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Candidate
Luca De Angelis

Tutor
Dr. Franco Pezzella

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Contents

Introduction

Among the symmetries exhibited by string theory, T-duality plays a fundamental role in the stringy descriptions of both gauge interactions and gravity.

String theory arose in the late 1960s as an attempt to describe the hadrons, the strong interacting particles. Some problems emerged out that prevented this program to be completely successful. In the early 1970s quantum chromodynamics was developed. It was recognized as the 'correct' theory to deal with the strong nuclear force. Nevertheless, in the subsequent years, it was realized that string theory could describe the Universe in a deeper way. In fact such theory has the potential of unifying gravity with the other forces (weak, electromagnetic and color forces) and all forms of matter in a single mathemathical framework [1, 2].

In string theories, the fundamental objects are not point-like, as it happens in quantum field theories, but one-dimensional. The myriad of observed particles are identified as particular vibrational modes of microscopic strings. They are of two types: open strings and closed strings. Open strings are topologically equivalent to line intervals and so they have two endpoints, while closed strings are topologically equivalent to circles and they have no endpoints. In every string theory, closed strings are always present. This is because open strings can close to form the closed ones. In the massless spectrum of closed strings there is a spin-2 particle, which was proposed to be identified with the graviton, the quantum of gravitation. Since string theory is a quantum theory and it naturally includes gravitons, it constitutes one of the most promising candidates for a unified theory of the fundamental interactions of Nature as well as a prototype of a complete theory of Physics. String theory can, in principle, reconcile Einstein's General Relativity (a classical theory) and Quantum Mechanics, two of the pillars of the 20^{th} century theoretical physics, and so provide a consistent quantum theory of gravity.

In string theory there is a unique dimensionful parameter, the string length ℓ_s, defined by the string tension. It can be thought as the typical length of the fundamental objects. It is natural to expect the string scale to be of the same order of magnitude as the Planck length

$$l_P = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.6 \times 10^{-33}\ \text{cm}\,.$$

A broad subdivision of string theories is the one between bosonic string theories and superstring theories. In the mass spectrum of bosonic string theories there are only integer spin particles, the bosons. These kinds of theories are unrealistic because of the lack of fermions, which, instead, have half-integer spin values. On the contrary, superstring theories describe bosons and fermions. All the potentially realistic models are based on superstring theories, since in Nature all the matter particles are fermions while all the force carriers are bosons.

In string theories there is an interesting symmetry relating bosons to fermions: the so-called supersymmetry. It links and unifies matter and forces and requires a fermion of the same mass for every boson. In the recent years, the Large Hadron Collider (LHC) reached energies up to a few TeV (8 TeV on march 2012), but it has not been possible to observe either the supersymmetry or the 'stringy' nature of particles. It means that the typical string energy scale as well as the characteristic energy scale of supersymmetry breaking are above the lower bounds set experimentally until now.

Another intriguing feature of string theories is that their consistency fixes the number of dimensions of the space-time they live in. In particular, a calculation shows that bosonic string theories are consistent only in a 26-dimensional space-time, while superstring theories are in a 10-dimensional one. Under certain assumptions, an eleventh dimension is also possible (M-theory). In order to make contact with the everyday world of our experience, in which there is one time-dimension and three spatial dimensions, a straightforward solution is that the 'extra'-dimensions are curled up. In this picture, they form a compact space the typical size of which is so small to have escaped detection in experiments done at too low energies.

If string theory has to do with real world, then it must be possible to deduce from it the other theories that have been empirically tested and are well-known up to same rank of energies or distances. More precisely it should recover in a suitable low-energy limit the Standard Model of Particle Physics and General Relativity. The latter is naturally incorporated in the theory and gets modified only at very short distances/high energies.

The ambitious task of finding the Standard Model inside string theory constitutes the research branch known as string phenomenology. An important role for accomplishing this project is played by Dp-branes. A Dp-brane is a nonperturbative object behaving as a soliton in field theory with a tension per unit p-volume inversally proportional to the coupling constant g_s. One of its basic properties is to have a $(p+1)$-dimensional gauge theory living on its world-volume, since open strings have their endpoints on it. On a stack of N parallel Dp-branes, there are N^2 different kinds of open strings having endpoints on them. Hence a $U(N)$ gauge theory lives on their world-volume, being the massless open-string states correspondent to the gauge fields of $U(N)$. In order to get the Standard Model from string theory, intersecting branes are required, because the open strings attached to them provide chiral fermions in the four-dimensional space-time as expected for the fermions described in the Standard Model. The use of intersecting branes can be involved. It results to be much more convenient to deal with another kind of branes, the magnetized branes, which are connected to the intersecting ones by T-duality. This property relates the intersection angle of intersecting branes into a constant magnetic field characterizing parallel magnetized branes [3, 4].

T-duality is one of the exotic and peculiar features of string theory. For a recent review, see Ref. [5] and references therein. T-duality implies that in many cases different geometries for the extra dimensions are equivalent. In the presence of compactifications, new modes, besides the usual ones, emerge out in closed string theories: a closed string wraps around the compactified dimensions leading to the introduction of a new meaningful quantity, the winding number. In the simplest case of compactification, the so-called circle compactification, one of the spatial dimensions is curled up to form a circle of radius R. T-duality relates closed strings compactified on such circle to the ones compactified on a dual circle of radius proportional to $1/R$. In other words, T-duality is a clear indication that geometrical concepts can break down at the string scale. One is led to introduce a new kind of coordinate, the T-dual coordinate with respect to the compactified coordinate, to parametrize the position of the string along the dual circle. This new coordinate is linked to the winding number in the same way as the ordinary coordinate is linked to the momentum, meaning that they are conjugated variables. Of course, this procedure can be generalized to the case of an arbitrary number of compactified dimensions (up to 22 or 6, depending on the theory under study) and in the presence of an antisymmetric background field.

T-duality remains an exact symmetry of the mass spectrum of closed strings, but not of the action. Recently, many efforts have been done in order to construct a model which is manifestly T-duality invariant. In order to do that, it is necessary to introduce the dual coordinates at the level of the sigma-model action. The main goal of this new action would be to explore more closely the gravity implied by string theory. In fact, if interested in writing down the effective field theory of this generalized sigma-model, one should consider, correspondently to the introduction of the ordinary coordinates and their duals, a dependence on these *doubled* coordinates of the fields associated with string states. Indeed, the effective field theory of this formulation is a double field theory [6, 7, 8, 9, 10, 11, 12, 13]. In particular, this has to be true for the well-known effective gravitational action of a closed bosonic string that involves the fields associated with the massless states: the gravitational field G, the Kalb-Ramond B and the dilaton ϕ. So, it would be interesting to understand what this action becomes in light of the fact that all those fields are dependent on doubled coordinates, trying to shed light on string gravity, not yet explored. But, of course, before solving this very crucial question, a preliminary step is to achieve a formulation of closed bosonic string with T-duality made manifest in its sigma-model. This is the main aim of this work.

This thesis is structured as follows.

In Chapter 1 the standard formulation of string theory is recalled. The Nambu-Goto and the Polyakov actions are briefly discussed, together with their symmetries. The equations of motion and the boundary conditions to be satisfied by the string coordinates are given in the conformal gauge. In particular, attention is paid on the classical and quantum closed string theory. We provide the explicit expansions for the string coordinates, the constraints deriving from the energy-momentum tensor, the Virasoro algebra and generators, the Poisson brackets and the conditions to select the physical states.

In Chapter 2, in order to explore T-duality of closed strings, the effects of compactification of one or more spatial dimensions are analyzed in some detail. We start from the simplest case, the circle compactification. The modified topology of the target space implies new modes to appear (the windings). Moreover, one is naturally led to introduce a dual circle and a 'new' string coordinate along that circle. The mass formula and the level matching-condition for the string states get slightly modified with respect to the noncompact case and they can be written in a matricial form, by introducing two 2×2 matrices, so *doubling* the number of coordinates involved. This procedure can be generalized in two directions: it can be extended for $n > 1$ compact dimensions (toroidal compactification) forming a torus described by a nonorthogonal metric G and in the presence of an antisymmetric Kalb-Ramond field B. These are two of the usual massless background fields that one finds in the closed-string spectrum. As for the circle compactification, also for the compactification on an n-torus, n dual coordinates can be introduced and the mass formula and the level matching condition can be written through two $2n \times 2n$ matrices. One is a block-matrix, the blocks of which are constructed out of some combinations of G and B. The second one can be seen as the metric of the group $O(n, n; \mathbb{Z})$, which plays a fundamental role. The explicit expansions of the coordinates are given and then the Poisson brackets are computed both for the ordinary coordinates and their duals and for the left and right coordinates. It is worth noticing that all these coordinates behave like noncommuting variables.

In Chapter 3, in order to construct and develop the T-duality symmetric formulation of closed string theory, requiring the introduction in the action of the dual coordinates besides the usual ones, we analyze the dynamics of a free bidimensional scalar field defined on a flat backgroud. In fact, a single string coordinate, independently of the target space index it carries, behaves like a scalar field defined on the world-sheet. The (Hodge-)dual scalar field is introduced. The action/Lagrangian obtained is characterized by a doubled set of variables (the scalar field and its dual) and is not manifestly local Lorentz invariant. A suitable choice of basis (and so of coordinates) permits to write the Lagrangian of the system as a sum of a particular class of first order Lagrangians, the Floreanini-Jackiw ones. Since they describe constrained systems, the Dirac method of quantization is required. Dirac brackets and commutators are computed. The local Lorentz invariance is recovered on-shell.

In Chapter 4, the results of Chapter 3 are generalized. The dynamics of a scalar field defined on a curved background (as the string world-sheet) is studied. There are little differences from the flat case. One of the most important is the necessity of introducing a 2-bein in the sigma-model action to balance the lack of local Lorentz invariance. The symmetries of the action describing the scalar field and its dual are illustrated. They permit to gauge-fix the 2-bein to a flat form, so to recover the results of the previous Chapter.

In Chapter 5, the manifestly T-duality invariant formulation of closed string theory is finally introduced and studied in some detail. Starting from a generalized sigma-model action, one can derive the formulation including a doubled set of coordinates as well as two background fields: the metric G and the Kalb-Ramond field B. This formulation, inspired by A. A. Tseytlin, is not local Lorentz invariant. As in the scalar field case, a change of basis introduces new coordinates. Their dynamics are encoded in first order Lagrangians. One of the results of this work is that, after performing the Dirac quantization procedure, the doubled coordinates, as well as the chiral ones, behave like noncommuting phase-space variables. Nevertheless, their expansions in terms of modes generate the same Virasoro algebra as the one in the standard formulation. A section devoted to the covariant formulation, proposed by C. Hull, concludes the Chapter.

Five Appendices complete this work. In Appendix A notation is set and some useful identities are given. In Appendix B, the 2-bein and some of its properties are analyzed in some detail. Appendix C is the one in which Hodge-duals are studied. In Appendix D some relations between the background fields are demonstrated. Appendix E is devoted to the study of the first order Floreanini-Jackiw Lagrangians. Lagrangians with a finite number of degrees of freedom are discussed, then they are generalized to the case of an infinite number of degrees of freedom.

Chapter 1

String theory - Usual formulation

In this first Chapter, the basics of string theory are faced. In particular, the Nambu-Goto action is showed and its classical equivalence to the Polyakov one is demonstrated. The Polyakov action is more suitable for the development of the theory itself. The global and local symmetries of the Polyakov action are discussed. The local symmetries permit to gauge-fix the world-sheet metric to get the so-called conformal gauge. The equations of motion and the boundary conditions for the string coordinates are computed. The vanishing of the components of the energy-momentum tensor has to be kept as a constraint to be solved together with the equations of motion. Then the classical closed string theory is analyzed: the expansions for the string coordinates are given and, moreover, also the Poisson brackets and the Virasoro generators are computed. The quantization procedure is performed. Normal-ordering ambiguities are taken into account. Finally, the conditions to select the physical string states are illustrated.

1.1 On string theory

In analogy with the action describing a relativistic point particle moving in a curved space-time, the integrand of which is proportional to the invariant length of the world-line of the particle itself, the Nambu-Goto action, describing a string in a space-time with flat minkowskian metric $\eta_{\mu\nu}$ with mostly plus signature, is proportional to the area of the world-sheet swept by the string during its motion:

$$\begin{aligned} S_{NG} &= -T \int d^2\xi \sqrt{- \det(\partial_\alpha X \cdot \partial_\beta X)} \\ &= -T \int d^2\xi \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}\,, \end{aligned} \tag{1.1}$$

where, as usual, $\dot{X}^\mu \equiv \partial_0 X^\mu \equiv \partial X^\mu / \partial\tau$, $X'^\mu \equiv \partial_1 X^\mu \equiv \partial X^\mu / \partial\sigma$ and $A \cdot B$ stands for $\eta_{\mu\nu} A^\mu B^\nu$. The object $\gamma_{\alpha\beta} \equiv \partial_\alpha X \cdot \partial_\beta X$ is the so-called induced metric. A set of two (adimensional) coordinates $\xi^\alpha = (\xi^0, \xi^1)$, with $\xi^0 \equiv \tau$ (time-like) and

$\xi^1 \equiv \sigma$ (space-like), is introduced on the world-sheet to parametrize its points. We choose $-\infty < \tau < +\infty$ and $0 \leq \sigma \leq \pi$. Of course, $d^2\xi \equiv d\tau\, d\sigma$. The dimensionful constant T in front of the action is the *string tension*. It makes the action dimensionless ($[T] = L^{-2}$ and $[X^\mu] = L$). The string tension can also be expressed in terms of the *string length* $l_s \equiv \sqrt{2\alpha'}$:

$$T = \frac{1}{2\pi\alpha'} = \frac{1}{\pi l_s^2}\,. \tag{1.2}$$

The action (1.1) is invariant under reparametrizations of the world-sheet coordinates $\xi^\alpha \to \xi'^\alpha = \xi'^\alpha(\xi)$, but it is awkward to quantize because of the square root [1].

The Polyakov action is

$$S_\sigma[h_{\alpha\beta}; X^\mu] = -\frac{T}{2}\int d^2\xi\,\sqrt{-h}\,h^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu\,. \tag{1.3}$$

It is a sigma-model action classically equivalent to the Nambu-Goto one (they give the same equation of motion for all the fields X^μ, $\mu = 0, 1, \ldots, D-1$). We stressed that S_σ is a functional of $h_{\alpha\beta}$, the world-sheet metric, and of X^μ, the string coordinates on the target space. Moreover, $h \equiv \det(h_{\alpha\beta})$ and $h^{\alpha\beta} \equiv (h^{-1})_{\alpha\beta}$.

The lack of a kinetic term for the world-sheet metric $h_{\alpha\beta}$ has the consequence that the equations of motion for the metric itself are equivalent to the vanishing of the world-sheet energy-momentum tensor $T^{\alpha\beta}$, so defined

$$T^{\alpha\beta} \equiv \frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_\sigma}{\delta h_{\alpha\beta}}\,. \tag{1.4}$$

We will consider the covariant version of $T^{\alpha\beta}$:

$$T_{\alpha\beta} = h_{\alpha\gamma}h_{\beta\delta}T^{\gamma\delta}\,.$$

Since

$$\frac{\delta h_{\gamma\delta}}{\delta h^{\alpha\beta}} = -h_{\alpha\gamma}h_{\beta\delta}\,,$$

$T_{\alpha\beta}$ can be written also as

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_\sigma}{\delta h^{\alpha\beta}}\,,$$

constituting a definition for the covariant tensor itself.

By remembering that $\delta\sqrt{-h} = -(1/2)\sqrt{-h}\,h_{\alpha\beta}\delta h^{\alpha\beta}$, $T_{\alpha\beta}$ can be easily computed:

$$T_{\alpha\beta} = \partial_\alpha X\cdot\partial_\beta X - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X\cdot\partial_\delta X = 0\,. \tag{1.5}$$

In order to demonstrate the equivalence between S_{NG} and S_σ, the easiest way is to take the determinant of both sides of the equality

$$\partial_\alpha X \cdot \partial_\beta X = \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X \,,$$

implying, in a bidimensional space, as the one we are dealing with,

$$\det(\partial_\alpha X \cdot \partial_\beta X) = \frac{1}{4} \det(h_{\alpha\beta}) \, (h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X)^2$$

and

$$\sqrt{-\det(\partial_\alpha X \cdot \partial_\beta X)} = \frac{1}{2} \sqrt{-h} \, h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X \,.$$

Let us notice that the tensor $T_{\alpha\beta}$ is traceless:

$$\mathrm{Tr}(T_{\alpha\beta}) \equiv h^{\alpha\beta} T_{\alpha\beta} = h^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} h^{\alpha\beta} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma X \cdot \partial_\delta X = 0 \,,$$

being $h^{\alpha\beta} h_{\alpha\beta} \equiv \mathrm{Tr}(\delta^\alpha{}_\beta) = 2$ (again) in a bidimensional space.

The equation of motion and the boundary terms for the field X^μ are obtained by varying the action S_σ and by imposing this variation to vanish. They are, respectively,

$$\partial_\alpha (\sqrt{-h} \, h^{\alpha\beta} \partial_\beta X^\mu) = 0 \tag{1.6}$$

and

$$-T \int_0^\pi d\sigma \left[(\sqrt{-h} \, h^{0\beta} \partial_\beta X_\mu) \delta X^\mu \right]_{\tau=-\infty}^{\tau=+\infty} - T \int_{-\infty}^{+\infty} d\tau \left[(\sqrt{-h} \, h^{1\beta} \partial_\beta X_\mu) \delta X^\mu \right]_{\sigma=0}^{\sigma=\pi} .$$

Actually, the boundary terms reduce to

$$-T \int_{-\infty}^{+\infty} d\tau \left[(\sqrt{-h} \, h^{1\beta} \partial_\beta X_\mu) \delta X^\mu \right]_{\sigma=0}^{\sigma=\pi} \,, \tag{1.7}$$

because $\delta X^\mu(\tau = \pm\infty, \sigma) = 0$, $\forall \, \sigma$ and $\forall \, \mu$.

1.2 Global and local symmetries of S_σ

The Polyakov action (1.3) exhibits a series of global and local symmetries:

- invariance under (global) Poincaré transformations of the world-sheet fields

$$\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu \quad \text{and} \quad \delta h^{\alpha\beta} = 0 \,, \tag{1.8}$$

where $a_{\mu\nu} = -a_{\nu\mu}$. The constant parameters $a^\mu{}_\nu$ and b^μ represent, respectively, space-time rotations and translations. The corresponding variation of the action is

$$\begin{aligned}\delta S_\sigma &= -T \int d^2\xi \sqrt{-h}\, h^{\alpha\beta} \eta_{\mu\nu} a^\mu{}_\rho \partial_\alpha X^\rho \partial_\beta X^\nu \\ &= -T \int d^2\xi \sqrt{-h}\, h^{\alpha\beta} a_{\nu\rho} \partial_\alpha X^\rho \partial_\beta X^\nu = 0\end{aligned}$$

being the integrand equal to the product of a symmetric tensor and an antisymmetric one;

- invariance under (local) reparametrizations of the world-sheet coordinates:

$$\xi^\alpha \to \xi'^\alpha = \xi'^\alpha(\xi) \quad \text{and} \quad h_{\alpha\beta} = \frac{\partial \xi'^\gamma}{\partial \xi^\alpha} \frac{\partial \xi'^\delta}{\partial \xi^\beta} h'_{\gamma\delta}\,. \tag{1.9}$$

As it is well-known, these kinds of reparametrizations generate the following transformations:

$$\begin{aligned} d^2\xi &\;\to\; d^2\xi'\, J \\ h \to J^{-2} h' \quad &\Rightarrow\quad \sqrt{-h} \to J^{-1}\sqrt{-h'}\,, \end{aligned}$$

where $J \equiv \det(\partial\xi^\alpha/\partial\xi'^\beta)$ is the Jacobian of the transformation. In this way $d^2\xi\sqrt{-h} \to d^2\xi'\sqrt{-h'}$, hence the name of 'invariant measure' for the quantity $d^2\xi\sqrt{-h}$. Moreover,

$$h^{\alpha\beta} = \frac{\partial \xi^\alpha}{\partial \xi'^\gamma} \frac{\partial \xi^\beta}{\partial \xi'^\delta} h'^{\gamma\delta} \quad \text{and} \quad \frac{\partial}{\partial \xi^\alpha} = \frac{\partial \xi'^\beta}{\partial \xi^\alpha} \frac{\partial}{\partial \xi'^\beta}$$

with $h^{\alpha\beta}\partial_\alpha X \cdot \partial_\beta X$ left invariant;

- invariance under Weyl transformations. They affect the world-sheet fields as follows:

$$h_{\alpha\beta} \to e^{\phi(\tau,\sigma)} h_{\alpha\beta} \quad \text{and} \quad \delta X^\mu = 0\,. \tag{1.10}$$

The first of (1.10) implies $h \to e^{2\phi}h$ and, consequently, $\sqrt{-h} \to e^\phi\sqrt{-h}$ and $h^{\alpha\beta} \to e^{-\phi}h^{\alpha\beta}$. So they make the quantity $\sqrt{-h}\, h^{\alpha\beta}$ invariant. It is a peculiarity of bidimensional spaces as it can be easily verified. In fact, in a generic N-dimensional space, a Weyl transformation of the type $h_{\alpha\beta} \to \Lambda(\tau,\sigma)h_{\alpha\beta}$ generates

$$h \to \Lambda^N h \quad \Rightarrow \quad \left\{ \begin{array}{ccc} \sqrt{-h} & \to & \Lambda^{N/2}\sqrt{-h} \\ h^{\alpha\beta} & \to & \Lambda^{-1} h^{\alpha\beta} \end{array} \right.$$

and so

$$\sqrt{-h}\, h^{\alpha\beta} \quad \to \quad \Lambda^{\frac{N-2}{2}} \sqrt{-h}\, h^{\alpha\beta}\,,$$

confirming the invariance only for $N = 2$.

The (three) local symmetries allow to choose a particular form for the world-sheet metric $h_{\alpha\beta}$, which has three independent entries ($h_{01} = h_{10}$). This procedure is also known as the *gauge-fixing*. Hereafter in this Chapter the *conformal gauge* $h_{\alpha\beta} = \eta_{ab}$ is performed. It is worth noticing that, after the gauge-fixing procedure, there are still other residual gauge transformations. These are called conformal transformations, a mixture of Weyl rescalings and reparametrizations.

1.3 Theory in the conformal gauge

In the conformal gauge, the Polyakov action assumes the form

$$
\begin{aligned}
S_\sigma[X^\mu] &= -\frac{T}{2}\int d^2\xi\, \eta^{ab}\, \eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu \\
&= \frac{T}{2}\int d^2\xi\, \left[(\partial_0 X)^2 - (\partial_1 X)^2\right] \\
&= \frac{T}{2}\int d^2\xi\, (\dot{X}^2 - X'^2)\,.
\end{aligned}
\tag{1.11}
$$

So, the Lagrangian density $\mathcal{L}$ is

$$
\mathcal{L} = \frac{T}{2}(\dot{X}^2 - X'^2)\,. \tag{1.12}
$$

The energy-momentum tensor $T_{\alpha\beta}$ becomes

$$
\begin{aligned}
T_{ab} &\equiv T_{\alpha\beta}|_{h_{\alpha\beta}=\eta_{ab}} \\
&= \partial_a X \cdot \partial_b X + \frac{1}{2}\eta_{ab}(\dot{X}^2 - X'^2)
\end{aligned}
\tag{1.13}
$$

and its components are easily computed:

$$
\begin{aligned}
T_{00} &= T_{11} = (1/2)(\dot{X}^2 + X'^2) \\
T_{01} &= T_{10} = \dot{X}\cdot X'\,.
\end{aligned}
$$

The condition of the vanishing of the trace becomes $\eta^{ab}T_{ab} = -T_{00} + T_{11} = 0$. Once one has gauge-fixed the metric, the vanishing of T_{ab} has to be kept as an additional constraint to be solved together with the equations of motion for the string coordinates X^μ. In the conformal gauge, the equation of motion (1.6) is

$$
\partial_a(\eta^{ab}\partial_b X^\mu) = \eta^{ab}\partial_a\partial_b X^\mu = \partial_a\partial^a X^\mu = 0\,, \tag{1.14}
$$

while the boundary term (1.7) is

$$
-T\int_{-\infty}^{+\infty} d\tau\, \left[\partial_1 X_\mu \delta X^\mu\right]_{\sigma=0}^{\sigma=\pi}\,. \tag{1.15}
$$

Let us introduce a new set of world-sheet coordinates, the *light-cone* coordinates: $\sigma^{\pm} \equiv \tau \pm \sigma$. The relations between the derivatives with respect the old and the new coordinates are

$$\left\{ \begin{array}{l} \partial_0 = \partial_+ + \partial_- \\ \partial_1 = \partial_+ - \partial_- \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \partial_+ = (1/2)(\partial_0 + \partial_1) \\ \partial_- = (1/2)(\partial_0 - \partial_1) \end{array} \right. , \tag{1.16}$$

while, in the light-cone basis, the metric and its inverse become, respectively

$$\begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{-+} & \eta_{--} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \eta^{++} & \eta^{+-} \\ \eta^{-+} & \eta^{--} \end{pmatrix} = -2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Moreover, the equation of motion can be written as $\partial_+\partial_- X^\mu = 0$ and it is classically solved by writing the solution as a linear combination of one left-moving and one right-moving wave: $X^\mu(\tau, \sigma) = c_1 X_L^\mu(\tau + \sigma) + c_2 X_R^\mu(\tau - \sigma)$, with $c_1,\ c_2 \in R$.

By using the law of tensor transformation under a change of coordinates, we can write the components of the energy-momentum tensor in the light-cone basis:

$$\begin{aligned} T_{++} &= \frac{1}{4}(T_{00} + T_{01} + T_{10} + T_{11}) \;=\; \frac{1}{2}(T_{00} + T_{01}) \\ &= \frac{1}{4}(\dot{X} + X')^2 \;=\; \partial_+ X \cdot \partial_+ X \end{aligned} \tag{1.17}$$

$$\begin{aligned} T_{+-} &= \frac{1}{4}(T_{00} - T_{01} + T_{10} - T_{11}) \;=\; \frac{1}{4}(T_{00} - T_{11}) \\ &= -\frac{1}{4}\mathrm{Tr}(T_{\alpha\beta}) \;=\; 0 \end{aligned} \tag{1.18}$$

$$T_{-+} = \frac{1}{4}(T_{00} + T_{01} - T_{10} - T_{11}) \;=\; \frac{1}{4}(T_{00} - T_{11}) \;=\; T_{+-} \tag{1.19}$$

$$\begin{aligned} T_{--} &= \frac{1}{4}(T_{00} - T_{01} - T_{10} + T_{11}) \;=\; \frac{1}{2}(T_{00} - T_{01}) \\ &= \frac{1}{4}(\dot{X} - X')^2 \;=\; \partial_- X \cdot \partial_- X\,. \end{aligned} \tag{1.20}$$

The components of the tensor, which are not automatically vanishing, are $T_{\pm\pm} = \partial_\pm X \cdot \partial_\pm X$.

1.4 Classical closed string theory

In the conformal gauge, the equation of motion for the string coordinates is $\partial_a \partial^a X^\mu = 0$. For a closed string, which is the one we are treating, the boundary term vanishes by putting

$$X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma) \,. \tag{1.21}$$

The equation of motion for the fields X^μ and the periodicity condition just quoted constitute a sufficient condition to ensure the stationarity of (1.11). We want here to observe that the most general closed-string boundary condition which let the boundary term vanish is a quasi-periodicity condition. It will be used in the next Chapters, where the theory in the presence of compactified spatial dimensions will be analyzed. As we already stressed at the end of the previous section, before using eq. (1.21), the general solution of the equation of motion is $X^\mu(\tau,\sigma) = c_1 X_L^\mu(\tau+\sigma) + c_2 X_R^\mu(\tau-\sigma)$. Because of the arbitrariness in the choice of c_1 and c_2, we can write the string coordinate as

$$X^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}} \left[X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma) \right] \,,$$

through a redefinition of X_L^μ and X_R^μ. On this kind of solutions, the components of the energy-momentum tensor are:

$$\begin{aligned} T_{++} &= \frac{1}{2}\partial_+ X_L \cdot \partial_+ X_L = \frac{1}{2}(\partial_+ X_L)^2 && (1.22) \\ T_{--} &= \frac{1}{2}\partial_- X_R \cdot \partial_- X_R = \frac{1}{2}(\partial_- X_R)^2 \,. && (1.23) \end{aligned}$$

In order to find the explicit form of the functions $X_{L,R}$, we have to write the periodicity condition (1.21) in terms of these fields. It takes the form

$$X_L^\mu(\sigma^+ + \pi) - X_L^\mu(\sigma^+) = X_R^\mu(\sigma^-) - X_R^\mu(\sigma^- - \pi) \,. \tag{1.24}$$

By deriving the last equality once with respect to σ^+ and once with respect to σ^- (being them independent variables), we find that

$$\partial_+ X_L^\mu(\sigma^+ + \pi) = \partial_+ X_L^\mu(\sigma^+)$$

and that

$$\partial_- X_R^\mu(\sigma^- - \pi) = \partial_- X_R^\mu(\sigma^-) \,.$$

In other words, these derivatives are periodic functions with period π, and hence, they have the following Fourier expansions:

$$\begin{aligned} \partial_+ X_L^\mu &= 2\sqrt{\alpha'} \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n^\mu e^{-2in\sigma^+} && (1.25) \\ \partial_- X_R^\mu &= 2\sqrt{\alpha'} \sum_{n\in\mathbb{Z}} \alpha_n^\mu e^{-2in\sigma^-} && (1.26) \end{aligned}$$

(this choice of normalization will be clearified later). The coefficients of the expansions $\tilde{\alpha}_n^\mu$ and α_n^μ will be interpreted as the string oscillation modes.

By integrating over $\sigma^{\pm}$ the equations (1.25 - 1.26), we get

$$X_L^\mu(\sigma^+) = x_L^\mu + 2\sqrt{\alpha'}\tilde{\alpha}_0^\mu\sigma^+ + i\sqrt{\alpha'}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in\sigma^+} \tag{1.27}$$

$$X_R^\mu(\sigma^-) = x_R^\mu + 2\sqrt{\alpha'}\alpha_0^\mu\sigma^- + i\sqrt{\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in\sigma^-}\,, \tag{1.28}$$

where x_L^μ and x_R^μ emerge as integration constants. (Let us observe that the fields $X_{L,R}$ are quasi-periodic.)

The periodicity condition $X^\mu(\tau,\sigma+\pi) = X^\mu(\tau,\sigma)$ imposes $\tilde{\alpha}_0^\mu = \alpha_0^\mu$. It means that in the X^μ expansion there is no linear term in σ and that, once quantized the theory, there will be only one momentum operator as well as one coordinate for the center-of-mass.

So the expansion for the field X^μ is

$$\begin{aligned}
X^\mu(\tau,\sigma) &= \frac{1}{\sqrt{2}}\left[X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma)\right] \\
&= \frac{1}{\sqrt{2}}(x_L^\mu + x_R^\mu) + \sqrt{2\alpha'}(\tilde{\alpha}_0^\mu + \alpha_0^\mu)\tau + \sqrt{2\alpha'}(\tilde{\alpha}_0^\mu - \alpha_0^\mu)\sigma \\
&\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^\mu e^{-2in\sigma} + \alpha_n^\mu e^{+2in\sigma}] \\
&= \frac{1}{\sqrt{2}}(x_L^\mu + x_R^\mu) + \sqrt{2\alpha'}(\tilde{\alpha}_0^\mu + \alpha_0^\mu)\tau \\
&\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^\mu e^{-2in\sigma} + \alpha_n^\mu e^{+2in\sigma}]\,.
\end{aligned}$$

The momentum density is, by definition,

$$\mathcal{P}_\mu \equiv \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu} = T\dot{X}_\mu\,. \tag{1.29}$$

We will use its contravariant version:

$$\mathcal{P}^\mu = T\left\{\sqrt{2\alpha'}(\tilde{\alpha}_0^\mu + \alpha_0^\mu) + \sqrt{2\alpha'}\sum_{n\neq 0}e^{-2in\tau}[\tilde{\alpha}_n^\mu e^{-2in\sigma} + \alpha_n^\mu e^{+2in\sigma}]\right\}. \tag{1.30}$$

The total momentum, which a priori is a function of τ, turns out to be

$$P^\mu(\tau) = \int_0^\pi d\sigma\mathcal{P}^\mu(\tau,\sigma) = T\sqrt{2\alpha'}(\tilde{\alpha}_0^\mu + \alpha_0^\mu)\pi = \frac{1}{\sqrt{2\alpha'}}(\tilde{\alpha}_0^\mu + \alpha_0^\mu) \equiv p^\mu\,, \tag{1.31}$$

and it coincides with the constant momentum of the center-of-mass of the string. The last equality in eq. (1.31) implies $\tilde{\alpha}_0^\mu = \alpha_0^\mu = (\sqrt{\alpha'/2})p^\mu$, slightly generalized into $p_L^\mu \equiv (1/\sqrt{\alpha'})\tilde{\alpha}_0^\mu = p_R^\mu \equiv (1/\sqrt{\alpha'})\alpha_0^\mu = (1/\sqrt{2})p^\mu$. Moreover, it is convenient to impose $x_L^\mu = x_R^\mu = (1/\sqrt{2})x^\mu$.

With these definitions, the expansion for the field X^μ is

$$X^\mu(\tau,\sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^\mu e^{-2in\sigma} + \alpha_n^\mu e^{+2in\sigma}]\,. \tag{1.32}$$

Requiring the fields X_L^μ, X_R^μ and their sum X^μ to be real, implies that x^μ and p^μ are real, while, for the oscillator modes, $\tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^*$ and $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$, where $*$ denotes complex conjugation.

Let us now calculate $\dot{X}^\mu$ and X'^μ:

$$\dot{X}^\mu = \sqrt{2\alpha'}\left(\sum_{n\in\mathbb{Z}}\tilde{\alpha}_n^\mu e^{-2in\sigma^+} + \sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-2in\sigma^-}\right) = \frac{1}{\sqrt{2}}\left(\partial_+ X_L^\mu + \partial_- X_R^\mu\right) \tag{1.33}$$

$$X'^\mu = \sqrt{2\alpha'}\left(\sum_{n\in\mathbb{Z}}\tilde{\alpha}_n^\mu e^{-2in\sigma^+} - \sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-2in\sigma^-}\right) = \frac{1}{\sqrt{2}}\left(\partial_+ X_L^\mu - \partial_- X_R^\mu\right) \tag{1.34}$$

equivalent to

$$\begin{aligned}\dot{X}^\mu + X'^\mu &= \sqrt{2}\partial_+ X_L^\mu \\ \dot{X}^\mu - X'^\mu &= \sqrt{2}\partial_- X_R^\mu\,.\end{aligned}$$

The Hamiltonian density (not written in terms of the Hamiltonian variables X and $\mathcal{P}$) is

$$\mathcal{H} \equiv \mathcal{P}_\mu \dot{X}^\mu - \mathcal{L} = T\dot{X}^2 - \frac{T}{2}(\dot{X}^2 - X'^2) = \frac{T}{2}(\dot{X}^2 + X'^2)\,, \tag{1.35}$$

while the Hamiltonian is

$$H = \int_0^\pi d\sigma \mathcal{H} = \frac{T}{2}\int_0^\pi d\sigma(\dot{X}^2 + X'^2)\,. \tag{1.36}$$

By inserting the mode expansion of the X fields in eq. (1.36), we get

$$H = \sum_{n\in\mathbb{Z}}(\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n + \alpha_{-n}\cdot\alpha_n)\,. \tag{1.37}$$

1.4.1 Some Poisson brackets

This section is devoted to the study of the Poisson brackets among the dynamical variables in the theory. In analogy with the Poisson brackets holding for generalized coordinates and their conjugate momenta in classical mechanics, we define

$$\{X^\mu(\tau,\sigma), \mathcal{P}^\nu(\tau,\sigma')\}_{PB} \equiv \eta^{\mu\nu}\delta(\sigma - \sigma') \tag{1.38}$$

$$\{X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')\}_{PB} = \{\mathcal{P}^\mu(\tau,\sigma), \mathcal{P}^\nu(\tau,\sigma')\}_{PB} \equiv 0\,. \tag{1.39}$$

The insertion of the mode expansion of the field X^μ in (1.38 - 1.39) gives

$$\{\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n\}_{PB} = \{\alpha^\mu_m, \alpha^\nu_n\}_{PB} = -i\, m\, \delta_{m+n,0}\, \eta^{\mu\nu} \quad \forall\, m, n \tag{1.40}$$

as well as

$$\begin{aligned} \{\tilde{\alpha}^\mu_m, \alpha^\nu_n\}_{PB} &= 0 \\ \{x^\mu, p^\nu\}_{PB} &= \eta^{\mu\nu} \\ \{x^\mu, \tilde{\alpha}^\nu_n\}_{PB} = \{x^\mu, \alpha^\nu_n\}_{PB} &= 0\,, \quad n \neq 0\,. \end{aligned}$$

In the derivation of these brackets we used the Fourier expansion of the Dirac δ-function:

$$\delta(x) = \frac{1}{\pi} \sum_{n\in\mathbb{Z}} e^{2inx}\,. \tag{1.41}$$

1.4.2 The mass formula and the Virasoro algebra

On-shell, the vanishing of the components of the energy-momentum tensor (1.22 - 1.23) can be translated into the vanishing of their Fourier coefficients:

$$\begin{aligned} T_{++} &= \frac{1}{2}\partial_+ X_L \cdot \partial_+ X_L = 4\alpha' \sum_{m\in\mathbb{Z}} \tilde{L}_m e^{-2im\sigma^+} = 0 \\ T_{--} &= \frac{1}{2}\partial_- X_R \cdot \partial_- X_R = 4\alpha' \sum_{m\in\mathbb{Z}} L_m e^{-2im\sigma^-} = 0\,. \end{aligned}$$

The coefficients are given by the following integrals valued at $\tau = 0$, being T_{++} and T_{--} quantities conserved in τ [14]:

$$\tilde{L}_m = \frac{T}{2}\int_0^\pi d\sigma\, e^{+2im\sigma}\, T_{++} = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n = 0 \tag{1.42}$$

$$L_m = \frac{T}{2}\int_0^\pi d\sigma\, e^{-2im\sigma}\, T_{--} = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\cdot\alpha_n = 0\,. \tag{1.43}$$

In particular, for $m = 0$, we have

$$\tilde{L}_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n = 0 \tag{1.44}$$

$$L_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\alpha_n = 0\,, \tag{1.45}$$

and, by comparing with (1.37), we get $H = 2(\tilde{L}_0 + L_0)$.

