified space volume. Using the representation of the projector Eq. (3.29) and the heat kernel expansion one can show, that[5]

$$\begin{aligned}\mathrm{Tr}\left\{\phi\chi\left(\frac{\not{D}}{\Lambda}\right)\right\} = \int d^4x\,\sqrt{g}\phi\Bigg(&\frac{45\Lambda^4}{8\pi^2} + \frac{15\Lambda^2}{16\pi^2}\left(R - \frac{8}{5}y^2H^2\right)\\ +\frac{1}{4\pi^2}\Bigg(&\frac{3}{8}R_{;\mu}{}^{\mu} + \frac{11}{32}G_B - y^2H^2_{;\mu}{}^{\mu} + 3y^2\left(\nabla_\mu H\nabla^\mu H - \frac{1}{6}RH^2\right)\\ &+3z^2H^4 + G^i_{\mu\nu}G^{\mu\nu i} + W^\alpha_{\mu\nu}W^{\mu\nu\alpha} + \frac{5}{3}B_{\mu\nu}B^{\mu\nu} - \frac{9}{16}C_{\mu\nu\rho\lambda}C^{\mu\nu\rho\lambda}\Bigg)\Bigg), \end{aligned} \qquad (3.30)$$

where G_B denotes the Gauss-Bonnet density:

$$G_B \equiv \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}_{\mu\nu}R^{\gamma\delta}_{\rho\sigma}. \qquad (3.31)$$

In Eq. (3.30) and below indexes placed after the symbol ";" denote covariant derivatives with respect to corresponding coordinates. In the next section we present some technical details, we needed to reach the final answer.

[5]See the details for a more general case in the next chapter.

3.4 Computational details

R contribution.

Under the Weyl transformation of the metric tensor given by Eq. (3.10), the scalar curvature transforms as follows[6]:

$$R \to \left(\tilde{R}\right)_{\phi} \equiv e^{-2\phi}\left(R + 6\left(\phi_{;\mu}^{\ \ \mu} + \phi_{;\mu}\phi_{;}^{\ \mu}\right)\right), \tag{3.32}$$

and integrating by parts one can easily show, that

$$\begin{aligned} &-\int d^4x\,\phi \int_0^1 dt \left(\sqrt{\tilde{g}}\tilde{R}\right)_{\phi t} \\ &= \int d^4x\,\sqrt{g}\left(-\frac{1}{2}\left(e^{2\phi} - 1\right)R + 3\cdot e^{2\phi}\left(\phi_{;\mu}\phi_{;}^{\ \mu}\right)\right). \end{aligned} \tag{3.33}$$

$\left(H^2\right)_{;\mu}^{\ \ \mu}$ and $R_{;\mu}^{\ \ \mu}$ contributions.

For a scalar quantity f, that transforms under the Weyl transformation Eq. (3.10) as

$$f \to \left(\tilde{f}\right)_{\phi}, \tag{3.34}$$

its Laplacian $\Delta f \equiv \nabla_\mu \nabla^\mu f$ transforms as follows:

$$\Delta f \to \left(\tilde{\Delta}\tilde{f}\right)_{\phi} = e^{-4\phi}\nabla^\mu e^{2\phi}\nabla_\mu \left(\tilde{f}\right)_{\phi}. \tag{3.35}$$

For $f = H^2$ and $f = R$, using the relations Eq. (3.32) and Eq. (3.35), we obtain the following contributions to the anomaly:

$$-\int d^4x\,\phi \int_0^1 dt\,\sqrt{\tilde{g}}\left(\tilde{\Delta}\tilde{H}^2\right)_{\phi t} = -\int d^4x\,\sqrt{g}\left(\phi_{;\mu}^{\ \ \mu} + \phi_{;\mu}\phi_{;}^{\ \mu}\right)H^2 \tag{3.36}$$

and

$$\begin{aligned} &-\int d^4x\,\phi \int_0^1 dt \left(\sqrt{\tilde{g}}\tilde{\Delta}\tilde{R}\right)_{\phi t} \\ &= -\int d^4x\,\sqrt{g}\left(\left(\phi_{;\mu}^{\ \ \mu} + \phi_{;\mu}\phi_{;}^{\ \mu}\right)R + 3\left(\phi_{;\mu}^{\ \ \mu} + \phi_{;\mu}\phi_{;}^{\ \mu}\right)^2\right). \end{aligned} \tag{3.37}$$

[6]Here and below the notation $\widetilde{(A)}_B$ stands for the quantity A transformed under Eq. (3.10) with $\phi = B$.

G_B contribution.

It is known from the differential geometry, that the Gauss-Bonnet density G_B in a four dimensional space-time can be presented in the following form, convenient for the forthcoming analysis:

$$G_B = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - 2\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right). \tag{3.38}$$

For the transformed Ricci tensor under the Weyl transformation Eq. (3.10) we have:

$$\left(\tilde{R}_{\mu\nu}\right)_\phi = R_{\mu\nu} + 2\left(\phi_{;\mu\nu} - \phi_{;\mu}\phi_{;\nu}\right) + \left(\phi_{;\lambda}^{;\lambda} + 2\phi_{;\lambda}\phi_{;}^{\ \lambda}\right)g_{\mu\nu} \tag{3.39}$$

Using laws of transformations of the Ricci tensor and the scalar curvature Eq. (3.39), Eq. (3.32) and also Weyl invariance of the Weyl tensor contribution after some simple computations we obtain:

$$\sqrt{g}G_B \to \left(\sqrt{\tilde{g}}\tilde{R}^*\tilde{R}^*\right)_\phi = \sqrt{g}\left(G_B + \nabla_\mu J^\mu\right). \tag{3.40}$$

where the current J^μ is defined as follows:

$$J^\mu \equiv 8\left(-\phi_\nu G^{\nu\mu} + \left(\phi_{;\lambda}^{;\lambda} + \phi_{;\lambda}\phi_{;}^{\ \lambda}\right)\phi_{;}^{\ \mu}\right) - 4\left(\phi_{;\lambda}\phi_{;}^{\ \lambda}\right)_{;}^{\ \mu}, \tag{3.41}$$

Contribution of the Gauss-Bonnet term to the anomaly potential is proportional (with the sign plus) to the following expression:

$$- \int d^4x\phi(x)\int_0^1 dt\left(\sqrt{\tilde{g}}\tilde{R}^*\tilde{R}^*\right)_{\phi t} =$$
$$\int d^4x\,\sqrt{g}\left(-\phi G_B - 4G^{\mu\nu}\phi_{;\mu}\phi_{;\nu} + 2\left(\phi_{;\mu}\phi_{;}^{\ \mu}\right)^2 + 4\left(\phi_{;\mu}\phi_{;}^{\ \mu}\right)\phi_{;\lambda}^{;\lambda}\right). \tag{3.42}$$

3.5 The final result

Substituting the expression Eq. (3.30) into Eq. (3.28) and using results Eq. (3.33), Eq. (3.36), Eq. (3.37) and Eq. (3.42) of the previous subsection we finally get:

$$
\begin{aligned}
&W - W_\phi \equiv \\
&\int d^4x \sqrt{g}\Big(A\left(\mathrm{e}^{4\phi}-1\right) + BH^2\left(\mathrm{e}^{2\phi}-1\right) - C\phi H^4 - \alpha_1\left(\mathrm{e}^{2\phi}-1\right)R \\
&+\alpha_2 \mathrm{e}^{2\phi}\left(\phi_{;\mu}\phi_{;}^{\ \mu}\right) - \alpha_3\,\phi\bigg(3y^2\left(\nabla_\mu H\nabla^\mu H - \frac{1}{6}RH^2\right) \\
&+G^i_{\mu\nu}G^{\mu\nu i} + W^\alpha_{\mu\nu}W^{\mu\nu\alpha} + \frac{5}{3}B_{\mu\nu}B^{\mu\nu} - \frac{9}{16}C_{\mu\nu\rho\lambda}C^{\mu\nu\rho\lambda}\bigg) \\
&-\alpha_4\Big(12R\left(\phi_{;\mu}^{\ \mu} + \phi_{;\mu}\phi_{;}^{\ \mu}\right) + 11\phi G_B + 44G^{\mu\nu}\phi_{;\mu}\phi_{;\nu} \\
&+14\left(\phi_{;\mu}^{\ \mu} + \phi_{;\mu}\phi_{;}^{\ \mu}\right)^2 + 22\left(\phi_{;\mu}^{\ \mu}\right)^2\Big) + \alpha_5\, y^2\left(2\phi_{;\mu}^{\ \mu} - \phi_{;\mu}\phi^{;\mu}\right)H^2\Big),
\end{aligned} \tag{3.43}
$$

where $G_{\mu\nu}$ stands for the Einstein tensor and the constants A, B, C, $\alpha_1..\alpha_5$, are defined as follows:

$$
\begin{aligned}
A &= \left(2\log\frac{\Lambda^2}{\mu^2} - 1\right)\frac{45\Lambda^4}{32\pi^2}, \quad B = \left(1-\log\frac{\Lambda^2}{\mu^2}\right)\frac{15\Lambda^2 y^2}{20\pi^2}, \quad C = \frac{3z^2}{4\pi^2}, \\
\alpha_1 &= \left(1-\log\frac{\Lambda^2}{\mu^2}\right)\frac{15\Lambda^2}{32\pi^2}, \quad \alpha_2 = \left(1-\log\frac{\Lambda^2}{\mu^2}\right)\frac{45\Lambda^2}{16\pi^2}, \quad \alpha_3 = \frac{1}{4\pi^2}, \\
\alpha_4 &= \frac{1}{128\pi^2}, \quad \alpha_5 = \frac{1}{8\pi^2}.
\end{aligned} \tag{3.44}
$$

Remark

At this point, let us emphasize that in case $\Lambda = \mu$ the infinitesimal Weyl anomaly, obtained within QFT with spectral regularization, is the bosonic spectral Lagrangian. Indeed, by definition,

$$
S_\mathrm{B} \equiv \mathrm{Tr}\left(\chi\left(\frac{D\!\!\!/}{\Lambda}\right)\right) \simeq \int d^4x\ \sqrt{g}\, L_\mathrm{BS}(x), \tag{3.45}
$$

where $L_\mathrm{BS}(x)$ stands for the bosonic spectral Lagrangian, computed via the heat-kernel technique (see Eq. (2.19)). Performing a similar computation and inserting $\phi(x)$ under the sign of the trace, we get

$$
\mathrm{Tr}\left(\phi\left[\chi\left(\frac{D\!\!\!/}{\Lambda}\right)\right]\right) \simeq \int d^4x\ \sqrt{g}\,\phi\, L_\mathrm{BS}(x)\,, \tag{3.46}
$$

and the bosonic spectral Lagrangian reads

$$L_{\mathrm{BS}}(x) = \frac{1}{\sqrt{g}}\ \frac{\delta}{\delta\phi(x)}\ \mathrm{Tr}\left(\phi\left[\chi\left(\frac{D\!\!\!\!/}{\Lambda}\right)\right]\right)\,. \tag{3.47}$$

Expanding Eq. (3.28) up to linear order in ϕ and taking the functional derivative in the infinitesimal limit, we obtain:

$$\begin{aligned}
\text{infinitesimal Weyl anomaly} \quad &\equiv \quad \lim_{\phi\to 0}\ \frac{1}{\sqrt{g}}\ \frac{\delta}{\delta\phi(x)}\ (\widetilde{W_{\mathrm{F}}})_{\phi} \\
&= \quad \lim_{\phi\to 0}\ \frac{1}{\sqrt{g}}\ \frac{\delta}{\delta\phi(x)}\ \int_0^1 \mathrm{d}t\ \mathrm{Tr}\left(\phi\left[\widetilde{\chi\left(\frac{D\!\!\!\!/}{\Lambda}\right)}\right]_{\phi\cdot t}\right) \\
&= \quad \lim_{\phi\to 0}\ \frac{1}{\sqrt{g}}\ \frac{\delta}{\delta\phi(x)}\ \left[\mathrm{Tr}\left(\phi\left[\chi\left(\frac{D\!\!\!\!/}{\Lambda}\right)\right]\right) + O\left(\phi^2\right)\right] \\
&= \quad \frac{1}{\sqrt{g}}\ \frac{\delta}{\delta\phi(x)}\ \mathrm{Tr}\left(\phi\left[\chi\left(\frac{D\!\!\!\!/}{\Lambda}\right)\right]\right) \\
&= \quad L_{\mathrm{BS}}(x)\,,
\end{aligned} \tag{3.48}$$

where in the last step we used Eq. (3.47).

3.6 Weyl Anomaly generating functional and collective dilaton

Using the result Eq. (3.43) we will obtain the expression for Weyl noninvariant part of the fermionic partition function in terms of the bosonic spectral action, coupled with the quantized "collective dilaton". Integrating over all possible dilatations one obtains (generalised) Weyl invariant functional

$$Z_{\mathrm{inv}}(D\!\!\!\!/,\mu) = \left(\int \mathrm{d}\phi \frac{1}{Z(D\!\!\!\!/_{\phi},\mu)}\right)^{-1}. \tag{3.49}$$

If we consider non Weyl invariant partition function we can split it in the product of a term invariant for Weyl transformations, and another not invariant.

$$Z(D\!\!\!\!/,\mu) = Z_{\mathrm{inv}}(D\!\!\!\!/,\mu) Z_{\mathrm{not}}(D\!\!\!\!/,\mu) \tag{3.50}$$

The terms in Z_{not} is due to the Weyl anomaly and we can calculate it. Consider the identity

$$Z(D\!\!\!\!/) = \left(\int [d\phi] \frac{1}{Z(D\!\!\!\!/_{\phi})}\right)^{-1} \int [d\phi] \frac{Z(D\!\!\!\!/)}{Z(D\!\!\!\!/_{\phi})} \tag{3.51}$$

Since the first term is invariant by construction, the second is the not invariant one:

$$Z_{\rm not}(\not{D}) = \int [d\phi] e^{-S_{\rm coll}} = \int [d\phi] \frac{Z(\not{D})}{Z(\not{D}_\phi)} \tag{3.52}$$

$$S_{\rm coll} = \log\left(\frac{Z(\not{D},\mu)}{Z(\not{D}_\phi,\mu)}\right) = W - W_\phi, \tag{3.53}$$

but the righthand side of Eq. (3.53) is already known, it is given by the the final result Eq. (3.43) of the previous section as a slightly modified bosonic spectral action coupled to the field ϕ.

Finally we obtain the following bosonisation relation:

$$\int [d\Psi][d\bar{\Psi}] \mathrm{e}^{-S_F[\bar{\Psi},\Psi,\text{bosonic background}]} = \int [d\phi] \mathrm{e}^{-S_{coll}[\phi,\text{bosonic background}]+W_{inv}}, \tag{3.54}$$

where W_{inv} is (nonlocal) Weyl invariant functional of background fields, and S_{coll} is a *local* functional of background fields and the dilation ϕ. For a flat space-time and coordinate independent fields S_{coll} was computed in the previous section. In equation Eq. (3.54) in the lefthand side bosonic background interects with quantized fermions, while in the right hand side the same bosonic background interects with a single scalar field. In this sense such a scalar field, can be considered a collective scalar degree of freedom, related with the breaking of Weyl invariance.

Let us clarify some aspects of the introduction of the collective degree of freedom of all fermions, or bosonization. In our context the term “bosonisation” does not mean that some composite operator $O_\phi(x)$, constructed from the scalar field ϕ and its derivatives, equals another composite operator $O_\Psi(x)$, constructed from the fermionic fields Ψ and $\bar{\Psi}$. More generally it means that the vacuum expectation of the product of n bosonic composite operators $O_\phi(x)$ equals the vacuum expectation of the product of n fermionic composite operators $O_\Psi(x)$ for $n = 1, 2, ...$, i.e. equality of corresponding classes of Green functions.

$$\langle O_\Psi(x_1), ..., O_\Psi(x_n)\rangle_{\textbf{ferm. vacuum}} = \langle O_\phi(x_1), ..., O_\phi(x_n)\rangle_{\textbf{bos. vacuum}}, \qquad n = 1, 2, ... \tag{3.55}$$

Now we will specify the mentioned classes of Green functions. Substitute $g_{\mu\nu} = \mathrm{e}^{2\alpha} g_{\mu\nu}$ and $H = \mathrm{e}^{-\alpha} H$ in Eq. (3.54) and consider α as a source. Since the invariant part W_{inv} in the right hand side of Eq. (3.54) remains unchanged under this substitution, it will not give contribution, one has:

$$\left(\frac{\delta^n}{\delta\alpha(x_1)...\delta\alpha(x_n)} \log Z_F^\alpha\right)\Bigg|_{\alpha_1,...\alpha_n=0} = \left(\frac{\delta^n}{\delta\alpha(x_1)...\delta\alpha(x_n)} \log Z_{coll}^\alpha\right)\Bigg|_{\alpha_1,...\alpha_n=0}, \tag{3.56}$$

where

$$Z_F^\alpha \equiv \int [d\Psi][d\bar\Psi]\mathrm{e}^{-S_F[\bar\Psi,\Psi,\mathrm{e}^{2\alpha}g_{\mu\nu},\mathrm{e}^{-\alpha}H]}, \quad Z_{coll}^\alpha \equiv \int [d\phi]\mathrm{e}^{-S_{coll}[\phi,\mathrm{e}^{2\alpha}g_{\mu\nu},\mathrm{e}^{-\alpha}H]} \quad (3.57)$$

In our case the composite fermionic operator O_Ψ, which we bosonize, and the corresponding bosonic operator O_ϕ poses are given correspondingly by:

$$O_\Psi(x) \;=\; \left(\frac{\delta}{\delta\alpha(x)}S_F[\bar\Psi,\Psi,\mathrm{e}^{2\alpha}g_{\mu\nu},\mathrm{e}^{-\alpha}H]\right)_{\alpha=0}, \quad (3.58)$$

$$O_\phi(x) \;=\; \left(\frac{\delta}{\delta\alpha(x)}S_{coll}[\phi,\mathrm{e}^{2\alpha}g_{\mu\nu},\mathrm{e}^{-\alpha}H]\right)_{\alpha=0}. \quad (3.59)$$

Notice that in the absence of the Higgs field, $H = 0$, these operators are (up to a $\sqrt{g}$ factor) nothing but traces of corresponding stress energy tensors $T^{\mu\nu}_{F,coll}(x) = \frac{2\delta}{\sqrt{g}\delta g_{\mu\nu}(x)}S_{F,coll}$. It is remarkable, that in this case the classical $T^{\mu}_{\mu\ F}$ vanishes on the equations of motion, however the quantum vacuum average

$$\langle T^{\mu}_{\mu\ F}(x)\rangle_{\mathbf{ferm.vac.}} \neq 0, \quad (3.60)$$

due to the trace anomaly. The collective action describes the trace anomaly already on classical level:

$$\langle T^{\mu}_{\mu\ F}(x)\rangle_{\mathbf{ferm.vac.}} = \langle T^{\mu}_{\mu\ coll}(x)\rangle_{\mathbf{bos.vac.}} \simeq T^{\mu}_{\mu\ coll}(x)\Big|_{\phi=\phi_{class}} + \text{loop corrections}, \quad (3.61)$$

where $\phi_{class}(x)$ solves the classical equations of motion $\frac{\delta S_{coll}[\phi]}{\delta\phi(x)} = 0$. In contrast to the fermionic partition function, the bosonic partition function doesn't possess the trace anomaly, and the Weyl non invariance of action appears already at classical level.

In the presence of the Higgs field, i.e. when the Dirac operator is given by Eq. (2.8), the operator $O_\Psi(x)$, given by Eq. (3.58) equals to

$$O_\Psi = \sqrt{g}\left(T^{\mu}_{\mu\ F} - \gamma_5\otimes S(H)\bar\Psi\Psi\right), \quad T^{\mu}_{\mu\ F} \equiv \frac{2g_{\mu\nu}}{\sqrt{g}}\frac{\delta}{\delta g_{\mu\nu}}S_F, \quad (3.62)$$

besides $\langle T^\mu_\mu\rangle$ now $\langle O_\Psi\rangle$ contains an additional fermionic condensate $\langle\bar\Psi(x)\Psi(x)\rangle$ contribution.

The computation of the collective action S_{coll} is strongly based on the use of the heat-kernel expansion, that being an asymptotic expansion, strictly makes sense in the low momenta approximation, while for the large momenta regime one should take into account *all* heat kernel coefficients i.e. perform a *summation* of the heat kernel expansion, see the fifth chapter. Nevertheless the bosonization that we discuss is also valid in low momenta region, this justifies the use of the first three nontrivial heat kernel coefficients in our treatment.

In the next chapter we generalize the spectral regularization for the bosonic degrees of freedom and apply it for the selfconsistent description of the induced gravity and the onset of inflation, driven by the trace anomaly.

Chapter 4

Spectral regularization: Induced gravity and the onset of inflation.

In this chapter we consider one more interesting application of the spectral regularization related with cosmology and modified gravity following ref. [24]. Generalizing the spectral regularization discussed in previous chapter on bosonic degrees of freedom for weakly coupled QFT's we will describe phenomena of induced gravity and cosmological inflation driven by trace anomaly on equal footing.

4.1 Motivation

It is commonly accepted that, at the early stages of our Universe, there was a phase of rapid acceleration, known as *inflation* [60–62], during which the length scales increased by approximately e^{75} times within a relatively short time of less than about $10^6 t_{\rm Pl}$ (with $t_{\rm Pl}$ denoting the Planck time). Such a scenario usually requires the presence of a (scalar) field, called the *inflaton*. Hence, generally speaking, one has to enlarge the field content of the theory. Although there are approaches based on exploiting of the Higgs scalar as an inflaton [63], this models face several difficulties [48, 64–66]. Another way out is to modify the gravitational action, without adding the inflaton, e.g. one can add an R^2-term to the Einstein - Hilbert action [67]. Nevertheless it is definitely interesting to minimize the amount of ingredients and try to manage just with QFT.

