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# The dark side of finite geometry: $\sigma$-quadrics, $m$-ovoids and non-linear MRD codes

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# Contents

# Preface

The aim of this PhD Thesis is to collect the results which I obtained during my PhD course at University of Naples Federico II under the supervision of Prof. Nicola Durante.

In the following, we briefly describe the structure of this thesis.

In Chapter 1, we recall some definitions and notions regarding projective spaces, polarities and polar spaces, and we also give an overview on sesquilinear forms on vector spaces. Here a classification result regarding non-degenerate, reflexive sesquilinear forms is shown. These ones are in 1 to 1 correspondence with polarities and give rise to the so called *finite classical polar spaces*. Since the classification of sesquilinear forms does not include quadrics when the field has even characteristic, quadratic forms on vector spaces are described. Finally, we introduce the concept of $m$-ovoid of a finite classical polar space and describe the link between the ovoids of the *Klein quadric* $Q^+(5, q)$ and the line spreads of a projective space $\mathrm{PG}(3, q)$.

In Chapter 2, we deal with the theory of *rank distance codes* (or for short *RD-codes*). An RD-code attaining the *Singleton-like bound* is known as *maximum rank distance code* (or for short *MRD-code*). We present the setting of $\sigma$-linearized polynomials, the state of art of the known families of MRD codes and we show how to construct families of RD-codes from *exterior sets*. Finally, we describe in detail the family of MRD codes obtained in [28].

In Chapter 3, Chapter 4, Chapter 5, the so called *$\sigma$-quadrics* are described. In particular, in Chapter 3 we give a complete classification of correlations, associated to non-reflexive sesquilinear forms, of a projective line $\mathrm{PG}(1, q^n)$ [23] and a projective plane $\mathrm{PG}(2, q^n)$ [23, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56] in the degenerate and the non-degenerate case. Also, the classification of correlations, associated to degenerate non-reflexive sesquilinear forms, of a projective space $\mathrm{PG}(3, q^n)$ is given [29]. In Chapter 4 and Chapter 5, we study correlations associated to degenerate non-reflexive sesquilinear forms, of the projective spaces $\mathrm{PG}(4, q^n)$ and $\mathrm{PG}(5, q^n)$, respectively. Moreover, we show how $\sigma$-quadrics are related to ovoids of polar spaces. We highlight that nothing is known about classification of correlations associated to non-degenerate non-reflexive sesquilinear forms, of a projective space $\mathrm{PG}(k, q^n)$ with $k \in \{3, 4, 5\}$.

In Chapter 6 and Chapter 7, we deal the classification of "low-degree" ovoids of the hyperbolic quadric $Q^+(5, q)$ and the parabolic quadric $Q(6, q)$. This research is inspired by the paper of Bartoli and Durante [7], in which they obtain classification results regarding ovoids of the parabolic quadric $Q(4,q)$. Ovoids of the Klein quadric $Q^+(5,q)$ of $\mathrm{PG}(5,q)$ have been studied in the last 40 year, also because of their connection with spreads of $\mathrm{PG}(3,q)$ and hence translation planes. Beside the classical example given by a three dimensional elliptic quadric (corresponding to the regular spread of $\mathrm{PG}(3,q)$) many other classes of examples are known. First of all the other examples (beside the elliptic quadric) of ovoids of $Q(4,q)$ give also examples of ovoids of $Q^+(5,q)$. Another important class of ovoids of $Q^+(5,q)$ is given by the ones associated to a flock of a three dimensional quadratic cone. To every ovoid of $Q^+(5,q)$ two bivariate polynomials $f_1(x,y)$ and $f_2(x,y)$ can be associated. We classify ovoids of $Q^+(5,q)$ such that $f_1(x,y) = y + g(x)$ and $\max\{\deg(f_1), \deg(f_2)\} < (\frac{1}{6.3}q)^{\frac{3}{13}} - 1$, that is $f_1(x,y)$ and $f_2(x,y)$ have "low total degree" compared with $q$. Ovoids of the parabolic quadric $Q(6,q)$ of $\mathrm{PG}(6,q)$ have been largely studied in the last 40 years. They can only occur if $q$ is an odd prime power and there are two known families of ovoids of $Q(6,q)$, the Thas-Kantor ovoids and the Ree-Tits ovoids, both for $q$ a power of 3. It is well known that to any ovoid of $Q(6,q)$ two polynomials $f_1(x,y,z)$, $f_2(x,y,z)$ can be associated. We classify ovoids of $Q(6,q)$ with $\max\{\deg(f_1), \deg(f_2)\} < (\frac{1}{6.3}q)^{\frac{3}{13}} - 1$.

In Chapter 8, we provide a construction of $(q+1)$-ovoids of the hyperbolic quadric $Q^+(7,q)$, $q$ an odd prime power, by "glueing" $(q+1)/2$-ovoids of the elliptic quadric $Q^-(5,q)$. This is possible by controlling some intersection properties of (putative) $m$-ovoids of elliptic quadrics. Secondly, we also construct $m$-ovoids for $m \in \{2,4,6,8,10\}$ in $Q^+(7,3)$. Therefore we first investigate how to construct spreads of $\mathrm{PG}(3,q)$ that have as many secants to an elliptic quadric as possible.

In Chapter 9, we give a geometric construction of a new family of non-linear MRD codes. These codes are obtained from a cone of a projective space whose vertex is a proper subspace and whose base is a special subset of a subspace skew with the vertex. The base of such a cone is linked to the non-linear MRD codes constructed by Cossidente et al. [17], Durante and Siciliano [33] and Donati and Durante [28]. These latter can be obtained in turn by puncturing codes in relevant family. Finally, we show that this class of codes is not equivalent to the one constructed by Otal and Özbudak in [69].

# Chapter 1

# Finite classical polar spaces

## 1.1 The projective space $\mathrm{PG}(V)$

We will denote a field by $\mathbb{F}$ and a finite field with $q$ elements, $q$ a prime power, by $\mathbb{F}_q$. For any subset $S$ of an $\mathbb{F}$-vector space $V$, we will denote the subspace spanned by the elements of $S$ by $\langle S\rangle$. We recall the definitions of linear and semilinear map between vector spaces.

**Definition 1.1.1.** Let $V$ and $W$ be two $\mathbb{F}$-vector spaces. A map $f : V \longrightarrow W$ is called *linear* if

$$f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}) \text{ and } f(a\mathbf{v}) = af(\mathbf{v})$$

for any $\mathbf{u}, \mathbf{v} \in V$ and $a \in \mathbb{F}$. If $f$ is bijective, then $f$ is an *isomorphism* between $V$ and $W$; in this case $V$ and $W$ are *isomorphic*. If $f$ is an isomorphism of $V$ into itself, then $f$ is an *automorphism* of $V$.
The set of all automorphisms of $V$

$$\mathrm{GL}(V) = \{g : V \longrightarrow V : g \text{ is an automorphism of } V\}.$$

is a group with respect to the map composition and it is called *general linear group* of $V$; if $\dim V = n+1$ then it is isomorphic to the group of invertible $(n+1)\times(n+1)$ matrices over $\mathbb{F}$. If $V = \mathbb{F}_q^{n+1}$, then the linear group will be denoted by $\mathrm{GL}(n+1, q)$.

The next definition generalizes the notion of linear map.

**Definition 1.1.2.** Let $V$ and $W$ be two $\mathbb{F}$-vector spaces. A map $f : V \longrightarrow W$ is called *semilinear* or *$\sigma$-linear* if there exists an automorphism $\sigma$ of $\mathbb{F}$ such that

$$f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v}) \text{ and } f(a\mathbf{v}) = a^{\sigma}f(\mathbf{v})$$

for any $\mathbf{u}, \mathbf{v} \in V$ and $a \in \mathbb{F}$. If $f$ is bijective, then $f$ is a *semilinear isomorphism* between $V$ and $W$. If $f$ is a semilinear isomorphism of $V$ into itself, then $f$ is a *semilinear automorphism* of $V$.
The set of all semilinear automorphisms of $V$

$$\Gamma\mathrm{L}(V) = \{g : V \longrightarrow V : g \text{ is a semilinear automorphism of } V\}$$

is a group with respect to the map composition and it is called *semilinear group* of $V$. If $V = \mathbb{F}_q^{n+1}$, this group will be denoted by $\Gamma\mathrm{L}(n+1,q)$.

The following proposition shows how many $k$-dimensional subspaces are contained in an $n$-dimensional vector space over a finite field $\mathbb{F}_q$ and it is easily proved.

**Proposition 1.1.3.** *Let $V$ be an $n$-dimensional $\mathbb{F}_q$-vector space and $1 \leq k \leq n$. The number of $k$-dimensional subspaces of $V$ equals*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^{n-i}-1}{q^{k-i}-1}.$$

Let $V$ be an $(n+1)$-dimensional $\mathbb{F}$-vector space. We will denote the set of all 1-dimensional subspaces of $V$ by PG($V$), and we will call them *points* of PG($V$). Any point $P$ of PG($V$) will be also denoted by $\langle \mathbf{v} \rangle$, where $\mathbf{v}$ is a non-zero vector of $V$. Let $S$ be a $k$-dimensional subspace of $V$, $0 \leq k \leq n+1$, the set of all 1-dimensional subspaces contained in $S$ will be called *$(k-1)$-dimensional projective subspace of* PG($V$), or shortly *$(k-1)$-space of* PG($V$). In the sequel if $S$ is a vector subspace of $V$, we will denote with the same symbol $S$ the associated projective subspace of PG($V$). A projective subspace of dimension $1, 2, n-1$ will be also called *line*, *plane*, *hyperplane*, respectively. We will assume that the empty set is a projective subspace of dimension $-1$. For any $-1 \leq j \leq n$, let $\mathcal{S}_j$ be the set of all $j$-dimensional projective subspaces of PG($V$).

**Definition 1.1.4.** Let $V$ be an $(n+1)$-dimensional vector space over $\mathbb{F}_q$. The pair $(\mathrm{PG}(V), (\mathcal{S}_{-1}, \mathcal{S}_0, \ldots, \mathcal{S}_n))$ is called either *$n$-dimensional projective space with underlying vector space $V$* or *$n$-dimensional projective space over $\mathbb{F}$* or *$n$-dimensional Desarguesian projective space.*

We will denote the $n$-dimensional projective space with underlying vector space $V = \mathbb{F}^{n+1}$ by PG($n, \mathbb{F}$). In the case $\mathbb{F} = \mathbb{F}_q$ is a finite field, the projective space will be denoted by PG($n, q$) and it will be also called *$n$-dimensional finite projective space.*

The number of $k$-dimensional projective subspaces contained in PG($n, q$) is given by the following proposition:

**Proposition 1.1.5.** *Let $0 \leq k \leq n$. The number of $k$-dimensional projective subspaces of* PG($n, q$) *equals*

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q = \prod_{i=0}^{k} \frac{q^{n+1-i}-1}{q^{k+1-i}-1}.$$

**Definition 1.1.6.** The points $P_1 = \langle \mathbf{v}_1 \rangle, P_2 = \langle \mathbf{v}_2 \rangle, \ldots, P_r = \langle \mathbf{v}_r \rangle$ of PG($V$) are *linearly independent* (resp. *linearly dependent*) if, and only if, the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ of $V$ are linearly independent (resp. linearly dependent).

**Definition 1.1.7.** An $(n+2)$-tuple

$$\mathcal{R} = (P_1, P_2, \ldots, P_{n+1}, P)$$

of points of PG($V$) such that any $n+1$ of them are linearly independent is a *projective frame* of PG($V$). The points $P_1, P_2, \dots, P_{n+1}$ and $P$ are called the *fundamental points* and the *unit point* of $\mathcal{R}$, respectively.

It is easy to see that if $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n+1}\}$ is a basis of $V$, then

$$\mathcal{R}(\mathcal{B}) = (\langle \mathbf{e}_1 \rangle, \langle \mathbf{e}_2 \rangle, \dots, \langle \mathbf{e}_{n+1} \rangle, \langle \mathbf{e}_1 + \mathbf{e}_2 + \dots + \mathbf{e}_{n+1} \rangle)$$

is a projective frame of PG($V$) called the *associated frame* of PG($V$). For any point

$$P = \langle \sum_{i=1}^{n+1} x_i \mathbf{e}_i \rangle$$

the vector $X = (x_1, \dots, x_{n+1})$, defined up to a non-zero scalar multiple, is the vector of the *projective coordinates* of $P$ with respect to the frame $\mathcal{R}(\mathcal{B})$. This yields there is a bijective map

$$\gamma_{\mathcal{R}} : P = \langle \sum_{i=1}^{n+1} x_i \mathbf{e}_i \rangle \in \mathrm{PG}(V) \longrightarrow \langle (x_1, \dots, x_{n+1}) \rangle \in \mathrm{PG}(n, \mathbb{F}).$$

So, fixing a projective frame in PG($V$), we can identify any point $P$ of PG($V$) with its projective coordinates $(x_1, \dots, x_{n+1})$ and we will write $P = (x_1, \dots, x_{n+1})$.

**Definition 1.1.8.** Let $f : V \longrightarrow W$ be an isomorphism between two $\mathbb{F}$-vector spaces $V$ and $W$. The bijective map

$$\tilde{f} : \langle \mathbf{v} \rangle \in \mathrm{PG}(V) \longrightarrow \langle f(\mathbf{v}) \rangle \in \mathrm{PG}(W)$$

is called *projectivity* between PG($V$) and PG($W$) *induced by* $f$. The set of all projectivity of PG($V$) into itself

$$\mathrm{PGL}(V) = \{g : \mathrm{PG}(V) \longrightarrow \mathrm{PG}(V) : \text{g is a projectivity of } \mathrm{PG}(V)\}$$

is a group with respect to the map composition and it is called *projective linear group* of PG($V$). If $V = \mathbb{F}_q^{n+1}$, this group will be denoted by PGL($n+1, q$).

**Definition 1.1.9.** Let $f : V \longrightarrow W$ be a linear map with $kerf \neq \{\mathbf{0}\}$ and let $\mathcal{K} = \{\langle \mathbf{v} \rangle \in \mathrm{PG}(V) : f(\mathbf{v}) \neq 0\}$. The map

$$f_{\mathcal{K}} : \langle \mathbf{v} \rangle \in \mathrm{PG}(V) \setminus \mathcal{K} \longrightarrow \langle f(\mathbf{v}) \rangle \in \mathrm{PG}(W)$$

is called *degenerate projectivity* between PG($V$) and PG($W$) *induced by* $f$.

**Definition 1.1.10.** Let $V$ and $W$ be two $\mathbb{F}$-vector spaces such that $\dim V = \dim W > 2$. A *collineation* between PG($V$) and PG($W$) is a bijective map $g : \mathrm{PG}(V) \longrightarrow \mathrm{PG}(W)$ sending lines into lines. If $\dim V = \dim W = 2$, a *collineation* is a bijective map $g : \langle \mathbf{v} \rangle \in \mathrm{PG}(V) \longrightarrow \langle f(\mathbf{v}) \rangle \in \mathrm{PG}(W)$ induced by a semilinear isomorphism $f : V \longrightarrow W$. The group of all collineations of PG($V$) into itself will be denoted by P$\Gamma$L($V$). If $V = \mathbb{F}_q^{n+1}$, we will use the following notation P$\Gamma$L($n+1, q$).

**Theorem 1.1.11** (Fundamental Theorem of Projective Geometry)**.** *Let $V$ and $W$ be two $\mathbb{F}$-vector spaces, with* $\dim V = \dim W > 2$. *Any collineation between* PG($V$) *and* PG($W$) *has the form*

$$g : P = \langle \boldsymbol{v} \rangle \in \mathrm{PG}(V) \longrightarrow P' = \langle f(\boldsymbol{v}) \rangle \in \mathrm{PG}(W),$$

*with $f : V \longrightarrow W$ a semilinear isomorphism.*

*Proof.* See [14, Theorem 1.5]. □

**Definition 1.1.12.** Let $f : V \longrightarrow W$ be a semilinear map with $ker f \neq \{\mathbf{0}\}$ and let $\mathcal{K} = \{\langle \mathbf{v} \rangle \in \mathrm{PG}(V) : f(\mathbf{v}) \neq 0\}$. The map

$$f_{\mathcal{K}} : \langle \mathbf{v} \rangle \in \mathrm{PG}(V) \setminus \mathcal{K} \longrightarrow \langle f(\mathbf{v}) \rangle \in \mathrm{PG}(W)$$

is called *degenerate collineation* between PG($V$) and PG($W$) *induced by $f$*.

## 1.2 The dual space of PG($V$)

Let $V$ be an $(n+1)$-dimensional $\mathbb{F}$-vector space and let $V^*$ be its *dual space*, that is the vector space of the linear maps from $V$ to $\mathbb{F}$. If $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}\}$ is a basis of $V$, then $\mathcal{B}^* = \{\mathbf{e}_1^*, \ldots, \mathbf{e}_{n+1}^*\}$ defined by

$$\mathbf{e}_i^*(\mathbf{e}_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

is the *dual basis* of $\mathcal{B}$. For every $\mathbf{v} \in V$ such that $\mathbf{v} = a_1\mathbf{e}_1 + \ldots + a_{n+1}\mathbf{e}_{n+1}$, it is

$$\mathbf{e}_i^*(\mathbf{v}) = a_i.$$

For any subspace $U$ of $V$ the corresponding subspace $U^*$ of $V^*$ is given by

$$U^* = \{f \in V^* : f(\mathbf{u}) = 0, \forall \mathbf{u} \in U\}.$$

So, we have a bijective correspondence between the subspaces of $V$ and the subspaces of $V^*$ satisfying:

- $(U^*)^*$ and $U$ are isomorphic;
- $U \subset W$ if, and only if, $W^* \subset U^*$;
- $\dim U^* = n + 1 - \dim U$.

The projective space PG($V^*$) is the dual space of PG($V$) and it will be also denoted by PG$(V)^*$.

**Definition 1.2.1.** If $\mathcal{R} = (\langle \mathbf{e}_1 \rangle, \langle \mathbf{e}_2 \rangle, \ldots, \langle \mathbf{e}_{n+1} \rangle, \langle \mathbf{e}_1 + \mathbf{e}_2 + \ldots + \mathbf{e}_{n+1} \rangle)$ is a projective frame of PG($V$), then $\mathcal{R}^* = (\langle \mathbf{e}_1^* \rangle, \langle \mathbf{e}_2^* \rangle, \ldots, \langle \mathbf{e}_{n+1}^* \rangle, \langle \mathbf{e}_1^* + \mathbf{e}_2^* + \ldots + \mathbf{e}_{n+1}^* \rangle)$ is the *dual projective frame* of PG$(V)^*$.

We can think at the dual space PG$(V)^*$ as the projective space whose points are the hyperplanes of PG($V$) and whose $k$-dimensional subspaces are the $(n-k-1)$-dimensional spaces of PG($V$).

## 1.3 Sesquilinear forms, correlations and polarities

**Definition 1.3.1.** Let $V$ be an $\mathbb{F}$-vector space. A map

$$\langle\, ,\, \rangle : (\mathbf{v}, \mathbf{v}') \in V \times V \longrightarrow \langle \mathbf{v}, \mathbf{v}' \rangle \in \mathbb{F}$$

is a *sesquilinear form* on $V$ if it is a linear map on the first argument and a $\sigma$-linear map on the second argument, i.e.

$$\langle \mathbf{v} + \mathbf{v}', \mathbf{v}'' \rangle = \langle \mathbf{v}, \mathbf{v}'' \rangle + \langle \mathbf{v}', \mathbf{v}'' \rangle, \;\; \langle a\mathbf{v}, \mathbf{v}' \rangle = a\langle \mathbf{v}, \mathbf{v}' \rangle$$

$$\langle \mathbf{v}, \mathbf{v}' + \mathbf{v}'' \rangle = \langle \mathbf{v}, \mathbf{v}' \rangle + \langle \mathbf{v}, \mathbf{v}'' \rangle, \;\; \langle \mathbf{v}, a\mathbf{v}' \rangle = a^\sigma \langle \mathbf{v}, \mathbf{v}' \rangle$$

for any $\mathbf{v}, \mathbf{v}', \mathbf{v}'' \in V$, $a \in \mathbb{F}$ and $\sigma$ an automorphism of $V$. If $\sigma$ is the identity map, then $\langle\, ,\, \rangle$ is a *bilinear form* on $V$.

The term *sequi* comes from the Latin, and it means one and a half.
Now, let $V$ be an $(n+1)$-dimensional $\mathbb{F}$-vector space. Let $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{n+1}\}$ be a basis of $V$. Let $X = (x_1, x_2, \dots, x_{n+1})$ and $Y = (y_1, y_2, \dots, y_{n+1})$ denote the vectors of coordinates of two elements $\mathbf{x}, \mathbf{y} \in V$ with respect to $\mathcal{B}$. Then we have

$$\langle \mathbf{x}, \mathbf{y} \rangle = X^t A Y^\sigma,$$

where $A = (\langle \mathbf{e}_i, \mathbf{e}_j \rangle)$ is the *associated matrix* to the sesquilinear form with respect to the basis $\mathcal{B}$, the superscript $t$ denotes the matrix transposition and $Y^\sigma = (y_1^\sigma, y_2^\sigma, \dots, y_{n+1}^\sigma)$.

For any subspace $U$ of $V$, we define the *right* and the *left orthogonal subspaces* of $U$ as

$$U^\top = \{\mathbf{y} \in V : \langle \mathbf{x}, \mathbf{y} \rangle = 0, \forall \mathbf{x} \in U\}$$

and

$$U^\perp = \{\mathbf{y} \in V : \langle \mathbf{y}, \mathbf{x} \rangle = 0, \forall \mathbf{x} \in U\},$$

respectively.

**Proposition 1.3.2.** *Let $U, W$ be two subspaces of $V$. We have:*

*i) both $U^\top$ and $U^\perp$ are subspaces of $V$;*

*ii) $U^\perp \cap W^\perp = (U \cap W)^\perp$, $U^\top \cap W^\top = (U \cap W)^\top$;*

*iii) $W \subset U \Longrightarrow U^\top \subset W^\top$, $U^\perp \subset W^\perp$.*

*Proof.* We prove the statement for the right orthogonal subspaces. A similar argument can be applied to the left orthogonal subspaces.

i) Let $\mathbf{y}, \mathbf{z} \in U^\top$ and $a, b \in \mathbb{F}$, then for any $\mathbf{x} \in U$

$$\langle \mathbf{x}, a\mathbf{y} + b\mathbf{z} \rangle = a^\sigma \langle \mathbf{x}, \mathbf{y} \rangle + b^\sigma \langle \mathbf{x}, \mathbf{z} \rangle = 0$$

and so $a\mathbf{y} + b\mathbf{z} \in U^\top$.

ii) If $\mathbf{v} \in U^\top \cap W^\top$ then $\langle \mathbf{x}, \mathbf{v} \rangle = \langle \mathbf{y}, \mathbf{v} \rangle = 0$ for any $\mathbf{x} \in U$ and $\mathbf{y} \in W$. Hence $\langle \mathbf{z}, \mathbf{v} \rangle = 0$ for any $\mathbf{z} \in U + W$ and so $U^\top \cap W^\top \subset (U+W)^\top$. If $\langle \mathbf{x} + \mathbf{y}, \mathbf{v} \rangle = 0$ for any $\mathbf{x} \in U$ and $\mathbf{y} \in W$, then $\langle \mathbf{x}, \mathbf{v} \rangle = 0$ for any $\mathbf{x} \in U$ and $\langle \mathbf{y}, \mathbf{v} \rangle = 0$ for any $\mathbf{y} \in W$, and so $\mathbf{v} \in U^\top \cap W^\top$, that is $(U+W)^\top \subset U^\top \cap W^\top$.

iii) Let $W \subset U$ and $\mathbf{y} \in U^\top$. Hence, for any $\mathbf{x} \in U$ we have $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ and, in particular, this holds for any $\mathbf{x} \in W$, that is $\mathbf{y} \in W^\top$.

□

The subspaces $V^\top$ and $V^\perp$ are called the *right* and the *left radicals* of $\langle\, , \,\rangle$, respectively.

**Proposition 1.3.3.** *Let $V$ be an $(n+1)$-dimensional $\mathbb{F}$-vector space. The right and the left radicals of a $\sigma$-sesquilinear form on $V$ have the same dimension given by*

$$\dim V^\top = \dim V^\perp = n + 1 - \mathrm{rk}(A),$$

*where $A$ is the associated matrix with respect to a basis of $V$.*

*Proof.* Note that the right and the left radicals have the following matrix equations

$$V^\top : A^{\sigma^{-1}} X = 0 \text{ and } V^\perp : A^t X = 0,$$

and $\mathrm{rk}(A^{\sigma^{-1}}) = \mathrm{rk}(A^t) = \mathrm{rk}(A)$. □

**Definition 1.3.4.** A sesquilinear form on $V$ is called *non-degenerate* if $V^\top = V^\perp = \{\mathbf{0}\}$. Otherwise, it is called *degenerate*.

**Proposition 1.3.5** ([4, Lemma 3.1])**.** *Let $V$ be an $(n+1)$-dimensional $\mathbb{F}$-vector space. If $\langle\, , \,\rangle$ is a non-degenerate sesquilinear form on $V$, then*

$$\dim U + \dim U^\perp = \dim V = \dim U + \dim U^\top.$$

*Proof.* We will prove the claim for $U^\top$. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ be a basis of $U$. Define the linear maps $\alpha_i$, $i = 1, \ldots, r$, by

$$\alpha_i(\mathbf{v}) = \langle \mathbf{e}_i, \mathbf{v} \rangle.$$

If $\sum_{i=1}^r \lambda_i \alpha_i = 0$, then $\sum_{i=1}^r \lambda_i \alpha_i(\mathbf{v}) = 0$ for any $\mathbf{v} \in V$. Therefore,

$$0 = \sum_{i=1}^r \lambda_i \alpha_i(\mathbf{v}) = \sum_{i=1}^r \lambda_i \langle \mathbf{e}_i, \mathbf{v} \rangle = \langle \sum_{i=1}^r \lambda_i \mathbf{e}_i, \mathbf{v} \rangle,$$

and so $\sum_{i=1}^r \lambda_i e_i = 0$, since $\langle\, , \,\rangle$ is non-degenerate. Thus, $\lambda_1 = \ldots = \lambda_r = 0$, which implies that $\alpha_1, \ldots, \alpha_r$ are linearly independents. Now, note that

$$U^\top = \bigcap_{i=1}^r ker(\alpha_i).$$

It is easy to see that $\{\alpha_1, \ldots, \alpha_r\}$ is a basis of the dual subspace $(U^\top)^*$, and so

$$\dim U^\top = n + 1 - r.$$

□

**Definition 1.3.6.** Let $\langle\, ,\, \rangle$ be a sesquilinear form on $V$. A vector $\mathbf{u} \in V$ is *isotropic* if $\langle \mathbf{u}, \mathbf{u} \rangle = 0$. A *totally isotropic subspace* $U$ of $V$ has the property

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0$$

for any $\mathbf{u}, \mathbf{v} \in U$. A *maximum totally isotropic subspace* is a totally isotropic subspace which is not contained in a larger totally isotropic subspace.

**Theorem 1.3.7** ([4, Theorem 3.3])**.** *A totally isotropic subspace, with respect to a non-degenerate sesquilinear form on an $(n+1)$-dimensional $\mathbb{F}$-vector space $V$ has dimension at most $\left\lfloor \frac{n+1}{2} \right\rfloor$.*

*Proof.* Let $U$ be a totally isotropic subspace of $V$, then by definition $U \subset U^{\perp}, U^{\top}$. Hence, by Proposition 1.3.5

$$2 \dim U \leq n + 1.$$

□

**Definition 1.3.8.** A sesquilinear form $\langle\, ,\, \rangle$ is *reflexive* if for any $\mathbf{u}, \mathbf{v} \in V$ it holds

$$\langle \mathbf{u}, \mathbf{v} \rangle = 0 \Longleftrightarrow \langle \mathbf{v}, \mathbf{u} \rangle = 0.$$

In this case, for any subspace $U$ of $V$ the subspace $U^{\perp} = U^{\top}$ will be called *orthogonal subspace* of $U$.

**Proposition 1.3.9** ([4, Theorem 3.4])**.** *Let $\langle\, ,\, \rangle$ be a reflexive sesquilinear form. For any subspace $U$ of $V$*

$$U \subset (U^{\perp})^{\perp}.$$

*Moreover, if $\langle\, ,\, \rangle$ is non-degenerate then*

$$U = (U^{\perp})^{\perp}.$$

*Proof.* Let $\mathbf{u} \in U$. For any $\mathbf{v} \in U^{\perp}$, we have $\langle \mathbf{u}, \mathbf{v} \rangle = 0$. Since $\langle\, ,\, \rangle$ is reflexive, $\langle \mathbf{v}, \mathbf{u} \rangle{=}0$ and so $\mathbf{u} \in (U^{\perp})^{\perp}$, that is $U \subset (U^{\perp})^{\perp}$.

Now, suppose $\langle\, ,\, \rangle$ is non-degenerate. Then, by Proposition 1.3.5,

$$\dim U = n + 1 - \dim U^{\perp} = \dim (U^{\perp})^{\perp},$$

and so $U = (U^{\perp})^{\perp}$. □

**Definition 1.3.10.** A bilinear form $\langle\, ,\, \rangle$ on $V$ is called *symmetric* if for any $\mathbf{x}, \mathbf{y} \in V$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle,$$

and *alternating* if for any $\mathbf{x} \in V$

$$\langle \mathbf{x}, \mathbf{x} \rangle = 0.$$

A $\sigma$-sesquilinear form $\langle\, ,\, \rangle$ on $V$ is called *Hermitian* if $\sigma^2 = 1$, $\sigma \neq 1$ and for any $\mathbf{x}, \mathbf{y} \in V$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle^{\sigma}.$$

Let $A$ denote the matrix associated to the form $\langle\, , \rangle$ with respect to a basis $\mathcal{B}$ of an $(n+1)$-dimensional $\mathbb{F}$-vector space $V$. If $\langle\, , \rangle$ is symmetric then $A$ is a symmetric matrix, i.e. $A = A^t$. If $\langle\, , \rangle$ is alternating then for any $\mathbf{x}, \mathbf{y} \in V$ we have

$$0 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{x}\rangle,$$

that is $\langle \mathbf{x}, \mathbf{y}\rangle = -\langle \mathbf{y}, \mathbf{x}\rangle$. This yields that $A$ is a skew-symmetric matrix, i.e. $A = -A^t$. If $\langle\, , \rangle$ is non-degenerate, since $|A| = (-1)^{n+1}|A^t|$, then $n$ must be odd. Note that if the field $\mathbb{F}$ has even characteristic then any alternating form is also symmetric (but the converse is not true). Therefore, in the following we will assume that a symmetric bilinear form is not alternating when $\mathbb{F}$ has even characteristic. Finally, if $\langle\, , \rangle$ is Hermitian then $A$ is a Hermitian matrix, i.e. $A^t = A^\sigma$.

**Theorem 1.3.11** ([4, Theorem 3.6]). *Let $\langle\, , \rangle$ be a non-degenerate, reflexive $\sigma$-sesquilinear form of an $(n+1)$-dimensional $\mathbb{F}$-vector space $V$. Then, up to a scalar factor, the form $\langle\, , \rangle$ is one of the following:*

- *an alternating form.*
- *a symmetric form. Moreover, if $\mathbb{F}$ has even characteristic then there is a non isotropic vector.*
- *a Hermitian form.*

**Definition 1.3.12.** Two $\sigma$-sesquilinear forms $\langle\, , \rangle$ and $(\, , )$ on an $\mathbb{F}$-vector space $V$ are *equivalent* if there is $f \in \Gamma\mathrm{L}(V)$ such that $\langle f(\mathbf{x}), f(\mathbf{y})\rangle = (\mathbf{x}, \mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in V$.
A linear map $f \in \mathrm{GL}(V)$ is called an *isometry* of $\langle\, , \rangle$ if $\langle f(\mathbf{x}), f(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$ for any $\mathbf{x}, \mathbf{y} \in V$.

In [4, Chapter 3], it is shown the following:

**Theorem 1.3.13.**

- *In $\mathbb{F}_q^{n+1}$, $q$ odd, there are, up to equivalence, one or two of non-degenerate, reflexive symmetric bilinear forms according as $n$ is even or odd, respectively.*
- *In $\mathbb{F}_q^{n+1}$, $q$ even, all non-degenerate, reflexive symmetric and not alternating bilinear forms are equivalent.*
- *In $\mathbb{F}_q^{n+1}$, $n$ odd, all non-degenerate, reflexive alternating bilinear forms are equivalent.*
- *In $\mathbb{F}_q^{n+1}$, all non-degenerate, reflexive Hermitian forms are equivalent.*

**Remark 1.3.14.** Let $\langle\, , \rangle$ be a non-degenerate, reflexive $\sigma$-sesquilinear form on $\mathbb{F}_q^{n+1}$. By Theorem 1.3.13, we may assume that:

- if $\langle\, , \rangle$ is symmetric, $q$ odd, there is a suitable basis $\mathcal{B}$ of $\mathbb{F}_q^{n+1}$ such that either

$$\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_2 + \ldots + x_{n-1}y_n + x_{n+1}y_{n+1}$$

if $n$ is even, or

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_2 + \ldots + x_{n-2} y_{n-1} + f(x_n, y_{n+1}),$$

with $f(x_n, y_{n+1})$ an irreducible polynomial of degree two over $\mathbb{F}_q$, or

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_2 + \ldots + x_n y_{n+1}$$

if $n$ is odd.
The group of all isometries of $\langle \, , \, \rangle$ is called *orthogonal group* and it is denoted by $\mathrm{GO}(n+1, q)$, $\mathrm{GO}^-(n+1, q)$ and $\mathrm{GO}^+(n+1, q)$, respectively. The groups of all projectivities of $\mathrm{PG}(n, q)$ induced by elements of $\mathrm{GO}(n+1, q)$, $\mathrm{GO}^-(n+1, q)$, $\mathrm{GO}^+(n+1, q)$ are denoted by $\mathrm{PGO}(n+1, q)$, $\mathrm{PGO}^-(n+1, q)$, $\mathrm{PGO}^+(n+1, q)$, respectively.