An object with total momentum p^μ has a relativistic mass given by $M^2 = -p_\mu p^\mu = -p^2$. Classically, one has

$$\tilde{L}_0 = \frac{1}{2}\sum_{n\neq 0}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n + \frac{1}{2}\tilde{\alpha}_0^2 = \sum_{n\in\mathbb{N}}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n + \frac{\alpha'}{4}p^2 = \tilde{N} + \frac{\alpha'}{4}p^2 = 0$$

$$L_0 = \frac{1}{2}\sum_{n\neq 0}\alpha_{-n}\cdot\alpha_n + \frac{1}{2}\alpha_0^2 = \sum_{n\in\mathbb{N}}\alpha_{-n}\cdot\alpha_n + \frac{\alpha'}{4}p^2 = N + \frac{\alpha'}{4}p^2 = 0\,,$$

where $\tilde{N} \equiv \sum_{n\in\mathbb{N}}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n$ and $N \equiv \sum_{n\in\mathbb{N}}\alpha_{-n}\cdot\alpha_n$, in the quantum version of the theory, are the so-called *number operators*, because they have integer eigenvalues. The sum of the last two equations leads to

$$M^2 = -p^2 = \frac{2}{\alpha'}\left(\sum_{n\in\mathbb{N}}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n + \sum_{n\in\mathbb{N}}\alpha_{-n}\cdot\alpha_n\right) = \frac{2}{\alpha'}(\tilde{N}+N)\,. \tag{1.46}$$

The Poisson brackets between L_m and L_n, defined in eq. (1.43), are

$$\{L_m, L_n\}_{PB} = -i\,(m-n)\,L_{m+n}\,, \tag{1.47}$$

where we have used the identity

$$\{AB, CD\} = A\{B,C\}D + \{A,C\}BD + CA\{B,D\} + C\{A,D\}B\,.$$

Eq. (1.47) represents the *Virasoro algebra*, from which the interpretation of the L_m's as the generators of such algebra. The considerations done for L_m of course hold also for $\tilde{L}_m$.

Almost the totality of the results so far collected will be slightly modified in the quantum theory.

1.5 Quantum closed string theory

In this section, we will translate the results obtained in the previous sections in a quantum context. To this aim, the Poisson brackets have to be substituted by the commutators between operators (which, with abuse of notations, will be denoted by the same symbols used for the classical quantities they described) acting on a suitable Hilbert space, and all the operators will be "normal-ordered".

The prescription

$$\{\cdot\,,\cdot\}_{PB} \to -i[\cdot\,,\cdot]\,, \tag{1.48}$$

yields

$$[X^\mu(\tau,\sigma), \mathcal{P}^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{1.49}$$

$$[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = [\mathcal{P}^\mu(\tau,\sigma), \mathcal{P}^\nu(\tau,\sigma')] = 0 \tag{1.50}$$

$$[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = [\alpha^\mu_m, \alpha^\nu_n] = m\,\delta_{m+n,0}\,\eta^{\mu\nu} \quad \forall\, m, n \tag{1.51}$$

and

$$\begin{aligned} [\tilde{\alpha}^\mu_m, \alpha^\nu_n] &= 0 \\ [x^\mu, p^\nu] &= i\eta^{\mu\nu} \\ [x^\mu, \tilde{\alpha}^\nu_n] = [x^\mu, \alpha^\nu_n] &= 0\,, \quad n \neq 0\,. \end{aligned}$$

The ambiguity in the normal-ordered operators emerges out only into $\tilde{L}_0$ and L_0, where the product of two zero-modes appears. The critical string theory is obtained by replacing $\tilde{L}_0$ and L_0 with the quantities $\tilde{L}_0 - a$ and $L_0 - a$, being $a = 1$ in a 26-dimensional space-time. This implies for (1.47) (after some calculations)

$$[L_m, L_n] = (m-n)\,L_{m+n} + \frac{c}{12} m(m^2-1)\,\delta_{m+n,0}\,, \tag{1.52}$$

where the term proportional to c (the *central charge*) is a quantum effect.

The presence of a afflicts the mass and the Hamiltonian, too. In fact, the identities $\tilde{L}_0 - 1 = 0$ and $L_0 - 1 = 0$ impose the following modifications:

$$M^2 = \frac{2}{\alpha'}(\tilde{N} + N) \quad \to \quad M^2 = \frac{2}{\alpha'}(\tilde{N} + N - 2) \tag{1.53}$$

and

$$H = 2(\tilde{L}_0 + L_0) \quad \to \quad H = 2(\tilde{L}_0 + L_0 - 2)\,. \tag{1.54}$$

In the quantum theory, we have to implement new conditions to select the physical states, denoted by $|\phi\rangle$. They have to be annihilated by the operators $\tilde{L}_0 - 1$ and $L_0 - 1$. The equations $(\tilde{L}_0 - 1)|\phi\rangle = (L_0 - 1)|\phi\rangle = 0$ can be translated into

$$(L_0 + \tilde{L}_0 - 2)|\phi\rangle = 0 \tag{1.55}$$

and the so-called *level-matching condition*, linking the left and right sectors:

$$(L_0 - \tilde{L}_0)|\phi\rangle = 0 \quad \Rightarrow \quad N = \tilde{N}\,. \tag{1.56}$$

From eqs. (1.53 - 1.56) one can deduce the closed string spectrum. In particular, at the first two mass levels, one gets

- the ground state $|0;k\rangle$, eigenstate of the operator p^μ associated with the eigenvalue k^μ and obtained for $N = \tilde{N} = 0$. It is a tachyon with $\alpha' M^2 = -4$;
- a set of states having the form $|\Omega^{\mu\nu}\rangle = \alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|0;k\rangle$ obtained for $N = \tilde{N} = 1$. They are massless, being $\alpha' M^2 = 0$. The symmetric and traceless part of $|\Omega^{\mu\nu}\rangle$ is interpreted as the *graviton*, a massless spin-2 particle, the trace term is a massless scalar named *dilaton*, while the antisymmetric part corresponds to the so-called *Kalb-Ramond field*.

Chapter 2

Compactification in string theory

The aim of this second Chapter is to describe in some detail what is meant by compactification in string theory. We will start from the simplest case, the compactification on a circle of radius R, to arrive at the compactification on an n-torus in the presence of a B-field, focussing on the intermediate steps. In the circle compactification, the basic ingredients as the winding number and the dual coordinate are introduced. The presence of a 'new' coordinate constitutes a signal that the correct *arena* to analyze such theory should have doubled spatial dimensions. The same idea holds also in the case of toroidal compactification. As in the previous Chapter, the mass formula and the level-matching conditions are computed. They exhibit a manifest $O(n,n;\mathbb{Z})$ invariance. The Poisson brackets are also constructed. In particular, although the ordinary coordinates and their duals live in completely different spaces and the left and right 'auxiliary variables' have not the role of true coordinates, their Poisson brackets show that they behave like noncommuting phase-space variables.

2.1 Circle compactification

Let us consider the case of a closed string compactified on a circle: the target space is now factorized into $\mathbb{R}^{1,24} \times S^1$. R is the radius of the circle. It will be clear in a moment that, under proper assumptions, such theory is equivalent to the one compactified on a circle of radius $\tilde{R} \equiv \alpha'/R$.

We want to underline that the component of the metric on the 25^{th} direction (the one compactified) is $\eta_{(25)(25)} = 1$, and so objects like the action, the energy-momentum tensor and so on, do not change at all with respect to the ones computed in Chapter 1. In the next section, the same theory will be studied with a straightforward generalization of the concept of "metric" on such a compactified dimension.

In the presence of compactification, the periodicity condition introduced to satisfy the boundary term for a closed string (1.21) gets modified into a quasi-periodicity condition:

$$X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma) + 2\pi R W^{25}\,, \tag{2.1}$$

where $W^{25} \in Z$ is the so-called *winding number*. Its meaning is the number of times the closed string wraps around the compact coordinate and in which direction.

The equation of motion for the field X^{25} is $\partial_+\partial_- X^{25} = 0$ and it is solved by putting $X^{25}(\tau,\sigma) = (1/\sqrt{2})\,[X_L^{25}(\tau+\sigma) + X_R^{25}(\tau-\sigma)]$. The insertion of the last equality in the quasi-periodicity condition (2.1) this time leads to

$$X_L^{25}(\sigma^+ + \pi) - X_L^{25}(\sigma^+) = X_R^{25}(\sigma^-) - X_R^{25}(\sigma^- - \pi) + 2\sqrt{2}\pi R W^{25}\,. \tag{2.2}$$

Eq. (2.2), as in the noncompact case, leads to the periodicity of the light-cone derivatives:

$$\partial_+ X_L^{25}(\sigma^+ + \pi) = \partial_+ X_L^{25}(\sigma^+)$$

$$\partial_- X_R^{25}(\sigma^- - \pi) = \partial_- X_R^{25}(\sigma^-)\,.$$

This means that the expansions of the $X_{L,R}^{25}$ fields are the same as (1.27 - 1.28):

$$X_L^{25}(\sigma^+) = x_L^{25} + 2\sqrt{\alpha'}\tilde{\alpha}_0^{25}\sigma^+ + i\sqrt{\alpha'}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^{25} e^{-2in\sigma^+} \tag{2.3}$$

$$X_R^{25}(\sigma^-) = x_R^{25} + 2\sqrt{\alpha'}\alpha_0^{25}\sigma^- + i\sqrt{\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25} e^{-2in\sigma^-}\,, \tag{2.4}$$

but $\sqrt{2\alpha'}(\tilde{\alpha}_0^{25} - \alpha_0^{25}) = 2RW^{25}$. The fact that $\tilde{\alpha}_0^{25} \neq \alpha_0^{25}$ has a significative consequence: in the corresponding quantum theory, there will be two kinds of momenta and two independent coordinates for the left and right sectors: $x_L^{25} \neq x_R^{25}$.

The field X^{25} has the following expansion:

$$\begin{aligned} X^{25}(\tau,\sigma) &= \frac{1}{\sqrt{2}}\left(x_L^{25} + x_R^{25}\right) + \sqrt{2\alpha'}(\tilde{\alpha}_0^{25} + \alpha_0^{25})\tau + \sqrt{2\alpha'}(\tilde{\alpha}_0^{25} - \alpha_0^{25})\sigma \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^{25}e^{-2in\sigma} + \alpha_n^{25}e^{+2in\sigma}]\,. \end{aligned}$$

As in the noncompact case, the momentum density is

$$\mathcal{P}_{25} = \mathcal{P}^{25} = T\dot{X}^{25} \tag{2.5}$$

and the total momentum is

$$P^{25} = \int_0^\pi d\sigma \mathcal{P}^{25} = \frac{1}{\sqrt{2\alpha'}}(\tilde{\alpha}_0^{25} + \alpha_0^{25}) \equiv p^{25}\,. \tag{2.6}$$

In analogy with the center-of-mass momentum p^{25} just defined, we are naturally led to introduce another object which is similar to a momentum (they have the same physical dimensions), the *winding*:

$$w^{25} \equiv \frac{1}{\sqrt{2\alpha'}}(\tilde{\alpha}_0^{25} - \alpha_0^{25})\,. \tag{2.7}$$

It is easy to verify that $w^{25} = RW^{25}/\alpha'$.

The compactification along a circle of radius R implies the string momentum to be quantized, so that $p_{25} = K_{25}/R$, with $K_{25} \in Z$. The integer number K_{25} is known as a *Kaluza-Klein excitation.* As we already said, in this theory x_L^{25} and x_R^{25} are independent variables and we write them in the following form:

$$\begin{aligned} x_L^{25} &= \frac{1}{\sqrt{2}}(x^{25} + \tilde{x}^{25}) \\ x_R^{25} &= \frac{1}{\sqrt{2}}(x^{25} - \tilde{x}^{25})\,. \end{aligned}$$

So we get for X^{25}

$$\begin{aligned} X^{25}(\tau,\sigma) &= x^{25} + 2\alpha' p^{25}\tau + 2\alpha' w^{25}\sigma \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^{25}e^{-2in\sigma} + \alpha_n^{25}e^{+2in\sigma}] \\ &= x^{25} + 2\alpha'\frac{K^{25}}{R}\tau + 2RW^{25}\sigma \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^{25}e^{-2in\sigma} + \alpha_n^{25}e^{+2in\sigma}]\,. \end{aligned} \tag{2.8}$$

For later convenience, it is useful to write the zero-modes in terms of the Kaluza-Klein excitations and the winding numbers. To this aim, let us recall that

$$\begin{aligned} \sqrt{2\alpha'}(\tilde{\alpha}_0^{25} + \alpha_0^{25}) &= 2\alpha' p_{25} \\ \sqrt{2\alpha'}(\tilde{\alpha}_0^{25} - \alpha_0^{25}) &= 2\alpha' w^{25} \end{aligned}$$

which are equivalent to

$$\begin{aligned} \sqrt{2\alpha'}(\tilde{\alpha}_0^{25} + \alpha_0^{25}) &= 2\alpha'\frac{K_{25}}{R} \\ \sqrt{2\alpha'}(\tilde{\alpha}_0^{25} - \alpha_0^{25}) &= 2RW^{25}\,. \end{aligned}$$

The last couple of equations are 'solved' by

$$\sqrt{2\alpha'}\tilde{\alpha}_0^{25} = \alpha'\left(\frac{K_{25}}{R} + \frac{RW^{25}}{\alpha'}\right) \tag{2.9}$$

$$\sqrt{2\alpha'}\alpha_0^{25} = \alpha'\left(\frac{K_{25}}{R} - \frac{RW^{25}}{\alpha'}\right) \tag{2.10}$$

The main difference with the noncompact version of the theory is that the mass receives contributions only from the noncompact dimensions. So

$$M^2 = -\sum_{\mu=0}^{24} p_\mu p^\mu \,. \tag{2.11}$$

On the other hand, in the quantum theory, the Virasoro generators remain the same and the condition of their vanishing still holds:

$$\begin{aligned}
\tilde{L}_0 - 1 &= \frac{1}{2}\sum_{n\neq 0}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n + \frac{1}{2}\tilde{\alpha}_0^2 - 1 \\
&= \sum_{n\in\mathbb{N}}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n + \frac{1}{2}\sum_{\mu,\nu=0}^{24}\eta_{\mu\nu}\tilde{\alpha}_0^\mu\tilde{\alpha}_0^\nu + \frac{1}{2}(\tilde{\alpha}_0^{25})^2 - 1 \\
&= \tilde{N} + \frac{\alpha'}{4}\sum_{\mu=0}^{24} p_\mu p^\mu + \frac{1}{2}(\tilde{\alpha}_0^{25})^2 - 1 = 0
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
L_0 - 1 &= \frac{1}{2}\sum_{n\neq 0}\alpha_{-n}\cdot\alpha_n + \frac{1}{2}\alpha_0^2 - 1 \\
&= \sum_{n\in\mathbb{N}}\alpha_{-n}\cdot\alpha_n + \frac{1}{2}\sum_{\mu,\nu=0}^{24}\eta_{\mu\nu}\alpha_0^\mu\alpha_0^\nu + \frac{1}{2}(\alpha_0^{25})^2 - 1 \\
&= N + \frac{\alpha'}{4}\sum_{\mu=0}^{24} p_\mu p^\mu + \frac{1}{2}(\alpha_0^{25})^2 - 1 = 0\,.
\end{aligned} \tag{2.13}$$

These equations imply

$$\begin{aligned}
\alpha' M^2 &= 2\tilde{N} + 2N - 4 + \frac{1}{2}[(\tilde{\alpha}_0^{25})^2 + (\alpha_0^{25})^2] \\
&= 2\tilde{N} + 2N - 4 + \alpha'\left[\left(\frac{K_{25}}{R}\right)^2 + \left(\frac{RW^{25}}{\alpha'}\right)^2\right]\,.
\end{aligned} \tag{2.14}$$

The quantity defined as

$$\begin{aligned}
\alpha' M_0^2 &\equiv \frac{1}{2}[(\tilde{\alpha}_0^{25})^2 + (\alpha_0^{25})^2] \\
&= \alpha'\left[\left(\frac{K_{25}}{R}\right)^2 + \left(\frac{RW^{25}}{\alpha'}\right)^2\right]
\end{aligned} \tag{2.15}$$

will play a fundamental role in what follows.

It is worth noticing that the mass spectrum is invariant under the following simultaneous transformations:

$$W^{25} \leftrightarrow K_{25} \quad \text{and} \quad R \leftrightarrow \alpha'/R\,, \tag{2.16}$$

representing the simplest example of a *T-duality transformation.*

The level-matching condition is now

$$(L_0 - \tilde{L}_0)|\phi\rangle = 0 \quad \Rightarrow \quad N - \tilde{N} = \frac{1}{2}[(\tilde{\alpha}_0^{25})^2 - (\alpha_0^{25})^2] = W^{25}K_{25}\,. \tag{2.17}$$

The difference with (1.56) is evident. It is due to the different topology of the space-time in which the periodicity condition is formulated.

What happens if we perform the transformations $W^{25} \leftrightarrow K_{25}$ and $R \leftrightarrow \alpha'/R$ on the zero-modes? They change as follows:

$$\tilde{\alpha}_0^{25} \rightarrow \tilde{\alpha}_0^{25} \tag{2.18}$$
$$\alpha_0^{25} \rightarrow -\alpha_0^{25}\,. \tag{2.19}$$

If we promote this kind of behaviour to the other modes,

$$\begin{aligned} \tilde{\alpha}_n^{25} &\rightarrow \tilde{\alpha}_n^{25} \\ \alpha_n^{25} &\rightarrow -\alpha_n^{25}\,, \end{aligned}$$

we recognize that the couple of fields $X_{L,R}^{25}$ transform as

$$\begin{aligned} X_L^{25} &\rightarrow X_L^{25} \\ X_R^{25} &\rightarrow -X_R^{25}\,. \end{aligned}$$

It implies that we can define another kind of field: $\tilde{X}^{25}$, the *T-dual* of X^{25}. It is given by

$$\begin{aligned} \tilde{X}^{25}(\tau,\sigma) &\equiv \frac{1}{\sqrt{2}}\left[X_L^{25}(\tau+\sigma) - X_R^{25}(\tau-\sigma)\right] \\ &= \frac{1}{\sqrt{2}}\left(x_L^{25} - x_R^{25}\right) + \sqrt{2\alpha'}(\tilde{\alpha}_0^{25} - \alpha_0^{25})\tau + \sqrt{2\alpha'}(\tilde{\alpha}_0^{25} + \alpha_0^{25})\sigma \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^{25}e^{-2in\sigma} - \alpha_n^{25}e^{+2in\sigma}] \\ &= \tilde{x}^{25} + 2\alpha' w^{25}\tau + 2\alpha' p^{25}\sigma \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^{25}e^{-2in\sigma} - \alpha_n^{25}e^{+2in\sigma}] \\ &= \tilde{x}^{25} + 2RW^{25}\tau + 2\alpha'\frac{K^{25}}{R}\sigma \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^{25}e^{-2in\sigma} - \alpha_n^{25}e^{+2in\sigma}]\,. \end{aligned} \tag{2.20}$$

From the last equation, it appears that the winding w^{25} has the role of a momentum (the quantity multiplied by τ in the expansion) for $\tilde{X}^{25}$.

2.1.1 Poisson brackets

The presence of the new momentum w^{25}, of $\tilde{x}^{25}$ and of the entire T-dual field $\tilde{X}^{25}$ itself, changes the Poisson brackets (and the commutators). In fact, besides the ones holding for the noncompact dimensions, we have:

$$\{X^{25}(\tau,\sigma),\mathcal{P}^{25}(\tau,\sigma')\}_{PB}=\delta(\sigma-\sigma') \tag{2.21}$$

$$\{X^{25}(\tau,\sigma),X^{25}(\tau,\sigma')\}_{PB}=\{\mathcal{P}^{25}(\tau,\sigma),\mathcal{P}^{25}(\tau,\sigma')\}_{PB}=0 \tag{2.22}$$

$$\{\tilde{\alpha}^{25}_m,\tilde{\alpha}^{25}_n\}_{PB}=\{\alpha^{25}_m,\alpha^{25}_n\}_{PB}=-i\,m\,\delta_{m+n,0}\quad\forall\, m,n \tag{2.23}$$

$$\begin{aligned}\{\tilde{\alpha}^{25}_m,\alpha^{25}_n\}_{PB}&=0\\ \{x^{25},p^{25}\}_{PB}&=1\\ \{x^{25},\tilde{\alpha}^{25}_n\}_{PB}=\{x^{25},\alpha^{25}_n\}_{PB}&=0\,,\quad n\neq 0\,,\end{aligned}$$

as well as

$$\{x^{25},w^{25}\}_{PB}=0\,.$$

Even if X^{25} is usually used as the proper coordinate, nevertheless it is useful to calculate the Poisson brackets relative to the fields $X_{L,R}$ for reasons that will be clear later on. They are:

$$\{X^{25}_L(\tau+\sigma),X^{25}_L(\tau+\sigma')\}_{PB}=-2\pi\alpha'\epsilon(\sigma-\sigma') \tag{2.24}$$

$$\{X^{25}_R(\tau-\sigma),X^{25}_R(\tau-\sigma')\}_{PB}=2\pi\alpha'\epsilon(\sigma-\sigma') \tag{2.25}$$

and

$$\{X^{25}_L(\tau+\sigma),X^{25}_R(\tau-\sigma')\}_{PB}=0\,.$$

It is worth stressing that the fields $X^{25}_{L,R}$ behave like noncommuting variables.

In the previous calculations, we used

$$\{x^{25}_L,\tilde{\alpha}^{25}_0\}_{PB}=\{x^{25}_R,\alpha^{25}_0\}_{PB}=\sqrt{\alpha'}\,,$$

equivalent to

$$\{x^{25}_L,p^{25}_L\}_{PB}=\{x^{25}_R,p^{25}_R\}_{PB}=1\,,$$

being $p^{25}_L=(1/\sqrt{\alpha'})\tilde{\alpha}^{25}_0$ and $p^{25}_R=(1/\sqrt{\alpha'})\alpha^{25}_0$ (the other Poisson brackets are vanishing), and the following expression of the ϵ-function:

$$\epsilon(x)=\frac{2x}{2\pi}-\frac{i}{2\pi}\sum_{n\neq 0}\frac{1}{n}e^{2inx} \tag{2.26}$$

(see eq. (A.9)).

The surprising aspect is that the Poisson brackets of the fields X^{25} and $\tilde{X}^{25}$ are the same as the ones computed for $X^{25}_{L,R}$:

$$\{X^{25}(\tau,\sigma),\tilde{X}^{25}(\tau,\sigma')\}_{PB} = -2\pi\alpha'\epsilon(\sigma-\sigma')\,. \tag{2.27}$$

In fact, due to definitions (and omitting the dependence on the variables),

$$\begin{aligned}\{X^{25},\tilde{X}^{25}\}_{PB} &= (1/2)\{X^{25}_L,X^{25}_L\}_{PB} - (1/2)\{X^{25}_R,X^{25}_R\}_{PB}\\ &= \{X^{25}_L,X^{25}_L\}_{PB}\,.\end{aligned}$$

In order to be consistent with the result in eq. (2.27), obtained through an undirect calculation, we have to admit that

$$\begin{aligned}\{x^{25},\tilde{x}^{25}\}_{PB} &= 0\\ \{\tilde{x}^{25},p^{25}\}_{PB} &= 0\\ \{\tilde{x}^{25},w^{25}\}_{PB} &= 1\\ \{\tilde{x}^{25},\tilde{\alpha}^{25}_n\}_{PB} = \{\tilde{x}^{25},\alpha^{25}_n\}_{PB} &= 0\,,\quad n\neq 0\,.\end{aligned}$$

In analogy with the momentum density (2.5), we define $\tilde{\mathcal{P}}^{25} \equiv T(\dot{\tilde{X}})^{25}$:

$$\tilde{\mathcal{P}}^{25} = T\left\{\sqrt{2\alpha'}(\tilde{\alpha}^{25}_0-\alpha^{25}_0)+\sqrt{2\alpha'}\sum_{n\neq 0}e^{-2in\tau}[\tilde{\alpha}^{25}_n e^{-2in\sigma}-\alpha^{25}_n e^{+2in\sigma}]\right\}. \tag{2.28}$$

With this new dynamical variable, we can construct the following Poisson brackets:

$$\{\tilde{X}^{25}(\tau,\sigma),\tilde{\mathcal{P}}^{25}(\tau,\sigma')\}_{PB} = \delta(\sigma-\sigma') \tag{2.29}$$

$$\{\tilde{X}^{25}(\tau,\sigma),\tilde{X}^{25}(\tau,\sigma')\}_{PB} = \{\tilde{\mathcal{P}}^{25}(\tau,\sigma),\tilde{\mathcal{P}}^{25}(\tau,\sigma')\}_{PB} = 0\,, \tag{2.30}$$

in complete agreement with the previous analogous brackets involving X^{25} and $\mathcal{P}^{25}$.

In conclusion of this section, let us observe that (see (1.33 - 1.34) and (2.20))

$$\left\{\begin{array}{rcl}(\dot{\tilde{X}})^{25} &=& X'^{25}\\ (\tilde{X}')^{25} &=& \dot{X}^{25}\end{array}\right. \quad\Leftrightarrow\quad \left\{\begin{array}{rcl}\partial_0\tilde{X}^{25} &=& \partial_1 X^{25}\\ \partial_1\tilde{X}^{25} &=& \partial_0 X^{25}\end{array}\right., \tag{2.31}$$

summed up in

$$\partial_a\tilde{X}^{25} = -\epsilon_{ab}\,\partial^b X^{25}\,, \tag{2.32}$$

which is the typical relation holding for Hodge-dual forms defined on a bidimensional flat space (see eq. (C.8)).

2.2 More on circle compactification

In this section we will deal again with the compactification on a single dimension, writing down the analogue of the results so far obtained, but trying to analyze the role of the component of the metric along the compactified dimension, that will be denoted by $G_{(25)(25)}$.

In order to fix the ideas, the target space can be thought as constituted of the product of a minkowskian part $\mathbb{R}^{1,24}$ and the 1-torus $T^1 = S^1$. Strictly speaking, the Polyakov action, the Lagrangian density and other variables get modified. As an example, the Lagrangian density is

$$\mathcal{L} = \frac{T}{2}\left[\sum_{\mu,\nu=0}^{24} \eta_{\mu\nu}(\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu) + G_{(25)(25)}(\dot{X}^{25}\dot{X}^{25} - X'^{25}X'^{25})\right]$$

and the line element on the target space is constructed as follows:

$$ds^2 = \sum_{\mu,\nu=0}^{24} \eta_{\mu\nu} dX^\mu dX^\nu + G_{(25)(25)} dX^{25} dX^{25} \,.$$

As we will see at the end of this section, $G_{(25)(25)}$ must be equal to R^2/α' (to be dimensionless and) to be consistent with the previous section.

The quasi-periodicity condition (2.1) gets slightly modified in the constant term:

$$X^{25}(\tau, \sigma + \pi) = X^{25}(\tau, \sigma) + 2\pi\sqrt{\alpha'}W^{25} \,. \tag{2.33}$$

The mode expansions for the fields $X^{25}_{L,R}$ remain the same as (2.3 - 2.4). The momentum density is

$$\mathcal{P}_{25} \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^{25}} = T G_{(25)(25)} \dot{X}^{25} \,, \tag{2.34}$$

while the momentum itself is

$$P_{25} = \int_0^\pi d\sigma \mathcal{P}_{25} = \frac{1}{\sqrt{2\alpha'}} G_{(25)(25)}(\tilde{\alpha}_0^{25} + \alpha_0^{25}) \equiv p_{25} \,. \tag{2.35}$$

Eq. (2.33) suggests the following new definition for the winding: $w^{25} \equiv W^{25}/\sqrt{\alpha'}$. It is a more convenient and symmetrical way to define it, because of the similarity with $p_{25} = K_{25}/\sqrt{\alpha'}$ due to the quantization of the momentum. In conclusion, the expansion for the field X^{25} is

$$\begin{aligned} X^{25}(\tau,\sigma) &= x^{25} + 2\alpha' G^{(25)(25)} p_{25}\tau + 2\alpha' w^{25}\sigma \\ &\quad + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^{25} e^{-2in\sigma} + \alpha_n^{25} e^{+2in\sigma}] \\ &= x^{25} + 2\sqrt{\alpha'} G^{(25)(25)} K_{25}\tau + 2\sqrt{\alpha'} W^{25}\sigma \\ &\quad + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^{25} e^{-2in\sigma} + \alpha_n^{25} e^{+2in\sigma}] \,, \end{aligned} \tag{2.36}$$

where $G^{(25)(25)}$ represents the inverse of $G_{(25)(25)}$.