During the very early stages of our universe, matter can be described by a set of massless fields with negligible interactions. Such fields, studied in the context of QFT in curved space-time, may lead to an inflationary era. More precisely trace (conformal) anomaly, resulting from the renormalisation of the conformal part of the vacuum action, becomes the dominant quantum effect and can drive an inflationary era in the absence of an inflaton field. Such a proposal was first introduced by Starobinsky [60], then studied by Vilenkin [68], and more recently it has been further investigated by various authors (see for instance Ref. [69] and references therein, and Refs. [70, 71]). The proposal of Starobinsky can be regarded as a modified gravity scenario,

$$\text{Starobinski} : S_{\textbf{infl}}\left[g_{\mu\nu}\right] = \underbrace{\frac{M_{\textbf{Pl}}^2}{16\pi}\int d^4x\,\sqrt{-g}R}_{\text{Einstein-Hilbert action one puts by hand}} \underbrace{+W\left[g_{\mu\nu}\right]}_{\text{Quantum effective action describes Weyl anomaly}} , \tag{4.1}$$

having however QFT origin.

It is remarkable, that the Einstein-Hilbert action itself can be also seen as an induced quantum effect [72, 73], however one cannot — to our knowledge — find in the literature a consistent mathematical scheme allowing to describe simultaneously induced quantum gravity and anomaly-induced effective action. Standard computation of trace anomaly in curved space-time usually relies on ζ-functional regularization [74], that does not exploit the ultraviolet cutoff scale, thereby missing the effect of Sakharov's induced gravity. In contrast to that, the frequently used Fock-Schwinger proper time regularization [75, 76], that gives immediately the Einstein-Hilbert action as a quantum effect, is not suitable to describe the Weyl anomaly, since it leads to a local Weyl noninvariant expression, while anomaly generating functional is however known to be nonlocal [77]. As we will see, using the spectral regularization, one can describe the onset of inflation driven by the trace anomaly of the quantum effective action in the absence a "bare" Einstein-Hilbert action i.e. our formalism allows to study induced gravity and anomalous inflation in a self consistent way.

$$\text{Our approach} : S_{\textbf{infl}}\left[g_{\mu\nu}\right] \underbrace{= W_\Lambda}_{\text{describes Weyl anomaly + induced Einstein-Hilbert action}} \tag{4.2}$$

In other words, we show that a cosmological arrow of time can result from a purely quantum effect.

In what follows we derive a mathematical description of anomaly using spectral regularization for classically Weyl invariant scalar and gauge theories. Then we show how the induced gravitational Einstein-Hilbert action appears in our formalism. After we discuss the trace anomaly induced inflation. Our main conclusion is that, requiring stability of the cosmological constant under loop corrections, the condition of Sakharov's induced gravity becomes equivalent to the condition for the existence of a stable inflationary solution.

4.2 Spectral Regularisation and Collective Dilaton Lagrangian

In this section we will derive a mathematical description of anomaly generalizing the spectral regularization, introduced before, also on bosonic degrees of freedom. We will first compute the anomaly and then present the anomaly generating functional. The latter is achieved through the introduction of an auxilary field, that can be considered as a collective degree of freedom of vacuum fluctuations of all fields, dual to conformal anomaly.

4.2.1 Spectral Regularisation: generalization for bosons.

Our main aim is to compute the influence of vacuum fluctuations of quantised fields on the dynamics of the metric tensor in the context of QFT with an ultraviolet cutoff.

Since in asymptotically free QFT, the interactions — non-abelian interactions, Yukawa interactions and Higgs self-interactions — can be considered as perturbative, the effect we are interested in is, at leading order, given by one-loop vacuum energy of free fields. However, even this simple approximation may lead, in curved space-time, to non-trivial effects like Sakharov's induced gravity [72] and Starobinsky's anomaly-induced inflation [60].

Let us consider a theory of free quantised fields of various spins moving in a gravitational background. The classical action reads

$$S_{\mathbf{cl}} = \int \mathrm{d}^4x\,\sqrt{g}\left(\sum_{j=1}^{N_{\mathcal{H}}}\,\mathcal{H}_j D_{\mathcal{H}}\mathcal{H}_j + \sum_{j=1}^{N_{\mathrm{F}}}\,\bar{\psi}_j \not{D}\psi_j + \frac{1}{4}\sum_{j=1}^{N_{\mathrm{V}}}\,F_{\mu\nu j}F^{\mu\nu}{}_j\right)\,, \tag{4.3}$$

where

$$\not{D} = ie^{\mu}_{k}\gamma^{k}\left(\partial_{\mu} - \frac{i}{2}\omega^{mn}_{\mu}\sigma_{mn}\right) , \tag{4.4}$$

$$D_{\mathcal{H}} = -\nabla^{2} - \frac{R}{6} , \tag{4.5}$$

where $N_{\mathrm{F}}, N_{\mathrm{V}}, N_{\mathcal{H}}$ stand for the number of Dirac four component fermions, gauge vector bosons and real Higgs-like scalars, respectively. The classical action Eq. (4.3) is conformally invariant. This setup may be considered as a good description of the Standard Model (or its generalizations) when all masses are much smaller than the Planck mass and the scalar fields are conformally coupled to gravity.

In order to quantise the theory we follow Faddeev-Popov gauge fixing procedure. In Feynman-t'Hooft gauge (a type of an R_{ξ} gauge, as a generalization of the Lorentz gauge, with $\xi = 1$), the action reads

$$\begin{aligned} S_{\mathbf{cl,gf}} &= \int d^{4}x\sqrt{g}\Bigg[\sum_{j=1}^{N_{\mathcal{H}}} \mathcal{H}_{j}D_{\mathcal{H}}\mathcal{H}_{j} + \sum_{j=1}^{N_{\mathrm{F}}} \bar{\psi}_{j}\not{D}\psi_{j} \\ &+ \sum_{j=1}^{N_{\mathrm{V}}}\left(\frac{1}{2}A^{\mu}{}_{j}\left(D_{\mathbf{vec}}\right)^{\nu}_{\mu}A_{\nu j} + \bar{c}_{j}D_{\mathbf{gh}}c_{j}\right)\Bigg] , \end{aligned} \tag{4.6}$$

where

$$D_{\mathbf{gh}} \equiv -\nabla^{2} , \tag{4.7}$$

$$\left(D_{\mathbf{vec}}\right)^{\nu}{}_{\mu} \equiv -\delta^{\nu}_{\mu}\nabla^{2} - R^{\nu}_{\mu} . \tag{4.8}$$

The object we are interested in, is a quantum partition function that (up to irrelevant constant) is given by

$$\begin{aligned} Z &\equiv \int[\mathrm{d}\bar{\psi}][\mathrm{d}\psi][\mathrm{d}\mathcal{H}][\mathrm{d}A][\mathrm{d}\bar{c}][\mathrm{d}c]e^{-S_{\mathbf{cl}}[\bar{\psi},\psi,\mathcal{H},A,\bar{c},c,g_{\mu\nu}]} \\ &= Z_{F}^{N_{\mathrm{F}}}\cdot Z_{\mathcal{H}}^{N_{\mathcal{H}}}\cdot Z_{\mathbf{vec}}^{N_{\mathrm{V}}}\cdot Z_{\mathbf{gh}}^{N_{\mathrm{V}}} , \end{aligned} \tag{4.9}$$

and is formally equal to:

$$Z = \frac{\left(\det\left(\not{D}^{2}\right)\right)^{\frac{N_{\mathrm{F}}}{2}}}{\left(\det\left(D_{\mathcal{H}}\right)\right)^{\frac{N_{\mathcal{H}}}{2}}}\frac{\left(\det\left(D_{\mathbf{gh}}\right)\right)^{N_{\mathrm{V}}}}{\left(\det\left(D_{\mathbf{vec}}\right)\right)^{\frac{N_{\mathrm{V}}}{2}}} . \tag{4.10}$$

Note that in a theory with $N_{\mathrm{F}}^{\mathbf{w}}$ two-component Weyl fermions one should replace N_{F} by $N_{\mathrm{F}}^{\mathbf{w}}/2$ in q. (4.10). Each operator O, appearing as $\det(O)$ in Eq. (4.10), is of a Laplacian type and unbounded; hence each determinant is infinite, rendering the whole partition function ill-defined. Now we regularize each determinant in Eq (4.10) in the same way like we did in the previous chapter, namely we take into account eigenvalues of corresponding operators smaller than Λ^2

$$\det \mathcal{O} = \prod \lambda_n \xrightarrow[\text{spectral regularization}]{} \det\left(\frac{\mathcal{O}_\Lambda}{\mu^2}\right) = \prod_{\lambda_n \leq \Lambda^2} \frac{\lambda_n}{\mu^2} , \tag{4.11}$$

where

$$\mathcal{O}_\Lambda \equiv \mathcal{O} \cdot P_\Lambda , \tag{4.12}$$

with

$$P_\Lambda \equiv \Theta\left(\Lambda^2 - \mathcal{O}\right) , \tag{4.13}$$

the projector on the subspace of eigenfunctions of $\mathcal{O}$ with eigenvalues smaller than Λ. The parameter μ is again introduced in order to have a dimensionless expression under the sign of determinant and in what follows, we consider $\Lambda = \mu$; other choices of μ will not affect substantially the regularization scheme.[1]

Although the procedure of spectral regularization can be easily understood and has nice properties, like preserving gauge invariance and general covariance, technically it is not easy to handle (in contrast to the Fock-Schwinger proper time formalism). Nevertheless, one can address both, induced quantum gravity and anomaly-induced inflation, using spectral regularization. Indeed, they are both related with Weyl non-invariance of the effective quantum action (or Weyl anomaly), since the classical theory is Weyl invariant. In the following, we compute Weyl anomaly and present the anomaly generating functional.

4.2.2 Spectral regularization: computation of the Weyl anomaly

Let us consider a conformal transformation of the metric tensor (3.10) Since the classical action Eq. (4.3) is Weyl invariant, the Weyl non-invariant contribution comes out, by definition, from Weyl anomaly. Let us compute the difference between the initial and the Weyl transformed quantum effective action, namely

$$W - \widetilde{(W)}_\phi = \log\left(\frac{\widetilde{(Z)}_\phi}{Z}\right) . \tag{4.14}$$

[1]The case of an arbitrary choice of μ is discussed in the previous chapter for the fermionic determinant and can be easily generalized for scalar or vector fields.

For the fermionic effective action $W_{\rm F}$, this difference, Eq. (4.14), reads (see Eq. (3.28) at $\mu = \Lambda$)

$$W_{\rm F} - (\widetilde{W_{\rm F}})_\phi = -\int_0^1 {\rm d}t\, {\rm Tr}\left(\phi\left[\chi\left(\frac{\widetilde{\not{D}^2}}{\Lambda^2}\right)\right]_{\phi\cdot t}\right) , \tag{4.15}$$

where

$$\chi(z) \equiv \Theta(1-z) . \tag{4.16}$$

Repeating the same computation for the case of a scalar field, one can easily show that

$$W_{\mathcal{H}} - (\widetilde{W_{\mathcal{H}}})_\phi = \int_0^1 {\rm d}t\, {\rm Tr}\left(\phi\left[\chi\left(\frac{\widetilde{D_{\mathcal{H}}}}{\Lambda^2}\right)\right]_{\phi\cdot t}\right) . \tag{4.17}$$

Indeed, under conformal transformation the Laplacian $D_{\mathcal{H}}$ transforms as

$$D_{\mathcal{H}} \to (\widetilde{D_{\mathcal{H}}})_\phi \equiv e^{-3\phi} D_{\mathcal{H}} e^{\phi} , \tag{4.18}$$

and thus one obtains

$$\begin{aligned}
W_{\mathcal{H}} - (\widetilde{W_{\mathcal{H}}})_\phi &= \log\left(\frac{(\widetilde{Z_{\mathcal{H}}})_\phi}{Z_{\mathcal{H}}}\right) \\
&= \int_0^1 {\rm d}t\, \partial_t \log(\widetilde{Z_{\mathcal{H}}})_{\phi(x)\cdot t} \\
&= -\frac{1}{2}\int_0^1 {\rm d}t\, \partial_t\, {\rm Tr}\left\{\log\left(\frac{\widetilde{D_{\mathcal{H}}}}{\Lambda^2}\widetilde{P_\Lambda}\right)_{\phi(x)\cdot t}\right\} \\
&= -\frac{1}{2}\int_0^1 {\rm d}t\;\; {\rm Tr}\left\{\widetilde{D_{\mathcal{H}}}^{-1}\left(-3\phi\widetilde{D_{\mathcal{H}}} + \widetilde{D_{\mathcal{H}}\phi}\right)\widetilde{P_\Lambda} + \underbrace{\left(\partial_t\Theta\left[\Lambda^2 - \widetilde{D_{\mathcal{H}}}\right]\right)\cdot\log\frac{\widetilde{D_{\mathcal{H}}}}{\Lambda^2}}_{0}\right\}_{\phi\cdot t} \\
&= \int_0^1 {\rm d}t\, {\rm Tr}\left\{\phi\widetilde{P_\Lambda}\right\}_{\phi\cdot t} ,
\end{aligned} \tag{4.19}$$

with

$$P_\Lambda \equiv \Theta\left(\Lambda^2 - D_{\mathcal{H}}\right) . \tag{4.20}$$

Since the Laplacians $D_{\mathbf{vec}}$ and $D_{\mathbf{gh}}$ do not transform in a homogeneous way, like $D_{\mathcal{H}}$ (see, Eq. (4.18)), one cannot write a straightforward generalization of Eq. (4.19) for $D_{\mathbf{vec}}$ and $D_{\mathbf{gh}}$. Nevertheless, there is a non-trivial interplay between gauge and ghost modes and using the computation presented in the next subsection one can generalize Eqs. (4.15), (4.17). Hence, defining

$$W_{\mathbf{gauge}} \equiv W_{\mathbf{vec}} + W_{\mathbf{gh}} , \tag{4.21}$$

one obtains

$$
\begin{aligned}
& W_{\mathbf{gauge}} - \left(\widetilde{W_{\mathbf{gauge}}}\right)_{\phi} = \\
& \int_0^1 \mathrm{d}t \left\{ \mathrm{Tr}\left(\phi\left[\chi\left(\frac{\widetilde{D_{\mathbf{vec}}}}{\Lambda^2}\right)\right]_{\phi\cdot t}\right) - 2\,\mathrm{Tr}\left(\phi\left[\chi\left(\frac{\widetilde{D_{\mathbf{gh}}}}{\Lambda^2}\right)\right]_{\phi\cdot t}\right)\right\} .
\end{aligned}
\tag{4.22}
$$

In the next subsection we will carefully discuss the case of gauge bosons and derive the formula Eq. (4.22).

4.2.3 Gauge-Ghost's Contribution: the computation

Starting from the Maxwell action for gauge fields

$$
S_{\mathbf{M}} = \int \mathrm{d}^4x\,\sqrt{g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) , \tag{4.23}
$$

we perform a Faddeev-Popov quantisation procedure, adding the gauge fixing term

$$
S_{\mathbf{gf}} \equiv \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{g}\left(\nabla_\mu A^\mu\right) , \tag{4.24}
$$

and the ghost action $S_{\mathbf{gh}}$

$$
S_{\mathbf{gh}} = \int \mathrm{d}^4x\,\sqrt{g}\bar{c}D_{\mathbf{gh}}c , \tag{4.25}
$$

where

$$
D_{\mathbf{gh}} \equiv -\nabla^2 . \tag{4.26}
$$

The overall gauge fixed Maxwell-ghost action then reads

$$
\begin{aligned}
S_{\mathbf{vec-gh}} &\equiv S_{\mathbf{M}} + S_{\mathbf{gf}} + S_{\mathbf{gh}} \\
&= \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{g}\left(A^\mu(D_{\mathbf{vec}})_\mu^{\ \nu}A_\nu\right) + \int \mathrm{d}^4x\,\sqrt{g}\bar{c}D_{\mathbf{gh}}c ,
\end{aligned}
\tag{4.27}
$$

where

$$
\begin{aligned}
(D_{\mathbf{vec}})^{\nu}_{\ \mu} &\equiv -\delta^\nu_\mu\nabla^2 - \left[\nabla_\mu, \nabla^\nu\right] \\
&= -\delta^\nu_\mu\nabla^2 - R^\nu_\mu .
\end{aligned}
\tag{4.28}
$$

The partition function describing a contribution of the quantised vector fields and ghosts to the vacuum energy is given by the functional integral

$$
\begin{aligned}
Z_{\mathbf{vec-gh}} &= \int [\mathrm{d}A][\mathrm{d}\bar{c}][\mathrm{d}c]e^{-S_{\mathbf{vec-gh}}} \\
&= \frac{\det D_{\mathbf{gh}}}{\sqrt{\det D_{\mathbf{vec}}}} .
\end{aligned}
\tag{4.29}
$$

Since both operators $D_{\mathbf{vec}}$ and $D_{\mathbf{gh}}$ are unbounded, the last equality is formal and thus we perform a spectral regularization. Following the general prescription we introduce the cutoff scale Λ and the two projectors

$$\begin{aligned} P^{\Lambda}_{\mathbf{vec}} &= \Theta\left(\Lambda^2 - D^2_{\mathbf{vec}}\right) , \\ P^{\Lambda}_{\mathbf{gh}} &= \Theta\left(\Lambda^2 - D^2_{\mathbf{gh}}\right) , \end{aligned} \tag{4.30}$$

in order to truncate the spectrum of the $D_{\mathbf{vec}}$ and $D_{\mathbf{gh}}$ operators, respectively.

The regularization is based on replacing the unbounded operators $D_{\mathbf{vec}}$ and $D_{\mathbf{gh}}$ by the truncated operators $D^{\Lambda}_{\mathbf{vec}}$ and $D^{\Lambda}_{\mathbf{gh}}$, respectively, denoted by

$$\begin{aligned} D_{\mathbf{vec}} &\to D^{\Lambda}_{\mathbf{vec}} \equiv \left(\frac{D_{\mathbf{vec}}}{\Lambda}\right) P^{\Lambda}_{\mathbf{vec}} + 1 - P^{\Lambda}_{\mathbf{vec}} , \\ D_{\mathbf{gh}} &\to D^{\Lambda}_{\mathbf{gh}} \equiv \left(\frac{D_{\mathbf{gh}}}{\Lambda}\right) P^{\Lambda}_{\mathbf{gh}} + 1 - P^{\Lambda}_{\mathbf{gh}} , \end{aligned} \tag{4.31}$$

in the determinants appearing in the partition function Eq. (4.29). Hence, the regularized partition function reads

$$\begin{aligned} Z^{\Lambda}_{\mathbf{vec-gh}} &\equiv \frac{\det D^{\Lambda}_{\mathbf{gh}}}{\sqrt{\det D^{\Lambda}_{\mathbf{vec}}}} \\ &= -\exp\left\{ \mathrm{Tr}\left(\frac{1}{2} P^{\Lambda}_{\mathbf{vec}} \log D_{\mathbf{vec}}\right) - \mathrm{Tr}\left(P^{\Lambda}_{\mathbf{gh}} \log D_{\mathbf{gh}}\right)\right\} . \end{aligned} \tag{4.32}$$

We are interested in computing the contribution of the quantised vector fields and ghosts to the anomaly $S_{\mathbf{coll}}$. We will impose the above discussed regularization and use Eq. (4.32) for the regularized partition function. Note that we denote a quantity $Q[g_{\mu\nu}]$ computed on the transformed metric tensor $g_{\mu\nu}e^{2\phi}$ by

$$\left(\widetilde{Q}\right)_{\phi} \equiv Q\left[e^{2\phi} g_{\mu\nu}\right] . \tag{4.33}$$

For the anomaly we have

$$
\begin{aligned}
S_{\mathrm{coll}} &= \log\left(\frac{\left(\widetilde{Z^{\Lambda}_{\mathbf{vec-gh}}}\right)_{\phi}}{Z^{\Lambda}_{\mathbf{vec-gh}}}\right)\\
&= \int_0^1 \mathrm{d}t\, \partial_t \log\left(\widetilde{Z^{\Lambda}_{\mathbf{vec-gh}}}\right)_{\phi\cdot t}\\
&= -\int_0^1 \mathrm{d}t\, \partial_t \left\{ \mathrm{Tr}\left(\frac{1}{2}\widetilde{P^{\Lambda}_{\mathbf{vec}}}\log\widetilde{D_{\mathbf{vec}}}\right)_{\phi\cdot t} - \mathrm{Tr}\left(\widetilde{P^{\Lambda}_{\mathbf{gh}}}\log\widetilde{D_{\mathbf{gh}}}\right)_{\phi\cdot t}\right\}\\
&= -\int_0^1 \mathrm{d}t \left(\frac{1}{2}\,\mathrm{Tr}\left[\widetilde{P^{\Lambda}_{\mathbf{vec}}}\left(\widetilde{D^{\Lambda}_{\mathbf{vec}}}\right)^{-1}\partial_t\widetilde{D^{\Lambda}_{\mathbf{vec}}}\right]_{\phi\cdot t}\right.\\
&\qquad\qquad \left. -\,\mathrm{Tr}\left[\widetilde{P^{\Lambda}_{\mathbf{gh}}}\left(\widetilde{D^{\Lambda}_{\mathbf{gh}}}\right)^{-1}\partial_t\widetilde{D^{\Lambda}_{\mathbf{gh}}}\right]_{\phi\cdot t}\right) . \qquad (4.34)
\end{aligned}
$$