- if $\langle \, , \, \rangle$ is symmetric and not alternating, $q$ even, there is a suitable basis $\mathcal{B}$ of $\mathbb{F}_q^{n+1}$ such that
$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \ldots + x_{n+1} y_{n+1}.$$

- if $\langle \, , \, \rangle$ is alternating, $n$ odd, there is a suitable basis $\mathcal{B}$ of $\mathbb{F}_q^{n+1}$ such that
$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_2 - x_2 y_1 + \ldots + x_n y_{n+1} - x_{n+1} y_n.$$
The group of all isometries of $\langle \, , \, \rangle$ is called *symplectic group* and it is denoted by $\mathrm{Sp}(n+1, q)$. The group of all projectivities of $\mathrm{PG}(n, q)$ induced by elements of $\mathrm{Sp}(n+1, q)$ is denoted by $\mathrm{PSp}(n+1, q)$.

- if $\langle \, , \, \rangle$ is Hermitian, $q$ square, there is a suitable basis $\mathcal{B}$ of $\mathbb{F}_q^{n+1}$ such that
$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1^{\sqrt{q}} + \ldots + x_{n+1} y_{n+1}^{\sqrt{q}}.$$
The group of all isometries of $\langle \, , \, \rangle$ is called *unitary group* and it is denoted by $\mathrm{GU}(n+1, q)$. The group of all projectivities of $\mathrm{PG}(n, q)$ induced by elements of $\mathrm{GU}(n+1, q)$ is denoted by $\mathrm{PGU}(n+1, q)$.

Usually symplectic, orthogonal and unitary groups are called *finite classical groups.* Regarding the dimension of a maximum totally isotropic subspace, the following have been proved:

**Theorem 1.3.15** ([4, Theorem 3.8])**.** *A maximum totally isotropic subspace, with respect to a non-degenerate alternating form defined on* $\mathbb{F}_q^{n+1}$*, has dimension* $\frac{n+1}{2}$*.*

**Theorem 1.3.16** ([4, Theorem 3.11])**.** *A maximum totally isotropic subspace, with respect to a non-degenerate hermitian form defined on* $\mathbb{F}_q^{n+1}$*, has dimension* $\lfloor \frac{n+1}{2} \rfloor$*.*

**Definition 1.3.17.** A *(degenerate) correlation* of $\mathrm{PG}(n, \mathbb{F})$ is a (degenerate) collineation between $\mathrm{PG}(n, \mathbb{F})$ and its dual space $\mathrm{PG}(n, \mathbb{F})^*$.

**Remark 1.3.18.** A correlation of $\mathrm{PG}(n, \mathbb{F})$ can be seen as a bijective map of $\mathrm{PG}(n, \mathbb{F})$ that maps $k$-dimensional subspaces into $(n-k-1)$-dimensional subspaces reversing inclusion.

**Theorem 1.3.19.** *Any (possibly degenerate) correlation of* $\mathrm{PG}(n, \mathbb{F})$, $n > 1$, *is induced by a $\sigma$-sesquilinear form of the underlying vector space $\mathbb{F}^{n+1}$. Moreover, every $\sigma$-sesquilinear form of $\mathbb{F}^{n+1}$ induces two (possibly degenerate) correlations of* $\mathrm{PG}(n, \mathbb{F})$*. The two correlations coincide if, and only if, the $\sigma$-sesquilinear form is reflexive.*

*Proof.* A (possibly degenerate) correlation of $\mathrm{PG}(n, \mathbb{F})$, $n > 1$, is induced by a $\sigma$-linear map $f$ of $\mathbb{F}^{n+1}$ into its dual, since it is a (possibly degenerate) collineation. Define a map

$$\langle \, , \, \rangle : \mathbb{F}^{n+1} \times \mathbb{F}^{n+1} \longrightarrow \mathbb{F}$$

in the following way:

$$\langle \mathbf{u}, \mathbf{v} \rangle = f(\mathbf{v})(\mathbf{u}),$$

that is the result of applying the element $f(\mathbf{v})$ of $(\mathbb{F}^{n+1})^*$ to $\mathbf{u}$. It follows that $\langle \, , \, \rangle$ is a $\sigma$-sesquilinear form. Indeed it is easy to see that $\langle \, , \, \rangle$ is linear on the first argument and since

$$\langle \mathbf{u}, \mathbf{v}_1 + \mathbf{v}_2 \rangle = f(\mathbf{v}_1 + \mathbf{v}_2)(\mathbf{u}) = f(\mathbf{v}_1)(\mathbf{u}) + f(\mathbf{v}_2)(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v}_1 \rangle + \langle \mathbf{u}, \mathbf{v}_2 \rangle$$

and

$$\langle \mathbf{u}, a\mathbf{v} \rangle = f(a\mathbf{v})(\mathbf{u}) = a^\sigma f(\mathbf{v})(\mathbf{u}) = a^\sigma \langle \mathbf{u}, \mathbf{v} \rangle,$$

it is semilinear on the second argument. Moreover $\langle \, , \, \rangle$ is non-degenerate if, and only if, $f$ is a bijection.
Conversely, any $\sigma$-sesquilinear form on $V = \mathbb{F}^{n+1}$ induces two (possibly degenerate) correlations of $\mathrm{PG}(n, \mathbb{F})$ given by

$$\perp : P = \langle \mathbf{u} \rangle \in \mathrm{PG}(n, \mathbb{F}) \setminus V^\perp \longrightarrow P^\perp = \{ \langle \mathbf{v} \rangle : \langle \mathbf{u}, \mathbf{v} \rangle = 0 \} \in \mathrm{PG}(n, \mathbb{F})^* \tag{1.1}$$

and

$$\top : P = \langle \mathbf{v} \rangle \in \mathrm{PG}(n, \mathbb{F}) \setminus V^\top \longrightarrow P^\top = \{ \langle \mathbf{u} \rangle : \langle \mathbf{u}, \mathbf{v} \rangle = 0 \} \in \mathrm{PG}(n, \mathbb{F})^*. \tag{1.2}$$

□

**Definition 1.3.20.** Let $\langle \, , \, \rangle$ be a sesquilinear form of $\mathbb{F}^{n+1}$. A point $P = \langle \mathbf{v} \rangle$ of $\mathrm{PG}(n, \mathbb{F})$ is called *absolute* if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$, i.e. if the vector $\mathbf{v}$ is isotropic.

Note that a correlation of $\mathrm{PG}(n, \mathbb{F})$ applied twice gives a collineation of $\mathrm{PG}(n, \mathbb{F})$.

**Definition 1.3.21.** A *(degenerate) polarity* is a (degenerate) correlation whose square is the identity.

Hence, if $\rho$ is a (possibly degenerate) polarity, then for every pair of points $P$ and $R$ the following holds:

$$P \in R^\rho \Longleftrightarrow R \in P^\rho,$$

and in this case the points $P$ and $R$ are said *conjugate points*. A subspace $U$ is called *totally isotropic* with respect to $\rho$ if $U \subset U^\rho$. In this case, a totally isotropic subspace $U$ of $\mathrm{PG}(n,\mathbb{F})$ has dimension at most $\lfloor \frac{n-1}{2} \rfloor$.

**Definition 1.3.22.** Let $\rho$ be a (degenerate) reflexive polarity of $\mathrm{PG}(n,q)$, the set of totally isotropic subspaces with respect to $\rho$ is called a *(degenerate) finite classical polar space.*

**Proposition 1.3.23.** *A (degenerate) correlation is a (degenerate) polarity if, and only if, the induced sesquilinear form is reflexive.*

*Proof.* Let $\langle\, , \rangle$ be a reflexive $\sigma$-sesquilinear form of $\mathbb{F}^{n+1}$. If $\mathbf{u} \in \langle \mathbf{v} \rangle^\perp$, then $\mathbf{v} \in \langle \mathbf{u} \rangle^\perp$. Hence the map $\langle \mathbf{u} \rangle \mapsto \langle \mathbf{u} \rangle^\perp$ defines a (possibly degenerate) polarity. Conversely, given a (possibly degenerate) polarity $\rho$, then $\langle \mathbf{u} \rangle \in \langle \mathbf{v} \rangle^\rho$ if, and only if, $\langle \mathbf{v} \rangle \in \langle \mathbf{u} \rangle^\rho$. So, the induced sesquilinear form is reflexive. □

## 1.4 Quadrics, Hermitian varieties, Symplectic geometries of $\mathrm{PG}(d,q)$

By Proposition 1.3.23, we have seen that polarities of $\mathrm{PG}(n,\mathbb{F})$ are in one to one correspondence with non-degenerate, reflexive $\sigma$-sesquilinear forms on $\mathbb{F}^{n+1}$. Hence, to every polarity of $\mathrm{PG}(n,\mathbb{F})$ there is an associated pair $(A,\sigma)$, with $A$ a non-singular matrix of order $n+1$ and $\sigma$ an automorphism of $\mathbb{F}$. From Theorem 1.3.11, we have the following:

**Corollary 1.4.1.** *Let $(A,\sigma)$ be a polarity of $\mathrm{PG}(n,q)$, one of the following holds:*

- *$\sigma = 1$, $A$ is a symmetric matrix, $q$ is odd. The polarity is called an* orthogonal polarity.
- *$\sigma = 1$, $A$ is a symmetric matrix, $q$ is even. There is a non absolute point and the polarity is called a* pseudo polarity.
- *$\sigma = 1$, $A$ is a skew-symmetric matrix, $n$ is odd. Every point is an absolute point and the polarity is called a* symplectic polarity.
- *$\sigma^2 = 1$, so $\sigma : x \mapsto x^{\sqrt{q}}$, $q$ is a square, $A$ is a Hermitian matrix. The polarity is called a* Hermitian *or* unitary polarity.

**Remark 1.4.2.** Let $\langle\, , \rangle$ be a $\sigma$-sesquilinear form on an $(n+1)$-dimensional $\mathbb{F}_q$-vector space $V$. The restriction of $\langle\, , \rangle$ on a subspace $U$ of $V$ is a $\sigma$-sesquilinear form that is symmetric, symmetric and not alternating, alternating or Hermitian according as $\langle\, , \rangle$ is symmetric, symmetric and not alternating, alternating or Hermitian. This yields that the restriction of a polarity $\rho$ of $\mathrm{PG}(n,q)$ on a projective subspace of $\mathrm{PG}(n,q)$ is a polarity that is orthogonal, pseudo-polarity, symplectic or Hermitian, according as $\rho$ is orthogonal, pseudo-polarity, symplectic or Hermitian.

Each of the above polarities of PG$(n, q)$ determines a set $\Gamma : X^t A X^\sigma = 0$, as set of absolute points. The three type of polarities give rise to the following, well known, subsets of PG$(n, q)$:

**Definition 1.4.3.** If $(A, \sigma)$ is a polarity of PG$(d, q)$, then one of the following holds:

- $\Gamma$ is called a *quadric* of PG$(n, q)$ ($q$ odd, orthogonal polarity).
- $\Gamma$ is a hyperplane of PG$(n, q)$ ($q$ even, pseudo polarity).
- $\Gamma$ is the full pointset of PG$(n, q)$ ($n$ odd, symplectic polarity) and the geometry determined is a *symplectic polar space*.
- $\Gamma$ is a *Hermitian variety* of PG$(n, q)$ ($q$ a square, $\sigma^2 = 1$, $\sigma \neq 1$, unitary polarity).

In all the cases of the previous definition, the set is often called also *non-degenerate* since the associated $\sigma$-sesquilinear form is non-degenerate. All the above sets have been classified and for each of them it is possible to give a canonical equation. By Remark 1.3.14, we have:

**Theorem 1.4.4.** *If $(A, \sigma)$ is a polarity of* PG$(n, q)$*, then let $\Gamma : X^t A X^\sigma = 0$ be the set of its absolute points. The following holds:*

*1) If $\Gamma$ is a quadric, so $q$ is odd, then we have:*

*i) If $n$ is even, then $\Gamma = Q(n, q)$ is a* parabolic quadric *and has the following canonical equation*

$$x_1x_2 + \ldots + x_{n-1}x_n + x_{n+1}^2 = 0.$$

*ii) If $n$ is odd, then either $\Gamma = Q^-(n, q)$ is an* elliptic quadric *and has the following canonical equation*

$$x_1x_2 + \ldots + x_{n-2}x_{n-1} + f(x_n, x_{n+1}) = 0,$$

*with $f(x_n, x_{n+1})$ irreducible polynomial of degree two over $\mathbb{F}_q$, or $\Gamma = Q^+(n, q)$ is a* hyperbolic quadric *and has the following canonical equation*

$$x_1x_2 + \ldots + x_nx_{n+1} = 0.$$

*2) If $\Gamma$ is a hyperplane, so $q$ is even, then $\Gamma : x_1 + \ldots + x_{n+1} = 0$.*

*3) If $\Gamma = W(n, q)$ is a symplectic polar space, so $n$ is odd, then the associated bilinear form $\langle\, ,\, \rangle$ has the following canonical equation*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1y_2 - x_2y_1 + \ldots + x_ny_{n+1} - x_{n+1}y_n.$$

*4) If $\Gamma = H(n,q)$ is a Hermitian variety, then $q$ is a square and $H(n,q)$ has the following canonical equation*

$$x_1^{\sqrt{q}} + x_2^{\sqrt{q}} + \ldots + x_{n+1}^{\sqrt{q}} = 0.$$

**Remark 1.4.5.** Let $\langle\, , \rangle$ be a degenerate, reflexive $\sigma$-sesquilinear form on $V = \mathbb{F}_q^{n+1}$ and let $A$ be the associated matrix with respect to a basis $\mathcal{B}$. Let $\Gamma$ be the set of absolute points with respect to the associated degenerate polarity $\rho$. Then $\dim V^{\perp} = r+1$, $r \geq 0$, and let $\mathcal{V}_r$ be the associated projective subspace of $\mathrm{PG}(n,q)$. Now, let $\mathcal{S}_{n-1-r}$ be a skew subspace with $\mathcal{V}_r$, i.e. $\mathcal{S}_{n-1-r} \cap \mathcal{V}_r = \emptyset$, and let $S$ denote the underlying vector subspace of $\mathcal{S}_{n-1-r}$. Since $\mathrm{rk}(A) = n-r$, then the restriction of $\langle\, , \rangle$ on $S$ is non-degenerate and induces a non-degenerate polarity $\rho'$ on $\mathcal{S}_{n-1-r}$. Hence, the set of absolute points $\Gamma'$ of $\mathcal{S}_{n-1-r}$ with respect to $\rho'$ is either a quadric ($q$ odd) or a hyperplane ($q$ even) or a Hermitian variety ($q$ square) or a symplectic polar space ($n$ and $r$ are both either odd or even). Then, the set $\Gamma$ is the cone with vertex $\mathcal{V}_r$ and base $\Gamma'$, i.e. it is the union of the lines joining a point of $\mathcal{V}_r$ and a point of $\Gamma'$, and it will be called either *degenerate quadric* or *degenerate Hermitian variety* or *degenerate symplectic polar space* accordingly to $\Gamma'$ is either a quadric or a Hermitian variety or a symplectic polar space.

## 1.5 Quadratic forms

We see, from Section 1.4, that we have missed quadrics in the $q$ even case. In order to include quadrics for the $q$ even case, we give the following definition:

**Definition 1.5.1.** Let $V$ be an $\mathbb{F}$-vector space. A *quadratic form on $V$* is any map

$$f : V \longrightarrow \mathbb{F}$$

such that

- $f(\lambda \mathbf{v}) = \lambda^2 f(\mathbf{v})$ for any $\mathbf{v} \in V$ and $\lambda \in \mathbb{F}$;
- $\langle \mathbf{u}, \mathbf{v} \rangle = f(\mathbf{u}+\mathbf{v}) - f(\mathbf{u}) - f(\mathbf{v})$ is a bilinear form on $V$ and it is called the *polarisation* of the quadratic form $f$.

The quadratic form $f$ is *non-degenerate* if $\langle\, , \rangle$ has the property that $\langle \mathbf{u}, \mathbf{v} \rangle = f(\mathbf{u}) = 0$ for all $\mathbf{v} \in V$ implies $\mathbf{u} = \mathbf{0}$, otherwise it is *degenerate*. A vector $\mathbf{u}$ is *singular* if $f(\mathbf{u}) = 0$. A subspace $U$ is *totally singular* if $\mathbf{u}$ is singular for all $\mathbf{u} \in U$. A *maximum totally singular subspace* is a totally singular subspace which is not contained in a larger totally singular subspace. A subspace $U$ is called *non-singular* if for any non-zero vector $\mathbf{u}$ of $U$ one has $f(\mathbf{u}) \neq 0$.

Clearly, the bilinear form $\langle\, , \rangle$ given in Definition 1.5.1 is symmetric. More precisely, if $\mathbb{F}$ has odd characteristic the following characterization holds:

**Proposition 1.5.2.** *Let $V$ be a vector space over a field $\mathbb{F}$ with odd characteristic. Then, any symmetric bilinear form $\langle\, , \rangle$ on $V$ induces a quadratic form on $V$.*

*Proof.* Let $f : V \longrightarrow \mathbb{F}$ be a map defined by

$$f(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v} \rangle$$

for any $\mathbf{v} \in V$. Clearly, $f$ is a quadratic form. □

Let $f$ be a quadratic form on $V$ and $\langle \, , \, \rangle$ the polarization of $f$. Regarding the subspaces of $V$, the following hold:

**Proposition 1.5.3.** *If $U$ is a totally singular subspace then $U \subset U^{\perp}$.*

*Proof.* For any $\mathbf{u}, \mathbf{v} \in U$ we get $\mathbf{u} + \mathbf{v} \in U$ and so

$$\langle \mathbf{u}, \mathbf{v} \rangle = f(\mathbf{u} + \mathbf{v}) - f(\mathbf{u}) - f(\mathbf{v}) = 0.$$

□

**Proposition 1.5.4.** *Let $U$ be a subspace of $V$. If $\mathbb{F}$ has odd characteristic and $U \subset U^{\perp}$ then $U$ is a totally singular subspace.*

*Proof.* For any $\mathbf{u} \in U$, we have

$$0 = \langle \mathbf{u}, \mathbf{u} \rangle = 2f(\mathbf{u}).$$

Hence, if $\mathbb{F}$ has odd characteristic then $f(\mathbf{u}) = 0$. □

**Definition 1.5.5.** A *(degenerate) quadric* of $\mathrm{PG}(V)$ is any set

$$\mathcal{Q} = \{\langle \mathbf{v} \rangle \in \mathrm{PG}(V) : f(\mathbf{v}) = 0\},$$

for some (degenerate) quadratic form $f$ on $V$. A point $P \in \mathcal{Q}$ will be called *singular*.

**Remark 1.5.6.** Let $V$ be an $(n+1)$-dimensional vector space over $\mathbb{F}$.

- If $\mathbb{F}$ has odd characteristic, then there is a bijective correspondence between (possibly degenerate) quadratic form on $V$ and (possibly degenerate) symmetric bilinear form on $V$. Hence, the definitions of quadric given in Definition 1.4.3 and Definition 1.5.5 are the same, and a subspace $U$ of $V$ is totally singular with respect to a quadratic form $f$ if, and only if, $U$ is totally isotropic with respect to the polarisation of $f$.
- If $\mathbb{F}$ has even characteristic, then the polarisation of the quadratic form $f$ on $V$ is alternating and is non-degenerate unless $n$ is odd.

Regarding the dimension of a maximum totally singular subspace, the following has been proved:

**Theorem 1.5.7** ([4, Theorem 3.28])**.** *Let $f$ be a non-singular quadratic form on an $(n+1)$-dimensional $\mathbb{F}$-vector space $V$. Then*

$$n \in \{2r-1, 2r, 2r+1\},$$

*where $r$ is the dimension of a maximum totally singular subspace.*

## 1.6 Properties of polar spaces

By Theorems 1.5.7, 1.3.15, 1.3.16, any two maximum totally isotropic subspaces of a polar space $\Gamma$ have the same dimension and are called *generators* of $\Gamma$. This common vector space dimension is called the *rank* of $\Gamma$. In Table 1.1, polar spaces of rank $r$ and parameter $e$ are listed. The meaning of the parameter $e$ is explained in Proposition 1.6.2.

Table 1.1: $\Gamma_{r,e}$ polar space of rank $r$ and parameter $e$

| $\Gamma_{r,e}$ | $Q^+(2r-1,q)$ | $H(2r-1,q)$ | $W(2r-1,q)$ | $Q(2r,q)$ | $H(2r,q)$ | $Q^-(2r+1,q)$ |
|---|---|---|---|---|---|---|
| $e$ | 0 | 1/2 | 1 | 1 | 3/2 | 2 |

The term $k$-*space* of $\Gamma_{r,e}$ will be used to denote a totally isotropic $k$-dimensional projective space of $\Gamma_{r,e}$.

**Lemma 1.6.1.** *Let $\Gamma_{r,e}$ be a finite classical polar space in* $\mathrm{PG}(n,q)$.

- *For any point $P$ of $\Gamma_{r,e}$, the set $P^\perp \cap \Gamma_{r,e}$ is a cone with base $\Gamma_{r-1,e}$ and vertex $P$, with $\Gamma_{r-1,e}$ a finite classical polar space of rank $r-1$ of the same type as $\Gamma_{r,e}$.*
- *Let $\ell = \langle P, Q\rangle$ where $P, Q$ are two distinct points of $\Gamma_{r,e}$. Then $\ell^\perp$ is an $(n-2)$-space of $\mathrm{PG}(n,q)$ intersecting $\Gamma_{r,e}$ in a polar space $\Gamma_{r-1,e}$ if $P \notin Q^\perp$, or in a cone having as vertex the line $\ell$ and as base a polar space $\Gamma_{r-2,e}$, if $P \in Q^\perp$.*
- *Let $\Pi_{k-1}$ be a $(k-1)$-space of $\Gamma_{r,e}$. Then $\Pi_{k-1}^\perp$ is an $(n-k)$-space of $\mathrm{PG}(n,q)$ intersecting $\Gamma_{r,e}$ in a cone having as vertex $\Pi_{k-1}$ and as base a polar space $\Gamma_{r-k,e}$, where $\Gamma_{r-k,e}$ is of the same type of $\Gamma_{r,e}$.*

*Proof.* It follows from Remark 1.4.5. □

**Proposition 1.6.2** ([4, Theorem 4.10])**.** *The number of points of a finite polar space $\Gamma_{r,e}$ is*

$$\frac{(q^r-1)(q^{r+e-1}+1)}{q-1},$$

*of which $q^{2(r-1)+e}$ are not conjugate with a given point of $\Gamma_{r,e}$. In particular, any finite polar space of rank one has $q^e+1$ points.*

**Proposition 1.6.3.** *The number of $(k-1)$-spaces of $\Gamma_{r,e}$ is given by*

$$\begin{bmatrix} r \\ k \end{bmatrix}_q \prod_{i=1}^{k}(q^{r+e-i}+1).$$

*Proof.* By induction on $k$. The case $k=1$ holds true by Proposition 1.6.2. Suppose that the formula is valid up to $k$. Count in two ways the pairs $(\Pi, \Pi')$, where $\Pi$ is a $(k-1)$-space of $\Gamma_{r,e}$, $\Pi'$ is a $k$-space of $\Gamma_{r,e}$ and $\Pi \subset \Pi'$. For a fixed $\Pi'$, we have $\left[\begin{smallmatrix} k+1 \\ k \end{smallmatrix}\right]_q$

possibilities for $\Pi$. On the other hand, $\Pi \subset \Pi' \subset \Pi'^{\perp}$, hence $\Pi \subset \Pi'^{\perp}$ and $\Pi' \subset \Pi^{\perp}$. By Lemma 1.6.1, the number of $k$-spaces through a fixed $(k-1)$-space $\Pi$ equals the number of point of $\Gamma_{r-k,e}$. Therefore,

$$\#k\text{-spaces of } \Gamma_{r,e} \cdot \begin{bmatrix} k+1 \\ k \end{bmatrix}_q = \#(k-1)\text{-spaces of } \Gamma_{r,e} \cdot \#\text{points of } \Gamma_{r-k,e}.$$

By the induction hypothesis we have

$$\begin{aligned} \#k\text{-spaces of } \Gamma_{r,e} &= \begin{bmatrix} r \\ k \end{bmatrix}_q \prod_{i=1}^{k}(q^{r+e-i}+1)\frac{(q^{r-k}-1)(q^{r-k+e-1}+1)}{q-1}\frac{q-1}{q^{k+1}-1} \\ &= \begin{bmatrix} r \\ k \end{bmatrix}_q \frac{q^{r-k}-1}{q^{k+1}-1}\prod_{i=1}^{k+1}(q^{r+e-i}+1) \\ &= \begin{bmatrix} r+1 \\ k \end{bmatrix}_q \prod_{i=1}^{k+1}(q^{r+e-i}+1). \end{aligned}$$

□

Let $\mathcal{P}(n,q)$ be a polar space of $\mathrm{PG}(n,q)$. In Table 1.2 the possible intersections of a $k$-space $\Sigma$ of $\mathrm{PG}(n,q)$ and the $(n-k-1)$-space $\Sigma^{\perp}$ with a polar space $\mathcal{P}(n,q)$ are described. Let $\Pi_{k-t-1}\mathcal{P}(t,q)$ denote a cone of $\Sigma$ having as vertex a $(k-t-1)$-dimensional subspace $\Pi_{k-t-1}$ and as base a polar space $\mathcal{P}(t,q)$ contained in a $t$-dimensional subspace skew with $\Pi_{k-t-1}$.

A $k$-space $\Sigma$ of $\mathrm{PG}(n,q)$ will be also called *hyperbolic*, *elliptic*, *parabolic*, *symplectic* or *hermitian* $k$-space of $\mathrm{PG}(n,q)$ accordingly to $\Sigma$ meets the polar space $\Gamma$ in $Q^+(k,q)$, $Q^-(k,q)$, $Q(k,q)$, $W(k,q)$ or $H(k,q)$.

## 1.7 $m$-ovoids of polar spaces

$m$-Ovoids in finite classical polar spaces have been defined in e.g. [78].

**Definition 1.7.1.** Let $\Gamma_{r,e}$ be a finite classical polar space. A set of points $\mathcal{O}$ of $\Gamma_{r,e}$ is an *$m$-ovoid* if each generator of $\Gamma_{r,e}$ meets $\mathcal{O}$ in $m$ points.

The following lemma is well known and easily proved, see e.g. [79].

**Proposition 1.7.2.** Suppose that $\mathcal{O}$ is an $m$-ovoid, then $|\mathcal{O}| = m(q^{r+e-1}+1)$.

**Lemma 1.7.3** ([5, Lemma 1]). *Let $\mathcal{O}$ be an $m$-ovoid of $\Gamma_{r,e}$, then for any point $P \in \Gamma_{r,e}$*

$$|P^{\perp} \cap \mathcal{O}| = \begin{cases} (m-1)(q^{r+e-2}+1)+1 & \text{if } P \in \mathcal{O}, \\ m(q^{r+e-2}+1) & \text{if } P \notin \mathcal{O}. \end{cases}$$

Table 1.2: Intersections of a $k$-space $\Sigma$ of $\mathrm{PG}(n,q)$ and the $(n-k-1)$-space $\Sigma^\perp$ with a polar space $\mathcal{P}(n,q)$ of $\mathrm{PG}(n,q)$.

| | | |
|---|---|---|
| $n$ odd | $\Sigma \cap W(n,q)$ | $\Sigma^\perp \cap W(n,q)$ |
| $t$ odd | $\Pi_{k-t-1}W(t,q)$ | $\Pi_{k-t-1}W(n-2k+t-1,q)$ |
| $n$ odd | $\Sigma \cap Q^+(n,q)$ | $\Sigma^\perp \cap Q^+(n,q)$ |
| $t$ odd | $\Pi_{k-t-1}Q^+(t,q)$ | $\Pi_{k-t-1}Q^+(n-2k+t-1,q)$ |
| $t$ odd | $\Pi_{k-t-1}Q^-(t,q)$ | $\Pi_{k-t-1}Q^-(n-2k+t-1,q)$ |
| $t$ even | $\Pi_{k-t-1}Q(t,q)$ | $\Pi_{k-t-1}Q(n-2k+t-1,q)$ |
| $n$ odd | $\Sigma \cap Q^-(n,q)$ | $\Sigma^\perp \cap Q^-(n,q)$ |
| $t$ odd | $\Pi_{k-t-1}Q^+(t,q)$ | $\Pi_{k-t-1}Q^-(n-2k+t-1,q)$ |
| $t$ odd | $\Pi_{k-t-1}Q^-(t,q)$ | $\Pi_{k-t-1}Q^+(n-2k+t-1,q)$ |
| $t$ even | $\Pi_{k-t-1}Q(t,q)$ | $\Pi_{k-t-1}Q(n-2k+t-1,q)$ |
| $n$ even | $\Sigma \cap Q(n,q)$ | $\Sigma^\perp \cap Q(n,q)$ |
| $t$ odd | $\Pi_{k-t-1}Q^+(t,q)$ | $\Pi_{k-t-1}Q(n-2k+t-1,q)$ |
| $t$ odd | $\Pi_{k-t-1}Q^-(t,q)$ | $\Pi_{k-t-1}Q(n-2k+t-1,q)$ |
| $t$ even | $\Pi_{k-t-1}Q(t,q)$ | $\begin{cases}\Pi_{k-t-1}Q^+(n-2k+t-1,q)\\ \Pi_{k-t-1}Q^-(n-2k+t-1,q)\end{cases}$ |
| | $\Sigma \cap H(n,q^2)$ | $\Sigma^\perp \cap H(n,q^2)$ |
| | $\Pi_{k-t-1}H(t,q^2)$ | $\Pi_{k-t-1}H(n-2k+t-1,q^2)$ |

An $m$-ovoid of $\Gamma_{r,e}$ is called *trivial* if it is the empty set or consists of all points of $\Gamma_{r,e}$, and hence if $m=0$ or $m=\frac{q^r-1}{q-1}$, respectively. A 1-ovoid will be simply called *ovoid*. Let $\mathcal{O}$ and $\mathcal{O}'$ be an $m$-ovoid and an $m'$-ovoid of $\Gamma_{r,e}$. If $\mathcal{O}$ and $\mathcal{O}'$ are disjoint, then $\mathcal{O}\cup\mathcal{O}'$ is an $(m+m')$-ovoid of $\mathcal{P}_{r,e}$. Furthermore the complement of an $m$-ovoid is a $\left(\frac{q^r-1}{q-1}-m\right)$-ovoid.

$m$-Ovoids are rare objects, see e.g. [19, 43] for an overview on known examples and some non-existence results for ovoids. Bounds on $m$ for the existence of $m$-ovoids of three classes of finite classical polar spaces can be found in [5], with some recent improvements in [20].

Also in [78], $l$-systems of finite classical polar are introduced. An $l$-system is a generalization of ovoids and spreads, in fact unifying the two concepts in one definition.

**Definition 1.7.4.** Let $\Gamma_{r,e}$ be a finite polar space of rank $r\geq 2$. A *$l$-system* of $\Gamma_{r,e}$, with $0\leq l\leq r-1$, is any set $\mathcal{S}=\{\pi_1,\ldots,\pi_k\}$ of $k=q^{r+e-1}+1$ $l$-spaces of $\Gamma_{r,e}$ such that no generator of $\Gamma_{r,e}$ containing $\pi_i$ has a point in common with $\cup_{j\neq i}\pi_j$, $i=1,\ldots,k$.

**Definition 1.7.5.** Let $\Gamma_{r,e}$ be a finite polar space of rank $r\geq 2$. A spread $\mathcal{S}$ of $\Gamma_{r,e}$ is a set of generators, which constitutes a partition of the pointset.

By Definition 1.7.4, we have that a 0-system is an ovoid and a $(r-1)$-system is a spread. The following theorem shows that a putative $l$-system will yield an $m$-ovoid.