The zero-modes expressed in terms of W^{25} and K_{25} are obtained from

$$\begin{aligned}\sqrt{2\alpha'}(\tilde{\alpha}_0^{25}+\alpha_0^{25}) &= 2\sqrt{\alpha'}G^{(25)(25)}K_{25}\\ \sqrt{2\alpha'}(\tilde{\alpha}_0^{25}-\alpha_0^{25}) &= 2\sqrt{\alpha'}W^{25}\end{aligned}$$

implying

$$\begin{aligned}\sqrt{2\alpha'}\tilde{\alpha}_0^{25} &= \sqrt{\alpha'}(G^{(25)(25)}K_{25}+W^{25})\\ \sqrt{2\alpha'}\alpha_0^{25} &= \sqrt{\alpha'}(G^{(25)(25)}K_{25}-W^{25})\end{aligned}$$

or, by eliminating the common factor $\sqrt{\alpha'}$,

$$\sqrt{2}\tilde{\alpha}_0^{25} = G^{(25)(25)}K_{25}+W^{25} \tag{2.37}$$

$$\sqrt{2}\alpha_0^{25} = G^{(25)(25)}K_{25}-W^{25}\,. \tag{2.38}$$

The conditions of vanishing Virasoro generators ($\tilde{L}_0-1=L_0-1=0$) are

$$\begin{aligned}\tilde{L}_0-1 &= \frac{1}{2}\sum_{n\neq 0}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n+\frac{1}{2}\tilde{\alpha}_0^2-1\\ &= \sum_{n\in\mathbb{N}}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n+\frac{1}{2}\sum_{\mu,\nu=0}^{24}\eta_{\mu\nu}\tilde{\alpha}_0^{\mu}\tilde{\alpha}_0^{\nu}+\frac{1}{2}G_{(25)(25)}(\tilde{\alpha}_0^{25})^2-1\\ &= \tilde{N}+\frac{\alpha'}{4}\sum_{\mu=0}^{24}p_\mu p^\mu+\frac{1}{2}G_{(25)(25)}(\tilde{\alpha}_0^{25})^2-1=0 \qquad (2.39)\\ L_0-1 &= \frac{1}{2}\sum_{n\neq 0}\alpha_{-n}\cdot\alpha_n+\frac{1}{2}\alpha_0^2-1\\ &= \sum_{n\in\mathbb{N}}\alpha_{-n}\cdot\alpha_n+\frac{1}{2}\sum_{\mu,\nu=0}^{24}\eta_{\mu\nu}\alpha_0^{\mu}\alpha_0^{\nu}+\frac{1}{2}G_{(25)(25)}(\alpha_0^{25})^2-1\\ &= N+\frac{\alpha'}{4}\sum_{\mu=0}^{24}p_\mu p^\mu+\frac{1}{2}G_{(25)(25)}(\alpha_0^{25})^2-1=0\,. \qquad (2.40)\end{aligned}$$

The mass-squared reads

$$\alpha' M^2 = 2\tilde{N}+2N-4+G_{(25)(25)}[(\tilde{\alpha}_0^{25})^2+(\alpha_0^{25})^2]\,, \tag{2.41}$$

while the quantity $\alpha' M_0^2$ is

$$\begin{aligned}\alpha' M_0^2 &\equiv G_{(25)(25)}[(\tilde{\alpha}_0^{25})^2+(\alpha_0^{25})^2]\\ &= G_{(25)(25)}[(G^{(25)(25)}K_{25})^2+(W^{25})^2]\\ &= W^{25}\,G_{(25)(25)}\,W^{25}+K_{25}\,G^{(25)(25)}\,K_{25}\,.\end{aligned} \tag{2.42}$$

It is evident that, in this case, the mass spectrum is invariant under the T-duality transformations

$$W^{25} \leftrightarrow K_{25} \quad \text{and} \quad G_{(25)(25)} \leftrightarrow G^{(25)(25)} \,. \tag{2.43}$$

The level-matching condition is now

$$(L_0 - \tilde{L}_0)|\phi\rangle = 0 \Rightarrow N - \tilde{N} = \frac{1}{2} G_{(25)(25)}[(\tilde{\alpha}_0^{25})^2 - (\alpha_0^{25})^2] = W^{25} K_{25} \,. \tag{2.44}$$

It is very interesting to observe that the mass spectrum and the level-matching condition can be written by introducing a couple of 2×2 matrices (so doubling the number of coordinates involved):

$$\alpha' M_0^2 = (W^{25}\, K_{25})\, \mathcal{G} \begin{pmatrix} W^{25} \\ K_{25} \end{pmatrix} , \tag{2.45}$$

where

$$\mathcal{G} \equiv \begin{pmatrix} G_{(25)(25)} & 0 \\ 0 & G^{(25)(25)} \end{pmatrix} \tag{2.46}$$

and

$$N - \tilde{N} = \frac{1}{2} (W^{25}\, K_{25}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W^{25} \\ K_{25} \end{pmatrix} . \tag{2.47}$$

Let us now observe that the exchange $R \leftrightarrow \alpha'/R$ is completely equivalent to $G_{(25)(25)} \leftrightarrow G^{(25)(25)}$ by choosing $G_{(25)(25)} = R^2/\alpha'$, as we stressed in the introduction of this section. In fact, it is evident that, with such a position, the result of sect. 2.1 is recovered starting from equation (2.42):

$$\begin{aligned} \alpha' M_0^2 &= \left(\sqrt{G^{(25)(25)}} K_{25}\right)^2 + \left(\sqrt{G_{(25)(25)}} W^{25}\right)^2 \\ &= \alpha' \left[\left(\frac{K_{25}}{R}\right)^2 + \left(\frac{R W^{25}}{\alpha'}\right)^2 \right] . \end{aligned} \tag{2.48}$$

If we perform the transformations $W^{25} \leftrightarrow K_{25}$ and $G_{(25)(25)} \leftrightarrow G^{(25)(25)}$ on the zero-modes, we get for them the following behaviour

$$\tilde{\alpha}_0^{25} \rightarrow G_{(25)(25)} \tilde{\alpha}_0^{25} \tag{2.49}$$

$$\alpha_0^{25} \rightarrow -G_{(25)(25)} \alpha_0^{25} \tag{2.50}$$

and something very similar for the entire left and right fields:

$$\begin{aligned} X_L^{25} &\rightarrow G_{(25)(25)} X_L^{25} \\ X_R^{25} &\rightarrow -G_{(25)(25)} X_R^{25} \,. \end{aligned}$$

The T-dual field, in this case, manifests its "covariant nature", being explicitly defined in terms of $G_{(25)(25)}$:

$$\begin{aligned}
\tilde{X}_{25}(\tau,\sigma) &\equiv \frac{1}{\sqrt{2}}G_{(25)(25)}\left[X_L^{25}(\tau+\sigma) - X_R^{25}(\tau-\sigma)\right] \\
&= G_{(25)(25)}\left\{\frac{1}{\sqrt{2}}\left(x_L^{25} - x_R^{25}\right) + \sqrt{2\alpha'}(\tilde{\alpha}_0^{25} - \alpha_0^{25})\tau + \sqrt{2\alpha'}(\tilde{\alpha}_0^{25} + \alpha_0^{25})\sigma\right. \\
&\left. +i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^{25}e^{-2in\sigma} - \alpha_n^{25}e^{+2in\sigma}]\right\} \\
&= \tilde{x}_{25} + 2\sqrt{\alpha'}G_{(25)(25)}W^{25}\tau + 2\sqrt{\alpha'}K_{25}\sigma \\
&+i\sqrt{\frac{\alpha'}{2}}G_{(25)(25)}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^{25}e^{-2in\sigma} - \alpha_n^{25}e^{+2in\sigma}]\,,
\end{aligned}$$

where

$$\tilde{x}_{25} \equiv \frac{1}{\sqrt{2}}G_{(25)(25)}(x_L^{25} - x_R^{25})\,.$$

2.2.1 Poisson brackets

The Poisson brackets involving the variables $X^{25}, \mathcal{P}^{25}$ and X_L^{25}, X_R^{25} are slightly modified with respect to those of section 2.1.1. One gets

$$\{X^{25}(\tau,\sigma), \mathcal{P}^{25}(\tau,\sigma')\}_{PB} = G^{(25)(25)}\delta(\sigma-\sigma') \tag{2.51}$$

$$\{X^{25}(\tau,\sigma), X^{25}(\tau,\sigma')\}_{PB} = \{\mathcal{P}^{25}(\tau,\sigma), \mathcal{P}^{25}(\tau,\sigma')\}_{PB} = 0 \tag{2.52}$$

and

$$\{X_L^{25}(\tau+\sigma), X_L^{25}(\tau+\sigma')\}_{PB} = -2\pi\alpha' G^{(25)(25)}\epsilon(\sigma-\sigma') \tag{2.53}$$

$$\{X_R^{25}(\tau-\sigma), X_R^{25}(\tau-\sigma')\}_{PB} = 2\pi\alpha' G^{(25)(25)}\epsilon(\sigma-\sigma')\,. \tag{2.54}$$

They lead to

$$\{\tilde{\alpha}_m^{25}, \tilde{\alpha}_n^{25}\}_{PB} = \{\alpha_m^{25}, \alpha_n^{25}\}_{PB} = -i\,m\,\delta_{m+n,0}G^{(25)(25)} \quad \forall\, m,n \tag{2.55}$$

The interesting Poisson brackets are those involving the tilded quantities. By defining $\tilde{\mathcal{P}}_{25} \equiv T(\dot{\tilde{X}})_{25}$, we can construct

$$\{\tilde{X}_{25}(\tau,\sigma), \tilde{\mathcal{P}}_{25}(\tau,\sigma')\}_{PB} = G_{(25)(25)}\delta(\sigma-\sigma') \tag{2.56}$$

$$\{\tilde{X}_{25}(\tau,\sigma), \tilde{X}_{25}(\tau,\sigma')\}_{PB} = \{\tilde{\mathcal{P}}_{25}(\tau,\sigma), \tilde{\mathcal{P}}_{25}(\tau,\sigma')\}_{PB} = 0\,. \tag{2.57}$$

Moreover

$$\begin{aligned}
\{\tilde{x}_{25}, p^{25}\}_{PB} &= 0 \\
\{\tilde{x}_{25}, w^{25}\}_{PB} &= 1 \\
\{\tilde{x}_{25}, \tilde{\alpha}_n^{25}\}_{PB} = \{\tilde{x}_{25}, \alpha_n^{25}\}_{PB} &= 0\,, \quad n \neq 0\,.
\end{aligned}$$

The fields X^{25} and $\tilde{X}_{25}$ behave like noncommuting variables

$$\{X^{25}(\tau,\sigma), \tilde{X}_{25}(\tau,\sigma')\}_{PB} = -2\pi\alpha'\epsilon(\sigma-\sigma')\,, \tag{2.58}$$

because

$$\begin{aligned}\{X^{25}, \tilde{X}_{25}\}_{PB} &= (1/2)G_{(25)(25)}\{X_L^{25}, X_L^{25}\}_{PB} - (1/2)G_{(25)(25)}\{X_R^{25}, X_R^{25}\}_{PB} \\ &= G_{(25)(25)}\{X_L^{25}, X_L^{25}\}_{PB}\,.\end{aligned}$$

As in the previous section, $\{x^{25}, \tilde{x}_{25}\}_{PB} = 0$.

2.3 Toroidal compactification

In this section we will investigate the case of compactification on an n-torus T^n (*toroidal compactification*). The physical sizes and the angles characterizing the various circles are encoded in a constant metric, called G_{IJ}. We will denote the noncompact coordinates by X^μ ($\mu = 0, \dots, d-1$), while the compactified ones by Y^I ($I = 1, \dots, n$). As it happened in the case of a single compactified dimension, the action, the Lagrangian density (and so on), are modified with respect to the noncompact case. As an example, let us write the Lagrangian density

$$\mathcal{L} = \frac{T}{2}\left[\sum_{\mu,\nu=0}^{d-1}\eta_{\mu\nu}(\dot{X}^\mu\dot{X}^\nu - X'^\mu X'^\nu) + \sum_{I,J=1}^{n} G_{IJ}(\dot{Y}^I\dot{Y}^J - Y'^I Y'^J)\right].$$

It implies that the Polyakov action S_σ can be split into two components: $S_\eta + S_G$, where (we will omit the summation symbols)

$$S_\eta = -\frac{T}{2}\int d^2\xi\,\sqrt{-h}\,h^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu$$

and

$$S_G = -\frac{T}{2}\int d^2\xi\,\sqrt{-h}\,h^{\alpha\beta}G_{IJ}\partial_\alpha Y^I\partial_\beta Y^J\,. \tag{2.59}$$

The noncompact and the compact spaces do not interact.

The line element in the target space $\mathbb{R}^{1,d-1}\times T^n$, with $d+n = D = 26$ is

$$ds^2 = \sum_{\mu,\nu=0}^{d-1}\eta_{\mu\nu}dX^\mu dX^\nu + \sum_{I,J=1}^{n} G_{IJ}dY^I dY^J\,. \tag{2.60}$$

Since now on, we will focus only on the compactified coordinates, being the expression of the fields X^μ in terms of the modes the same as the ones calculated in the first Chapter.

The energy-momentum tensor has the same structure as the one computed in the first section via the substitution $\eta_{\mu\nu} \leftrightarrow G_{IJ}$:

$$T_{\alpha\beta} = \partial_\alpha Y \cdot \partial_\beta Y - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma Y \cdot \partial_\delta Y \,, \tag{2.61}$$

with $A \cdot B \equiv G_{IJ} A^I B^J$. Let us recall that $T_{\alpha\beta}$ vanishes on the equations of motion for the world-sheet metric.

The action S_G exhibits the Poincaré, reparametrizations and Weyl invariances.

- Poincaré transformations:

$$\delta Y^I = a^I{}_J Y^J + b^I \quad \text{and} \quad \delta h^{\alpha\beta} = 0 \tag{2.62}$$

  imply

$$\begin{aligned} \delta S_G &= -T \int d^2\xi \, \sqrt{-h}\, h^{\alpha\beta} G_{IJ} a^I{}_K \partial_\alpha Y^K \partial_\beta Y^J \\ &= -T \int d^2\xi \, \sqrt{-h}\, h^{\alpha\beta} a_{JK} \partial_\alpha Y^K \partial_\beta Y^J = 0 \,; \end{aligned}$$

- reparametrizations of the world-sheet coordinates. There is no difference from section 1.2;
- Weyl transformations: they regard the world-sheet metric structure only.

These considerations guarantee the possibility of gauge-fixing $h_{\alpha\beta} = \eta_{ab}$. The equation of motion and the boundary term for the field Y^I deriving from the action (2.59) are

$$\partial_a \partial^a Y^I = 0 \quad \text{and} \quad -T \int_{-\infty}^{+\infty} d\tau \left[\left(G_{IJ} \partial_1 Y^J \right) \delta Y^I \right]_{\sigma=0}^{\sigma=\pi} = 0 \,. \tag{2.63}$$

The first one is surely satisfied by requiring the solution to be

$$Y^I(\tau,\sigma) = \frac{1}{\sqrt{2}} \left[Y_L^I(\tau+\sigma) + Y_R^I(\tau-\sigma) \right] ,$$

while the second one is satisfied by imposing a quasi-periodicity conditions of the type

$$Y^I(\tau, \sigma+\pi) = Y^I(\tau,\sigma) + 2\pi\sqrt{\alpha'} W^I \,. \tag{2.64}$$

The mode expansions for the fields $Y^I_{L,R}$ can be found in analogy with the procedure carried out in section 2.2:

$$Y_L^I(\sigma^+) = y_L^I + 2\sqrt{\alpha'}\tilde{\alpha}_0^I \sigma^+ + i\sqrt{\alpha'} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^I e^{-2in\sigma^+} \tag{2.65}$$

$$Y_R^I(\sigma^-) = y_R^I + 2\sqrt{\alpha'}\alpha_0^I \sigma^- + i\sqrt{\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^I e^{-2in\sigma^-} \,. \tag{2.66}$$

The field Y^I becomes

$$\begin{aligned} Y^I(\tau,\sigma) &= \frac{1}{\sqrt{2}}(y_L^I + y_R^I) + \sqrt{2\alpha'}(\tilde{\alpha}_0^I + \alpha_0^I)\tau + \sqrt{2\alpha'}(\tilde{\alpha}_0^I - \alpha_0^I)\sigma \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^I e^{-2in\sigma} + \alpha_n^I e^{+2in\sigma}]\,, \end{aligned}$$

The momentum density is

$$\mathcal{P}_I \equiv \frac{\partial \mathcal{L}}{\partial \dot{Y}^I} = T G_{IJ}\dot{Y}^J\,, \tag{2.67}$$

while the momentum itself is

$$P_I = \int_0^\pi d\sigma \mathcal{P}_I = \frac{1}{\sqrt{2\alpha'}}G_{IJ}(\tilde{\alpha}_0^J + \alpha_0^J) \equiv p_I\,. \tag{2.68}$$

Momentum quantization implies $p_I = K_I/\sqrt{\alpha'}$, while the boundary condition implies $w^I = W^I/\sqrt{\alpha'}$. With these identities, the compact coordinate Y^I has the following expansion:

$$\begin{aligned} Y^I(\tau,\sigma) &= y^I + 2\alpha' G^{IJ}p_J\tau + 2\alpha' w^I\sigma \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^I e^{-2in\sigma} + \alpha_n^I e^{+2in\sigma}] \\ &= y^I + 2\sqrt{\alpha'}G^{IJ}K_J\tau + 2\sqrt{\alpha'}W^I\sigma \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^I e^{-2in\sigma} + \alpha_n^I e^{+2in\sigma}]\,, \end{aligned} \tag{2.69}$$

with

$$y^I \equiv \frac{1}{\sqrt{2}}(y_L^I + y_R^I)\,.$$

An expression of $\tilde{\alpha}_0^I$ and α_0^I in terms of the winding number W^I and the Kaluza-Klein excitation K_I is obtained from the two conditions

$$\begin{aligned} \sqrt{2\alpha'}(\tilde{\alpha}_0^I + \alpha_0^I) &= 2\sqrt{\alpha'}G^{IJ}K_J \\ \sqrt{2\alpha'}(\tilde{\alpha}_0^I - \alpha_0^I) &= 2\sqrt{\alpha'}W^I \end{aligned}$$

implying

$$\begin{aligned} \sqrt{2\alpha'}\tilde{\alpha}_0^I &= \sqrt{\alpha'}(G^{IJ}K_J + W^I) \\ \sqrt{2\alpha'}\alpha_0^I &= \sqrt{\alpha'}(G^{IJ}K_J - W^I)\,, \end{aligned}$$

or, by cancelling the common factor,

$$\sqrt{2}\tilde{\alpha}_0^I = G^{IJ}K_J + W^I \tag{2.70}$$

$$\sqrt{2}\alpha_0^I = G^{IJ}K_J - W^I \,. \tag{2.71}$$

The vanishing of the Virasoro generators $\tilde{L}_0 - 1$ and $L_0 - 1$ explicitly reads

$$\begin{aligned}
\tilde{L}_0 - 1 &= \frac{1}{2}\sum_{n\neq 0}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n + \frac{1}{2}\tilde{\alpha}_0^2 - 1 \\
&= \sum_{n\in\mathbb{N}}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n + \frac{1}{2}\sum_{\mu,\nu=0}^{d-1}\eta_{\mu\nu}\tilde{\alpha}_0^\mu\tilde{\alpha}_0^\nu + \frac{1}{2}\sum_{I,J=1}^{n}G_{IJ}\tilde{\alpha}_0^I\tilde{\alpha}_0^J - 1 \\
&= \tilde{N} + \frac{\alpha'}{4}\sum_{\mu=0}^{d-1}p_\mu p^\mu + \frac{1}{2}\sum_{I,J=1}^{n}G_{IJ}\tilde{\alpha}_0^I\tilde{\alpha}_0^J - 1 = 0 \qquad (2.72) \\
L_0 - 1 &= \frac{1}{2}\sum_{n\neq 0}\alpha_{-n}\cdot\alpha_n + \frac{1}{2}\alpha_0^2 - 1 \\
&= \sum_{n\in\mathbb{N}}\alpha_{-n}\cdot\alpha_n + \frac{1}{2}\sum_{\mu,\nu=0}^{d-1}\eta_{\mu\nu}\alpha_0^\mu\alpha_0^\nu + \frac{1}{2}\sum_{I,J=1}^{n}G_{IJ}\alpha_0^I\alpha_0^J - 1 \\
&= N + \frac{\alpha'}{4}\sum_{\mu=0}^{d-1}p_\mu p^\mu + \frac{1}{2}\sum_{I,J=1}^{n}G_{IJ}\alpha_0^I\alpha_0^J - 1 = 0\,. \qquad (2.73)
\end{aligned}$$

In the number operators there are contributions both from the noncompact and the compact oscillators. The mass-squared reads

$$\begin{aligned}
\alpha' M^2 &= -\alpha'\sum_{\mu=0}^{d-1}p_\mu p^\mu \\
&= 2\tilde{N} + 2N - 4 + G_{IJ}[\tilde{\alpha}_0^I\tilde{\alpha}_0^J + \alpha_0^I\alpha_0^J]\,, \qquad (2.74)
\end{aligned}$$

while

$$\begin{aligned}
\alpha' M_0^2 &\equiv G_{IJ}[\tilde{\alpha}_0^I\tilde{\alpha}_0^J + \alpha_0^I\alpha_0^J] \\
&= G_{IJ}W^I W^J + G^{IJ}K_I K_J \\
&= (W^I\, K_I)\begin{pmatrix} G_{IJ} & 0 \\ 0 & G^{IJ}\end{pmatrix}\begin{pmatrix} W^J \\ K_J\end{pmatrix}. \qquad (2.75)
\end{aligned}$$

As in the case of a single compact dimension, we have introduced a square matrix

$$\mathcal{G} = \begin{pmatrix} G_{IJ} & 0 \\ 0 & G^{IJ}\end{pmatrix}, \tag{2.76}$$

with *doubled* dimensions: $2n \times 2n$.

The level-matching condition can be written in a similar fashion, too. In fact, $(L_0 - \tilde{L}_0)|\phi\rangle = 0 \Rightarrow$

$$\begin{aligned} N - \tilde{N} &= \frac{1}{2} G_{IJ} [\tilde{\alpha}_0^I \tilde{\alpha}_0^J - \alpha_0^I \alpha_0^J] \\ &= W^I K_I \\ &= \frac{1}{2} (W^I \; K_I) \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} W^J \\ K_J \end{pmatrix} \\ &= \frac{1}{2} (W^I \; K_I) \begin{pmatrix} 0 & \delta_I{}^J \\ \delta^I{}_J & 0 \end{pmatrix} \begin{pmatrix} W^J \\ K_J \end{pmatrix}. \end{aligned}$$

It is evident that, in this case, the mass spectrum and the level-matching condition are invariant under the simultaneous transformations

$$W^I \leftrightarrow K_I \quad \text{and} \quad \mathcal{G} \leftrightarrow \mathcal{G}^{-1}, \tag{2.77}$$

the last of which is equivalent to $G_{IJ} \leftrightarrow G^{IJ}$. In this case the T-duality transformation shows its "nongeometrical" nature, meaning that the metric is not transformed through a diffeomorphism.

The exchanges $W^I \leftrightarrow K_I$, $G_{IJ} \leftrightarrow G^{IJ}$ induce on the zero-modes the transformations

$$\tilde{\alpha}_0^I \rightarrow G_{IJ} \tilde{\alpha}_0^J \tag{2.78}$$
$$\alpha_0^I \rightarrow -G_{IJ} \alpha_0^J, \tag{2.79}$$

which can be promoted for $Y^I_{L,R}$:

$$\begin{aligned} Y_L^I &\rightarrow G_{IJ} Y_L^J \\ Y_R^I &\rightarrow -G_{IJ} Y_R^J. \end{aligned}$$

In this way the dual field $\tilde{Y}_I(\tau, \sigma)$ is

$$\begin{aligned} \tilde{Y}_I(\tau,\sigma) &\equiv \frac{1}{\sqrt{2}} \left[G_{IJ} Y_L^J(\tau+\sigma) - G_{IJ} Y_R^J(\tau-\sigma) \right] \\ &= \tilde{y}_I + 2\sqrt{\alpha'} G_{IJ} W^J \tau + 2\sqrt{\alpha'} K_I \sigma \\ &\quad + i\sqrt{\frac{\alpha'}{2}} G_{IJ} \sum_{n \neq 0} \frac{e^{-2in\tau}}{n} [\tilde{\alpha}_n^J e^{-2in\sigma} - \alpha_n^J e^{+2in\sigma}], \end{aligned}$$

where, as before,

$$\tilde{y}_I \equiv \frac{1}{\sqrt{2}} G_{IJ} (y_L^J - y_R^J).$$

2.3.1 Poisson brackets

The Poisson brackets relative to the coordinates $Y^I, \mathcal{P}^I$ and $Y^I_{L,R}$ are, respectively,

$$\{Y^I(\tau,\sigma), \mathcal{P}^J(\tau,\sigma')\}_{PB} = G^{IJ}\delta(\sigma-\sigma') \tag{2.80}$$

$$\{Y^I(\tau,\sigma), Y^J(\tau,\sigma')\}_{PB} = \{\mathcal{P}^I(\tau,\sigma), \mathcal{P}^J(\tau,\sigma')\}_{PB} = 0 \tag{2.81}$$

and

$$\begin{aligned}
\{Y^I_L(\tau+\sigma), Y^J_L(\tau+\sigma')\}_{PB} &= -2\pi\alpha' G^{IJ}\epsilon(\sigma-\sigma') &(2.82)\\
\{Y^I_R(\tau-\sigma), Y^J_R(\tau-\sigma')\}_{PB} &= 2\pi\alpha' G^{IJ}\epsilon(\sigma-\sigma')\,. &(2.83)
\end{aligned}$$

For the oscillators one gets:

$$\{\tilde{\alpha}^I_m, \tilde{\alpha}^J_n\}_{PB} = \{\alpha^I_m, \alpha^J_n\}_{PB} = -i\, m\, \delta_{m+n,0} G^{IJ} \quad \forall m, n\,. \tag{2.84}$$

The Poisson brackets involving the tilded quantities are (by defining $\tilde{\mathcal{P}}_I \equiv T(\dot{\tilde{Y}})_I$):

$$\{\tilde{Y}_I(\tau,\sigma), \tilde{\mathcal{P}}_J(\tau,\sigma')\}_{PB} = G_{IJ}\delta(\sigma-\sigma') \tag{2.85}$$

$$\{\tilde{Y}_I(\tau,\sigma), \tilde{Y}_J(\tau,\sigma')\}_{PB} = \{\tilde{\mathcal{P}}_I(\tau,\sigma), \tilde{\mathcal{P}}_J(\tau,\sigma')\}_{PB} = 0 \tag{2.86}$$

as well as

$$\begin{aligned}
\{\tilde{y}_I, p^J\}_{PB} &= 0\\
\{\tilde{x}_I, w^J\}_{PB} &= \delta_I{}^J\\
\{\tilde{y}_I, \tilde{\alpha}^J_n\}_{PB} = \{\tilde{y}_I, \alpha^J_n\}_{PB} &= 0\,, \quad n \neq 0\,.
\end{aligned}$$

The surprising aspect is that the fields Y and $\tilde{Y}$ have the same kind of behaviour of $Y_{L,R}$:

$$\{Y^I(\tau,\sigma), \tilde{Y}_J(\tau,\sigma')\}_{PB} = -2\pi\alpha'\delta^I{}_J\epsilon(\sigma-\sigma')\,. \tag{2.87}$$

In fact, due to definitions,

$$\begin{aligned}
\{Y^I, \tilde{Y}_J\}_{PB} &= (1/2)G_{JK}\{Y^I_L, Y^K_L\}_{PB} - (1/2)G_{JK}\{Y^I_R, Y^K_R\}_{PB}\\
&= G_{JK}\{Y^I_L, Y^K_L\}_{PB}\,.
\end{aligned}$$

Moreover, $\{y^I, \tilde{y}_J\}_{PB} = 0$.

2.4 Toroidal compactification in the presence of a B-field

In this section we will focus on the case of n compactified dimensions (compactification on an T^n-torus) in the presence of a constant antisymmetric Kalb-Ramond field B_{IJ}.

The action describing the compact target space is the following:

$$\begin{aligned} S &= S_G + S_B \\ &= -\frac{T}{2}\int d^2\xi\,\sqrt{-h}\,h^{\alpha\beta}G_{IJ}\partial_\alpha Y^I\partial_\beta Y^J + \frac{T}{2}\int d^2\xi\,\epsilon^{\alpha\beta}B_{IJ}\partial_\alpha Y^I\partial_\beta Y^J \\ &= -\frac{T}{2}\int d^2\xi\,\left(\sqrt{-h}\,h^{\alpha\beta}G_{IJ} - \epsilon^{\alpha\beta}B_{IJ}\right)\partial_\alpha Y^I\partial_\beta Y^J\,. \end{aligned} \tag{2.88}$$

The equation of motion for the field Y^I, deriving from the previous action is due only to S_G. In fact, S_B is a sum of surface integrals:

$$\begin{aligned} S_B &= \frac{T}{2}\int d^2\xi\,\epsilon^{\alpha\beta}B_{IJ}\partial_\alpha Y^I\partial_\beta Y^J \\ &= \frac{T}{2}\int d^2\xi\,\partial_\alpha\left(\epsilon^{\alpha\beta}B_{IJ}Y^I\partial_\beta Y^J\right) - \frac{T}{2}\int d^2\xi\,\epsilon^{\alpha\beta}B_{IJ}Y^I\partial_\alpha\partial_\beta Y^J \\ &= \frac{T}{2}\int d^2\xi\,\partial_\alpha\left(\epsilon^{\alpha\beta}B_{IJ}Y^I\partial_\beta Y^J\right) \\ &= \frac{T}{2}\int d^2\xi\,\left\{\partial_0\left(\epsilon^{01}B_{IJ}Y^I\partial_1 Y^J\right) + \partial_1\left(\epsilon^{10}B_{IJ}Y^I\partial_0 Y^J\right)\right\} \\ &= \frac{T}{2}\int_0^\pi d\sigma\,\left[B_{IJ}Y^I\partial_1 Y^J\right]_{\tau=-\infty}^{\tau=+\infty} - \frac{T}{2}\int_{-\infty}^{+\infty} d\tau\,\left[B_{IJ}Y^I\partial_0 Y^J\right]_{\sigma=0}^{\sigma=\pi}\,. \end{aligned}$$

The energy-momentum tensor has the same structure as the one computed in the previous section since

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta(S_G+S_B)}{\delta h^{\alpha\beta}} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S_G}{\delta h^{\alpha\beta}}\,. \tag{2.89}$$

The action S exhibits the following invariances under:

- Poincaré transformations

  $$\delta Y^I = a^I{}_J Y^J + b^I \quad \text{and} \quad \delta h^{\alpha\beta} = 0 \tag{2.90}$$

  implying

  $$\begin{aligned} \delta S_G &= -T\int d^2\xi\,\sqrt{-h}\,h^{\alpha\beta}G_{IJ}a^I{}_K\partial_\alpha Y^K\partial_\beta Y^J = 0 \\ \delta S_B &= T\int d^2\xi\,\epsilon^{\alpha\beta}B_{IJ}a^I{}_K\partial_\alpha Y^K\partial_\beta Y^J = 0\,; \end{aligned}$$

- reparametrizations (it is sufficient to notice that $\epsilon^{\alpha\beta}/\sqrt{-h}$ transforms as a tensor being $\epsilon^{\alpha\beta}$ a tensor density);
- Weyl rescalings: they do not act on S_B.

The three local invariances permit to gauge-fix the world-sheet metric: $h_{\alpha\beta} = \eta_{ab}$.

Let us now rewrite the action $S = S_G + S_B$ in the conformal gauge:

$$
\begin{aligned}
S &= -\frac{T}{2}\int d^2\xi\,(\eta^{ab}G_{IJ} - \epsilon^{ab}B_{IJ})\partial_a Y^I \partial_b Y^J \\
&= \frac{T}{2}\int d^2\xi\,(G_{IJ}\dot{Y}^I\dot{Y}^J - G_{IJ}Y'^I Y'^J + 2B_{IJ}\dot{Y}^I Y'^J)\,. \qquad (2.91)
\end{aligned}
$$

The equation of motion and the boundary term deriving from the action (2.91) are

$$
\partial_a\partial^a Y^I = 0 \quad \text{and} \quad -T\int_{-\infty}^{+\infty} d\tau\,\left[\left(G_{IJ}\partial_1 Y^J + B_{IJ}\partial_0 Y^J\right)\delta Y^I\right]_{\sigma=0}^{\sigma=\pi} = 0\,. \quad (2.92)
$$

The boundary term vanishes once one requires a quasi-periodicity condition for the string coordinates

$$
Y^I(\tau,\sigma+\pi) = Y^I(\tau,\sigma) + 2\pi\sqrt{\alpha'}W^I\,. \qquad (2.93)
$$

As in the previous section, the expansions for the fields $Y^I_{L,R}$ are the ones in eqs. (2.65 - 2.66) and Y^I can be formally written as

$$
\begin{aligned}
Y^I(\tau,\sigma) &= \frac{1}{\sqrt{2}}(y^I_L + y^I_R) + \sqrt{2\alpha'}(\tilde{\alpha}^I_0 + \alpha^I_0)\tau + \sqrt{2\alpha'}(\tilde{\alpha}^I_0 - \alpha^I_0)\sigma \\
&\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}^I_n e^{-2in\sigma} + \alpha^I_n e^{+2in\sigma}]\,, \qquad (2.94)
\end{aligned}
$$

while the difference is in the momentum density:

$$
\mathcal{P}_I \equiv \frac{\partial\mathcal{L}}{\partial\dot{Y}^I} = T(G_{IJ}\dot{Y}^J + B_{IJ}Y'^J)\,. \qquad (2.95)
$$

The momentum is

$$
P_I = \int_0^\pi d\sigma\mathcal{P}_I = \frac{1}{\sqrt{2\alpha'}}[G_{IJ}(\tilde{\alpha}^J_0 + \alpha^J_0) + B_{IJ}(\tilde{\alpha}^J_0 - \alpha^J_0)] \equiv p_I\,. \qquad (2.96)
$$

Quantization of the momentum implies $p_I = K_I/\sqrt{\alpha'}$. As before, the winding is $w^I = W^I/\sqrt{\alpha'}$.

The zero-modes expressed in terms of W^I and K_I are obtained from

$$
\begin{aligned}
\sqrt{2\alpha'}(\tilde{\alpha}^I_0 + \alpha^I_0) &= 2\sqrt{\alpha'}G^{IJ}(K_J - B_{JK}W^K) \\
\sqrt{2\alpha'}(\tilde{\alpha}^I_0 - \alpha^I_0) &= 2\sqrt{\alpha'}W^I
\end{aligned}
$$

implying

$$
\begin{aligned}
\sqrt{2\alpha'}\tilde{\alpha}_0^I &= \sqrt{\alpha'}\left[G^{IJ}K_J + G^{IJ}(G_{JK} - B_{JK})W^K\right] \\
&= \sqrt{\alpha'}G^{IJ}(K_J + E^T_{JK}W^K)
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{2\alpha'}\alpha_0^I &= \sqrt{\alpha'}\left[G^{IJ}K_J - G^{IJ}(G_{JK} + B_{JK})W^K\right] \\
&= \sqrt{\alpha'}G^{IJ}(K_J - E_{JK}W^K)\,,
\end{aligned}
$$

or

$$\sqrt{2}\tilde{\alpha}_0^I = G^{IJ}(K_J + E^T_{JK}W^K) \tag{2.97}$$

$$\sqrt{2}\alpha_0^I = G^{IJ}(K_J - E_{JK}W^K)\,, \tag{2.98}$$

where we have introduced the constant $n \times n$ matrix $E_{IJ} \equiv G_{IJ} + B_{IJ}$.

The expansion of the compact coordinate Y^I in terms of K_I and W^I is

$$
\begin{aligned}
Y^I(\tau,\sigma) &= y^I + 2\sqrt{\alpha'}(G^{IJ}K_J - G^{IK}B_{KJ}W^J)\tau + 2\sqrt{\alpha'}W^I\sigma \\
&\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}_n^I e^{-2in\sigma} + \alpha_n^I e^{+2in\sigma}]\,.
\end{aligned} \tag{2.99}
$$

The conditions on the Virasoro generators $\tilde{L}_0 - 1 = L_0 - 1 = 0$ are the same as in eqs. (2.72 - 2.73).

The mass-squared reads

$$\alpha' M^2 = 2\tilde{N} + 2N - 4 + G_{IJ}[\tilde{\alpha}_0^I\tilde{\alpha}_0^J + \alpha_0^I\alpha_0^J]\,, \tag{2.100}$$

while

$$
\begin{aligned}
\alpha' M_0^2 &\equiv G_{IJ}[\tilde{\alpha}_0^I\tilde{\alpha}_0^J + \alpha_0^I\alpha_0^J] \\
&= (G - BG^{-1}B)_{IJ}W^IW^J + B_{IK}G^{KJ}W^IK_J \\
&\quad - G^{IK}B_{KJ}K_IW^J + G^{IJ}K_IK_J \\
&= (W^I\, K_I)\begin{pmatrix} (G - BG^{-1}B)_{IJ} & B_{IK}G^{KJ} \\ -G^{IK}B_{KJ} & G^{IJ} \end{pmatrix}\begin{pmatrix} W^J \\ K_J \end{pmatrix}.
\end{aligned} \tag{2.101}
$$

The $2n \times 2n$ matrix

$$\mathcal{G} \equiv \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \tag{2.102}$$

is the generalization, in the presence of a B field, of the one in eq. (2.76) and it plays a fundamental role.

The level-matching condition can be written through the introduction of another $2n \times 2n$ matrix.

$(L_0 - \tilde{L}_0)|\phi\rangle = 0 \Rightarrow$

$$\begin{aligned}
N - \tilde{N} &= \frac{1}{2} G_{IJ}[\tilde{\alpha}_0^I \tilde{\alpha}_0^J - \alpha_0^I \alpha_0^J] \\
&= W^I K_I \\
&= \frac{1}{2}(W^I\, K_I) \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} W^J \\ K_J \end{pmatrix} \\
&= \frac{1}{2}(W^I\, K_I) \begin{pmatrix} 0 & \delta_I{}^J \\ \delta^I{}_J & 0 \end{pmatrix} \begin{pmatrix} W^J \\ K_J \end{pmatrix}. \qquad (2.103)
\end{aligned}$$

In this case, a T-duality transformation affecting both the mass and the level-macthing condition, can be expressed in a series of equivalent ways:

- $W^I \leftrightarrow K_I$ and $\mathcal{G} \leftrightarrow \mathcal{G}^{-1}$ (involving $2n \times 2n$ matrices);
- $W^I \leftrightarrow K_I$ and $E \leftrightarrow E^{-1}$ (involving $n \times n$ matrices);
- $W^I \leftrightarrow K_I$ and $(G - BG^{-1}B)_{IJ} \leftrightarrow G^{IJ}$, $B_{IK}G^{KJ} \leftrightarrow -G^{IK}B_{KJ}$ (again involving $n \times n$ matrices).