In contrast to the fermionic and scalar cases, the next step in the computation of the anomaly S_{coll} is not a trivial task, because both operators $D_{\mathbf{vec}}$ and $D_{\mathbf{gh}}$ do not transform in a covariant way, namely

$$
\begin{aligned}
\left[\widetilde{(D_{\mathbf{vec}})_{\mu}}^{\lambda}\right]_{\phi} &= e^{-2\phi}\left((D_{\mathbf{vec}})_{\mu}{}^{\lambda} + 2\phi_{\mu}\nabla^{\lambda} - 2\phi^{\lambda}\nabla_{\mu} - 2\phi_{\mu}{}^{\lambda} + 4\phi_{\mu}\phi^{\lambda}\right),\\
\left[\widetilde{D_{\mathbf{gh}}}\right]_{\phi} &= e^{-2\phi}\left(D_{\mathbf{gh}} - 2\phi^{\mu}\nabla_{\mu}\right) . \qquad (4.35)
\end{aligned}
$$

From the transformation law, Eq. (4.35) above, we derive

$$
\begin{aligned}
\partial_t\left[\widetilde{(D_{\mathbf{vec}})_{\mu}}^{\lambda}\right]_{\phi\cdot t} &= -2\phi\left[\widetilde{(D_{\mathbf{vec}})_{\mu}}^{\lambda}\right]_{\phi\cdot t} + 2\left[\widetilde{\phi_{\mu}\nabla^{\lambda}}\right]_{\phi\cdot t} - 2\left[\widetilde{\phi^{\lambda}\nabla_{\mu}}\right]_{\phi\cdot t} - 2\left[\widetilde{\phi_{\mu}{}^{\lambda}}\right]_{\phi\cdot t} ,\\
\partial_t\left[\widetilde{D_{\mathbf{gh}}}\right]_{\phi\cdot t} &= -2\phi\left[\widetilde{D_{\mathbf{gh}}}\right]_{\phi\cdot t} - 2\left[\widetilde{\phi^{\mu}\nabla_{\mu}}\right]_{\phi\cdot t} . \qquad (4.36)
\end{aligned}
$$

Substituting Eq. (4.36) in Eq. (4.34) we obtain the expression for the anomaly; it has a part similar to that of the fermonic and bosonic cases and in addition there are some “bad terms”, namely

$$
S_{\mathrm{coll}} = \int_0^1 \mathrm{d}t \left\{ \mathrm{Tr}_{\mathbf{vec}}\left(\phi\left[\widetilde{P^{\Lambda}_{\mathbf{vec}}}\right]_{\phi\cdot t}\right) - 2\,\mathrm{Tr}_{\mathbf{gh}}\left(\phi\left[\widetilde{P^{\Lambda}_{\mathbf{gh}}}\right]_{\phi\cdot t}\right) + (\widetilde{\text{“bad terms”}})_{\phi\cdot t}\right\} , \qquad (4.37)
$$

where the “bad terms” are given by

$$
\begin{aligned}
\text{“bad terms”} \equiv\ & 2\Big\{ \mathrm{Tr}_{\mathbf{vec}}\left[P^{\Lambda}_{\mathbf{vec}}\,(D_{\mathbf{vec}})^{-1}\left(\phi_{\mu}\nabla^{\lambda} - \phi^{\lambda}\nabla_{\mu} - \phi_{\mu}{}^{\lambda}\right)\right]\\
& + 2\,\mathrm{Tr}_{\mathbf{gh}}\left[P^{\Lambda}_{\mathbf{gh}}\left(D_{\mathbf{gh}}\right)^{-1}\left(\phi^{\mu}\nabla_{\mu}\right)\right]\Big\} . \qquad (4.38)
\end{aligned}
$$

In what follows we will show that the "bad terms" cancel.
Let us first introduce a complete set $\Phi_n,\ n = 1, 2...$ of orthonormal eigenfunctions of the ghost operator $D_{\mathbf{gh}}$, as

$$D_{\mathbf{gh}}\Phi_n = \lambda_n\Phi_n \quad \text{with} \int d^4x\,\sqrt{g}\Phi_n\Phi_m = \delta_{nm}\ . \tag{4.39}$$

One can easily check that the set of functions $\xi_n^\mu \equiv \frac{\nabla^\mu\Phi_n}{\sqrt{\lambda_n}}$ satisfies

$$(D_{\mathbf{vec}})^\mu{}_\nu\,\xi_n^\nu = \lambda_n\xi_n^\mu \quad \text{with} \int d^4x\,\sqrt{g}\xi_{n\mu}\xi_m^\mu = \delta_{nm}, \tag{4.40}$$

namely it forms an orthonormal basis in a space of longitudinal eigenvectors of the the operator $D_{\mathbf{vec}}$. Let us also introduce the orthonormal set of transversal eigenvectors of

$$(D_{\mathbf{vec}})^\mu{}_\nu\,B_n^\nu = \beta_n B_n^\mu \quad \text{with} \quad \int d^4x\,\sqrt{g}B_{n\mu}B_m^\mu = \delta_{nm} \quad \text{and} \quad \nabla_\mu B_n^\mu = 0\ , \tag{4.41}$$

so the set $\{\xi_n^\mu, B_m^\mu\}$ with $n, m = 1, 2, \cdots$ forms a basis in the space of all gauge potentials. The gauge contribution to the "bad terms" is

$$\begin{aligned}
& \mathrm{Tr}\left[P^\Lambda_{\mathbf{vec}}\,(D_{\mathbf{vec}})^{-1}\left(\phi_\mu\nabla^\lambda - \phi^\lambda\nabla_\mu - \phi_\mu{}^{;\lambda}\right)\right] \\
= & \sum_{n:\ \lambda_n\le\Lambda}\int d^4x\,\sqrt{g}\left(\xi_n^\mu\left(D^{-1}_{\mathbf{vec}}\right)_\mu^{\ \eta}\left(\phi_\eta\nabla^\lambda - \phi^\lambda\nabla_\eta - \phi_\eta{}^{;\lambda}\right)\xi_{n\lambda}\right) \\
& + \sum_{n:\ \beta_n\le\Lambda}\underbrace{\int d^4x\,\sqrt{g}\left(B_n^\mu\left(D^{-1}_{\mathbf{vec}}\right)_\mu^{\ \eta}\left(\phi_\eta\nabla^\lambda - \phi^\lambda\nabla_\eta - \phi_\eta{}^{;\lambda}\right)B_{n\lambda}\right)}_{0} \\
= & -2\sum_{n:\ \lambda_n\le\Lambda}\frac{1}{\lambda_n}\int d^4x\,\sqrt{g}\,(\Phi_n\phi^\nu\nabla_\nu\Phi_n)\ ,
\end{aligned} \tag{4.42}$$

and the ghost contribution to the "bad terms" reads

$$\begin{aligned}
& 2\,\mathrm{Tr}\left[P^\Lambda_{\mathbf{gh}}\left(D_{\mathbf{gh}}\right)^{-1}\left(\phi^\mu\nabla_\mu\right)\right] \\
= \quad & 2\sum_{n:\ \lambda_n\le\Lambda}\frac{1}{\lambda_n}\int d^4x\,\sqrt{g}\,(\Phi_n\phi^\nu\nabla_\nu\Phi_n) \\
= \quad & -\mathrm{Tr}\left[P^\Lambda_{\mathbf{vec}}\,(D_{\mathbf{vec}})^{-1}\left(\phi_\mu\nabla^\lambda - \phi^\lambda\nabla_\mu - \phi_\mu{}^{;\lambda}\right)\right]\ .
\end{aligned} \tag{4.43}$$

Clearly, the "bad terms" cancel mode by mode.

Hence, the final answer for the gauge-ghost contribution to the anomaly is

$$S_{\mathbf{coll}} = \int_0^1 \mathrm{d}t \left\{ \mathrm{Tr}\left(\phi\left[\chi\left(\frac{\widetilde{D_{\mathbf{vec}}}}{\Lambda^2}\right)\right]_{\phi \cdot t}\right) - 2\,\mathrm{Tr}\left(\phi\left[\chi\left(\frac{\widetilde{D_{\mathbf{gh}}}}{\Lambda^2}\right)\right]_{\phi \cdot t}\right)\right\} , \tag{4.44}$$

where the cutoff function χ is the Heaviside step-function

$$\chi(z) \equiv \Theta(1 - z) . \tag{4.45}$$

4.2.4 Weyl anomaly upon the spectral regularization: final result

To complete the computation of the scalar and gauge contributions to the anomaly, Eqs. (4.17) and (4.22) respectively, we will follow the same procedure as in Refs. [10, 12, 13, 17, 18].

Let us first perform a decomposition of the projector P_Λ:

$$\begin{aligned} P_\Lambda &= \Theta\left(\Lambda^2 - O\right) \\ &= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\mathrm{d}s}{s - i\epsilon} e^{is} e^{-\left(\frac{is}{\Lambda^2}\right)O} , \end{aligned} \tag{4.46}$$

and then do a heat kernel expansion[2] in terms of the heat kernel (Schwinger-De Witt) coefficients [44]:

$$\mathrm{Tr}\left(\phi\, e^{-zO}\right) \simeq \sum_{n=0}^{\infty} z^{\frac{1}{2}(n-4)} a_n(\phi, O) , \tag{4.47}$$

where

$$z = \frac{is}{\Lambda^2} , \tag{4.48}$$

and

$$a_n(\phi, O) = \int d^4x \, \sqrt{g}\, \phi\, a_n(O, x) . \tag{4.49}$$

The main advantage of the heat kernel method is that it provides the required information in terms of only a few geometric invariants. Since Eq. (4.49) relies on the asymptotic heat kernel expansion, it makes sense only when the background field invariants, appearing in the heat kernel coefficients $a_n(O, x)$, are smaller than the corresponding powers of the ultraviolet cutoff Λ. This requirement defines the

[2] More precisely a Schrödinger kernel expansion, since the argument z is purely imaginary.

applicability of our approach; we assume this requirement to be satisfied.[3] Since we are working on a manifold without boundary, only even heat kernel coefficients a_{2k} are non-zero.

Performing the integration over s in Eq. (4.49), namely

$$\int_{-\infty}^{+\infty} \mathrm{d}s \; s^{k-3} e^{is} = \begin{cases} \frac{1}{2\pi i}\,\mathrm{Res}_{s=0}\left(s^{k-2}e^{is}\right) & \text{for} \quad k = 0, 1, 2\;; \\ (2\pi)\, i^{k-3}\left(\partial^{(k-3)}\delta\right)(1) = 0 & \text{for} \quad k \geq 3\;, \end{cases} \tag{4.50}$$

we obtain

$$\mathrm{Tr}\left(\phi\,\Theta\left(\Lambda^2 - O\right)\right) = \int \mathrm{d}^4x\,\sqrt{g}\left(\frac{a_0(O,x)}{2}\Lambda^4 + a_2(O,x)\Lambda^2 + a_4(O,x)\right)\,. \tag{4.51}$$

Using the expansion Eq. (4.51) for the total anomaly we obtain

$$\begin{aligned} W - \widetilde{(W)}_\phi \;\; = \;\; & \int \mathrm{d}^4x\,\phi(x) \int_0^1 \mathrm{d}t\,\sqrt{\tilde{g}_{\phi t}}\left\{\frac{\Lambda^4}{2}\left(N_{\mathcal{H}} a_0^{\mathcal{H}} + N_V\left[a_0^{\mathbf{vec}} - 2a_0^{\mathbf{gh}}\right] - \frac{N_{\mathrm{F}}^{\mathbf{w}}}{2} a_0^{\mathrm{F}}\right)\right. \\ & + \Lambda^2\left(N_{\mathcal{H}}\widetilde{\left(a_2^{\mathcal{H}}\right)}_{\phi\cdot t} + N_V\left[\widetilde{\left(a_2^{\mathbf{vec}}\right)}_{\phi\cdot t} - 2\widetilde{\left(a_2^{\mathbf{gh}}\right)}_{\phi\cdot t}\right] - \frac{N_{\mathrm{F}}^{\mathbf{w}}}{2}\widetilde{\left(a_2^{\mathrm{F}}\right)}_{\phi\cdot t}\right) \\ & \left. + \left(N_{\mathcal{H}}\widetilde{\left(a_4^{\mathcal{H}}\right)}_{\phi\cdot t} + N_{\mathrm{V}}\left[\widetilde{\left(a_4^{\mathbf{vec}}\right)}_{\phi\cdot t} - 2\widetilde{\left(a_4^{\mathbf{gh}}\right)}_{\phi\cdot t}\right] - \frac{N_{\mathrm{F}}^{\mathbf{w}}}{2}\widetilde{\left(a_4^{\mathrm{F}}\right)}_{\phi\cdot t}\right)\right\}. \end{aligned} \tag{4.52}$$

We give in Tables 4.1 and 4.2 the values of the heat kernel coefficients a_0, a_2 and a_4, respectively, for free massless fields of different spin. In what follows we will

Table 4.1: Heat kernel coefficients a_0 and a_2 for free massless fields of various spin; we have calculated them using Ref. [44].

Spin	a_0	a_2
0, conformal coupling	$\frac{1}{16\pi^2}\cdot 1$	0
1/2, Dirac fermion	$\frac{1}{16\pi^2}\cdot 4$	$\frac{1}{16\pi^2}\left(\frac{R}{3}\right)$
1, without ghosts	$\frac{1}{16\pi^2}\cdot 4$	$\frac{1}{16\pi^2}\left(\frac{R}{3}\right)$
0, minimal coupling	$\frac{1}{16\pi^2}\cdot 1$	$-\frac{1}{16\pi^2}\left(\frac{R}{6}\right)$
1, gauge (i.e., with ghosts)	$\frac{1}{16\pi^2}\cdot 2$	$\frac{1}{16\pi^2}\left(\frac{2R}{3}\right)$

use the shorthand notations listed below:

$$\phi_\mu \equiv \partial_\mu\phi, \quad X \equiv \phi_\mu\phi^\mu, \quad Y \equiv \nabla^\mu\phi_\mu\,. \tag{4.53}$$

Table 4.2: Heat kernel coefficient a_4 for free massless fields of various spin [44].

$a_4 = \frac{1}{2880\pi^2}\left(a \cdot C^2 + b \cdot \mathbf{GB} + c \cdot R_{;\mu}^{\ \mu}\right)$			
Spin	a	b	c
0, conformal coupling	3/2	-1/2	-1
1/2, Dirac fermion	-9	11/2	6
1, gauge (i.e., with ghosts)	18	-31	18

Using Eq. (3.33), Eq. (3.37) and Eq. (3.42) the total anomaly Eq. (4.52) reads

$$\begin{aligned} W - \widetilde{(W)}_\phi \;&=\; \int d^4x \sqrt{g}\left\{\alpha_1\left(e^{4\phi}-1\right) + \alpha_2\left(\frac{1}{2}\left(e^{2\phi}-1\right)R - 3\,e^{2\phi}X\right) + \alpha_3\phi\, C^2 \right.\\ &\quad +\alpha_4\left(\phi\,\mathbf{GB} + 4G^{\mu\nu}\phi_\mu\phi_\nu - 4XY - 2X^2\right)\\ &\quad \left. +\,\alpha_5\left((X+Y)R + 3(X+Y)^2\right)\right\}\,, \end{aligned} \tag{4.54}$$

where

$$\begin{aligned} \alpha_1 &\equiv \frac{\Lambda^4}{128\pi^2}\left(N_{\mathcal{H}} + 2N_{\rm V} - 2N_{\rm F}^{\rm w}\right),\\ \alpha_2 &\equiv \frac{\Lambda^2}{16\pi^2}\left(-\frac{1}{6}N_{\rm F}^{\rm w} + \frac{2}{3}N_{\rm V}\right),\\ \alpha_3 &\equiv \frac{1}{2880\pi^2}\left(\frac{3}{2}N_{\mathcal{H}} + \frac{9}{2}N_{\rm F}^{\rm w} + 18N_{\rm V}\right),\\ \alpha_4 &\equiv -\frac{1}{2880\pi^2}\left(\frac{1}{2}N_{\mathcal{H}} + \frac{11}{4}N_{\rm F}^{\rm w} + 31N_{\rm V}\right),\\ \alpha_5 &\equiv \frac{1}{2880\pi^2}\left(-N_{\mathcal{H}} - 3N_{\rm F}^{\rm w} + 18N_{\rm V}\right)\,. \end{aligned} \tag{4.55}$$

At this point, one can make a remark:

Remark

We would like to compare results for the trace anomaly obtained via the spectral and the ζ-function regularizations. An infinitesimal anomaly reads

$$\lim_{\phi\to 0}\frac{1}{\sqrt{g}}\frac{\delta}{\delta\phi(x)}\widetilde{(W)}_\phi = -\left(\frac{\Lambda^4}{2}\cdot A_0(x) + \Lambda^2\cdot A_2(x) + \Lambda^0\cdot A_4(x)\right)\,, \tag{4.56}$$

[3]In the case of anomaly-induced inflation, one must check that the scalar curvature R is small enough with respect to the ultraviolet cutoff scale Λ^2; this is indeed the case.

where

$$
\begin{aligned}
A_0 &\equiv N_{\mathcal{H}}\, a_0^{\mathcal{H}} + N_V \left[a_0^{\mathbf{vec}} - 2a_0^{\mathbf{gh}}\right] - \frac{N_{\mathrm{F}}^{\mathbf{w}}}{2}\, a_0^{\mathrm{F}}\,, \\
A_2 &\equiv N_{\mathcal{H}}\, a_2^{\mathcal{H}} + N_V \left[a_2^{\mathbf{vec}} - 2a_2^{\mathbf{gh}}\right] - \frac{N_{\mathrm{F}}^{\mathbf{w}}}{2}\, a_2^{\mathrm{F}}\,, \\
A_4 &\equiv N_{\mathcal{H}} a_4^{\mathcal{H}} + N_{\mathrm{V}} \left[(a_4^{\mathbf{vec}}) - 2a_4^{\mathbf{gh}}\right] - \frac{N_{\mathrm{F}}^{\mathbf{w}}}{2} a_4^{\mathrm{F}}\,,
\end{aligned}
\tag{4.57}
$$

and the heat kernel coefficients a_0, a_2, a_4 are given in Tables 4.1 and 4.2. The A_4-contribution coincides with the result for anomaly obtained via ζ-function regularization and the dimensional one [2]. Quadric and quadratic in Λ terms can be interpreted as an ultraviolet divergence and hence subtracted through the addition of the corresponding local counter terms. Indeed, one can define the renormalised effective action

$$
W^{\mathrm{ren}} \equiv W + \int \mathrm{d}^4x\, \sqrt{g}\left(\alpha_1 + \alpha_2\left(\frac{R}{2}\right)\right), \tag{4.58}
$$

with α_1, α_2 defined in Eq. (4.55). One can easily check (see computations in subsection 4.3.2) that

$$
\lim_{\phi\to 0} \frac{1}{\sqrt{g}}\, \frac{\delta}{\delta\phi(x)}\, (\widetilde{W^{\mathrm{ren}}})_\phi = -A_4(x), \tag{4.59}
$$

with A_4 defined in Eq. (4.56). However in this way, spectral regularization does not lead to any new result.

In what follows, we will not subtract the divergent terms and we will keep Λ finite and of order of the Planck scale. We will thus be able to describe simultaneously both, the induced gravitational action and the onset of (trace) anomaly-induced inflation. We will hence conclude that all terms in the Lagrangian, leading to a period of an accelerated expansion of the universe, may be considered as the outcome of a quantum effect.

4.2.5 Anomaly generating functional and collective dilaton

Although – in contrast to proper time regularization – spectral regularization does not allow one to compute the partition function explicitly, there is a formalism based on the introduction of *a collective dilaton* (see the previous chapter) that allows one to express the Weyl non-invariant part of such a regularized determinant, as an integral over an auxilary field ϕ of some local action that depends on ϕ and the background fields.

Repeating the steps of the previous chapter i.e. substituting the conformally transformed metric tensor $g_{\mu\nu}e^{2\phi}$ in Eq. (4.10) and integrating over all possible $\phi(x)$, one can write the identity

$$Z = \left(\int [\mathrm{d}\phi] \widetilde{(Z^{-1})}_{\phi} \right)^{-1} \cdot \int [\mathrm{d}\phi] \left(\frac{Z}{\widetilde{(Z)}_{\phi}} \right) . \tag{4.60}$$

Since the first term above is the integral over the Weyl group of a Weyl transformed quantity, it is Weyl invariant under the action of the Weyl group, so we denote it by $Z_{\mathbf{inv}}$. Hence, Eq. (4.60) can be rewritten as

$$Z \equiv Z_{\mathbf{inv}} \cdot \int [\mathrm{d}\phi] e^{-S_{\mathbf{coll}}} , \tag{4.61}$$

where

$$S_{\mathbf{coll}} \equiv \log \left(\frac{\widetilde{(Z)}_{\phi}}{Z} \right) . \tag{4.62}$$

Thus, the non-Weyl invariant partition function Z in Eq. (4.61) is written as the product of a term Z_{inv} invariant under Weyl transformations and another one, non-invariant, which depends on the auxiliary field ϕ and is due to Weyl anomaly. The introduction of the auxiliary field, representing the collective degree of freedom of all fermions, can be seen as *bononisation*. As we will later show, there exists a *local* Lagrangian $L_{\mathbf{coll}}$ depending on ϕ and background fields, such that $S_{\mathbf{coll}} = \int d^4x \sqrt{g} L$. Hence, instead of computing Z, we can use a bosonisation-like relation

$$\begin{aligned} Z[g_{\mu\nu}] &= \int [\mathrm{d}\bar{\psi}][\mathrm{d}\psi][\mathrm{d}\mathcal{H}][\mathrm{d}A][\mathrm{d}\bar{c}][\mathrm{d}c] e^{-S_{\mathbf{cl}}[\bar{\psi},\psi,\mathcal{H},A,\bar{c},c,g_{\mu\nu}]} \\ &\simeq Z_{\mathbf{inv}} \cdot \int [D\phi] e^{-S_{\mathbf{coll}}[\phi, g_{\mu\nu}]} . \end{aligned} \tag{4.63}$$

Clearly, ϕ stands for a collective degree of freedom of vacuum fluctuations of all fields dual to conformal anomaly, hence the term "collective action".