**Theorem 1.7.6** ([78, 6.1])**.** *Let $\mathcal{S}$ be an $l$-system of the polar space $\Gamma_{r,e}$, and $\mathcal{O}$ the set of points given by the union of the $l$-dimensional subspaces in $\mathcal{S}$. Then $\mathcal{O}$ is an $m$-ovoid of $\Gamma_{r,e}$ with $m = \frac{q^{l+1}-1}{q-1}$.*

## 1.8 Spreads of PG$(3, q)$ and ovoids of $Q^+(5, q)$

Now, we briefly describe the relation between ovoids of the hyperbolic quadric $Q^+(5, q)$, also called the *Klein quadric*, and line spreads of PG$(3, q)$.
Let $u = (u_1, u_2, u_3, u_4), v = (v_1, v_2, v_3, v_4)$ be two linearly independent vectors of $V = \mathbb{F}_q^4$. The *Plücker coordinates* of $(u, v)$ are $p_{ij} = u_i v_j - u_j v_i$ for every $i \neq j$. Let $\tau$ be the map from pairs of linearly independent vectors of $V$ to points of PG$(5, q)$ defined by

$$\tau(u, v) = \langle (p_{12}, p_{42}, p_{14}, p_{23}, p_{13}, p_{34}) \rangle.$$

The map $\tau$ is well defined from the lines of PG$(3, q)$ into the points of the quadric $Q^+(5, q) : X_0X_5 + X_1X_4 + X_2X_3 = 0$ and it is called the *Klein correspondence*. It follows that there is a bijective correspondence between the line spreads of PG$(3, q)$ and the ovoids of $Q^+(5, q)$; see [4, Section 4.6], [11, Section 4.8]. Hence, $Q^+(5, q)$ has ovoids for all values of $q$. We can always assume that an ovoid of $Q^+(5, q)$ contains the points $(1, 0, 0, 0, 0, 0)$ and $(0, 0, 0, 0, 0, 1)$ and hence it can be written in the following form:

$$\mathcal{O}(f_1, f_2) = \{(1, x, y, f_1(x, y), f_2(x, y), -yf_1(x, y) - xf_2(x, y))\}_{x,y \in \mathbb{F}_q} \cup \{(0, 0, 0, 0, 0, 1)\}$$

for some functions $f_i : \mathbb{F}_q^2 \longrightarrow \mathbb{F}_q$ with $f_i(0, 0) = 0$, $i \in \{1, 2\}$; see [85, Chapter 1].
The set $\mathcal{O}(f_1, f_2)$ is an ovoid if and only if

$$\begin{aligned} &\langle (1, x_1, y_1, f_1(x_1, y_1), f_2(x_1, y_1), -x_1 f_2(x_1, y_1) - y_1 f_1(x_1, y_1)), \\ &(1, x_2, y_2, f_1(x_2, y_2), f_2(x_2, y_2), -x_2 f_2(x_2, y_2) - y_2 f_1(x_2, y_2)) \rangle \neq 0, \end{aligned} \tag{1.3}$$

for every $(x_1, y_1) \neq (x_2, y_2)$ in $\mathbb{F}_q^2$, where $\langle \, , \, \rangle$ is the symmetric for $q$ odd (or alternating for $q$ even) form associated to the quadratic form of the quadric $Q^+(5, q)$. Note that

$$\langle (1, x_1, y_1, f_1(x_1, y_1), f_2(x_1, y_1), -x_1 f_2(x_1, y_1) - y_1 f_1(x_1, y_1), (0, 0, 0, 0, 0, 1) \rangle = 1.$$

Using the Klein correspondence, we can translate an ovoid $\mathcal{O}(f_1, f_2)$ of $Q^+(5, q)$ to the corresponding line spread of PG$(3, q)$. This results in the set $\mathcal{S}(f_1, f_2) = \{\ell_{x,y} : (x, y) \in \mathbb{F}_q^2\} \cup \{\ell_\infty\}$, where $\ell_\infty : X_0 = X_1 = 0$ and

$$\ell_{x,y} : \begin{cases} X_2 = -f_1(x, y)X_0 + f_2(x, y)X_1 \\ X_3 = xX_0 + yX_1. \end{cases}$$

Now, let $\mathcal{K}$ be a quadratic cone of PG$(3, q)$ with vertex the point $(0, 0, 0, 1)$ and base the conic with equations $X_0X_1 - X_2^2 = X_3 = 0$. A *flock* $\mathcal{F}$ of $\mathcal{K}$ is a partition of the points of the cone, different from the vertex, into $q$ conics. Without loss of generality, we can suppose that the planes containing the conics of the flock $\mathcal{F}$ are given by

$$\pi_t : tX_0 - f(t)X_1 + g(t)X_2 + X_3 = 0, \;\; t \in \mathbb{F}_q,$$

for some functions $f, g : \mathbb{F}_q \longrightarrow \mathbb{F}_q$ with $f(0) = g(0) = 0$. We denote this flock by $\mathcal{F}(f, g)$. In particular, the set $\mathcal{S}(\mathcal{F}) = \{r_{x,y} : (x, y) \in \mathbb{F}_q^2\} \cup \{\ell_\infty\}$, where

$$r_{x,y} : \begin{cases} X_2 = -(y + g(x))X_0 - f(x)X_1 \\ X_3 = xX_0 + yX_1, \end{cases}$$

is a line spread of PG$(3, q)$; see [37] and also [44, Theorem 54.2]. We will say that the ovoid of $Q^+(5, q)$ corresponding to the line spread $\mathcal{S}(\mathcal{F})$ is *associated to the flock* $\mathcal{F}(f, g)$.

# Chapter 2

# Rank distance codes

## 2.1 Preliminary results and notions

Let $\mathbb{F}_q$ be the finite field of $q$ elements, $q$ a prime power, and let $\mathbb{F}_q^{m\times n}$ be the set of $m \times n$ matrices with entries over $\mathbb{F}_q$ endowed with *rank distance*

$$d : (A, B) \in \mathbb{F}_q^{m\times n} \times \mathbb{F}_q^{m\times n} \mapsto \mathrm{rk}(A - B) \in \{0, 1, \dots, \min\{m, n\}\}.$$

A subset of $\mathbb{F}_q^{m\times n}$, including at least two elements is called a *rank distance code*. The *minimum distance* $d(\mathcal{C})$ of a code $\mathcal{C}$ is naturally defined by

$$d(\mathcal{C}) = \min\{d(A, B) : A, B \in \mathcal{C}, A \neq B\}.$$

If $d = d(\mathcal{C})$, we will say that $\mathcal{C}$ is an $(m, n, q; d)$-rank distance code. An $(m, n, q; d)$-rank distance code is *additive* if it is an additive subgroup of $\mathbb{F}_q^{m\times n}$. An additive code is $\mathbb{F}_q$-*linear* if it is a subspace of $\mathbb{F}_q^{m\times n}$ seen as a vector space over $\mathbb{F}_q$. A code that it is neither linear nor additive is called a *non-linear* code.

It is well-known that the size of an $(m, n, q; d)$-rank distance code $\mathcal{C}$ satisfies the *Singleton-like bound*, see e.g. [21, Theorem 5.4]:

$$|\mathcal{C}| \leq q^{\max\{m,n\}(\min\{m,n\}-d+1)}.$$

When this bound is achieved, $\mathcal{C}$ is called an $(m, n, q; d)$-*maximum rank distance* code, or shortly $(m, n, q; d)$-*MRD* code. The *adjoint code* of $\mathcal{C}$ is defined as the code consisting of transposed matrices belonging to $\mathcal{C}$:

$$\mathcal{C}^\top = \{X^t : X \in \mathcal{C}\}.$$

As in [84, Theorem 3.5], two rank distance codes $\mathcal{C}, \mathcal{C}' \subset \mathbb{F}_q^{m\times n}$, $m, n \geq 2$, are called *equivalent* if there exist $P \in \mathrm{GL}(m, q)$, $Q \in \mathrm{GL}(n, q)$, $R \in \mathbb{F}_q^{m\times n}$ and a field automorphism $\rho \in \mathrm{Aut}(\mathbb{F}_q)$ such that

$$\mathcal{C}' = P\mathcal{C}^\rho Q + R = \{PX^\rho Q + R : X \in \mathcal{C}\}.$$

When $m = n$, in addition to being equivalent, two codes are said *adjointly equivalent* if

$$\mathcal{C}' = P(\mathcal{C}^\top)^\rho Q + R = \left\{P(X^t)^\rho Q + R : X \in \mathcal{C}\right\}.$$

If both $\mathcal{C}$ and $\mathcal{C}'$ are additive, then one may assume that $R$ is the zero matrix. Similarly, if both $\mathcal{C}$ and $\mathcal{C}'$ are linear, then $\rho$ can be chosen, without loss of generality, as the identity. The code $\mathcal{C}' \subset \mathbb{F}_q^{(m-u)\times n}$ obtained by $\mathcal{C} \subset \mathbb{F}_q^{m\times n}$ deleting the last $u$ rows, $1 \leq u \leq m-1$, is called the *punctured code* of $\mathcal{C}$ . In [13, Corollary 7.3], it is showed that if $\mathcal{C}$ is an MRD code then $\mathcal{C}'$ is MRD as well.
The *left* and *right idealisers* of a code $\mathcal{C} \subset \mathbb{F}_q^{m\times n}$ are defined as the sets

$$I_L(\mathcal{C}) = \left\{P \in \mathbb{F}_q^{m\times m} : PX \in \mathcal{C}, \forall X \in \mathcal{C}\right\},$$
$$I_R(\mathcal{C}) = \left\{Q \in \mathbb{F}_q^{n\times n} : XQ \in \mathcal{C}, \forall X \in \mathcal{C}\right\},$$

respectively.
Although the rank distance codes are subsets of matrices, they can be represented in a particular setting, that of $\sigma$-*linearized polynomials*. From now on, suppose $m = n$ and let $\sigma : x \in \mathbb{F}_{q^n} \mapsto x^{q^s} \in \mathbb{F}_{q^n}$ be a field automorphism of $\mathbb{F}_{q^n}$ with $(s, n) = 1$. A $\sigma$-*linearized polynomial* over $\mathbb{F}_{q^n}$ is a polynomial of the form $\alpha(x) = \sum_{i=0}^{\ell} \alpha_i x^{\sigma^i} \in \mathbb{F}_{q^n}[x]$, $\ell \in \mathbb{N}$. If $\alpha_\ell \neq 0$, the integer $\ell$ is called the $\sigma$-*degree* of $\alpha(x)$ and it will be denoted by $\deg_\sigma \alpha(x)$ or $\deg_{q^s} \alpha(x)$. The set

$$\mathcal{L}_{n,q,\sigma} = \left\{\sum_{i=0}^{n-1} \alpha_i x^{\sigma^i} : \alpha_i \in \mathbb{F}_{q^n}\right\} \tag{2.1}$$

with the usual sum, the scalar multiplication by an element of $\mathbb{F}_{q^n}$ and the map composition modulo $x^{q^n} - x$ is an algebra isomorphic to the algebra of the endomophisms of $\mathbb{F}_{q^n}$ seen as vector space over $\mathbb{F}_q$. Indeed, it is well known that any $\mathbb{F}_q$-linear endomorphism of $\mathbb{F}_{q^n}$ can be represented uniquely as a $\sigma$-polynomial beloging to $\mathcal{L}_{n,q,\sigma}$, [59, Chapter 3]. Then any rank distance code $\mathcal{C}$, $|\mathcal{C}| \geq 2$, can be seen as a suitable subset of $\mathcal{L}_{n,q,\sigma}$.

So, the definitions of transposed matrix, adjoint code and equivalence between codes can be naturally reformulated in this environment. Let $\alpha(x) = \sum_{i=0}^{n-1} \alpha_i x^{\sigma^i} \in \mathcal{L}_{n,q,\sigma}$, then the *adjoint polynomial* of $\alpha(x)$ is defined as

$$\hat{\alpha}(x) = \sum_{i=0}^{n-1} \alpha_i^{\sigma^{n-i}} x^{\sigma^{n-i}}$$

and if $\mathcal{C} \subset \mathcal{L}_{n,q,\sigma}$ is a rank distance code, $\mathcal{C}^\top = \{\hat{\alpha}(x) : \alpha \in \mathcal{C}\}$. Moreover, two rank codes $\mathcal{C}$ and $\mathcal{C}'$ are equivalent or adjointly equivalent if and only if there exists $(f, \rho, g, h)$ such that $f, g, h \in \mathcal{L}_{n,q,s}$, with $f$ and $g$ permutation polynomials, and $\rho \in \mathrm{Aut}(\mathbb{F}_q)$ such that

$$\mathcal{C}' = \{f \circ \alpha^\rho \circ g + h \colon \alpha \in \mathcal{C}\} \quad \text{or} \quad \mathcal{C}' = \{f \circ \alpha^\rho \circ g + h \colon \alpha \in \mathcal{C}^\top\},$$

respectively, where the automophism $\rho$ acts only over the coefficients of a polynomial $\alpha$ in $\mathcal{C}$ or $\mathcal{C}^\top$, respectively. A map

$$(f, \rho, g, h) : \alpha \in \mathcal{C} \mapsto f \circ \alpha^\rho \circ g + h \in \mathcal{C}'$$

is called an *equivalence* between $\mathcal{C}$ and $\mathcal{C}'$. If $\mathcal{C} = \mathcal{C}'$, an equivalence is called an *automorphism* of $\mathcal{C}$. The set $\mathrm{Aut}(\mathcal{C})$ of all automorphisms of $\mathcal{C}$ is a group with respect to the product

$$(f, \rho, g, h) \star (f', \rho', g', h') = (f' \circ f^{\rho'}, \rho\rho', g^{\rho'} \circ g', f' \circ h^{\rho'} \circ g' + h')$$

where the composition is taken modulo $x^{q^n} - x$.

Now, we shall present the state of the art regarding the known $(n, n, q; d)$-MRD codes in the linearized polynomials setting. We shall start from maximum rank distance code that are $\mathbb{F}_q$-linear subspaces of $\mathcal{L}_{n,q,\sigma}$.

*2.1 Linear MRD codes*

The first class of linear MRD codes, discovered by Delsarte [21] and Gabidulin [34], are known in literature as *Delsarte-Gabidulin codes.* Later in [35], Gabidulin and Kshevetskiy provided a generalization of them, called the *generalized Gabidulin code*: let $1 \leq k \leq n$ be integers, then generalized Gabidulin code is the set of linearized polynomials

$$\mathcal{G}_{k,\sigma} = \left\{ \sum_{i=0}^{k-1} \alpha_i x^{\sigma^i} : \alpha_i \in \mathbb{F}_{q^n} \right\}$$

and it is an $(n, n, q; n - k + 1)$-MRD code.

In [76], Sheekey exhibited a wider class of linear MRD codes, called *twisted Gabidulin codes* and later generalized in [66] by Lunardon, Trombetti and Zhou. Let $n, k, h$ be positive integers, $k < n$ and let $\eta$ be in $\mathbb{F}_{q^n}$ such that $\mathrm{N}_{q^{sn}/q^s}(\eta) \neq (-1)^{nk}$ [1], then the *generalized twisted Gabidulin codes* is the set

$$\mathcal{H}_{k,\sigma}(\eta, h) = \left\{ \sum_{i=0}^{k-1} \alpha_i x^{\sigma^i} + \eta \alpha_0^{q^h} x^{\sigma^k} : \alpha_i \in \mathbb{F}_{q^n} \right\}$$

and it is an $(n, n, q; n - k + 1)$-MRD code.

Puchinger et al. provided a further generalization of generalized twisted Gabidulin codes in [73].

Finally, a family of maximum rank distance codes in $\mathcal{L}_{n,q,\sigma}$, $n = 2t$, $2 \leq d \leq n$ and $q$ odd is discovered by Trombetti and Zhou in 2019 and described in [82]: let $\xi \in \mathbb{F}_{q^n}$ satisfying that $\mathrm{N}_{q^n/q^t}(\xi)$ is a non-square in $\mathbb{F}_q$, then

$$\mathcal{D}_{k,\sigma}(\xi) = \left\{ \sum_{i=0}^{k-1} \alpha_i x^{\sigma^i} + \xi \alpha_k x^{\sigma^k} : \alpha_1, \ldots, \alpha_{k-1} \in \mathbb{F}_{q^n}, \alpha_0, \alpha_k \in \mathbb{F}_{q^t} \right\}$$

[1] The symbol $\mathrm{N}_{q^m/q^\ell}(x) = x^{\frac{q^m-1}{q^\ell-1}}$ denote the *norm* of $x \in \mathbb{F}_{q^m}$ over $\mathbb{F}_{q^\ell}$, where $m, \ell$ are integers with $\ell \mid m$.

and it is an $\mathbb{F}_q$-linear $(n, n, q; n-k+1)$-MRD code.

*2.2 Additive MRD codes*

In [68], Otal and Özbudak showed a family of non-linear additive MRD codes as a generalization of generalized twisted Gabidulin codes. Let $n, k, u, h$ be positive integers satisfying $q = q_0^u$ and $k < n$. Let $\eta \in \mathbb{F}_{q^n}$ such that $\mathrm{N}_{q^{sn}/q_0^s}(\eta) \neq (-1)^{nku}$. Then the set

$$\mathcal{A}_{k,\sigma,q_0}(\eta, h) = \left\{ \sum_{i=0}^{k-1} \alpha_i x^{\sigma^i} + \eta \alpha_0^{q_0^h} x^{\sigma^k} : \alpha_i \in \mathbb{F}_{q^n} \right\}$$

is an $\mathbb{F}_{q_0}$-linear $(n, n, q; n-k+1)$-MRD code.

*2.3 Non-linear MRD codes*

In finite geometry $(n, n, q; n)$-MRD codes are known as *spread sets* (see e.g. [22]) and there are examples for both the linear and the non-linear case. The first class of non-linear MRD codes whose codewords are not invertible and so different from spread sets, are the $(3, 3, q; 2)$-MRD codes constructed by Cossidente et al. [17]. These arise from a geometrical context and are generalized in two steps: firstly, by Durante and Siciliano [33] obtaining a family of $(n, n, q; n-1)$-MRD codes, $n \geq 3$, and then by Donati and Durante obtaining a family of $(d+1, n, q; d)$-MRD codes, $d \leq n-1$, see [28]. As we shall extend this family in Chapter 9, we will describe it in detail in Section 2.3.

The second family of non-additive MRD codes for all $n, d$ has been constructed by Otal and Özbudak in [69]: let $I$ be a subset of $\mathbb{F}_q$, $1 \leq k \leq n-1$ and consider

$$\mathcal{C}^{(1)}_{n,k,\sigma,I} = \left\{ \sum_{i=0}^{k-1} \alpha_i x^{\sigma^i} : \alpha_0, \ldots, \alpha_{k-1} \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\alpha_0) \in I \right\}, \tag{2.2}$$

$$\mathcal{C}^{(2)}_{n,k,\sigma,I} = \left\{ \sum_{i=1}^{k} \beta_i x^{\sigma^i} : \beta_1, \ldots, \beta_k \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\beta_k) \notin (-1)^{n(k+1)} I \right\}.$$

Then $\mathcal{C}_{n,k,\sigma,I} = \mathcal{C}^{(1)}_{n,k,\sigma,I} \cup \mathcal{C}^{(2)}_{n,k,\sigma,I} \subset \mathcal{L}_{n,q,\sigma}$ is an $(n, n, q; n-k+1)$-MRD code. In [69, Corollary 2.1], they proved

1. if $q = 2$ or $I \in \{\{0\}, \mathbb{F}_q^*, \mathbb{F}_q, \emptyset\}$ then $\mathcal{C}_{n,k,\sigma,I}$ is equivalent to a generalized Gabidulin code
2. if $q > 2$ and $I \notin \{\{0\}, \mathbb{F}_q^*, \mathbb{F}_q, \emptyset\}$, then $\mathcal{C}_{n,k,\sigma,I}$ is not an affine code (i.e. not a translated version of an additive code).

## 2.2 The geometric setting

Let $\mathbb{E} = \mathrm{End}(\mathbb{F}_{q^n}, \mathbb{F}_q)$ be the vector space of all endomorphisms of $\mathbb{F}_{q^n}$ seen as a vector space over the field $\mathbb{F}_q$. As any element of $\mathbb{E}$ corresponds 1-to-1 to a linearized polynomial

$f(x) = \sum_{i=0}^{n-1} \alpha_i x^{\sigma^i} \in \mathcal{L}_{n,q,\sigma}$, it will be called *rank* of $f : \mathbb{F}_{q^n} \longrightarrow \mathbb{F}_{q^n}$, $\mathrm{rk} f = \dim_{\mathbb{F}_q} \mathrm{im} f$. Also, the $\mathbb{F}_q$-linear map $f(x)$ has rank $r$ if and only if the *Dickson matrix*

$$D_f = \begin{pmatrix} \alpha_0 & \alpha_1 & \dots & \alpha_{n-1} \\ \alpha_{n-1}^{\sigma} & \alpha_0^{\sigma} & \dots & \alpha_{n-2}^{\sigma} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_1^{\sigma^{n-1}} & \alpha_2^{\sigma^{n-1}} & \dots & \alpha_0^{\sigma^{n-1}} \end{pmatrix}$$

has rank $r$, see for more details [59].

Let $V$ be a $v$-dimensional vector space over the field $\mathbb{F}_{q^n}$ and let $\mathrm{PG}(v-1, q^n) = \mathrm{PG}(V, \mathbb{F}_{q^n})$. A set $\Delta$ of points of $\mathrm{PG}(v-1, q^n)$ is an $\mathbb{F}_q$-*linear set* if each point is defined by a non-zero vector of an $\mathbb{F}_q$-linear vector space $U \subset V$, in symbol

$$\Delta = \left\{ \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U, \mathbf{u} \neq \mathbf{0} \right\}.$$

If $\dim_{\mathbb{F}_q} U = u$, we say that $\Delta$ has *rank* $u$. The size of $\Delta$ can be at most $\frac{q^u-1}{q-1}$ and if it is attained, $\Delta$ is said to be *scattered.* If $u = v$ and $\langle \Delta \rangle = \mathrm{PG}(v-1, q^n)$, then $\Delta$ is a *(canonical) subgeometry* of $\mathrm{PG}(v-1, q^n)$. It follows that $\Delta$ is a canonical subgeometry if and only if any its frame is also a frame of $\mathrm{PG}(v-1, q^n)$.

Let $(X_0, X_1, \dots, X_{n-1})$ denote the homogeneous projective coordinates of $\mathrm{PG}(n-1, q^n)$ and let $\hat{\sigma}$ be the collineation of $\mathrm{PG}(n-1, q^n)$ defined by

$$(X_0, X_1, \dots, X_{n-1})^{\hat{\sigma}} = (X_{n-1}^{\sigma}, X_0^{\sigma}, \dots, X_{n-2}^{\sigma}).$$

Then, the collineation $\hat{\sigma}$ fixes pointwise the canonical subgeometry

$$\Sigma = \{(\alpha, \alpha^{\sigma}, \dots, \alpha^{\sigma^{n-1}}) : \alpha \in \mathbb{F}_{q^n}^*\} \cong \mathrm{PG}(n-1, q). \tag{2.3}$$

Let $\overline{S} = \mathrm{PG}(W, q^n)$ be a subspace of $\mathrm{PG}(n-1, q^n)$, the integer $\dim_{\mathbb{F}_{q^n}} W = w$ will be called *rank* of $\overline{S}$. Then $S = \overline{S} \cap \Sigma$ is a subspace of $\Sigma$ of rank at most $w$. We will say that $\overline{S}$ is a *subspace of* $\Sigma$ if $S$ and $\overline{S}$ have the same rank. In particular, this holds if and only if $\overline{S}$ is fixed by the collineation $\hat{\sigma}$ (see e.g. [62]).

Any point $P$ of $\mathrm{PG}(n-1, q^n)$ defines the subspace

$$L(P) = \langle P, P^{\hat{\sigma}}, \dots, P^{\hat{\sigma}^{n-1}} \rangle.$$

Note that $L(P)$ has rank at most $n$ and it is fixed by $\hat{\sigma}$, and so $L(P)$ is a subspace of $\Sigma$. If $P(\alpha_0, \alpha_1, \dots, \alpha_{n-1})$ is a point of $\mathrm{PG}(n-1, q^n)$, it will be of *type* $r$ *with respect to* $\Sigma$ if $L(P)$ has rank $r$, or equivalently if $f(x) = \sum_{i=0}^{n-1} \alpha_i x^{\sigma^i}$ has rank $r$, or also the Dickson matrix $D_f$ has rank $r$.

Let $\mathrm{PG}(\mathbb{F}_q^{m \times n}, \mathbb{F}_q) = \mathrm{PG}(mn-1, q)$, $m \leq n$, and let $\mathcal{S}_{m-1,n-1}$ be the *Segre variety* of $\mathrm{PG}(\mathbb{F}_q^{m \times n}, \mathbb{F}_q)$, i.e. $\mathcal{S}_{m-1,n-1}$ is the set of all points $\langle M \rangle$ in $\mathrm{PG}(\mathbb{F}_q^{m \times n}, \mathbb{F}_q)$ such that $\mathrm{rank}\, M = 1$, see [43, Section 4.5]. This can be seen as the $\mathbb{F}_q$-field reduction of the set of points

$$\Sigma_{m,n} = \{(x, x^{\sigma}, \dots, x^{\sigma^{m-1}}) \colon x \in \mathbb{F}_{q^n}^*\} \cong \mathrm{PG}(n-1, q)$$

of $\mathrm{PG}(m-1,q^n)$ in $\mathrm{PG}(mn-1,q)$, see [58].
Let $\mathcal{A}$ be a subset of $\mathrm{PG}(n-1,q)$ and denote the *h-secant variety* of $\mathcal{A}$ by $\Omega_h(\mathcal{A})$, i.e. the union of the $h$-dimensional projective subspaces spanned by collections of $h+1$ independent points of $\mathcal{A}$, [41]. A set of points $\mathcal{E} \subset \mathrm{PG}(n-1,q)$ is called an *exterior set* with respect to $\Omega_h(\mathcal{A})$ if any line joining two points of $\mathcal{E}$ is disjoint from $\Omega_h(\mathcal{A})$. The following theorem proves an upper bound for the size of an exterior set.

**Theorem 2.2.1.** *Let $\mathcal{A} \subset \mathrm{PG}(n-1,q)$ such that $\langle \mathcal{A} \rangle = \mathrm{PG}(n-1,q)$. Let $\mathcal{E} \subset \mathrm{PG}(n-1,q)$ be an exterior set with respect to $\Omega_h(\mathcal{A})$, then*

$$|\mathcal{E}| \leq \frac{q^{n-h-1}-1}{q-1}$$

*for any $0 \leq h \leq n-1$.*

*Proof.* Firstly, we may assume $h \leq n-3$. Indeed, if $h \in \{n-2, n-1\}$ the claim is trivially satisfied: if $h = n-1$, then $\Omega_{n-1}(\mathcal{A}) = \mathrm{PG}(n-1,q)$ and $\mathcal{E} = \emptyset$; if $h = n-2$, then $\Omega_{n-2}(\mathcal{A})$ contains hyperplanes and $|\mathcal{E}| \leq 1$.
Now, let $h \leq n-3$ and let $S_h$ be an $h$-dimensional subspace contained in $\Omega_h(\mathcal{A})$. The number of $(h+1)$-dimensional projective subspaces of $\mathrm{PG}(n-1,q)$ containing $S_h$ equals the number of points of an $(n-h-2)$-dimensional projective subspace of $\mathrm{PG}(n-1,q)$. Since $\mathcal{E}$ is an exterior set, then it meets any $(h+1)$-dimensional projective subspace containing $S_h$ in at most one point. This proves the claim. □

Given $M$ and $N$ a subspace and a subset of points of $\mathrm{PG}(n-1,q)$, respectively, with $M \cap N = \emptyset$, we will denote the *cone with vertex $M$ and base $N$* by $\mathcal{K}(M,N)$.

**Corollary 2.2.2.** *Let $\mathcal{A} \subset \mathrm{PG}(n-1,q)$ such that $\langle \mathcal{A} \rangle = \mathrm{PG}(t-1,q)$, $1 \leq t < n$, and let $\mathcal{E} \subset \mathrm{PG}(n-1,q)$ be an exterior set with respect to $\Omega_h(\mathcal{A})$. Then $\mathcal{E}$ is contained in a cone $\mathcal{K} = \mathcal{K}(S_{n-t-1}, \bar{\mathcal{E}})$, with base $\bar{\mathcal{E}} = \mathcal{E} \cap \langle \mathcal{A} \rangle$ and vertex an $(n-t-1)$-dimensional subspace $S_{n-t-1}$ complementary with $\langle \mathcal{A} \rangle$. Moreover,*

$$|\mathcal{E}| \leq \frac{q^{n-h-1}-1}{q-1},$$

*for any $0 \leq h \leq t-1$.*

*Proof.* Clearly, $\bar{\mathcal{E}} \subset \langle \mathcal{A} \rangle$ is an exterior set with respect to $\Omega_h(\mathcal{A}) \subset \langle \mathcal{A} \rangle$. By Theorem 2.2.1, one gets

$$|\bar{\mathcal{E}}| \leq \frac{q^{t-h-1}-1}{q-1}$$

for any $0 \leq h \leq t-1$. Then

$$\begin{aligned} |\mathcal{K}| &= |\bar{\mathcal{E}}| \cdot |S_{n-t-1}| \cdot (q-1) + |\bar{\mathcal{E}}| + |S_{n-t-1}| \\ &\leq \frac{q^{t-h-1}-1}{q-1} \cdot \frac{q^{n-t}-1}{q-1} \cdot (q-1) + \frac{q^{t-h-1}-1}{q-1} + \frac{q^{n-t}-1}{q-1} \\ &= \frac{q^{n-h-1}-1}{q-1}. \end{aligned}$$

$\square$

An exterior set $\mathcal{E} \subset \mathrm{PG}(n-1,q)$ with respect to $\Omega_h(\mathcal{A})$, $\langle \mathcal{A} \rangle = \mathrm{PG}(t-1,q)$, $1 \leq h+1 \leq t \leq n$, is called *maximum* if $|\mathcal{E}| = \frac{q^{n-h-1}-1}{q-1}$. Note that the image of $\Omega_h(\Sigma_{m,n}) \subset \mathrm{PG}(m-1,q^n)$ under the $\mathbb{F}_q$-field reduction is the $h$-secant variety $\Omega_h(\mathcal{S}_{m-1,n-1})$ of the points whose representative matrices in $\mathbb{F}_q^{m\times n}$ have rank at most $h+1$.

The (maximum) exterior sets with respect to $\Omega_h(\Sigma_{m,n})$ are related to (maximum) rank distance codes. More precisely,

**Theorem 2.2.3.** *Let $\mathcal{E} \subset \mathrm{PG}(m-1,q^n)$ be an exterior set with respect to $\Omega_h(\Sigma_{m,n})$ and denote the image of $\mathcal{E}$ under the $\mathbb{F}_q$-field reduction by $\mathcal{E}'$. Then, the set*

$$\mathcal{C} = \left\{\rho M : \langle M \rangle_{\mathbb{F}_q} \in \mathcal{E}', \rho \in \mathbb{F}_q\right\} \tag{2.4}$$

*is an $(m,n,q;h+2)$-RD code. In addition, if $\mathcal{E}$ is maximum then $\mathcal{C}$ is an MRD-code.*

See [16, 17, 29] and the references therein for more details on exterior sets.

## 2.3 Non-linear MRD codes arising from $C_F^\sigma$-sets

In [28], G. Donati and N. Durante exhibited a class of non-linear MRD codes with parameters $(d+1,n,q;d)$ with $q>2$, $n\geq 3$ and $2\leq d\leq n-1$ by means of a family of maximum exterior sets with respect the $(d-2)$-secant variety of a subgeometry isomorphic to $\mathrm{PG}(d,q)$ in $\Sigma_{d+1,n}$. Now, we recall briefly their construction. So, let $\mathrm{PG}(d,q^n)$ be a $d$-dimensional projective space and consider $A$ and $B$ two distinct points of $\mathrm{PG}(d,q^n)$, $\mathcal{S}_A$ and $\mathcal{S}_B$ be the stars of lines (pencils of lines if $d=2$) through $A$ and $B$, respectively. Let $\sigma$ be the Frobenius automorphism of $\mathbb{F}_{q^n}$ defined by $\sigma : x \mapsto x^{q^s}$ with $(n,s)=1$ and consider $\Phi$ a $\sigma$-collineation between $\mathcal{S}_A$ and $\mathcal{S}_B$ which does not map the line $AB$ into itself and such that the subspace spanned by the lines $\Phi^{-1}(AB), AB, \Phi(AB)$ has dimension $\min\{3,d\}$. The set $\mathcal{X}$ of points of intersection of corresponding lines under the collineation $\Phi$ is called $C_F^\sigma$*-set* of $\mathrm{PG}(d,q^n)$ and the points $A$ and $B$ are called the *vertices* of $\mathcal{X}$, see [27, 28]. There, it is proved that every $C_F^\sigma$-set $\mathcal{X}$ of $\mathrm{PG}(d,q^n)$ with vertices $A$ and $B$ is the union of $\{A,B\}$ and $q-1$ pairwise disjoint subsets, called the *components* of $\mathcal{X}$, each of which is a scattered $\mathbb{F}_q$-linear set of rank $n$, for more details on linear sets see e.g. [72].

Let $N_a = \{y \in \mathbb{F}_{q^n} : \mathrm{N}_{q^n/q}(y) = a\}$ for any $a \in \mathbb{F}_q^*$. Without loss of generality, we may assume that $A=(0,\dots,0,1)$ and $B=(1,0,\dots,0)$ and then the sets

$$\mathcal{X}_a = \left\{(1,t,t^{\sigma+1},\dots,t^{\sigma^{d-1}+\dots+\sigma+1}) : t \in N_a\right\},$$

with $a \in \mathbb{F}_q^*$, are the components of $\mathcal{X}$. Since every $\mathcal{X}_a$ has $(q^n-1)/(q-1)$ points, then $\mathcal{X}$ has $q^n+1$ points. In particular, every $\mathcal{X}_a$ is isomorphic to $\mathrm{PG}(n-1,q)$, see [29, Remark 3.5]. For any $a \in \mathbb{F}_q^*$, the line $AB$ of $\mathrm{PG}(d,q^n)$ is partitioned into $\{A,B\}$ and the $q-1$ sets

$$J_a = \left\{(1,0,\dots,0,(-1)^{d+1}t) : t \in N_a\right\},$$

each of which is an $\mathbb{F}_q$-scattered linear set (of pseudoregulus type with transversal points $A$ and $B$), see [27, Remark 2.2] and [64]. Note that $\mathcal{X}_1 = \Sigma_{d+1,n}$ and let $\Pi$ be a subgeometry of $\Sigma_{d+1,n}$ isomorphic to $\mathrm{PG}(d,q)$. Then for any $T \subset \mathbb{F}_q^*$, $1 \in T$, the set

$$\mathcal{E} = \left( \mathcal{X} \setminus \bigcup_{a \in T} \mathcal{X}_a \right) \cup \bigcup_{a \in T} J_a$$

is a maximum exterior set with respect to $\Omega_{d-2}(\Pi)$, see [28, Theorem 5.1]. Hence, to the set $\mathcal{E}$ corresponds a $(d+1,n,q;d)$-MRD code.