They imply on the zero-modes the following transformations:

$$\tilde{\alpha}_0^I \to E_{IJ}\tilde{\alpha}_0^J \qquad (2.104)$$
$$\alpha_0^I \to -E^T_{IJ}\alpha_0^J\,. \qquad (2.105)$$

The quickest way to verify them is to use the identities $(G \pm B)G^{-1}(G \mp B) = G - BG^{-1}B$, which, translated for the matrix E, read: $EG^{-1}E^T = E^T G^{-1} E = G - BG^{-1}B$. These kinds of transformations hold for Y_L^I and Y_R^I, too:

$$\begin{aligned}
Y_L^I &\to E_{IJ} Y_L^J \\
Y_R^I &\to -E^T_{IJ} Y_R^J\,.
\end{aligned}$$

The dual field $\tilde{Y}_I(\tau, \sigma)$ is

$$\begin{aligned}
\tilde{Y}_I(\tau,\sigma) &\equiv \frac{1}{\sqrt{2}}\left[E_{IJ} Y_L^J(\tau+\sigma) - E^T_{IJ} Y_R^J(\tau-\sigma)\right] \\
&= \tilde{y}_I + 2\sqrt{\alpha'}[(G - BG^{-1}B)_{IJ}W^J + B_{IK}G^{KJ}K_J]\tau + 2\sqrt{\alpha'}K_I\sigma \\
&\quad + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-2in\tau}}{n}[E_{IJ}\tilde{\alpha}_n^J e^{-2in\sigma} - E^T_{IJ}\alpha_n^J e^{+2in\sigma}]\,.
\end{aligned}$$

This time

$$\tilde{y}_I \equiv \frac{1}{\sqrt{2}}(E_{IJ}\, y_L^J - E^T_{IJ}\, y_R^J)\,.$$

We learned that a T-duality transformation acts on the fields Y and $\tilde{Y}$, as well as on the couplings E and $\tilde{E} \equiv E^{-1}$. We want to stress that this symmetry can be made more evident in the action describing such a model. In fact, to this aim, let us slightly modify the action given at the beginning of this Chapter, eq. (2.88): because of the symmetry of $h^{\alpha\beta}$ and G_{IJ} and the antisymmetry of $\epsilon^{\alpha\beta}$ and B_{IJ}, the actions S_G and S_B can be written as

$$S_G \to S'_G = -\frac{T}{2}\int d^2\xi\, \sqrt{-h}\, h^{\alpha\beta}(G_{IJ}+B_{IJ})\partial_\alpha Y^I \partial_\beta Y^J$$

and

$$S_B \to S'_B = \frac{T}{2}\int d^2\xi\, \epsilon^{\alpha\beta}(G_{IJ}+B_{IJ})\partial_\alpha Y^I \partial_\beta Y^J\,.$$

So their sum reads:

$$\begin{aligned} S &= S'_G + S'_B \\ &= -\frac{T}{2}\int d^2\xi\, \left(\sqrt{-h}\, h^{\alpha\beta} - \epsilon^{\alpha\beta}\right)(G_{IJ}+B_{IJ})\partial_\alpha Y^I \partial_\beta Y^J \\ &= -\frac{T}{2}\int d^2\xi\, \left(\sqrt{-h}\, h^{\alpha\beta} - \epsilon^{\alpha\beta}\right) E_{IJ}\partial_\alpha Y^I \partial_\beta Y^J\,. \end{aligned} \tag{2.106}$$

Applying the duality transformation, we get the T-dual sigma-model action

$$\tilde{S} = -\frac{T}{2}\int d^2\xi\, \left(\sqrt{-h}\, h^{\alpha\beta} - \epsilon^{\alpha\beta}\right)\tilde{E}^{IJ}\partial_\alpha \tilde{Y}_I \partial_\beta \tilde{Y}_J\,. \tag{2.107}$$

2.4.1 Poisson brackets

The Poisson brackets relative to the coordinates $Y^I, \mathcal{P}^I$ and $Y^I_{L,R}$ are:

$$\{Y^I(\tau,\sigma), \mathcal{P}^J(\tau,\sigma')\}_{PB} = G^{IJ}\delta(\sigma-\sigma') \tag{2.108}$$

$$\{Y^I(\tau,\sigma), Y^J(\tau,\sigma')\}_{PB} = \{\mathcal{P}^I(\tau,\sigma), \mathcal{P}^J(\tau,\sigma')\}_{PB} = 0 \tag{2.109}$$

$$\{Y^I_L(\tau+\sigma), Y^J_L(\tau+\sigma')\}_{PB} = -2\pi\alpha' G^{IJ}\epsilon(\sigma-\sigma') \tag{2.110}$$

$$\{Y^I_R(\tau-\sigma), Y^J_R(\tau-\sigma')\}_{PB} = 2\pi\alpha' G^{IJ}\epsilon(\sigma-\sigma')\,. \tag{2.111}$$

The oscillators satisfy

$$\{\tilde{\alpha}^I_m, \tilde{\alpha}^J_n\}_{PB} = \{\alpha^I_m, \alpha^J_n\}_{PB} = -i\, m\, \delta_{m+n,0} G^{IJ} \quad \forall\, m,n\,. \tag{2.112}$$

For the fields Y and $\tilde{Y}$ it happens that:

$$\{Y^I(\tau,\sigma), \tilde{Y}_J(\tau,\sigma')\}_{PB} = -2\pi\alpha' \delta^I{}_J \epsilon(\sigma-\sigma')\,, \tag{2.113}$$

as it is very simple to verify:

$$\begin{aligned} \{Y^I, \tilde{Y}_J\}_{PB} &= (1/2)E_{JK}\{Y^I_L, Y^K_L\}_{PB} - (1/2)E^T_{JK}\{Y^I_R, Y^K_R\}_{PB} \\ &= (1/2)(E_{JK}+E^T_{JK})\{Y^I_L, Y^K_L\}_{PB} \\ &= G_{JK}\{Y^I_L, Y^K_L\}_{PB}\,. \end{aligned}$$

2.5 On the matrix $\mathcal{G}$

The block matrix $\mathcal{G}$ is symmetric because $(G-BG^{-1}B)^T = G-BG^{-1}B$, $(G^{-1})^T = G^{-1}$ and $(BG^{-1})^T = -G^{-1}B$. It can be decomposed into the following product of matrices:

$$\begin{aligned}\mathcal{G} &= \begin{pmatrix} G-BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix}\begin{pmatrix} G & 0 \\ -G^{-1}B & 1_n \end{pmatrix}\begin{pmatrix} 1_n & 0 \\ 0 & G^{-1} \end{pmatrix}.\end{aligned}$$

So, $\det\mathcal{G} = \det G \cdot \det G^{-1} = 1$.

The inverse matrix can be computed in more than one fashion. It reads

$$\mathcal{G}^{-1} = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G-BG^{-1}B \end{pmatrix}. \tag{2.114}$$

The T-duality transformation of the background fields is encoded into the exchange $E \leftrightarrow \tilde{E} \equiv E^{-1}$, which fixes $\tilde{G} = (G-BG^{-1}B)^{-1}$ and $\tilde{B} = -G^{-1}B\tilde{G}$ (see Appendix D). Due to these equalities, the matrix $\mathcal{G}^{-1}$ can be also written as

$$\mathcal{G}^{-1} = \begin{pmatrix} \tilde{G}-\tilde{B}\tilde{G}^{-1}\tilde{B} & \tilde{B}\tilde{G}^{-1} \\ -\tilde{G}^{-1}\tilde{B} & \tilde{G}^{-1} \end{pmatrix} \equiv \tilde{\mathcal{G}}\,. \tag{2.115}$$

The equality $\tilde{\mathcal{G}} = \mathcal{G}^{-1}$ is another way to express part of the T-duality transformation.

2.6 The $O(n,n;\mathbb{Z})$ duality group

As we already stressed, the bosonic string compactified on an T^n-torus has a symmetry under the group $O(n,n;\mathbb{Z})$. It generalizes the T-duality symmetry on a single compact direction (circle compactification). This symmetry is best described in terms of the matrix $\mathcal{G}$, as we saw in the previous section. Indeed, for a nonorthogonal torus the $R \leftrightarrow \alpha'/R$ duality of the circle compactification generalizes to the inversion symmetry

$$W^I \leftrightarrow K_I \quad \text{and} \quad \mathcal{G} \leftrightarrow \mathcal{G}^{-1}\,.$$

A further discrete shift symmetry, leaving the zero-modes $\tilde{\alpha}_0^I$ and α_0^I invariant, appears only when $n > 1$. It is given by

$$B_{IJ} \to B_{IJ} + \frac{1}{2}N_{IJ} \;\; \text{with} \;\; W^I \to W^I,\; K_I \to K_I + N_{IJ}W^J\,, \tag{2.116}$$

where N_{IJ} is an antisymmetric matrix of integers.

By definition, a $2n \times 2n$ matrix A belongs to the group $O(n,n;\mathbb{R})$ if

$$A^T \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}, \tag{2.117}$$

or

$$A^T \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix} A = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}. \tag{2.118}$$

The group $O(n,n;\mathbb{Z})$ is the subgroup of $O(n,n;\mathbb{R})$ consisting of those matrices the elements of which are integers. If the matrix $\mathcal{G}$ has integral entries, then its inverse $\mathcal{G}^{-1}$ has automatically integral entries, too.

The symmetry under a T-duality transfomation is realized as

$$\mathcal{G} \to A\mathcal{G}A^T \quad \text{and} \quad \begin{pmatrix} W \\ K \end{pmatrix} \to \begin{pmatrix} W' \\ K' \end{pmatrix} = A \begin{pmatrix} W \\ K \end{pmatrix}. \tag{2.119}$$

This preserves the result for the mass spectrum in eq. (2.101) as well as the level-matching condition in eq. (2.103).

In terms of $O(n,n;\mathbb{Z})$ transformations, the inversion symmetry corresponds to the matrix

$$\begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \tag{2.120}$$

(which in the next Chapters will be called Ω) and the shift symmetry corresponds to the matrix

$$\begin{pmatrix} 1_n & 0 \\ N_{IJ} & 1_n \end{pmatrix}. \tag{2.121}$$

Chapter 3

T-dual invariant formulation: 'flat' scalar field

In this Chapter we introduce the fundamental ingredients to construct the T-duality invariant formulation of closed string theory, which is examined in Chapter 5. The starting point is a free Lagrangian/action describing a bidimensional scalar field φ. Through the introduction of an auxiliary variable, it is inserted in the action a new scalar field $\tilde{\varphi}$ which, on-shell, turns out to be Hodge-dual of the 'original' field φ. The action is then symmetrized to make both the variables appear on equal footing. A suitable rotation introduce a couple of new coordinates in terms of which the symmetrized Lagrangian splits into two decoupled first order Floreanini-Jackiw Lagrangians. Their invariances are then discussed and quantization is performed, following Dirac's procedure [15, 16, 17, 18, 19].

3.1 Scalar field symmetric action

On a bidimensional manifold $\mathcal{M}$, let us consider the flat minkowskian metric $\eta_{ab} =$ diag$(-1, +1)$ and the coordinates $\xi^a = (\xi^0 \equiv \tau, \xi^1 \equiv \sigma)$, with $-\infty < \tau < +\infty$ e $0 \leq \sigma \leq \pi$. Let us introduce a scalar field $\varphi = \varphi(\tau, \sigma)$ the dynamics of which is described by the free Lagrangian density

$$\begin{aligned} \mathcal{L}^{(0)} &= -\frac{1}{2}\partial_a\varphi\partial^a\varphi \\ &= -\frac{1}{2}\eta^{ab}\partial_a\varphi\partial_b\varphi \\ &= \frac{1}{2}\left[(\partial_0\varphi)^2 - (\partial_1\varphi)^2\right] \end{aligned} \tag{3.1}$$

or by the action

$$S^{(0)}\left[\varphi\right] = \int d^2\xi\, \mathcal{L}^{(0)} = \frac{1}{2}\int d^2\xi\, \left[(\partial_0\varphi)^2 - (\partial_1\varphi)^2\right]. \tag{3.2}$$

The action (3.2) describes a single degree of freedom.

The equation of motion and the boundary terms for the field φ, coming from the variation of the action (3.2), are

$$\partial_a \partial^a \varphi = 0 \tag{3.3}$$

and

$$\int_0^\pi d\sigma \left[\partial_0 \varphi \delta\varphi\right]_{\tau=-\infty}^{\tau=+\infty} - \int_{-\infty}^{+\infty} d\tau \left[\partial_1 \varphi \delta\varphi\right]_{\sigma=0}^{\sigma=\pi} = 0\,.$$

By remembering that $\delta\varphi(\tau = \pm\infty, \sigma) = 0\; \forall\, \sigma$, the boundary terms reduce to

$$-\int_{-\infty}^{+\infty} d\tau \left[\partial_1 \varphi \delta\varphi\right]_{\sigma=0}^{\sigma=\pi} = 0\,. \tag{3.4}$$

The boundary term (3.4) is of the same type as the one encountered in Chapter 1 for the closed string coordinates in the conformal gauge (eq. (1.15)) and it is sufficient to require a periodicity condition as

$$\varphi(\tau, \sigma + \pi) = \varphi(\tau, \sigma) \tag{3.5}$$

to satisfy it (even if this in not the unique possibility).

The Lagrangian can be put in a 'first order form', in which it is linear in the time derivative of the field φ. At this aim, let us introduce an auxiliary field $p(\tau, \sigma) = \partial_0\varphi$ (which, in this simple theory, coincides with $\partial\mathcal{L}^{(0)}/\partial(\partial_0\varphi)$, the conjugate momentum with respect to φ). So we obtain $\mathcal{L}^{(0)} \to \mathcal{L}'$, where

$$\mathcal{L}' = p\, \partial_0 \varphi - \frac{1}{2}p^2 - \frac{1}{2}(\partial_1\varphi)^2\,. \tag{3.6}$$

Moreover let us make the 'recast' $p(\tau, \sigma) = \partial_1\tilde{\varphi}$, which requires the introduction of another scalar field $\tilde{\varphi} = \tilde{\varphi}(\tau, \sigma)$ and the transition from the Lagrangian density $\mathcal{L}'$ to the one named $\hat{\mathcal{L}}$:

$$\hat{\mathcal{L}} = \partial_0\varphi\partial_1\tilde{\varphi} - \frac{1}{2}(\partial_1\varphi)^2 - \frac{1}{2}(\partial_1\tilde{\varphi})^2 \tag{3.7}$$

and similarly for the actions S' and $\hat{S}$:

$$S' = \int d^2\xi \left[p\, \partial_0\varphi - \frac{1}{2}p^2 - \frac{1}{2}(\partial_1\varphi)^2\right] \tag{3.8}$$

$$\hat{S}\,[\varphi, \tilde{\varphi}] = \int d^2\xi \left[\partial_0\varphi\partial_1\tilde{\varphi} - \frac{1}{2}(\partial_1\varphi)^2 - \frac{1}{2}(\partial_1\tilde{\varphi})^2\right]. \tag{3.9}$$

In order to 'symmetrize' the action (3.9), in the sense that the fields φ and $\tilde{\varphi}$ will be on equal footing, let us make the following observations: one half of the integral $\int d^2\xi\, \partial_0\varphi\partial_1\tilde{\varphi}$ can be written as

$$\frac{1}{2}\int d^2\xi\, \partial_0\varphi\partial_1\tilde{\varphi} = \frac{1}{2}\int d^2\xi\, \epsilon^{ab}\partial_a\varphi\partial_b\tilde{\varphi} + \frac{1}{2}\int d^2\xi\, \partial_1\varphi\partial_0\tilde{\varphi}$$

and the term involving the pseudo-tensor ϵ^{ab} reduces to surface integrals

$$\begin{aligned}
\frac{1}{2}\int d^2\xi\,\epsilon^{ab}\partial_a\varphi\partial_b\tilde{\varphi} &= \begin{cases} \frac{1}{2}\int d^2\xi\,\partial_a(\epsilon^{ab}\varphi\partial_b\tilde{\varphi}) - \frac{1}{2}\int d^2\xi\,\varphi\epsilon^{ab}\partial_a\partial_b\tilde{\varphi} \\ \\ \frac{1}{2}\int d^2\xi\,\partial_b(\epsilon^{ab}\tilde{\varphi}\partial_a\varphi) - \frac{1}{2}\int d^2\xi\,\tilde{\varphi}\epsilon^{ab}\partial_b\partial_a\varphi \end{cases} \\
&= \begin{cases} \frac{1}{2}\int d^2\xi\,\partial_a(\epsilon^{ab}\varphi\partial_b\tilde{\varphi}) \\ \\ -\frac{1}{2}\int d^2\xi\,\partial_a(\epsilon^{ab}\tilde{\varphi}\partial_b\varphi) \end{cases}
\end{aligned}$$

(being $\epsilon^{ab}\partial_a\partial_b\varphi = \epsilon^{ab}\partial_a\partial_b\tilde{\varphi} = 0$). It is easy to see that the action (3.9) can be written as

$$\hat{S} = \hat{S}_{sym} + \begin{cases} \frac{1}{2}\int d^2\xi\,\partial_a(\epsilon^{ab}\varphi\partial_b\tilde{\varphi}) \\ \\ -\frac{1}{2}\int d^2\xi\,\partial_a(\epsilon^{ab}\tilde{\varphi}\partial_b\varphi) \end{cases}, \tag{3.10}$$

where

$$\hat{S}_{sym}[\varphi,\tilde{\varphi}] = \frac{1}{2}\int d^2\xi\,\left[\partial_0\varphi\partial_1\tilde{\varphi} + \partial_0\tilde{\varphi}\partial_1\varphi - (\partial_1\varphi)^2 - (\partial_1\tilde{\varphi})^2\right]. \tag{3.11}$$

$\hat{S}$ and $\hat{S}_{sym}$ are (essentially) equivalent, differing by surfaces terms, which explicitly are

$$\begin{aligned}
\frac{1}{2}\int d^2\xi\,\partial_a(\epsilon^{ab}\varphi\partial_b\tilde{\varphi}) &= \frac{1}{2}\int_0^{\pi} d\sigma\,[\varphi\partial_1\tilde{\varphi}]_{\tau=-\infty}^{\tau=+\infty} - \frac{1}{2}\int_{-\infty}^{+\infty} d\tau\,[\varphi\partial_0\tilde{\varphi}]_{\sigma=0}^{\sigma=\pi} \\
&= -\frac{1}{2}\int_{-\infty}^{+\infty} d\tau\,[\varphi\partial_0\tilde{\varphi}]_{\sigma=0}^{\sigma=\pi}
\end{aligned} \tag{3.12}$$

or

$$\begin{aligned}
-\frac{1}{2}\int d^2\xi\,\partial_a(\epsilon^{ab}\tilde{\varphi}\partial_b\varphi) &= -\frac{1}{2}\int_0^{\pi} d\sigma\,[\tilde{\varphi}\partial_1\varphi]_{\tau=-\infty}^{\tau=+\infty} + \frac{1}{2}\int_{-\infty}^{+\infty} d\tau\,[\tilde{\varphi}\partial_0\varphi]_{\sigma=0}^{\sigma=\pi} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} d\tau\,[\tilde{\varphi}\partial_0\varphi]_{\sigma=0}^{\sigma=\pi},
\end{aligned} \tag{3.13}$$

where we have used the conditions $\varphi(\tau=\pm\infty,\sigma) = \tilde{\varphi}(\tau=\pm\infty,\sigma) = 0\ \forall\,\sigma$.

By adding the hypothesis of periodicity in σ of the field $\tilde{\varphi}$, too

$$\tilde{\varphi}(\tau,\sigma+\pi) = \tilde{\varphi}(\tau,\sigma) \tag{3.14}$$

(something which is reasonable and that will be justified later on), we see that the surviving surface integrals (3.12 - 3.13) vanish.

Let us now make an interesting observation: the variation of the action $\hat{S}_{sym}$, calculated when $\delta\varphi = f(\tau)$, is the following surface integral

$$\delta\hat{S}_{sym}|_{\delta\varphi=f(\tau)} = \frac{1}{2}\int d^2\xi\,\partial_1\left[\partial_0 f(\tau)\,\tilde{\varphi}\right]$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-\infty}^{+\infty} d\tau \left[\partial_0 f(\tau)\, \tilde{\varphi}\right]_{\sigma=0}^{\sigma=\pi} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} d\tau \partial_0 f(\tau)\, [\tilde{\varphi}]_{\sigma=0}^{\sigma=\pi} = 0\,, \qquad (3.15)
\end{aligned}
$$

which is vanishing because of the periodicity of $\tilde{\varphi}$. So the action is invariant under the shift $\varphi \to \varphi + f(\tau)$. We get a similar result for $\delta\tilde{\varphi} = \tilde{f}(\tau) \Leftrightarrow \tilde{\varphi} \to \tilde{\varphi} + \tilde{f}(\tau)$ due to the periodicity of φ:

$$
\begin{aligned}
\delta \hat{S}_{sym}|_{\delta\varphi = f(\tau)} &= \frac{1}{2}\int d^2\xi\, \partial_1 \left[\partial_0 \tilde{f}(\tau)\, \varphi\right] \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} d\tau \left[\partial_0 \tilde{f}(\tau)\, \varphi\right]_{\sigma=0}^{\sigma=\pi} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} d\tau \partial_0 \tilde{f}(\tau)\, [\varphi]_{\sigma=0}^{\sigma=\pi} = 0\,. \qquad (3.16)
\end{aligned}
$$

The equation of motion and the boundary term for φ coming from the variation of $\hat{S}_{sym}$ are

$$\partial_1(\partial_0\tilde{\varphi} - \partial_1\varphi) = 0 \qquad (3.17)$$

and

$$
\begin{aligned}
&\frac{1}{2}\int_0^{\pi} d\sigma \left[\partial_1\tilde{\varphi}\delta\varphi\right]_{\tau=-\infty}^{\tau=+\infty} + \frac{1}{2}\int_{-\infty}^{+\infty} d\tau \left[\partial_0\tilde{\varphi}\delta\varphi\right]_{\sigma=0}^{\sigma=\pi} - \int_{-\infty}^{+\infty} d\tau \left[\partial_1\varphi\delta\varphi\right]_{\sigma=0}^{\sigma=\pi} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} d\tau \left[\partial_0\tilde{\varphi}\delta\varphi\right]_{\sigma=0}^{\sigma=\pi} - \int_{-\infty}^{+\infty} d\tau \left[\partial_1\varphi\delta\varphi\right]_{\sigma=0}^{\sigma=\pi} = 0\,, \quad (3.18)
\end{aligned}
$$

because $\delta\varphi(\tau = \pm\infty, \sigma) = 0\ \forall\, \sigma$.

Eq. (3.17) is of the second order and it is solved by $\partial_0\tilde{\varphi} - \partial_1\varphi = F(\tau)$. It can be made of the first order by using the invariance of the action (and hence of the equation of motion) under $\tilde{\varphi} \to \tilde{\varphi} + \tilde{f}(\tau)$. In this way, with a suitable choice of $\tilde{f}$, the equation of motion and the boundary term reduce to

$$\partial_0\tilde{\varphi} - \partial_1\varphi = 0 \quad \text{and} \quad -\frac{1}{2}\int_{-\infty}^{+\infty} d\tau \left[\partial_1\varphi\delta\varphi\right]_{\sigma=0}^{\sigma=\pi} = 0 \qquad (3.19)$$

for

$$\tilde{f} = \int d\tau\, F\,.$$

On the other hand, we get for the field $\tilde{\varphi}$:

$$\partial_1(\partial_0\varphi - \partial_1\tilde{\varphi}) = 0 \qquad (3.20)$$

and

$$
\begin{aligned}
&\frac{1}{2}\int_0^{\pi} d\sigma \left[\partial_1\varphi\delta\tilde{\varphi}\right]_{\tau=-\infty}^{\tau=+\infty} + \frac{1}{2}\int_{-\infty}^{+\infty} d\tau \left[\partial_0\varphi\delta\tilde{\varphi}\right]_{\sigma=0}^{\sigma=\pi} - \int_{-\infty}^{+\infty} d\tau \left[\partial_1\tilde{\varphi}\delta\tilde{\varphi}\right]_{\sigma=0}^{\sigma=\pi} \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} d\tau \left[\partial_0\varphi\delta\tilde{\varphi}\right]_{\sigma=0}^{\sigma=\pi} - \int_{-\infty}^{+\infty} d\tau \left[\partial_1\tilde{\varphi}\delta\tilde{\varphi}\right]_{\sigma=0}^{\sigma=\pi} = 0\,, \qquad (3.21)
\end{aligned}
$$

because $\delta\tilde{\varphi}(\tau = \pm\infty, \sigma) = 0\ \forall\, \sigma$.

Eqs. (3.20 - 3.21) become

$$\partial_0\varphi - \partial_1\tilde{\varphi} = 0 \quad \text{and} \quad -\frac{1}{2}\int_{-\infty}^{+\infty} d\tau\, [\partial_1\tilde{\varphi}\delta\tilde{\varphi}]_{\sigma=0}^{\sigma=\pi} = 0 \tag{3.22}$$

if we choose

$$f = \int d\tau\, \tilde{F}\,.$$

The equations of motion for φ and $\tilde{\varphi}$ in (3.19) and (3.22) can be summarized into a single mathemathical statement:

$$\begin{cases} \partial_0\tilde{\varphi} = \partial_1\varphi \\ \partial_1\tilde{\varphi} = \partial_0\varphi \end{cases} \quad \Leftrightarrow \quad \partial_a\tilde{\varphi} = -\epsilon_{ab}\,\partial^b\varphi. \tag{3.23}$$

On-shell the fields φ and $\tilde{\varphi}$ are (Hodge-)dual of each other, as it happened is section 2.1.1 for the string coordinate X^{25} and its T-dual $\tilde{X}^{25}$ (see eqs. (2.32) and (C.8)). It is worth noticing that the duality condition in eq. (3.23) is, as it is natural, equivalent to $\partial_a\varphi = -\epsilon_{ab}\,\partial^b\tilde{\varphi}$ and so it implies a wave equation for both φ and $\tilde{\varphi}$:

$$\partial^a\partial_a\tilde{\varphi} = -\epsilon_{ab}\partial^a\partial^b\varphi = 0\ ; \quad \partial^a\partial_a\varphi = -\epsilon_{ab}\partial^a\partial^b\tilde{\varphi} = 0\,.$$

Moreover, by interpreting eq. (3.23) as an off-shell condition, its insertion in the action

$$\hat{S} = \hat{S}_{sym} + \frac{1}{2}\int d^2\xi\ \epsilon^{ab}\partial_a\varphi\partial_b\tilde{\varphi}\,,$$

permits to recover the starting action (3.2) ($\epsilon^{ab}\,\epsilon_{ac} = -\delta^b{}_c$, see Appendix B).

3.1.1 Matricial form of $\hat{S}_{sym}$

The action $\hat{S}_{sym}$ can be written in a 'matricial form' by introducing the constant 2×2 matrices

$$C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ ; \quad M = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.24}$$

and the vector $\Phi \equiv (\varphi, \tilde{\varphi})$ ($\Phi^{\mathcal{A}=1} = \varphi$, $\Phi^{\mathcal{A}=2} = \tilde{\varphi}$):

$$\begin{aligned} \hat{S}_{sym}\left[\Phi^{\mathcal{A}}\right] &= \frac{1}{2}\int d^2\xi\ \left[\partial_0\Phi^T C\partial_1\Phi + \partial_1\Phi^T M\partial_1\Phi\right] \\ &= \frac{1}{2}\int d^2\xi\ \left[C_{\mathcal{AB}}\,\partial_0\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}} + M_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}}\right]. \end{aligned} \tag{3.25}$$

Obviously, the equations of motion and the boundary terms coming from the variation of (3.25) with respect to $\delta\Phi^{\mathcal{A}}$

$$\partial_1(C_{\mathcal{AB}}\,\partial_0\Phi^{\mathcal{B}} + M_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{B}}) = 0 \tag{3.26}$$

and

$$\frac{1}{2}\int d^2\xi\,\left[\partial_0(C_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}})+\partial_1(C_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_0\Phi^{\mathcal{B}})+2\partial_1(M_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}})\right]$$
$$=\frac{1}{2}\int d^2\xi\,\partial_a(\epsilon^{ab}C_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_b\Phi^{\mathcal{B}})+\int d^2\xi\,\partial_1\left[(C_{\mathcal{AB}}\,\partial_0\Phi^{\mathcal{B}}+M_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{B}})\delta\Phi^{\mathcal{A}}\right]=0 \tag{3.27}$$

are completely equivalent to (3.19) and (3.22).

The action $\hat{S}_{sym}$ exhibits the invariance under $\Phi^{\mathcal{A}}\to\Phi^{\mathcal{A}}+\mathbf{f}^{\mathcal{A}}(\tau)$. The corresponding variation of the action is:

$$\begin{aligned}\delta\hat{S}_{sym}|_{\delta\Phi=f(\tau)} &= \frac{1}{2}\int d^2\xi\,\partial_1\left[C_{\mathcal{AB}}\,\partial_0\mathbf{f}^{\mathcal{A}}(\tau)\,\Phi^{\mathcal{B}}\right]\\ &= \frac{1}{2}\int_{-\infty}^{+\infty}d\tau\,\left[C_{\mathcal{AB}}\,\partial_0\mathbf{f}^{\mathcal{A}}(\tau)\,\Phi^{\mathcal{B}}\right]_{\sigma=0}^{\sigma=\pi}\\ &= \frac{1}{2}\int_{-\infty}^{+\infty}d\tau\,C_{\mathcal{AB}}\,\partial_0\mathbf{f}^{\mathcal{A}}(\tau)\,\left[\Phi^{\mathcal{B}}\right]_{\sigma=0}^{\sigma=\pi}=0\,,\end{aligned} \tag{3.28}$$

because of the periodicity of φ and $\tilde{\varphi}$.

The solutions of (3.26) are $C_{\mathcal{AB}}\,\partial_0\Phi^{\mathcal{B}}+M_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{B}}=\mathcal{F}_{\mathcal{A}}(\tau)$ and can be made

$$C_{\mathcal{AB}}\,\partial_0\Phi^{\mathcal{B}}+M_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{B}}=0 \tag{3.29}$$

by choosing

$$\mathbf{f}^{\mathcal{A}}=\int d\tau\,C^{\mathcal{AB}}\mathcal{F}_{\mathcal{B}}$$

(let us recall that $C^{\mathcal{AB}}\equiv(C^{-1})_{\mathcal{AB}}$). The results of the previous section are recovered once we have

$$\mathbf{f}(\tau)=\begin{pmatrix}\tilde{f}(\tau)\\ f(\tau)\end{pmatrix}\quad\text{and}\quad\mathcal{F}(\tau)=\begin{pmatrix}\tilde{F}(\tau)\\ F(\tau)\end{pmatrix}.$$

The boundary terms, evaluated on the equations of motion, read:

$$\begin{aligned}&\tfrac{1}{2}\int d^2\xi\,\partial_a(\epsilon^{ab}C_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_b\Phi^{\mathcal{B}})\\ &=\tfrac{1}{2}\int d^2\xi\,\partial_0(C_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}})-\tfrac{1}{2}\int d^2\xi\,\partial_1(C_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_0\Phi^{\mathcal{B}})\\ &=\tfrac{1}{2}\int_0^{\pi}d\sigma\,\left[C_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}}\right]_{\tau=-\infty}^{\tau=+\infty}-\tfrac{1}{2}\int_{-\infty}^{+\infty}d\tau\,\left[C_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_0\Phi^{\mathcal{B}}\right]_{\sigma=0}^{\sigma=\pi}\\ &=-\tfrac{1}{2}\int_{-\infty}^{+\infty}d\tau\,\left[C_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_0\Phi^{\mathcal{B}}\right]_{\sigma=0}^{\sigma=\pi}=0\,,\end{aligned} \tag{3.30}$$

because $\delta\Phi^{\mathcal{A}}(\tau=\pm\infty,\sigma)=0\;\forall\,\sigma$.

Let us observe that the matrix C is the one involved in the definition of the group $O(1,1)$, called Ω (see section 2.6). It has the following properties: $C \equiv \Omega = \Omega^T = \Omega^{-1}$.

The action (3.25) is invariant under $O(1,1)$-transformations acting both on the vectors (Φ) and on the 'couplings' (M) according to the following scheme:

$$\Lambda \in O(1,1) \Leftrightarrow \Lambda^T \Omega \Lambda = \Omega$$

$$\Phi' = \Lambda^{-1}\Phi \ ; \quad C' = \Lambda^T C \Lambda = \Lambda^T \Omega \Lambda = \Omega \ ; \quad M' = \Lambda^T M \Lambda \, . \tag{3.31}$$

It is worth noticing that the action (3.11) is manifestly invariant under the exchange $\varphi \leftrightarrow \tilde{\varphi}$; this symmetry is realized through the "inversion" $O(1,1)$ matrix $\Lambda = \Omega$ implying

$$\Phi' = \Omega\Phi = \begin{pmatrix} \tilde{\varphi} \\ \varphi \end{pmatrix} \ ; \quad M' = \Omega M \Omega = M^{-1} (= M) \, . \tag{3.32}$$

3.2 The chiral coordinates

We can introduce a pair of new fields $\varphi_+ = \varphi_+(\tau,\sigma)$ and $\varphi_- = \varphi_-(\tau,\sigma)$, such that

$$\begin{cases} \varphi = \frac{1}{\sqrt{2}}(\varphi_+ + \varphi_-) \\ \tilde{\varphi} = \frac{1}{\sqrt{2}}(\varphi_+ - \varphi_-) \end{cases} \Leftrightarrow \begin{cases} \varphi_+ = \frac{1}{\sqrt{2}}(\varphi + \tilde{\varphi}) \\ \varphi_- = \frac{1}{\sqrt{2}}(\varphi - \tilde{\varphi}) \end{cases} \left(\Leftrightarrow \varphi_\pm = \frac{1}{\sqrt{2}}(\varphi \pm \tilde{\varphi}) \right) . \tag{3.33}$$

In this new basis the fields φ_+ and φ_- are arranged into the vector $\chi \equiv (\varphi_+, \varphi_-)$. The vectors Φ and χ are linked through

$$\chi = \mathcal{Z}\Phi \quad \Leftrightarrow \quad \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \varphi \\ \tilde{\varphi} \end{pmatrix} \tag{3.34}$$

and the matrices C and M transform as follows:

$$C' = (\mathcal{Z}^{-1})^T C \mathcal{Z}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.35}$$

and

$$M' = (\mathcal{Z}^{-1})^T M \mathcal{Z}^{-1} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{3.36}$$

The action, consequently, transforms in

$$\begin{aligned} \hat{S}_{sym}\left[\chi^{\mathcal{A}}\right] &= \frac{1}{2}\int d^2\xi \ \left[\partial_0 \chi^T C' \partial_1 \chi + \partial_1 \chi^T M' \partial_1 \chi\right] \\ &= \frac{1}{2}\int d^2\xi \ \left[C'_{\mathcal{A}\mathcal{B}}\, \partial_0 \chi^{\mathcal{A}} \partial_1 \chi^{\mathcal{B}} + M'_{\mathcal{A}\mathcal{B}}\, \partial_1 \chi^{\mathcal{A}} \partial_1 \chi^{\mathcal{B}}\right] \\ &= \int d^2\xi \ \left[\mathcal{L}_+(\partial_a \varphi_+) + \mathcal{L}_-(\partial_a \varphi_-)\right] , \end{aligned} \tag{3.37}$$

with

$$\mathcal{L}_{\pm}(\partial_a\varphi_{\pm}) = \pm\frac{1}{2}\partial_0\varphi_{\pm}\partial_1\varphi_{\pm} - \frac{1}{2}(\partial_1\varphi_{\pm})^2\,. \tag{3.38}$$

The dynamics of the fields φ_+ and φ_- are completely decoupled.

The Lagrangians in (3.38) are the so-called *Floreanini-Jackiw Lagrangians* [20], a class of first order Lagrangians extensively studied in literature. The equations of motion for φ_+ and φ_- are easily computed from (3.29):

$$C_{\mathcal{AB}}\,\partial_0\Phi^{\mathcal{B}} + M_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{B}} = 0 \quad \rightarrow \quad C'_{\mathcal{AB}}\,\partial_0\chi^{\mathcal{B}} + M'_{\mathcal{AB}}\,\partial_1\chi^{\mathcal{B}} = 0 \quad \Rightarrow$$

$$\Rightarrow \begin{cases} \partial_0\varphi_+ = \ \ \partial_1\varphi_+ \\ \partial_0\varphi_- = -\partial_1\varphi_- \end{cases} \Rightarrow \begin{cases} \partial_-\varphi_+ = 0 \\ \partial_+\varphi_- = 0 \end{cases} \Rightarrow \begin{cases} \varphi_+ = \varphi_+(\tau+\sigma) \\ \varphi_- = \varphi_-(\tau-\sigma) \end{cases}. \tag{3.39}$$

The previous equations show that φ_+ and φ_- are, respectively, functions of $\sigma^+ \equiv \tau+\sigma$ and $\sigma^- \equiv \tau-\sigma$ only on-shell.

The boundary terms, on-shell, are (see eq. (3.30))

$$\begin{aligned} -\frac{1}{2}\int_{-\infty}^{+\infty} d\tau\,\left[C_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_0\Phi^{\mathcal{B}}\right]_{\sigma=0}^{\sigma=\pi} &\rightarrow -\frac{1}{2}\int_{-\infty}^{+\infty} d\tau\,\left[C'_{\mathcal{AB}}\,\delta\chi^{\mathcal{A}}\partial_0\chi^{\mathcal{B}}\right]_{\sigma=0}^{\sigma=\pi} \\ = -\frac{1}{2}\int_{-\infty}^{+\infty} d\tau\,[\delta\varphi_+\partial_0\varphi_+]_{\sigma=0}^{\sigma=\pi} &+ \frac{1}{2}\int_{-\infty}^{+\infty} d\tau\,[\delta\varphi_-\partial_0\varphi_-]_{\sigma=0}^{\sigma=\pi} = 0\,. \end{aligned} \tag{3.40}$$

and they are satisfied once one imposes periodicity conditions in σ.