Since all our computations were carried in Euclidean QFT, in order to apply our result in a physical context one should perform a Wick rotation back to Minkowski signature in Eq. (4.54). Hence, for the anomaly generating functional we have

$$Z_{\mathbf{coll}} \equiv \int [d\phi] e^{-S_{\mathbf{coll}}} \xrightarrow[\text{Wick rotation back}]{} Z_{\mathbf{coll\ M}} \equiv \int [d\phi] e^{iS_{\mathbf{coll\ M}}} , \tag{4.64}$$

and the Minkowskian version of the collective action reads

$$
\begin{aligned}
S_{\mathbf{coll\ M}} &= \int \mathrm{d}^4x\,\sqrt{-g}\Big(-\alpha_1\left(e^{4\phi}-1\right)+\alpha_2\left(\frac{1}{2}\left(\mathrm{e}^{2\phi}-1\right)R-3\,e^{2\phi}X\right)-\alpha_3\phi\,C^2 \\
&\qquad -\alpha_4\left(\phi\,\mathbf{GB}+4G^{\mu\nu}\phi_\mu\phi_\nu-4XY-2X^2\right) \\
&\qquad -\alpha_5\left((X+Y)R+3(X+Y)^2\right)\Big)\,, \qquad (4.65)
\end{aligned}
$$

with the coefficients given in Eq. (4.55).

In what follows, we will show that Weyl anomaly in QFT with spectral regularization reproduces Sakharov's induced gravity, as well as Starobinsky's anomaly-induced inflation. This is the main message of our study.

4.3 Sakharov's Induced Gravity and Spectral Regularisation

4.3.1 Standard Approach: Proper Time Regularisation

The standard approach to the Sakharov's induced gravity is based on Fock-Schwinger proper time regularization [73]. In this formalism, one first selects a convenient reference metric $\tilde{g}_{\mu\nu}$ and then computes the difference in the one-loop contribution to the effective action which results from comparing two different metrics defined on the same topological manifold. Hence, we consider the difference $W[g_{\mu\nu}] - W[\tilde{g_{\mu\nu}}]$, with W defined as $W \equiv -\log Z$.

Let us write the formal equality

$$
\begin{aligned}
\mathrm{Tr}\left(\log\frac{D}{\tilde{D}}\right) &= \sum_{n=0}^{\infty}\log\frac{\lambda_n}{\tilde{\lambda}_n} \\
&\qquad\qquad (4.66) \\
&= -\sum_{n=0}^{\infty}\int_0^\infty \mathrm{d}s\left(\frac{e^{-s\lambda_n}}{s}-\frac{e^{-s\tilde{\lambda}_n}}{s}\right) \\
&\qquad\qquad (4.67) \\
&= -\int_0^\infty\frac{\mathrm{d}s}{s}\,\mathrm{Tr}\left(e^{-sD}-e^{s\tilde{D}}\right)\,, \qquad (4.68)
\end{aligned}
$$

and then perform a heat kernel expansion for the $\mathrm{Tr}\,(e^{-sD})$ and $\mathrm{Tr}\,(e^{s\tilde{D}})$ terms to get

$$
\mathrm{Tr}\left(\log\frac{D}{\tilde{D}}\right) = -\int_0^\infty\frac{\mathrm{d}s}{s}\sum_{k=0}^{\infty}s^{k-2}\left(a_{2k}(D)-a_{2k}(\tilde{D})\right)\,, \qquad (4.69)
$$

where the coefficients a_k are the Seeley-De Witt coefficients, universal functions of the space-time geometry. In order to perform the integration over s in Eq. (4.69) for $k = 0, 1$ one needs an ultraviolet regulator $\mu_{\mathbf{uv}}$; integration over s for all other values of k, namely for all $k > 1$, is ultraviolet finite but it requires the infrared regulator $\mu_{\mathbf{ir}} \ll \mu_{\mathbf{uv}}$. It is worth noting that the heat kernel expansion has allowed us to identify the potential divergences.
We obtain

$$\begin{aligned}
\mathrm{Tr}\left(\log\frac{D}{\tilde{D}}\right) &= -\int_{\mu_{\mathbf{uv}}^{-2}}^{\mu_{\mathbf{ir}}^{-2}}\frac{\mathrm{d}s}{s}\sum_{k=0}^{\infty}s^{k-2}\left(a_{2k}(D)-a_{2k}(\tilde{D})\right)\\
&= -\frac{\mu_{\mathbf{uv}}^{4}}{2}\left(a_0\left(D\right)-a_0\left(\tilde{D}\right)\right)-\mu_{\mathbf{uv}}^{2}\left(a_2\left(D\right)-a_2\left(\tilde{D}\right)\right)\\
&\quad -\log\left(\frac{\mu_{\mathbf{uv}}^{2}}{\mu_{\mathbf{ir}}^{2}}\right)\left(a_4\left(D\right)-a_4\left(\tilde{D}\right)\right)+\cdots
\end{aligned} \tag{4.70}$$

Let us emphasise that the regulators $\mu_{\mathbf{uv}}$ and $\mu_{\mathbf{ir}}$ are not ultaviolet and infrared, respectively, cutoff scales for the spectrum of D; they are attributes to make the regularization scheme finite.[4]
Using Eq. (4.10) with log det = Tr log and Eq. (4.70) we get

$$W^{\mathbf{pt}} \equiv -\log Z = \int d^4x\,\sqrt{g}\left(\lambda_{\mathbf{ind}}^{\mathbf{pt}}+\frac{M_{\mathrm{Pl}}^{2\ \mathbf{ind}}}{16\pi}R+\left\{O\left(R^2\right)\right\}\right), \tag{4.71}$$

where

$$\lambda_{\mathbf{ind}}^{\mathbf{pt}} = \frac{\mu_{\mathbf{uv}}^{4}}{64\pi^2}\left(2N_{\mathrm{F}}^{\mathbf{w}}-N_{\mathcal{H}}-2N_{\mathrm{V}}\right), \tag{4.72}$$

and

$$M_{\mathrm{Pl}}^{2\ \mathbf{ind}} = \frac{\mu_{\mathbf{uv}}^{2}}{2\pi}\left(\frac{N_{\mathrm{F}}^{\mathbf{w}}}{6}-\frac{2N_{\mathrm{V}}}{3}\right). \tag{4.73}$$

The main idea of Sakharov's induced gravity lies in attributing a physical meaning to the ultraviolet cutoff scale, so that it denotes the upper scale for which the considered QFT is a valid effective theory. In this way, it is not necessary to subtract divergences, and setting $\Lambda \sim M_{\mathrm{Pl}} \sim 10^{19}$GeV, the term $\mu_{\mathbf{uv}}^{2}R$ can be considered as an induced gravitational action.

Hence, starting from a classically Weyl invariant theory, quantisation implied a Weyl non-invariant Einstein-Hilbert action. One may thus conclude that, under proper time regularization, the Weyl anomaly contains operators of dimension

[4] Considering $D = -\partial^2 + m^2$ or $D = -\partial^2$ and a finite volume of Euclidean spacetime, the spectrum D has an infrared cutoff, however the integration over s in Eq. (4.69) is still infrared divergent.

two, in contrast to the (standard) dimensional regularization or the ζ-function regularization, where anomaly contains just operators of dimension four.

Nevertheless, the considered *proper time* regularization procedure does not reproduce correctly the a_4-contribution to the anomaly (c.f. Eq. (4.59)), and therefore it cannot be used to investigate the trace anomaly induced inflation. Indeed, substituting in $W^{\mathbf{pt}}$, defined in (4.71) above, the conformally transformed metric tensor $e^{2\phi} g_{\mu\nu}$ and then taking the derivative over $\phi(x)$, one immediately finds

$$\lim_{\phi\to 0} \frac{1}{\sqrt{g}} \frac{\delta}{\delta\phi(x)} \widetilde{(W^{\mathbf{pt}})}_\phi = 4\, \lambda^{\mathbf{pt}}_{\mathbf{ind}} + \frac{1}{8\pi} M_{\mathrm{Pl}}^{2\ \mathbf{ind}} R\,, \tag{4.74}$$

taking into account that the $\left\{O\left(R^2\right)\right\}$-terms in Eq. (4.71) are given by

$$\begin{aligned} \log\left(\frac{\mu^2_{\mathbf{uv}}}{\mu^2_{\mathbf{ir}}}\right) a_4\,(D) &= \log\left(\frac{\mu^2_{\mathbf{uv}}}{\mu^2_{\mathbf{ir}}}\right) \frac{1}{2880\pi^2} \left[\frac{3}{2} N_{\mathcal{H}} + \frac{9}{2} N_{\mathrm{F}}^{\mathbf{w}} + 18 N_{\mathrm{V}}\right] \int d^4x\, \sqrt{g} C^2 \\ &= \textbf{Weyl inv.}\,, \end{aligned} \tag{4.75}$$

and thus do not contribute in Eq. (4.74).

Let us remind to the reader that as we have previously shown (see the remark, Eq. (4.56)), the spectral regularization reproduces correctly the a_4-contribution to the anomaly. We will next show that it also reproduces correctly the induced Einstein-Hilbert action; it can be thus used to describe both.

4.3.2 The Spectral Regularisation Approach

The effective action

$$W_{\mathbf{eff}}[g_{\mu\nu}] = -\log Z[g_{\mu\nu}]\,, \tag{4.76}$$

is known to be a non-local functional of the metric tensor $g_{\mu\nu}$ and in particular, of the Lagrangian density $L_{\mathbf{eff}}[g_{\mu\nu}]$, so that

$$W_{\mathbf{eff}}[g_{\mu\nu}] = \int d^4x\, \sqrt{g} L_{\mathbf{eff}}[g_{\mu\nu}] \tag{4.77}$$

does not exist and correspondingly the local collective action $S_{\mathbf{coll}}$, once integrated over ϕ, captures all non-locality of the Weyl non-invariant part of the effective action.

Nevertheless, the terms with coefficients α_1, α_2 and α_5 in the anomaly, Eq. (4.54), can be generated by local terms in the effective action $W_{\mathbf{eff}}$.
Indeed, let us consider

$$W_{\mathbf{loc}}[g_{\mu\nu}] = \int \mathrm{d}^4x\, \sqrt{g} \left(-\alpha_1 - \alpha_2 \left(\frac{R}{2}\right) - \alpha_5 \left(\frac{R^2}{12}\right)\right)\,, \tag{4.78}$$

and

$$W_{\mathbf{loc}}[g_{\mu\nu}] - W_{\mathbf{loc}}[g_{\mu\nu}e^{2\phi}] \;=\; \int \mathrm{d}^4x\,\sqrt{g}\Bigg(\alpha_1\left(e^{4\phi}-1\right) + \alpha_2\left(\frac{1}{2}\left(\mathrm{e}^{2\phi}-1\right)R - 3\,e^{2\phi}X\right) + \alpha_5\left((X+Y)R + 3(X+Y)^2\right)\Bigg)\ . \tag{4.79}$$

Comparing Eqs. (4.54) and (4.79), we conclude that the total effective action can be rewritten as

$$W = W_{\mathbf{inv}} + W_{\mathbf{loc}} + W_{\mathbf{nonloc}}\ , \tag{4.80}$$

where $W_{\mathbf{inv}}$ is some Weyl invariant functional of $g_{\mu\nu}$ and $W_{\mathbf{nonloc}}$ is a non-local functional, generating α_3 and α_4 terms in the collective action Eq. (4.54) (we refer the reader to Ref. [77]). Equivalently, one can say that QFT with spectral regularization leads to Sakharov's induced gravity:

$$\begin{aligned} W_{\mathbf{ind}}\left[g_{\mu\nu}\right] &= \int \mathrm{d}^4x\,\sqrt{g}\Bigg(\frac{\Lambda^4}{128\pi^2}\left(2N_{\mathrm{F}}^{\mathbf{w}} - N_{\mathcal{H}} - 2N_{\mathrm{V}}\right) \\ &+ \frac{\Lambda^2}{32\pi^2}\left(\frac{1}{6}N_{\mathrm{F}}^{\mathbf{w}} - \frac{2}{3}N_{\mathrm{V}}\right)R + O\left(\left\{R^2\right\}\right)\Bigg) \\ &= \int \mathrm{d}^4x\,\sqrt{g}\left(\lambda^{\mathbf{ind}} + \frac{1}{16\pi}\left(M_{\mathrm{Pl}}^{\mathbf{ind}}\right)^2 R\right) + O\left(\left\{R^2\right\}\right)\ , \end{aligned} \tag{4.81}$$

where

$$\left(M_{\mathrm{Pl}}^{\mathbf{ind}}\right)^2 = \frac{\Lambda^2}{12\pi}\left(N_{\mathrm{F}}^{\mathbf{w}} - 4N_{\mathrm{V}}\right)\ , \tag{4.82}$$

and

$$\lambda^{\mathbf{ind}} = \frac{\Lambda^4}{128\pi^2}\left(2N_{\mathrm{F}}^{\mathbf{w}} - N_{\mathcal{H}} - 2N_{\mathrm{V}}\right)\ , \tag{4.83}$$

with $O\left(\left\{R^2\right\}\right)$ denoting all local and non-local terms responsible for Λ^0-contributions in the anomaly-induced effective action. The latter is much smaller in the low energy regime ($R << \Lambda$) and hence it can be neglected at energies much smaller than the cutoff scale. It however plays a significant role during the inflationary era; it will be studied in the next section within the isotropic approximation.

In order to identify the induced Planck mass with the real one at $\sim 10^{19}$GeV one should impose the cutoff scale Λ at the Planck energy scale. This however automatically leads to a huge value of the induced cosmological constant, namely $\lambda^{\mathbf{ind}} \sim M_{\mathrm{Pl}}^4$. One may consider the presence of bare cosmological constant with the opposite sign, namely $\lambda^{\mathbf{bare}} \sim -\,M_{\mathrm{Pl}}^4$ and impose the fine-turning:

$$\lambda^{\mathbf{observable}} = \lambda^{\mathbf{bare}} + \lambda^{\mathbf{ind}}\ .$$

To avoid such a fine-tuning, we will adopt an alternative approach and hence, we will impose the Pauli compensation principle, i.e we require, that the Λ^4 fermonic and bosonic contributions to the vacuum energy cancel each other. The latter implies that the numbers of physical fermonic and bosonic degrees of freedom are equal, namely

$$N_{\mathcal{H}} = 2\left(N_{\mathrm{F}}^{\mathbf{w}} - N_{\mathrm{V}}\right) , \tag{4.84}$$

on the number of scalars, spinors and vectors, so that all quartic divergences cancel.

Thus, under spectral regularization we obtain:

$$W_{\mathbf{ind}}\left[g_{\mu\nu}\right] = \int \mathrm{d}^4x\,\sqrt{g}\left(\frac{\Lambda^2}{32\pi^2}\left(\frac{1}{6}N_{\mathrm{F}}^{\mathbf{w}} - \frac{2}{3}N_{\mathrm{V}}\right)R + O\left(\left\{R^2\right\}\right)\right) . \tag{4.85}$$

The above equation, Eq. (4.85), agrees with the one obtained following the Fock-Schwinger proper time formalism [73]. It is worth noting that the Pauli compensation condition $N_{\mathcal{H}} = 2\left(N_{\mathrm{F}}^{\mathbf{w}} - N_{\mathrm{V}}\right)$ is not just a property of the spectral regularization; it holds in all regularization procedures with an ultraviolet cutoff scale and in that sense it is universal.

In the next section, we will consider a high energy region, but $R < \Lambda^2$, i.e. where the spectral regularization is still applicable. We will show that, imposing the Pauli compensation condition, the Λ^0-contribution to the anomaly (which we have neglected here), together with the Λ^2-contribution, leads automatically to Starobinsky's anomaly-induced inflation.

4.4 Inflation Induced from Trace Anomaly: The Isotropic Approximation

We will explore the dynamics of a metric tensor in the isotropic approximation. The spacetime is considered to be spatially flat, namely $g_{\mu\nu} = e^{\beta(\tau)}\eta_{\mu\nu}$; the cases of closed and open universes can be studied along similar lines. Although one should first derive an equation of motion for the metric tensor $g_{\mu\nu}$ and only afterwards substitute the conformally flat anzatz, it is possible to avoid the first step following the procedure described in Ref. [78].

Hence, to get equations of motion in the isotropic case for an arbitrary[5] general

[5] Although the procedure discussed is Ref. [78] deals with a local action W, repeating the same analysis one obtains the same result also in a nonlocal situation, provided after the substitution of the conformally flat anzatz, the action can be written as the right-hand-side of Eq. (4.86). As we will see, our model belongs to this case.

covariant action $W[g_{\mu\nu}]$ one should [78]:

- Firstly, substitute the conformally flat anzatz $\mathrm{d}s^2 = \mathrm{d}t^2 - a(t)^2 \mathrm{d}\vec{x}^2$ in the action $W[g_{\mu\nu}]$.
- Secondly, rewrite the result of the substitution in the form

$$W[a(t)] = \mathrm{vol} \cdot \int \mathrm{d}t\, a^3\, \Im(H, \dot{H}), \tag{4.86}$$

  where vol is a three-dimensional volume and H stands for the Hubble parameter, $H \equiv \dot{a}/a$. In principle, the function $\Im$ can also depend on higher derivatives of the Hubble parameter, but we will restrict ourselves to the minimal needed case.
- Thirdly, obtain the following equation for the scale factor:

$$\Im - H\frac{\partial \Im}{\partial H} + \left(-\dot{H} + 3H^2\right)\frac{\partial \Im}{\partial \dot{H}} + H\frac{d}{dt}\frac{\partial \Im}{\partial \dot{H}} = 0\,, \tag{4.87}$$

  which is just the generalization of the Friedmann equation.

Remark

Equation (4.87) is third order in a, while the equation of motion $\delta W/\delta a = 0$ is of fourth order. One can easily check that the above prescription is just a formulation of energy conservation. Indeed, since $W[a(t)]$ in Eq. (4.86) does not depend explicitly on time, Nother's theorem implies conservation of the quantity:

$$E \equiv \frac{\partial L}{\partial a_t}a_t - L + \frac{\partial L}{\partial a_{tt}}a_{tt} - \left(\frac{d}{dt}\frac{\partial L}{\partial a_{tt}}\right)\,. \tag{4.88}$$

If in addition, one imposes that the overall (gravity+fields) energy E vanishes, then the resulting equation will be exactly Eq. (4.87).

In our case the action is given by[6]

$$W_{\text{total}}\left[g_{\mu\nu}\right] = W + W_\lambda, \quad W_\lambda \equiv \int \mathrm{d}^4x\,\sqrt{-g}\,(-\lambda) \tag{4.89}$$

where[7] W is a quantum effective action $\frac{1}{i}\log Z$ with Z defined by Eq. (4.10) and spectral regularization.

[6]The cosmological constant λ is known to be much smaller than all other constants of dimension four, in particular M_{Pl}^4. We do not expect that λ is generated dynamically through quantum anomalies and we can make no comment on its origin. In the following, we are interested to check that the cosmological constant will not destabilise the inflationary solution.

[7]In what follows we skip the index **M** for brevity, keeping in mind, that we are working in a Minkowski space-time.

Following the prescription described above, we must substitute the conformally flat anzatz for the metric tensor in comoving coordinates in Eq. (4.54). This is done in two steps: firstly, we substitute the conformally flat anzatz in the conformal coordinates $g_{\mu\nu} = e^{2\beta(\tau)}\eta_{\mu\nu}$ and secondly, we perform the corresponding change of variables to the comoving frame. Hence, substituting $g_{\mu\nu} = \eta_{\mu\nu}$ and $\phi = \beta(\tau)$ in Eq. (4.54), we get[8]

$$W_{\text{total}}\left[\beta(\tau)\right] = \text{vol} \cdot \int \mathrm{d}\tau \left(3\alpha_2 e^{2\beta}\beta_\tau^2 + (3\alpha_5 - 2\alpha_4)\beta_\tau^4 + 3\alpha_5\beta_{\tau\tau}^2 - \lambda e^{4\beta}\right). \quad (4.90)$$

Performing the change of variables $\beta(\tau) \to a(t)$ with

$$\tau = \int_{t_0}^{t} a^{-1}(z)\, dz, \quad \beta(\tau) = \log a(t), \quad (4.91)$$

we arrive to the following expression for the effective action:

$$W_{\text{total}}[(a(t))] = \text{vol} \cdot \int dt\, a^3\, \Im\left(H, \dot{H}\right), \quad (4.92)$$

where

$$\Im(H, \dot{H}) \equiv 3\alpha_2 H^2 + (6\alpha_5 - 2\alpha_4) H^4 + 3\alpha_5 \dot{H}^2 + 6\alpha_5 H^2 \dot{H} - \lambda\,. \quad (4.93)$$

Substituting the above expression for $\Im$, Eq. (4.93), in Eq. (4.87), we arrive to the following equation of motion in terms of the Hubble parameter H:

$$\ddot{H} + 3\,H\dot{H} - \frac{1}{2}\frac{\dot{H}^2}{H} + \frac{3}{4}\frac{H^3}{Q} - 3\,H\Lambda^2 + \frac{\lambda\,P}{H} = 0\,, \quad (4.94)$$

where

$$\begin{aligned} Q &\equiv \frac{N_{\text{F}} - 4N_{\text{V}}}{N_{\text{F}} + 8N_{\text{V}}}\,, \\ P &\equiv \frac{96\pi^2}{N_{\text{F}} - 4N_{\text{V}}}\,. \end{aligned} \quad (4.95)$$

Equation (4.94) for the Hubble parameter $H(t)$ is of second order, so we write it as a system of two first order equations, in order to use the phase portrait technique. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v \\ H \end{pmatrix} = \begin{pmatrix} -3\,Hv + \frac{1}{2}\frac{v^2}{H} - \frac{3}{4}\frac{H^3}{Q} + 3\,H\Lambda^2 - \frac{\lambda P}{H} \\ v \end{pmatrix}. \quad (4.96)$$

[8]We use the fact, that $W[\eta_{\mu\nu}] = 0$, that can be easily checked by direct computation, since in this case the spectrum of each Laplacian, appearing under the sign of determinant is trivial.