# Chapter 3

# Absolute points of correlations of $\mathrm{PG}(d, q^n)$, $d \in \{1, 2, 3\}$

## 3.1 The $\sigma$-quadrics of $\mathrm{PG}(d, q^n)$

Let $V = \mathbb{F}_{q^n}^{d+1}$ be a vector space. If $V$ is equipped with a $\sigma$-sesquilinear form $\langle\ ,\ \rangle$ we may consider in $\mathrm{PG}(d, q^n)$ the set $\Gamma$ of absolute points of the associated correlations $\perp$, $\top$, defined as in (1.1), (1.2) respectively, i.e. the set of points $P$ such that $P \in P^{\perp}$ (or equivalently $P \in P^{\top}$). If $A$ is the associated matrix to the $\sigma$-sesquilinear form $\langle\ ,\ \rangle$ w.r.t. a basis of $V$, then the set $\Gamma$ consists of the points satisfying the equation $X^t A X^{\sigma} = 0$. Throughout we can assume $\sigma \neq 1$, since we have seen that if $\sigma = 1$, then $\Gamma : X^t A X = 0$ is either a (possibly degenerate) quadric or a (possibly degenerate) symplectic geometry. Let $L = V^{\perp}$ and $R = V^{\top}$ denote the left and right radicals of $\langle\ ,\ \rangle$ respectively, seen as subspaces of $\mathrm{PG}(d, q^n)$ that will be called the *vertices* of $\Gamma$.

**Definition 3.1.1.** A $\sigma$-*quadric* of $\mathrm{PG}(d, q^n)$ is the set of the absolute points of a (possibly degenerate) $\sigma$-correlation, $\sigma \neq 1$, of $\mathrm{PG}(d, q^n)$. A $\sigma$-quadric of $\mathrm{PG}(2, q^n)$ will be called a $\sigma$-*conic*.

The $\sigma$-quadrics include Hermitian varieties and have been considered in [23, 29].

**Proposition 3.1.2.** *Let $\Gamma : X^t A X^{\sigma} = 0$ be a $\sigma$-quadric of $\mathrm{PG}(d, q^n)$. Every subspace $S$ intersects $\Gamma$ either in a $\sigma$-quadric of $S$ or it is contained in $\Gamma$.*

*Proof.* Let $(x_1, \ldots, x_{d+1})$ be the homogeneous projective coordinates of $\mathrm{PG}(d, q^n)$. Let $S$ be an $h$-dimensional subspace of $\mathrm{PG}(d, q^n)$. We may asssume, w.l.o.g., that

$$S : x_{h+2} = \ldots = x_{d+1} = 0.$$

Let $A'$ be the submatrix of $A$ obtained by deleting the last $d - h$ rows and columns of $A$. If $A' = 0$, then $S \subset \Gamma$. If $A' \neq 0$, then $S \cap \Gamma$ is a $\sigma$-quadric of $S$. $\square$

In [29] the following result regarding subspaces contained in $\sigma$-quadrics has been proved.

**Proposition 3.1.3.** *If $S_h$ is an $h$-dimensional subspace of $\mathrm{PG}(d,q^n)$ contained in a $\sigma$-quadric $\Gamma$ with equation $X^tAX^\sigma = 0$, then $h \leq \left\lfloor d - \frac{\mathrm{rk}(A)}{2} \right\rfloor$. Moreover, for any $\sigma$ there exists a $\sigma$-quadric with equation $X^tAX^\sigma = 0$, containing a subspace with dimension $\left\lfloor d - \frac{\mathrm{rk}(A)}{2} \right\rfloor$.*

**Definition 3.1.4.** Let $\Gamma : X^tAX^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(d,q^n)$. A line $\ell$ is *$i$-secant* to $\Gamma$ if it has exactly $i \geq 1$ points in common with $\Gamma$. In particular, if $i = 1$ the line is also called *tangent*.

**Proposition 3.1.5** ([29, Proposition 2.3])**.** Let $\Gamma : X^tAX^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(d,q^n)$.

- For every point $Y \in \Gamma \setminus R$, the hyperplane $Y^\top$ is the union of lines through $Y$ either contained or 1-secant or 2-secant to $\Gamma$.
- For every point $Y \in \Gamma \setminus L$, the hyperplane $Y^\perp$ is the union of lines through $Y$ either contained or 1-secant or 2-secant to $\Gamma$.

**Corollary 3.1.6** ([29, Corollary 2.4])**.** *Let $\Gamma : X^tAX^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(d,q^n)$ and let $L = V^\perp$, $R = V^\top$.*

- *For every point $Y \in L$, the set $Y^\top \cap \Gamma$ is the union of lines through $Y$ contained in $\Gamma$.*
- *For every point $Y \in R$, the set $Y^\perp \cap \Gamma$ is the union of lines through $Y$ contained in $\Gamma$.*

**Proposition 3.1.7.** *Let $\Gamma : X^tAX^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(d,q^n)$, $n \geq 2$. A line through a point $Y$ of $L = V^\perp$ (or $R = V^\top$) and contained in $Y^\perp$ (or $Y^\top$) has one or all points in common with $\Gamma$.*

*Proof.* Let $Y \in L$ and let $Z$ be a point of $Y^\top$. Let $X = \lambda Y + \mu Z$, $\lambda, \mu \in \mathbb{F}_{q^n}$, $(\lambda,\mu) \neq (0,0)$, be a point on the line joining $Y$ and $Z$. Then $X \in \Gamma$ if and only if

$$Y^tAY^\sigma\lambda^{\sigma+1} + Y^tAZ^\sigma\lambda\mu^\sigma + Z^tAY^\sigma\lambda^\sigma\mu + Z^tAZ^\sigma\mu^{\sigma+1} = 0 \tag{3.1}$$

where $Y^tAY^\sigma = Y^tAZ^\sigma = 0$, since $Y \in L$ and $Z^tAY^\sigma = 0$, since $Z \in Y^\top$. Hence, Equation (3.1) becomes $Z^tAZ^\sigma\mu^{\sigma+1} = 0$ and two cases occur: either $Z^tAZ^\sigma = 0$ and the line $YZ$ is contained in $\Gamma$ or $Z^tAZ^\sigma \neq 0$ and the line $YZ$ intersects $\Gamma$ exactly in the point $Y$.
If $Y \in R$, the result follows in a similar way. □

## 3.2 The $\sigma$-quadrics of $\mathrm{PG}(1,q^n)$ and $\mathrm{PG}(2,q^n)$

Now, we can start with the study of $\sigma$-quadrics. First we describe the set $\Gamma$ in $\mathrm{PG}(1,q^n)$, for more details see [23].

**Proposition 3.2.1.** *Let $\Gamma$ be a $\sigma$-quadric of $\mathrm{PG}(1,q^n)$. Then $\Gamma$ is either the empty set or a point or two points or an $\mathbb{F}_q$-subline.*

Now, we describe the set $\Gamma$ in $\mathrm{PG}(2,q^n)$ when $|A|=0$, i.e. the $\sigma$-sesquilinear form $\langle\, ,\, \rangle$ on $\mathbb{F}^3_{q^n}$ is degenerate, see [23].

**Proposition 3.2.2.** *Let $\Gamma : X^tAX = 0$, $|A| = 0$, be a $\sigma$-conic of $\mathrm{PG}(2,q^n)$ with an associated degenerate $\sigma$-sesquilinear form. Then $\Gamma$ is one of the following:*

- *a cone with vertex a point $P$ projecting a $\sigma$-quadric on a line not through $P$ (hence either the point $P$ or a line, or two lines, or $q+1$ lines through $R$);*
- *a (possibly degenerate) $C^\sigma_F$-set (see [24, 25, 26, 27]).*

The sets of absolute points with respect to a non-degenerate, non-reflexive $\sigma$-sequilinear form, $\sigma \neq 1$, of $\mathbb{F}^3_{q^n}$ have been completely determined by B.C. Kestenband in 11 papers. We refer the interested reader to [46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56]. In what follows we will use the following definition.

**Definition 3.2.3.** A *Kestenband $\sigma$-conic* is the set of absolute points of a non-degenerate, non-reflexive $\sigma$-sequilinear form, $\sigma \neq 1$, of $\mathbb{F}^3_{q^n}$.

In the next two theorems let $\sigma : x \mapsto x^{q^m}$ with $(2n+1, m) = 1$.

**Theorem 3.2.4.** *Let $\Gamma : X^tAX^\sigma = 0$ be a Kestenband $\sigma$-conic of $\mathrm{PG}(2,q^{2n+1})$, $n>0$, $q$ odd. Let $f$ be the collineation defined by $X \mapsto (A^t)^{-1}A^\sigma X^{\sigma^2}$. Then one of the following holds:*

- *if $f$ fixes no points of $\mathrm{PG}(2,q^{2n+1}) \setminus \Gamma$, then $|\Gamma| = q^{2n+1}+1$. Moreover, $f$ fixes one point of $\Gamma$ and $\Gamma$ is projectively equivalent to the set satisfying*
$$x_3x_1^\sigma + x_2^{\sigma+1} + (x_1 + ex_3)x_3^\sigma = 0,$$
*where $\frac{e^{q^{2n+1}+1}-1}{e-1} \neq 0$.*
- *if $f$ fixes more than one point of $\mathrm{PG}(2,q^{2n+1}) \setminus \Gamma$, then $|\Gamma| = q^{2n+1}+1$. Moreover, $\Gamma$ is projectively equivalent to the set satisfying*
$$x_1^{\sigma+1} + x_2^{\sigma+1} + x_3^{\sigma+1} = 0,$$
*the collineation $f$ fixes $\mathrm{PG}(2,q)$ pointwise and hence it fixes the $q+1$ points of $\Gamma$ on the set*
$$\{(x_1,x_2,x_3) \in \mathrm{PG}(2,q) : x_1^2 + x_2^2 + x_3^2 = 0\}$$
*and $q^2$ points of $\mathrm{PG}(2,q) \setminus \Gamma$.*
- *if $f$ fixes one point of $\mathrm{PG}(2,q^{2n+1}) \setminus \Gamma$, then $\Gamma$ is projectively equivalent to the set satisfying*
$$ax_1^{\sigma+1} + (bx_1 + cx_2)x_2^\sigma + dx_3^{\sigma+1} = 0,$$
*with $a,b,c,d \neq 0$. Moreover, $f$ fixes either $0,1,2$ or $q+1$ points of $\Gamma$. In particular, we have:*

– *if $f$ fixes either $0, 2$ or $q+1$ points of $\Gamma$, then $|\Gamma| = q^{2n+1}+1$. In the last case the $q+1$ points of $\Gamma$ fixed by $f$ are collinear.*
– *if $f$ fixes 1 point of $\Gamma$, then $|\Gamma| \in \{q^{2n+1}-q^{n+1}+1, q^{2n+1}+q^{n+1}+1\}$.*

**Theorem 3.2.5.** *Let $\Gamma : X^tAX^\sigma = 0$ be a Kestenband $\sigma$-conic of $\mathrm{PG}(2,q^{2n+1})$, $n>0$, $q$ even. Let $f$ be the collineation defined by $X \mapsto (A^t)^{-1}A^\sigma X^{\sigma^2}$. Then one of the following holds:*

- *if $f$ fixes no points of $\mathrm{PG}(2,q^{2n+1})\backslash\Gamma$, then $|\Gamma| \in \{q^{2n+1}-q^{n+1}+1, q^{2n+1}+q^{n+1}+1\}$. Moreover, $f$ fixes one point of $\Gamma$ and $\Gamma$ is projectively equivalent to the set satisfying*

$$x_3x_1^\sigma + x_2^{\sigma+1} + (x_1+ex_3)x_3^\sigma = 0,$$

  *where $\frac{e^{q^{2n+1}+1}-1}{e-1} \neq 0$.*

- *if $f$ fixes at least one point of $\mathrm{PG}(2,q^{2n+1}) \setminus \Gamma$, then $|\Gamma| = q^{2n+1}+1$. Moreover, $\Gamma$ is projectively equivalent to the set satisfying*

$$x_1^{\sigma+1} + \rho x_1x_2^\sigma + x_2^{\sigma+1} + x_3^{\sigma+1} = 0,$$

  *for some $\rho \in \mathbb{F}_{q^{2n+1}}$. The trinomial $x^{q^m+1}+\rho x+1$ can have one, two, $q+1$ or no zeros. In particular:*

  – *if it has one or $q+1$ zeros, then $\rho = 0$ and the collineation $f$ fixes $\mathrm{PG}(2,q)$ pointwise and hence it fixes the $q+1$ points of $\Gamma$ on the $\mathbb{F}_q$-subline*

$$\{(x_1,x_2,x_3) \in \mathrm{PG}(2,q) : x_1+x_2+x_3 = 0\}$$

  *and $q^2$ points of $\mathrm{PG}(2,q) \setminus \Gamma$.*
  – *if it has two zeros, then the collineation $f$ fixes 2 points of $\Gamma$.*
  – *if it has no zero, then the collineation $f$ fixes no point of $\Gamma$.*

In the next theorems let $\sigma : x \mapsto x^{q^m}$ with $(2n,m)=1$.

**Theorem 3.2.6.** *Let $\Gamma : X^tAX^\sigma = 0$, with $A$ a diagonal matrix, be a Kestenband $\sigma$-conic of $\mathrm{PG}(2,q^{2n})$, $n>0$. Then*

$$|\Gamma| \in \{q^{2n}+1+(-q)^{n+1}(q-1), q^{2n}+1+(-q)^n(q-1), q^{2n}+1-2(-q)^n\}.$$

*Moreover:*

- *if $|\Gamma| = q^{2n}+1+(-q)^{n+1}(q-1)$, then $\Gamma$ is projectively equivalent to the set satisfying*

$$x_1^{\sigma+1} + x_2^{\sigma+1} + x_3^{\sigma+1} = 0.$$

- *if $|\Gamma| = q^{2n}+1+(-q)^n(q-1)$, then $\Gamma$ is projectively equivalent to the set satisfying*

$$x_1^{\sigma+1} + x_2^{\sigma+1} + ax_3^{\sigma+1} = 0,$$

  *with $a \notin \{x^{q+1} : x \in \mathbb{F}_{q^{2n}}\}$.*

- *if* $|\Gamma| = q^{2n} + 1 - 2(-q)^n$, *then* $\Gamma$ *is projectively equivalent to the set satisfying*

$$x_1^{\sigma+1} + bx_2^{\sigma+1} + x_3^{\sigma+1} = 0,$$

*with* $a, b, a/b \notin \{x^{q+1} : x \in \mathbb{F}_{q^{2n}}\}$.

**Theorem 3.2.7.** *Let* $\Gamma : X^t A X^\sigma = 0$, *with* $A$ *a non-diagonal matrix, be a Kestenband* $\sigma$*-conic of* $\mathrm{PG}(2, q^{2n})$, $n > 0$, $q$ *even. Then*

$$|\Gamma| \in \{q^{2n} - (-q)^{n+1} + 1, q^{2n} - (-q)^n + 1, q^{2n} + 1\}.$$

**Theorem 3.2.8.** *Let* $\Gamma : X^t A X^\sigma = 0$, *with* $A$ *a non-diagonal matrix, be a Kestenband* $\sigma$*-conic of* $\mathrm{PG}(2, q^{2n})$, $n > 0$, $q$ *odd. Then one of the following holds:*

- $\Gamma$ *is projectively equivalent to the set satisfying*

$$rx_1^{\sigma+1} + \rho x_1 x_2^\sigma + sx_2^{\sigma+1} + x_3^{\sigma+1} = 0.$$

*Moreover, the trinomial* $rx^{q^m+1} + \rho x + s$ *can have* $0, 1, 2$ *or* $q + 1$ *roots and*

$$|\Gamma| \in \{q^{2n} + 1 + (-q)^{n+1}(q - 1), q^{2n} + 1 + (-q)^n(q - 1), q^{2n} + 1 - 2(-q)^n\}.$$

*Moreover:*

  - *if it has* $q+1$ *zeros or no zero, then* $\Gamma$ *is projectively equivalent to one already studied in the diagonal case, see [3, Corollary 7, Proposition 29, Proposition 30].*
  - *if it has one zero, then either*

$$|\Gamma| \in \{q^{2n} - q^{n+1} + 1, q^{2n} + q^n + 1\}$$

*if* $n$ *is odd, or*

$$|\Gamma| \in \{q^{2n} + q^{n+1} + 1, q^{2n} - q^n + 1\}$$

*if* $n$ *is even.*
  - *if it has two zeros, then* $|\Gamma| = q^{2n} + 1$.

- $\Gamma$ *is projectively equivalent to the set satisfying*

$$x_1^{\sigma+1} + x_1 x_2^\sigma + rx_2^{\sigma+1} + \tau x_2 x_3^\sigma + sx_3^{\sigma+1} = 0.$$

*Moreover,* $|\Gamma| \in \{q^{2n} - q^n + 1, q^{2n} + 1, q^{2n} + q^n + 1\}$.

## 3.3 The $\sigma$-quadrics of $\mathrm{PG}(3, q^n)$

We describe the set $\Gamma$ in $\mathrm{PG}(3, q^n)$ when $|A| = 0$; see [29]. We distinguish three cases according to the rank of $A$.

**Proposition 3.3.1.** *Let $\Gamma : X^t AX = 0$, $\mathrm{rk}(A) = 3$, be a $\sigma$-quadric of $\mathrm{PG}(3, q^n)$ with an associated degenerate $\sigma$-sesquilinear form, with*

$$\sigma : x \in \mathbb{F}_{q^n} \mapsto x^{q^m} \in \mathbb{F}_{q^n},$$

*and $(n, m) = 1$. If $q$ is even and $n$ is even, put $r = (q^n - 1, q^m + 1)$. Then the vertices $R$ and $L$ are points and $\Gamma$ is one of the following:*

1. *If $L \neq R$, then:*
   - *$\Gamma$ is an* elliptic $\sigma$-quadric *of $\mathrm{PG}(3, q^n)$ with vertex points $R$ and $L$ and has canonical equation $x_2^{\sigma+1} - bx_3^{\sigma+1} + x_1x_4^{\sigma} = 0$, with $b$ a non-square if $q$ is odd and $b^{(q^n-1)/r} \neq 1$ if $q$ is even and $n$ is even. Moreover, $|\Gamma| = q^{2n} + 1$ and contains no line.*
   - *$\Gamma$ is a* parabolic $\sigma$-quadric *of $\mathrm{PG}(3, q^n)$ with vertex points $R$ and $L$, $q$ is even and has canonical equation $x_2^{\sigma+1} - bx_3^{\sigma+1} + x_1x_4^{\sigma} = 0$, where the equation $x^{\sigma+1} = b$ has a unique solution. Moreover, $|\Gamma| = q^{2n} + q^n + 1$ and contains a unique line through $R$ and a unique line through $L$.*
   - *$\Gamma$ is a* hyperbolic $\sigma$-quadric *of $\mathrm{PG}(3, q^n)$ with vertex points $R$ and $L$, both $q$ and $n$ are odd and has canonical equation $x_2^{\sigma+1} - bx_3^{\sigma+1} + x_1x_4^{\sigma} = 0$, where $x^{\sigma+1} = b$ has exactly two solutions. Moreover, $|\Gamma| = q^{2n} + 2q^n + 1$ and contains exactly two lines through $R$ and exactly two lines through $L$.*
   - *$\Gamma$ is a* $(q+1)$-hyperbolic $\sigma$-quadric *of $\mathrm{PG}(3, q^n)$ with vertex points $R$ and $L$, $n$ is even and has canonical equation $x_2^{\sigma+1} - bx_3^{\sigma+1} + x_1x_4^{\sigma} = 0$, where $x^{\sigma+1} = b$ has exactly $q+1$ solutions. Moreover, $|\Gamma| = q^{2n} + 2q^n + 1$ and contains exactly $q + 1$ lines through $R$ and exactly $q + 1$ lines through $L$.*
2. *If $L = R$, then $\Gamma$ is a cone with vertex the point $R$ and base a Kestenband $\sigma$-conic of a plane $\pi$ not through $R$.*

**Proposition 3.3.2.** *Let $\Gamma : X^t AX = 0$, $\mathrm{rk}(A) = 2$, be a $\sigma$-quadric of $\mathrm{PG}(3, q^n)$ with an associated degenerate $\sigma$-sesquilinear form. Then the vertices $R$ and $L$ are lines and $\Gamma$ is one of the following:*

- *If $R \cap L = \emptyset$, then $\Gamma$ is a $\sigma$-*quadric of pseudoregulus type *with vertex lines $R$ and $L$, and has canonical equation $x_1x_4^{\sigma} - x_2x_3^{\sigma} = 0$. The only lines contained in $\Gamma$ are $R$, $L$ and the $q^n + 1$ lines of a pseudoregulus with transversal lines $R$ and $L$ (see [64]).*
- *If $R \cap L = \{P\}$ is a point, then $\Gamma$ is a cone with vertex the point $P$ projecting a (possibly degenerate) $C_F^{\sigma}$-set in a plane not through $P$.*

- *If $R = L$, then $\Gamma$ is a cone with vertex the line $R$ over a $\sigma$-quadric of a line skew with $R$.*

**Proposition 3.3.3.** *Let $\Gamma : X^t AX = 0$, $\mathrm{rk}(A) = 1$, be a $\sigma$-quadric of $\mathrm{PG}(3, q^n)$ with an associated degenerate $\sigma$-sesquilinear form. Then the vertices $R$ and $L$ are planes and $\Gamma$ is one of the following:*

- *If $L \neq R$, then $\Gamma = L \cup R$.*
- *If $L = R$, then $\Gamma = L$.*

**Remark 3.3.4.** In contrast to the case of $\mathrm{PG}(2, q^n)$, nothing is known for the sets of absolute points in $\mathrm{PG}(3, q^n)$ of a non-degenerate, non-reflexive sesquilinear form of $\mathbb{F}_{q^n}^4$.

# Chapter 4

# Absolute points of correlations of $\mathrm{PG}(4, q^n)$

We recall that $\sigma$-quadrics have been completely classified in $\mathrm{PG}(d, q^n)$ for $d \in \{1, 2\}$ (see [23]) and partially classified for $d = 3$ (see [29]). Here we will deal with the four dimensional case. As in [29], we will divide the study $\sigma$-quadrics of $\mathrm{PG}(4, q^n)$ into several cases, depending on the rank of the associated matrix. We will say that $\Gamma : X^t A X^\sigma = 0$ is a $\sigma$-quadric of *rank* $i$ if $\mathrm{rk}(A) = i \in \{1, 2, 3, 4\}$. We will fix

$$\sigma : x \in \mathbb{F}_{q^n} \mapsto x^{q^m} \in \mathbb{F}_{q^n}$$

with $(n, m) = 1$. In the last section, we will show that these objects are related to ovoids of the parabolic quadric $Q(4, q)$. The results of this chapter have appeared in [30].

## 4.1 $\sigma$-quadrics of rank 4 in $\mathrm{PG}(4, q^n)$

Let $\Gamma : X^t A X^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(4, q^n)$ associated to a $\sigma$-sesquilinear form $\langle\, , \,\rangle$. In this section we assume that $\mathrm{rk}(A) = 4$. Therefore the vertices $L$ and $R$ are points of $\mathrm{PG}(4, q^n)$. We distinguish several cases:

1) $L \neq R$.
We may assume w.l.o.g. that the point $R = (1, 0, 0, 0, 0)$ is the right radical and the point $L = (0, 0, 0, 0, 1)$ is the left radical. It follows that

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & a_{32} & a_{33} & a_{34} & a_{35} \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and
$\Gamma : (a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{43}x_4)x_2^\sigma + (a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + a_{43}x_4)x_3^\sigma + (a_{14}x_1 +$

$a_{24}x_2 + a_{34}x_3 + a_{44}x_4)x_4^\sigma + (a_{15}x_1 + a_{25}x_2 + a_{35}x_3 + a_{45}x_4)x_5^\sigma = 0.$
The degenerate collineation

$$\top : Y \in \mathrm{PG}(4,q^n) \setminus \{R\} \mapsto X^t A Y^\sigma = 0 \in \mathrm{PG}(4,q^n)^*$$

associated to the sesquilinear form maps points onto hyperplanes through the point $L$. Points that are on a common line through $R$ are mapped onto the same hyperplane through $L$. Therefore $\top$ induces a collineation $\Phi : \mathcal{S}_R \longrightarrow \mathcal{S}_L^*$, where

$$\mathcal{S}_R = \{l_{\alpha,\beta,\gamma,\delta} : (\alpha,\beta,\gamma,\delta) \in \mathrm{PG}(3,q^n)\} \text{ with}$$

$$l_{\alpha,\beta,\gamma,\delta} : \begin{cases} x_1 = \lambda \\ x_2 = \mu\alpha \\ x_3 = \mu\beta \\ x_4 = \mu\gamma \\ x_5 = \mu\delta \end{cases}, \ (\lambda,\mu) \in \mathrm{PG}(1,q^n)$$

and

$$\mathcal{S}_L^* = \{\Sigma_{\alpha',\beta',\gamma',\delta'} : (\alpha',\beta',\gamma',\delta') \in \mathrm{PG}(3,q^n)\} \text{ with}$$

$$\Sigma_{\alpha',\beta',\gamma',\delta'} : \alpha' x_1 + \beta' x_2 + \gamma' x_3 + \delta' x_4 = 0.$$

The collineation $\Phi$ is given by $\Phi(l_{\alpha,\beta,\gamma,\delta}) = \Sigma_{\alpha',\beta',\gamma',\delta'}$, with

$$(\alpha',\beta',\gamma',\delta')^t = A'\left[(\alpha,\beta,\gamma,\delta)^t\right]^\sigma,$$

where $A'$ is the matrix obtained by $A$ by deleting the last row and the first column. Note that $|A'| \neq 0$ since $\mathrm{rk}(A) = 4$. It is easy to see that $\Gamma$ is the set of points of intersection of corresponding elements under the collineation $\Phi$.
If $Y = (y_1,y_2,y_3,y_4,y_5)$ is a point of $\Gamma \setminus \{R\}$, then the tangent hyperplane to $\Gamma$ at the point $Y$ is the hyperplane $\Sigma_Y = Y^\top$ with equation $X^t A Y^\sigma = 0$. It follows that, for every point $Y$ of $\Gamma \setminus \{R\}$ the hyperplane $\Sigma_Y$ contains the point $L$. The tangent hyperplane $\Sigma_L = L^\top$ to $\Gamma$ at the point $L$ is the hyperplane with equation $X^t A L^\sigma = 0$, that is:

$$\Sigma_L : a_{15}x_1 + a_{25}x_2 + a_{35}x_3 + a_{45}x_4 = 0.$$

In what follows, we will assume that $\Phi$ maps the lines of a fixed projective frame of $\mathcal{S}_R$ into the hyperplanes of a fixed projective frame of $\mathcal{S}_L^*$. We again distinguish some cases.

- First assume that $\Sigma_L$ contains the line $RL$. We have the following:

  **Proposition 4.1.1.** *Let $\Gamma : X^t A X^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(4,q^n)$, with $\mathrm{rk}(A) = 4$, $L \neq R$ and $RL \subset \Sigma_L$, then one of the folowing holds:*

  *i)* $\Gamma$ *is a* degenerate elliptic $\sigma$-quadric of $\mathrm{PG}(4,q^n)$ with collinear vertex points $R$ and $L$. *Then $\Gamma$ satisfies $x_1x_4^\sigma + x_2^{\sigma+1} - cx_3^{\sigma+1} + x_4x_5^\sigma = 0$, with $c$ a non square if $q$ is odd and $c^{(q^n-1)/r} \neq 1$, $r = (q^n-1, q^m+1)$, if $q$ and $n$ are even. Moreover, $|\Gamma| = q^{3n} + q^n + 1$ and $\Gamma$ contains only the line $RL$.*

*ii)* $\Gamma$ *is a* degenerate $q^n$-parabolic $\sigma$-quadric of $\mathrm{PG}(4, q^n)$ with collinear vertex points $R$ and $L$. *Then $q$ is even and $\Gamma$ satisfies* $x_1x_4^\sigma + x_2^{\sigma+1} - cx_3^{\sigma+1} + x_4x_5^\sigma = 0$, *where the equation* $x^{\sigma+1} = c$ *has a unique solution. Moreover,* $|\Gamma| = q^{3n} + q^{2n} + q^n + 1$ *and $\Gamma$ contains $q^n$ lines through $R$ and $q^n$ lines through $L$ (beside $RL$).*

*iii)* $\Gamma$ *is a* degenerate $2q^n$-hyperbolic $\sigma$-quadric of $\mathrm{PG}(4, q^n)$ with collinear vertex points $R$ and $L$. *Then $q$ and $n$ are odd and $\Gamma$ satisfies* $x_1x_4^\sigma + x_2^{\sigma+1} - cx_3^{\sigma+1} + x_4x_5^\sigma = 0$, *where* $x^{\sigma+1} = c$ *has exactly two solutions. Moreover,* $|\Gamma| = q^{3n} + 2q^{2n} + q^n + 1$ *and $\Gamma$ contains $2q^n$ lines through $R$ and $2q^n$ lines through $L$ (beside $RL$).*

*iv)* $\Gamma$ *is a* degenerate $(q+1)q^n$-hyperbolic $\sigma$-quadric of $\mathrm{PG}(4, q^n)$ with collinear vertex points $R$ and $L$. *Then $n$ is even and $\Gamma$ satisfies* $x_1x_4^\sigma + x_2^{\sigma+1} - cx_3^{\sigma+1} + x_4x_5^\sigma = 0$, $x^{\sigma+1} = c$ *has exactly $q+1$ solutions. Moreover,* $|\Gamma| = q^{3n} + (q+1)q^{2n} + q^n + 1$ *and $\Gamma$ contains $(q+1)q^n$ lines through $R$ and $(q+1)q^n$ lines through $L$ (beside $RL$).*

*v)* $\Gamma$ *is a* non-degenerate parabolic $\sigma$-quadric of $\mathrm{PG}(4, q^n)$ with collinear vertex points $R$ and $L$. *Then $\Gamma$ satisfies* $-x_2^{\sigma+1} + x_1x_3^\sigma + x_3x_4^\sigma + x_4x_5^\sigma = 0$. *Moreover,* $|\Gamma| = q^{3n} + q^{2n} + q^n + 1$ *and $\Gamma$ contains $q^n$ lines through $R$ and $q^n$ lines through $L$ (beside $RL$).*

*Proof.* Since $\Sigma_L$ contains the line $RL$, w.l.o.g., we may put $\Sigma_L : x_4 = 0$. Hence $a_{15} = a_{25} = a_{35} = 0$ and we can put $a_{45} = 1$, obtaining
$\Gamma : (a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4)x_2^\sigma + (a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + a_{43}x_4)x_3^\sigma + (a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44}x_4)x_4^\sigma + x_4x_5^\sigma = 0$.
With this assumption the collineation $\Phi$ maps the line $RL$ into the hyperplane $\Sigma_L$. Consider now the star $\mathcal{S}_{R,\Sigma_L}$ of lines through $R$ in $\Sigma_L$. We distinguish two cases.

a) Suppose that $\Phi$ maps the lines of $\mathcal{S}_{R,\Sigma_L}$ into the hyperplanes through the line $RL$.
In this case, we can assume that $\Phi$ maps the line $x_3 = x_4 = x_5 = 0$ into the hyperplane $x_2 = 0$, the line $x_2 = x_4 = x_5 = 0$ into the hyperplane $x_3 = 0$ and the line $x_2 = x_3 = x_5 = 0$ into the hyperplane $x_1 = 0$ obtaining

$$\Gamma : \tilde{a}x_1x_4^\sigma + \tilde{b}x_2^{\sigma+1} - \tilde{c}x_3^{\sigma+1} + x_4x_5^\sigma = 0,$$

with $\tilde{a}\tilde{b}\tilde{c} \neq 0$. Then, via the projectivity

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{pmatrix} = \begin{pmatrix} \tilde{a}\tilde{b}^\sigma & 0 & 0 & 0 & 0 \\ 0 & \tilde{b} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \tilde{b} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$\Gamma$ is projectively equivalent to the set satisfying

$$x_1x_4^\sigma + x_2^{\sigma+1} - cx_3^{\sigma+1} + x_4x_5^\sigma = 0,$$

where $c = \tilde{c}\tilde{b}^{\sigma}$. By Corollary 3.1.6, we know that $R^{\perp} \cap \Gamma$ is the union of lines through $R$. Since $R^{\perp}$ has equation $x_4 = 0$, then a line $l_{\alpha,\beta,\gamma,\delta}$ through $R$ is contained in $\Gamma$ if, and only if, $\alpha^{\sigma+1} - c\beta^{\sigma+1} = 0$. Hence, the number of lines through $R$ contained in $\Gamma$, different from the line $RL$, depends on the cardinality of the set $\{l_{x,1,0,y} \in \mathcal{S}_R : x^{\sigma+1} = c\}$, and this is given by $q^n|\{x \in \mathbb{F}_{q^n} : x^{\sigma+1} = c\}|$. Moreover, the number of lines through $L$ contained in $\Gamma$, different from the line $RL$, is equal to $q^n|\{x \in \mathbb{F}_{q^n} : x^{\sigma+1} = c\}|$. Indeed, let $\mathcal{S}_L = \{t_{\alpha,\beta,\gamma,\delta} : (\alpha, \beta, \gamma, \delta) \in \mathrm{PG}(3, q^n)\}$ be the set of lines through $L$, where

$$t_{\alpha,\beta,\gamma,\delta} : \begin{cases} x_1 = \lambda\alpha \\ x_2 = \lambda\beta \\ x_3 = \lambda\gamma \\ x_4 = \lambda\delta \\ x_5 = \mu \end{cases}, (\lambda, \mu) \in \mathrm{PG}(1, q^n).$$

By Corollary 3.1.6, $\Sigma_L \cap \Gamma$ is the union of lines through $L$. Then a line $t_{\alpha,\beta,\gamma,\delta}$ is contained in $\Gamma$ if, and only if, $\beta^{\sigma+1} - c\gamma^{\sigma+1} = 0$. This yields that the number of lines through $L$ contained in $\Gamma$, different from the line $RL$, depends on the cardinality of the set $\{t_{x,y,1,0} \in \mathcal{S}_L : y^{\sigma+1} = c\}$. The number of solutions of the equation $x^{\sigma+1} = c$ is either $0, 1, 2$ or $q + 1$ depending upon $q$ even or odd and $n$ even or odd. We distinguish several cases:

- If $q$ and $n$ are even, then there are either $0$ or $1$ or $q + 1$ solutions giving either $0$ or $q^n$ or $(q+1)q^n$ lines through $R$ (and hence through $L$) contained in $\Gamma$.
- If $q$ is even and $n$ is odd, then there is a unique solution of the equation giving $q^n$ lines through $R$ (and through $L$) contained in $\Gamma$.
- If $q$ is odd and $n$ is even, then there are either $0$ or $q + 1$ solutions of the equation giving either $0$ or $(q + 1)q^n$ lines through $R$ (and through $L$) contained in $\Gamma$.
- If $q$ and $n$ are odd, then there are either $0$ or $2$ solutions of the equation giving either $0$ or $2q^n$ lines through $R$ (and through $L$) contained in $\Gamma$.