3.3 Invariances of the Lagrangians and quantization

The Lagrangians

$$\mathcal{L}_{\pm} = \pm\frac{1}{2}\partial_0\varphi_{\pm}\partial_1\varphi_{\pm} - \frac{1}{2}(\partial_1\varphi_{\pm})^2$$

and

$$\hat{\mathcal{L}}_{sym} \equiv \frac{1}{2}\partial_0\varphi\partial_1\tilde{\varphi} + \frac{1}{2}\partial_0\tilde{\varphi}\partial_1\varphi - \frac{1}{2}(\partial_1\varphi)^2 - \frac{1}{2}(\partial_1\tilde{\varphi})^2 \tag{3.41}$$

exhibit a series of invariances. Let us examine $\mathcal{L}_{\pm}$ first:

- space-time translations (we do not indicate the constant parameters of the transformations)

$$\delta_\tau\varphi_{\pm} = \partial_0\varphi_{\pm} \qquad \Rightarrow \qquad \delta_\tau\mathcal{L}_{\pm} = \partial_0\mathcal{L}_{\pm}\,; \tag{3.42}$$

$$\delta_\sigma\varphi_{\pm} = \partial_1\varphi_{\pm} \qquad \Rightarrow \qquad \delta_\sigma\mathcal{L}_{\pm} = \partial_1\mathcal{L}_{\pm}\,; \tag{3.43}$$

- an analogue of Lorentz transformations

$$\delta_L\varphi_{\pm} = (\tau\pm\sigma)\partial_1\varphi_{\pm} \qquad \Rightarrow \qquad \delta_L\mathcal{L}_{\pm} = \partial_1[(\tau\pm\sigma)\mathcal{L}_{\pm}]\,; \tag{3.44}$$

- "conformal" transformations

$$(\delta\varphi_\pm)_{conf} = f(\tau \pm \sigma)\partial_1\varphi_\pm \;\;\Rightarrow\;\; (\delta\mathcal{L}_\pm)_{conf} = \partial_1[f(\tau \pm \sigma)\mathcal{L}_\pm]\,; \qquad (3.45)$$

- scale symmetry

$$(\delta\varphi_\pm)_{scale} = \tau\partial_0\varphi_\pm + \sigma\partial_1\varphi_\pm \;\;\Rightarrow\;\; (\delta\mathcal{L}_\pm)_{scale} = \partial_0(\tau\mathcal{L}_\pm) + \partial_1(\sigma\mathcal{L}_\pm)\,. \quad (3.46)$$

The invariances of $\hat{\mathcal{L}}_{sym}$ are:

- space-time translations

$$\begin{cases} \delta_\tau\varphi = \partial_0\varphi \\ \delta_\tau\tilde{\varphi} = \partial_0\tilde{\varphi} \end{cases} \qquad \Rightarrow \qquad \delta_\tau\hat{\mathcal{L}}_{sym} = \partial_0\hat{\mathcal{L}}_{sym}\,; \qquad (3.47)$$

$$\begin{cases} \delta_\sigma\varphi = \partial_1\varphi \\ \delta_\sigma\tilde{\varphi} = \partial_1\tilde{\varphi} \end{cases} \qquad \Rightarrow \qquad \delta_\sigma\hat{\mathcal{L}}_{sym} = \partial_1\hat{\mathcal{L}}_{sym}\,; \qquad (3.48)$$

- eq. (3.44) implies the corresponding Lorentz symmetry

$$\delta_L\varphi = \tau\partial_1\varphi + \sigma\partial_1\tilde{\varphi} \qquad \text{and} \qquad \delta_L\tilde{\varphi} = \tau\partial_1\tilde{\varphi} + \sigma\partial_1\varphi\,. \qquad (3.49)$$

It is worth noticing that eqs. (3.44) and (3.49), on-shell ($\partial_0\varphi_\pm = \pm\partial_1\varphi_\pm$ and $\partial_0\tilde{\varphi} = \partial_1\varphi$, $\partial_1\tilde{\varphi} = \partial_0\varphi$), reproduce the usual Lorentz rotations:

$$\delta\varphi_\pm = \tau\partial_1\varphi_\pm + \sigma\partial_0\varphi_\pm \qquad (3.50)$$

$$\delta\varphi = \tau\partial_1\varphi + \sigma\partial_0\varphi \qquad (3.51)$$

$$\delta\tilde{\varphi} = \tau\partial_1\tilde{\varphi} + \sigma\partial_0\tilde{\varphi}\,. \qquad (3.52)$$

So, on-shell, Lorentz invariance is completely restored.

In the chiral basis, it is straightforward to quantize the Lagrangians $\mathcal{L}_\pm$ (see Appendix E) and, consequently, $\hat{\mathcal{L}}_{sym}$. It results that

$$[\varphi_\pm(\tau,\sigma), \varphi_\pm(\tau,\sigma')] = \mp i\epsilon(\sigma - \sigma') \qquad (3.53)$$

$$[\mathcal{P}_\pm(\tau,\sigma), \mathcal{P}_\pm(\tau,\sigma')] = \pm\frac{i}{4}\delta'(\sigma - \sigma') \qquad (3.54)$$

$$[\varphi_\pm(\tau,\sigma), \mathcal{P}_\pm(\tau,\sigma')] = \frac{i}{2}\delta(\sigma - \sigma')\,, \qquad (3.55)$$

where

$$\mathcal{P}_\pm \equiv \frac{\partial\mathcal{L}_\pm}{\partial(\partial_0\varphi_\pm)} = \pm\frac{1}{2}\partial_1\varphi_\pm\,. \qquad (3.56)$$

Eq. (3.33) implies

$$[\varphi(\tau,\sigma), \varphi(\tau,\sigma')] = [\tilde{\varphi}(\tau,\sigma), \tilde{\varphi}(\tau,\sigma')] = 0\,, \qquad (3.57)$$

$$[\varphi(\tau,\sigma), \tilde{\varphi}(\tau,\sigma')] = -i\epsilon(\sigma - \sigma')\,. \qquad (3.58)$$

3.4 Hamiltonian formulation

In this section we want to study the form of the Hamiltonian densities relative to the models described by the Lagrangian densities $\hat{\mathcal{L}}_{sym}$ and $\mathcal{L}_{\pm}$. Let us start from

$$\begin{aligned}\hat{\mathcal{L}}_{sym} &= \frac{1}{2}\partial_0\varphi\partial_1\tilde{\varphi} + \frac{1}{2}\partial_0\tilde{\varphi}\partial_1\varphi - \frac{1}{2}(\partial_1\varphi)^2 - \frac{1}{2}(\partial_1\tilde{\varphi})^2 \\ &= \frac{1}{2}\left[C_{\mathcal{AB}}\,\partial_0\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}} + M_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}}\right].\end{aligned}$$

We can compute the conjugate momenta with respect to φ and $\tilde{\varphi}$ or, equivalently, the ones conjugate to $\Phi^{\mathcal{A}}$:

$$\mathcal{P} \equiv \frac{\partial\hat{\mathcal{L}}_{sym}}{\partial(\partial_0\varphi)} = \frac{1}{2}\partial_1\tilde{\varphi} \tag{3.59}$$

$$\tilde{\mathcal{P}} \equiv \frac{\partial\hat{\mathcal{L}}_{sym}}{\partial(\partial_0\tilde{\varphi})} = \frac{1}{2}\partial_1\varphi \tag{3.60}$$

$$\mathcal{P}_{\mathcal{A}} \equiv \frac{\partial\hat{\mathcal{L}}_{sym}}{\partial(\partial_0\Phi^{\mathcal{A}})} = \frac{1}{2}C_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{B}}\,. \tag{3.61}$$

With the definitions (3.59 - 3.60), we can compute other Dirac Brackets (or commutators) involving φ, $\tilde{\varphi}$, $\mathcal{P}$ and $\tilde{\mathcal{P}}$:

$$[\mathcal{P}(\tau,\sigma),\mathcal{P}(\tau,\sigma')] = [\tilde{\mathcal{P}}(\tau,\sigma),\tilde{\mathcal{P}}(\tau,\sigma')] = 0 \tag{3.62}$$

$$[\varphi(\tau,\sigma),\mathcal{P}(\tau,\sigma')] = [\tilde{\varphi}(\tau,\sigma),\tilde{\mathcal{P}}(\tau,\sigma')] = \frac{i}{2}\delta(\sigma-\sigma')\,. \tag{3.63}$$

The Hamiltonian density is now obtained through the Legendre transformation:

$$\begin{aligned}\mathcal{H} &\equiv \mathcal{P}\partial_0\varphi + \tilde{\mathcal{P}}\partial_0\tilde{\varphi} - \hat{\mathcal{L}}_{sym} \\ &= \frac{1}{2}(\partial_1\varphi)^2 + \frac{1}{2}(\partial_1\tilde{\varphi})^2 \end{aligned} \tag{3.64}$$

or

$$\begin{aligned}\mathcal{H} &\equiv \mathcal{P}_{\mathcal{A}}\partial_0\Phi^{\mathcal{A}} - \hat{\mathcal{L}}_{sym} \\ &= -\frac{1}{2}M_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}}\,. \end{aligned} \tag{3.65}$$

It consists in the 'potential' part of the Lagrangian density, as it happens for the Floreanini-Jackiw Lagrangians (see Appendix E). The Hamiltonian density results positive-definite, although the presence of the minus sign in eq. (3.65). A way to avoid this matter consists in defining $M = -\mathcal{G}$ ($M_{\mathcal{AB}} = -\mathcal{G}_{\mathcal{AB}}$). This little change leads to

$$\mathcal{H} = \frac{1}{2}\mathcal{G}_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}} \geq 0\,.$$

In the chiral basis, $\mathcal{H}$ assumes the form

$$\begin{aligned}
\mathcal{H} &= -\frac{1}{2}M'_{\mathcal{AB}}\,\partial_1\chi^{\mathcal{A}}\partial_1\chi^{\mathcal{B}} \\
&= \frac{1}{2}(\partial_1\varphi_+)^2 + \frac{1}{2}(\partial_1\varphi_-)^2 \\
&= \frac{1}{2}(\partial_+\varphi_+)^2 - \partial_+\varphi_+\partial_-\varphi_+ + \frac{1}{2}(\partial_-\varphi_+)^2 \\
& \frac{1}{2}(\partial_+\varphi_-)^2 - \partial_+\varphi_-\partial_-\varphi_- + \frac{1}{2}(\partial_-\varphi_-)^2 \,. \qquad (3.66)
\end{aligned}$$

The same result can be obtained by using the conjugate momenta with respect to φ_+, φ_- defined in eq. (3.56) or by introducing the ones conjugate to $\chi^{\mathcal{A}}$:

$$\mathcal{P}^{(\pm)}_{\mathcal{A}} \equiv \frac{\partial\hat{\mathcal{L}}_{sym}}{\partial(\partial_0\chi^{\mathcal{A}})} = \frac{1}{2}C'_{\mathcal{AB}}\,\partial_1\chi^{\mathcal{B}} \qquad (3.67)$$

and by performing a Legendre transformation on the sum of $\mathcal{L}_+$ and $\mathcal{L}_-$.

On-shell for φ_+ and φ_-, the Hamiltonian density reduces to

$$\mathcal{H} = \frac{1}{2}(\partial_+\varphi_+)^2 + \frac{1}{2}(\partial_-\varphi_-)^2 \,. \qquad (3.68)$$

The quantities computed in this section will find their generalizations and a deeper meaning in Chapter 5.

Chapter 4

T-dual invariant formulation: 'curved' scalar field

Since the string world-sheet is *-a priori-* a curved manifold (before choosing the conformal gauge), then it is necessary to generalize the results found in Chapter 3 for a scalar field defined on a flat background to encompass the possibility to deal with a curved one [16, 17].

4.1 Scalar field on a curved background

If one wants to couple a scalar field φ to an "external" metric $h_{\alpha\beta}$, one has to substitute partial derivatives with the covariant ones ($\partial_a \to \nabla_\alpha$) and to multiply by $\sqrt{-h}$. In our bidimensional theory, once a 2-bein is introduced, the prescriptions just quoted get modified: $\partial_a \to \nabla_a \equiv e_a{}^\alpha \nabla_\alpha$ and $\sqrt{-h} = e$ (see Appendix B).

In order to describe the dynamics of a scalar field $\varphi = \varphi(\tau, \sigma)$ defined on a curved manifold, let us introduce the action

$$S^{(0)}\left[h_{\alpha\beta};\varphi\right] = -\frac{1}{2}\int d^2\xi\,\sqrt{-h}\,h^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi\,, \tag{4.1}$$

which takes the following form when a 2-bein is introduced:

$$\begin{aligned}
S^{(0)}\left[e^a{}_\alpha;\varphi\right] &= -\frac{1}{2}\int d^2\xi\, e\,\eta^{ab}\, e_a{}^\alpha\nabla_\alpha\varphi\, e_b{}^\beta\nabla_\beta\varphi \\
&= -\frac{1}{2}\int d^2\xi\, e\,\eta^{ab}\,\nabla_a\varphi\nabla_b\varphi \\
&= \frac{1}{2}\int d^2\xi\, e\left[(\nabla_0\varphi)^2 - (\nabla_1\varphi)^2\right].
\end{aligned} \tag{4.2}$$

At this level, since the action is manifestly local Lorentz invariant, the equation of motion and the boundary terms for the field φ can be equivalently computed from (4.1) or (4.2), to obtain

$$\partial_\alpha(\sqrt{-h}\, h^{\alpha\beta}\partial_\beta\varphi) = 0 \tag{4.3}$$

and

$$-\int_{-\infty}^{+\infty} d\tau \left[(\sqrt{-h}\, h^{1\beta}\partial_\beta\varphi)\delta\varphi\right]_{\sigma=0}^{\sigma=\pi} = 0\,, \tag{4.4}$$

where $\delta\varphi(\tau = \pm\infty, \sigma) = 0\ \forall\,\sigma$ (see (1.6) and (1.7)).

In analogy with the previous Chapter, let us define the auxiliary field $p(\tau,\sigma) = \nabla_0\varphi$, compatible with $p \equiv e^{-1}\partial\mathcal{L}^{(0)}/\partial(\nabla_0\varphi)$, to get the action

$$S' = \int d^2\xi\, e\left[p\,\nabla_0\varphi - \frac{1}{2}p^2 - \frac{1}{2}(\nabla_1\varphi)^2\right]. \tag{4.5}$$

Then we put $p(\tau,\sigma) = \nabla_1\tilde{\varphi}$ and so the action becomes:

$$\begin{aligned} \hat{S}\,[e^a{}_\alpha;\varphi,\tilde{\varphi}] &= \int d^2\xi\, e\left[\nabla_0\varphi\nabla_1\tilde{\varphi} - \frac{1}{2}(\nabla_1\varphi)^2 - \frac{1}{2}(\nabla_1\tilde{\varphi})\right] \qquad (4.6)\\ &= \int d^2\xi\, e\left[e_0{}^\alpha e_1{}^\beta\nabla_\alpha\varphi\nabla_\beta\tilde{\varphi} - \frac{1}{2}e_1{}^\alpha e_1{}^\beta(\nabla_\alpha\varphi\nabla_\beta\varphi + \nabla_\alpha\tilde{\varphi}\nabla_\beta\tilde{\varphi})\right]. \end{aligned}$$

As in the flat case, the action (4.6) can be symmetrized by recalling that $e\,e_0{}^\alpha e_1{}^\beta = \epsilon^{\alpha\beta} + e\,e_1{}^\alpha e_0{}^\beta$ (see Appendix B). The result is

$$\begin{aligned} \hat{S} &= \hat{S}_{sym} + \frac{1}{2}\int d^2\xi\,\epsilon^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\tilde{\varphi} \\ &= \hat{S}_{sym} + \frac{1}{2}\int d^2\xi\,\epsilon^{\alpha\beta}\,\partial_\alpha\varphi\,\partial_\beta\tilde{\varphi}\,, \qquad (4.7) \end{aligned}$$

being

$$\hat{S}_{sym}\,[e_a{}^\alpha;\varphi,\tilde{\varphi}] = \frac{1}{2}\int d^2\xi\, e\left[\nabla_0\varphi\nabla_1\tilde{\varphi} + \nabla_0\tilde{\varphi}\nabla_1\varphi - (\nabla_1\varphi)^2 - (\nabla_1\tilde{\varphi})^2\right]. \tag{4.8}$$

4.2 Local symmetries of the action $\hat{S}_{sym}$

The action $\hat{S}_{sym}$ exhibits invariance under the following transformations:

- Weyl transformations: they act on the 2-bein as follows

$$e^a{}_\alpha \to \lambda(\tau,\sigma)e^a{}_\alpha \quad \text{or} \quad \delta e^a{}_\alpha = \lambda(\tau,\sigma)e^a{}_\alpha\,. \tag{4.9}$$

  These kinds of transformations imply $e \to \lambda^2 e$ and, consequently, $e_a{}^\alpha \to \lambda^{-1}e_a{}^\alpha$. Terms like $e\,e_a{}^\alpha e_b{}^\beta$, involved in (4.8), are left invariant;

- bidimensional diffeomorphisms: $\xi^a \to \xi'^a(\xi)$. By introducing the notation $J \equiv \det\left(\partial\xi^a/\partial\xi'^b\right)$, it happens that $d^2\xi \to d^2\xi'\, J$, $e \to e' J^{-1}$ and $\nabla_a \to \nabla'_a = \nabla_a$. The fields φ and $\tilde{\varphi}$ do not change at all.

As it can be immediately verified, (4.8) is not manifestly invariant under local Lorentz transformations:

$$\delta e^a_{\ \alpha} = \omega^a_{\ b}(\tau,\sigma) e^b_{\ \alpha}\,, \tag{4.10}$$

with $\omega_{ab} = -\omega_{ba}$. In the following, the choice $\omega_{ab}(\tau,\sigma) = \alpha(\tau,\sigma)\epsilon_{ab}$ will be made.

As in the previous Chapter, we can write the action $\hat{S}_{sym}$ in a matricial form, by introducing the constant matrices C and M and the vector $\Phi \equiv (\varphi, \tilde{\varphi})$:

$$\begin{aligned}\hat{S}_{sym}\left[e^a_{\ \alpha};\Phi^{\mathcal{A}}\right] &= \frac{1}{2}\int d^2\xi\, e\left[\nabla_0\Phi^T C\nabla_1\Phi + \nabla_1\Phi^T M\nabla_1\Phi\right]\\ &= \frac{1}{2}\int d^2\xi\, e\left[C_{\mathcal{AB}}\nabla_0\Phi^{\mathcal{A}}\nabla_1\Phi^{\mathcal{B}} + M_{\mathcal{AB}}\nabla_1\Phi^{\mathcal{A}}\nabla_1\Phi^{\mathcal{B}}\right] \\ &= \frac{1}{2}\int d^2\xi\, e\left[e_0^{\ \alpha}e_1^{\ \beta}C_{\mathcal{AB}} + e_1^{\ \alpha}e_1^{\ \beta}M_{\mathcal{AB}}\right]\nabla_\alpha\Phi^{\mathcal{A}}\nabla_\beta\Phi^{\mathcal{B}}\,.\end{aligned} \tag{4.11}$$

Weyl invariance of $\hat{S}_{sym}$ is equivalent to

$$\frac{\delta\hat{S}_{sym}}{\delta e^a_{\ \alpha}}\left(\delta e^a_{\ \alpha}\right)_{Weyl} = \frac{\delta\hat{S}_{sym}}{\delta e^a_{\ \alpha}}\left[\lambda(\tau,\sigma)e^a_{\ \alpha}\right] = \lambda(\tau,\sigma)\frac{\delta\hat{S}_{sym}}{\delta e^a_{\ \alpha}}e^a_{\ \alpha} = 0 \tag{4.12}$$

and the variation of the action under local Lorentz transformations has the form

$$\frac{\delta\hat{S}_{sym}}{\delta e^a_{\ \alpha}}\left(\delta e^a_{\ \alpha}\right)_{Lorentz} = \frac{\delta\hat{S}_{sym}}{\delta e^a_{\ \alpha}}\left[\alpha(\tau,\sigma)\epsilon^a_{\ b}e^b_{\ \alpha}\right] = \alpha(\tau,\sigma)\epsilon^a_{\ b}\frac{\delta\hat{S}_{sym}}{\delta e^a_{\ \alpha}}e^b_{\ \alpha}\,. \tag{4.13}$$

The last two equations strongly suggest to introduce a tensor so defined:

$$t_a^{\ b} \equiv e^{-1}\frac{\delta\hat{S}_{sym}}{\delta e^a_{\ \alpha}}e^b_{\ \alpha}\,. \tag{4.14}$$

In this way, the variations (4.12) and (4.13) are proportional, respectively, to the trace $t \equiv t_a^{\ a}$ and the ϵ-trace $\hat{t} \equiv \epsilon^a_{\ b}t_a^{\ b}$ of the tensor $t_a^{\ b}$.

The components of the tensor $t_a^{\ b}$ can be easily computed from (4.11) to obtain

$$\begin{aligned}t_a^{\ b} &= \frac{1}{2}\left\{\delta_a^{\ b}\left[C_{\mathcal{AB}}\nabla_0\Phi^{\mathcal{A}}\nabla_1\Phi^{\mathcal{B}} + M_{\mathcal{AB}}\nabla_1\Phi^{\mathcal{A}}\nabla_1\Phi^{\mathcal{B}}\right] - \delta_0^{\ b}C_{\mathcal{AB}}\nabla_a\Phi^{\mathcal{A}}\nabla_1\Phi^{\mathcal{B}}\right.\\ &\quad\left. -\delta_1^{\ b}C_{\mathcal{AB}}\nabla_0\Phi^{\mathcal{A}}\nabla_a\Phi^{\mathcal{B}} - 2\delta_1^{\ b}M_{\mathcal{AB}}\nabla_a\Phi^{\mathcal{A}}\nabla_1\Phi^{\mathcal{B}}\right\}\end{aligned} \tag{4.15}$$

leading to

$$t_0^{\ 0} = \frac{1}{2}M_{\mathcal{AB}}\nabla_1\Phi^{\mathcal{A}}\nabla_1\Phi^{\mathcal{B}} \tag{4.16}$$

$$
\begin{aligned}
&= -\frac{1}{2}\left[(\nabla_1\varphi)^2+(\nabla_1\tilde{\varphi})^2\right] \\
t_0{}^1 &= \frac{1}{2}\left[-C_{\mathcal{AB}}\nabla_0\Phi^{\mathcal{A}}\nabla_0\Phi^{\mathcal{B}}-2M_{\mathcal{AB}}\nabla_0\Phi^{\mathcal{A}}\nabla_1\Phi^{\mathcal{B}}\right] \qquad (4.17)\\
&= -\nabla_0\varphi\nabla_0\tilde{\varphi}+\nabla_0\varphi\nabla_1\varphi+\nabla_0\tilde{\varphi}\nabla_1\tilde{\varphi}
\end{aligned}
$$

$$
\begin{aligned}
t_1{}^0 &= -\frac{1}{2}C_{\mathcal{AB}}\nabla_1\Phi^{\mathcal{A}}\nabla_1\Phi^{\mathcal{B}} \qquad (4.18)\\
&= -\nabla_1\varphi\nabla_1\tilde{\varphi}
\end{aligned}
$$

$$
\begin{aligned}
t_1{}^1 &= -\frac{1}{2}M_{\mathcal{AB}}\nabla_1\Phi^{\mathcal{A}}\nabla_1\Phi^{\mathcal{B}} \qquad (4.19)\\
&= \frac{1}{2}\left[(\nabla_1\varphi)^2+(\nabla_1\tilde{\varphi})^2\right] = -t_0{}^0\,.
\end{aligned}
$$

It is immediate to see that $t \equiv t_a{}^a = t_0{}^0 + t_1{}^1 = 0$.

Let us now compute $t_{ab} = t_a{}^c\,\eta_{cb}$:

$$
\begin{aligned}
t_{00} &= -t_0{}^0 \qquad (4.20)\\
t_{01} &= +t_0{}^1 \qquad (4.21)\\
t_{10} &= -t_1{}^0 \qquad (4.22)\\
t_{11} &= +t_1{}^1\,. \qquad (4.23)
\end{aligned}
$$

On the equations of motion for the 2-bein, $\delta\hat{S}_{sym}/\delta e^a{}_\alpha = 0$, the components of $t_a{}^b$ are vanishing. This implies also the vanishing of the ϵ-trace: $\hat{t} \equiv \epsilon^a{}_b t_a{}^b = \epsilon^{ab}t_{ab} = t_{01} - t_{10} = t_0{}^1 + t_1{}^0 = 0$. Equivalently, in matricial form,

$$
\hat{t} = -\frac{1}{2}\left[\nabla_0\Phi^T C\nabla_0\Phi + \nabla_0\Phi^T M\nabla_1\Phi + \nabla_1\Phi^T M\nabla_0\Phi + \nabla_1\Phi^T C\nabla_1\Phi\right]. \qquad (4.24)
$$

By adding and subtracting the term $(1/2)\nabla_1\Phi^T MC^{-1}M\nabla_1\Phi$, the ϵ-trace $\hat{t}$ becomes

$$
\begin{aligned}
\hat{t} &= -\frac{1}{2}\left[(\nabla_0\Phi^T C+\nabla_1\Phi^T M)C^{-1}(C\nabla_0\Phi+M\nabla_1\Phi)\right.\\
&\quad \left.+\nabla_1\Phi^T(C-MC^{-1}M)\nabla_1\Phi\right]\\
&= -\frac{1}{2}(\nabla_0\Phi^T C+\nabla_1\Phi^T M)C^{-1}(C\nabla_0\Phi+M\nabla_1\Phi)\,, \qquad (4.25)
\end{aligned}
$$

because $C = MC^{-1}M$ (see Section 3.1.1).

Moreover, because of the form of the matrices C and M, also in the curved case the action $\hat{S}_{sym}$ is $O(1,1)$-invariant. In particular the duality transformation $\Lambda = \Omega$ still holds.

As in the flat case, the action $\hat{S}_{sym}$ remains unchanged by performing the shift $\Phi^{\mathcal{A}} \to \Phi^{\mathcal{A}} + \mathbf{f}^{\mathcal{A}}(\tau,\sigma) \Leftrightarrow \delta\Phi^{\mathcal{A}} = \mathbf{f}^{\mathcal{A}}(\tau,\sigma)$, with the functions $\mathbf{f}^{\mathcal{A}}$ satisfying the conditions $\nabla_1 \mathbf{f}^{\mathcal{A}} = 0$ and $\mathbf{f}^{\mathcal{A}}(\tau,\sigma+\pi) = \mathbf{f}^{\mathcal{A}}(\tau,\sigma)$. In fact,

$$\begin{aligned} \delta\hat{S}_{sym}|_{\delta\Phi=f(\tau,\sigma)} &= \frac{1}{2}\int_{-\infty}^{+\infty} d\tau \left[C_{\mathcal{AB}}\,\partial_0\mathbf{f}^{\mathcal{A}}\Phi^{\mathcal{B}}\right]_{\sigma=0}^{\sigma=\pi} \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} d\tau\, C_{\mathcal{AB}} \left[\partial_0\mathbf{f}^{\mathcal{A}}\Phi^{\mathcal{B}}\right]_{\sigma=0}^{\sigma=\pi} = 0\,. \end{aligned} \tag{4.26}$$

The equations of motion and the boundary terms for the fields $\Phi^{\mathcal{A}}$ are the following:

$$\partial_\alpha\left[e\, e_1{}^\alpha (C_{\mathcal{AB}}\nabla_0\Phi^{\mathcal{B}} + M_{\mathcal{AB}}\nabla_1\Phi^{\mathcal{B}})\right] = 0\,, \tag{4.27}$$

$$\begin{aligned} -\tfrac{1}{2}\textstyle\int_{-\infty}^{+\infty} d\tau \left[C_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_0\Phi^{\mathcal{B}}\right]_{\sigma=0}^{\sigma=\pi} \\ +\textstyle\int_{-\infty}^{+\infty} d\tau \left[e\, e_1{}^1 (C_{\mathcal{AB}}\nabla_0\Phi^{\mathcal{B}} + M_{\mathcal{AB}}\nabla_1\Phi^{\mathcal{B}})\delta\Phi^{\mathcal{A}}\right]_{\sigma=0}^{\sigma=\pi} = 0\,. \end{aligned} \tag{4.28}$$

The 2-bein has four independent components. In this theory, on-shell (for the 2-bein itself), there are sufficient invariances to gauge-fix everyone of its components (Weyl, diffeomoerphisms (2) and Lorentz invariances) and to make the choice $e^a{}_\alpha = \delta^a{}_\alpha$ (*flat gauge*). Once the 2-bein has been gauge-fixed, we have the following "reductions":

$$\hat{S}_{sym}\left[e^a{}_\alpha;\Phi^{\mathcal{A}}\right] \;\to\; \hat{S}_{sym}\left[\Phi^{\mathcal{A}}\right] = \frac{1}{2}\int d^2\xi\, \left[C_{\mathcal{AB}}\,\partial_0\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}} + M_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}}\right];$$

$$\text{eqs. of motion} \;\to\; \partial_1(C_{\mathcal{AB}}\,\partial_0\Phi^{\mathcal{B}} + M_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{B}}) = 0\,;$$

$$\text{boundary terms} \to \left\{ \begin{array}{c} -\frac{1}{2}\int_{-\infty}^{+\infty} d\tau \left[C_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_0\Phi^{\mathcal{B}}\right]_{\sigma=0}^{\sigma=\pi} \\ +\int_{-\infty}^{+\infty} d\tau \left[(C_{\mathcal{AB}}\,\partial_0\Phi^{\mathcal{B}} + M_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{B}})\delta\Phi^{\mathcal{A}}\right]_{\sigma=0}^{\sigma=\pi} = 0 \end{array}\right.;$$

$$\epsilon - \text{trace} \;\to\; \hat{t} = -\frac{1}{2}(\partial_0\Phi^T C + \partial_1\Phi^T M)C^{-1}(C\partial_0\Phi + M\partial_1\Phi) = 0\,;$$

$$\mathbf{f}^{\mathcal{A}}(\tau,\sigma)\ \text{s.t.} \left\{ \begin{array}{l} \nabla_1\mathbf{f}^{\mathcal{A}} = 0 \\ \mathbf{f}^{\mathcal{A}}(\tau,\sigma+\pi) = \mathbf{f}^{\mathcal{A}}(\tau,\sigma) \end{array}\right. \to \left\{ \begin{array}{l} \partial_1\mathbf{f}^{\mathcal{A}} = 0 \Leftrightarrow \mathbf{f}^{\mathcal{A}} = \mathbf{f}^{\mathcal{A}}(\tau) \\ \text{no correspondent} \end{array}\right. .$$

and the totality of the results of the previous Chapter still hold. So the order of the equations of motion can be reduced and the equations themselves become

$$C_{\mathcal{AB}}\,\partial_0\Phi^{\mathcal{B}} + M_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{B}} = 0\,. \tag{4.29}$$

This form of the equations of motion has interesting consequences

- on the boundary terms, reducing to

$$-\frac{1}{2}\int_{-\infty}^{+\infty} d\tau \left[C_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_0\Phi^{\mathcal{B}}\right]_{\sigma=0}^{\sigma=\pi} = 0\,; \tag{4.30}$$

- on the ϵ-trace, which automatically vanishes $\hat{t}=0$.

4.3 Change of basis

As in the previous section, it is possible to write the action as a sum of Floreanini-Jackiw Lagrangians through a change of basis: from the nonchiral one to the chiral one:

$$\hat{S}_{sym}\left[e^a{}_\alpha;\varphi_+,\varphi_-\right] = \int d^2\xi\ \left[\mathcal{L}_+(e^a{}_\alpha;\nabla_a\varphi_+) + \mathcal{L}_-(e^a{}_\alpha;\nabla_a\varphi_-)\right]\,, \tag{4.31}$$

with

$$\begin{aligned}\mathcal{L}_\pm(e^a{}_\alpha;\nabla_a\varphi_\pm) &= e\left[\pm\frac{1}{2}\nabla_0\varphi_\pm\nabla_1\varphi_\pm - \frac{1}{2}(\nabla_1\varphi_\pm)^2\right]\\ &= \frac{1}{2}e\left[\pm e_0{}^\alpha e_1{}^\beta - e_1{}^\alpha e_1{}^\beta\right]\partial_\alpha\varphi_\pm\partial_\beta\varphi_\pm\,. \end{aligned} \tag{4.32}$$

In the flat gauge the Lagrangians in (4.32) reduce to (3.38).

Chapter 5

Double string theory actions

In this Chapter a manifestly T-duality invariant formulation of closed bosonic string theory will be finally faced. What we showed in Chapters 3 and 4 will be the starting point to construct the generalized sigma-model action depending on a doubled set of variables: the string coordinates along the compactified dimensions and their T-duals. In the Tseytlin's noncovariant formulation, the role of the local Lorentz invariance, recovered on-shell, is crucial in order to derive the symmetry group $O(n, n; \mathbb{Z})$. After choosing a suitable basis provided by the left and right coordinates, the Dirac's quantization procedure is performed since this model contains primary second class constraints, being described by a sum of Floreanini-Jackiw Lagrangians. The chiral coordinates (and, consequently, the original ones) behave like noncommuting variables. In the Hull's covariant formulation, instead, the generalized sigma-model action already involving a doubled number of coordinates is analyzed together with a 'self-duality' constraint halving the degrees of freedom. The two formulations are equivalent as it can be demonstrated.

5.1 Tseytlin's noncovariant formulation

Let us start from the following generalized sigma-model action functional

$$S\left[e^a{}_\alpha;\chi^i\right] = \frac{T}{2}\int d^2\xi\, e\, \mathcal{C}^{ab}_{ij}(\chi)\nabla_a\chi^i\nabla_b\chi^j\,, \tag{5.1}$$

where the χ^i's are bidimensional scalar fields and components of an N-dimensional vector in the target space (N still undetermined at this level). The usual formulation of string theory is recovered once one interpretes χ^i as the string coordinates and takes for $\mathcal{C}^{ab}_{ij}$ the form $\mathcal{C}^{ab}_{ij} = -(\eta^{ab}G_{ij} - \epsilon^{ab}B_{ij})$ (see eq. (2.88) and Appendix B).

The action (5.1) explicitly reads

$$S = \frac{T}{2}\int d^2\xi\, e\left[\mathcal{C}^{00}_{ij}\nabla_0\chi^i\nabla_0\chi^j + \mathcal{C}^{01}_{ij}\nabla_0\chi^i\nabla_1\chi^j + \mathcal{C}^{10}_{ij}\nabla_1\chi^i\nabla_0\chi^j + \mathcal{C}^{11}_{ij}\nabla_1\chi^i\nabla_1\chi^j\right]$$

$$
\begin{aligned}
&= \frac{T}{2}\int d^2\xi\, e\left[\mathcal{C}^{00}_{ij}\nabla_0\chi^i\nabla_0\chi^j + (\mathcal{C}^{01}_{ij}+\mathcal{C}^{10}_{ji})\nabla_0\chi^i\nabla_1\chi^j + \mathcal{C}^{11}_{ij}\nabla_1\chi^i\nabla_1\chi^j\right] \\
&= \frac{T}{2}\int d^2\xi\, e\left[\mathcal{C}^{00}_{ij}\nabla_0\chi^i\nabla_0\chi^j + \mathbf{C}_{ij}\nabla_0\chi^i\nabla_1\chi^j + \mathcal{C}^{11}_{ij}\nabla_1\chi^i\nabla_1\chi^j\right], \qquad (5.2)
\end{aligned}
$$

where we have defined

$$
\mathbf{C}_{ij}(\chi) \equiv \mathcal{C}^{01}_{ij}(\chi) + \mathcal{C}^{10}_{ji}(\chi)\,.
$$

Moreover, the matrices $\mathcal{C}^{00}$ and $\mathcal{C}^{11}$ can be chosen symmetric.