We are looking for special points of the vector field on the $[H, v]$-plane defined by the right-hand side of Eq. (4.96). Solving this algebraic equation we find two special points[9], H_1 and H_2:

$$\begin{aligned} H_1 &= \sqrt{2Q}\Lambda \sqrt{1 - \sqrt{1 - \frac{\lambda P}{3\Lambda^4 Q}}} \\ &\simeq \frac{1}{\Lambda} \sqrt{\frac{\lambda P}{3}} \left(1 + O\left(\frac{\lambda}{\Lambda^4}\right)\right) , \end{aligned} \tag{4.97}$$

describing a slowly expanding universe, and

$$\begin{aligned} H_2 &= \sqrt{2Q}\Lambda \sqrt{1 + \sqrt{1 - \frac{\lambda P}{3\Lambda^4 Q}}} \\ &\simeq 2\sqrt{Q}\Lambda \left(1 + O\left(\frac{\lambda}{\Lambda^4}\right)\right) , \end{aligned} \tag{4.98}$$

describing a rapidly expanding universe and hence offering a good candidate for an inflationary model.

Linearising the system Eq. (4.96) in the vicinity of each special point, we draw the following conclusions:

- The rapidly expanding solution H_2 is stable (stable focus on $[H, V]$ plane).
- The slowly expanding solution H_1 is unstable (unstable focus on $[H, V]$ plane).

In conclusion, if Pauli compensation condition is satisfied, namely if all quartic divergences cancel each other, then Sakharov's induced gravity leads to Starobinsky's anomaly-induced inflation, and vice versa.

In this and previous chapters we considered the QFT under the spectral regularization i.e. in a presence of the ultraviolet cutoff Λ. We did not discuss the reason of an introduction of such a cutoff scale. We did not discuss the question: "What sort of new physics or phase transition one expects to meet at high energies?" In the next two sections we will consider two models naturally exhibiting a presence of the ultraviolet cutoff scale. The former is based on the bosonic spectral action, reviewed in the first chapter while the latter will be just a generalization of the SM motivated by the idea of strong unification at the Planck scale.

[9]There are four special points, but since we are interested in expanding solutions we only consider positive values of H.

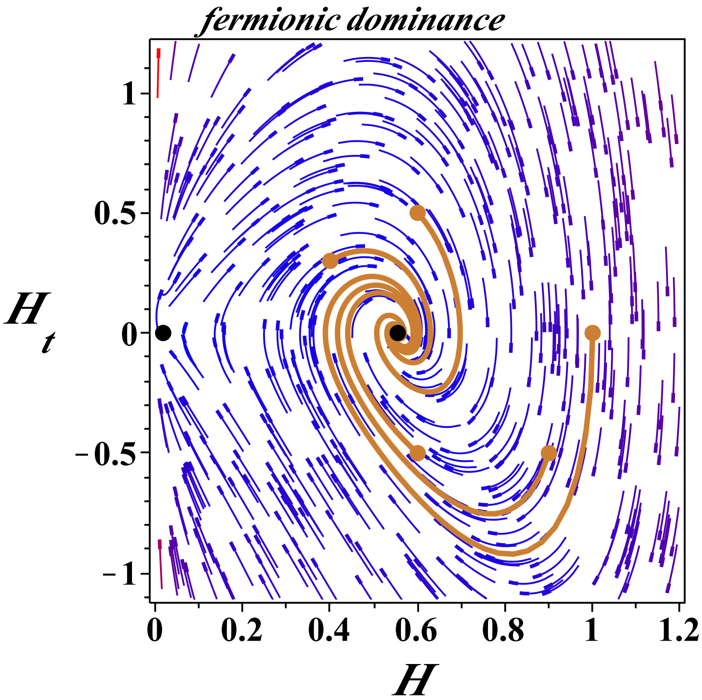

Figure 4.1: Phase portrait showing the dynamics of the scale factor in the case of the Sakharov's induced gravity with the Pauli compensation condition for the quartic divergences cancellation. The parameters are taken as follows, $N_{\mathrm{V}} = 12$, $N_{\mathrm{V}} = 5N_{\mathrm{V}}$.This system shows the existence of a stable de Sitter solution with the scalar curvature smaller or equal to the M_{Pl}; corresponds to a rapidly expanding universe.

Chapter 5

High vs. low momenta behavior of the bosonic spectral action

In previous chapters we discussed some applications of the ultraviolet spectral regularization in QFT. Now we consider some QFTs naturally exhibiting the ultraviolet cutoff scale. We are talking about a situation, when the cutoff scale has the physical meaning of an energy in which the theory (seen as effective) has a phase transition. Following ref. [23] we will show that the bosonic spectral action, discussed so far, exhibits two qualitatively different regimes of behavior, the and transition scale is give by the cutoff scale Λ. While the low energy regime of the BSA reproduces the Standard Model non minimally coupled with gravity, the high energetic behavior appears to be drastically different and as we will see exchange of high momenta bosons is impossible in this theory. The latter, due to uncertainty principle, makes impossible measurement of distances smaller than the inverse cutoff scale Λ^{-1}, thereby introducing the minimal length scale in this theory.

Traditional approach to BSA, that we discussed before is based on the heat kernel expansion

$$S_B = \sum_n \Lambda^{4-2n} a_{2n}(\not{D}), \tag{5.1}$$

that is an expansion in inverse powers of the cutoff scale Λ. A contribution, proportional to Λ^{-2n} has the following structure,

$$\Lambda^{-2n}(\text{contribution}) = \int d^4x\, \sqrt{g}\, \left(\frac{\text{powers of fields, powers of } \partial}{\Lambda^{2n}}\right) \tag{5.2}$$

where powers of the cutoff scale Λ in denominator are compensated by powers of fields and their derivatives in numerator. As a matter of fact, higher heat kernel coefficients contain higher derivatives of fields and at low energies their contribution is small, so BSA recovers Standard Model bosonic Lagrangian.

$$\text{high momenta = large derivatives,} \quad \text{or symbolically} \quad \frac{\partial}{\Lambda} > 1 \tag{5.3}$$

Nevertheless high momenta behavior of BSA is also of interest, since as we will see it is *qualitatively different.* In order to study propagation of high momenta (i.e. large frequency) bosons one should compute BSA up to quadratic order in fields, summing all derivatives. One needs such a resummation of the heat kernel expansion, that allows to derive linear equations of motion, valid for both high and low momenta regions. The solution was obtained by Barvinsky and Vilkovisky [79]:

$$\begin{aligned}
&\mathrm{Tr}\ \exp\left(-\frac{D^2}{\Lambda^2}\right) \simeq \\
&\simeq \frac{\Lambda^4}{(4\pi)^2}\int d^4x\,\sqrt{g}\,\mathrm{tr}\,\Big[1+\Lambda^{-2}P \\
&+\Lambda^{-4}(R_{\mu\nu}f_1\left(-\frac{\nabla^2}{\Lambda^2}\right)R^{\mu\nu}+Rf_2\left(-\frac{\nabla^2}{\Lambda^2}\right)R \\
&-Pf_3\left(-\frac{\nabla^2}{\Lambda^2}\right)R+Pf_4\left(-\frac{\nabla^2}{\Lambda^2}\right)P+\Omega_{\mu\nu}f_5\left(-\frac{\nabla^2}{\Lambda^2}\right)\Omega^{\mu\nu})\Big] \\
&+O(R^3,\Omega^3,E^3),
\end{aligned} \tag{5.4}$$

where $P \equiv E - \frac{1}{6}R$, and $f_1, ..., f_5$ are known functions:

$$\begin{aligned}
&f_1(\xi) = \frac{h(\xi)-1+\frac{1}{6}\xi}{\xi^2}, \qquad f_5(\xi) = -\frac{h(\xi)-1}{2\xi}, \\
&f_2(\xi) = \tfrac{1}{288}h(\xi) - \tfrac{1}{12}f_5(\xi) - \tfrac{1}{8}f_1(\xi), \qquad f_3(\xi) = \tfrac{1}{12}h(\xi) - f_5(\xi), \\
&f_4 = \tfrac{1}{2}h(\xi).
\end{aligned}$$

and

$$h(z) := \int_0^1 d\alpha\, e^{-\alpha(1-\alpha)\,z}.$$

For illustrative purposes in what follows we will discuss simplified bosonic spectral action corresponding a single fermion, interacting with gauge Higgs and gravitational fields. We will compute for this special case the righthand side of (5.4) and will show that at low energies (5.4) reproduces BSA, discussed above (i.e. heat kernel result[1]) completely. In the next section we will introduce the Dirac operator for our simplified case and compute relevant curvatures E and Ω appearing in the Barvinsky - Vilkovisky expansion.

5.1 Dirac operator and relevant curvatures.

We remind, that we work with Euclidean fermions, and impose the fermionic doubling discussed in the second chapter, the Hilbert space is split into left and right parts,

$$H = H_{\mathrm{L}} \oplus H_{\mathrm{R}} \tag{5.5}$$

where H_{L} and H_{R} are spaces of left and right (four component) fermions, and the Higgs field ϕ *connects* left and right fermions. In case of a single massive fermion, the classical action reads

$$\begin{aligned} S_{\mathrm{F}} &= \int d^4x\ \sqrt{g}\,\Psi^{\dagger}\not{D}\Psi, \\ \Psi &\equiv \begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix}, \end{aligned}$$

where the Dirac operator

$$\not{D} = \begin{pmatrix} i\gamma^{\mu}\nabla_{\mu} & \gamma_5\phi \\ \gamma_5\phi & i\gamma^{\mu}\nabla_{\mu} \end{pmatrix} = i\gamma^{\mu}\nabla_{\nu} \otimes 1_2^{\mathrm{L-R}} + \gamma_5\phi \otimes \sigma_1^{\mathrm{L-R}}, \tag{5.6}$$

and

$$1_2^{\mathrm{L-R}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \sigma_1^{\mathrm{L-R}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.7}$$

are matrices, acting on L and R indices. In the following we will skip this indexes for brevity.

Now we have to present the relevant Laplacian in a canonical form. Square of the Dirac operator reads:

$$\not{D}^2 = \begin{pmatrix} \left(i\gamma^{\mu}\nabla_{\mu}\right)^2 + (\gamma_5\phi)^2 & i\gamma^{\mu}\nabla_{\mu}\gamma_5\phi + \gamma_5\phi\, i\gamma^{\mu}\nabla_{\mu} \\ i\gamma^{\mu}\nabla_{\mu}\gamma_5\phi + \gamma_5\phi\, i\gamma^{\mu}\nabla_{\mu} & \left(i\gamma^{\mu}\nabla_{\mu}\right)^2 + (\gamma_5\phi)^2 \end{pmatrix} \tag{5.8}$$

Using known formula [44]

$$\left(i\gamma^{\mu}\nabla_{\mu}\right)^2 = -\left(\nabla^2 + \frac{i}{4}\left[\gamma^{\mu}, \gamma^{\nu}\right] F_{\mu\nu} + \frac{R}{4}\right) \tag{5.9}$$

[1]We mean a_0, a_2 and a_4 contributions.

and the simple identity

$$i\gamma^\mu \nabla_\mu \gamma_5 \phi + \gamma_5 \phi \, i\gamma^\mu \nabla_\mu = i\gamma^\mu \nabla_\mu \gamma_5 \phi - \gamma^\mu \gamma_5 \phi \, i\nabla_\mu = i\gamma^\mu \gamma_5 \left[\nabla_\mu, \phi\right] = i\gamma^\mu \gamma_5 \phi_{;\mu}$$

we obtain

$$\not{D}^2 = -\left(\nabla^2 \otimes 1_2 + \left[\frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] F_{\mu\nu} + \frac{R}{4} - \phi^2\right] \otimes 1_2 - \left[i\gamma^\mu \gamma_5 \phi_{;\mu}\right] \otimes \sigma_1\right) \tag{5.10}$$

Canonical form of the Laplace type operator

$$\not{D}^2 = -\left(\nabla_{\mathrm{tot}}^2 + E\right), \tag{5.11}$$

where the total derivative

$$\nabla_{\mathrm{tot}} \equiv \nabla_\mu \otimes 1_2 \tag{5.12}$$

Comparing equations (5.10) and (5.11) we conclude that

$$E = \left[\frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] F_{\mu\nu} + \frac{R}{4} - \phi^2\right] \otimes 1_2 - \left[i\gamma^\mu \gamma_5 \phi_{;\mu}\right] \otimes \sigma_1 \tag{5.13}$$

Since the following relation takes place

$$\left[\nabla_\mu, \nabla_\nu\right] = iF_{\mu\nu} - \frac{1}{4}\gamma^\sigma \gamma^\rho R_{\sigma\rho\mu\nu}, \tag{5.14}$$

the curvature $\Omega^{\mu\nu}$ reads

$$\Omega^{\mu\nu} \equiv [\nabla_{\mathrm{tot}}^\mu, \nabla_{\mathrm{tot}}^\nu] = \left(iF_{\mu\nu} - \frac{1}{4}\gamma^\sigma \gamma^\rho R_{\sigma\rho\mu\nu}\right) \otimes 1_2. \tag{5.15}$$

At this point we have all ingredients needed to perform the Barvinsky - Vilkovisky expansion.

5.2 Barvinsky-Vilkovisky expansion

Now we have to substitute curvatures E and Ω given correspondingly by (5.13) (5.15) in the righthand side of (5.4). Since the computation is not so trivial, in this section we will give some intermediate technical details. The only nontrivial consideration deserve f_3, f_4 and f_5 terms in (5.4). After straightforward computation we arrive to the following answer for f_3 contribution

$$-\mathrm{tr}\, P \, f_3\left(-\frac{\nabla^2}{\Lambda^2}\right) R = -\frac{2}{3} R \, f_3\left(-\frac{\nabla^2}{\Lambda^2}\right) R + 8\,\phi^2 \, f_3\left(-\frac{\nabla^2}{\Lambda^2}\right) R \tag{5.16}$$

The f_4 and f_5 contributions are more technically involved, since one has to take traces over gamma matrices. Product of two commutators of gamma matrices equals to

$$\left[\gamma^D,\gamma^E\right]\left[\gamma^A,\gamma^B\right] = -\frac{1}{6}\varepsilon^{DEAB}\underbrace{\varepsilon_{FGHM}\gamma^F\gamma^G\gamma^H\gamma^M}_{4!\gamma_5}$$
$$+2\left\{\eta^{DB}\left[\gamma^E,\gamma^A\right] + \eta^{EA}\left[\gamma^D,\gamma^B\right] - \eta^{DA}\left[\gamma^E,\gamma^B\right] - \eta^{EB}\left[\gamma^D,\gamma^A\right]\right\}$$
$$+4\underbrace{\left(\eta^{DB}\eta^{AE} - \eta^{DA}\eta^{EB}\right)}_{\text{contributes in tr}}$$

Hence

$$\mathrm{tr}\left[\gamma^D,\gamma^E\right]\left[\gamma^A,\gamma^B\right] = 16\left(\eta^{DB}\eta^{AE} - \eta^{DA}\eta^{EB}\right) \tag{5.17}$$

Using the identity (5.17) we obtain the following expressions for f_4 and f_5 contributions

$$\mathrm{tr}\, P\, f_4\left(-\frac{\nabla^2}{\Lambda^2}\right)P = 4\,F_{\mu\nu}\,f_4\left(-\frac{\nabla^2}{\Lambda^2}\right)F^{\mu\nu} + \frac{1}{18}\,R\,f_4\left(-\frac{\nabla^2}{\Lambda^2}\right)R \tag{5.18}$$
$$+8\,\phi^2\,f_4\left(-\frac{\nabla^2}{\Lambda^2}\right)\phi^2 + 8\,\phi_{;\mu}\,f_4\left(-\frac{\nabla^2}{\Lambda^2}\right)\phi^{;\mu} - \frac{4}{3}\,R\,f_4\left(-\frac{\nabla^2}{\Lambda^2}\right)\phi^2$$

$$\mathrm{tr}\,\Omega_{\mu\nu}\,f_5\left(-\frac{\nabla^2}{\Lambda^2}\right)\Omega^{\mu\nu}$$
$$= -8\,F_{\mu\nu}\,f_5\left(-\frac{\nabla^2}{\Lambda^2}\right)F^{\mu\nu} - R_{\mu\nu\rho\sigma}\,f_5\left(-\frac{\nabla^2}{\Lambda^2}\right)R^{\mu\nu\rho\sigma} \tag{5.19}$$

Substituting our intermediate results (5.16), (5.18), (5.19) for f_3, f_4 and f_5 contributions in the general formula (5.4) we obtain

$$\mathrm{Tr}\,\exp\left(-\frac{\not{D}^2}{\Lambda^2}\right) \simeq \frac{1}{16\pi^2}\int d^4x\,\sqrt{g}\,\{\,8\Lambda^4 - \Lambda^2\left(8\phi^2 - \frac{2}{3}R\right)$$
$$+R\left[8f_2\left(-\frac{\nabla^2}{\Lambda^2}\right) - \frac{2}{3}f_3\left(-\frac{\nabla^2}{\Lambda^2}\right) + \frac{1}{18}f_4\left(-\frac{\nabla^2}{\Lambda^2}\right)\right]R$$
$$+8\,R_{\mu\nu}f_1\left(-\frac{\nabla^2}{\Lambda^2}\right)R^{\mu\nu} - R_{\mu\nu\rho\sigma}\,f_5\left(-\frac{\nabla^2}{\Lambda^2}\right)R^{\mu\nu\rho\sigma}$$
$$-\phi^2\left[-8f_3\left(-\frac{\nabla^2}{\Lambda^2}\right) + \frac{4}{3}f_4\left(-\frac{\nabla^2}{\Lambda^2}\right)\right]R$$
$$+8\,\phi\left[-\nabla^2 f_4\left(-\frac{\nabla^2}{\Lambda^2}\right)\right]\phi + 8\,\phi^2 f_4\left(-\frac{\nabla^2}{\Lambda^2}\right)\phi^2$$
$$+F_{\mu\nu}\left[4f_4\left(-\frac{\nabla^2}{\Lambda^2}\right) - 8f_5\left(-\frac{\nabla^2}{\Lambda^2}\right)\right]F^{\mu\nu}\Big\} \tag{5.20}$$

The formula (5.20) is much more informative rather then the heat kernel anzatz

$$
\begin{aligned}
&\mathrm{Tr}\ \exp\left(-\frac{\not{D}^2}{\Lambda^2}\right) \simeq \Lambda^4 a_0(\not{D}) + \Lambda^2 a_2(\not{D}) + \Lambda^0 a_4(\not{D}) \\
&= \frac{1}{16\pi^2}\int d^4x\ \sqrt{g}\left\{8\Lambda^4 - \Lambda^2\left(8\phi^2 - \frac{2R}{3}\right)\right. \\
&+4\phi\left(-\nabla^2 - \frac{R}{6}\right)\phi + 4\phi^4 + \frac{4}{3}F_{\mu\nu}F^{\mu\nu} \\
&\left.-\frac{1}{10}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{11}{1080}R^*R^*\right\} \qquad (5.21)
\end{aligned}
$$

Although as we will see in the next section the formula (5.20) reproduces[2] correctly "standard" BSA anzatz (5.21) at low energy, it is valid for *all* energy region in the quadratic in fields approximation.