In these cases we will call the set $\Gamma$ either an *elliptic* or a *$q^n$-parabolic* or a *$2q^n$-hyperbolic* or a *$(q + 1)q^n$-hyperbolic $\sigma$-quadric with collinear vertex points $R$ and $L$* according as the number of lines through $R$ (different from the line $RL$) contained in $\Gamma$ is either $0$ or $q^n$ or $2q^n$ or $(q + 1)q^n$.
Now, let $l$ a line through $R$. If $l \notin \mathcal{S}_{R,\Sigma_L}$ then $\Phi(l) \not\supset RL$ and so $l \cap \Phi(l)$ is a point. If $l \in \mathcal{S}_{R,\Sigma_L}$ then $l \cap \Phi(l)$ is either the point $R$ or the line $l$. Recalling that $\Gamma$ contains the line $RL$, we get $|\Gamma| = q^{3n} + q^n \cdot q^n|\{x \in \mathbb{F}_{q^n} : x^{\sigma+1} = c\}| + q^n + 1$.

b) Now, suppose that $\Phi$ does not map the lines of $\mathcal{S}_{R,\Sigma_L}$ into the hyperplanes through the line $RL$.
In this case, there exists a hyperplane $\Sigma$ containing $RL$ such that the lines of

the star $\mathcal{S}_{R,\Sigma}$ are mapped, under $\Phi$, into the hyperplanes through $RL$. Hence there is another line through $R$ (together with $RL$) contained in $\Gamma$. In this case we may assume that $\Sigma : x_3 = 0$ and $\Phi$ maps the line $x_3 = x_4 = x_5 = 0$ into the hyperplane $x_2 = 0$, the line $x_2 = x_4 = x_5 = 0$ into the hyperplane $x_1 = 0$, and the line $x_2 = x_3 = x_5 = 0$ into the hyperplane $x_3 = 0$. Hence:

$$\Gamma : -ax_2^{\sigma+1} + bx_1x_3^{\sigma} + cx_3x_4^{\sigma} + x_4x_5^{\sigma} = 0,$$

with $abc \neq 0$. Then, via the projectivity

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{pmatrix} = \begin{pmatrix} ba^{\sigma}(ca^{\sigma})^{-\sigma} & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & ca^{\sigma} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$\Gamma$ is projectively equivalent to the set satisfying

$$-x_2^{\sigma+1} + x_1x_3^{\sigma} + x_3x_4^{\sigma} + x_4x_5^{\sigma} = 0.$$

Since $R^{\perp}$ has equation $x_3 = 0$, then a line $l_{\alpha,\beta,\gamma,\delta}$ through $R$ is contained in $\Gamma$ if, and only if, $\alpha^{\sigma+1} = \gamma\delta^{\sigma}$. Observe that if $\alpha = 0$ then either $\gamma = 0$, which gives the line $RL$, or $\delta = 0$, which gives the line $l_{0,0,1,0}$. So, the number of lines through $R$ contained in $\Gamma$, different from the lines $RL$ and $l_{0,0,1,0}$, depends on the cardinality of the set $\{l_{1,0,x,y} \in \mathcal{S}_R : xy^{\sigma} = 1\}$. A pair $(x,y)$ is a solution of $xy^{\sigma} = 1$ if, and only if, $y = x^{-\sigma^{-1}}$. Hence there are $q^n - 1$ solutions of the equation giving $q^n - 1$ lines through $R$ contained in $\Gamma$. We will call the set $\Gamma$ a *non-degenerate parabolic $\sigma$-quadric with collinear vertex points $R$ and $L$.* □

- Next, assume that $\Sigma_L$ does not contain the line $RL$ (or equivalently $\Phi$ does not map the line $RL$ into a hyperplane through the line $RL$). We have the following:

**Proposition 4.1.2.** *Let $\Gamma : X^tAX^{\sigma} = 0$ be a $\sigma$-quadric of* $\mathrm{PG}(4,q^n)$, *with* $\mathrm{rk}(A) = 4$, *$L \neq R$ and $RL \not\subset \Sigma_L$, then one of the following holds:*

*i)* $\Gamma$ *is a* non-degenerate $\sigma$-quadric of type 1 of $\mathrm{PG}(4,q^n)$ with vertex points $R$ and $L$. *Then $q$ is odd and $\Gamma$ satisfies $x_2^{\sigma+1} + x_3^{\sigma+1} + x_4^{\sigma+1} + x_1x_5^{\sigma} = 0$, where the Kestenband $\sigma$-conic of* $\mathrm{PG}(2,q^n)$ *given by the equation $x^{\sigma+1} + y^{\sigma+1} + z^{\sigma+1} = 0$ has $q^n + 1$ points. Moreover, $|\Gamma| = q^{3n} + q^{2n} + q^n + 1$ and $\Gamma$ contains exactly $q^n + 1$ lines through $R$ and exactly $q^n + 1$ lines through $L$.*

*ii)* $\Gamma$ *is a* non-degenerate $\sigma$-quadric of type 2 of $\mathrm{PG}(4,q^n)$ with vertex points $R$ and $L$. *Then $n$ is even and $\Gamma$ satisfies $x_2^{\sigma+1} + x_3^{\sigma+1} + x_4^{\sigma+1} + x_1x_5^{\sigma} = 0$, where the Kestenband $\sigma$-conic of* $\mathrm{PG}(2,q^n)$ *given by the equation $x^{\sigma+1} + y^{\sigma+1} + z^{\sigma+1} = 0$ has $q^n + 1 + (-q)^{n/2+1}(q-1)$ points. Moreover, $|\Gamma| = q^{3n} + q^{2n} + q^n + q^n(-q)^{n/2+1}(q-1) + 1$ and $\Gamma$ contains exactly $q^n + 1 + (-q)^{n/2+1}(q-1)$ lines through $R$ and exactly $q^n + 1 + (-q)^{n/2+1}(q-1)$ lines through $L$.*

*iii)* $\Gamma$ *is a* non-degenerate $\sigma$-quadric of type 3 of $\mathrm{PG}(4,q^n)$ with vertex points $R$ and $L$. *Then $n$ is even and $\Gamma$ satisfies* $x_2^{\sigma+1}+x_3^{\sigma+1}+cx_4^{\sigma+1}+x_1x_5^{\sigma}=0$, *where the Kestenband $\sigma$-conic of* $\mathrm{PG}(2,q^n)$ *given by the equation* $x^{\sigma+1}+y^{\sigma+1}+cz^{\sigma+1}=0$ *has* $q^n+1+(-q)^{n/2}(q-1)$ *points, with* $c\notin\{x^{q+1}:x\in\mathbb{F}_{q^n}\}$. *Moreover,* $|\Gamma|=q^{3n}+q^{2n}+q^n+q^n(-q)^{n/2}(q-1)+1$ *and $\Gamma$ contains exactly* $q^n+1+(-q)^{n/2}(q-1)$ *lines through $R$ and exactly* $q^n+1+(-q)^{n/2}(q-1)$ *lines through $L$.*

*iv)* $\Gamma$ *is a* non-degenerate $\sigma$-quadric of type 4 of $\mathrm{PG}(4,q^n)$ with vertex points $R$ and $L$. *Then $n$ is even and $\Gamma$ satisfies* $x_2^{\sigma+1}+bx_3^{\sigma+1}+cx_4^{\sigma+1}+x_1x_5^{\sigma}=0$, *where the Kestenband $\sigma$-conic of* $\mathrm{PG}(2,q^n)$ *given by the equation* $x^{\sigma+1}+by^{\sigma+1}+cz^{\sigma+1}=0$ *has* $q^n+1-2(-q)^{n/2}$ *points, with* $b$, $c$, $b/c\notin\{x^{q+1}:x\in\mathbb{F}_{q^n}\}$. *Moreover,* $|\Gamma|=q^{3n}+q^{2n}+q^n-2q^n(-q)^{n/2}+1$ *and $\Gamma$ contains exactly* $q^n+1-2(-q)^{n/2}$ *lines through $R$ and exactly* $q^n+1-2(-q)^{n/2}$ *lines through $L$.*

*Proof.* W.l.o.g. we may put $\Sigma_L : x_1=0$. In this case there is a hyperplane through $R$ (not containing $L$), say $\Sigma_R$, such that the star of lines through $R$ in $\Sigma_R$ is mapped, under $\Phi$, into the hyperplanes through $RL$. We may assume that $\Sigma_R : x_5=0$. Hence $\Phi$ maps the lines $l_{\alpha,\beta,\gamma,0}$ into the hyperplanes $\Sigma_{0,\beta',\gamma',\delta'}$ so we may assume that $\Phi$ maps the line $l_{1,0,0,0}$ into the hyperplanes $\Sigma_{0,1,0,0}$, the line $l_{0,1,0,0}$ into the hyperplanes $\Sigma_{0,0,1,0}$ and the line $l_{0,0,1,0}$ into the hyperplanes $\Sigma_{0,0,0,1}$. Hence the points of $\Gamma$ satisfy the equation

$$\Gamma : \tilde{a}x_2^{\sigma+1}+\tilde{b}x_3^{\sigma+1}+\tilde{c}x_4^{\sigma+1}+x_1x_5^{\sigma}=0,$$

with $abc\neq 0$. Then, via the projectivity

$$\begin{pmatrix} x_1'\\ x_2'\\ x_3'\\ x_4'\\ x_5'\end{pmatrix}=\begin{pmatrix}1&0&0&0&0\\0&\tilde{a}&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&\tilde{a}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}$$

$\Gamma$ is projectively equivalent to the set satisfying

$$x_2^{\sigma+1}+bx_3^{\sigma+1}+cx_4^{\sigma+1}+x_1x_5^{\sigma}=0,$$

where $b=\tilde{b}\tilde{a}^{\sigma}$ and $c=\tilde{c}\tilde{a}^{\sigma}$. Since $R^{\perp}$ has equation $x_5=0$, then a line $l_{\alpha,\beta,\gamma,\delta}$ through $R$ is contained in $\Gamma$ if, and only if, $\alpha^{\sigma+1}+b\beta^{\sigma+1}+c\gamma^{\sigma+1}=0$. Hence, the numbers of lines through $R$ contained in $\Gamma$ depends on the number of points of the Kestenband $\sigma$-conic of $\mathrm{PG}(2,q^n)$ given by the equation $x^{\sigma+1}+by^{\sigma+1}+cz^{\sigma+1}=0$. We distinguish several cases:

- If $q$ and $n$ are odd, then there are $q^n+1$ points giving $q^n+1$ lines through $R$ (and through $L$) contained in $\Gamma$.

- If $q$ is even and $n$ is odd, then there are $q^n + 1$ points giving $q^n + 1$ lines through $R$ (and through $L$) contained in $\Gamma$.
- If $n$ is even, then there are either $q^n + 1 + (-q)^{n/2+1}(q-1)$ or $q^n + 1 + (-q)^{n/2}(q-1)$ or $q^n+1-2(-q)^{n/2}$ points giving either $q^n+1+(-q)^{n/2+1}(q-1)$ or $q^n+1+(-q)^{n/2}(q-1)$ or $q^n+1-2(-q)^{n/2}$ lines through $R$ (and through $L$) contained in $\Gamma$.

In these cases we will call the set $\Gamma$ either *of type* 1 or *of type* 2 or *of type* 3 or *of type* 4 *with vertex points* $R$ *and* $L$ according as the number of lines through $R$ contained in $\Gamma$ is either $q^n+1$ or $q^n+1+(-q)^{n/2+1}(q-1)$ or $q^n+1+(-q)^{n/2}(q-1)$ or $q^n+1-2(-q)^{n/2}$.
Now, let $l$ a line through $R$. If $l$ is not contained in $\Sigma_R$, then $\Phi(l) \not\supset RL$ and so $l \cap \Phi(l)$ is a point. If $l$ is contained in $\Sigma_R$, then $l \cap \Phi(l)$ is either the point $R$ or the line $l$. Recalling that $\Gamma$ does not contain the line $RL$, we get $|\Gamma| = q^{3n} + q^n|\{(x,y,z) \in \mathrm{PG}(2,q^n) : x^{\sigma+1} + by^{\sigma+1} + cz^{\sigma+1} = 0\}| + 1$. $\square$

2) $L = R$.
The following proposition holds:

**Proposition 4.1.3.** *Let* $\Gamma : X^tAX^\sigma = 0$ *be a* $\sigma$*-quadric of* $\mathrm{PG}(5, q^n)$*, with* $\mathrm{rk}(A) = 4$*,* $L = R$*, then* $\Gamma$ *is a cone with vertex the point* $R$ *projecting a* $\sigma$*-quadric of rank* 4 *in a hyperplane* $\Sigma$*, with* $R \notin \Sigma$*.*

*Proof.* We may assume w.l.o.g. that the point $R = L = (1,0,0,0,0)$ is both the left radical and the right radical. It follows that, in this case, $\Gamma$ is a cone with vertex the point $R$ projecting a $\sigma$-quadric of rank 4 in a hyperplane not through $R$. Indeed, since the matrix $A$ has rank four with first column and last row equal to 0, by choosing a hyperplane not through the point $R$, e.g. $\Sigma : x_1 = 0$, we get that the set $\Gamma \cap \Sigma$ is a $\sigma$-quadric of the hyperplane $\Sigma$ with associated matrix of rank 4. $\square$

## 4.2 $\sigma$-quadrics of rank 3 in $\mathrm{PG}(4, q^n)$

Let $\Gamma : X^tAX^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(4, q^n)$ with $\mathrm{rk}(A) = 3$. Therefore the vertices $R$ and $L$ are lines of $\mathrm{PG}(4, q^n)$.

**Proposition 4.2.1.** *Let* $\Gamma : X^tAX^\sigma = 0$ *be a* $\sigma$*-quadric of* $\mathrm{PG}(4, q^n)$*, with* $\mathrm{rk}(A) = 3$*. Then one of the following holds:*

1) *If* $R \cap L = \emptyset$*, then* $\Gamma$ *is one of the following:*

   *i)* *a* degenerate hyperbolic $\sigma$-quadric with skew vertex lines $R$ and $L$ *satisfying the equation* $x_1x_3^\sigma + x_2x_4^\sigma + x_3x_5^\sigma = 0$*. Moreover,* $|\Gamma| = q^{3n} + 2q^{2n} + q^n + 1$ *and* $\Gamma$ *contains* $q^n$ *trasversal lines with* $R$ *and* $L$ *and* $q^{2n}$ *lines incident with* $R$ *and skew with* $L$*.*

*ii)* *a* non-degenerate parabolic $\sigma$-quadric with skew vertex lines $R$ and $L$ *satisfying the equation* $x_3^{\sigma+1}+x_2x_4^\sigma+x_1x_5^\sigma=0$. *Moreover,* $|\Gamma|=q^{3n}+q^{2n}+q^n+1$ *and* $\Gamma$ *is the union of* $q^{2n}+q^n+1$ *lines whose* $q^n+1$ *lines are trasversal with* $R$ *and* $L$ *and* $q^{2n}$ *lines are incident with* $R$ *and skew with* $L$.

2) $R\cap L=\{V\}$ *and* $\Gamma$ *is a cone with vertex the point* $V$ *projecting a* $\sigma$*-quadric of rank* 3 *in a hyperplane* $\Sigma$ *with two (collinear or not) vertex points, with* $V\notin\Sigma$*;*

3) $R=L$ *and* $\Gamma$ *is a cone with vertex the line* $R$ *projecting a Kestenband* $\sigma$*-conic of a plane* $\pi$ *not through* $R$.

*Proof.* Since $\mathrm{rk}(A)=3$, the vertices $L$ and $R$ are lines of $\mathrm{PG}(4,q^n)$. We distinguish three cases:

1) $R\cap L=\emptyset$.
We may assume w.l.o.g. that $R: x_3=x_4=x_5=0$ and $L: x_1=x_2=x_3=0$. Then:

$$\Gamma:(a_{13}x_1+a_{23}x_2+a_{33}x_3)x_3^\sigma+(a_{14}x_1+a_{24}x_2+a_{34}x_3)x_4^\sigma+(a_{15}x_1+a_{25}x_2+a_{35}x_3)x_5^\sigma=0.$$

Let

$$\mathcal{P}_R=\{\pi_{a,b,c}:(a,b,c)\in\mathrm{PG}(2,q^n)\},$$

where

$$\pi_{a,b,c}=\begin{cases}x_1=\lambda\\x_2=\mu\\x_3=\gamma a\\x_4=\gamma b\\x_5=\gamma c\end{cases},(\lambda,\mu,\gamma)\in\mathrm{PG}(2,q^n)$$

and

$$\mathcal{S}_L=\{\Sigma_{a,b,c}:(a,b,c)\in\mathrm{PG}(2,q^n)\},\text{ where }\Sigma_{a,b,c}:ax_1+bx_2+cx_3=0.$$

The $\sigma$-quadric $\Gamma$ is the set of points of $\mathrm{PG}(4,q^n)$ of intersection of corresponding elements under a collineation $\Phi:\mathcal{P}_R\longrightarrow\mathcal{S}_L$. Let $\Sigma_{RL}$ be the hyperplane spanned by the lines $R$ and $L$, it follows that:

$$\Sigma_{RL}:x_3=0.$$

Let $\pi$ the plane through $R$ s.t. $\Sigma_{RL}=\Phi(\pi)$. We distinguish two cases.

a) First assume that $\Sigma_{RL}$ contains the plane $\pi$.
W.l.o.g., we may put $\pi: x_3=x_4=0$. Then $\Phi(\pi_{0,0,1})=\Sigma_{0,0,1}$. We may assume that $\Phi$ maps the plane $\pi_{1,0,0}$ into the hyperplane $\Sigma_{1,0,0}$, the plane $\pi_{0,1,0}$ into the hyperplane $\Sigma_{0,1,0}$ and the plane $\pi_{1,1,1}$ into the hyperplane $\Sigma_{1,1,1}$ and hence a canonical equation of $\Gamma$ in this case is given by

$$\Gamma:x_1x_3^\sigma+x_2x_4^\sigma+x_3x_5^\sigma=0.$$

The set $\Gamma$ is the union of the plane $\pi$ and $q^{2n} + q^n$ lines. Let $P$ be a point of the plane $\pi_{a,b,c} \in \mathcal{P}_R$, then $P = (\lambda, \mu, \gamma a, \gamma b, \gamma c)$. Observe that $P$ belongs to $L$ if, and only if, $\lambda = \mu = a = 0$ and $\gamma \neq 0$. It follows that the plane $\pi_{a,b,c}$ is skew with $L$ if, and only if, $a \neq 0$. Therefore, $\Gamma$ contains $q^n$ lines which are trasversal with $R$ and $L$, and $q^{2n}$ which are incident with $R$ and skew with $L$. Hence $\Gamma$ has $q^{3n}+2q^{2n}+q^n+1$ points. We will call this set a *degenerate hyperbolic $\sigma$-quadric with skew vertex lines $R$ and $L$.*

b) Now assume that $\Sigma_{RL}$ does not contain the plane $\pi$.
It follows that, w.l.o.g., we may put $\pi : x_4 = x_5 = 0$. Then $\Phi(\pi_{1,0,0}) = \Sigma_{0,0,1}$. We may assume that $\Phi$ maps the plane $\pi_{0,1,0}$ into the hyperplane $\Sigma_{0,1,0}$, the plane $\pi_{0,0,1}$ into the hyperplane $\Sigma_{1,0,0}$ and the plane $\pi_{1,1,1}$ into the hyperplane $\Sigma_{1,1,1}$ and hence a canonical equation of $\Gamma$ in this case is given by

$$\Gamma : x_3^{\sigma+1} + x_2 x_4^{\sigma} + x_1 x_5^{\sigma} = 0.$$

The set $\Gamma$ is the union of $q^{2n} + q^n + 1$ lines whose $q^n + 1$ lines are transversal with $R$ and $L$, and $q^{2n}$ lines are incident with $R$ and skew with $L$. Observing that $\pi \cap \Sigma_{RL} = R$, it follows that $\Gamma$ has $q^{3n} + q^{2n} + q^n + 1$ points. We will call this set a *non-degenerate parabolic $\sigma$-quadric with skew vertex lines $R$ and $L$.*

2) $R \cap L = \{V\}$ is a point.
We may assume w.l.o.g. that $R : x_3 = x_4 = x_5 = 0$ and $L : x_2 = x_3 = x_4 = 0$. In this case the $\sigma$-quadric $\Gamma$ is a cone with vertex the point $V$. Since the matrix $A$ has rank three with first two columns and first and last rows equal to 0, by choosing a hyperplane not through the point $V$, e.g. $\Sigma : x_1 = 0$, we get that the set $\Gamma \cap \Sigma$ is a $\sigma$-quadric of the hyperplane $\Sigma$ with associated matrix of rank three and two (collinear or not) vertex points given by $R \cap \Sigma$ and $L \cap \Sigma$. It follows that $\Gamma$ is a cone with vertex the point $V$ projecting a $\sigma$-quadric of rank 3 in a hyperplane not through $V$ with two (collinear or not) vertex points.

3) $R = L$.
We may assume w.l.o.g. that $R = L : x_3 = x_4 = x_5 = 0$. It follows that, in this case, $\Gamma$ is a cone with vertex the line $R$. Since the matrix $A$ has rank three with first two columns and first two rows equal to 0, by choosing a plane not through the line $R$, e.g. $\pi : x_1 = x_2 = 0$, we get that the set $\Gamma \cap \pi$ is a $\sigma$-conic of the plane $\pi$ with associated matrix of rank three. Hence it is a Kestenband $\sigma$-conic of $\pi$. It follows that $\Gamma$ is a cone with vertex the line $R$ projecting a Kestenband $\sigma$-conic in a plane not through $R$. $\square$

**Remark 4.2.2.** Let $\Gamma : X^t A X^{\sigma} = 0$ be a $\sigma$-quadric of $\mathrm{PG}(4, q^2)$ with $\mathrm{rk}(A) = 3$ and $\sigma^2 = 1$. If $R = L$ then $\Gamma$ is a Hermitian cone with vertex the line $R$.

## 4.3 $\sigma$-quadrics of rank 2 in $\mathrm{PG}(4, q^n)$

Let $\Gamma : X^t A X^{\sigma} = 0$ be a $\sigma$-quadric of $\mathrm{PG}(4, q^n)$ with $\mathrm{rk}(A) = 2$. Therefore the vertices $R$ and $L$ are planes of $\mathrm{PG}(4, q^n)$.

**Proposition 4.3.1.** *Let $\Gamma : X^tAX^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(5,q^n)$, with $\mathrm{rk}(A) = 2$, then one of the following holds:*

1) *$R \cap L = \{V\}$ is a point and $\Gamma$ is a cone with vertex the point $V$ projecting a $\sigma$-quadric of pseudoregulus type in a hyperplane $\Sigma$ with skew vertex lines, with $V \notin \Sigma$;*

2) *$R \cap L = t$ is a line and $\Gamma$ is a cone with vertex the line $t$ projecting a (possibly degenerate) $C_F^\sigma$-set of a plane $\pi$, with $t \cap \pi = \emptyset$;*

3) *$R = L$ and $\Gamma$ is a cone with vertex the plane $R$ projecting a $\sigma$-quadric of a line $v$, with $v \cap \pi = \emptyset$. Hence $\Gamma$ is either the plane $\pi$ or one, two or $q+1$ hyperplanes through $\pi$ forming an $\mathbb{F}_q$-subpencil.*

*Proof.* Since $\mathrm{rk}(A) = 2$, the vertices $L$ and $R$ are two planes of $\mathrm{PG}(4,q^n)$. We distinguish three cases:

1) $R \cap L = \{V\}$ is a point.
We may assume w.l.o.g. that $R : x_4 = x_5 = 0$ and $L : x_1 = x_2 = 0$. It follows that, in this case, $\Gamma$ is a cone with vertex the point $V$. Since the matrix $A$ has rank two with first three columns and last three rows equal to 0, by choosing a hyperplane not through the point $V$, e.g. $\Sigma : x_3 = 0$, we get that the set $\Gamma \cap \Sigma$ is $\sigma$-quadric of the hyperplane $\Sigma$ with associated matrix of rank 2 and vertices the lines

$$r : x_3 = x_4 = x_5 = 0 \text{ and } l : x_1 = x_2 = x_3 = 0.$$

Hence it is a $\sigma$-quadric of pseudoregulus type of $\Sigma$ with skew vertex lines $r$ and $l$ (see [29] and also Proposition 3.3.2). It follows that $\Gamma$ is a cone with vertex the point $V$ projecting a $\sigma$-quadric of pseudoregulus type in a hyperplane not through $V$ with skew vertex lines.

2) $R \cap L = t$ is a line.
We may assume w.l.o.g. that $R : x_4 = x_5 = 0$, $L : x_3 = x_4 = 0$. In this case the $\sigma$-quadric $\Gamma$ is a cone with vertex the line $t$ projecting a (possibly degenerate) $C_F^\sigma$-set in a plane not through $t$. Indeed let $\pi$ a plane not through the line $t$ and let $A = R \cap \pi$, $B = L \cap \pi$. It follows that $\Gamma \cap \pi$ is a set of points of $\pi$ generated by a collineation between the pencils of lines of $\pi$ with center the points $A$ and $B$ induced by the collineation between the pencil of hyperplanes $\mathcal{P}_R$ and $\mathcal{P}_L$ that is associated to $\Gamma$.

3) $R = L$.
We may assume w.l.o.g. that $R = L : x_4 = x_5 = 0$. In this case the $\sigma$-quadric is a cone with vertex the plane $R$ over a $\sigma$-quadric of a line skew with $R$. That is $\Gamma$ is either the plane $R$ or a hyperplane through $R$ or a pair of distinct hyperplanes through $R$ or $q+1$ hyperplanes through $R$ forming an $\mathbb{F}_q$-subpencil of hyperplanes through $R$. □

## 4.4 $\sigma$-quadrics of rank 1 in $\mathrm{PG}(4,q^n)$

Let $\Gamma : X^tAX^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(4,q^n)$ with $\mathrm{rk}(A) = 3$. Therefore the vertices $R$ and $L$ are hyperplanes of $\mathrm{PG}(4,q^n)$.

**Proposition 4.4.1.** *Let* $\Gamma : X^t A X^\sigma = 0$ *be a* $\sigma$*-quadric* $\Gamma$ *of* $\mathrm{PG}(4,q^n)$*, with* $\mathrm{rk}(A) = 1$*, then one of the following holds:*

1) $L \neq R$ *and* $\Gamma = L \cup R$.

2) $L = R$ *and* $\Gamma = R$.

*Proof.* Since $\mathrm{rk}(A) = 1$, the vertices $L$ and $R$ of $\Gamma$ are two hyperplanes of $\mathrm{PG}(4,q^n)$. We distinguish two cases:
1) $L \neq R$.
We may assume that $R : x_5 = 0$ and $L : x_1 = 0$. Hence $\Gamma : x_1 x_5^\sigma = 0$, that is the union of two different hyperplanes.
2) $L = R$.
We may assume that $R = L : x_5 = 0$. Hence $\Gamma : x_5^{\sigma+1} = 0$, that is a hyperplane of $\mathrm{PG}(4,q^n)$. □

## 4.5 $\sigma$-quadrics of $\mathrm{PG}(4,q)$ and ovoids of $Q(4,q)$

In this final section we will show, as an application of $\sigma$-quadrics of $\mathrm{PG}(4,q)$, that two of the known ovoids of $Q(4,q)$ can be obtained as intersection of a suitable $\sigma$-quadric with $Q(4,q)$.
We recall that an ovoid of $Q(4,q)$ is a set of $q^2+1$ points intersecting any generator in exactly one point. Let $(X_1, X_2, X_3, X_4, X_5)$ denote the homogeneous coordinates of $\mathrm{PG}(4,q)$. Let $Q(4,q) : X_1X_5 + X_2X_4 + X_3^2 = 0$. We can always assume that an ovoid of $Q(4,q)$ contains the points $(1,0,0,0,0)$ and $(0,0,0,0,1)$ and hence it can be written in the following form:

$$\mathcal{O}_4(f) = \{(0,0,0,0,1)\} \cup \{(1, x, y, f(x,y), -y^2 - xf(x,y)) : x, y \in \mathbb{F}_q\}$$

for some function $f : \mathbb{F}_q^2 \longrightarrow \mathbb{F}_q$ with $f(0,0) = 0$; see [85, Chapter 2]. They are rare objects and, beside the classical example given by an elliptic quadric, only three classes are known for $q$ odd, one class for $q$ even and a sporadic example for $q = 3^5$. They have been studied since the end of the '80s also because of their connections with many other important and well studied objects such as semifield flocks of a three dimensional quadratic cone, ovoids of $\mathrm{PG}(3,q)$, eggs of finite projective spaces, translation generalized quadrangles, rank 2 commutative semifields etc. etc. In Table 4.1 the known ovoids of $Q(4,q)$ are listed; see [45, 70, 71, 81].

In the following we show that two classes of ovoids of $Q(4,q)$ are related to $\sigma$-quadrics of $\mathrm{PG}(4,q)$.

**Proposition 4.5.1.** *Let* $n$ *be a non-square of* $\mathbb{F}_q$*,* $q = p^h$*,* $q$ *odd and* $h > 1$*, and let* $\sigma \neq 1$ *be an automorphism of* $\mathbb{F}_q$*. The* $\sigma$*-quadric* $\Gamma$ *of rank* 2 *of* $\mathrm{PG}(4,q)$ *given by the equation* $X_4X_1^\sigma + nX_1X_2^\sigma = 0$ *meets the quadric* $Q(4,q) : X_1X_5 + X_2X_4 + X_3^2 = 0$ *in the union of a Kantor ovoid and a quadratic cone contained in a hyperplane of* $\mathrm{PG}(4,q)$*.*

Table 4.1: Known ovoids of $Q(4,q)$.

| Name | $f(x,y)$ | Restrictions |
|---|---|---|
| *Elliptic quadric* | $-nx$ | $q$ odd, $n$ non-square |
| *Elliptic quadric* | $ax+y$ | $q$ even |
| *Penttila-Williams* | $-x^9-y^{81}$ | $q=3^5$ |
| *Thas-Payne* | $-nx-(n^{-1}x)^{1/9}-y^{1/3}$ | $q=3^h$, $h>2$, $n$ non-square |
| *Ree-Tits slice* | $-x^{2\sigma+3}-y^{\sigma}$ | $q=3^h$, $h>1$, $\sigma=\sqrt{3q}$ |
| *Kantor* | $-nx^{\sigma}$ | $q=p^h$, $q$ odd $h>1$, $\sigma\neq 1$ |
| *Tits* | $x^{\sigma+1}+y^{\sigma}$ | $q=2^{2h+1}$, $\sigma=2^{h+1}$ |

*Proof.* Observe that $\Gamma=\{X_1=0\}\cup\{(1,x,y,-nx^{\sigma},z):x,y,z\in\mathbb{F}_q\}$.
First let $P=(1,x,y,-nx^{\sigma},z)$ be a point in $\Gamma\setminus\{X_1=0\}$. It follows that $P$ belongs to $Q(4,q)$ if, and only if, $z=-y^2+nx^{\sigma+1}$.
Now observe that the intersection $\{X_1=0\}\cap Q(4,q)$ is given by

$$\begin{cases} X_1=0 \\ X_2X_4+X_3^2=0, \end{cases}$$

that is a quadratic cone of the hyperplane $X_1=0$. □

**Proposition 4.5.2.** *Let $q=2^{2h+1}$ and let $\sigma=2^{h+1}$. The $\sigma$-quadric $\Gamma$ of rank 3 of $\mathrm{PG}(4,q)$ given by the equation $X_4X_1^{\sigma}+X_2^{\sigma+1}+X_1X_3^{\sigma}=0$ meets the quadric $Q(4,q)$ : $X_1X_5+X_2X_4+X_3^2=0$ in the union of a Tits ovoid and the line $X_1=X_2=X_3=0$.*

*Proof.* Observe that

$$\Gamma=\{X_1=X_2=X_3=0\}\cup\{(1,x,y,x^{\sigma+1}+y^{\sigma},z):x,y,z\in\mathbb{F}_q\}.$$

First let $P=(1,x,y,x^{\sigma+1}+y^{\sigma},z)$ be a point in $\Gamma\setminus\{X_1=X_2=X_3=0\}$. It follows that $P$ belongs to $Q(4,q)$ if, and only if, $z=y^2+x^{\sigma+2}+xy^{\sigma}$.
Now observe that the quadric $Q(4,q)$ contains the lines $\{X_1=X_2=X_3=0\}$. □

# Chapter 5

# Absolute points of correlations of $\mathrm{PG}(5, q^n)$

In this chapter, we will deal with the five dimensional case. We will divide the $\sigma$-quadrics of $\mathrm{PG}(5, q^n)$ according to the rank of the associated matrix.
Let $V = \mathbb{F}_{q^n}^6$, let $\langle\ ,\ \rangle$ be a degenerate $\sigma$-sesquilinear form and let $\Gamma : X^t A X^\sigma = 0$ be the associated $\sigma$-quadric with $\mathrm{rk}(A) = i \in \{1, 2, 3, 4, 5\}$. We will say that $\Gamma$ is a $\sigma$-quadric of *rank* $i$. We will fix

$$\sigma : x \in \mathbb{F}_{q^n} \mapsto x^{q^m} \in \mathbb{F}_{q^n}$$

with $(n, m) = 1$. As an application, in the last section we will show that $\sigma$-quadrics of $\mathrm{PG}(5, q)$ are related to ovoids of the hyperbolic quadric $Q^+(5, q)$. The results of this chapter have appeared in [31].