The action S can be always written in a 'first order form' (no quadratic term in time derivatives) such as

$$
S = \frac{T}{2}\int d^2\xi\, e\left[\mathbf{C}_{ij}\nabla_0\chi^i\nabla_1\chi^j + M_{ij}\nabla_1\chi^i\nabla_1\chi^j\right]. \qquad (5.3)
$$

Let us examine the two following possibilities:

1) $\mathcal{C}^{00}_{ij} = 0\ \forall\, i,j$, implies that the action (5.2) reduces to (5.3) by defining $M_{ij} \equiv \mathcal{C}^{11}_{ij}$. The matrix $\mathbf{C}_{ij}$ can be separated into its symmetric and antisymmetric parts: $\mathbf{C}_{ij} = \mathbf{C}_{(ij)} + \mathbf{C}_{[ij]} \equiv C_{ij} + H_{ij}$, where $C_{ij} = (1/2)[(\mathcal{C}^{01}_{ij}+\mathcal{C}^{10}_{ji}) + (\mathcal{C}^{01}_{ji}+\mathcal{C}^{10}_{ij})]$ and $H_{ij} = (1/2)[(\mathcal{C}^{01}_{ij}+\mathcal{C}^{10}_{ji}) - (\mathcal{C}^{01}_{ji}+\mathcal{C}^{10}_{ij})]$;

2) $\mathcal{C}^{00}_{ij} \neq 0$ (and these matrices are taken invertible). We can introduce a set of new variables: $p_i \equiv \mathcal{C}^{00}_{ij}\nabla_0\chi^j$, implying $\nabla_0\chi^i = (\mathcal{C}^{00})^{ij}p_j$. So the action (5.2) gets modified into

$$
S' = \frac{T}{2}\int d^2\xi\, e\left[2\nabla_0\chi^i p_i - (\mathcal{C}^{00})^{ij}p_ip_j + \mathbf{C}_{ij}\nabla_0\chi^i\nabla_1\chi^j + \mathcal{C}^{11}_{ij}\nabla_1\chi^i\nabla_1\chi^j\right].
$$

If we introduce a set of new fields (the Ψ's), by imposing $p_i = \nabla_1\Psi_i$, the action becomes

$$
\begin{aligned}
\hat{S} &= \frac{T}{2}\int d^2\xi\, e\left[2\nabla_0\chi^i\nabla_1\Psi_i - (\mathcal{C}^{00})^{ij}\nabla_1\Psi_i\nabla_1\Psi_j\right. \\
&\qquad\qquad \left. + \mathbf{C}_{ij}\nabla_0\chi^i\nabla_1\chi^j + \mathcal{C}^{11}_{ij}\nabla_1\chi^i\nabla_1\chi^j\right] \\
&= \frac{T}{2}\int d^2\xi\, e\left[\mathbf{C}_{\mathcal{I}\mathcal{J}}(\chi)\nabla_0\mathcal{X}^{\mathcal{I}}\nabla_1\mathcal{X}^{\mathcal{J}} + M_{\mathcal{I}\mathcal{J}}(\chi)\nabla_1\mathcal{X}^{\mathcal{I}}\nabla_1\mathcal{X}^{\mathcal{J}}\right],
\end{aligned}
$$

where $\mathcal{X} \equiv (\chi, \Psi)$ is a $2N$ vector and $\mathbf{C}_{\mathcal{I}\mathcal{J}}$ and $M_{\mathcal{I}\mathcal{J}}$ are $2N$ square matrices. In particular

$$
\mathbf{C}_{\mathcal{I}\mathcal{J}} = \begin{pmatrix} \mathbf{C}_{ij} & 2\delta_i{}^j \\ 0 & 0 \end{pmatrix}
$$

$\mathbf{C}_{\mathcal{I}\mathcal{J}}$ can be separated into its symmetric and antisymmetric parts:

$$
\mathbf{C}_{\mathcal{I}\mathcal{J}} = \mathbf{C}_{(\mathcal{I}\mathcal{J})} + \mathbf{C}_{[\mathcal{I}\mathcal{J}]} \equiv C_{\mathcal{I}\mathcal{J}} + H_{\mathcal{I}\mathcal{J}},
$$

where

$$\mathbf{C}_{(\mathcal{I}\mathcal{J})} = \begin{pmatrix} \mathbf{C}_{(ij)} & 2\delta_i^{\ j} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad C_{[\mathcal{I}\mathcal{J}]} = \begin{pmatrix} \mathbf{C}_{[ij]} & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$M_{\mathcal{I}\mathcal{J}} = \begin{pmatrix} \mathcal{C}^{11}_{ij} & 0 \\ 0 & -(\mathcal{C}^{00})^{ij} \end{pmatrix}.$$

We obtain the action (5.3) by redefining $2N \to N'$, $\mathcal{I},\mathcal{J},\ldots \to i,j,\ldots$, $\mathcal{X} \to \chi$, $\mathbf{C}_{\mathcal{I}\mathcal{J}} \to \mathbf{C}_{ij}$ and $M_{\mathcal{I}\mathcal{J}} \to M_{ij}$.

In the following, we will choose the matrices $\mathbf{C}$ and M to be constant.

5.2 Local symmetries of the action

As we just noticed, we can assume as our starting point the action:

$$\begin{aligned} S &= \frac{T}{2}\int d^2\xi\, e\left[\mathbf{C}_{ij}\nabla_0\chi^i\nabla_1\chi^j + M_{ij}\nabla_1\chi^i\nabla_1\chi^j\right] \qquad (5.4)\\ &= \frac{T}{2}\int d^2\xi\, e\left[C_{ij}\nabla_0\chi^i\nabla_1\chi^j + H_{ij}\nabla_0\chi^i\nabla_1\chi^j + M_{ij}\nabla_1\chi^i\nabla_1\chi^j\right]\\ &= \frac{T}{2}\int d^2\xi\, e\left[C_{ij}\nabla_0\chi^i\nabla_1\chi^j + \frac{1}{2}\epsilon^{ab}H_{ij}\nabla_a\chi^i\nabla_b\chi^j + M_{ij}\nabla_1\chi^i\nabla_1\chi^j\right]. \end{aligned}$$

This action has the same structure as the one analyzed in the previous Chapter for the scalar field on the curved background (see (4.11)). So almost all the considerations done will hold with minimal changes.

The action (5.4) is invariant under the already mentioned local transformations (Weyl transformations and bidimensional diffeomorphisms), while it is not manifestly invariant under local Lorentz transformations. As in the previous Chapter, we are led to introduce the tensor $t_a^{\ b}$, which has a slightly modified definition:

$$t_a^{\ b} \equiv \frac{1}{T}e^{-1}\frac{\delta S}{\delta e^a_{\ \alpha}}e^b_{\ \alpha}\,. \qquad (5.5)$$

The structure of the tensor $t_a^{\ b}$ can be computed from the action (5.4):

$$\begin{aligned} t_a^{\ b} &= \frac{1}{2}\left\{\delta_a^{\ b}\left[C_{ij}\nabla_0\chi^i\nabla_1\chi^j + M_{ij}\nabla_1\chi^i\nabla_1\chi^j\right] - \delta_0^{\ b}C_{ij}\nabla_a\chi^i\nabla_1\chi^j\right.\\ &\quad\left. -\delta_1^{\ b}C_{ij}\nabla_0\chi^i\nabla_a\chi^j - 2\delta_1^{\ b}M_{ij}\nabla_a\chi^i\nabla_1\chi^j\right\}. \qquad (5.6) \end{aligned}$$

It is now easy to read the components of $t_a^{\ b}$:

$$\begin{aligned} t_0^{\ 0} &= \frac{1}{2}M_{ij}\nabla_1\chi^i\nabla_1\chi^j \qquad (5.7)\\ t_0^{\ 1} &= -\frac{1}{2}\mathbf{C}_{ij}\nabla_0\chi^i\nabla_0\chi^j - M_{ij}\nabla_0\chi^i\nabla_1\chi^j \end{aligned}$$

$$= -\frac{1}{2}C_{ij}\nabla_0\chi^i\nabla_0\chi^j - M_{ij}\nabla_0\chi^i\nabla_1\chi^j \tag{5.8}$$

$$\begin{aligned} t_1{}^0 &= -\frac{1}{2}\mathbf{C}_{ij}\nabla_1\chi^i\nabla_1\chi^j \\ &= -\frac{1}{2}C_{ij}\nabla_1\chi^i\nabla_1\chi^j \end{aligned} \tag{5.9}$$

$$t_1{}^1 = -\frac{1}{2}M_{ij}\nabla_1\chi^i\nabla_1\chi^j = -t_0{}^0\,. \tag{5.10}$$

In these components the antisymmetric part of $\mathbf{C}_{ij}$ has disappeared.

The completely covariant components of $t_a{}^b$ are:

$$t_{00} = -t_0{}^0 \tag{5.11}$$
$$t_{01} = +t_0{}^1 \tag{5.12}$$
$$t_{10} = -t_1{}^0 \tag{5.13}$$
$$t_{11} = +t_1{}^1\,. \tag{5.14}$$

Let us now write the ϵ-trace of the tensor $t_a{}^b$:

$$\begin{aligned} \hat{t} &= -\frac{1}{2}\left[(\nabla_0\chi^T C + \nabla_1\chi^T M)C^{-1}(C\nabla_0\chi + M\nabla_1\chi)\right. \\ &\qquad \left. +\nabla_1\chi^T(C - MC^{-1}M)\nabla_1\chi\right]. \end{aligned} \tag{5.15}$$

In the following, we will neglect the term $(1/2)\epsilon^{ab}H_{ij}\nabla_a\chi^i\nabla_b\chi^j$, which is the only one manifestly Lorentz invariant. So the action simplifies in

$$S\left[e^a{}_\alpha;\chi^i\right] = \frac{T}{2}\int d^2\xi\, e\left[C_{ij}\nabla_0\chi^i\nabla_1\chi^j + M_{ij}\nabla_1\chi^i\nabla_1\chi^j\right]. \tag{5.16}$$

On the equations of motion for the 2-bein, the ϵ-trace vanishes and there are sufficient invariances to fix the flat gauge. After this choice, the equations of motion for the fields χ^i, once reduced to the first order, are

$$C_{ij}\partial_0\chi^j + M_{ij}\partial_1\chi^j = 0\,. \tag{5.17}$$

In this way, the vanishing of $\hat{t}$ on the equations of motion for χ^i

$$\begin{aligned} 2\,\hat{t} &= -(\partial_0\chi^T C + \partial_1\chi^T M)C^{-1}(C\partial_0\chi + M\partial_1\chi) - \partial_1\chi^T(C - MC^{-1}M)\partial_1\chi \\ &= -\partial_1\chi^T(C - MC^{-1}M)\partial_1\chi = 0 \end{aligned}$$

imposes a condition on the matrices C and M

$$C - MC^{-1}M = 0\,. \tag{5.18}$$

Let us observe that the equations of motion (5.17) and the "Lorentz" constraint (5.18) can be combined to obtain

$$\left\{\begin{array}{l} C\partial_0\chi + M\partial_1\chi = 0 \\ C - MC^{-1}M = 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} C\partial_0\chi + M\partial_1\chi = 0 \\ M\partial_0\chi + C\partial_1\chi = 0 \end{array}\right. \Leftrightarrow C\partial_a\chi - \epsilon_{ab}M\partial^b\chi = 0\,, \tag{5.19}$$

a sort of "covariantized" constraint. Its role will be important in the covariant formulation by Hull illustrated at the end of this Chapter.

Suitably rotating and rescaling χ (and keeping the name χ for the rotated and rescaled fields), the matrix C can be put into the form

$$C = \ \mathrm{diag}(\underbrace{+1,\ldots,+1}_{p\ \text{times}},\underbrace{-1,\ldots,-1}_{q\ \text{times}})\ , \qquad p+q=N\,. \tag{5.20}$$

The world-sheet quantum Lorentz anomaly (as it is shown in [21, 22]) is absent only if C has zero signature, i.e. $p = q \equiv n$, which implies $N = 2n$. In this way

$$C = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix} \tag{5.21}$$

and $C = MC^{-1}M$ becomes one of the two possible definitions of the $O(n,n)$-matrices. In other words, the compact target space of the action (5.16) can be considered as the doubled torus T^{2n} by adding periodicity conditions for the components of χ^i. In order to restore notations, at this point, we redefine $i, j, \ldots \to \mathcal{A}, \mathcal{B}, \ldots$.

5.3 A first change of basis

If we choose for $\chi^{\mathcal{A}}$ the form $\chi^{\mathcal{A}} \equiv (\chi_+^I, \chi_-^I)$, we can introduce the new variables $\Phi^{\mathcal{A}} \equiv (Y^I, \tilde{Y}_I)$ through

$$\left\{\begin{array}{l} Y^I = \frac{1}{\sqrt{2}}(\chi_+^I + \chi_-^I) \\ \tilde{Y}_I = \frac{1}{\sqrt{2}}\delta_{IJ}(\chi_+^J - \chi_-^J) \end{array}\right. , \tag{5.22}$$

implying

$$\Phi = \mathcal{Z}\chi \ \text{ with } \quad \mathcal{Z} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1_n & 1_n \\ 1_n & -1_n \end{pmatrix}. \tag{5.23}$$

The transformed of C is

$$C' = (\mathcal{Z}^{-1})^T C \mathcal{Z}^{-1} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} = \Omega = \Omega^T = \Omega^{-1}\,. \tag{5.24}$$

The Lorentz constraint $C = MC^{-1}M$ becomes

$$\Omega = M'\Omega M'\,, \tag{5.25}$$

a quadratic matricial equation for M', determining it up to a sign. (Let us recall the M' is a symmetric matrix.)

The $2n \times 2n$ matrix $M'_{\mathcal{AB}}$ can be determined in terms of two $n \times n$ matrices: one symmetric, which will be called G_{IJ} in analogy with the torus metric, and one antisymmetric, which will be denoted by B_{IJ} in analogy with the Kalb-Ramond background field. The proof is very simple. By suppressing indices, M' can be written using square blocks as

$$M' = \pm \begin{pmatrix} a & b \\ b^T & c \end{pmatrix}, \tag{5.26}$$

where the $n \times n$ matrices a, b and c have the following properties: $a = a^T$, $c = c^T$. The Lorentz constraint in the nonchiral basis (5.25) gives the four conditions (of which only three are independent, being the third one the transposition of the second one)

$$\begin{array}{rcl} a\,b^T + b\,a & = & 0 \\ a\,c + b\,b & = & 1_n \\ b^T b^T + c\,a & = & 1_n \\ b^T\,c + c\,b & = & 0 \end{array} . \tag{5.27}$$

All the previous equalities are satisfied by putting $a = G - BG^{-1}B$, $c = G^{-1}$ (both symmetric) and $b = BG^{-1}$. In this way

$$M' = \pm \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{5.28}$$

We choose the minus sign in eq. (5.28) to recover the results of the scalar field developed in Chapter 3.

Finally, in the flat gauge, the action

$$\begin{aligned} S &= \frac{T}{2}\int d^2\xi\ \left[C'_{\mathcal{AB}}\,\partial_0\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}} + M'_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}}\right] \\ &= \frac{T}{2}\int d^2\xi\ \Big[\partial_0 Y^I\partial_1\tilde{Y}_I + \partial_0\tilde{Y}_I\partial_1 Y^I - (G - BG^{-1}B)_{IJ}\partial_1 Y^I\partial_1 Y^J \\ &\qquad -(BG^{-1})_I{}^J\partial_1 Y^I\partial_1\tilde{Y}_J + (G^{-1}B)^I{}_J\partial_1\tilde{Y}_I\partial_1 Y^J - G^{IJ}\partial_1\tilde{Y}_I\partial_1\tilde{Y}_J\Big] \end{aligned}$$

exhibits the manifestly T-duality invariance. In fact, this action is $O(n,n)$-invariant. In particular, by performing a transformation of the fields and the couplings through the inversion matrix $\Lambda = \Omega$, we get (see section 2.5)

$$\Phi' = \begin{pmatrix} \tilde{Y} \\ Y \end{pmatrix}$$

and

$$M' = M^{-1} = -\begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} = -\begin{pmatrix} \tilde{G} - \tilde{B}\tilde{G}^{-1}\tilde{B} & \tilde{B}\tilde{G}^{-1} \\ -\tilde{G}^{-1}\tilde{B} & \tilde{G}^{-1} \end{pmatrix}.$$

The equations of motion for the fields Y^I and $\tilde{Y}_I$ are:

$$C'_{\mathcal{AB}}\,\partial_0\Phi^{\mathcal{B}} + M'_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{B}} = 0 \quad \Rightarrow$$

$$\Rightarrow \quad \begin{cases} \partial_0 Y = -G^{-1}B\,\partial_1 Y + G^{-1}\,\partial_1\tilde{Y} \\ \partial_0\tilde{Y} = \quad (G - BG^{-1}B)\,\partial_1 Y + BG^{-1}\,\partial_1\tilde{Y} \end{cases} . \tag{5.29}$$

The above system can be arranged by suitably combining the equations in the following form:

$$\begin{cases} \partial_0\tilde{Y} = G\,\partial_1 Y + B\,\partial_0 Y \\ \partial_1\tilde{Y} = G\,\partial_0 Y + B\,\partial_1 Y \end{cases} \tag{5.30}$$

resembling the scalar duality condition (3.23). As in that model, eq. (5.30) can be summarized into

$$\partial_a\tilde{Y} = -\epsilon_{ab}\,G\,\partial^b Y + B\,\partial_a Y\,. \tag{5.31}$$

Eq. (5.31) is a generalization of the duality condition (3.23) in the presence of two constant background fields: a generic metric G and a Kalb-Ramond field B. It can be seen as an extension of the standard duality condition due to a magnetic field.

Let us observe that, in constrast with the scalar case, the equation of motion for a dual field $\tilde{Y}_I$ induced by eq. (5.31) is

$$\partial^a\partial_a\tilde{Y}_I = -\epsilon_{ab}\,G_{IJ}\,\partial^a\partial^b Y^J + B_{IJ}\,\partial^a\partial_a Y^J = B_{IJ}\,\partial^a\partial_a Y^J\,, \tag{5.32}$$

showing that $\tilde{Y}_I$ does not automatically satisfy a wave equation, but, instead, this is true if and only if all the Y^J, with $J \neq I$, do.

5.4 On the role of chiral coordinates

We can simultaneously block-diagonalize the matrices C' and M' through the matrix $\mathcal{T}$, so defined:

$$\mathcal{T} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} G^{-1}E^T & G^{-1} \\ G^{-1}E & -G^{-1} \end{pmatrix}, \tag{5.33}$$

where $E \equiv G + B$.

The transformed matrices are

$$C'' = (\mathcal{T}^{-1})^T C' \mathcal{T}^{-1} = \begin{pmatrix} G & 0 \\ 0 & -G \end{pmatrix} \tag{5.34}$$

and

$$M'' = (\mathcal{T}^{-1})^T M' \mathcal{T}^{-1} = -\begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix}. \tag{5.35}$$

At the same time, a new pair of coordinates remains defined: $\mathcal{Y} \equiv (Y_L, Y_R)$ satisfying

$$\mathcal{Y} = \mathcal{T}\Phi \qquad \Leftrightarrow \qquad \begin{cases} Y_L = \frac{1}{\sqrt{2}} G^{-1}(E^T Y + \tilde{Y}) \\ Y_R = \frac{1}{\sqrt{2}} G^{-1}(EY - \tilde{Y}) \end{cases} , \tag{5.36}$$

or, equivalently,

$$\Phi = \mathcal{T}^{-1}\mathcal{Y} \qquad \Leftrightarrow \qquad \begin{cases} Y = \frac{1}{\sqrt{2}}(Y_L + Y_R) \\ \tilde{Y} = \frac{1}{\sqrt{2}}(EY_L - E^T Y_R) \end{cases} . \tag{5.37}$$

The transformed equations of motion are $C''\partial_0\mathcal{Y} + M''\partial_1\mathcal{Y} = 0$, which explicitly read

$$\begin{cases} G\,\partial_0 Y_L = \quad G\,\partial_1 Y_L \\ G\,\partial_0 Y_R = -G\,\partial_1 Y_R \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} Y_L = Y_L(\tau+\sigma) \\ Y_R = Y_R(\tau-\sigma) \end{cases} , \tag{5.38}$$

showing that, on-shell, $Y_{L,R}$ are functions of $\sigma^\pm = \tau \pm \sigma$ (chiral functions).

In the new chiral basis, the Lagrangian density is the sum of Floreanini-Jackiw Lagrangians

$$\begin{aligned} \mathcal{L} &= \frac{T}{2}\left(C''_{\mathcal{AB}}\,\partial_0\mathcal{Y}^{\mathcal{A}}\partial_1\mathcal{Y}^{\mathcal{B}} + M''_{\mathcal{AB}}\,\partial_1\mathcal{Y}^{\mathcal{A}}\partial_1\mathcal{Y}^{\mathcal{B}}\right) \\ &= \mathcal{L}_+(\partial_a Y_L) + \mathcal{L}_-(\partial_a Y_R)\,, \end{aligned} \tag{5.39}$$

where

$$\mathcal{L}_+(\partial_a Y_L) = \frac{T}{2}G_{IJ}\partial_0 Y_L^I\partial_1 Y_L^J - \frac{T}{2}G_{IJ}\partial_1 Y_L^I\partial_1 Y_L^J \tag{5.40}$$

$$\mathcal{L}_-(\partial_a Y_R) = -\frac{T}{2}G_{IJ}\partial_0 Y_R^I\partial_1 Y_R^J - \frac{T}{2}G_{IJ}\partial_1 Y_R^I\partial_1 Y_R^J\,. \tag{5.41}$$

The conjugate momenta are the following ones:

$$\mathcal{P}_{L,I} \equiv \frac{\partial\mathcal{L}_+}{\partial(\partial_0 Y_L^I)} = \frac{T}{2}G_{IJ}\partial_1 Y_L^J \tag{5.42}$$

$$\mathcal{P}_{R,I} \equiv \frac{\partial\mathcal{L}_-}{\partial(\partial_0 Y_R^I)} = -\frac{T}{2}G_{IJ}\partial_1 Y_R^J \tag{5.43}$$

and define primary second class constraints

$$\Psi_{L,I} = \mathcal{P}_{L,I} - \frac{T}{2}G_{IJ}\partial_1 Y_L^J \approx 0 \tag{5.44}$$

$$\Psi_{R,I} = \mathcal{P}_{R,I} + \frac{T}{2}G_{IJ}\partial_1 Y_R^J \approx 0\,. \tag{5.45}$$

The other constraints to be implemented are the ones connected with the world-sheet energy-momentum tensor, which we recall:

$$
\begin{aligned}
t_{00} &= -\frac{1}{2}M'_{\mathcal{AB}}\,\partial_1\mathcal{Y}^{\mathcal{A}}\partial_1\mathcal{Y}^{\mathcal{B}} \;=\; t_{11} \\
t_{01} &= -\frac{1}{2}C'_{\mathcal{AB}}\,\partial_0\mathcal{Y}^{\mathcal{A}}\partial_0\mathcal{Y}^{\mathcal{B}} - M'_{\mathcal{AB}}\,\partial_0\mathcal{Y}^{\mathcal{A}}\partial_1\mathcal{Y}^{\mathcal{B}} \\
t_{10} &= \frac{1}{2}C'_{\mathcal{AB}}\,\partial_1\mathcal{Y}^{\mathcal{A}}\partial_1\mathcal{Y}^{\mathcal{B}}\,.
\end{aligned}
$$

It is convenient to write them in the light-cone basis:

$$
t_{++} = \frac{1}{4}(t_{00}+t_{01}+t_{10}+t_{11}) = \frac{1}{2}t_{00}+\frac{1}{4}(t_{01}+t_{01}) \tag{5.46}
$$

$$
\begin{aligned}
t_{+-} &= \frac{1}{4}(t_{00}-t_{01}+t_{10}-t_{11}) = -\frac{1}{4}(t_{01}-t_{10}) \\
&= -\frac{1}{4}\epsilon^{ab}t_{ab}
\end{aligned} \tag{5.47}
$$

$$
t_{-+} = \frac{1}{4}(t_{00}+t_{01}-t_{10}-t_{11}) = \frac{1}{4}(t_{01}-t_{10}) = -t_{+-} \tag{5.48}
$$

$$
t_{--} = \frac{1}{4}(t_{00}-t_{01}-t_{10}+t_{11}) = \frac{1}{2}t_{00}-\frac{1}{4}(t_{01}+t_{10})\,. \tag{5.49}
$$

The above components, written in terms of Y_L and Y_R, read

$$
\begin{aligned}
t_{++} &= \frac{1}{2}\partial_+Y_L^TG\partial_+Y_L - \partial_+Y_L^TG\partial_-Y_L + \frac{1}{2}\partial_+Y_R^TG\partial_+Y_R \\
&= \frac{1}{2}G_{IJ}(\partial_+Y_L^I\partial_+Y_L^J - 2\partial_+Y_L^I\partial_-Y_L^J + \partial_+Y_R^I\partial_+Y_R^J)
\end{aligned} \tag{5.50}
$$

$$
\begin{aligned}
t_{+-} &= \frac{1}{2}(\partial_-Y_L^TG\partial_-Y_L - \partial_+Y_R^TG\partial_+Y_R) \\
&= \frac{1}{2}G_{IJ}(\partial_-Y_L^I\partial_-Y_L^J - \partial_+Y_R^I\partial_-Y_R^J)
\end{aligned} \tag{5.51}
$$

$$
\begin{aligned}
t_{--} &= \frac{1}{2}\partial_-Y_L^TG\partial_-Y_L + \frac{1}{2}\partial_-Y_R^TG\partial_-Y_R - \partial_+Y_R^TG\partial_-Y_R \\
&= \frac{1}{2}G_{IJ}(\partial_-Y_L^I\partial_-Y_L^J + \partial_-Y_R^I\partial_-Y_R^J - 2\partial_+Y_R^I\partial_-Y_R^J)\,.
\end{aligned} \tag{5.52}
$$

On-shell, their expressions are

$$
t_{++} = \frac{1}{2}\partial_+Y_L^TG\partial_+Y_L
$$

$$
\begin{aligned}
&= \frac{1}{2} G_{IJ} \partial_+ Y_L^I \partial_+ Y_L^J \\
&= \frac{1}{2} (\partial_+ Y_L)^2
\end{aligned} \tag{5.53}
$$

$$
t_{+-} = -t_{-+} = 0 \tag{5.54}
$$

$$
\begin{aligned}
t_{--} &= \frac{1}{2} \partial_- Y_R^T G \partial_- Y_R \\
&= \frac{1}{2} G_{IJ} \partial_- Y_R^I \partial_- Y_R^J \\
&= \frac{1}{2} (\partial_- Y_R)^2 \, .
\end{aligned} \tag{5.55}
$$

It is worth noticing the strict analogy with eqs. (1.22 - 1.23). This is not surprising since in theories with a full manifest local Lorentz invariance, the energy-momentum tensors computed by using the metric (and projected on the local tangent space) and the one obtained by using the 2-bein are the same, as the following calculations show (see Appendix B):

$$
\begin{aligned}
t_a{}^b &= \frac{1}{T} e^{-1} \frac{\delta S}{\delta e^a{}_\alpha} e^b{}_\alpha \\
&= \frac{1}{T} \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h^{\beta\gamma}} \frac{\delta h^{\beta\gamma}}{\delta e^a{}_\alpha} e^b{}_\alpha \\
&= \frac{1}{T} \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h^{\beta\gamma}} \eta^{cd} \frac{\delta (e_c{}^\beta e_d{}^\gamma)}{\delta e^a{}_\alpha} e^b{}_\alpha \\
&= -\frac{1}{2} T_{\beta\gamma} \eta^{cd} \left(\frac{\delta e_c{}^\beta}{\delta e^a{}_\alpha} e_d{}^\gamma + e_c{}^\beta \frac{\delta e_d{}^\gamma}{\delta e^a{}_\alpha} \right) e^b{}_\alpha \\
&= -\frac{1}{2} T_{\beta\gamma} \eta^{cd} \left[(-e_a{}^\beta e_c{}^\alpha) e_d{}^\gamma + e_c{}^\beta (-e_a{}^\gamma e_d{}^\alpha) \right] e^b{}_\alpha \\
&= \frac{1}{2} T_{\beta\gamma} \eta^{cd} \left[\delta_c{}^b e_a{}^\beta e_d{}^\gamma + \delta_d{}^b e_c{}^\beta e_a{}^\gamma \right] \\
&= T_{\beta\gamma} \eta^{bc} e_a{}^\beta e_c{}^\gamma
\end{aligned}
$$

and

$$
\begin{aligned}
t_{ab} &= t_a{}^c \eta_{cb} \\
&= T_{\beta\gamma} \eta^{cd} e_a{}^\beta e_d{}^\gamma \eta_{cb} \\
&= T_{\beta\gamma} \delta^d{}_b e_a{}^\beta e_d{}^\gamma \\
&= T_{\beta\gamma} e_a{}^\beta e_b{}^\gamma \\
&= T_{ab} \, .
\end{aligned}
$$

In order to quantize the Lagrangians in (5.40), and applying the procedure described in Appendix E, we have to compute a series of Poisson brackets and to analyze the behaviour of the constraints. Let us start from the following Poisson brackets:

$$\begin{aligned} \{Y_L^I(\tau,\sigma), Y_L^J(\tau,\sigma')\}_{PB} &= 0 && (5.56)\\ \{\mathcal{P}_{L,I}(\tau,\sigma), \mathcal{P}_{L,J}(\tau,\sigma')\}_{PB} &= 0 && (5.57)\\ \{Y_L^I(\tau,\sigma), \mathcal{P}_{L,J}(\tau,\sigma')\}_{PB} &= \delta^I{}_J\delta(\sigma-\sigma') && (5.58) \end{aligned}$$

$$\begin{aligned} \{Y_R^I(\tau,\sigma), Y_R^J(\tau,\sigma')\}_{PB} &= 0 && (5.59)\\ \{\mathcal{P}_{R,I}(\tau,\sigma), \mathcal{P}_{R,J}(\tau,\sigma')\}_{PB} &= 0 && (5.60)\\ \{Y_R^I(\tau,\sigma), \mathcal{P}_{R,J}(\tau,\sigma')\}_{PB} &= \delta^I{}_J\delta(\sigma-\sigma'). && (5.61) \end{aligned}$$

Moreover, in order to construct Dirac brackets, it is fundamental to find

$$\begin{aligned} \{\Psi_{L,I}(\tau,\sigma), \Psi_{L,J}(\tau,\sigma')\}_{PB} &= -TG_{IJ}\delta'(\sigma-\sigma')\\ \{\Psi_{R,I}(\tau,\sigma), \Psi_{R,J}(\tau,\sigma')\}_{PB} &= TG_{IJ}\delta'(\sigma-\sigma') \end{aligned}$$

leading to

$$\begin{aligned} [\{\Psi_{L,I}(\tau,\sigma), \Psi_{L,J}(\tau,\sigma')\}_{PB}]^{-1} &= -\frac{1}{T}G^{IJ}\epsilon(\sigma-\sigma')\\ [\{\Psi_{R,J}(\tau,\sigma), \Psi_{R,J}(\tau,\sigma')\}_{PB}]^{-1} &= \frac{1}{T}G^{IJ}\epsilon(\sigma-\sigma'). \end{aligned}$$

The Poisson brackets involving any two of the light-cone components of the energy-momentum tensor are vanishing. The non-vanishing ones are:

$$\begin{aligned} \{\Psi_{L,I}(\tau,\sigma), t_{++}(\tau,\sigma')\}_{PB} &= \frac{1}{2}G_{IJ}\,\delta'(\sigma-\sigma')\,[2\partial'_+Y_L^J(\tau,\sigma') - \partial'_-Y_L^J(\tau,\sigma')]\\ \{\Psi_{L,I}(\tau,\sigma), t_{+-}(\tau,\sigma')\}_{PB} &= -\frac{1}{2}G_{IJ}\,\delta'(\sigma-\sigma')\,\partial'_-Y_L^J(\tau,\sigma')\\ \{\Psi_{L,I}(\tau,\sigma), t_{--}(\tau,\sigma')\}_{PB} &= -\frac{1}{2}G_{IJ}\,\delta'(\sigma-\sigma')\,\partial'_-Y_L^J(\tau,\sigma')\\ \{\Psi_{R,I}(\tau,\sigma), t_{++}(\tau,\sigma')\}_{PB} &= \frac{1}{2}G_{IJ}\,\delta'(\sigma-\sigma')\,\partial'_+Y_R^J(\tau,\sigma')\\ \{\Psi_{R,I}(\tau,\sigma), t_{+-}(\tau,\sigma')\}_{PB} &= -\frac{1}{2}G_{IJ}\,\delta'(\sigma-\sigma')\,\partial'_+Y_R^J(\tau,\sigma')\\ \{\Psi_{R,I}(\tau,\sigma), t_{--}(\tau,\sigma')\}_{PB} &= -\frac{1}{2}G_{IJ}\,\delta'(\sigma-\sigma')\,[2\partial'_-Y_R^J(\tau,\sigma') - \partial'_+Y_R^J(\tau,\sigma')]\,. \end{aligned}$$

The constraint algebra closes on-shell.

At this point, we can write down the Dirac brackets between the chiral coordinates and their conjugate momenta:

$$\begin{aligned} \{Y_L^I(\tau,\sigma), Y_L^J(\tau,\sigma')\}_{DB} &= -\frac{1}{T}G^{IJ}\epsilon(\sigma-\sigma') && (5.62)\\ \{\mathcal{P}_{L,I}(\tau,\sigma), \mathcal{P}_{L,J}(\tau,\sigma')\}_{DB} &= \frac{T}{4}G_{IJ}\delta'(\sigma-\sigma') && (5.63) \end{aligned}$$

$$\{Y_L^I(\tau,\sigma),\mathcal{P}_{L,J}(\tau,\sigma')\}_{DB} = \frac{1}{2}\delta^I{}_J\delta(\sigma-\sigma') \tag{5.64}$$

$$\{Y_R^I(\tau,\sigma),Y_R^J(\tau,\sigma')\}_{DB} = \frac{1}{T}G^{IJ}\epsilon(\sigma-\sigma') \tag{5.65}$$

$$\{\mathcal{P}_{R,I}(\tau,\sigma),\mathcal{P}_{R,J}(\tau,\sigma')\}_{DB} = -\frac{T}{4}G_{IJ}\delta'(\sigma-\sigma') \tag{5.66}$$

$$\{Y_R^I(\tau,\sigma),\mathcal{P}_{R,J}(\tau,\sigma')\}_{DB} = \frac{1}{2}\delta^I{}_J\delta(\sigma-\sigma')\,. \tag{5.67}$$

For computing the Dirac brackets of the 'original' variables Y^I and $\tilde{Y}_I$, let us recall eq. (5.37):

$$\{Y^I(\tau,\sigma),Y^J(\tau,\sigma')\}_{DB} = 0 \tag{5.68}$$

$$\{\tilde{Y}_I(\tau,\sigma),\tilde{Y}_J(\tau,\sigma')\}_{DB} = 0 \tag{5.69}$$

$$\{Y^I(\tau,\sigma),\tilde{Y}_J(\tau,\sigma')\}_{DB} = -\frac{1}{T}\delta^I{}_J\epsilon(\sigma-\sigma'). \tag{5.70}$$

Eqs. (5.62), (5.65) and (5.70) show that the manifestly T-duality invariant formulation of closed string theory gives the same results of the standard formulation (see eqs. (2.110), (2.111), (2.113) and (1.2)).

In analogy with the one computed in Chapter 3, the Hamiltonian density is

$$\begin{aligned}\mathcal{H} &= \frac{T}{2}\mathcal{G}_{\mathcal{AB}}\,\partial_1\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}}\\ &= \frac{T}{2}\Big[(G-BG^{-1}B)_{IJ}\partial_1Y^I\partial_1Y^J+(BG^{-1})_I{}^J\partial_1Y^I\partial_1\tilde{Y}_J\\ &\qquad -(G^{-1}B)^I{}_J\partial_1\tilde{Y}_I\partial_1Y^J+G^{IJ}\partial_1\tilde{Y}_I\partial_1\tilde{Y}_J\Big]\,,\end{aligned}$$

while, in the chiral basis, it reads

$$\begin{aligned}\mathcal{H} &= -\frac{T}{2}M''_{\mathcal{AB}}\,\partial_1\mathcal{Y}^{\mathcal{A}}\partial_1\mathcal{Y}^{\mathcal{B}}\\ &= \frac{T}{2}\big(\partial_+Y_L^TG\partial_+Y_L-2\partial_+Y_L^TG\partial_-Y_L+\partial_+Y_R^TG\partial_+Y_R\\ &\qquad \partial_-Y_L^TG\partial_-Y_L-2\partial_+Y_R^TG\partial_-Y_R+\partial_-Y_R^TG\partial_-Y_R\big)\,.\end{aligned}$$

On-shell, it reduces to

$$\mathcal{H}=\frac{T}{2}\left(\partial_+Y_L^TG\partial_+Y_L+\partial_-Y_R^TG\partial_-Y_R\right). \tag{5.71}$$

We recognize this Hamiltonian density to be the same as the standard one $\mathcal{H}=(1/2)(\dot{X}^2+X'^2)$ evaluated on the solutions of the equations of motion $X=(1/\sqrt{2})(X_L+X_R)$.