We also notice that in order to apply BSA in particle physics one should subtract huge "cosmological constant" $\sim \Lambda^4$ and enormously large Higgs mass term, $\sim \Lambda^2 H^2$ normalizing both on their physical values, that are known to be much smaller than corresponding powers of the cutoff scale Λ. Renormalized bosonic spectral action reads

$$
(\mathrm{BSA})_{\mathrm{ren}} \equiv (\mathrm{BSA}) + (\lambda - \mathrm{counterterm}) + \left(H^2 - \mathrm{counterterm}\right) \qquad (5.22)
$$

In our simplified case[3]

$$
(\mathrm{BSA})_{\mathrm{ren}} = \mathrm{Tr}\ \exp\left(-\frac{\not{D}^2}{\Lambda^2}\right) - \frac{1}{16\pi^2}\int d^4x\ \sqrt{g}\left\{8\Lambda^4 - 8\Lambda^2\phi^2\right\} \qquad (5.23)
$$

5.3 Low momenta limit

Before we go ahead and consider high momenta behavior of BSA, first we would like to show, how at low momenta regime the general formula (5.20) reduces to the frequently used heat kernel result, based on the anzatz (5.21), describing the Standard Model bosonic action coupled with gravity. At low momenta regime combinations of form factors $f_1, .., f_5$ appearing in (5.20) have the asymptotics

[2]Up to a constant related with Gauss-Bonnet contribution, that however does not effect on dynamics

[3]We consider for simplicity the cosmological constant $\lambda = 0$ and the Higgs vev also equal to zero, since each of them is much smaller than the uv cutoff scale Λ, since the latter is of the order of the Planck mass.

listed below:

$$
\begin{aligned}
&8f_1(\xi) - \frac{2}{3}f_3(\xi) + \frac{1}{18}f_4(\xi) \simeq -\frac{1}{60} + O(\xi)\\
&8f_1(\xi) \simeq \frac{2}{15} + O(\xi)\\
&-f_5(\xi) \simeq -\frac{1}{12} + O(\xi)\\
&-8f_3(\xi) + \frac{4}{3}f_4(\xi) \simeq \frac{2}{3} + O(\xi)\\
&8f_4(\xi) \simeq 4 + O(\xi)\\
&4f_4(\xi) - 8f_5(\xi) \simeq \frac{4}{3} + O(\xi)
\end{aligned}
$$

Substituting these formulas in the Barvinsky-Vilkovisky expansion (5.20) we find the following behavior of the right hand side of (5.20) at low momenta:

$$
\begin{aligned}
&\mathrm{Tr}\,\exp\left(-\frac{D\!\!\!\!/^2}{\Lambda^2}\right) \simeq \frac{1}{16\pi^2}\int d^4x\,\sqrt{g}\left\{8\Lambda^4 - \Lambda^2\left(8\phi^2 - \frac{2R}{3}\right)\right.\\
&+4\phi\left(-\nabla^2 - \frac{R}{6}\right)\phi + 4\phi^4 + \frac{4}{3}F_{\mu\nu}F^{\mu\nu}\\
&\left.-\frac{1}{60}R^2 + \frac{2}{15}R_{\mu\nu}R^{\mu\nu} - \frac{1}{12}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right\}
\end{aligned}
\tag{5.24}
$$

Terms, appearing in the right hand side of (5.24) proportional to Λ^4 and Λ^2 coincide with a_0 and a_2 contributions in (5.21) correspondingly. Higgs and gauge actions, appearing in (5.24) again coincide with a_4 contribution in (5.21), nevertheless one have to be careful with the terms, quadratic in Riemann tensor. This combination can be expressed via Gauss-Bonnet density and Weyl square:

$$
\begin{aligned}
&R^*R^* = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\\
&C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\\
&-\frac{1}{60}R^2 + \frac{2}{15}R_{\mu\nu}R^{\mu\nu} - \frac{1}{12}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = -\frac{1}{10}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{1}{60}R^*R^*
\end{aligned}
$$

It is remarkable, that GB contribution, being a topological term does not depend on the metric tensor $g_{\mu\nu}$ and thereby does not contribute to equations of motion:

$$
\frac{\delta}{\delta g_{\mu\nu}(x)}\int d^4x\,\sqrt{g}\,R^*R^* = 0, \tag{5.25}
$$

while Weyl tensor contribution coincides with the one, given by a_4. Finally we conclude, that *up to a constant, that does not effect on equations of motion, low* *momenta behavior of Barvinsky - Vilkovisky expansion* (5.20) *coincides the result* (5.21), coming from first three heat kernel coefficients.

At low energies the renormalized BSA reads[4]

$$(\mathrm{BSA})_{\mathrm{ren}} \simeq \frac{1}{16\pi^2} \int d^4x \, \sqrt{g} \left\{ \frac{2\Lambda^2}{3} R \right. \tag{5.26}$$
$$\left. +4\phi \left(-\nabla^2 - \frac{R}{6} \right) \phi + 4\phi^4 + \frac{4}{3} F_{\mu\nu} F^{\mu\nu} - \frac{1}{10} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right\} + O\left(\frac{1}{\Lambda^2} \right)$$

In what follows we would like to study high momenta behavior of BSA. We are interested in propagation of *free* particles, therefore for our purposes quadratic in fields approximation in (5.20) is sufficient. On the other side if one wants to study *interaction* of particles at high momenta, using BSA, a knowledge of the ansatz (5.20) is not enough: one should take into account cubic and higher powers of curvatures E and Ω but this goes beyond the scope of present project.

Extracting the quadratic in fields contribution in (5.20) is straightforward for gauge and scalar fields, however in gravitational sector one should perform some computations, and the next section is devoted to this issue. Although some of this formulas one can find in the literature, we present here detailed computations, for pedagogical purposes and in order to avoid mistakes in signs due to mixing of notations.

5.4 Gravitational sector: weak fields

Now we consider gravitons i.e. fluctuations of the metric tensor, imposing the transverse and traceless gauge fixing condition

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}, \quad h^{\mu}_{\mu} \equiv \delta^{\mu\nu} h_{\mu\nu} = 0, \quad \partial_\mu h^{\mu\nu} = 0 \tag{5.27}$$

First we focus our attention on the contribution in (5.20), linear in R, i.e. Einstein Hilbert action.

R - contribution.

Rewriting the Einstein-Hilbert action as a quadratic combination of Christoffel symbols (see the Appendix)

$$\int d^4x \, \sqrt{g} \, R = \int d^4x \, \sqrt{g} \left(\Gamma^{\gamma}_{\gamma\sigma} \Gamma^{\sigma}_{\mu\nu} - \Gamma^{\sigma}_{\gamma\nu} \Gamma^{\gamma}_{\sigma\mu} \right) g^{\mu\nu} \tag{5.28}$$

[4]We do not write Gauss-Bonnet contribution since it does not effect on the dynamics

and expanding the Christoffel symbols at linear order

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\left(\partial_\rho h^{\mu}_{\nu} + \partial_\nu h^{\mu}_{\rho} - \partial^{\mu} h_{\nu\rho}\right) \tag{5.29}$$

and substituting the result in (5.28) we find, that due to the gauge fixing condition, only one term $\Gamma\Gamma$ of the two contributes in (5.28)

$$\int d^4x\ \sqrt{g}\,R \simeq \frac{1}{4}\int d^4x\,\delta^{\mu\nu}\left(\partial_\gamma h^{\sigma}_{\nu} + \partial_\nu h^{\sigma}_{\gamma} - \partial^{\sigma} h_{\gamma\nu}\right)\left(-\partial_\sigma h^{\gamma}_{\mu} - \partial_\mu h^{\gamma}_{\sigma} + \partial^{\gamma} h_{\sigma\mu}\right)$$

Again due to the gauge fixing condition, only three terms of nine in the previous formula are nonzero

$$= \frac{1}{4}\{\,\partial_\gamma h^{\sigma}_{\nu}\,\partial^{\gamma} h^{\nu}_{\sigma} - \underbrace{\partial^{\mu} h^{\sigma}_{\gamma}\,\partial_\mu h^{\gamma}_{\sigma} + \partial^{\sigma} h_{\gamma\nu}\,\partial_\sigma h^{\gamma\nu}}_{\text{cancel each other}}\} = \frac{1}{4}\int d^4x\, h_{\nu\sigma}\left(-\partial^2\right) h^{\nu\sigma}, \tag{5.30}$$

and the final result for Einstein-Hilbert action at the leading order reads

$$\int d^4x\ \sqrt{g}\,R \simeq \frac{1}{4}\int d^4x\, h_{\nu\sigma}\left(-\partial^2\right) h^{\nu\sigma} \tag{5.31}$$

Riemann and Ricci tensors

Using our notations and sign conventions for Riemann and Ricci tensors in Appendix, and the linearized Christoffel symbols (5.29) we obtain the following expression for the Riemann tensor at the leading (i.e. linear) order

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\sigma\Gamma^{\mu}_{\nu\rho} - \partial_\rho\Gamma^{\mu}_{\nu\sigma} + O(\Gamma^2) \simeq \frac{1}{2}\left(\partial_\sigma\partial_\nu h^{\mu}_{\rho} - \partial_\rho\partial_\nu h^{\mu}_{\sigma} + \partial^{\mu}\partial_\rho h_{\nu\sigma} - \partial^{\mu}\partial_\sigma h_{\nu\rho}\right) \tag{5.32}$$

Due to the gauge fixing condition only one term of four in the the linearized Riemann tensor (5.32) survives when one computes the leading asymptotic of the Ricci tensor

$$R_{\nu\rho} \simeq \frac{1}{2}\partial^2 h_{\nu\rho} + O(h^2) \tag{5.33}$$

Square root of g contribution

Expanding $\sqrt{g}$ in Taylor series up to quadratic order in metric perturbations we have

$$\sqrt{g} = 1 + \left[\left(\frac{\partial\sqrt{g}}{\partial g_{\mu\nu}}\right)\Bigg|_{g_{\mu\nu}=\delta_{\mu\nu}} h_{\mu\nu}\right] + \frac{1}{2}\left[\left(\frac{\partial^2\sqrt{g}}{\partial g_{\mu\nu}\partial g_{\rho\sigma}}\right)\Bigg|_{g_{\mu\nu}=\delta_{\mu\nu}} h_{\mu\nu}h_{\rho\sigma}\right] + O(h^3)$$

It is known, that,

$$\frac{\partial g}{\partial g_{\mu\nu}} = g g^{\mu\nu} \tag{5.34}$$

and

$$\left(\frac{\partial\, g^{\lambda\xi}}{\partial\, g_{\mu\nu}}\right) = -\frac{1}{2}\left(g^{\mu\sigma} g^{\nu\rho} + g^{\nu\sigma} g^{\mu\rho}\right). \tag{5.35}$$

Using the identity (5.34) we obtain, that the first derivative in (5.34) vanishes

$$\left.\frac{\partial\sqrt{g}}{\partial g_{\mu\nu}}\right|_{g_{\mu\nu}=\delta_{\mu\nu}} = \frac{1}{2}\left.\sqrt{g}\, g^{\mu\nu}\right|_{g_{\mu\nu}=h_{\mu\nu}} = \frac{1}{2} h^{\mu}_{\mu} = 0. \tag{5.36}$$

Formulas (5.34) and (5.35) allow us to compute the second derivative in (5.34)

$$\frac{\partial^2\sqrt{g}}{\partial g_{\mu\nu}\partial g_{\rho\sigma}} = \frac{1}{2}\sqrt{g}\left(\frac{1}{2} g^{\mu\nu} g^{\rho\sigma} - \frac{1}{2}\left[g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}\right],\right) \tag{5.37}$$

and the final answer reads

$$\sqrt{g} = 1 - \frac{1}{4} h_{\mu\nu} h^{\mu\nu} + O(h^3) \tag{5.38}$$

5.5 High momenta behavior

Now we have all ingredients, needed to finish the computation of the quadratic part of BSA at high momenta.

One can easily find large momenta asymptotic of the form factors:

$$\begin{aligned}
f_1(\xi) &\simeq \frac{1}{6}\xi^{-1} - \xi^{-2} + O\left(\xi^{-3}\right)\\
f_2(\xi) &\simeq -\frac{1}{18}\xi^{-1} + \frac{2}{9}\xi^{-2} + O\left(\xi^{-3}\right)\\
f_3(\xi) &\simeq -\frac{1}{3}\xi^{-1} + \frac{4}{3}\xi^{-2} + O\left(\xi^{-3}\right)\\
f_4(\xi) &\simeq \xi^{-1} + 2\xi^{-2} + O\left(\xi^{-3}\right)\\
f_5(\xi) &\simeq \frac{1}{2}\xi^{-1} - \xi^{-2} + O\left(\xi^{-3}\right)
\end{aligned}$$

In quadratic approximation gauge, Higgs and gravitational fields being *free* do not interact with each other so we consider each sector separately and than combine all together.

Gravitational sector

Using formulas (5.31), (5.32), (5.33) and (5.38) from the previous section we obtain that:

$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ contribution

$$-\frac{1}{16\pi^2}\int d^4x\ \sqrt{g}\,R_{\mu\nu\rho\sigma}f_5\left(-\frac{\nabla^2}{\Lambda^2}\right)R^{\mu\nu\rho\sigma}$$
$$\simeq \frac{\Lambda^4}{16\pi^2}\int d^4x\,h_{\mu\nu}\left[-\left(-\frac{\partial^2}{\Lambda^2}\right)^2 f_5\left(-\frac{\partial^2}{\Lambda^2}\right)\right]h^{\mu\nu}$$

$R_{\mu\nu}R^{\mu\nu}$ contribution

$$\frac{1}{16\pi^2}\int d^4x\ \sqrt{g}\,8\,R_{\mu\nu}f_1\left(-\frac{\nabla^2}{\Lambda^2}\right)R^{\mu\nu} \simeq \frac{\Lambda^4}{16\pi^2}\int d^4x\,h_{\mu\nu}\left[2\left(-\frac{\partial^2}{\Lambda^2}\right)^2 f_1\left(-\frac{\partial^2}{\Lambda^2}\right)\right]h^{\mu\nu}$$

R contribution

$$\frac{1}{16\pi^2}\,\frac{2}{3}\int d^4x\ \sqrt{g}\,\Lambda^2 R = \frac{\Lambda^4}{16\pi^2}\int d^4x\,h_{\mu\nu}\left[\frac{1}{6}\left(-\frac{\partial^2}{\Lambda^2}\right)\right]h^{\mu\nu} \tag{5.39}$$

$\sqrt{g}$ contribution

$$\frac{1}{16\pi^2}\int\ \sqrt{g}\,8\Lambda^4 = \frac{\Lambda^4}{16\pi^2}\int d^4x\left(8 - 2h_{\mu\nu}h^{\mu\nu}\right) \tag{5.40}$$

Overall gravitational contribution

$$\frac{\Lambda^4}{16\pi^2}\int d^4x\left[8 + h_{\mu\nu}\mathcal{F}_{\rm gr}\left(-\frac{\partial^2}{\Lambda^2}\right)h^{\mu\nu} + O\left(h^4\right)\right]$$
$$\mathcal{F}_{\rm gr}(\xi) \equiv -2 + \frac{\xi}{6} + 2\,\xi^2 f_1(\xi) - \xi^2 f_5(\xi) \simeq -3 + O\left(\frac{1}{\xi}\right) \tag{5.41}$$

For the renormalized BSA we obtain:

$$(\mathrm{BSA})^{\rm gr}_{\rm ren} \simeq \frac{\Lambda^4}{16\pi^2}\int d^4x\,h_{\mu\nu}\mathcal{F}^{\rm ren}_{\rm gr}\left(-\frac{\partial^2}{\Lambda^2}\right)h^{\mu\nu}$$
$$\mathcal{F}^{\rm ren}_{\rm gr}(\xi) \equiv \frac{\xi}{6} + 2\,\xi^2 f_1(\xi) - \xi^2 f_5(\xi) \simeq -1 + O\left(\frac{1}{\xi}\right) \tag{5.42}$$

Gauge sector

The contribution in (5.20), quadratic in $F_{\mu\nu}$ reads[5]

$$\frac{1}{16\pi^2}\int d^4x\, F_{\mu\nu}\left[\mathcal{F}_{\text{vec}}\left(-\frac{\partial^2}{\Lambda^2}\right)\right]F^{\mu\nu}$$
$$\mathcal{F}_{\text{vec}} \equiv 4f_4\,(\xi) - 8f_5\,(\xi) \simeq \frac{16}{\xi^2} + O\left(\frac{1}{\xi^3}\right) \tag{5.43}$$

Imposing transversal gauge fixing condition

$$\partial_\mu A^\mu = 0 \tag{5.44}$$

we get the following gauge contribution to the gauge fixed BSA

$$(\text{BSA})^{\text{vec}}_{\text{gf}} = \frac{\Lambda^2}{16\pi^2}\int d^4x\, A_\mu\left[\mathcal{F}^{\text{gf}}_{\text{vec}}\left(-\frac{\partial^2}{\Lambda^2}\right)\right]A^\mu$$
$$\mathcal{F}^{\text{gf}}_{\text{vec}} \equiv 2\xi\,(4f_4\,(\xi) - 8f_5\,(\xi)) \simeq \frac{32}{\xi} + O\left(\frac{1}{\xi^2}\right) \tag{5.45}$$

Scalar sector

Collecting all terms, quadratic in ϕ in (5.20) we get the following scalar contribution

$$\frac{\Lambda^2}{16\pi^2}\int d^4x\,\phi\left[\mathcal{F}_{\text{sc}}\left(-\frac{\partial^2}{\Lambda^2}\right)\right]\phi$$
$$\mathcal{F}_{\text{sc}} \equiv -8 + 8\,\xi\, f_4(\xi) \simeq \frac{16}{\xi} + O\left(\frac{1}{\xi^2}\right) \tag{5.46}$$

And subtracting unphysical terms in (5.20) for the renormalized scalar sector of BSA we have

$$(\text{BSA})^{\text{sc}}_{\text{ren}} = \frac{\Lambda^2}{16\pi^2}\int d^4x\,\phi\left[\mathcal{F}^{\text{ren}}_{\text{sc}}\left(-\frac{\partial^2}{\Lambda^2}\right)\right]\phi$$
$$\mathcal{F}^{\text{ren}}_{\text{sc}} \equiv 8\,\xi\, f_4(\xi) \simeq 8 + O\left(\frac{1}{\xi}\right) \tag{5.47}$$

[5] Results regarding the gauge sector were first obtained in [80, 81].

Overall contribution

Collecting together results of (5.41), (5.43) and (5.46) we arrive to the following large momenta asymptotic of the BSA in quadratic in fields approximation

$$\mathrm{Tr}\,\exp\left(-\frac{D\!\!\!\!/^2}{\Lambda^2}\right) \simeq \frac{\Lambda^4}{16\pi^2}\int d^4x\left[8 - 3h_{\mu\nu}h^{\mu\nu} + 16\phi\frac{1}{-\partial^2}\phi + 16F_{\mu\nu}\frac{1}{(-\partial^2)^2}F^{\mu\nu}\right]$$

Correspondingly from (5.42), (5.45) and (5.47) we derive the large momenta asymptotic of the renormalized BSA in quadratic approximation in fields[6]

$$(\mathrm{BSA})^{\mathrm{high}}_{\mathrm{ren}} \simeq \frac{1}{16\pi^2}\int d^4x\left[-\Lambda^4 h_{\mu\nu}h^{\mu\nu} + 8\Lambda^2\phi^2 + 32\Lambda^4 A_\mu\frac{1}{(-\partial^2)}A^\mu\right] \tag{5.48}$$

It is interesting to compare the latter with low momenta asymptotic:

$$(\mathrm{BSA})^{\mathrm{low}}_{\mathrm{ren}} \simeq \frac{1}{16\pi^2}\int d^4x\left[\frac{\Lambda^2}{6}\,h_{\mu\nu}\left(-\partial^2\right)h^{\mu\nu}\right.$$
$$\left.+4\phi\left(-\partial^2\right)\phi + \frac{8}{3}A_\mu\left(-\partial^2\right)A^\mu\right], \tag{5.49}$$

that can be obtained from formulas from (5.42), (5.45) and (5.47) using low momenta asymptotics of functions $\mathcal{F}^{\mathrm{ren}}_{\mathrm{gr}}$, $\mathcal{F}^{\mathrm{ren}}_{\mathrm{sc}}$ $\mathcal{F}^{\mathrm{gf}}_{\mathrm{vec}}$ or directly from formula (5.26), expanding its righthand side up to a quadratic order in fields.

As we can see the low and high energy regimes of BSA, given by formulas (5.49) and (5.48) correspondingly are completely different. The low energy regime leads to wave equation of motion and thereby propagating particles, at high momenta the action does not contain positive powers of derivatives, so one has two understand what it means physically. In the next section we will give a physical interpretation of the result (5.48).

5.6 Physical interpretation

In order to interpret these results, and understand their physical meaning, we take the point of view that the cutoff is a physical scale up to which we may trust our theory, the natural candidate would be Planck's length. There is physical cutoff on length, which is imposed as a cutoff on the eigenvalues of the Dirac operator. This does not necessarily mean that there is a minimal length[7], although this is a possible interpretation.

[6] Transversal gauge fixing condition is imposed.

[7] For example the presence of Λ_{QCD} does not mean than in chromodynamics there is a maximal energy. There is however a phase transition, related with confinement.

We will see that a cutoff on the eigenvalues of the Dirac operator, and hence of the Laplacian, has profound consequences on the propagation of the fields. We are considering free fields (i.e. plane waves), they are the ones one should use to probe spacetime. The propagator in position space $\Delta_F(x, y)$ has a meaning: the probability amplitude that a particle is created at position x, and later annihilated at position y. The probing of spacetime, in whichever scheme of realistic or gedanken experiment, involves always the interaction of particles, which are "created" in some apparatus, then interact with another particle at some position in space, and then are "annihilated" in a detector.

Due to homogeneity and isotropy, a two-point Green's functions depends on the difference between positions: $G(x - y)$. These are distributions acting on the space of test functions which physically are the sources $J(x)$. The latter are classical, and we consider them to be the probes of spacetime. Let us now consider two situations, long and short distances. To probe short distances one requires high energetic sources. Mathematically this means that, in momentum space, the support of $J(k)$ is located in the large k region. Using Eq. (5.48) it turns out, as we will discuss in more detail below, that asymptotically, in the high energy region, the Green's function becomes $\delta(x - y)$, or its derivatives. A source in x has no effect on any other point, except x itself. Heuristically, usually you have the vacuum, you "disturb" it with a source, and this disturbance propagates in a certain way, usually as a particle, generally a virtual one. Now instead we have that what happens in a point has no effect on neighbouring points. Points do not talk to each other.