## 5.1 $\sigma$-quadrics of rank 5 in $\mathrm{PG}(5, q^n)$

In this section a $\sigma$-quadric $\Gamma$ of $\mathrm{PG}(5, q^n)$ will have equation $X^t A X^\sigma = 0$ with $\mathrm{rk}(A) = 5$. Hence the vertices $L$ and $R$ of $\Gamma$ are two points of $\mathrm{PG}(5, q^n)$. We distinguish two cases.

1) $L \neq R$.
We may assume w.l.o.g. that $R = (1, 0, 0, 0, 0, 0)$ and $L = (0, 0, 0, 0, 0, 1)$. It follows that

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ 0 & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and
$\Gamma : (a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)x_2^\sigma + (a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + a_{43}x_4 + a_{53}x_5)x_3^\sigma +$
$(a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44}x_4 + a_{54}x_5)x_4^\sigma + (a_{15}x_1 + a_{25}x_2 + a_{35}x_3 + a_{45}x_4 + a_{55}x_5)x_5^\sigma +$

$(a_{16}x_1 + a_{26}x_2 + a_{36}x_3 + a_{46}x_4 + a_{56}x_5)x_6^\sigma = 0$.
The degenerate collineation

$$\top : Y \in \mathrm{PG}(5, q^n) \setminus \{R\} \mapsto X^t AY^\sigma = 0 \in \mathrm{PG}(5, q^n)^*$$

associated to the sesquilinear form maps points onto hyperplanes through the point $L$. Points that are on a common line through $R$ are mapped onto the same hyperplane through $L$. Let $\mathcal{S}_R = \{l_{\alpha,\beta,\gamma,\delta,\epsilon} : (\alpha, \beta, \gamma, \delta, \epsilon) \in \mathrm{PG}(4, q^n)\}$ be the set of lines through $R$, where

$$l_{\alpha,\beta,\gamma,\delta,\epsilon} : \begin{cases} x_1 = \lambda \\ x_2 = \mu\alpha \\ x_3 = \mu\beta \\ x_4 = \mu\gamma \\ x_5 = \mu\delta \\ x_6 = \mu\epsilon \end{cases} , (\lambda, \mu) \in \mathrm{PG}(1, q^n)$$

and let $\mathcal{S}_L^* = \{\Lambda_{\alpha',\beta',\gamma',\delta',\epsilon'} : (\alpha', \beta', \gamma', \delta', \epsilon') \in \mathrm{PG}(4, q^n)\}$ be the set of hyperplanes through $L$, where

$$\Lambda_{\alpha',\beta',\gamma',\delta',\epsilon'} : \alpha' x_1 + \beta' x_2 + \gamma' x_3 + \delta' x_4 + \epsilon' x_5 = 0.$$

Therefore $\top$ induces a collineation $\Phi : \mathcal{S}_R \longrightarrow \mathcal{S}_L^*$ given by $\Phi(l_{\alpha,\beta,\gamma,\delta,\epsilon}) = \Lambda_{\alpha',\beta',\gamma',\delta',\epsilon'}$, with

$$(\alpha', \beta', \gamma', \delta', \epsilon')^t = A'(\alpha^\sigma, \beta^\sigma, \gamma^\sigma, \delta^\sigma, \epsilon^\sigma)^t,$$

where $A'$ is the matrix obtained by $A$ by deleting the last row and the first column. Note that $|A'| \neq 0$ since $\mathrm{rk}(A) = 5$. It is easy to see that $\Gamma$ is the set of points of intersection of corresponding elements under the collineation $\Phi$, i.e. a point $P$ belongs to $\Gamma$ if and only if $\{P\} = l_{\alpha,\beta,\gamma,\delta,\epsilon} \cap \Phi(l_{\alpha,\beta,\gamma,\delta,\epsilon})$ for some $l_{\alpha,\beta,\gamma,\delta,\epsilon} \in \mathcal{S}_R$.
If $Y = (y_1, y_2, y_3, y_4, y_5, y_6)$ is a point of $\Gamma \setminus \{R\}$, then the tangent hyperplane to $\Gamma$ at the point $Y$ is the hyperplane $\Lambda_Y = Y^\top$ with equation $X^t AY^\sigma = 0$. It follows that, for every point $Y$ of $\Gamma \setminus \{R\}$ the hyperplane $\Lambda_Y$ contains the point $L$. The tangent hyperplane $\Lambda_L = L^\top$ to $\Gamma$ at the point $L$ is the hyperplane with equation $X^t AL^\sigma = 0$, that is:

$$\Lambda_L : a_{16}x_1 + a_{26}x_2 + a_{36}x_3 + a_{46}x_4 + a_{56}x_5 = 0.$$

In what follows, we will assume that $\Phi$ maps the lines of a fixed projective frame of $\mathcal{S}_R$ into the hyperplanes of a fixed projective frame of $\mathcal{S}_L^*$.

We distinguish two cases.

- First assume that $\Lambda_L$ contains the line $RL$.

  We will prove the following proposition:

  **Proposition 5.1.1.** *Let $\Gamma : X^t AX^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(5, q^n)$, with $\mathrm{rk}(A) = 5$, $R \neq L$ and $RL \subset \Delta_L = L^\top$, then one of the following holds:*

*i)* *$q$ odd and $\Gamma$ satisfies $x_1x_5^\sigma + x_2^{\sigma+1} + x_3^{\sigma+1} + x_4^{\sigma+1} + x_5x_6^\sigma = 0$. Moreover, $\Gamma$ has $q^{4n} + q^{3n} + q^{2n} + q^n + 1$ points and contains exactly $(q^n + 1)q^n$ lines through $R$ (or $L$), beside $RL$.*

*ii)* *$n$ even and $\Gamma$ satisfies $x_1x_5^\sigma + x_2^{\sigma+1} + x_3^{\sigma+1} + x_4^{\sigma+1} + x_5x_6^\sigma = 0$. Moreover, $\Gamma$ has $q^{4n} + q^{3n} + q^{2n} + q^{2n}(-q)^{n/2+1}(q-1) + q^n + 1$ points and contains exactly $(q^n + 1 + (-q)^{n/2+1}(q-1))q^n$ lines through $R$ (or $L$), beside $RL$.*

*iii)* *$n$ even and $\Gamma$ satisfies $x_1x_5^\sigma + x_2^{\sigma+1} + x_3^{\sigma+1} + x_4^{\sigma+1} + x_5x_6^\sigma = 0$. Moreover, $\Gamma$ has $q^{4n} + q^{3n} + q^{2n} + q^{2n}(-q)^{n/2}(q-1) + q^n + 1$ points and contains exactly $(q^n + 1 + (-q)^{n/2}(q-1))q^n$ lines through $R$ (or $L$), beside $RL$.*

*iv)* *$n$ even and $\Gamma$ satisfies $x_1x_5^\sigma + x_2^{\sigma+1} + x_3^{\sigma+1} + x_4^{\sigma+1} + x_5x_6^\sigma = 0$. Moreover, $\Gamma$ has $q^{4n} + q^{3n} + q^{2n} - 2q^{2n}(-q)^{n/2} + q^n + 1$ points and contains exactly $(q^n + 1 - 2(-q)^{n/2})q^n$ lines through $R$ (or $L$), beside $RL$.*

*v)* *either $q$ odd and $c$ a non-square or $q, n$ even and $c^{(q^n-1)/r} \neq 1$, with $r = (q^n - 1, q^m + 1)$, and $\Gamma$ satisfies $x_1x_4^\sigma + x_2^{\sigma+1} - cx_3^{\sigma+1} + x_4x_5^\sigma + x_5x_6^\sigma = 0$. Moreover, $\Gamma$ has $q^{4n} + q^{3n} + q^n + 1$ points and contains exactly $q^{2n} + 1$ lines through $R$ (or $L$), beside $RL$.*

*vi)* *$q$ even and $x^{\sigma+1} = c$ has a unique solution and $\Gamma$ satisfies $x_1x_4^\sigma + x_2^{\sigma+1} - cx_3^{\sigma+1} + x_4x_5^\sigma + x_5x_6^\sigma = 0$. Moreover, $\Gamma$ has $q^{4n} + q^{3n} + q^{2n} + q^n + 1$ points and contains exactly $q^{2n} + q^n + 1$ lines through $R$ (or $L$), beside $RL$.*

*vii)* *$q, n$ odd and $x^{\sigma+1} = c$ has two solutions and $\Gamma$ satisfies $x_1x_4^\sigma + x_2^{\sigma+1} - cx_3^{\sigma+1} + x_4x_5^\sigma + x_5x_6^\sigma = 0$. Moreover, $\Gamma$ has $q^{4n} + q^{3n} + 2q^{2n} + q^n + 1$ points and contains exactly $q^{2n} + 2q^n + 1$ lines through $R$ (or $L$), beside $RL$.*

*viii)* *$n$ even and $x^{\sigma+1} = c$ has $q+1$ solutions and $\Gamma$ satisfies $x_1x_4^\sigma + x_2^{\sigma+1} - cx_3^{\sigma+1} + x_4x_5^\sigma + x_5x_6^\sigma = 0$. Moreover, $\Gamma$ has $q^{4n} + q^{3n} + (q+1)q^{2n} + q^n + 1$ points and contains exactly $q^{2n} + (q+1)q^n + 1$ lines through $R$ (or $L$), beside $RL$.*

*Proof.* Since $RL \subset \Lambda_L$, we get $a_{16} = 0$. Up to projectivity, we may put $\Lambda_L : x_5 = 0$. Hence $a_{26} = a_{36} = a_{46} = 0$ and we can put $a_{56} = 1$, obtaining
$\Gamma : (a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5)x_2^\sigma + (a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + a_{43}x_4 + a_{53}x_5)x_3^\sigma + (a_{14}x_1 + a_{24}x_2 + a_{34}x_3 + a_{44}x_4 + a_{54}x_5)x_4^\sigma + (a_{15}x_1 + a_{25}x_2 + a_{35}x_3 + a_{45}x_4 + a_{55}x_5)x_5^\sigma + x_5x_6^\sigma = 0$.
With this assumption the collineation $\Phi$ maps the line $RL$ into the hyperplane $\Lambda_L$. Consider now the star $\mathcal{S}_{R,\Lambda_L}$ of lines through $R$ in $\Lambda_L$. We distinguish two cases.

I) $\Phi$ maps the lines of $\mathcal{S}_{R,\Lambda_L}$ into the hyperplanes through the line $RL$.
In this case, we can assume that $\Phi$ maps the line $x_3 = x_4 = x_5 = x_6 = 0$ into the hyperplane $x_2 = 0$, the line $x_2 = x_4 = x_5 = x_6 = 0$ into the hyperplane $x_3 = 0$, the line $x_2 = x_3 = x_5 = x_6 = 0$ into the hyperplane $x_4 = 0$ and the line $x_2 = x_3 = x_4 = x_6 = 0$ into the hyperplane $x_1 = 0$ obtaining

$$\Gamma : ax_1x_5^\sigma + bx_2^{\sigma+1} + cx_3^{\sigma+1} + dx_4^{\sigma+1} + x_5x_6^\sigma = 0,$$

with $abcd \neq 0$. Then, via the projectivity

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \\ x'_4 \\ x'_5 \\ x'_6 \end{pmatrix} = \begin{pmatrix} ab^\sigma & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

$\Gamma$ is projectively equivalent to the set satisfying

$$x_1 x_5^\sigma + x_2^{\sigma+1} + \tilde{c} x_3^{\sigma+1} + \tilde{d} x_4^{\sigma+1} + x_5 x_6^\sigma = 0,$$

where $\tilde{c} = cb^\sigma$ and $\tilde{d} = db^\sigma$. By Corollary 3.1.6, we know that $R^\perp \cap \Gamma$ is the union of lines through R. Since $R^\perp$ has equation $x_5 = 0$, then a line $l_{\alpha,\beta,\gamma,\delta,\epsilon}$ through $R$ is contained in $\Gamma$ if, and only if, $\alpha^{\sigma+1} + \tilde{c}\beta^{\sigma+1} + \tilde{d}\gamma^{\sigma+1} = 0$. Hence the numbers of lines through $R$ contained in $\Gamma$, different from the line $RL$, depends on the cardinality of the set

$$\bigcup_{h \in \mathbb{F}_{q^n}} \{(h, x, y, z) : x^{\sigma+1} + \tilde{c} y^{\sigma+1} + \tilde{d} z^{\sigma+1} = 0, (x, y, z) \in \mathrm{PG}(2, q^n)\}.$$

Now, it is easy to see that the cardinality of the set above is also equal to the number of lines through $L$ contained in $\Gamma$, different from the line $RL$. Observe that the set $\mathcal{K} = \{(x, y, z) \in \mathrm{PG}(2, q^n) : x^{\sigma+1} + \tilde{c} y^{\sigma+1} + \tilde{d} z^{\sigma+1} = 0\}$ is a Kestenband $\sigma$-conic of $\mathrm{PG}(2, q^n)$ We distinguish two cases:

- If $n$ is odd, then $\mathcal{K}$ has $q^n + 1$ points.
- If $n$ is even, then $\mathcal{K}$ has either $q^n + 1 + (-q)^{n/2+1}(q-1)$ or $q^n + 1 + (-q)^{n/2}(q-1)$ or $q^n + 1 - 2(-q)^{n/2}$ points.

Now, let $l$ be a line through $R$. If $l \notin \mathcal{S}_{R,\Lambda_L}$, then $\Phi(l) \not\supset RL$ and $l \cap \Phi(l)$ is either the point $R$ or the line $l$. If $l \in \mathcal{S}_{R,\Lambda_L}$, then $l \cap \Phi(l)$ is either a point or the line $l$. Recalling that $\Gamma$ contains the line $RL$, we have $|\Gamma| = q^{4n} + q^n \cdot q^n |\{(x, y, z) \in \mathrm{PG}(2, q^n) : x^{\sigma+1} + \tilde{c} y^{\sigma+1} + \tilde{d} z^{\sigma+1} = 0\}| + q^n + 1$.

II) $\Phi$ does not map the lines of $\mathcal{S}_{R,\Lambda_L}$ into the hyperplanes through the line $RL$. In this case, there exists a hyperplane $\Lambda$ containing $RL$ such that the lines of the star $\mathcal{S}_{R,\Lambda}$ are mapped, under $\Phi$, into the hyperplanes through $RL$. Up to projectivity, we may assume $\Lambda : x_4 = 0$. Hence there is another line through $R$ (together with $RL$) contained in $\Gamma$. In this case, we may assume that $\Phi$ maps the line $x_3 = x_4 = x_5 = x_6 = 0$ into the hyperplane $x_2 = 0$, the line $x_2 = x_4 = x_5 = x_6 = 0$ into the hyperplane $x_3 = 0$, the line $x_2 = x_3 = x_4 = x_6 = 0$ into the hyperplane $x_4 = 0$, and the line $x_2 = x_3 = x_5 = x_6 = 0$ into the hyperplane $x_1 = 0$. Hence:

$$\Gamma : a x_1 x_4^\sigma + b x_2^{\sigma+1} - c x_3^{\sigma+1} + d x_4 x_5^\sigma + x_5 x_6^\sigma = 0,$$

with $abcd \neq 0$. Then, via the projectivity

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \\ x_6' \end{pmatrix} = \begin{pmatrix} ab^\sigma (db^\sigma)^{-\sigma} & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & db^\sigma & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

$\Gamma$ is projectively equivalent to the set satisfying

$$x_1 x_4^\sigma + x_2^{\sigma+1} - \tilde{c} x_3^{\sigma+1} + x_4 x_5^\sigma + x_5 x_6^\sigma = 0,$$

where $\tilde{c} = cb^\sigma$. Since $R^\perp$ has equation $x_4 = 0$, then a line $l_{\alpha,\beta,\gamma,\delta,\epsilon}$ through $R$ is contained in $\Gamma$ if, and only if, $\alpha^{\sigma+1} - \tilde{c}\beta^{\sigma+1} + \delta\epsilon^\sigma = 0$. Hence the numbers of lines through $R$ (and also through $L$) contained in $\Gamma$ depends on the number of points of the $\sigma$-quadric $\Gamma'$ of $\mathrm{PG}(3, q^n)$ given by the equation $\Gamma' : x^{\sigma+1} - \tilde{c} y^{\sigma+1} + zt^\sigma = 0$. We distinguish several cases:

– If $q$ and $n$ are even, then $\Gamma'$ has either $q^{2n}+1$ or $q^{2n}+q^n+1$ or $q^{2n}+(q+1)q^n+1$ points.
– If $q$ is even and $n$ is odd, then $\Gamma'$ has $q^{2n} + q^n + 1$ points.
– If $q$ is odd and $n$ is even, then $\Gamma'$ has either $q^{2n} + 1$ or $q^{2n} + (q + 1)q^n + 1$ points.
– If $q$ and $n$ are odd, then $\Gamma'$ has either $q^{2n} + 1$ or $q^{2n} + 2q^n + 1$ points.

Now, let $l$ be a line through $R$. If $l \notin \mathcal{S}_{R,\Lambda}$, then $\Phi(l) \not\supset l$ and $l \cap \Phi(l)$ is a point. If $l \in \mathcal{S}_{R,\Lambda}$, then $l \cap \Phi(l)$ is either the point R or the line $l$. We have $|\Gamma| = q^{4n} + q^n |\{(x, y, z, t) \in \mathrm{PG}(3, q^n) : x^{\sigma+1} - \tilde{c} y^{\sigma+1} + zt^\sigma = 0\}| + 1$. $\square$

- **OPEN CASE:** $\Lambda_L$ does not contain the line $RL$ (or equivalently $\Phi$ does not map the line $RL$ into a hyperplane through the line $RL$).
Since $RL \not\subset \Lambda_L$, then $a_{16} \neq 0$. Up to projectivity, we may put $\Lambda_L : x_1 = 0$. In this case there is a hyperplane through $R$ (not containing $L$), say $\Lambda_R$, such that the star of lines through $R$ in $\Lambda_R$ is mapped, under $\Phi$, into the hyperplanes through $RL$. We may assume, up to projectivity, $\Lambda_R : x_6 = 0$. Hence $\Phi$ maps the lines $l_{\alpha,\beta,\gamma,\delta,0}$ into the hyperplanes $\Lambda_{0,\beta',\gamma',\delta',\epsilon'}$ so we may assume that $\Phi$ maps the line $l_{1,0,0,0,0}$ into the hyperplanes $\Lambda_{0,1,0,0,0}$, the line $l_{0,1,0,0,0}$ into the hyperplanes $\Lambda_{0,0,1,0,0}$, the line $l_{0,0,1,0,0}$ into the hyperplanes $\Lambda_{0,0,0,1,0}$, and the line $l_{0,0,0,1,0}$ into the hyperplanes $\Lambda_{0,0,0,0,1}$. Hence the points of $\Gamma$ satisfy the equation

$$\Gamma : a x_2^{\sigma+1} + b x_3^{\sigma+1} + c x_4^{\sigma+1} + d x_5^{\sigma+1} + x_1 x_6^\sigma = 0.$$

Since $R^\perp$ has equation $x_6 = 0$, a line $l_{\alpha,\beta,\gamma,\delta,\epsilon}$ through $R$ is contained in $\Gamma$ if, and only if, $a\alpha^{\sigma+1} + b\beta^{\sigma+1} + c\gamma^{\sigma+1} + d\delta^{\sigma+1} = 0$. Hence the numbers of lines through $R$

contained in $\Gamma$ depends on the number of points of the $\sigma$-quadric $\Gamma'$ of $\mathrm{PG}(3,q^n)$ given by the equation $ax^{\sigma+1}+by^{\sigma+1}+cz^{\sigma+1}+dt^{\sigma+1}=0$ with associated matrix of rank four. In this case, we cannot determine the behaviour of the $\sigma$-quadric $\Gamma$ since nothing is known about the set of absolute points of a non-degenerate correlations of $\mathrm{PG}(3,q^n)$, different from a polarity.

2) $L=R$.
The following proposition holds:

**Proposition 5.1.2.** *Let $\Gamma : X^tAX^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(5,q^n)$, with $\mathrm{rk}(A)=5$, $R=L$, then $\Gamma$ is a cone with vertex $R$ and base a $\sigma$-quadric of rank 5 in a hyperplane $\Lambda$, with $R\notin\Lambda$.*

*Proof.* We may assume w.l.o.g. that $R=L=(1,0,0,0,0,0)$. It follows that, in this case, $\Gamma$ is a cone with vertex the point $R$ projecting a $\sigma$-quadric of rank five in a hyperplane not through $R$. Indeed, since the matrix $A$ has rank five with first column and first row equal to 0, by choosing a hyperplane not through the point $R$, e.g. $\Lambda : x_1 = 0$, we get that the set $\Gamma\cap\Lambda$ is a $\sigma$-quadric of the hyperplane $\Lambda$ with associated matrix of rank five. $\square$

## 5.2 $\sigma$-quadrics of rank 4 in $\mathrm{PG}(5,q^n)$

In this section a $\sigma$-quadric $\Gamma$ of $\mathrm{PG}(5,q^n)$ will have equation $X^tAX^\sigma=0$ with $\mathrm{rk}(A)=4$. Hence the vertices $L$ and $R$ of $\Gamma$ are two lines of $\mathrm{PG}(5,q^n)$. We distinguish three cases:

1) $R\cap L=\emptyset$.
We may assume that $R : x_3=x_4=x_5=x_6=0$ and $L : x_1=x_2=x_3=x_4=0$. Then:
$\Gamma : (a_{13}x_1+a_{23}x_2+a_{33}x_3+a_{34}x_4)x_3^\sigma+(a_{14}x_1+a_{24}x_2+a_{34}x_3+a_{44}x_4)x_4^\sigma+(a_{15}x_1+a_{25}x_2+a_{35}x_3+a_{45}x_4)x_5^\sigma+(a_{16}x_1+a_{26}x_2+a_{36}x_3+a_{46}x_4)x_6^\sigma=0$.
Let $\mathcal{P}_R=\{\pi_{a,b,c,d} : (a,b,c,d)\in\mathrm{PG}(3,q^n)\}$ be the set of planes through $R$, where

$$\pi_{a,b,c,d} = \begin{cases} x_1=\lambda \\ x_2=\mu \\ x_3=\gamma a \\ x_4=\gamma b \\ x_5=\gamma c \\ x_6=\gamma d \end{cases} ,(\lambda,\mu,\gamma)\in\mathrm{PG}(2,q^n)$$

and let $\mathcal{S}_L=\{\Lambda_{a,b,c,d} : (a,b,c,d)\in\mathrm{PG}(3,q^n)\}$ be the set of hyperplanes through $L$, where

$$\Lambda_{a,b,c,d} : ax_1+bx_2+cx_3+dx_4=0.$$

The collineation $\Phi$ is given by $\Phi(\pi_{a,b,c,d})=\Lambda_{a',b',c',d'}$, with

$$(a',b',c',d')^t = A'(a^\sigma,b^\sigma,c^\sigma,d^\sigma)^t,$$

where $A'$ is the matrix obtained by $A$ by deleting the first two columns and the last two rows. The $\sigma$-quadric $\Gamma$ is the set of points of $\mathrm{PG}(5, q^n)$ of intersection of corresponding elements under a collineation $\Phi : \mathcal{P}_R \longrightarrow \mathcal{S}_L$, i.e. a point $P$ belongs to $\Gamma$ if and only if $\{P\} = \pi_{a,b,c,d} \cap \Phi(\pi_{a,b,c,d})$ for some $\pi_{a,b,c,d} \in \mathcal{P}_R$. Note that the intersection of corresponding elements under a collineation $\Phi$ can be either a line or a plane.
Let $\pi_{a,b,c,d}$ be a plane through the line $R$. Note that $\pi_{a,b,c,d}$ meets the line $L$ at the point $(0, 0, 0, 0, c, d)$ if, and only if, $a = b = 0$. Now, suppose that $\pi_{a,b,c,d} \cap \Phi(\pi_{a,b,c,d}) = \pi_{a,b,c,d}$ and let $\Sigma : x_3 = x_4 = 0$ be the subspace spans by the lines $R$ and $L$. It follows that $\Sigma \subset \Phi(\pi_{a,b,c,d})$ and so we have $\Phi(\pi_{a,b,c,d}) : c'x_3 + d'x_4 = 0$, with $(c', d') \in \mathrm{PG}(1, q^n)$. Let $\mathcal{P}_{R,\Sigma} = \{\pi_{0,0,c,d} : (c, d) \in \mathrm{PG}(1, q^n)\}$ be the set of planes through the line $R$ contained in $\Sigma$. The following proposition holds:

**Proposition 5.2.1.** *Let $\Gamma : X^tAX^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(5, q^n)$, with $\mathrm{rk}(A) = 4$, $R \cap L = \emptyset$, then one of the following holds:*

*i)* *$\Gamma$ satisfies $x_1x_3^\sigma + x_2x_4^\sigma + x_3x_5^\sigma + x_4x_6^\sigma = 0$. Moreover, $\Gamma$ is the union of $q^n + 1$ planes through $R$ and incident with the line $L$ and $q^{3n} + q^{2n}$ lines each of whose is contained in a plane through $R$ and skew with $L$.*

*ii)* *either $q$ odd and $\alpha$ a non-square or $q, n$ even and $\alpha^{(q^n-1)/r} \neq 1$, with $r = (q^n - 1, q^m + 1)$, and $\Gamma$ satisfies $\alpha x_3^{\sigma+1} - x_4^{\sigma+1} + x_1x_5^\sigma + x_2x_6^\sigma = 0$. Moreover, $\Gamma$ is the union of the line $R$ and $q^{3n}+q^{2n}$ lines each of whose is contained in a plane through $R$. In particular, there exist exactly $q^n + 1$ lines incident with $R$ and $L$.*

*iii)* *$q$ even and $x^{\sigma+1} = \alpha$ has a unique solution and $\Gamma$ satisfies $\alpha x_3^{\sigma+1} - x_4^{\sigma+1} + x_1x_5^\sigma + x_2x_6^\sigma = 0$. Moreover, $\Gamma$ is the union of a plane through $R$ and $q^{3n} + q^{2n}$ lines each of whose is contained in a plane through $R$. In particular, there exist exactly $q^n + 1$ lines incident with $R$ and $L$.*

*iv)* *$q, n$ odd and $x^{\sigma+1} = \alpha$ has two solutions and $\Gamma$ satisfies $\alpha x_3^{\sigma+1} - x_4^{\sigma+1} + x_1x_5^\sigma + x_2x_6^\sigma = 0$. Moreover, $\Gamma$ is the union of two planes through $R$ and $q^{3n} + q^{2n}$ lines each of whose is contained in a plane through $R$. In particular, there exist exactly $q^n + 1$ lines incident with $R$ and $L$.*

*v)* *$n$ even and $x^{\sigma+1} = \alpha$ has $q+1$ solutions and $\Gamma$ satisfies $\alpha x_3^{\sigma+1} - x_4^{\sigma+1} + x_1x_5^\sigma + x_2x_6^\sigma = 0$. Moreover, $\Gamma$ is the union of $q + 1$ planes through $R$ and $q^{3n} + q^{2n}$ lines each of whose is contained in a plane through $R$. In particular, there exist exactly $q^n + 1$ lines incident with $R$ and $L$.*

*vi)* *$\Gamma$ satisfies $x_3^{\sigma+1} + x_1x_4^\sigma + x_2x_5^\sigma + x_4x_6^\sigma = 0$. Moreover, $\Gamma$ is the union of a plane through $R$ and $q^{3n} + q^{2n}$ lines each of whose is contained in a plane through $R$. In particular, there exist exactly $q^n$ lines incident with $R$ and $L$.*

*Proof.* We distinguish two cases.
• First, assume that $\Phi$ maps the planes of $\mathcal{P}_{R,\Sigma}$ into the hyperplanes of $\mathcal{S}_L$ containing the subspace $\Sigma$. W.l.o.g., we may assume that $\Phi$ maps the plane $\pi_{1,0,0,0}$ into the hyperplane $\Lambda_{1,0,0,0}$, the plane $\pi_{0,1,0,0}$ into the hyperplane $\Lambda_{0,1,0,0}$, the plane $\pi_{0,0,1,0}$ into the

hyperplane $\Lambda_{0,0,1,0}$, the plane $\pi_{0,0,0,1}$ into the hyperplane $\Lambda_{0,0,0,1}$ and the plane $\pi_{1,1,1,1}$ into the hyperplane $\Lambda_{1,1,1,1}$. Under these assumptions, it follows that $\Phi$ maps the plane $\pi_{a,b,c,d}$ into the hyperplane $\Lambda_{a^\sigma,b^\sigma,c^\sigma,d^\sigma}$ and hence

$$\Gamma : x_1x_3^\sigma + x_2x_4^\sigma + x_3x_5^\sigma + x_4x_6^\sigma = 0.$$

If $(a,b) \neq (0,0)$, then $\pi_{a,b,c,d} \cap \Lambda_{a^\sigma,b^\sigma,c^\sigma,d^\sigma}$ is a line different from the line $R$. If $a = b = 0$, then $\pi_{0,0,c,d} \subset \Lambda_{0,0,c^\sigma,d^\sigma}$.

• Next, assume that $\Phi$ does not map the planes of $\mathcal{P}_{R,\Sigma}$ into the hyperplanes of $\mathcal{S}_L$ containing the subspace $\Sigma$. Hence, there exists a 3-dimensional subspace $\Sigma'$ through the line $R$ not containing the line $L$ such that the planes of $\mathcal{P}_{R,\Sigma'}$ are mapped, under $\Phi$, into the hyperplanes of $\mathcal{S}_L$ containing $\Sigma$. We again distinguish two cases.

I) $\Sigma' \cap L = \emptyset$.

In this case, we may assume that $\Sigma' : x_5 = x_6 = 0$ and so $\mathcal{P}_{R,\Sigma'} = \{\pi_{a,b,0,0} : (a,b) \in \mathrm{PG}(1,q^n)\}$, and $\Phi$ maps the plane $\pi_{1,0,0,0}$ into the hyperplane $\Lambda_{0,0,1,0}$, the plane $\pi_{0,1,0,0}$ into the hyperplane $\Lambda_{0,0,0,1}$, the plane $\pi_{0,0,1,0}$ into the hyperplane $\Lambda_{1,0,0,0}$, the plane $\pi_{0,0,0,1}$ into the hyperplane $\Lambda_{0,1,0,0}$. Hence:

$$\Gamma : \alpha x_3^{\sigma+1} - x_4^{\sigma+1} + \beta x_1x_5^\sigma + \gamma x_2x_6^\sigma = 0,$$

with $\alpha\beta\gamma \neq 0$. Then, via the projectivity

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \\ x_6' \end{pmatrix} = \begin{pmatrix} \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

$\Gamma$ is projectively equivalent to the set satisfying

$$\alpha x_3^{\sigma+1} - x_4^{\sigma+1} + x_1x_5^\sigma + x_2x_6^\sigma = 0.$$

It follows that $\Phi$ maps the plane $\pi_{a,b,c,d}$ into the hyperplane $\Lambda_{c^\sigma,d^\sigma,\alpha a^\sigma,-b^\sigma}$. If $(c,d) \neq (0,0)$, then $\pi_{a,b,c,d} \cap \Lambda_{c^\sigma,d^\sigma,\alpha a^\sigma,-b^\sigma}$ is a line different from the line $R$. If $c = d = 0$, then $\pi_{a,b,0,0} \cap \Lambda_{0,0,\alpha a^\sigma,-b^\sigma}$ is either the line $R$ or the plane $\pi_{a,b,0,0}$. The number of planes through $R$ contained in $\Gamma$ depends on the number of points of solutions of the equation $x^{\sigma+1} = \alpha$ and hence:

- If $q$ and $n$ are even, then there are either 0 or 1 or $q+1$ solutions.
- If $q$ is even and $n$ is odd, then there is a unique solution.
- If $q$ is odd and $n$ is even, then there are either 0 or $q+1$ solutions.
- If $q$ and $n$ are odd, then there are either 0 or 2 solutions.