In conclusion, let us examine the explicit form of the equations of motion relative to the fields Y^I, $\tilde{Y}_I$, Y_L^I and Y_R^I. In order to find the expansions for the solutions, we have to recall that the torus structure imposes quasi-periodicity boundary conditions to be satisfied:

$$Y^I(\tau,\sigma+\pi) = Y^I(\tau,\sigma) + 2\pi\sqrt{\alpha'}W^I \quad (5.72)$$
$$\tilde{Y}_I(\tau,\sigma+\pi) = \tilde{Y}_I(\tau,\sigma) + 2\pi\sqrt{\alpha'}K_I \quad (5.73)$$

$$Y^I_L(\tau,\sigma+\pi) = Y^I_L(\tau,\sigma) + 2\pi\sqrt{\alpha'}\tilde{\alpha}^I_0 \quad (5.74)$$
$$Y^I_R(\tau,\sigma+\pi) = Y^I_R(\tau,\sigma) - 2\pi\sqrt{\alpha'}\alpha^I_0\,, \quad (5.75)$$

where the quantities named W^I, K_I, $\tilde{\alpha}^I_0$ and α^I_0 (in analogy with the usual formulation of string theory) are generic constants.

Let us start with the "chiral" fields $Y^I_{L,R}$. On shell, as we saw, they are respectively functions of σ^+ and σ^-. The quasi-periodicity conditions become

$$Y^I_L(\sigma^+ + \pi) = Y^I_L(\sigma^+) + 2\pi\sqrt{\alpha'}\tilde{\alpha}^I_0 \quad (5.76)$$
$$Y^I_R(\sigma^- - \pi) = Y^I_R(\sigma^-) - 2\pi\sqrt{\alpha'}\alpha^I_0\,. \quad (5.77)$$

The most general expansions for chiral fields satisfying boundary conditions such as the ones in eq. (5.76 - 5.77) are:

$$Y^I_L(\sigma^+) = y^I_L + 2\sqrt{\alpha'}\tilde{\alpha}^I_0\sigma^+ + i\sqrt{\alpha'}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^I_n e^{-2in\sigma^+} \quad (5.78)$$

$$Y^I_R(\sigma^-) = y^I_R + 2\sqrt{\alpha'}\alpha^I_0\sigma^- + i\sqrt{\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha^I_n e^{-2in\sigma^-}\,. \quad (5.79)$$

Even if they are of the same form as the ones encountered in Chapter 1, the sets of Fourier coefficients $\tilde{\alpha}^I_n$, α^I_n, with $n \in \mathbb{Z}$, are, in principle, completely different from those which appeared there. We can anyway define $p^I_L \equiv (1/\sqrt{\alpha})\tilde{\alpha}^I_0$ and $p^I_R \equiv (1/\sqrt{\alpha})\alpha^I_0$. The structure of the transformations (5.37) and the quasi-periodicity conditions (5.72 - 5.73) uniquely establish the expansions for Y^I and $\tilde{Y}_I$:

$$\begin{aligned} Y^I(\tau,\sigma) &= y^I + 2\sqrt{\alpha'}(G^{IJ}K_J - G^{IK}B_{KJ}W^J)\tau + 2\sqrt{\alpha'}W^I\sigma \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[\tilde{\alpha}^I_n e^{-2in\sigma} + \alpha^I_n e^{+2in\sigma}] \quad (5.80) \\ \tilde{Y}_I(\tau,\sigma) &= \tilde{y}_I + 2\sqrt{\alpha'}[(G - BG^{-1}B)_{IJ}W^J + B_{IK}G^{KJ}K_J]\tau + 2\sqrt{\alpha'}K_I\sigma \\ &\quad + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-2in\tau}}{n}[E_{IJ}\tilde{\alpha}^J_n e^{-2in\sigma} - E^T_{IJ}\alpha^J_n e^{+2in\sigma}]\,, \quad (5.81) \end{aligned}$$

where

$$y^I \equiv \frac{1}{\sqrt{2}}(y^I_L + y^I_R) \quad \text{and} \quad \tilde{y}_I \equiv \frac{1}{\sqrt{2}}(E_{IJ}\,y^J_L - E^T_{IJ}y^J_R)\,.$$

The vanishing of the energy-momentum tensor, on-shell, translates into the vanishing of the Fourier coefficients of the expansions:

$$t_{++} = \frac{1}{2}\partial_+ Y_L^T G \partial_+ Y_L = 4\alpha' \sum_{m\in\mathbb{Z}} \tilde{L}_m e^{-2im\sigma^+} = 0$$
$$t_{--} = \frac{1}{2}\partial_- Y_R^T G \partial_- Y_R = 4\alpha' \sum_{m\in\mathbb{Z}} L_m e^{-2im\sigma^-} = 0\,.$$

The coefficients are given by the following integrals:

$$\tilde{L}_m = \frac{T}{4}\int_0^\pi d\sigma e^{+2im\sigma} t_{++} = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}^T_{m-n} G \tilde{\alpha}_n = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n = 0 \qquad (5.82)$$
$$L_m = \frac{T}{4}\int_0^\pi d\sigma e^{-2im\sigma} t_{--} = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha^T_{m-n} G \alpha_n = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\cdot\alpha_n = 0\,, \qquad (5.83)$$

which have the same form as the usual Virasoro generators (see eqs. (1.42 - 1.43)) by using the matrix G_{IJ} to construct the scalar product. In particular, for $m = 0$, we have

$$\tilde{L}_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}^T_{-n} G \tilde{\alpha}_n = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n = 0 \qquad (5.84)$$
$$L_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha^T_{-n} G \alpha_n = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\alpha_n = 0\,. \qquad (5.85)$$

The Hamiltonian is obtained by integrating over σ the hamiltonian density once one has substituted the expressions for the light-cone derivatives of the chiral fields:

$$H = \int_0^\pi d\sigma\,\mathcal{H} = \sum_{n\in\mathbb{Z}} (\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n + \alpha_{-n}\cdot\alpha_n)\,. \qquad (5.86)$$

By comparing the Virasoro generators $\tilde{L}_0$ and L_0 with (5.86), we get again $H = 2(\tilde{L}_0 + L_0)$. The quantities here computed, in the quantum version of the theory, are subjected to the normal-ordering ambiguities.

The conjugate momentum densities (5.42 - 5.43) are:

$$\begin{aligned}\mathcal{P}_{L,I} &= TG_{IJ}\left\{\sqrt{\alpha'}\tilde{\alpha}^J_0 + \sqrt{\alpha'}\sum_{n\neq 0}\tilde{\alpha}^J_n e^{-2in(\tau+\sigma)}\right\} \\ &= TG_{IJ}\left\{\sqrt{\alpha'}\sum_{n\in\mathbb{Z}}\tilde{\alpha}^J_n e^{-2in(\tau+\sigma)}\right\}\end{aligned} \qquad (5.87)$$

$$\begin{aligned}\mathcal{P}_{R,I} &= TG_{IJ}\left\{\sqrt{\alpha'}\alpha^J_0 + \sqrt{\alpha'}\sum_{n\neq 0}\alpha^J_n e^{-2in(\tau-\sigma)}\right\} \\ &= TG_{IJ}\left\{\sqrt{\alpha'}\sum_{n\in\mathbb{Z}}\alpha^J_n e^{-2in(\tau-\sigma)}\right\}\,.\end{aligned} \qquad (5.88)$$

The insertion of the expansions for the fields Y_L^I and $\mathcal{P}_{L,I}$, for example, in the eqs. (5.62 - 5.64) leads to the following Dirac brackets:

$$\begin{aligned}
\{y_L^I, p_{L,J}\}_{DB} &= \delta^I{}_J \\
\{y_L^I, \tilde{\alpha}_n^J\}_{DB} &= 0\,, \quad n \neq 0 \\
\{\tilde{\alpha}_m^I, \tilde{\alpha}_n^J\}_{DB} &= -i\, m\, \delta_{m+n,0}\, G^{IJ} \quad \forall\, m, n
\end{aligned}$$

and similarly for the right sector:

$$\begin{aligned}
\{y_R^I, p_{R,J}\}_{DB} &= \delta^I{}_J \\
\{y_R^I, \alpha_n^J\}_{DB} &= 0\,, \quad n \neq 0 \\
\{\alpha_m^I, \alpha_n^J\}_{DB} &= -i\, m\, \delta_{m+n,0}\, G^{IJ} \quad \forall\, m, n\,.
\end{aligned}$$

The relations between the oscillators are the same as the usual formulation of string theory.

The results obtained so far are a clear indication that the noncovariant duoble string formulation is an extension of the usual formulation.

5.5 Hull's covariant formulation

In this section, we examine the covariant duality symmetric formulation of string theory proposed by Hull. Let us start from the sigma-model action

$$S\left[h_{\alpha\beta}; \Phi^{\mathcal{A}}\right] = -\frac{T}{4} \int d^2\xi \, \sqrt{-h}\, h^{\alpha\beta} \mathcal{G}_{\mathcal{A}\mathcal{B}}\, \partial_\alpha \Phi^{\mathcal{A}} \partial_\beta \Phi^{\mathcal{B}}\,, \tag{5.89}$$

describing the compact part of the target space. We will adopt the same notations as introduced in the previous sections of this Chapter. So the vector $\Phi^{\mathcal{A}}$ is $2n$-dimensional ($\mathcal{A} = 1, \ldots, 2n$). Its components can be split into two n-dimensioanl groups: $\Phi^{\mathcal{A}} \equiv (Y^I, \tilde{Y}_I)$ $(I = 1, \ldots, n)$. The action (5.89) has an explicit $GL(2n, \mathbb{R})$ invariance, besides the local ones connected with the structure of the action itself.

Let us recall the definition of the matrix $\mathcal{G}_{\mathcal{A}\mathcal{B}}$:

$$\mathcal{G} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}, \tag{5.90}$$

together with its relation with the symmetric matrix $M_{\mathcal{A}\mathcal{B}}$ defined in Chapter 5: $\mathcal{G}_{\mathcal{A}\mathcal{B}} = -M_{\mathcal{A}\mathcal{B}}$. In order to keep only the physical degrees of freedom, it is necessary to halve them through an appropriate constraint. It turns out that the right constraint to implement is

$$C_{\mathcal{A}\mathcal{B}}\, \partial_a \Phi^{\mathcal{B}} + \epsilon_{ab}\, \mathcal{G}_{\mathcal{A}\mathcal{B}}\, \partial^b \Phi^{\mathcal{B}} = 0 \qquad (C = \Omega)\,,$$

exactly reproducing the covariantized constraint (5.19) originating from the requirement of on-shell local Lorentz invariance in the noncovariant formulation.

It can be shown that the action (5.89), suitably combined with the covariantized constraint just quoted, is completely equivalent to the noncovariant action by Tseytlin as it can be more easily demonstrated in the chiral basis [23, 24, 25].

In analogy with the completely covariant energy-momentum tensor defined in Chapter 1, we can introduce

$$T_{\alpha\beta} \equiv -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}}\,, \tag{5.91}$$

leading to

$$2\,T_{\alpha\beta} = \partial_\alpha \Phi^T \mathcal{G} \partial_\beta \Phi - \frac{1}{2} h_{\alpha\beta} h^{\gamma\delta} \partial_\gamma \Phi^T \mathcal{G} \partial_\delta \Phi\,. \tag{5.92}$$

Since the action (5.89) exhibits all the invariances of the sigma-model actions already studied (reparametrizations and Weyl invariances), the conformal gauge $h_{\alpha\beta} = \eta_{ab}$ can be chosen.

In this way the action and the energy-momentum tensor become

$$\begin{aligned} S\left[h_{\alpha\beta};\Phi^{\mathcal{A}}\right] \quad \rightarrow \quad S\left[\Phi^{\mathcal{A}}\right] &= -\frac{T}{4}\int d^2\xi\ \eta^{ab}\,\mathcal{G}_{\mathcal{AB}}\,\partial_a\Phi^{\mathcal{A}}\partial_b\Phi^{\mathcal{B}} \\ &= \frac{T}{4}\int d^2\xi\ \mathcal{G}_{\mathcal{AB}}\left(\partial_0\Phi^{\mathcal{A}}\partial_0\Phi^{\mathcal{B}} - \partial_1\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}}\right) \end{aligned} \tag{5.93}$$

$$2\,T_{\alpha\beta} \quad \rightarrow \quad 2T_{ab} = \partial_a \Phi^T \mathcal{G} \partial_b \Phi - \frac{1}{2}\eta_{ab}\,\eta^{cd}\partial_c \Phi^T \mathcal{G} \partial_d \Phi\,. \tag{5.94}$$

The equation of motion for the field $\Phi^{\mathcal{A}}$ coming from the action written in the conformal guage is the wave equation $\partial_a\partial^a\Phi^{\mathcal{A}} = 0$, while the boundary term is

$$-\frac{T}{2}\int_{-\infty}^{+\infty} d\tau\,\left[\mathcal{G}_{\mathcal{AB}}\,\delta\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}}\right]_{\sigma=0}^{\sigma=\pi}\,, \tag{5.95}$$

where we used the fact that $\delta\Phi^{\mathcal{A}}(\tau = \pm\infty, \sigma) = 0\ \forall\,\sigma$.

In order to ensure the boundary term to vanish, we choose quasi-periodicity boundary conditions for the coordinates:

$$\Phi^{\mathcal{A}}(\tau, \sigma + \pi) = \Phi^{\mathcal{A}}(\tau, \sigma)\,.$$

The conjugate momentum $\mathcal{P}_{\mathcal{A}}$ is

$$\mathcal{P}_{\mathcal{A}} = \frac{T}{2}\mathcal{G}_{\mathcal{AB}}\,\partial_0\Phi^{\mathcal{B}}\,.$$

It permits to write the Hamiltonian density

$$\mathcal{H} = \mathcal{P}_{\mathcal{A}}\partial_0\Phi^{\mathcal{A}} - \mathcal{L} = \frac{T}{4}\mathcal{G}_{\mathcal{AB}}(\partial_0\Phi^{\mathcal{A}}\partial_0\Phi^{\mathcal{B}} + \partial_1\Phi^{\mathcal{A}}\partial_1\Phi^{\mathcal{B}})\,. \tag{5.96}$$

In what follows, the theory will be analyzed in the chiral basis: $\mathcal{Y}^{\mathcal{A}} = (Y_L^I, Y_R^I)$, where Y_L^I and Y_R^I are generic functions of τ and σ. The transformation matrix is the one in eq. (5.33). The $O(n,n)$ metric and the generalized metric transform as follows:

$$C = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \quad \rightarrow \quad C' = \begin{pmatrix} G & 0 \\ 0 & -G \end{pmatrix},$$
$$\mathcal{G} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \quad \rightarrow \quad \mathcal{G}' = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix}.$$

The action takes the form

$$S\left[\mathcal{Y}^{\mathcal{A}}\right] = \frac{T}{4}\int d^2\xi\, \mathcal{G}'_{\mathcal{AB}}\left[\partial_0\mathcal{Y}^{\mathcal{A}}\partial_0\mathcal{Y}^{\mathcal{B}} - \partial_1\mathcal{Y}^{\mathcal{A}}\partial_1\mathcal{Y}^{\mathcal{B}}\right]. \tag{5.97}$$

In this frame, any dependence on the Kalb-Ramond field has disappeared. It is worth noticing that the covariantized constraint, in the chiral basis, becomes a "self/antiself-duality" constraint for Y_L and Y_R:

$$C'_{\mathcal{AB}}\,\partial_a\mathcal{Y}^{\mathcal{B}} + \epsilon_{ab}\,\mathcal{G}'_{\mathcal{AB}}\,\partial^b\mathcal{Y}^{\mathcal{B}} = 0 \;\Leftrightarrow\; \begin{cases} G_{IJ}\partial_0 Y_L^J = G_{IJ}\partial_1 Y_L^J \\ G_{IJ}\partial_0 Y_R^J = -G_{IJ}\partial_1 Y_R^J \end{cases}, \tag{5.98}$$

equivalent to the couple of conditions $Y_L^I = Y_L^I(\tau+\sigma)$ and $Y_R^I = Y_R^I(\tau-\sigma)$. In this model, due to the presence of the constraint put by hand, the left and right coordinates are chiral functions also off-shell.

The conjugate momentum with respect to $\mathcal{Y}^{\mathcal{A}}$ is

$$\mathcal{P}_{\mathcal{A}}^{chiral} = \frac{T}{2}\mathcal{G}'_{\mathcal{AB}}\,\partial_0\mathcal{Y}^{\mathcal{B}}, \tag{5.99}$$

and it is equivalent to

$$\mathcal{P}_{L,I} = \frac{T}{2}G_{IJ}\partial_0 Y_L^J \quad \text{and} \quad \mathcal{P}_{R,I} = \frac{T}{2}G_{IJ}\partial_0 Y_R^J\,. \tag{5.100}$$

It is crucial to observe that, through the covariantized constaint, the left and right conjugate momenta can be written as

$$\mathcal{P}_{L,I} = \frac{T}{2}G_{IJ}\partial_1 Y_L^J \quad \text{and} \quad \mathcal{P}_{R,I} = -\frac{T}{2}G_{IJ}\partial_1 Y_R^J\,, \tag{5.101}$$

exactly reproducing those found in eqs. (5.42 - 5.43).

Moreover, by multiplying eq. (5.98) by the constant $T/2$, and defining

$$\Psi_{L,I} \equiv \frac{T}{2}G_{IJ}\partial_0 Y_L^J - \frac{T}{2}G_{IJ}\partial_1 Y_L^J = \mathcal{P}_{L,I} - \frac{T}{2}G_{IJ}\partial_1 Y_L^J = 0 \tag{5.102}$$
$$\Psi_{R,I} \equiv \frac{T}{2}G_{IJ}\partial_0 Y_R^J + \frac{T}{2}G_{IJ}\partial_1 Y_R^J = \mathcal{P}_{R,I} + \frac{T}{2}G_{IJ}\partial_1 Y_R^J = 0\,, \tag{5.103}$$

one recognizes the primary second class constaints of Tseytlin's formulation. The other constraints of the theory are the ones given by the light-cone components of the energy-momentum tensor written in the chiral basis:

$$2\,T_{ab} = \partial_a \mathcal{Y}^T \mathcal{G}' \partial_b \mathcal{Y} - \frac{1}{2}\eta_{ab}\,\eta^{cd}\partial_c \mathcal{Y}^T \mathcal{G}' \partial_d \mathcal{Y}\,, \tag{5.104}$$

implying

$$T_{++} = \frac{1}{2} G_{IJ} \partial_+ Y^I_L \partial_+ Y^J_L \tag{5.105}$$

$$T_{--} = \frac{1}{2} G_{IJ} \partial_- Y^I_R \partial_- Y^J_R\,. \tag{5.106}$$

The set of constraints $\Psi_{L,I}, \Psi_{R,I}, T_{++}$ and T_{--} is the same as the one of the noncovariant formulation. They satisfy the same algebra. This correspondence guarantees that the quantization of the theory performed through the Dirac procedure exactly reproduces the Dirac brackets already computed at the end of the previous Chapter.

Appendix A

Notation and useful relations

A.1 Indices notations

In this thesis we used both small and capital latin or greek indices, in agreement with the following scheme:

$a, b, \ldots$	$=$	$0, 1$	flat world-sheet indices;
$\alpha, \beta, \ldots$	$=$	$0, 1$	curved world-sheet indices;
$\mu, \nu, \ldots$	$=$	$0, 1, \ldots, d-1$	noncompact target space indices;
$I, J, \ldots$	$=$	$1, \ldots, n$	compact target space indices;
$\mathcal{A}, \mathcal{B}, \ldots$	$=$	$1, \ldots, 2n$	doubled compact target space indices.

A.2 Useful relations

The Heaviside θ-function is defined as follows

$$\theta(x) \equiv \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} . \tag{A.1}$$

It has an integral expression, too:

$$\theta(x) = \int_{-\infty}^{x} dt\, \delta(t)\,, \tag{A.2}$$

implying

$$\frac{d}{dx}\theta(x) = \delta(x)\,. \tag{A.3}$$

Let us now introduce the ϵ-function, so defined

$$\epsilon(x) \equiv \frac{1}{2}\left[\theta(x) - \theta(-x)\right]\,, \tag{A.4}$$

which, due to (A.1), leads to

$$\epsilon(x) = \frac{1}{2}\left[\left\{\begin{array}{ll} 1, & x \geq 0 \\ 0, & x < 0 \end{array}\right. - \left\{\begin{array}{ll} 0, & x \geq 0 \\ 1, & x < 0 \end{array}\right.\right] = \left\{\begin{array}{ll} 1/2, & x \geq 0 \\ -1/2, & x < 0 \end{array}\right. .$$

Equations (A.3 - A.4) permit to verify that

$$\frac{d}{dx}\epsilon(x) = \frac{1}{2}\left[\delta(x) + \delta(-x)\right] = \delta(x)\,, \tag{A.5}$$

being the Dirac δ-function symmetric.

Let us now show that

$$\delta'(x) \equiv \frac{d}{dx}\delta(x) = \left[\epsilon(x)\right]^{-1} \tag{A.6}$$

in the sense of the following integral identity

$$\int dy\, \delta'(x-y)\, \epsilon(y-z) = \delta(x-z)\,. \tag{A.7}$$

The proof is very simple:

$$\begin{aligned} \int dy\, \delta'(x-y)\, \epsilon(y-z) &= \int dy\, \partial_x \delta(x-y)\, \epsilon(y-z) \\ &= -\int dy\, \partial_y \delta(x-y)\, \epsilon(y-z) \\ &= -\int dy\, \partial_y \left[\delta(x-y)\, \epsilon(y-z)\right] + \int dy\, \delta(x-y)\, \partial_y \epsilon(y-z) \\ &= -\left[\delta(x-y)\, \epsilon(y-z)\right]_{-\infty}^{+\infty} + \int dy\, \delta(x-y)\, \delta(y-z) \\ &= \delta(x-z)\,. \end{aligned}$$

Let us recall the Fourier expansion of the Dirac δ-function

$$\delta(x) = \frac{1}{\pi}\sum_{n\in\mathbb{Z}} e^{2inx}\,, \tag{A.8}$$

which is very useful in the derivation of the Poisson brackets of the string coordinates.

A way for obtaining a fundamental relation widely used in this thesis to compute both the Poisson and the Dirac brackets, is to write down the integral expression of the ϵ-function, and to use the Fourier expansion of $\delta(x)$. So we get

$$\epsilon(x) = \frac{1}{2}\int_{-x}^{x} dt\, \delta(t) = \frac{2x}{2\pi} - \frac{i}{2\pi}\sum_{n\neq 0}\frac{1}{n}e^{2inx}\,. \tag{A.9}$$

In a similar fashion, we can write

$$\delta'(x) = \frac{2i}{\pi}\sum_{n\neq 0} n\, e^{2inx}\,. \tag{A.10}$$

Appendix B

On the 2-bein

In a bidimensional space, let us introduce the coordinates (x^0, x^1) and the following objects:

- the minkowskian metric

$$\eta_{ab} = \text{ diag}(-1, +1) \quad \Rightarrow \quad \eta \equiv \text{ det}(\eta_{ab}) = -1\,; \tag{B.1}$$

- a world-sheet metric

$$g_{\alpha\beta} = g_{\alpha\beta}(x^0, x^1)\,, \tag{B.2}$$

  the determinant of which is $g \equiv \text{ det}(g_{\alpha\beta}) = g_{00}\, g_{11} - g_{01}\, g_{10}\,;$

- a 2-bein

$$e^a{}_\alpha = e^a{}_\alpha(x^0, x^1)\,. \tag{B.3}$$

  Its components can be arranged in a 2×2 matrix

$$(e^a{}_\alpha) = \begin{pmatrix} e^{a=0}{}_{\alpha=0} & e^{a=0}{}_{\alpha=1} \\ e^{a=1}{}_{\alpha=0} & e^{a=1}{}_{\alpha=1} \end{pmatrix}. \tag{B.4}$$

  The determinant e of the matrix (B.4) is easily computed in

$$e \equiv \text{ det}(e^a{}_\alpha) = (e^{a=0}{}_{\alpha=0})(e^{a=1}{}_{\alpha=1}) - (e^{a=0}{}_{\alpha=1})(e^{a=1}{}_{\alpha=0}) > 0\,. \tag{B.5}$$

  (At this level the inequality in (B.5) is a choice.)

  The 2-bein $e^a{}_\alpha(x^0, x^1)$ is a basis for the vectors belonging to the world-sheet local tangent space and permits to decompose the metric $g_{\alpha\beta}$ as

$$g_{\alpha\beta} = \eta_{ab}\, e^a{}_\alpha e^b{}_\beta\,. \tag{B.6}$$

  From (B.6) we can see that $g = -e^2 \Leftrightarrow e = \sqrt{-g}$, confirming our choice on the sign of e.

It is possible to introduce the inverse 2-bein in one of the following equivalent ways:

$$e^a{}_\alpha e_b{}^\alpha = \delta^a{}_b \quad ; \qquad e_a{}^\alpha e^b{}_\alpha = \delta_a{}^b \tag{B.7}$$

or

$$e^a{}_\alpha e_a{}^\beta = \delta_\alpha{}^\beta \quad ; \qquad e_a{}^\alpha e^a{}_\beta = \delta^\alpha{}_\beta \,. \tag{B.8}$$

In order to find the explicit expressions of the $e_a{}^\alpha$'s in terms of the $e^a{}_\alpha$'s, let us write down explictly the first eq. in (B.7):

$$\left\{\begin{array}{l}
(e^{a=0}{}_{\alpha=0})(e_{b=0}{}^{\alpha=0}) + (e^{a=0}{}_{\alpha=1})(e_{b=0}{}^{\alpha=1}) = 1 \\[2ex]
(e^{a=0}{}_{\alpha=0})(e_{b=1}{}^{\alpha=0}) + (e^{a=0}{}_{\alpha=1})(e_{b=1}{}^{\alpha=1}) = 0 \\[2ex]
(e^{a=1}{}_{\alpha=0})(e_{b=0}{}^{\alpha=0}) + (e^{a=1}{}_{\alpha=1})(e_{b=0}{}^{\alpha=1}) = 0 \\[2ex]
(e^{a=1}{}_{\alpha=0})(e_{b=1}{}^{\alpha=0}) + (e^{a=1}{}_{\alpha=1})(e_{b=1}{}^{\alpha=1}) = 1
\end{array}\right. . \tag{B.9}$$

The first and the third equations in (B.9) contain $e_{a=0}{}^{\alpha=0}$ and $e_{a=0}{}^{\alpha=1}$, while the second and the fourth ones contain $e_{a=1}{}^{\alpha=0}$ and $e_{a=1}{}^{\alpha=1}$. So we get the systems

$$\left\{\begin{array}{l}
(e^{a=0}{}_{\alpha=0})(e_{b=0}{}^{\alpha=0}) + (e^{a=0}{}_{\alpha=1})(e_{b=0}{}^{\alpha=1}) = 1 \\[2ex]
(e^{a=1}{}_{\alpha=0})(e_{b=0}{}^{\alpha=0}) + (e^{a=1}{}_{\alpha=1})(e_{b=0}{}^{\alpha=1}) = 0
\end{array}\right.$$

and

$$\left\{\begin{array}{l}
(e^{a=0}{}_{\alpha=0})(e_{b=1}{}^{\alpha=0}) + (e^{a=0}{}_{\alpha=1})(e_{b=1}{}^{\alpha=1}) = 0 \\[2ex]
(e^{a=1}{}_{\alpha=0})(e_{b=1}{}^{\alpha=0}) + (e^{a=1}{}_{\alpha=1})(e_{b=1}{}^{\alpha=1}) = 1
\end{array}\right. ,$$

respectively solved by

$$\left\{\begin{array}{lcl}
e_{a=0}{}^{\alpha=0} & = & e^{-1} e^{a=1}{}_{\alpha=1} \\[2ex]
e_{a=0}{}^{\alpha=1} & = & -e^{-1} e^{a=1}{}_{\alpha=0}
\end{array}\right.$$

and by

$$\left\{\begin{array}{lcl}
e_{a=1}{}^{\alpha=0} & = & -e^{-1} e^{a=0}{}_{\alpha=1} \\[2ex]
e_{a=1}{}^{\alpha=1} & = & e^{-1} e^{a=0}{}_{\alpha=0}
\end{array}\right. .$$

The inverse 2-bein can be arranged in a 2×2 matrix, too:

$$(e_a{}^\alpha) = \begin{pmatrix} e_{a=0}{}^{\alpha=0} & e_{a=0}{}^{\alpha=1} \\ e_{a=1}{}^{\alpha=0} & e_{a=1}{}^{\alpha=1} \end{pmatrix} = \frac{1}{e} \begin{pmatrix} e^{a=1}{}_{\alpha=1} & -e^{a=1}{}_{\alpha=0} \\ -e^{a=0}{}_{\alpha=1} & e^{a=0}{}_{\alpha=0} \end{pmatrix} . \tag{B.10}$$

The determinant of this matrix is

$$\begin{aligned}
\det(e_a{}^\alpha) &= (e_{a=0}{}^{\alpha=0})(e_{a=1}{}^{\alpha=1}) - (e_{a=0}{}^{\alpha=1})(e_{a=1}{}^{\alpha=0}) \\
&= \frac{1}{e^2}\left[(e^{a=1}{}_{\alpha=1})(e^{a=0}{}_{\alpha=0}) - (-e^{a=1}{}_{\alpha=0})(-e^{a=0}{}_{\alpha=1})\right] = e^{-1} \,,
\end{aligned} \tag{B.11}$$

as it must be.

Let us now study the variation of the determinant of the 2-bein:

$$\begin{aligned}
\delta e &= \delta e^{a=0}{}_{\alpha=0}\, e^{a=1}{}_{\alpha=1} + e^{a=0}{}_{\alpha=0}\,\delta e^{a=1}{}_{\alpha=1} - \delta e^{a=0}{}_{\alpha=1}\, e^{a=1}{}_{\alpha=0} - e^{a=0}{}_{\alpha=1}\,\delta e^{a=1}{}_{\alpha=0} \\
&= e\,(e_{a=0}{}^{\alpha=0}\,\delta e^{a=0}{}_{\alpha=0} + e_{a=1}{}^{\alpha=1}\,\delta e^{a=1}{}_{\alpha=1} + e_{a=0}{}^{\alpha=1}\,\delta e^{a=0}{}_{\alpha=1} + e_{a=1}{}^{\alpha=0}\,\delta e^{a=1}{}_{\alpha=0}) \\
&= e\, e_a{}^{\alpha}\,\delta e^a{}_{\alpha} \\
&= \delta(e\, e_a{}^{\alpha} e^a{}_{\alpha}) - \delta e\, e_a{}^{\alpha} e^a{}_{\alpha} - e\,\delta e_a{}^{\alpha} e^a{}_{\alpha} = -e\, e^a{}_{\alpha}\delta e_a{}^{\alpha}\,.
\end{aligned}$$

So

$$\delta e = \begin{cases} e\, e_a{}^{\alpha}\delta e^a{}_{\alpha} \\ -e\, e^a{}_{\alpha}\delta e_a{}^{\alpha} \end{cases}. \tag{B.12}$$

Let us now examine how to write the variation of the inverse 2-bein $e_a{}^{\alpha}$ in terms of the $e^a{}_{\alpha}$'s and viceversa. Our starting point is again the first eq. in (B.7).

$$\begin{aligned}
&e^a{}_{\alpha} e_b{}^{\alpha} = \delta^a{}_b &&\Rightarrow\quad \delta e^a{}_{\alpha}\, e_b{}^{\alpha} + e^a{}_{\alpha}\delta e_b{}^{\alpha} = 0 \quad\Rightarrow \\
\Rightarrow\quad &e^a{}_{\alpha}\delta e_b{}^{\alpha} = -e_b{}^{\alpha}\delta e^a{}_{\alpha} &&\Rightarrow\quad e_a{}^{\beta}(e^a{}_{\alpha}\delta e_b{}^{\alpha}) = e_a{}^{\beta}(-e_b{}^{\alpha}\delta e^a{}_{\alpha}) \quad\Rightarrow \\
\Rightarrow\quad &e_a{}^{\beta} e^a{}_{\alpha}\delta e_b{}^{\alpha} = -e_a{}^{\beta} e_b{}^{\alpha}\delta e^a{}_{\alpha} &&\Rightarrow\quad \delta^{\beta}{}_{\alpha}\delta e_b{}^{\alpha} = -e_a{}^{\beta} e_b{}^{\alpha}\delta e^a{}_{\alpha} \quad\Rightarrow \\
\Rightarrow\quad &\delta e_b{}^{\beta} = -e_a{}^{\beta} e_b{}^{\alpha}\delta e^a{}_{\alpha} &&\Leftrightarrow\quad \delta e_a{}^{\alpha} = -e_a{}^{\beta} e_b{}^{\alpha}\delta e^b{}_{\beta}\,.
\end{aligned}$$

Summarizing:

$$\delta e_a{}^{\alpha} = -e_a{}^{\beta} e_b{}^{\alpha}\delta e^b{}_{\beta} \quad \text{or} \quad \delta e^a{}_{\alpha} = -e^a{}_{\beta} e^b{}_{\alpha}\delta e_b{}^{\beta}\,. \tag{B.13}$$

We choose the Levi-Civita symbol (the pseudo-tensor ϵ^{ab}) of the form:

$$\epsilon^{ab} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}. \tag{B.14}$$

If we want ϵ_{ab} to be such that $\epsilon_{01} = -1$, remembering the definition of the determinant of an arbitrary square matrix in D dimensions

$$g_{\mu_1\nu_1} g_{\mu_2\nu_2} \ldots g_{\mu_D\nu_D} \epsilon^{\mu_1\mu_2\ldots\mu_D} = -g\,\epsilon_{\nu_1\nu_2\ldots\nu_D}\,, \tag{B.15}$$

we have to write

$$\eta_{ac}\,\eta_{bd}\,\epsilon^{cd} = -\eta\,\epsilon_{ab} = \epsilon_{ab} \quad\Rightarrow\quad \epsilon_{ab} = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}. \tag{B.16}$$

A similar construction has to be followed in the curved case. Starting from

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}, \tag{B.17}$$

the object $\epsilon_{\alpha\beta}$ is

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \tag{B.18}$$

because of

$$g_{\alpha\gamma}\, g_{\beta\delta}\, \epsilon^{\gamma\delta} = -g\, \epsilon_{\alpha\beta} = e^2\, \epsilon_{\alpha\beta}\,. \tag{B.19}$$

Two very useful relations connecting the pseudo-tensors $\epsilon^{\alpha\beta}$ and ϵ^{ab} are:

$$\epsilon^{\alpha\beta} = e\, \epsilon^{ab} e_a{}^{\alpha} e_b{}^{\beta} \quad \text{and} \quad \epsilon^{ab} = e^{-1} \epsilon^{\alpha\beta} e^a{}_{\alpha} e^b{}_{\beta}\,, \tag{B.20}$$

constituting two sorts of decompositions, the first of which can be written as

$$e\, e_0{}^{\alpha} e_1{}^{\beta} = \epsilon^{\alpha\beta} + e\, e_1{}^{\alpha} e_0{}^{\beta}\,. \tag{B.21}$$

The analogous of (B.20) involving $\epsilon_{\alpha\beta}$ and ϵ_{ab} are

$$\epsilon_{\alpha\beta} = e^{-1} \epsilon_{ab} e^a{}_{\alpha} e^b{}_{\beta} \quad \text{and} \quad \epsilon_{ab} = e\, \epsilon_{\alpha\beta} e_a{}^{\alpha} e_b{}^{\beta}\,. \tag{B.22}$$

The last equalities can be derived from eqs. (B.6), (B.16) and (B.19):

$$g_{\alpha\gamma}\, g_{\beta\delta}\, \epsilon^{\gamma\delta} = e^2\, \epsilon_{\alpha\beta} \quad \Rightarrow$$

$$\Rightarrow \quad \left(\eta_{ac}\, e^a{}_{\alpha} e^c{}_{\gamma}\right) \left(\eta_{bd}\, e^b{}_{\beta} e^d{}_{\delta}\right) \left(e\, \epsilon^{mn} e_m{}^{\gamma} e_n{}^{\delta}\right) = e^2\, \epsilon_{\alpha\beta} \quad \Rightarrow$$

$$\Rightarrow \quad \eta_{ac}\, \eta_{bd}\, e^a{}_{\alpha} e^b{}_{\beta} \delta^c{}_m\, \delta^d{}_n\, \epsilon^{mn} = e\, \epsilon_{\alpha\beta} \quad \Rightarrow$$

$$\Rightarrow \quad \eta_{am}\, \eta_{bn}\, \epsilon^{mn}\, e^a{}_{\alpha} e^b{}_{\beta} = e\, \epsilon_{\alpha\beta} \quad \Rightarrow$$

$$\Rightarrow \quad \epsilon_{\alpha\beta} = e^{-1} \epsilon_{ab} e^a{}_{\alpha} e^b{}_{\beta}\,.$$

Some contractions (flat space or curved space):

$$\epsilon^{ab}\, \epsilon_{ab} = -2! = -2\,; \tag{B.23}$$

$$\epsilon^{ab}\, \epsilon_{ac} = -\delta^b{}_c\,; \tag{B.24}$$

$$\epsilon^{ab}\, \epsilon_{cd} = -\delta^a_{[c} \delta^b{}_{d]} = -\left(\delta^a{}_c \delta^b{}_d - \delta^a{}_d \delta^b{}_c\right)\,; \tag{B.25}$$

$$\epsilon^{\alpha\beta}\, \epsilon_{\alpha\beta} = -2! = -2\,; \tag{B.26}$$

$$\epsilon^{\alpha\beta}\, \epsilon_{\alpha\gamma} = -\delta^{\beta}{}_{\gamma}\,; \tag{B.27}$$

$$\epsilon^{\alpha\beta}\, \epsilon_{\gamma\delta} = -\delta^{\alpha}_{[\gamma} \delta^{\beta}{}_{\delta]} = -\left(\delta^{\alpha}{}_{\gamma} \delta^{\beta}{}_{\delta} - \delta^{\alpha}{}_{\delta} \delta^{\beta}{}_{\gamma}\right)\,. \tag{B.28}$$

In Chapters 4 and 5, we also introduced the following derivative

$$\nabla_a = e_a{}^{\alpha} \nabla_{\alpha}\,. \tag{B.29}$$

The action of such a derivative on a world-sheet scalar $f(x^0, x^1)$ is:

$$\nabla_a f = e_a{}^\alpha \nabla_\alpha f = e_a{}^\alpha \partial_\alpha f \,. \tag{B.30}$$

Let us notice that the nablas commute when applied on a world-sheet scalar as the following calculations show:

$$\begin{aligned}
[\nabla_a, \nabla_b] f &= \nabla_a(\nabla_b f) - \nabla_b(\nabla_a f) \\
&= e_a{}^\alpha \nabla_\alpha (e_b{}^\beta \nabla_\beta f) - e_b{}^\alpha \nabla_\alpha (e_a{}^\beta \nabla_\beta f) \\
&= e_a{}^\alpha e_b{}^\beta \left(\nabla_\alpha(\nabla_\beta f) - \nabla_\beta(\nabla_\alpha f)\right) \\
&= e_a{}^\alpha e_b{}^\beta [\nabla_\alpha, \nabla_\beta] f
\end{aligned}$$

and

$$\begin{aligned}
[\nabla_\alpha, \nabla_\beta] f &= \nabla_\alpha(\nabla_\beta f) - \nabla_\beta(\nabla_\alpha f) \\
&= \nabla_\alpha(\partial_\beta f) - \nabla_\beta(\partial_\alpha f) \\
&= \partial_\alpha \partial_\beta f - \Gamma^\gamma_{\alpha\beta} \partial_\gamma f - (\partial_\beta \partial_\alpha f - \Gamma^\gamma_{\beta\alpha} \partial_\gamma f) = 0 \,. \\
& \text{torsionless (Levi-Civita) connections}
\end{aligned}$$

In the previous calculation we have also used the condition $\nabla_\alpha e^b{}_\beta = 0 \ \forall \alpha, \beta, b$, known as the *tetrad postulate* in $D = 4$.