Let us be more detailed. The classical action reads (in the quadratic field approximation):

The classical action reads (in the quadratic field approximation):

$$S[J, \varphi] = \int d^4x \left(\frac{1}{2} \varphi(x) \mathcal{F}(\partial^2) \varphi(x) - J(x) \varphi(x) \right) , \tag{5.50}$$

where φ is any of the bosonic fields, ϕ, A, or h. The equation of motion is:

$$\mathcal{F}(\partial^2)\phi(x) = J(x) \tag{5.51}$$

The inverse of the differential operator, staying in lefthand side is a Green function $G(x - y)$

$$G = \frac{1}{F(\partial^2)} \tag{5.52}$$

allows us to write the solutions of (5.51) as

$$\varphi_J(x) = \int d^4y J(y) G(x - y) \tag{5.53}$$

It is more convenient to use a momenta representation, expending the field φ and the source J in Fourier series, i.e. in eigenfunctions of the momenta operator.

$$\varphi(x) = \frac{1}{(2\pi)^2}\int d^4k\, e^{ikx}\,\hat{\varphi}(k) \quad J(x) = \frac{1}{(2\pi)^2}\int d^4k\, e^{ikx}\,\hat{J}(k)$$
$$G(x-y) = \frac{1}{(2\pi)^2}\int d^4k\, e^{ik(x-y)}\,\hat{G}(k) \tag{5.54}$$

The Green function in momenta space reads

$$G(k) = \frac{1}{F(-k^2)} \tag{5.55}$$

and thus we obtain:

$$\varphi_J(x) = \frac{1}{(2\pi)^4}\int d^4k e^{ikx} J(k)\frac{1}{F(-k^2)} \tag{5.56}$$

At low energy $F(k) \sim k^2$, and everything is as we know. The Green's function is the usual one, leading to the normal propagation of particles. The calculation above shows that in the very high energy regime (the scale is given by Λ) the qualitative behavior has changed, and asymptotically $F(k) = 1/k^2$ vectors, and $F(k) = 1$ for scalars and gravitons. We now related this behavior of F with the nonpropagation, or better, to the impossibility to probe nearby points. Short distances require high momentum probes, let us therefore consider $J(k) \neq 0$ for $|k^2| \in [K^2, K^2 + \delta k^2]$, with K^2 very large.

$$\varphi_J(x) \xrightarrow[K\to\infty]{} \begin{cases} \frac{1}{(2\pi)^4}\int d^4k\, e^{ikx} J(k)k^2 = (-\partial^2)J(x) & \text{for scalars and vectors} \\ \frac{1}{(2\pi)^4}\int d^4k\, e^{ikx} J(k) = J(x) & \text{for gravitons} \end{cases} \tag{5.57}$$

What we find remarkable is the fact that the values of $\phi_j(x)$ depends only on J or its derivatives calculated at x itself. Compare with the standard case, in which to have the value at x the whole function J is required. In term of Green's function in position space, expression (5.57) means

$$G(x-y) \propto \begin{cases} (-\partial^2)\delta\,(x-y) & \text{for vectors} \\ \delta\,(x-y) & \text{for scalars and gravitons} \end{cases} \tag{5.58}$$

Remark on the fermionic case

Although BSA, exhibiting, as we have seen the minimal length scale, has to do with dynamics of bosons, one can naturally modify a theory of fermions in such a way that the minimal length will appear also there. The key idea is the spectral regularization.

The latter, we remind, is based on the replacement:

$$\not{D} \longrightarrow \not{D} P_\Lambda + (1 - P_\Lambda)\,\Lambda, \quad P_\Lambda = \Theta(\Lambda^2 - \not{D}^2)$$

Thus the classical fermionic action modifies as follows

$$S_F = \int d^4x\,\sqrt{g}\bar{\psi}\not{D}\psi \longrightarrow \int d^4x\,\sqrt{g}\,(\bar{\psi}\not{D}P_\Lambda\psi + \Lambda\bar{\psi}\,(1 - P_\Lambda)\,\psi)$$

The latter means, that the fermionic Green function in momenta space $G(k)$ becomes

$$G(k) = 1, k^2 > \Lambda$$

or in coordinate space it acts as a delta function on high momenta sources

$$G(x - y) = \delta(x - y).$$

It is remarkable, that similar result, i.e. presence of minimal length scale upon the spectral regularisation was obtained in the framework of spectral geometry in [82].

In this chapter we have seen, that the bosonic spectral action exhibits a phase transition at high momenta. While the low energy regime, described by the formula (5.49), leads to propagating particles, the high momenta regime is given by (5.48) that can be interpreted as the fact, high momenta bosons do not propagate. Since, due to uncertainty principle, to probe short distances one requires high momenta, we see, that in this theory appears a notion of the *minimal length scale*.

In the next chapter we discuss another model, naturally posing the phase transition at the Planck scale related with non propagating gauge bosons. In contrast to the one, discussed above the new one will not be related with the bosonic spectral action and will be just based on the addition of the fermionic multiplets to the Standard Model, motivated by some natural requirements.

Chapter 6

Universal Landau Pole

In the previous chapter we considered BSA as an example of the model, naturally exhibiting the ultraviolet cutoff scale. We have seen, that BSA reproduces QFT at energies, smaller than the corresponding cutoff scale $\Lambda \sim M_{\rm Pl}$, while at higher momenta the behavior is qualitatively different: the bosons do not propagate anymore. Nevertheless that model is based on the Spectral Action Principle, discussed in the second chapter. One can ask, if exists such a generalization of the SM that exhibits similar behavior i.e. the *physical* cutoff scale in the ultraviolet without referring to the Spectral Action Principle. In what follows we give the arguments towards this idea and propose physically reasonable realization, based on the Universal Landau Pole for all gauge couplings at the Planck scale. Let us start from the motivation.

6.1 Do we really need asymptotic freedom?

The guiding principle, one follows when constructs a theory is a *simplicity*, which states that "the less number of parameters - the better". Thats why unification theories claiming, that all interactions unify at some GUT scale $\sim 10^{16} Gev$ are so fascinating and attractive for a broad class of researches starting from mid 70-th [83] and till nowadays, see e.g. [84]. Such theories enlarging the gauge group naturally lead to asymptotic freedom at high energies. The latter means that there is no essential ultraviolet cutoff scale, and formally these models can be exploited at arbitrary high energies.

Nevertheless one should not forget about gravity: at the Planck scale the gravity becomes strongly coupled and one should not neglect by quantum gravitational effects anymore. Since no self consistent quantum gravitational theory is known one can not go beyond the Planck scale. In this sense the grand unification, leading to asymptotic freedom, may loose its motivation. Thus it seems natural to impose the *strong unification* of all gauge couplings at the Planck scale [31, 32] - in this case both principle of minimality and strong gravity at Planck scale are respected.

Under the renormalization group flow the coupling constants of the three fundamental gauge interactions behave quite differently [85]. While the couplings of the non-abelian interactions, weak and strong, constantly diminish with as the energy increases, the coupling of the abelian interaction grows, and eventually diverges, a phenomenon usually referred as *Landau pole* [86, 87]. In fact a precense of new multiplets of particles will alter this behavior. One may render the strongly coupled regime at the Planck scale, requiring that new physics is organized in such a way, that under the RG flow *all gauge couplings will have a common Landau Pole* at the Planck scale.

$$g_{1,2,3}(\mu) \to \infty \text{ at } \mu \to M_{Pl} \tag{6.1}$$

In this chapter, we are going to show, that such a Universal Landau Pole (ULP) model can be constructed, and under some essential assumptions, the *minimal* solution is *unique* and moreover, the ULP generalization of the Standard Model naturally solves the instability problem [91–93] of the Higgs potential, related with relatively light mass $\sim$ 125 GeV of the Higgs boson, recently discovered by LHC [89, 90].

It is remarkable, that when the energy approaches to ULP, kinetic terms of *all* gauge fields vanish, so gauge bosons can not propagate anymore.

$$\frac{1}{g(\mu)^2} F_{\mu\nu} F^{\mu\nu} \to 0 \text{ at } \mu \to M_{Pl}, \tag{6.2}$$

The situation is very similar to the one, discussed in the previous chapter, however we emphasize, that now we do not impose the Spectral Action principle, but add new particles to the Standard Model. In the next section we formulate the minimality requirements.

6.2 Minimal ULP: requirements

We are looking for a solution to render ULP based on some physical assumptions:

- Simplicity: We want to avoid the proliferation of parameters, and we do not want any fine tuning.

- Gauge Group: We want the gauge group of the Standard Model: $SU(3) \times SU(2) \times U(1)$ unchanged - enlarging the gauge group without a contradiction with the "minimality" requirement in principle can be motivated just by introduction of a GUT group. This scenarios however lead to ULP at $10^{16} GeV$ [see [88] for review] i.e. three order of magnitude smaller, than the Planck scale.

- Stability: quartic coupling of the Higgs field self interaction λ is always positive under the renormalization group flow. This is the most important discrimination.

- Higgs sector: We want it to remain unchanged. If the new particles are Dirac 4-component spinors with Dirac masses $\rightarrow$ automatically there are no axial anomalies. Use of the same Higgs field to generate masses of new heavy particles is problematic, because it requires huge Yukawa's and worsen the instability. We note, that if masses of new fermions were generated by (minimal SM) Higgs, with vertex $Y\bar{\psi}\psi H$ loop correction drives the quartic coupling λ to negative values, and the bigger Yukawa constant Y, the less stable is the Universe. Indeed consider the sign of the contribution of the Yukawa coupling into quartic coupling's beta function. It is negative, the larger the Yukawa, the more lambda is decreasing under RG running. Going beyond the minimal SM, and introduction of many Higgs doublets makes the situation much more complicated, and leads us out of the simplicity requirement.

- NO pathological electric charges $\rightarrow$ restrictions on the representations of new fermions.

6.3 Minimal working ULP: realization

In order to satisfy requirements, listed in the previous section we use Dirac mass terms $M\bar{\psi}\psi$ for new fermions and we are looking for a minimal number of them. To get rid of pathological electric charges we consider new fermions belonging to known (SM) representations of gauge group, however we stress, that new paricles are vector-like fermions. Thus we introduce

- L-quarkons: SU(3) - triplets, SU(2) - doublets, $Y = \frac{1}{3}$, i.e. under gauge transformations transform as left quarks

- R-quarkons: SU(3) - triplets, SU(2) - singlets, $Y = \frac{4}{3},\ -\frac{2}{3}$ i.e. under gauge transformations transform as right quarks
- L-leptos: SU(3) - singlets, SU(2) - doublets, $Y = -1$ i.e. under gauge transformations transform as right quarks
- R-leptos: SU(3) - singlets, SU(2) - singlets, $Y = -2,\ 0$ i.e. under gauge transformations transform as right quarks

The only new vertexes appearing in this theory with respect to SM couple Quarkons and Leptos to electro-weak (E-W) gauge bosons and gluons, see Fig. 6.1, and correspondingly the only new diagrams, modifying RG flow at one loop are presented on Fig. 6.2.

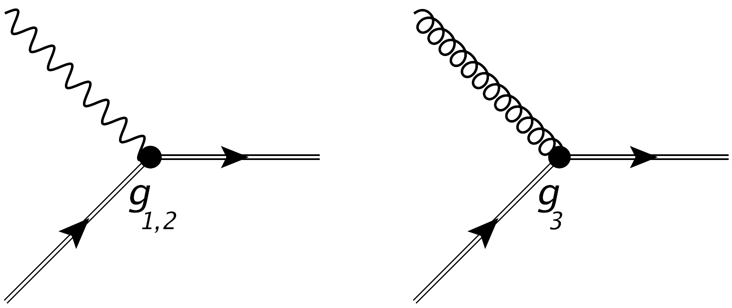

Figure 6.1: New vertexes appearing in the minimal ULP generalization of the Standard Model couple new fermions (double arrowed line) to the E-W gauge bosons (wavy line) and gluons (curly line)

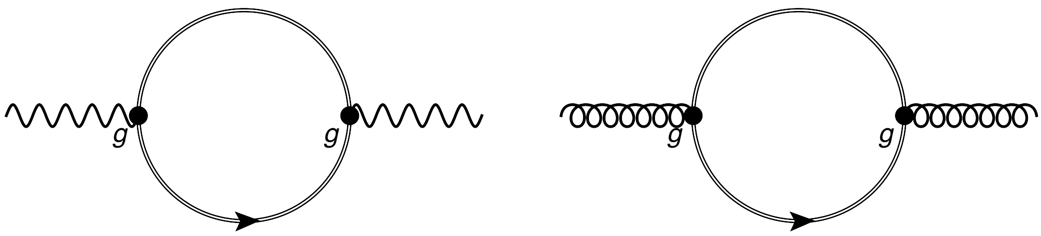

Figure 6.2: The only new one loop diagrams modifying the RG flow at one loop. Double arrowed line represents new fermions, E-W gauge bosons are depicted by wavy line and curly line presents gluons.

6.4 Scheme of the computation

It is remarkable, that in the Standard Model there is a perfect agreement between one and two loops approximations for the running of gauge couplings, see Fig. 6.3. Since in our ULP approach we add new particles in perturbative region, one may expect the same agreement, therefore in order to describe the gauge running we will use one loop approximation.

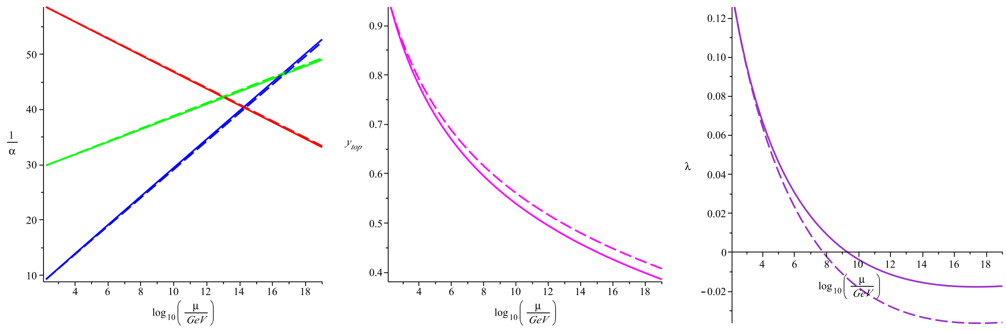

Figure 6.3: On the first picture one and two loops approximations for gauge coupling running are compared. The second picture represents one and two loops approximations for the top quark Yukawa coupling running. On the third picture one and two loops results for quartic coupling running are compared. On all plots solid lines represent two loops results, while one loop graphics are depicted with dash lines.

We will assume that the various particles have masses such that they contribute only when a particular threshold of energy is reached. The full evolution is therefore given by a set of straight segments and the solution is found matching the boundary conditions.

Running of the gauge couplings is given by:

$$\begin{aligned}
\frac{4\pi}{g_{1,2,3}^2(t)} &= -\frac{b_{1,2,3}\cdot(t-t_0)}{2\pi}+\frac{4\pi}{g_{1,2,3}^2(t_0)}, \quad t\equiv\log\frac{\mu}{GeV} \\
b_1 &= \frac{41}{6}+\frac{2}{3}N_{\mathbf{L-leptos}}+\frac{4}{3}N_{\mathbf{R-leptos}}+\frac{2}{9}N_{\mathbf{L-quarkon}}+\frac{20}{9}N_{\mathbf{R-quarkon}} \\
b_2 &= -\frac{19}{6}+\frac{2}{3}N_{\mathbf{L-leptos}}+2N_{\mathbf{L-quarkon}} \\
b_3 &= -7+\frac{4}{3}\left(N_{\mathbf{L-quarkon}}+N_{\mathbf{R-quarkon}}\right)
\end{aligned}$$

The integers N in these formulas refer to the number of quarkon and leptos multiplets contributing to beta functions.

Since the coefficients are piecewise constant, and change at the energies representing the scale at which the new particles, it is possible to do a systematic search. We have imposed as boundary condition of the differential equation that $1/\alpha_i = 0$ at the Planck scale $M_{\rm Pl}$. In any case the model cannot be trusted at energies approaching $M_{\rm Pl}$ for more than one reason. The perturbative approach will have broken down, not to speak of the one loop approximation, and moreover gravitational effects could not be ignored. Our setting a precise boundary condition giving a common pole at a particular scale is therefore just expedient to describe a common pole that the present theoretical tools cannot properly describe.

As we said above, there would be four kinds of particles that switch on at four scales, and the boundary conditions at the intermediate scales impose three constraints. We require that the scales be between the TeV region and the ULP, and that the evolution is monotonous (the curves must not intersect themselves). One can see, however, that the only allowed order of masses of new particles is this one: $Quarkon_L$, $Quarkon_R$, $Leptos_L$, $Leptos_R$. If one tries to change it, e.g ordering: $Quarkon_L$, $Quarkon_R$, $Leptos_R$, $Leptos_L$, one gets the Fig. 6.4

Actually one has a system of three linear equations with four unknowns: masses of left and right quarkons, masses of left and right leptos. Putting one of them by hand we have a system of three equations for three variables. If one requires the switching on of the leptos to be at the same scale one finds solutions. On the contrary setting the quarkons at the same scale does not provide a solution. This enables us to reduce the number of parameters to three, with three equations, and therefore find a unique solution. Since the scale for the leptos must be larger than the one of quarkons and therefore closer to the Planck scale the possibility of splitting the two scale of the leptos give just a little uncertainty at very high energies.

We are also able to fix the number n of generations. For $n = 1, 2, 3$ we don't have enough particles to change signs of all beta functions. $n = 4$ is our case. When one has $n \geq 5$, there appears a region of metastability (lambda becomes negative), that we would not like to have - see Fig. 6.4

Nevertheless in order to determine the quartic coupling running, one loop approximation is not good enough, see Fig. 6.3 and the reason is relatively large ~ 1 value of the top Yukawa quark constant at low energies. In order to maintain the precision, we proceed as follows: below the first threshold,where the top Yukawa coupling is largest ~ 1 and perturbation theory in the scalar sector worst we will use *two loops approximation*. Above this scale Y_t is smaller so we will perform our computation at one loop level.

In the low energy i.e. Standard Model region one should solve the following

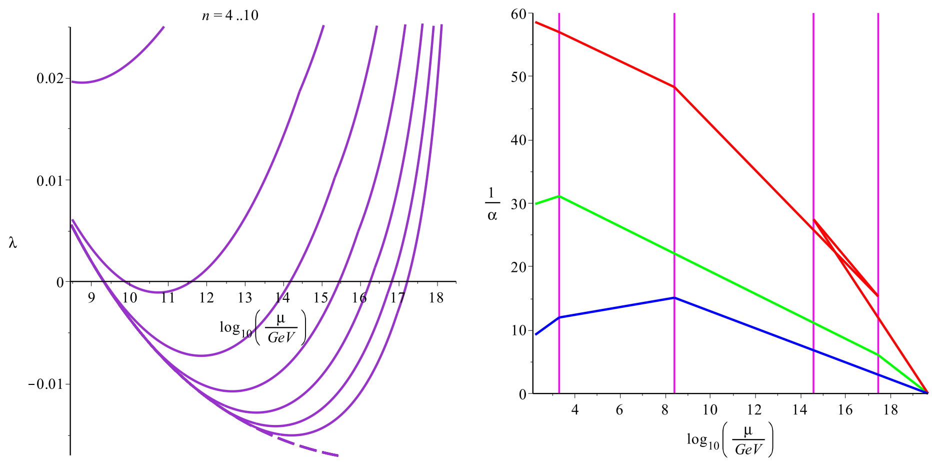

Figure 6.4: On the left: running of quatric coupling for different n. Only $n = 4$ does not exhibit the instability. On the right: an attempt to put the wrong order of the crossovers $Quarkon_L$, $Quarkon_R$, $Leptos_R$, $Leptos_L$ leads to selfintersections

system of (nonlinear) equations

$$\begin{cases} \frac{dX(t)}{dt} = \beta_X(X(t)), \quad X = \{g_1, g_2, g_3, y, \lambda\}, \quad t = \log\frac{\mu}{GeV} \\ t_0 = 172.9, \\ g_1(t_0) = 0.358729, \\ g_2(t_0) = 0.648382, \\ g_3(t_0) = 1.16471, \\ y(t_0) = 0.937982, \\ \lambda(t_0) = 0.125769 \end{cases} \tag{6.3}$$

where the initial conditions come out from experiment data and the beta functions are presented below. For the abelian gauge coupling g_1 the beta function is defined as follows [94]:

$$\begin{aligned} \beta_{g_1}^{(2)} &= \frac{1}{16\pi^2}\frac{41}{6}{g_1}^3 \\ &+\frac{1}{256\pi^4}{g_1}^3\left(\frac{199}{18}{g_1}^2+\frac{9}{2}{g_2}^2+\frac{44}{3}{g_3}^2-\frac{17}{6}y^2\right) \end{aligned} \tag{6.4}$$

For the $SU(2)$ gauge coupling g_2 the beta function reads [94]:

$$\begin{aligned} \beta_{g_2}^{(2)} &= \frac{1}{16\pi^2}\left(-\frac{19}{6}\right){g_2}^3 \\ &+\frac{1}{256\pi^4}{g_2}^3\left(\frac{3}{2}{g_1}^2+\frac{35}{6}{g_2}^2+12\,{g_3}^2-\frac{3}{2}y^2\right). \end{aligned} \tag{6.5}$$

For the strong coupling g_3 the corresponding beta function is given by [94]

$$\begin{aligned}
\beta^{(2)}_{g_3} &= \frac{1}{16\pi^2}(-7){g_3}^3 \\
&+\frac{1}{256\pi^4}\,{g_3}^3\left(\frac{11}{6}\,{g_1}^2+\frac{9}{2}\,{g_2}^2-26\,{g_3}^2-2\,y^2\right).
\end{aligned} \tag{6.6}$$

For the top quark's Yukawa coupling y we have [95]

$$\begin{aligned}
\beta^{(2)}_{y} &= \frac{1}{16\pi^2}\,y\left[-9/4\,{g_2}^2-\frac{17}{12}\,{g_1}^2-8\,{g_3}^2+9/2\,y^2\right] \\
&+\frac{1}{256\pi^4}\,y\left[-\frac{23}{4}\,{g_2}^4-3/4\,{g_2}^2{g_1}^2+\frac{1187}{216}\,{g_1}^4\right. \\
&+9\,{g_2}^2{g_3}^2+\frac{19}{9}\,{g_1}^2{g_3}^2-108\,{g_3}^4 \\
&+\left(\frac{225}{16}\,{g_2}^2+\frac{131}{16}\,{g_1}^2+36\,{g_3}^2\right)y^2 \\
&\left.-12\,y^4-12\,y^2\lambda+6\,\lambda^2\right].
\end{aligned} \tag{6.7}$$

For the quartic coupling λ the corresponding beta function reads [96]

$$\begin{aligned}
\beta^{(2)}_{\lambda} &= \frac{1}{16\pi^2}\left[24\,\lambda^2-6\,y^4+3/4\,{g_2}^4+3/8\,\left({g_2}^2+{g_1}^2\right)^2\right. \\
&\left.+\left(-9\,{g_2}^2-3\,{g_1}^2+12\,y^2\right)\lambda\right] \\
&+\frac{1}{256\pi^2}\left[\frac{305}{16}\,{g_2}^6-\frac{289}{48}\,{g_2}^4{g_1}^2-\frac{559}{48}\,{g_2}^2{g_1}^4-\frac{379}{48}\,{g_1}^6\right. \\
&+30\,y^6-y^4\left(8/3\,{g_1}^2+32\,{g_3}^2+3\,\lambda\right) \\
&+\lambda\left(-\frac{73}{8}\,{g_2}^4+\frac{39}{4}\,{g_2}^2{g_1}^2+\frac{629}{24}\,{g_1}^4\right. \\
&\left.+108\,{g_2}^2\lambda+36\,{g_1}^2\lambda-312\,\lambda^2\right) \\
&+y^2\left(-9/4\,{g_2}^4+21/2\,{g_2}^2{g_1}^2-\frac{19}{4}\,{g_1}^4\right. \\
&\left.\left.+\lambda\left\{\frac{45}{2}\,{g_2}^2+\frac{85}{6}\,{g_1}^2+80\,{g_3}^2-144\,\lambda\right\}\right)\right].
\end{aligned} \tag{6.8}$$

Solving numerically the nonlinear system (6.3) up to the first treshhold we generate initial conditions for the forthcoming computation, which is performed, as we said, with one loop precision.