II) $\Sigma' \cap L$ is a point.
In this case, we may assume that $\Sigma' : x_4 = x_5 = 0$ and so $\mathcal{P}_{R,\Sigma'} = \{\pi_{a,0,0,d} : (a, d) \in \mathrm{PG}(1, q^n)\}$, and $\Phi$ maps the plane $\pi_{1,0,0,0}$ into the hyperplane $\Lambda_{0,0,1,0}$, the plane $\pi_{0,1,0,0}$ into the hyperplane $\Lambda_{1,0,0,0}$, the plane $\pi_{0,0,1,0}$ into the hyperplane $\Lambda_{0,1,0,0}$, the plane $\pi_{0,0,0,1}$ into the hyperplane $\Lambda_{0,0,0,1}$ and the plane $\pi_{1,1,1,1}$ into the hyperplane $\Lambda_{1,1,1,1}$. Under these assumptions, it follows that $\Phi$ maps the plane $\pi_{a,b,c,d}$ into the hyperplane $\Lambda_{b^\sigma,c^\sigma,a^\sigma,d^\sigma}$ and hence:

$$\Gamma : x_3^{\sigma+1} + x_1 x_4^\sigma + x_2 x_5^\sigma + x_4 x_6^\sigma = 0.$$

If $(b, c) \neq (0, 0)$, then $\pi_{a,b,c,d} \cap \Lambda_{b^\sigma,c^\sigma,a^\sigma,d^\sigma}$ is a line different from the line $R$. If $b = c = 0$, then $\pi_{a,0,0,d} \cap \Lambda_{0,0,a^\sigma,d^\sigma}$ is either the line $R$ or the plane $\pi_{a,0,0,d}$. The plane $\pi_{a,0,0,d}$ is contained in $\Lambda_{0,0,a^\sigma,d^\sigma}$ if, and only if, $a = 0$.

□

2) $R \cap L$ is a point.
The following proposition holds:

**Proposition 5.2.2.** *Let $\Gamma : X^t A X^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(5, q^n)$, with $\mathrm{rk}(A) = 4$, $R \cap L = \{V\}$ is a point, then $\Gamma$ is a cone with vertex the point $V$ projecting a $\sigma$-quadric of rank 4 in a hyperplane $\Lambda$ with two (collinear or not) vertex points, with $V \notin \Lambda$.*

*Proof.* We may assume w.l.o.g. that $R : x_3 = x_4 = x_5 = x_6 = 0$ and $L : x_2 = x_3 = x_4 = x_5 = 0$. In this case the $\sigma$-quadric $\Gamma$ is a cone with vertex the point $V$. Since the matrix $A$ has rank four with first two columns and first and last row equal to 0, by choosing a hyperplane not through the point $V$, e.g. $\Lambda : x_1 = 0$, we get that the set $\Gamma \cap \Lambda$ is a $\sigma$-quadric of the hyperplane $\Lambda$ with associated matrix of rank four and two (collinear or not) vertex points given by $R' = R \cap \Lambda$ and $L' = L \cap \Lambda$. It follows that $\Gamma$ is a cone with vertex the point $V$ projecting a $\sigma$-quadric of rank 4 in a hyperplane not through $V$ with two (collinear or not) vertex points. □

3) $R = L$.
The following proposition holds:

**Proposition 5.2.3.** *Let $\Gamma : X^t A X^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(5, q^n)$, with $\mathrm{rk}(A) = 4$, $R = L$, then $\Gamma$ is a cone with vertex the line $R$ projecting a $\sigma$-quadric of rank 4 in a 3-dimensional subspace $\Sigma$, with $\Sigma \cap R = \emptyset$.*

*Proof.* We may assume w.l.o.g. that $R = L : x_3 = x_4 = x_5 = x_6 = 0$. It follows, that in this case, $\Gamma$ is a cone with vertex the line $R$ projecting a $\sigma$-quadric of rank 4 in a 3-dimensional subspace not through $R$. Indeed, since the matrix $A$ has rank four with first two columns and first two rows equal to 0, by choosing a 3-dimensional subspace not through the line $R$, e.g. $\Sigma : x_1 = x_2 = 0$, we get that the set $\Gamma \cap \Sigma$ is a $\sigma$-quadric of the subspace $\Sigma$ with associated matrix of rank 4. □

## 5.3 $\sigma$-quadrics of rank 3 in $\mathrm{PG}(5, q^n)$

Let $\Gamma : X^t A X^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(5, q^n)$ with $\mathrm{rk}(A) = 3$. Therefore the vertices $R$ and $L$ are planes of $\mathrm{PG}(5, q^n)$.

**Proposition 5.3.1.** *Let $\Gamma : X^t A X^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(5, q^n)$, with $\mathrm{rk}(A) = 3$, then one of the following holds:*

1) *$R \cap L = \emptyset$ and $\Gamma$ satisfies $x_1 x_4^\sigma + x_2 x_5^\sigma + x_3 x_6^\sigma = 0$. Moreover, $\Gamma$ is the union of $q^{2n} + q^n + 1$ planes each of which meets the plane $R$ in a line and the plane $L$ in a point.*

2) *$R \cap L = \{V\}$ is a point and $\Gamma$ is a cone with vertex the point $V$ projecting a $\sigma$-quadric of rank 3 in a hyperplane $\Lambda$ with skew vertex lines, with $V \notin \Lambda$.*

3) *$R \cap L = t$ is a line and $\Gamma$ is a cone with vertex the line $t$ projecting a $\sigma$-quadric of rank 3 in a 3-dimensional subspace $\Sigma$ with two (collinear or not) vertex points, with $\Sigma \cap t = \emptyset$.*

4) *$R = L$ and $\Gamma$ is a cone with vertex the plane $R$ projecting a Kestenband $\sigma$-conic of a plane $\pi$ skew with $R$.*

*Proof.* Since $\mathrm{rk}(A) = 3$, the vertices $L$ and $R$ of $\Gamma$ are two planes of $\mathrm{PG}(5, q^n)$. We distinguish four cases.
1) $R \cap L = \emptyset$.
We may assume w.l.o.g. that $R : x_4 = x_5 = x_6 = 0$ and $L : x_1 = x_2 = x_3 = 0$. Then:

$$\Gamma : (a_{14}x_1 + a_{24}x_2 + a_{34}x_3)x_4^\sigma + (a_{15}x_1 + a_{25}x_2 + a_{35}x_3)x_5^\sigma + (a_{16}x_1 + a_{26}x_2 + a_{36}x_3)x_6^\sigma = 0.$$

Let $\mathcal{P}_R = \{\Sigma_{a,b,c} : (a, b, c) \in \mathrm{PG}(2, q^n)\}$ be the set of three-dimensional subspaces through $R$, where

$$\Sigma_{a,b,c} = \begin{cases} x_1 = \lambda \\ x_2 = \mu \\ x_3 = \rho \\ x_4 = \gamma a \\ x_5 = \gamma b \\ x_6 = \gamma c \end{cases} , (\lambda, \mu, \rho, \gamma) \in \mathrm{PG}(3, q^n)$$

and let $\mathcal{S}_L = \{\Lambda_{a,b,c} : (a, b, c) \in \mathrm{PG}(2, q^n)\}$ be the set of hyperplanes through $L$, where

$$\Lambda_{a,b,c} : ax_1 + bx_2 + cx_3 = 0.$$

The $\sigma$-quadric $\Gamma$ is the set of points of $\mathrm{PG}(5, q^n)$ of intersection of corresponding elements under a collineation $\Phi : \mathcal{P}_R \longrightarrow \mathcal{S}_L$, i.e. a point $P$ belongs to $\Gamma$ if and only if $\{P\} = \Sigma_{a,b,c} \cap \Phi(\Sigma_{a,b,c})$ for some $\Sigma_{a,b,c} \in \mathcal{P}_R$. Observe that $\Sigma_{a,b,c}$ meets the plane $L$ at the point

$P=(0,0,0,a,b,c)$. So the set $\Gamma$ is the union of $q^{2n}+q^n+1$ planes each of which meets the plane $R$ in a line and the plane $L$ in a point.
Up to projectivity, we may assume that:

$$\Phi(\Sigma_{1,0,0})=\Lambda_{1,0,0},\Phi(\Sigma_{0,1,0})=\Lambda_{0,1,0},\Phi(\Sigma_{0,0,1})=\Lambda_{0,0,1},\Phi(\Sigma_{1,1,1})=\Lambda_{1,1,1}.$$

Hence a canonical equation of $\Gamma$ is given by

$$\Gamma: x_1x_4^\sigma+x_2x_5^\sigma+x_3x_6^\sigma=0.$$

2) $R\cap L=\{V\}$ is a point.
We may assume w.l.o.g. that $R: x_4=x_5=x_6=0$ and $L: x_2=x_3=x_4=0$. In this case the $\sigma$-quadric $\Gamma$ is a cone with vertex the point $V$. Since the matrix $A$ has rank three with first three columns and first and last two rows equal to 0, by choosing a hyperplane not through the point $V$, e.g. $\Lambda: x_1=0$, we get that the set $\Gamma\cap\Lambda$ is a $\sigma$-quadric of the hyperplane $\Lambda$ with associated matrix of rank three and skew vertex lines given by $r=R\cap\Lambda$ and $l=L\cap\Lambda$. It follows that $\Gamma$ is a cone with vertex the point $V$ projecting a $\sigma$-quadric of rank 3 in a hyperplane not through $V$ with skew vertex lines.
3) $R\cap L=t$ is a line.
We may assume w.l.o.g. that $R: x_4=x_5=x_6=0$ and $L: x_3=x_4=x_5=0$. In this case the $\sigma$-quadric $\Gamma$ is a cone with vertex the line $t$. Since the matrix $A$ has rank three with first three columns and first, second and last row equal to 0, by choosing a 3-dimensional subspace not through the line $t$, e.g. $\Sigma: x_1=x_2=0$, we get that the set $\Gamma\cap\Sigma$ is a $\sigma$-quadric of the subspace $\Sigma$ with associated matrix of rank three and two (collinear or not) vertex points given by $R'=R\cap\Sigma$ and $L'=L\cap\Sigma$. It follows that $\Gamma$ is a cone with vertex the line $t$ projecting a $\sigma$-quadric of rank 3 in a 3-dimensional subspace not through $t$ with two (collinear or not) vertex points.
4) $R=L$.
We may assume w.l.o.g. that $R=L: x_4=x_5=x_6=0$. It follows that, in this case, $\Gamma$ is a cone with vertex the plane $R$. Since the matrix $A$ has rank three with first three columns and first three rows equal to 0, by choosing a plane skew with the plane $R$, e.g. $\pi: x_1=x_2=x_3=0$, we get that the set $\Gamma\cap\pi$ is a $\sigma$-conic of the plane $\pi$ with associated matrix of rank three. Hence it is a Kestenband $\sigma$-conic of $\pi$. It follows that $\Gamma$ is a cone with vertex the plane $R$ projecting a Kestenband $\sigma$-conic in a plane skew with $R$. $\square$

**Remark 5.3.2.** Let $\Gamma: X^tAX^\sigma=0$ be a $\sigma$-quadric of $\mathrm{PG}(5,q^2)$ with $\mathrm{rk}(A)=3$ and $\sigma^2=1$. If $R=L$ then $\Gamma$ is a Hermitian cone with vertex the plane $R$.

## 5.4 $\sigma$-quadrics of rank 2 in $\mathrm{PG}(5,q^n)$

Let $\Gamma: X^tAX^\sigma=0$ be a $\sigma$-quadric of $\mathrm{PG}(5,q^n)$ with $\mathrm{rk}(A)=2$. Therefore the vertices $R$ and $L$ are three-dimensional subspaces of $\mathrm{PG}(5,q^n)$.

**Proposition 5.4.1.** *Let $\Gamma: X^tAX^\sigma=0$ be a $\sigma$-quadric of $\mathrm{PG}(5,q^n)$, with $\mathrm{rk}(A)=2$, then one of the following holds:*

*1) $R \cap L = t$ is a line and $\Gamma$ is a cone with vertex the line $t$ projecting a $\sigma$-quadric of pseudoregulus type in a 3-dimensional subspace $\Sigma$ with skew vertex lines, with $\Sigma \cap t = \emptyset$;*

*2) $R \cap L = \pi$ is a plane and $\Gamma$ is a cone with vertex the plane $\pi$ projecting a (possibly degenerate) $C_F^\sigma$-set of a plane $\pi'$, with $\pi \cap \pi' = \emptyset$;*

*3) $R = L$ and $\Gamma$ is a cone with vertex the subspace $R$ projecting a $\sigma$-quadric of a line $t$, with $R \cap t = \emptyset$. Hence $\Gamma$ is either the subspace $R$ or one, two or $q+1$ hyperplanes through $R$ forming an $\mathbb{F}_q$-subpencil.*

*Proof.* Since $\mathrm{rk}(A) = 2$, the vertices $L$ and $R$ of $\Gamma$ are two three-dimensional subspaces of $\mathrm{PG}(5, q^n)$. We distinguish three cases.
1) $R \cap L = t$ is a line.
We may assume w.l.o.g. that $R : x_5 = x_6 = 0$ and $L : x_1 = x_2 = 0$. It follows that, in this case, $\Gamma$ is a cone with vertex the line $t$. Since the matrix $A$ has rank two with first four columns and last four rows equal to 0, by choosing a 3-dimensional subspace not through the line $t$, e.g. $\Sigma : x_3 = x_4 = 0$, we get that the set $\Gamma \cap \Sigma$ is $\sigma$-quadric of the subspace $\Sigma$ with associated matrix of rank 2 and vertices the lines

$$r : x_3 = x_4 = x_5 = x_6 = 0 \quad \text{and} \quad l : x_1 = x_2 = x_3 = x_4 = 0.$$

Hence it is a $\sigma$-quadric of pseudoregulus type of $\Sigma$ with skew vertex lines $r$ and $l$ (see [29] and also Proposition 3.3.2). It follows that $\Gamma$ is a cone with vertex the line $t$ projecting a $\sigma$-quadric of pseudoregulus type in a 3-dimensional subspace not through $t$ with skew vertex lines.
2) $R \cap L = \pi$ is a plane.
We may assume w.l.o.g. that $R : x_5 = x_6 = 0$, $L : x_4 = x_5 = 0$. In this case the $\sigma$-quadric $\Gamma$ is a cone with vertex the plane $\pi$ projecting a (possibly degenerate) $C_F^\sigma$-set in a plane skew with $\pi$. Indeed let $\pi'$ a plane skew with the plane $\pi$ and let $R' = R \cap \pi'$, $L' = L \cap \pi'$. It follows that $\Gamma \cap \pi'$ is a set of points of $\pi'$ generated by a collineation between the pencils of lines of $\pi'$ with center the points $R'$ and $L'$ induced by the collineation between the pencil of hyperplanes $\mathcal{P}_R$ and $\mathcal{P}_L$ that is associated to $\Gamma$.
3) $R = L$.
We may assume w.l.o.g. that $R = L : x_5 = x_6 = 0$. In this case the $\sigma$-quadric is a cone with vertex the subspace $R$ over a $\sigma$-quadric of a line skew with $R$. That is $\Gamma$ is either the subspace $R$ or a hyperplane through $R$ or a pair of distinct hyperplanes through $R$ or $q+1$ hyperplanes through $R$ forming an $\mathbb{F}_q$-subpencil of hyperplanes through $R$. □

## 5.5 $\sigma$-quadrics of rank 1 in $\mathrm{PG}(5, q^n)$

Let $\Gamma : X^t A X^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(5, q^n)$ with $\mathrm{rk}(A) = 1$. Therefore the vertices $R$ and $L$ are hyperplanes of $\mathrm{PG}(5, q^n)$.

**Proposition 5.5.1.** *Let $\Gamma : X^t A X^\sigma = 0$ be a $\sigma$-quadric of $\mathrm{PG}(5, q^n)$, with $\mathrm{rk}(A) = 1$, then one of the following holds:*

*1)* $L \neq R$ *and* $\Gamma = L \cup R$.

*2)* $L = R$ *and* $\Gamma = R$.

*Proof.* Since $\mathrm{rk}(A) = 1$, the vertices $L$ and $R$ of $\Gamma$ are two hyperplanes of $\mathrm{PG}(5,q^n)$. We distinguish two cases.
1) $L \neq R$.
We may assume that $R : x_6 = 0$ and $L : x_1 = 0$. Hence $\Gamma : x_1 x_6^\sigma = 0$, that is the union of two different hyperplanes.
2) $L = R$.
We may assume that $R = L : x_6 = 0$. Hence $\Gamma : x_6^{\sigma+1} = 0$, that is a hyperplane of $\mathrm{PG}(5,q^n)$. □

## 5.6 $\sigma$-quadrics of $\mathrm{PG}(5,q)$ and ovoids of $Q^+(5,q)$

Let $(X_0, X_1, X_2, X_3, X_4, X_5)$ denote the homogeneous coordinates of $\mathrm{PG}(5,q)$. In the sequel we fix the following hyperbolic quadric:

$$Q^+(5,q) : X_0X_5 + X_1X_4 + X_2X_3 = 0.$$

We recall that an ovoid of $Q^+(5,q)$ is a set of $q^2+1$ points intersecting any generator in exactly one point. Any ovoid of $Q^+(5,q)$ can be written in the following form:

$$\mathcal{O}_5(f_1,f_2) = \{(1,x,y,f_1(x,y),f_2(x,y),-xf_2(x,y)-yf_1(x,y))\}_{x,y\in\mathbb{F}_q} \cup \{(0,0,0,0,0,1)\}$$

for some functions $f_i : \mathbb{F}_q^2 \longrightarrow \mathbb{F}_q$ with $f_i(0,0) = 0$, $i \in \{1,2\}$.

In this final section we will show, as an application of $\sigma$-quadrics of $\mathrm{PG}(5,q)$, that three of the known ovoids of $Q^+(5,q)$ can be obtained as intersection of two suitable $\sigma$-quadrics with $Q^+(5,q)$. We will also show that one of the known ovoids can be obtained as intersection of two suitable degenerate quadrics with $Q^+(5,q)$. Some known ovoids of $Q^+(5,q)$ are listed in the table below; see [37, 44, 80]. These ovoids are associated to flocks of a quadratic cone of $\mathrm{PG}(3,q)$ and are the union of $q$ conics having a common point.

| Name | $f_1(x,y)$ | $f_2(x,y)$ | Restrictions |
|---|---|---|---|
| *Fisher-Thas-Walker* | $y - x^2$ | $x^3/3$ | $q \equiv -1 \bmod 3$ |
| *Ganley* | $y - dx^3$ | $-n_1x^9 - n_1n_2^2x$ | $p = 3$, $\ell > 2$, $d^2 = n_1n_2$ |
| *Thas* | $y - bx$ | $cx$ | $t^2 + bt + c$ is irreducible over $\mathbb{F}_q$ |
| *Penttila-Williams* | $y + x^{27}$ | $-x^9$ | $p = 3$, $\ell = 5$ |

**Proposition 5.6.1.** *Let* $q = 2^{2k+1}$, *with* $k \geq 1$, *and let* $\sigma$ *be the automorphism of* $\mathbb{F}_q$ *defined by* $x \mapsto x^2$. *Let* $\Gamma_1$ *be a* $\sigma$*-quadric of rank* 2 *of* $\mathrm{PG}(5,q)$ *given by the equation* $X_4X_0^\sigma = X_1^{\sigma+1}/3$, *let* $\Gamma_2$ *be a* $\sigma$*-quadric of rank* 2 *of* $\mathrm{PG}(5,q)$ *given by the equation*

*$(X_3+X_2)X_0^\sigma+X_0X_1^\sigma=0$, and let $Q^+(5,q): X_0X_5+X_1X_4+X_2X_3=0$ be the Klein quadric of $\mathrm{PG}(5,q)$. Then $\Gamma_1\cap\Gamma_2\cap Q^+(5,q)$ is the union of a Fisher-Thas-Walker ovoid and two planes intersecting in a line.*

*Proof.* Observe that $\Gamma_1=\{X_0=X_1=0\}\cup\{(1,x,y,z,x^{\sigma+1}/3,t): x,y,z,t\in\mathbb{F}_q\}$ and $\Gamma_2=\{X_0=0\}\cup\{(1,x,y,y+x^\sigma,z,t): x,y,z,t\in\mathbb{F}_q\}$. A point $P_1=(1,x,y,z,x^{\sigma+1}/3,t)$ of $\Gamma_1\setminus\{X_0=X_1=0\}$ belongs to $Q^+(5,q)$ if, and only if, $t=x^{\sigma+2}/3+yz$. The 3-dimensional subspace $\{X_0=X_1=0\}$ meets the quadric $Q^+(5,q)$ in the union of two planes, say $\pi_1$ and $\pi_2$, given by the equations

$$\pi_1: X_0=X_1=X_2=0 \ \text{ and } \ \pi_2: X_0=X_1=X_3=0.$$

In particular $\pi_1$ and $\pi_2$ have in common the line $X_0=X_1=X_2=X_3=0$. So, we have proved that

$$\Gamma_1\cap Q^+(5,q)=\pi_1\cup\pi_2\cup\{(1,x,y,z,x^{\sigma+1}/3,x^{\sigma+2}/3+yz): x,y,z\in\mathbb{F}_q\}.$$

Now, a point $P_2=(1,x,y,z,x^{\sigma+1}/3,x^{\sigma+2}/3+yz)$ of $(\Gamma_1\cap Q^+(5,q))\setminus(\pi_1\cup\pi_2)$ belongs to $\Gamma_2$ if, and only if, $z=y+x^\sigma$. Finally, $\pi_1\cup\pi_2$ is contained in $\Gamma_2$. The claim follows. □

**Proposition 5.6.2.** *Let $q=3^l$, with $l>2$, let $n_1,n_2$ be non-squares of $\mathbb{F}_q$ and let $d^2=n_1n_2$. Let $\sigma_1$ and $\sigma_2$ be the two automorphisms of $\mathbb{F}_q$ defined by $x\mapsto x^3$ and $x\mapsto x^9$, respectively. Let $\Gamma_1$ be a $\sigma_1$-quadric of rank 2 of $\mathrm{PG}(5,q)$ given by the equation $(X_3-X_2)X_0^{\sigma_1}+dX_0X_1^{\sigma_1}=0$, let $\Gamma_2$ be a $\sigma_2$-quadric of $\mathrm{PG}(5,q)$ of rank 2 given by the equation $(X_4+n_1n_2^2X_1)X_0^{\sigma_2}+n_1X_0X_1^{\sigma_2}=0$, and let $Q^+(5,q): X_0X_5+X_1X_4+X_2X_3=0$ be the Klein quadric of $\mathrm{PG}(5,q)$. Then $\Gamma_1\cap\Gamma_2\cap Q^+(5,q)$ is the union of a Ganley ovoid and a hyperbolic cone contained in a hyperplane of $\mathrm{PG}(5,q)$.*

*Proof.* Observe that $\Gamma_1=\{X_0=0\}\cup\{(1,x,y,y-dx^{\sigma_1},z,t): x,y,z,t\in\mathbb{F}_q\}$ and $\Gamma_2=\{X_0=0\}\cup\{(1,x,y,z,-n_1n_2^2x-n_1x^{\sigma_2},t): x,y,z,t\in\mathbb{F}_q\}$. A point $P_1=(1,x,y,y-dx^{\sigma_1},z,t)$ of $\Gamma_1\setminus\{X_0=0\}$ belongs to $Q^+(5,q)$ if, and only if, $t=-xz-y^2+dyx^{\sigma_1}$. It follows that $\Gamma_1\cap Q^+(5,q)$ is given by

$$\{X_1X_4+X_2X_3=X_0=0\}\cup\{(1,x,y,y-dx^{\sigma_1},z,-xz-y^2+dyx^{\sigma_1}): x,y,z\in\mathbb{F}_q\},$$

where $\{X_1X_4+X_2X_3=X_0=0\}$ is a cone with vertex the point $(0,0,0,0,0,1)$ projecting a hyperbolic quadric in the 3-dimensional subspace $X_0=X_5=0$. Now, a point $P_2=(1,x,y,z,-n_1n_2^2x-n_1x^{\sigma_2},t)$ of $\Gamma_2\setminus\{X_0=0\}$ belongs to $\Gamma_1\cap Q^+(5,q)$ if, and only if, $z=y-dx^{\sigma_1}$ and $t=-y^2+dyx^{\sigma_1}+n_1n_2^2x^2+n_1x^{\sigma_2+1}$. The claim follows. □

**Proposition 5.6.3.** *Let $\sigma_1$ and $\sigma_2$ be the two automorphisms of $\mathbb{F}_q$, $q=3^5$, defined by $x\mapsto x^3$ and $x\mapsto x^9$, respectively. Let $\Gamma_1$ be a $\sigma_1$-quadric of $\mathrm{PG}(5,q)$ of rank 2 given by the equation $(X_3-X_2)X_0^{\sigma_1}+X_0X_4^{\sigma_1}=0$, let $\Gamma_2$ be a $\sigma_2$-quadric of $\mathrm{PG}(5,q)$ of rank 2 given by the equation $X_0X_1^{\sigma_2}+X_4X_0^{\sigma_2}=0$, and let $Q^+(5,q): X_0X_5+X_1X_4+X_2X_3=0$ be the Klein quadric of $\mathrm{PG}(5,q)$. Then $\Gamma_1\cap\Gamma_2\cap Q^+(5,q)$ is the union of a Penttila-Williams ovoid and a hyperbolic cone contained in a hyperplane of $\mathrm{PG}(5,q)$.*

*Proof.* Observe that $\Gamma_1 = \{X_0 = 0\} \cup \{(1, x, y, y - z^{\sigma_1}, z, t) : x, y, z, t \in \mathbb{F}_q\}$ and $\Gamma_2 = \{X_0 = 0\} \cup \{(1, x, y, z, -x^{\sigma_2}, t) : x, y, z, t \in \mathbb{F}_q\}$. A point $P_1 = (1, x, y, y - z^{\sigma_1}, z, t)$ of $\Gamma_1 \setminus \{X_0 = 0\}$ belongs to $Q^+(5,q)$ if, and only if, $t = -xz - y^2 + yz^{\sigma_1}$. It follows that $\Gamma_1 \cap Q^+(5,q)$ is given by

$$\{X_1X_4 + X_2X_3 = X_0 = 0\} \cup \{(1, x, y, y - z^{\sigma_1}, z, -xz - y^2 + yz^{\sigma_1}) : x, y, z \in \mathbb{F}_q\}.$$

Now, a point $P_2 = (1, x, y, z, -x^{\sigma_2}, t)$ of $\Gamma_2 \setminus \{X_0 = 0\}$ belongs to $\Gamma_1 \cap Q^+(5,q)$ if, and only if, $z = y + x^{\sigma_1\sigma_2}$ and $t = x^{\sigma_2+1} - y^2 - yx^{\sigma_1\sigma_2}$. The claim follows. $\square$

**Proposition 5.6.4.** *Let $b, c \in \mathbb{F}_q$, $q = p^l$, such that $X^2 + bX + c$ is irreducible over $\mathbb{F}_q$. Let $\Gamma_1$ be a quadric of $\mathrm{PG}(5,q)$ given by the equation $X_0(X_4 - cX_1) = 0$, let $\Gamma_2$ be a quadric of $\mathrm{PG}(5,q)$ given by the equation $X_0(X_3 - X_2 + bX_1) = 0$, and let $Q^+(5,q)$ : $X_0X_5 + X_1X_4 + X_2X_3 = 0$ be the Klein quadric of $\mathrm{PG}(5,q)$. Then $\Gamma_1 \cap \Gamma_2 \cap Q^+(5,q)$ is the union of a Thas ovoid and a hyperbolic cone contained in a hyperplane of $\mathrm{PG}(5,q)$.*

*Proof.* Observe that $\Gamma_1 \cap \Gamma_2 = \{X_0 = 0\} \cup \{X_4 - cX_1 = X_3 - X_2 + bX_1 = 0\}$ and $Q^+(5,q) \cap \{X_0 = 0\}$ is a cone with vertex the point $(0, 0, 0, 0, 0, 1)$ projecting a hyperbolic quadric in the 3-dimensional subspace $X_0 = X_5 = 0$. Now, a point $P = (1, x, y, y - bx, cx, z)$ of $(\Gamma_1 \cap \Gamma_2) \setminus \{X_0 = 0\}$, with $x, y, z \in \mathbb{F}_q$, belongs to $Q^+(5,q)$ if, and only if, $z = bxy - y^2 - cx^2$. The claim follows. $\square$

# Chapter 6

# Classification of low-degree ovoids of $Q^+(5,q)$

It is well known that to any ovoid of the Klein quadric $Q^+(5,q)$ two bivariate polynomials $f_1(x,y), f_2(x,y)$ can be associated. In this chapter, we classify ovoids of $Q^+(5,q)$ such that $f_1(x,y) = y + g(x)$ and $\max\{\deg(f_1), \deg(f_2)\} < (\frac{1}{6.3}q)^{\frac{3}{13}} - 1$. The results of this chapter can also be found in [9].

## 6.1 Introduction

Let $(X_0, X_1, X_2, X_3, X_4, X_5)$ denote the homogeneous coordinates of $\mathrm{PG}(5,q)$ and by $H_\infty$ the hyperplane at infinity $X_0 = 0$. We fix the following hyperbolic quadric:

$$Q^+(5,q) : X_0X_5 + X_1X_4 + X_2X_3 = 0.$$

An ovoid of $Q^+(5,q)$ is a set of $q^2+1$ points intersecting any generator in exactly one point. Any ovoid of $Q^+(5,q)$ can be written in the following form:

$$\mathcal{O}_5(f_1,f_2) = \{(1,x,y,f_1(x,y),f_2(x,y),-xf_2(x,y)-yf_1(x,y))\}_{x,y\in\mathbb{F}_q} \cup \{(0,0,0,0,0,1)\},$$

for some functions $f_i : \mathbb{F}_q^2 \longrightarrow \mathbb{F}_q$ with $f_i(0,0) = 0$, $i \in \{1,2\}$.

In this chapter we study ovoids of $Q^+(5,q)$ associated to flocks of a quadratic cone of $\mathrm{PG}(3,q)$ (see [80, 83]). These ovoids are the union of $q$ conics sharing a point and this property characterizes the ovoids of $Q^+(5,q)$ associated to flock of a quadratic cone of $\mathrm{PG}(3,q)$. Indeed the following holds:

**Theorem 6.1.1** ([37]). *For each ovoid $\mathcal{O}_5$ of $Q^+(5,q)$, which is the union of $q$ conics sharing a point, there is a flock $\mathcal{F}$ of a three-dimensional quadratic cone such that $\mathcal{O}_5$ is associated to $\mathcal{F}$.*

Let $tr(x) = x + x^p + \cdots + x^{p^{\ell n}-1}$ denote the *absolute trace* of an element $x$ of $\mathbb{F}_{q^n}$, $q = p^\ell$. The known ovoids of $Q^+(5,q)$ contained in a secant hyperplane section or arising from a flock of a three-dimensional quadratic cone are listed in Table 6.1, where $n$ is a non-square of $\mathbb{F}_q$, $q = p^\ell$ odd, $\sigma \neq 1$ is an automorphism of $\mathbb{F}_q$, $a \neq 1$ is an element with $tr(a) = 1$ of $\mathbb{F}_q$, $q$ even, and $n_1, n_2$ are non-squares of $\mathbb{F}_q$, $q = 3^\ell$; see [37, 44, 80].

Table 6.1: Known ovoids of $Q^+(5, q)$.

| Name | $f_1(x, y)$ | $f_2(x, y)$ | Restrictions |
|---|---|---|---|
| *Elliptic quadric* | $y$ | $-nx$ | $q$ odd |
| *Elliptic quadric* | $y$ | $ax + y$ | $q$ even |
| *Kantor* | $y$ | $-nx^\sigma$ | $q$ odd, $\ell > 1$ |
| *Penttila-Williams* | $y$ | $-x^9 - y^{81}$ | $p = 3$, $\ell = 5$ |
| *Thas-Payne* | $y$ | $-nx - (n^{-1}x)^{1/9} - y^{1/3}$ | $p = 3, \ell > 2$ |
| *Ree-Tits slice* | $y$ | $-x^{2\sigma+3} - y^\sigma$ | $p = 3, \ell > 1$,<br>$\ell$ odd, $\sigma = \sqrt{3q}$ |
| *Tits* | $y$ | $x^{\sigma+1} + y^\sigma$ | $p = 2, \ell > 1$<br>$\ell$ odd, $\sigma = \sqrt{2q}$ |
| *Fisher-Thas-Walker* | $y - x^2$ | $x^3/3$ | $q \equiv -1 \bmod 3$ |
| *Kantor-Payne* | $y - \beta x^3$ | $\gamma x^5$ | $p \equiv \pm 2 \bmod 5$<br>$\ell$ odd, $\beta^2 = 5\gamma$ |
| *Law-Penttila* | $y - x^4 - nx^2$ | $-n^{-1}x^9 + x^7$<br>$+n^2x^3 - n^3x$ | $p = 3$ |
| *Ganley* | $y - dx^3$ | $-n_1x^9 - n_1n_2^2x$ | $p = 3$, $\ell > 2$, $d^2 = n_1n_2$ |
| *Thas* | $y - bx$ | $cx$ | $t^2 + bt + c$ is<br>irreducible over $\mathbb{F}_q$ |
| *Kantor* | $y - x^2$ | $\frac{1}{3}x^3 - nx^5 - n^{-1}x$ | $p = 5$ |
| *Penttila-Williams* | $y + x^{27}$ | $-x^9$ | $p = 3$, $\ell = 5$ |

Note that Condition (1.3) reads

$$(x_1 - x_2)(f_2(x_2, y_2) - f_2(x_1, y_1)) + (y_1 - y_2)(f_1(x_2, y_2) - f_1(x_1, y_1)) \neq 0, \qquad (6.1)$$

for every $(x_1, y_1) \neq (x_2, y_2)$ in $\mathbb{F}_q^2$.