Appendix C

Hodge-duals

In a D-dimensional space, endowed with a metric tensor $g_{\alpha\beta}$, let us define the Hodge $*$-operator through its action on a particular p-form:

$$*(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}) \equiv \frac{\epsilon^{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_D}}{(D-p)!|g|^{1/2}} g_{\mu_{p+1}\nu_{p+1}} \ldots g_{\mu_D \nu_D} dx^{\nu_{p+1}} \wedge \ldots \wedge dx^{\nu_D} \,. \quad \text{(C.1)}$$

By reducing to $D = 2$, there are only two coordinates (x^0, x^1) and eq. (C.1) becomes:

$$*dx^\alpha = \frac{\epsilon^{\alpha\beta}}{|g|^{1/2}} \, g_{\beta\gamma} \, dx^\gamma \quad \text{(C.2)}$$

for a generic curved space, or

$$*dx^a = \epsilon^{ab} \, \eta_{bc} \, dx^c \quad \text{(C.3)}$$

for a flat minkowskian space.

Let us now introduce a bidimensional scalar field $\phi = \phi(x^0, x^1)$. In the flat minkowskian space the 1-form $d\phi$ is:

$$d\phi \equiv \partial_a \phi \, dx^a = \partial_0 \phi \, dx^0 + \partial_1 \phi \, dx^1 \,. \quad \text{(C.4)}$$

If the field ϕ depends on the combination $x^0 + x^1$ (i.e. $\phi = \phi(x^0 + x^1)$), it happens that $\partial_0 \phi = \partial_1 \phi$. The Hodge-dual of the 1-form $d\phi$ reads

$$\begin{aligned} *d\phi &= \partial_a \phi \, * \, dx^a \\ &= \epsilon^{ab} \, \eta_{bc} \, \partial_a \phi \, dx^c \\ &= \epsilon^{01} \, \eta_{11} \, \partial_0 \phi \, dx^1 + \epsilon^{10} \, \eta_{00} \, \partial_1 \phi \, dx^0 \\ &= \partial_0 \phi \, dx^1 + \partial_1 \phi \, dx^0 \\ &= \partial_0 \phi \, dx^0 + \partial_1 \phi \, dx^1 = d\phi \,. \end{aligned} \quad \text{(C.5)}$$

A field such that $*d\phi = d\phi$ is defined *self-dual*.

If, on the other hand, there exists a bidimensional scalar field $\psi = \psi(x^0, x^1)$ for which $\partial_0\psi = -\partial_1\psi$, (i.e. $\psi = \psi(x^0 - x^1)$), then

$$\begin{aligned} *d\psi &= \partial_a\psi * dx^a \\ &= \epsilon^{ab}\,\eta_{bc}\,\partial_a\psi\, dx^c \\ &= \epsilon^{01}\,\eta_{11}\,\partial_0\psi\, dx^1 + \epsilon^{10}\,\eta_{00}\,\partial_1\psi\, dx^0 \\ &= \partial_0\psi\, dx^1 + \partial_1\psi\, dx^0 \\ &= -(\partial_0\psi\, dx^0 + \partial_1\psi\, dx^1) = -d\psi\,. \end{aligned} \tag{C.6}$$

The field ψ, satisfying $*d\psi = -d\psi$, is said *antiself-dual.*

In a 2-dimensional flat minkowskian space, the following statements hold

$$\begin{aligned} \phi = \phi(x^0 + x^1) &\Leftrightarrow \phi \textit{ self-dual} \\ \psi = \psi(x^0 - x^1) &\Leftrightarrow \psi \textit{ antiself-dual}. \end{aligned}$$

Let us now introduce two scalar fields: $\varphi = \varphi(x^0, x^1)$ and $\tilde{\varphi} = \tilde{\varphi}(x^0, x^1)$. These fields are defined to be (*Hodge-*)*duals* if it happens that

$$d\tilde{\varphi} = *d\varphi, \tag{C.7}$$

which explicitly reads:

$$d\tilde{\varphi} = \partial_0\tilde{\varphi}\, dx^0 + \partial_1\tilde{\varphi}\, dx^1 = \partial_1\varphi\, dx^0 + \partial_0\varphi\, dx^1 = *d\varphi$$

and, hence, is equivalent to

$$\left\{ \begin{array}{l} \partial_0\tilde{\varphi} = \partial_1\varphi \\ \partial_1\tilde{\varphi} = \partial_0\varphi \end{array} \right. \Leftrightarrow \partial_a\tilde{\varphi} = -\epsilon_{ab}\,\partial^b\varphi. \tag{C.8}$$

In order to generalize the duality conditions between φ and $\tilde{\varphi}$ just found to a curved space, we have to follow the prescriptions of substituting ordinary derivatives with covariant ones and of multiplying by $\sqrt{-g}$. In so doing, (C.8) becomes

$$\nabla_\alpha\tilde{\varphi} = -\sqrt{-g}\,\epsilon_{\alpha\beta}\,\nabla^\beta\varphi \quad \Rightarrow \quad \partial_\alpha\tilde{\varphi} = -\sqrt{-g}\,\epsilon_{\alpha\beta}\,\partial^\beta\varphi, \tag{C.9}$$

because the covariant derivatives act on scalar fields ($\nabla_\alpha\phi = \partial_\alpha\phi$). Once a 2-bein is introduced (see Appendix B), eq. (C.9) is equivalent to the following relations:

$$\left\{ \begin{array}{l} \nabla_0\tilde{\varphi} = \nabla_1\varphi \\ \nabla_1\tilde{\varphi} = \nabla_0\varphi \end{array} \right. , \tag{C.10}$$

as it will be demonstrated in a moment. Recalling eq. (C.2), specified for $|g| = -g$, we have

$$d\tilde{\varphi} = *d\varphi \quad \Rightarrow \quad \partial_\alpha\tilde{\varphi}\, dx^\alpha = \frac{\epsilon^{\alpha\beta}}{\sqrt{-g}}\, g_{\beta\gamma}\,\partial_\alpha\varphi\, dx^\gamma$$

$$
\begin{aligned}
&\Rightarrow \quad \partial_\alpha \tilde{\varphi} = \frac{1}{\sqrt{-g}}\, \epsilon^{\beta\gamma}\, g_{\gamma\alpha}\, g_{\beta\sigma}\, \partial^\sigma \varphi \\
&\Rightarrow \quad \partial_\alpha \tilde{\varphi} = \frac{1}{\sqrt{-g}} (-g\, \epsilon_{\sigma\alpha})\, \partial^\sigma \varphi \\
&\Rightarrow \quad \partial_\alpha \tilde{\varphi} = \frac{1}{\sqrt{-g}}\, g\, \epsilon_{\alpha\sigma}\, \partial^\sigma \varphi \\
&\Rightarrow \quad \partial_\alpha \tilde{\varphi} = -\sqrt{-g}\, \epsilon_{\alpha\beta}\, \partial^\beta \varphi \\
&\Rightarrow \quad \partial_\alpha \tilde{\varphi} = -\epsilon_{ab}\, e^a{}_\alpha e^b{}_\beta\, \eta^{cd} e_c{}^\beta e_d{}^\gamma\, \partial_\gamma \varphi \\
&\Rightarrow \quad \partial_\alpha \tilde{\varphi} = -\epsilon_{ab}\, \eta^{cd}\, \delta^b{}_c\, e^a{}_\alpha e_d{}^\gamma\, \partial_\gamma \varphi \\
&\Rightarrow \quad e_a{}^\alpha\, \partial_\alpha \tilde{\varphi} = -\epsilon_{ab}\, \eta^{bc}\, e_c{}^\beta\, \partial_\beta \varphi \\
&\Rightarrow \quad \nabla_a \tilde{\varphi} = -\epsilon_{ab}\, \eta^{bc}\, \nabla_c \varphi \\
&\Rightarrow \quad \nabla_a \tilde{\varphi} = -\epsilon_{ab}\, \nabla^b \varphi \,,
\end{aligned}
$$

exactly reproducing eq. (C.10).

Appendix D

Background fields

In this Appendix we find some meaningful relations involving the T-duality-transformed background fields $\tilde{G}$ and $\tilde{B}$ in terms of the 'original' ones, G and B.

Let us start from the following definitions of the $n \times n$ matrices:

$$E \equiv G + B \; ; \quad \tilde{E} \equiv \tilde{G} + \tilde{B} \, , \tag{D.1}$$

in which G and $\tilde{G}$ are symmetric while B and $\tilde{B}$ are antisymmetric.

Obviously

$$E^T = G^T + B^T = G - B$$

and

$$\tilde{E}^T = \tilde{G}^T + \tilde{B}^T = \tilde{G} - \tilde{B} \, ,$$

leading to

$$G = \frac{1}{2}\left(E + E^T\right) \; , \quad B = \frac{1}{2}\left(E - E^T\right) ;$$

$$\tilde{G} = \frac{1}{2}\left(\tilde{E} + \tilde{E}^T\right) \; , \quad \tilde{B} = \frac{1}{2}\left(\tilde{E} - \tilde{E}^T\right) .$$

As we stressed in Chapters 2 and 5, a T-duality transformation acts on the string coordinates and their duals as well as on the background fields G and B, accordingly to the following law:

$$E \leftrightarrow \tilde{E} \equiv E^{-1} \, . \tag{D.2}$$

It implies

$$\tilde{G} = \frac{1}{2}\left[(G + B)^{-1} + (G - B)^{-1}\right] \tag{D.3}$$

and

$$\tilde{B} = \frac{1}{2}\left[(G + B)^{-1} - (G - B)^{-1}\right] . \tag{D.4}$$

The last equations can be used to derive more useful expressions for $\tilde{G}$ and $\tilde{B}$. In fact,

$$\begin{aligned}
\tilde{G}^{-1} &= \left\{\frac{1}{2}\left[(G+B)^{-1}+(G-B)^{-1}\right]\right\}^{-1} \\
&= \left\{\frac{1}{2}(G+B)^{-1}\left[I+(G+B)(G-B)^{-1}\right]\right\}^{-1} \\
&= \left\{\frac{1}{2}(G+B)^{-1}\left[(G-B)+(G+B)\right](G-B)^{-1}\right\}^{-1} \\
&= \left\{\frac{1}{2}(G+B)^{-1}(2G)(G-B)^{-1}\right\}^{-1} \\
&= (G-B)\,G^{-1}(G+B) \\
&= G+B-B-BG^{-1}B = G-BG^{-1}B
\end{aligned}$$

and, finally,

$$\tilde{G} = \left(G-BG^{-1}B\right)^{-1}. \tag{D.5}$$

Analogously,

$$\begin{aligned}
\tilde{B}^{-1} &= \left\{\frac{1}{2}\left[(G+B)^{-1}-(G-B)^{-1}\right]\right\}^{-1} \\
&= \left\{\frac{1}{2}(G+B)^{-1}\left[I-(G+B)(G-B)^{-1}\right]\right\}^{-1} \\
&= \left\{\frac{1}{2}(G+B)^{-1}\left[(G-B)-(G+B)\right](G-B)^{-1}\right\}^{-1} \\
&= \left\{\frac{1}{2}(G+B)^{-1}(-2B)(G-B)^{-1}\right\}^{-1} \\
&= -(G-B)\,B^{-1}(G+B) \\
&= -GB^{-1}G-G+G+B = B-GB^{-1}G
\end{aligned}$$

equivalent to

$$\tilde{B} = \left(B-GB^{-1}G\right)^{-1}. \tag{D.6}$$

An equivalent formulation of (D.6) is: $\tilde{B} = -G^{-1}B\tilde{G}$. In fact

$$\begin{aligned}
\left(-G^{-1}B\tilde{G}\right)^{-1} &= -\tilde{G}^{-1}B^{-1}G \\
&= -\left(G-BG^{-1}B\right)B^{-1}G \\
&= -GB^{-1}G+B = \tilde{B}^{-1}.
\end{aligned}$$

Moreover, the following identity holds $\tilde{B}\tilde{G}^{-1} = -G^{-1}B$ together with its transposition $-\tilde{G}^{-1}\tilde{B} = BG^{-1}$.

Appendix E

First order Lagrangians

In this Appendix, first order Lagrangians are studied.

E.1 Lagrangians with a discrete number of degrees of freedom

Let us consider the following first order Lagrangians:

$$L_{\pm}(q,\dot{q}) = \pm\frac{1}{2}\sum_{i,j} q^i c_{ij}\dot{q}^j - V(q)\,, \qquad i,j = 1,\ldots,N\,. \tag{E.1}$$

(Hereafter, we will use Einstein's summation convention.)

The constant matrix c_{ij} can be always chosen antisymmetric ($c_{ij} = -c_{ji}$). The presence of a symmetric part in c_{ij} leads to a total time derivative. In fact, if c_{ij} is decomposed into its symmetric and antisymmetric parts ($c_{ij} = c_{(ij)} + c_{[ij]}$), we have:

$$\begin{aligned}
\frac{1}{2}q^i c_{(ij)}\dot{q}^j &= \frac{1}{4}q^i c_{(ij)}\dot{q}^j + \frac{1}{4}q^i c_{(ij)}\dot{q}^j \\
&= \frac{1}{4}q^j c_{(ji)}\dot{q}^i + \frac{1}{4}q^i c_{(ij)}\dot{q}^j \\
&= \frac{1}{4}\dot{q}^i c_{(ij)}q^j + \frac{1}{4}q^i c_{(ij)}\dot{q}^j \\
&= \frac{1}{4}\frac{d}{dt}\left(q^i c_{(ij)}q^j\right).
\end{aligned}$$

The lagrangians in (E.1) are degenerate in the sense that

$$\det\left(\frac{\partial^2 L_{\pm}}{\partial\dot{q}^i\partial\dot{q}^j}\right) = 0$$

and so they describe constrained systems.

We want the matrix c_{ij} to be invertible. This requirement selects the values of N. In fact, if N is odd, then $\det(c_{ij}) = 0$ independently of the entries c_{ij}, while if N is even, then we must anyway impose the condition $\det(c_{ij}) \neq 0$. For example, for $N = 4$, it happens that

$$\det\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = (af - be + cd)^2\,,$$

which can vanish even if no one of the parameters a, b, c, d, e, f is zero.

In order to write down the Euler-Lagrange equations coming from $L_\pm$, let us compute some quantities:

$$\frac{\partial L_\pm}{\partial q^i} = \pm\frac{1}{2}c_{ij}\dot{q}^j - \frac{\partial V}{\partial q^i}$$

and

$$\frac{\partial L_\pm}{\partial \dot{q}^i} = \pm\frac{1}{2}q^j c_{ji} = \mp\frac{1}{2}c_{ij}q^j \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L_\pm}{\partial \dot{q}^i}\right) = \mp\frac{1}{2}c_{ij}\dot{q}^j\,.$$

So, the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L_\pm}{\partial \dot{q}^i}\right) - \frac{\partial L_\pm}{\partial q^i} = 0 \;\Rightarrow\; \mp\frac{1}{2}c_{ij}\dot{q}^j \mp \frac{1}{2}c_{ij}\dot{q}^j + \frac{\partial V}{\partial q^i} = 0 \;\Rightarrow$$

$$\Rightarrow\; \pm c_{ij}\dot{q}^j = \frac{\partial V}{\partial q^i} \;\Rightarrow\; \dot{q}^i = \pm c^{ij}\frac{\partial V}{\partial q^j}\,, \tag{E.2}$$

where c^{ij} is the element ij of the matrix c^{-1} ($c^{ij} \equiv (c^{-1})_{ij}$).

The conjugate momentum to q^i is, by definition,

$$p_i^{(\pm)} \equiv \frac{\partial L_\pm}{\partial \dot{q}^i} = \mp\frac{1}{2}c_{ij}q^j\,. \tag{E.3}$$

These conjugate momenta define N primary 2^{nd} class constraints:

$$\phi_i^{(\pm)}(q, p^{(\pm)}) \equiv p_i^{(\pm)} \pm \frac{1}{2}c_{ij}q^j \approx 0\,. \tag{E.4}$$

The Legendre transformation permits to compute the Hamiltonian

$$H(q, p^{(\pm)}) \equiv \left(\sum_{k=1}^{N} p_k^{(\pm)}\dot{q}^k - L_\pm(q, \dot{q})\right)_{\dot{q}=u(q,p^{(\pm)})} = V(q)\,. \tag{E.5}$$

Aimed to study the quantization of the observables described by the first order Lagrangians, let us introduce the Poisson brackets. They are defined as follows:

$$\{f, g\}_{PB} \equiv \sum_{k=1}^{N}\left[\frac{\partial f}{\partial q^k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q^k}\right]. \tag{E.6}$$

They are antisymmetric

$$\{g, f\}_{PB} = -\{f, g\}_{PB} \,. \tag{E.7}$$

By definition, let us calculate some Poisson brackets:

$$\{q^i, q^j\}_{PB} = 0 \,; \tag{E.8}$$

$$\{p_i^{(\pm)}, p_j^{(\pm)}\}_{PB} = 0 \,; \tag{E.9}$$

$$\{q^i, p_j^{(\pm)}\}_{PB} = \delta^i{}_j \,. \tag{E.10}$$

With these brackets we can construct the following ones

$$\begin{aligned} \{q^i, \phi_j^{(\pm)}\}_{PB} &= \left\{q^i, p_j^{(\pm)} \pm \frac{1}{2} c_{jk} q^k\right\}_{PB} \\ &= \{q^i, p_j^{(\pm)}\}_{PB} = \delta^i{}_j \,; \end{aligned} \tag{E.11}$$

$$\begin{aligned} \{\phi_i^{(\pm)}, q^j\}_{PB} &= \left\{p_i^{(\pm)} \pm \frac{1}{2} c_{ik} q^k, q^j\right\}_{PB} \\ &= \{p_i^{(\pm)}, q^j\}_{PB} = -\delta_i{}^j \,; \end{aligned} \tag{E.12}$$

$$\begin{aligned} \{p_i^{(\pm)}, \phi_j^{(\pm)}\}_{PB} &= \left\{p_i^{(\pm)}, p_j^{(\pm)} \pm \frac{1}{2} c_{jk} q^k\right\}_{PB} \\ &= \pm\frac{1}{2} c_{jk} \{p_i^{(\pm)}, q^k\}_{PB} \\ &= \pm\frac{1}{2} c_{jk} (-\delta_i{}^k) \\ &= \mp\frac{1}{2} c_{ji} = \pm\frac{1}{2} c_{ij} \,; \end{aligned} \tag{E.13}$$

$$\begin{aligned} \{\phi_i^{(\pm)}, p_j^{(\pm)}\}_{PB} &= \left\{p_i^{(\pm)} \pm \frac{1}{2} c_{ik} q^k, p_j^{(\pm)}\right\}_{PB} \\ &= \pm\frac{1}{2} c_{ik} \{q^k, p_j^{(\pm)}\}_{PB} \\ &= \pm\frac{1}{2} c_{ik}\, \delta^k{}_j = \pm\frac{1}{2} c_{ij} \,. \end{aligned} \tag{E.14}$$

And, moreover,

$$\begin{aligned} \{\phi_i^{(\pm)}, \phi_j^{(\pm)}\}_{PB} &= \left\{p_i^{(\pm)} \pm \frac{1}{2} c_{ik} q^k, \phi_j^{(\pm)}\right\}_{PB} \\ &= \{p_i^{(\pm)}, \phi_j^{(\pm)}\}_{PB} \pm \frac{1}{2} c_{ik} \{q^k, \phi_j^{(\pm)}\}_{PB} \\ &= \pm\frac{1}{2} c_{ij} \pm \frac{1}{2} c_{ik}\, \delta^k{}_j \\ &= \pm\frac{1}{2} c_{ij} \pm \frac{1}{2} c_{ij} = \pm c_{ij} \,, \end{aligned} \tag{E.15}$$

leading to

$$\left[\{\phi_i^{(\pm)}, \phi_j^{(\pm)}\}_{PB}\right]^{-1} = \pm c^{ij} \,, \tag{E.16}$$

which are fundamental to construct Dirac brackets, defined as follows:

$$\{f, g\}_{DB} \equiv \{f, g\}_{PB} - \sum_{k,l} \{f, \phi_k\}_{PB} \left[\{\phi_k, \phi_l\}_{PB}\right]^{-1} \{\phi_l, g\}_{PB} \,. \tag{E.17}$$

The antisymmetry of $\left[\{\phi_i^{(\pm)}, \phi_j^{(\pm)}\}_{PB}\right]^{-1}$ is a crucial property. In fact, only in this case, Dirac brackets are antisymmetric as it can be easily verified by direct inspection.

Let us now calculate some Dirac brackets by using the definition (E.17) and the results in eqs. (E.8 - E.14) and in (E.16):

$$\{q^i, q^j\}_{DB} = -\sum_{k,l} \delta^i{}_k \left(\pm c^{kl}\right) \left(-\delta_l{}^j\right) = \pm c^{ij} \,; \tag{E.18}$$

$$\begin{aligned} \{p_i^{(\pm)}, p_j^{(\pm)}\}_{DB} &= -\sum_{k,l} \left(\pm \frac{1}{2} c_{ik}\right) \left(\pm c^{kl}\right) \left(\pm \frac{1}{2} c_{lj}\right) \\ &= \mp \frac{1}{4} \sum_k c_{ik} \, \delta^k{}_j = \mp \frac{1}{4} c_{ij} \,; \end{aligned} \tag{E.19}$$

$$\begin{aligned} \{q^i, p_j^{(\pm)}\}_{DB} &= \delta^i{}_j - \sum_{k,l} \delta^i{}_k \left(\pm c^{kl}\right) \left(\pm \frac{1}{2} c_{lj}\right) \\ &= \delta^i{}_j - \frac{1}{2} \delta^i{}_j = \frac{1}{2} \delta^i{}_j \,. \end{aligned} \tag{E.20}$$

Every Dirac brackets containing a $\phi_i^{(\pm)}$ vanishes by definition:

$$\{q^i, \phi_j^{(\pm)}\}_{DB} = \{p_i^{(\pm)}, \phi_j^{(\pm)}\}_{DB} = \{\phi_i^{(\pm)}, \phi_j^{(\pm)}\}_{DB} = 0 \,.$$

Quantization: we use the correspondence

$$\{\cdot\,, \cdot\}_{PB} \to -i[\cdot\,, \cdot] \,; \tag{E.21}$$

which leads to

$$\{\cdot\,, \cdot\}_{DB} \to -i[\cdot\,, \cdot] \,. \tag{E.22}$$

So we get

$$[\hat{q}^i, \hat{q}^j] = \pm i \, c^{ij} \,; \tag{E.23}$$

$$[\hat{p}_i^{(\pm)}, \hat{p}_j^{(\pm)}] = \mp \frac{i}{4} c_{ij} \,; \tag{E.24}$$

$$[\hat{q}^i, \hat{p}_j^{(\pm)}] = \frac{i}{2} \delta^i{}_j \,. \tag{E.25}$$

E.2 Lagrangians with an infinite number of degrees of freedom

Let us now examine two kinds of integral Lagrangians:

$$\begin{aligned} L_\pm &= \pm\frac{1}{2}\int dx\,dy\,\chi_\pm(x,t)\,\epsilon(x-y)\,\dot\chi_\pm(y,t) - \frac{1}{2}\int dx\,\chi^2_\pm(x,t) \\ &= \int dx\left\{\pm\frac{1}{2}\int dy\,\epsilon(x-y)\,\dot\chi_\pm(y,t)\cdot\chi_\pm(x,t) - \frac{1}{2}\chi^2_\pm(x,t)\right\}. \end{aligned} \tag{E.26}$$

The conjugate momenta to $\chi_\pm$ are

$$\begin{aligned} \pi_\pm(x,t) &\equiv \frac{\delta L_\pm}{\delta\dot\chi_\pm(x,t)} = \pm\frac{1}{2}\int dy\,dz\,\chi_\pm(y,t)\,\epsilon(y-z)\,\frac{\delta\dot\chi_\pm(z,t)}{\delta\dot\chi_\pm(x,t)} \\ &= \pm\frac{1}{2}\int dy\,dz\,\chi_\pm(y,t)\,\epsilon(y-z)\,\delta(z-x) \\ &= \pm\frac{1}{2}\int dy\,\chi_\pm(y,t)\,\epsilon(y-x) \\ &= \mp\frac{1}{2}\int dy\,\epsilon(x-y)\,\chi_\pm(y,t)\,. \end{aligned} \tag{E.27}$$

As in the previous section, the conjugate momenta define primary 2^{nd} class constraints. They are

$$\phi_\pm(x,t) \equiv \pi_\pm(y,t) \mp \frac{1}{2}\int dy\,\chi_\pm(y,t)\epsilon(y-x) \approx 0\,, \tag{E.28}$$

and satisfy

$$\{\phi_\pm(x,t),\phi_\pm(y,t)\}_{PB} = \pm\epsilon(x-y) \neq 0\,. \tag{E.29}$$

We can generalize the results obtained in the previous section in agreement with the following discrete/continuous correspondences:

$$\begin{aligned} q^i(t) &\rightarrow \chi_\pm(x,t) \\ p_i^{(\pm)}(t) &\rightarrow \pi_\pm(x,t) \\ \sum_i &\rightarrow \int dx \\ \sum_{i,j} &\rightarrow \int dx\,dy \\ \delta^i{}_j &\rightarrow \delta(x-y) \\ c_{ij} &\rightarrow \epsilon(x-y) \\ V(q) &\rightarrow \frac{1}{2}\int dx\,\chi^2_\pm(x,t) \\ \phi_i^{(\pm)}(t) &\rightarrow \phi_\pm(x,t) \end{aligned}$$

leading to the fundamental Poisson brackets

$$\begin{aligned} \{q^i, q^j\}_{PB} = 0 \quad &\rightarrow \quad \{\chi_\pm(x,t), \chi_\pm(y,t)\}_{PB} = 0\,; \\ \{p_i^{(\pm)}, p_j^{(\pm)}\}_{PB} = 0 \quad &\rightarrow \quad \{\pi_\pm(x,t), \pi_\pm(y,t)\}_{PB} = 0\,; \\ \{q^i, p_j^{(\pm)}\}_{PB} = \delta^i{}_j \quad &\rightarrow \quad \{\chi_\pm(x,t), \pi_\pm(y,t)\}_{PB} = \delta(x-y)\,. \end{aligned} \tag{E.30}$$

On the other side, for the Dirac ones and for the commutators, we have

$$\begin{aligned} \{q^i, q^j\}_{DB} = \pm c^{ij} \quad &\rightarrow \quad \{\chi_\pm(x,t), \chi_\pm(y,t)\}_{DB} = \pm\delta'(x-y) \Rightarrow \\ \Rightarrow [\hat{q}^i, \hat{q}^j] = \pm i\, c^{ij} \quad &\rightarrow \quad [\hat{\chi}_\pm(x,t), \hat{\chi}_\pm(y,t)] = \pm i\, \delta'(x-y)\,; \end{aligned} \tag{E.31}$$

$$\begin{aligned} \{p_i^{(\pm)}, p_j^{(\pm)}\}_{DB} = \mp\tfrac{1}{4} c_{ij} \quad &\rightarrow \quad \{\pi_\pm(x,t), \pi_\pm(y,t)\}_{DB} = \mp\tfrac{1}{4}\epsilon(x-y) \Rightarrow \\ \Rightarrow [\hat{p}_i^{(\pm)}, \hat{p}_j^{(\pm)}] = \mp\tfrac{i}{4} c_{ij} \quad &\rightarrow \quad [\hat{\pi}_\pm(x,t), \hat{\pi}_\pm(y,t)] = \mp\tfrac{i}{4}\epsilon(x-y)\,; \end{aligned} \tag{E.32}$$

$$\begin{aligned} \{q^i, p_j^{(\pm)}\}_{DB} = \tfrac{1}{2}\delta^i{}_j \quad &\rightarrow \quad \{\chi_\pm(x,t), \pi_\pm(y,t)\}_{DB} = \tfrac{1}{2}\delta(x-y) \Rightarrow \\ \Rightarrow [\hat{q}^i, \hat{p}_j^{(\pm)}] = \tfrac{i}{2}\delta^i{}_j \quad &\rightarrow \quad [\hat{\chi}_\pm(x,t), \hat{\pi}_\pm(y,t)] = \tfrac{i}{2}\delta(x-y)\,. \end{aligned} \tag{E.33}$$

Let us recall that $\delta'(x) \equiv [\epsilon(x)]^{-1}$ (see Appendix A).

In order to recover the first order Lagrangians so extensively quoted in this thesis

$$\mathcal{L}_+(\dot{\varphi}_+, \varphi'_+) = \frac{1}{2}\dot{\varphi}_+\varphi'_+ - \frac{1}{2}\left(\varphi'_+\right)^2 \tag{E.34}$$

and

$$\mathcal{L}_-(\dot{\varphi}_-, \varphi'_-) = -\frac{1}{2}\dot{\varphi}_-\varphi'_- - \frac{1}{2}\left(\varphi'_-\right)^2 \tag{E.35}$$

(the so-called Floreanini-Jackiw Lagrangians), we have to introduce a new pair of fields $\varphi_\pm$ as

$$\varphi_\pm(x,t) \equiv \int dy\, \epsilon(x-y)\, \chi_\pm(y,t) = \mp 2\, \pi_\pm(x,t) \tag{E.36}$$

(the last equality derives from eq. (E.27)) and to admit that

$$L_\pm = \int dx\, \mathcal{L}_\pm\,.$$

The time and space derivatives of $\varphi_\pm$ are

$$\dot{\varphi}_\pm(x,t) = \int dy\, \epsilon(x-y)\, \dot{\chi}_\pm(y,t) \tag{E.37}$$

and

$$\varphi'_{\pm}(x,t) = \chi_{\pm}(x,t)\,. \tag{E.38}$$

Let us compute the conjugate momenta with respect to $\varphi_{\pm}$:

$$\mathcal{P}_{\pm} \equiv \frac{\partial \mathcal{L}_{\pm}}{\partial \dot{\varphi}_{\pm}} = \pm\frac{1}{2}\varphi'_{\pm}. \tag{E.39}$$

From eq. (E.38) we see that $\mathcal{P}_{\pm} = \pm(1/2)\chi_{\pm}$.

In order to write down the Hamiltonians corresponding to Floreanini-Jackiw Lagrangians, we can equivalently use the discrete/continuous correspondence on eq. (E.5) or compute them directly by a Legendre transformation. The result is

$$H = \frac{1}{2}\int dx\,\chi^2_{\pm}(x,t) = \frac{1}{2}\int dx\,[\varphi'_{\pm}(x,t)]^2\,. \tag{E.40}$$

Finally, we can compute the following Dirac brackets:

$$\begin{aligned}
\{\varphi_{\pm}(x,t),\varphi_{\pm}(y,t)\}_{DB} &= \{\mp 2\,\pi_{\pm}(x,t),\mp 2\,\pi_{\pm}(y,t)\}_{DB} \\
&= 4\{\pi_{\pm}(x,t),\pi_{\pm}(y,t)\}_{DB} \\
&= \mp\epsilon(x-y)\,;
\end{aligned} \tag{E.41}$$

$$\begin{aligned}
\{\mathcal{P}_{\pm}(x,t),\mathcal{P}_{\pm}(y,t)\}_{DB} &= \{\pm\frac{1}{2}\chi_{\pm}(x,t),\pm\frac{1}{2}\chi_{\pm}(y,t)\}_{DB} \\
&= \frac{1}{4}\{\chi_{\pm}(x,t),\chi_{\pm}(y,t)\}_{DB} \\
&= \pm\frac{1}{4}\delta'(x-y)\,;
\end{aligned} \tag{E.42}$$

$$\begin{aligned}
\{\varphi_{\pm}(x,t),\mathcal{P}_{\pm}(y,t)\}_{DB} &= \{\mp 2\,\pi_{\pm}(x,t),\pm\frac{1}{2}\chi_{\pm}(y,t)\}_{DB} \\
&= -\{\pi_{\pm}(x,t),\chi_{\pm}(y,t)\}_{DB} \\
&= \frac{1}{2}\delta(x-y)\,.
\end{aligned} \tag{E.43}$$

Consequently, for the commutators,

$$[\hat{\varphi}_{\pm}(x,t),\hat{\varphi}_{\pm}(y,t)] = \mp\frac{i}{2}\epsilon(x-y)\,; \tag{E.44}$$

$$[\hat{\mathcal{P}}_{\pm}(x,t),\hat{\mathcal{P}}_{\pm}(y,t)] = \pm\frac{i}{4}\delta'(x-y)\,; \tag{E.45}$$

$$[\hat{\varphi}_{\pm}(x,t),\hat{\mathcal{P}}_{\pm}(y,t)] = \frac{i}{2}\delta(x-y)\,. \tag{E.46}$$

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