6.5 The final result

Performing the computation, discussed in the previous section we arrive to the following result. New particles must be at the following scales:

- At 5.0 10^3 GeV the L-quarkons ($N_{\mathbf{L-quarkon}} = 4$).
- At 3.7 10^7 GeV the R-quarkons ($N_{\mathbf{R-quarkon}} = 4$).
- At 2.6 10^{14} GeV the L and R-leptos ($N_{\mathbf{L-leptos}} = N_{\mathbf{R-leptos}} = 4$).

On Fig. 6.5 we show the running of the gauge coupling. (the initial running shown is actually made with the two-loop equation). One can see that the hierarchy of the couplings is respected, the strong force remains stronger than the weak. The scale at which there is the appearance of the new particles is larger than the experimental bounds on the presence of new fermions, but not too much. This scenario shows that the ULP may exist with new physics at energies within reach. The top quark

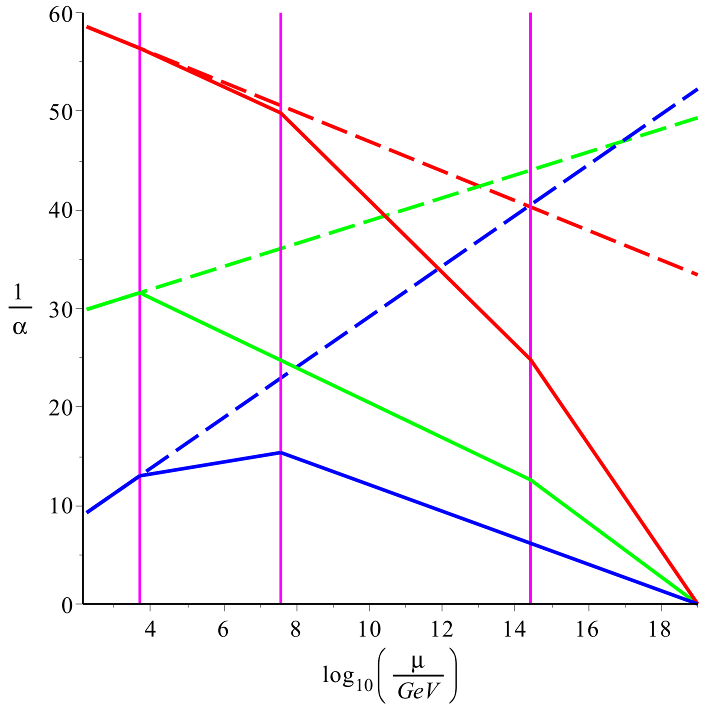

Figure 6.5: The running of α_i, the inverse of the gauge couplings. The dotted lines are the runnings in the absence of quarkons and leptos. The α_i are in descending order as i increases.

Yukawa coupling is undistinguishable from the standard model for energies up to 10^6 GeV, and vanishes at the ULP, see Fig. 6.6

The quartic coupling running is shown in Fig. 6.7. We see that the quartic coupling for our choice of new particles comes close to vanish, but never actually becomes negative. That means that the ULP generalization of the Standard model saves the world from the vacuum instability! In the next section we explain *why*

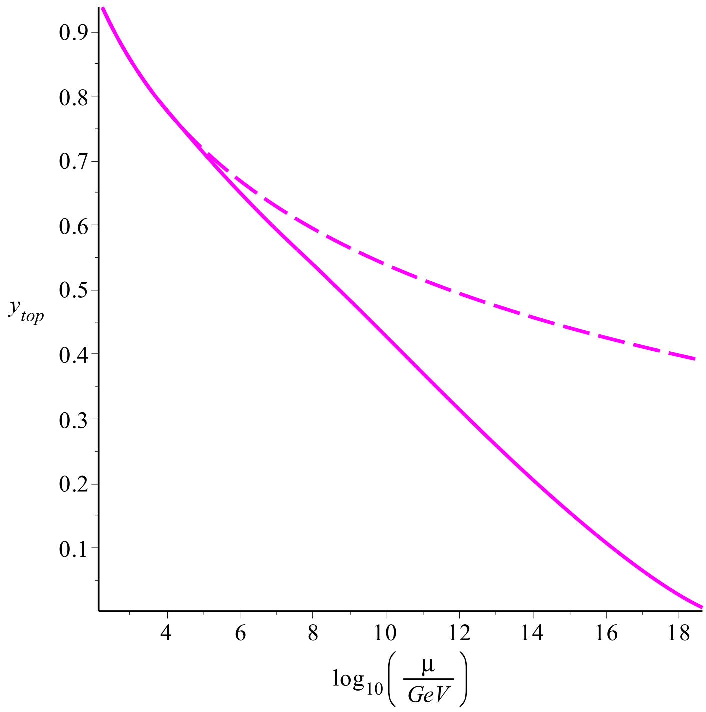

Figure 6.6: The running of y_t. The dotted line is the SM case.

ULP is a natural mechanism to resolve the instability problem

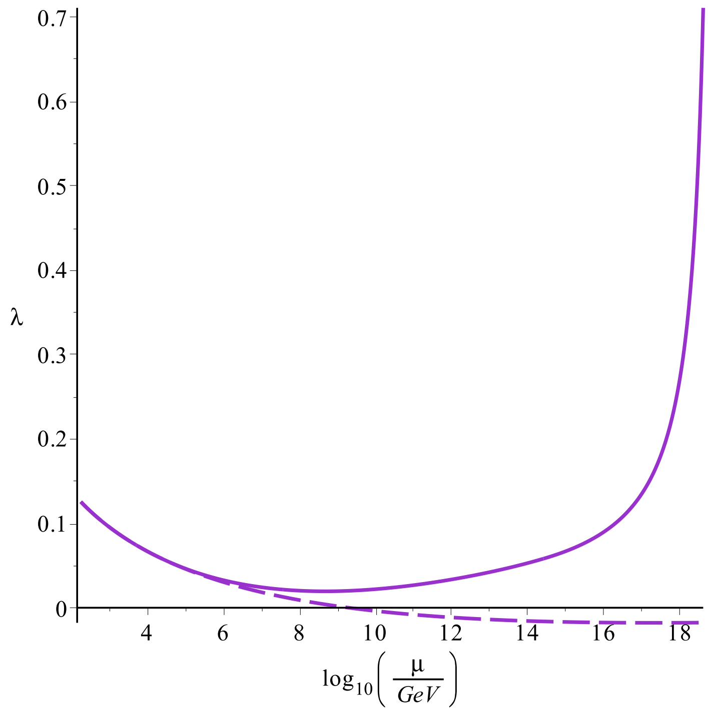

Figure 6.7: The running of quartic coupling of the Higgs field. The dotted line shows the instability that the standard model develops in the presence of a Higgs mass of 124-126 GeV.

6.6 Remark on the resolution of the vacuum instability problem.

Now we clarify how our vector-like fermions save the Universe from instability, i.e. how they don't let RG flow to drive the quartic coupling $\lambda(\mu)$ to negative values.

Let us trace at one loop level, how the quartic coupling becomes negative in the Standard Model. The one loop beta function is given by (6.9), and since the top quark Yukawa coupling is the biggest one at low energy region, we see, neglecting by other constants, that the whole beta function is negative.

$$\begin{aligned}\beta_\lambda^{(1)} &= \frac{1}{16\pi^2}\Big(24\,\lambda^2 - 6\,y^4 + \frac{3}{4}\,g_2^4 + \frac{3}{8}\,\left({g_2}^2 + {g_1}^2\right)^2 \\ &\quad + \left(-9\,{g_2}^2 - 3\,{g_1}^2 + 12\,y^2\right)\lambda\Big). \end{aligned} \tag{6.9}$$

As we have figured out after addition of new particles, the top Yukawa quark coupling decreases significantly faster with respect to the Standard Model case, see Fig. 6.6, correspondingly the contribution of the diagram, presented on Fig. 6.8 decreases with growth of energy. From another side, the contribution of bosonic

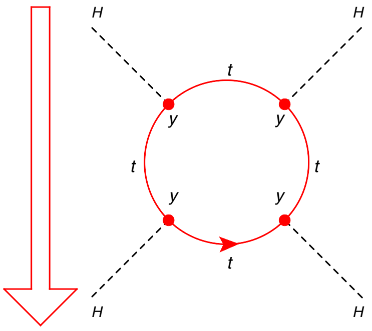

Figure 6.8: Feynman diagram, giving the biggest contribution in the one loop beta function of the quartic coupling. With growth of energy its contribution decreases.

loops increases, since *all* gauge couplings grow up. At some point bosonic dia-

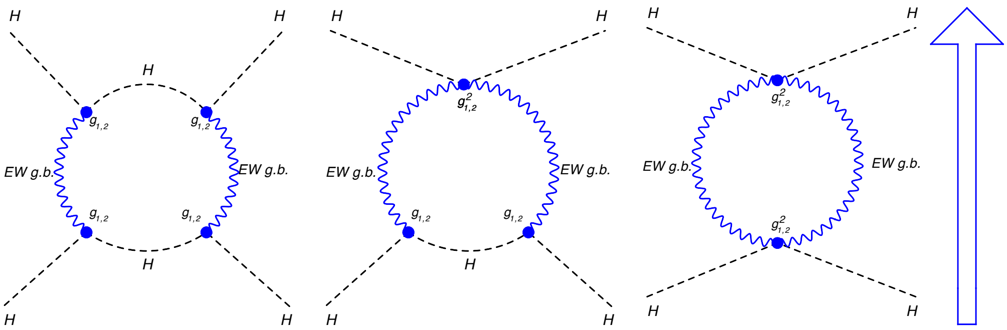

Figure 6.9: Feynman diagrams, giving the biggest contribution in ULP regime. Although their contribution is small at low energies, at high energies they dominate over fermionic loops.

grams start to dominate over the fermionic ones, curing the sign of beta function (6.9). Thats is how the ULP saves the world from the vacuum instability.

Chapter 7

Conclusions

In this thesis we studied several applications of the ultraviolet spectral regularisation in QFT to the bosonic spectral action, induced gravity and cosmology. Then we discussed some relevant physical models, requiring a presence of the ultraviolet cutoff. Now let us summarize, what has been done and what are future perspectives of this research.

First we have seen how the bosonic spectral action emerges form the fermionic action via (generalised) Weyl anomaly of the fermionic partition function under the spectral regularisation. This is valid considering the Standard Model as an effective field theory, valid for the energies smaller than a physical scale Λ. The procedure followed is spectral and therefore well suited for the noncommutative approach to the standard model. In such a setup the Weyl anomaly generating functional was expressed as a functional integral over an auxiliary dilaton field of a local action, and the latter comes out to be the Chamseddine Connes Bosonic Spectral Action introduced in the context of noncommutative geometry, coupled to the dilaton.

Another important result, that we obtained is related with a generalization of the spectral regularization on bosonic degrees of freedom. More precisely, imposing spectral regularization with the cutoff scale Λ in a classically Weyl invariant theory, we related Sakharov's induced gravity to the anomaly-induced effective action and thus obtained Starobinky's anomaly-induced inflation. We computed the anomaly and expressed the anomalous part of the quantum effective action through the quantized single collective scalar degree of freedom of all quantum vacuum fluctuations, dubbed the collective dilaton field ϕ, described by the local action Eq. (4.54). It is worth noticing that the condition of stability of the cosmological constant under Λ^4-corrections, namely $N_{\mathcal{H}} = 2\left(N_{\mathrm{F}}^{\mathrm{w}} - N_{\mathrm{V}}\right)$, appears naturally within our procedure. Our approach allowed us to treat the Sakharov's induced gravity on equal footing with the Starobinsky's anomaly-induced inflation, in a self-consistent way. More precisely, we found that

$$\begin{aligned} M_{\text{Pl}}^{2\ \mathbf{ind}} &= \frac{\Lambda^2}{12\pi}\left(N_{\text{F}}^{\mathbf{w}} - 4N_{\text{V}}\right) , \\ H_{\mathbf{inflat}} &\simeq 2\sqrt{\frac{N_{\text{F}}^{\mathbf{w}} - 4N_{\text{V}}}{N_{\text{F}} + 8N_{\text{V}}}} \cdot \Lambda \cdot \left(1 + O\left(\frac{\lambda}{\Lambda^4}\right)\right) . \end{aligned}$$

Provided the stability condition is satisfied, Sakharov's induced gravity and anomaly induced action leading to Starobinsky's anomaly-induced inflation appear simultaneously if $N_{\text{F}}^{\mathbf{w}} > 4N_{\text{V}}$. The fact that QFT gave rise to the Einstein-Hilbert action and the onset of an inflationary era, in the absence of an inflaton field, may indicate that the cosmological arrow of time results from quantum effects in a classically Weyl-invariant theory. In future research one should carefully elaborate the mass issue and also go beyond the isotropic approximation to realize how our approach feats the CMB data.

Then we discussed some models naturally exhibiting the transition scale in the ultraviolet. We explored the high momenta asymptotic of the bosonic spectral action. Using the covariant perturbation theory, invented by Barvinsky and Vilkovisky, which is suitable for studying of a high energy regime of Bosonic Spectral Action we found that high energy bosons do not propagate, that indicates the phase transition at the cutoff scale Λ. The fact, that kinetic terms quadratic in fields vanish at high momenta definitely looks fascinating, however it is also interesting to clarify what happens with cubic terms and higher: if they survive and we have some sort of highly nontrivially interacting fields that in no sense can be regarded as free, or the higher terms also vanish freezing all the dynamics completely. This is a still open question.

Finally we studied another model, naturally exhibiting the ultraviolet cutoff scale. Being motivated by presence of strong gravity at the Planck scale, we proposed a strong unification of all gauge interactions at the Planck scale, as an alternative to asymptotic freedom. The unification was achieved adding fermions with vector gauge couplings coming in multiplets and with hypercharges identical to those of the Standard Model. The presence of these particles also prevents the Higgs quartic coupling from becoming negative, thus avoiding the instability (or metastability) of the SM vacuum. The mechanism of the vacuum stabilization that we proposed, which is natural for the ULP model is interesting by itself and can be considered outside of the ULP context. One can rise a question, what are the restrictions on the vector-like fermionic multiplets needed to resolve the vacuum instability? How can we do it in a minimal way? These and other question we leave for future research, and at this point we declare

THE END.

A Appendix

Definitions

Riemann tensor

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\sigma \Gamma^{\mu}_{\nu\rho} - \partial_\rho \Gamma^{\mu}_{\nu\sigma} + \Gamma^{\lambda}_{\nu\rho}\Gamma^{\mu}_{\lambda\sigma} - \Gamma^{\lambda}_{\nu\sigma}\Gamma^{\mu}_{\lambda\rho} \tag{A.1}$$

Ricci tensor

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu} = \partial_\nu \Gamma^{\sigma}_{\mu\sigma} - \partial_\sigma \Gamma^{\sigma}_{\mu\nu} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma} \tag{A.2}$$

Scalar curvature

$$R = g^{\mu\nu}\left\{\partial_\nu \Gamma^{\sigma}_{\mu\sigma} - \partial_\sigma \Gamma^{\sigma}_{\mu\nu} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma}\right\} \tag{A.3}$$

Christoffel symbols, first of the second kind

$$\Gamma_{\mu,\nu\rho} \equiv \frac{1}{2}\left(\partial_\rho g_{\mu\nu} + \partial_\nu g_{\mu\rho} - \partial_\mu g_{\nu\rho}\right) \tag{A.4}$$

$$\Gamma^{\mu}_{\nu\rho} \equiv \frac{1}{2}\, g^{\mu\lambda}\left(\partial_\rho g_{\lambda\nu} + \partial_\nu g_{\lambda\rho} - \partial_\lambda g_{\nu\rho}\right) \tag{A.5}$$

Einstein-Hilbert action

First useful identity: derivation

$$\begin{aligned} 0 = \int d^4x\ \sqrt{g}\,\nabla_\mu A^\mu &= \int d^4x\ \sqrt{g}\left[\partial_\mu A^\mu + \Gamma^{\mu}_{\mu\lambda}A^\lambda\right] \\ &= \int d^4x \underbrace{\left[\sqrt{g}\,\Gamma^{\gamma}_{\gamma\mu} - \partial_\mu \sqrt{g}\right]}_{0} A^\mu \end{aligned} \tag{A.6}$$

First useful identity: result

$$\Gamma^{\gamma}_{\gamma\rho} = \partial_\rho \log\ \sqrt{g} \tag{A.7}$$

Second useful identity: derivation

$$0 = \nabla_\sigma g^{\mu\nu} = \partial_\sigma g^{\mu\nu} + \Gamma^\mu_{\sigma\xi}\, g^{\xi\nu} + \Gamma^\nu_{\sigma\xi}\, g^{\mu\xi} \tag{A.8}$$

Second useful identity: result

$$\partial_\sigma g^{\mu\nu} = -\left(\Gamma^\mu_{\sigma\xi}\, g^{\xi\nu} + \Gamma^\nu_{\sigma\xi}\, g^{\mu\xi}\right) \tag{A.9}$$

First intermediate step

$$-\partial_\sigma \{ \sqrt{g} g^{\mu\nu}\} \Gamma^\sigma_{\mu\nu} = -\sqrt{g}\left[\Gamma^\gamma_{\gamma\sigma}\Gamma^\sigma_{\mu\nu} - 2\Gamma^\sigma_{\gamma\nu}\Gamma^\gamma_{\sigma\mu}\right] g^{\mu\nu} \tag{A.10}$$

Second intermediate step

$$\Gamma^\sigma_{\sigma\mu}\partial_\nu \{ \sqrt{g} g^{\mu\nu}\} = -\sqrt{g}\,\Gamma^\gamma_{\mu\nu}\Gamma^\sigma_{\gamma\sigma}\, g^{\mu\nu} \tag{A.11}$$

Third intermediate step

$$\sqrt{g} R = \sqrt{g}\left\{\Gamma^\gamma_{\gamma\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\gamma\nu}\Gamma^\gamma_{\sigma\mu}\right\} g^{\mu\nu} - \partial_\sigma \left\{ \sqrt{g}\left[g^{\mu\nu}\Gamma^\sigma_{\mu\nu} - g^{\mu\sigma}\Gamma^\gamma_{\mu\gamma}\right]\right\} \tag{A.12}$$

Final result

$$\int d^4x\ \sqrt{g}\, R = \int d^4x\ \sqrt{g}\ \left(\Gamma^\gamma_{\gamma\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\gamma\nu}\Gamma^\gamma_{\sigma\mu}\right) g^{\mu\nu} \tag{A.13}$$

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