In Section 6.2, using (6.1), a hypersurface $\mathcal{S}_{f_1,f_2}$ of $\mathrm{PG}(5, q)$ will be associated to $\mathcal{O}_5(f_1, f_2)$. Investigating the existence of absolutely irreducible components defined over $\mathbb{F}_q$ in $\mathcal{S}_{f_1,f_2}$, we will provide the following classification result, which is also the main achievement of this chapter.

**Theorem 6.1.2.** *Let $f_1(X, Y) = Y + \sum_i a_i X^i$ and $f_2(X, Y) = \sum_{ij} b_{i,j} X^i Y^j$. Suppose that $q > 6.3(\max\{d_1, d_2\} + 1)^{13/3}$, where $d_1 = \deg(f_1), d_2 = \deg(f_2)$. Let $\Psi = \{i : b_{i,0} \neq 0, 0 < i < d_2\}$ and denote*

$$h = \begin{cases} \max(\Psi) & \text{if } \Psi \neq \emptyset; \\ \infty & \text{if } \Psi = \emptyset. \end{cases}$$

*If $\mathcal{O}_5(f_1,f_2)$ is an ovoid of $Q^+(5,q)$ then either $d_2=d_1=1$ or $b_{i,j}=0$ for any $j\neq 0$ and one of the following holds:*

1. $p=2$, $1<d_1<d_2\leq 2d_1-1$, $d_2$ *odd;*
2. $p=3$, $h+1<2d_1<d_2=3^j$ *and* $3\mid d_1$*;*
3. $p=3$, $h+1=2d_1<d_2=3^j$ *and* $3\mid (d_1-1)$*;*
4. $p\geq 3$ *and* $d_2=2d_1-1\leq 11$*;*
5. $p\geq 3$, $h=\infty$, $d_1=1$, $d_2=p^j$, $a_{d_1}=0$, *and* $\mathcal{O}_5(Y,f_2)$ *corresponds to an ovoid* $\mathcal{O}_4(f_2)$ *of* $Q(4,q)$*;*
6. $p>3$ *and* $(p,d_1,d_2,h)\in\{(5,2,5,3),(7,2,7,3),(7,3,7,5)\}$.

**Remark 6.1.3.** Note that when $f_1(X,Y)=Y$ (i.e. case 5. in the theorem above), the classification of $\mathcal{O}_5(Y,f_2)$ follows directly from the classification of $\mathcal{O}_4(f_2)$ of $Q(4,q)$ which has been achieved in [7, Main Theorem]. Therefore, the only examples whose associated polynomials have small degree are the elliptic quadric and the Kantor ovoid.

The other low degree examples in Table 6.1 fall in Case 1. (Fisher-Thas-Walker,Kantor-Payne), Case 2. (Ganley), Case 3. (Law-Penttila), Case 4. (Fisher-Thas-Walker, Kantor-Payne), Caso 6. (Kantor).

## 6.2 Link between ovoids of $Q^+(5,q)$ and algebraic varieties

See [6] for a survey on links between algebraic varieties over finite fields and relevant combinatorial objects.

An algebraic hypersurface $\mathcal{S}$ is an algebraic variety that may be defined by a single implicit equation and having dimension 1 less than the full space. An algebraic hypersurface defined over a field $\mathbb{F}$ is *absolutely irreducible* if the associated polynomial is irreducible over every algebraic extension of $\mathbb{F}$. An absolutely irreducible $\mathbb{F}$-rational component of a hypersurface $\mathcal{S}$, defined by the polynomial $F$, is simply an absolutely irreducible hypersurface such that the associated polynomial has coefficients in $\mathbb{F}$ and it is a factor of $F$. We will make use of the homogeneous equations for algebraic varieties. For a deeper introduction to algebraic varieties we refer the interested reader to [42].

As mentioned in the previous section, an ovoid of $Q^+(5,q): X_0X_5+X_1X_4+X_2X_3=0$ can be written in the form

$$\mathcal{O}_5(f_1,f_2)=\{(1,x,y,f_1(x,y),f_2(x,y),-xf_2(x,y)-yf_1(x,y)\}_{x,y\in\mathbb{F}_q}\cup\{(0,0,0,0,0,1)\},$$

for some functions $f_i$ with $f_i(0,0)=0$, $i\in\{1,2\}$ and

$$(x_1-x_2)(f_2(x_2,y_2)-f_2(x_1,y_1))+(y_1-y_2)(f_1(x_2,y_2)-f_1(x_1,y_1))\neq 0$$

for every $(x_1,y_1)\neq(x_2,y_2)$ in $\mathbb{F}_q^2$. Let $\widetilde{f_i}(X,Y,T)$ be the homogenization of $f_i(X,Y)$ and let $d_i=\deg(f_i)$. In order to obtain non-existence results and partial classifications for the ovoids of $Q^+(5,q)$, we will consider the hypersurface $\mathcal{S}_{f_1,f_2} \subset \mathrm{PG}(4,q)$ defined by $F(X_0,X_1,X_2,X_3,X_4)=0$, where

$$\begin{aligned} F(X_0,X_1,X_2,X_3,X_4) &= (X_1-X_3)\Big(\widetilde{f}_2(X_3,X_4,X_0)-\widetilde{f}_2(X_1,X_2,X_0)\Big)\\ &\quad +(X_2-X_4)\Big(\widetilde{f}_1(X_3,X_4,X_0)-\widetilde{f}_1(X_1,X_2,X_0)\Big). \end{aligned} \tag{6.2}$$

**Lemma 6.2.1.** *The set $\mathcal{O}_5(f_1,f_2)$ is an ovoid of $Q^+(5,q)$ if and only if $\mathcal{S}_{f_1,f_2}$ contains no affine $\mathbb{F}_q$-rational points off the three-dimensional subspace $X_1-X_3=0=X_2-X_4$.*

A fundamental tool to determine the existence of rational points in an algebraic variety over finite fields is the following result by Lang and Weil [57] in 1954, which can be seen as a generalization of the Hasse-Weil bound.

**Theorem 6.2.2** (Lang-Weil Theorem)**.** *Let $\mathcal{V} \subset \mathbb{P}^N(\mathbb{F}_q)$ be an absolutely irreducible variety of dimension $n$ and degree $d$. Then there exists a constant $C$ depending only on $N$, $n$, and $d$ such that*

$$\left|\#\mathcal{V}(\mathbb{F}_q)-\sum_{i=0}^{n} q^i\right| \leq (d-1)(d-2)q^{n-1/2}+Cq^{n-1}, \tag{6.3}$$

*where $\#\mathcal{V}(\mathbb{F}_q)$ denotes the number of $\mathbb{F}_q$-rational points of $\mathcal{V}$.*

Although the constant $C$ was not computed in [57], explicit estimates have been provided for instance in [12, 15, 38, 39, 59, 74] and they have the general shape $C=f(d)$ provided that $q>g(n,d)$, where $f$ and $g$ are polynomials of (usually) small degree. We refer to [15] for a survey on these bounds. Excellent surveys on Hasse-Weil and Lang-Weil type theorems are [36, 38]. We include here the following result by Cafure and Matera [15].

**Theorem 6.2.3** ([15, Theorem 7.1])**.** *Let $\mathcal{V} \subset \mathrm{AG}(n,q)$ be an absolutely irreducible $\mathbb{F}_q$-variety of dimension $r>0$ and degree $\delta$. If $q>2(r+1)\delta^2$, then the following estimate holds:*

$$|\#(\mathcal{V}\cap \mathrm{AG}(n,q))-q^r| \leq (\delta-1)(\delta-2)q^{r-1/2}+5\delta^{13/3}q^{r-1}.$$

For our purposes, the existence of an absolutely irreducible $\mathbb{F}_q$-rational component in $\mathcal{S}_{f_1,f_2}$ is enough to provide asymptotic non-existence results.

**Theorem 6.2.4.** *Let $\mathcal{S}_{f_1,f_2} : F(X_0,X_1,X_2,X_3,X_4)=0$, where $F$ is defined as in (6.2). Suppose that $q>6.3(d+1)^{13/3}$, $\max\{\deg(f_1),\deg(f_2)\}=d$, and $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component $\mathcal{V}$ defined over $\mathbb{F}_q$. Then $\mathcal{O}_5(f_1,f_2)$ is not an ovoid of $Q^+(5,q)$.*

*Proof.* Since $q>6.3(d+1)^{13/3}$, by Theorem 6.2.3, with $\delta=d+1$ and $r=3$, it is readily seen that $\mathcal{V}$ contains more than $q^2$ affine $\mathbb{F}_q$-rational points. This yields the existence of at least one affine $\mathbb{F}_q$-rational point $(a,b,c,d)$ in $\mathcal{V}\subset\mathcal{S}_{f_1,f_2}$ such that $a\neq c$ or $b\neq d$ and therefore $\mathcal{O}_5(f_1,f_2)$ is not an ovoid of $Q^+(5,q)$. □

We also include, for sake of completeness, the following results which will be useful in the sequel.

**Lemma 6.2.5** ([2, Lemma 2.1])**.** *Let $\mathcal{H}, \mathcal{S} \subset \mathrm{PG}(n,q)$ be two projective hypersurfaces. If $\mathcal{H} \cap \mathcal{S}$ has a reduced absolutely irreducible component defined over $\mathbb{F}_q$ then $\mathcal{S}$ has an absolutely irreducible component defined over $\mathbb{F}_q$.*

**Lemma 6.2.6** ([10, Lemma 2.9])**.** *Let $\mathcal{S}$ be a hypersurface containing $O = (0,0,\ldots,0)$ of affine equation $F(X_1, ..., X_n) = 0$, where*

$$F(X_1,\ldots,X_n) = F_d(X_1,\ldots,X_n) + F_{d+1}(X_1,\ldots,X_n) + \cdots ,$$

*with $F_i$ the homogeneous term of degree $i$ of $F(X_1,\ldots,X_n)$ for $i = d, d+1, \ldots$. Let $P$ be an $\mathbb{F}_q$-rational simple point of the variety $F_d(X_1,\ldots,X_n) = 0$. Then there exists an $\mathbb{F}_q$-rational plane $\pi$ through the line $\ell$ joining $O$ and $P$ such that $\pi \cap \mathcal{S}$ has $\ell$ as a non-repeated tangent $\mathbb{F}_q$-rational line at the origin and $\pi \cap \mathcal{S}$ has a non-repeated absolutely irreducible $\mathbb{F}_q$-rational component.*

We will make use a number of times of the following lemma to show that a polynomial $F(X_1,\ldots,X_n)$ is not square in $\overline{\mathbb{F}_q}[X_1,\ldots,X_n]$.

**Lemma 6.2.7.** *Let $F(X_1,\ldots,X_n)$ be a polynomial in $\mathbb{F}_q[X_1,\ldots,X_n]$. Assume that $F(X_1,\ldots,X_n)$ is a square in $\overline{\mathbb{F}_q}[X_1,\ldots,X_n]$. Then, the following hold:*

- *any specialization $X_i = \alpha_i$ of $F(X_1,\ldots,X_n)$ is a square;*
- *the homogeneous term in $F(X_1,\ldots,X_n)$ of highest degree is a square too;*
- *if the homogeneous terms of highest and second highest degree in $F(X_1,\ldots,X_n)$ are $G(X_1,\ldots,X_n)^\ell$ and $H(X_1,\ldots,X_n)$, for some even number $\ell$ then $G(X_1,\ldots,X_n)^{\ell/2}$ divides $H(X_1,\ldots,X_n)$.*

*Proof.* By hypothesis, there is a polynomial $W(X_1,\ldots,X_n)$ such that $F(X_1,\ldots,X_n) = (W(X_1,\ldots,X_n))^2$. It follows immediately that any specialization $X_i = \alpha_i$ of $F(X_1,\ldots,X_n)$ is a square. Now, write

$$W(X_1,\ldots,X_n) = \sum_{i \le m} L_i(X_1,\ldots,X_n),$$

where $m$ denotes the degree of $W(X_1,\ldots,X_n)$ and $L_i(X_1,\ldots,X_n)$ is the homogeneous term of $W(X_1,\ldots,X_n)$ of degree $i$. Then, we have

$$F(X_1,\ldots,X_n) = \sum_{i \le m} (L_i(X_1,\ldots,X_n))^2 + 2\sum_{i \ne j} L_i(X_1,\ldots,X_n) L_j(X_1,\ldots,X_n),$$

and so $F(X_1,\ldots,X_n)$ has degree $2m$. The homogeneous terms of highest and second highest degree in $F(X_1,\ldots,X_n)$ are given by

$$(L_m(X_1,\ldots,X_n))^2 \;\text{ and }\; 2L_m(X_1,\ldots,X_n)L_k(X_1,\ldots,X_n),$$

respectively, where $k = \max\{i : L_i(X_1,\ldots,X_n) \not\equiv 0, i \ne m\}$. The claim follows.

□

## 6.3 General results

In view of Theorem 6.2.4, our goal is to find conditions for which

$$\mathcal{S}_{f_1,f_2} : F(X_0, X_1, X_2, X_3, X_4) = 0,$$

where $F(X_0, X_1, X_2, X_3, X_4)$ is given by

$$(X_2-X_4)\left[\sum_{i,j} a_{ij}\left(X_3^iX_4^j - X_1^iX_2^j\right)X_0^{d-i-j}\right]+(X_1-X_3)\left[\sum_{i,j} b_{ij}\left(X_3^iX_4^j - X_1^iX_2^j\right)X_0^{d-i-j}\right],$$

possesses an absolutely irreducible component defined over $\mathbb{F}_q$. Recall that $f_1(0,0) = 0$ and $f_2(0,0) = 0$, therefore $a_{0,0} = 0$ and $b_{0,0} = 0$.

**Remark 6.3.1.** Note that we can always assume that at least one of $a_{1,0}$ and $b_{0,1}$ is zero. In the following we will always assume $b_{0,1} = 0$.

**Proposition 6.3.2.** *The hypersurface $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible $\mathbb{F}_q$-rational component in the following cases:*

1. *$d_1 = d_2$, $a_{0,d}b_{d,0} = 0$.*
2. *$d = d_1 > d_2$ and there exists $i \neq 0$ such that $a_{i,d-i} \neq 0$.*
3. *$d = d_2 > d_1$ and there exists $i \neq d$ such that $b_{i,d-i} \neq 0$.*

*Proof.* Consider the set $\mathcal{S}' = \mathcal{S}_{f_1,f_2} \cap (X_0 = 0)$.

1. Suppose $d_1 = d_2 = d$. Consider $\mathcal{S}'' = \mathcal{S}' \cap (X_3 = 0)$ given by

$$\begin{aligned} -(X_2 - X_4)\left[\sum_i a_{i,d-i}X_1^iX_2^{d-i}\right] - X_1\left[\sum_i b_{i,d-i}X_1^iX_2^{d-i}\right] & \\ +a_{0,d}(X_2 - X_4)X_4^d + b_{0,d}X_1X_4^d &= 0, \end{aligned}$$

which is of degree 1 in $X_4$ if $a_{0,d} = b_{0,d} = 0$.

If $a_{0,d} = 0$ and $b_{0,d} \neq 0$ then $X_1$ is a non-repeated factor of $\mathcal{S}''$.

Consider $\mathcal{S}'' = S' \cap (X_2 = 0)$ given by

$$\begin{aligned} -X_4\left[\sum_i a_{i,d-i}X_3^iX_4^{d-i}\right] + (X_1 - X_3)\left[\sum_i b_{i,d-i}X_3^iX_4^{d-i}\right] & \\ +a_{d,0}X_4X_1^d - b_{d,0}(X_1 - X_3)X_1^d &= 0, \end{aligned}$$

which is of degree 1 in $X_1$ if $a_{d,0} = b_{d,0} = 0$.

If $b_{d,0} = 0$ and $a_{d,0} \neq 0$ then $X_4$ is a non-repeated factor of $\mathcal{S}''$.

2. $d = d_1 > d_2$. The equation of $\mathcal{S}'$ reads
$$(X_2 - X_4) \sum_{i \le d} a_{i,d-i} \left( X_3^i X_4^{d-i} - X_1^i X_2^{d-i} \right) = 0.$$
Now, $(X_2 - X_4) \nmid G(X_1, X_2, X_3, X_4) = \sum_{i \le d} a_{i,d-i} \left( X_3^i X_4^{d-i} - X_1^i X_2^{d-i} \right)$ since there exists an element $a_{i,d-i} \neq 0$ with $i \neq 0$ and then $(X_2 - X_4)$ is a non-repeated component of $\mathcal{S}'$. By Lemma 6.2.5, $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$ (passing through $H_\infty \cap (X_2 = X_4)$).

3. $d = d_2 > d_1$. Now, the equation of $\mathcal{S}'$ reads
$$(X_1 - X_3) \sum_{i \le d} b_{i,d-i} \left( X_3^i X_4^{d-i} - X_1^i X_2^{d-i} \right) = 0.$$
Now, $(X_1 - X_3) \nmid G(X_1, X_2, X_3, X_4) = \sum_{i \le d} b_{i,d-i} \left( X_3^i X_4^{d-i} - X_1^i X_2^{d-i} \right)$ since there exists an element $b_{i,d-i} \neq 0$ with $i \neq d$ and then $(X_1 - X_3)$ is a non-repeated component of $\mathcal{S}'$. By Lemma 6.2.5, $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$ (passing through $H_\infty \cap (X_1 = X_3)$).

□

In what follows denote the non-negative integer $\max\{j : a_{i,j} \neq 0\}$ by $j_1$, the non-negative integer $\max\{j : b_{i,j} \neq 0\}$ by $j_2$, the non-negative integer $\max\{i : a_{i,j} \neq 0\}$ by $i_1$ and the non-negative integer $\max\{i : b_{i,j} \neq 0\}$ by $i_2$.

**Proposition 6.3.3.** *If $(j_2, j_1) \neq (1, 0)$ and $j_2 > j_1$ then $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.*

*Proof.* Let $F(X_4, X_1, X_2, X_3, X_0)$ be the polynomial
$$(X_2 - X_0) \left[ \sum_{i,j} a_{ij} \left( X_3^i X_0^j - X_1^i X_2^j \right) X_4^{d-i-j} \right] + (X_1 - X_3) \left[ \sum_{i,j} b_{ij} \left( X_3^i X_0^j - X_1^i X_2^j \right) X_4^{d-i-j} \right].$$
The surface $\widetilde{\mathcal{S}} : F(X_4, X_1, X_2, X_3, X_0) = 0$ is projectively equivalent to $\mathcal{S}_{f_1,f_2}$.

We study now the tangent cone $\Omega$ of $\widetilde{\mathcal{S}}$ at the origin, whose equation depends on $j_1$ and $j_2$. We distinguish several cases.

1. $j_1 = 0$.
   (i) $j_2 = 0$. $\Omega$ reads $\sum_i a_{i,0}(X_3^i - X_1^i) X_4^{d-i}$.
   (ii) $j_2 = 1$. $\Omega$ reads $(X_1 - X_3) \sum_i b_{i,1} X_3^i X_4^{d-i-1} + \sum_i a_{i,0}(X_1^i - X_3^i) X_4^{d-i}$.
   (iii) $j_2 \ge 2$. $\Omega$ reads $(X_1 - X_3) \sum_i b_{i,j_2} X_3^i X_4^{d-i-j_2}$.

2. $j_1 \geq 1$.

   (i) $j_2 < j_1 + 1$. $\Omega$ reads $\sum_i a_{i,j_1} X_3^i X_4^{d-i-j_1}$.

   (ii) $j_2 = j_1 + 1$. $\Omega$ reads $\sum_i a_{i,j_1} X_3^i X_4^{d-i-j_1} - (X_1 - X_3) \sum_i b_{i,j_2} X_3^i X_4^{d-i-j_2}$.

   (iii) $j_2 > j_1 + 1$. $\Omega$ reads $(X_1 - X_3) \sum_i b_{i,j_2} X_3^i X_4^{d-i-j_2}$.

If cases (1iii) and (2iii) hold, then $X_1 - X_3 = 0$ is a non-repeated $\mathbb{F}_q$-rational component of such a tangent cone. Also, there exists a non-singular point $P$ for $\Omega$. By Lemma 6.2.6 there exists an $\mathbb{F}_q$-rational plane $\pi$ through the line $\ell$ joining the origin and $P$ such that $\pi \cap \widetilde{\mathcal{S}}$ has $\ell$ as a non-repeated tangent $\mathbb{F}_q$-rational line at the origin and $\pi \cap \widetilde{\mathcal{S}}$ has a non-repeated absolutely irreducible $\mathbb{F}_q$-rational component. Using twice Lemma 6.2.5, one deduces the existence of an absolutely irreducible $\mathbb{F}_q$-rational component in $\widetilde{\mathcal{S}}$ and therefore in $\mathcal{S}_{f_1,f_2}$.

If case (2ii) holds, then $\Omega$ is a rational surface ($X_1$ has degree 1). This means that $\Omega$ contains an absolutely irreducible $\mathbb{F}_q$-rational component of degree 1 in $X_1$. It is readily seen that affine singular points of $\Omega$ satisfy $\sum_i a_{i,j_1} X_3^i X_4^{d-i-j_1} = 0$ and therefore there exists at least a non-singular point $P$ for $\Omega$. The same conclusion as above holds.

□

**Proposition 6.3.4.** *If $(i_1, i_2) \neq (1, 0)$ and $i_1 > i_2$ then $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.*

*Proof.* Let $F(X_3, X_1, X_2, X_0, X_4)$ be the polynomial

$$(X_2-X_4)\left[\sum_{i,j} a_{ij}\left(X_0^i X_4^j - X_1^i X_2^j\right) X_3^{d-i-j}\right] + (X_1-X_0)\left[\sum_{i,j} b_{ij}\left(X_0^i X_4^j - X_1^i X_2^j\right) X_3^{d-i-j}\right].$$

The surface $\widetilde{\mathcal{S}} : F(X_3, X_1, X_2, X_0, X_4) = 0$ is projectively equivalent to $\mathcal{S}_{f_1,f_2}$.

We study now the tangent cone $\Omega$ of $\widetilde{\mathcal{S}}$ at the origin, whose equation depends on $i_1$ and $i_2$. We distinguish several cases.

1. $i_2 = 0$.

   (i) $i_1 = 0$. $\Omega$ reads $\sum_j b_{0,j} (X_4^j - X_2^j) X_3^{d-j}$.

   (ii) $i_1 = 1$. $\Omega$ reads $(X_2 - X_4) \sum_j a_{1,j} X_4^j X_3^{d-j-1} + \sum_j b_{0,j} (X_2^j - X_4^j) X_3^{d-j}$.

   (iii) $i_1 \geq 2$. $\Omega$ reads $(X_2 - X_4) \sum_j a_{i_1,j} X_4^j X_3^{d-j-i_1}$.

2. $i_2 \geq 1$.

   (i) $i_1 < i_2 + 1$. $\Omega$ reads $\sum_j b_{i_2,j} X_4^j X_3^{d-j-i_2}$.

   (ii) $i_1 = i_2 + 1$. $\Omega$ reads $(X_2 - X_4) \sum_j a_{i_1,j} X_4^j X_3^{d-j-i_1} - \sum_j b_{i_2,j} X_4^j X_3^{d-j-i_2}$.

   (iii) $i_1 > i_2 + 1$. $\Omega$ reads $(X_2 - X_4) \sum_j a_{i_1,j} X_4^j X_3^{d-j-i_1}$.

If cases (1iii) and (2iii) hold, then $X_2 - X_4 = 0$ is a non-repeated $\mathbb{F}_q$-rational component of such a tangent cone. Also, there exists a non-singular point $P$ for $\Omega$. By Lemma 6.2.6 there exists an $\mathbb{F}_q$-rational plane $\pi$ through the line $\ell$ joining the origin and $P$ such that $\pi \cap \widetilde{\mathcal{S}}$ has $\ell$ as a non-repeated tangent $\mathbb{F}_q$-rational line at the origin and $\pi \cap \widetilde{\mathcal{S}}$ has a non-repeated absolutely irreducible $\mathbb{F}_q$-rational component. Using twice Lemma 6.2.5, one deduces the existence of an absolutely irreducible $\mathbb{F}_q$-rational component in $\widetilde{\mathcal{S}}$ and therefore in $\mathcal{S}_{f_1,f_2}$.

If case (2ii) holds, then $\Omega$ is a rational surface ($X_2$ has degree 1). This means that $\Omega$ contains an absolutely irreducible $\mathbb{F}_q$-rational component of degree 1 in $X_2$. It is readily seen that affine singular points of $\Omega$ satisfy $\sum_j b_{i_2,j} X_4^j X_3^{d-j-i_2} = 0$ and therefore there exists at least a non-singular point $P$ for $\Omega$. The same conclusion as above holds.

□

**Proposition 6.3.5.** *Let $f_1(X,Y) = \sum_{ij} a_{i,j} X^i Y^j$ and $f_2(X,Y) = \sum_{ij} b_{i,j} X^i Y^j$. Suppose that $q > 6.3(\max\{d_1, d_2\} + 1)^{13/3}$, where $d_1 = \deg(f_1), d_2 = \deg(f_2)$. Let*

$$i_1 = \max\{i : a_{i,j} \neq 0\}, i_2 = \max\{i : b_{i,j} \neq 0\}, j_1 = \max\{j : a_{i,j} \neq 0\}, j_2 = \max\{j : b_{i,j} \neq 0\}.$$

*If $\mathcal{O}_5(f_1, f_2)$ is an ovoid of $Q^+(5,q)$ then $b_{i,j} = 0$ if $j \neq 0$ and one of the following holds:*

1. *$d_1 = d_2$ and $a_{0,d} b_{d,0} \neq 0$;*
2. *$d_1 > d_2$ and $a_{i,d-i} = 0$ for all $i \neq 0$;*
3. *$d_1 < d_2$ and $b_{i,d-i} = 0$ for all $i \neq d$;*
4. *$(j_2, j_1) = (1,0)$ or $j_2 \leq j_1$;*
5. *$(i_1, i_2) = (1,0)$ or $i_1 \leq i_2$.*

## 6.4 Specific $f_1$

In what follows, $f_1(X,Y) = Y + \sum_i a_i X^i$, $f_2(X,Y) = \sum_{i,j} b_{i,j} X^i Y^j$ with $a_i, b_{i,j} \in \mathbb{F}_q$ for all $i, j$, $d_1 = \max\{1, i : a_i \neq 0\}$ and $d_2 = \max\{i + j : b_{i,j} \neq 0\}$. So,

$$\mathcal{S}_{f_1,f_2} : F(X_0,X_1,X_2,X_3,X_4) = (X_2 - X_4)^2 X_0^{d-1} - (X_2 - X_4)\left[\sum_i a_i \left(X_3^i - X_1^i\right) X_0^{d-i}\right] - \\ (X_1 - X_3)\left[\sum_{i,j} b_{ij}\left(X_3^i X_4^j - X_1^i X_2^j\right) X_0^{d-i-j}\right] = 0. \tag{6.4}$$

Let $\Psi = \{i : b_{i,0} \neq 0, 0 < i < d_2\}$ and denote

$$h = \begin{cases} \max(\Psi) & \text{if } \Psi \neq \emptyset; \\ \infty & \text{if } \Psi = \emptyset. \end{cases}$$

**Remark 6.4.1.** Let $d_1 > 1$. If $\mathcal{S}_{f_1,f_2}$ does not contain an absolutely irreducible component defined over $\mathbb{F}_q$, then by Proposition 6.3.2, $d_2 \geq d_1$. Also, if $d_2 > d_1$ then $j_2 \leq j_1$ by Proposition 6.3.3. If $d_2 = d_1 > 1$ then $a_{0,d} = 0$ and so by Proposition 6.3.2, $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$. So we have to deal with the following two cases

- $d_1 < d_2$ and $(j_1, j_2) \in \{(1,0), (1,1)\}$;
- $d = d_1 = d_2 = 1$.

**Remark 6.4.2.** Let

$$F(X_1, \ldots, X_n, Y) = Y^2 + Y f(X_1, \ldots, X_n) + g(X_1, \ldots, X_n) \in \mathbb{F}_q[X_1, \ldots, X_n],$$

$q$ odd. Then $F$ is reducible if and only if there exist $r, s \in \overline{\mathbb{F}_q}[X_1, \ldots, X_n]$ such that $f = r + s$, $g = rs$. Indeed, from

$$Y^2 + Y f(X_1, \ldots, X_n) + g(X_1, \ldots, X_n) = \left(Y + \frac{\alpha_1(X_1, \ldots, X_n)}{\alpha_2(X_1, \ldots, X_n)}\right)\left(Y + \frac{\beta_1(X_1, \ldots, X_n)}{\beta_2(X_1, \ldots, X_n)}\right),$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \overline{\mathbb{F}_q}(X_1, \ldots, X_n)$ and $\gcd(\alpha_1, \alpha_2) = \gcd(\beta_1, \beta_2) = 1$, one deduces $f = \alpha_1/\alpha_2 + \beta_1/\beta_2$, $g = (\alpha_1\beta_1)/(\alpha_2\beta_2)$.

## 6.5 $p = 2$

**Proposition 6.5.1.** *Let $\mathcal{S}_{f_1,f_2}$ be as in (6.4), $d = d_2 > d_1$. If $j_2 = 1$ then $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.*

*Proof.* Let $\mathcal{W} = \mathcal{S}_{f_1,f_2} \cap (X_2 = X_4)$ be defined by

$$\begin{aligned} &(X_1 + X_3)\left[\sum_{i,j} b_{ij} X_2^j X_0^{d-i-j}\left(X_1^i + X_3^i\right)\right] \\ = \;& (X_1 + X_3)\left(X_2 \sum_{0<i\leq d-2} b_{i,1}(X_1^i + X_3^i)X_0^{d-i-1} + \sum_{0<i\leq d} b_{i,0}(X_1^i + X_3^i)X_0^{d-i}\right). \end{aligned}$$

If some of the $b_{i,1}$'s, $0 < i \leq d-2$, is nonzero then $\mathcal{W}$ contains a rational component of degree 1 in $X_2$ and therefore $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$ through it. $\square$

**Proposition 6.5.2.** *Let $\mathcal{S}_{f_1,f_2}$ be as in (6.4), $d = d_2 > d_1$. If $d_1 = 1$ then $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.*

*Proof.* By Proposition 6.5.1, $j_2 = 0$ and therefore $F(X_0, X_1, X_2, X_3, X_4)$ reads

$$(X_2+X_4)^2X_0^{d-1} + a_1(X_1+X_3)(X_2+X_4)X_0^{d-1} + (X_1+X_3)\sum_{0<i\le d} b_{i,0}\left(X_1^i + X_3^i\right)X_0^{d-i}.$$

If $a_1 = 0$ then $\mathcal{S}_{f_1,f_2}$ is clearly a rational surface (in $Z = (X_2+X_4)^2$) and therefore it contains an absolutely irreducible component defined over $\mathbb{F}_q$.

Suppose now $a_1 \neq 0$ and consider the affine part of $\mathcal{S}_{f_1,f_2} : F(1, X_1, X_2, X_3, X_4) = 0$. It can be written as

$$Z^2 + Z + \frac{\sum_{0<i\le d} b_{i,0}\left(X_1^i + X_3^i\right)}{a_1^2(X_1+X_3)}, \tag{6.5}$$

where $Z = (X_2+X_4)/\left(a_1(X_1+X_3)\right)$. In what follows we will prove that the surface $\mathcal{Y}$ defined by the affine equation (6.5) is absolutely irreducible.

Note that the surface $\mathcal{Y}$ is reducible only if there exists $\overline{Z}(X_1,X_3) = \frac{\overline{Z}_1(X_1,X_3)}{\overline{Z}_2(X_1,X_3)}$, with $\overline{Z}_i(X_1,X_3) \in \overline{\mathbb{F}_q}[X_1,X_3]$, such that $(\overline{Z}(X_1,X_3))^2 + \overline{Z}(X_1,X_3) + \frac{\sum_{0<i\le d} b_{i,0}\left(X_1^i+X_3^i\right)}{a_1^2(X_1+X_3)} = 0$. This condition reads

$$(\overline{Z}_1(X_1,X_3))^2 + \overline{Z}_1(X_1,X_3)\overline{Z}_2(X_1,X_3) + (\overline{Z}_2(X_1,X_3))^2\frac{\sum_{0<i\le d} b_{i,0}\left(X_1^i + X_3^i\right)}{a_1^2(X_1+X_3)} = 0. \tag{6.6}$$

Now, $\deg(\overline{Z}_1) > \deg(\overline{Z}_2)$, otherwise $2\deg(\overline{Z}_1) \le \deg(\overline{Z}_1) + \deg(\overline{Z}_2) < 2\deg(\overline{Z}_2) + d - 1$ and (6.6) cannot be satisfied.

Also, the same argument shows that $2\deg(\overline{Z}_1) = 2\deg(\overline{Z}_2) + d - 1$ and therefore the highest homogeneous terms in $(\overline{Z}_1(X_1,X_3))^2$ and in $(\overline{Z}_2(X_1,X_3))^2\frac{\sum_{0<i\le d} b_{i,0}\left(X_1^i+X_3^i\right)}{a_1^2(X_1+X_3)}$ must be equal.

Write $d = 2^j k$, with $j \ge 0$ and $k$ odd. The above argument yields that, in particular,

$$L_d(X_1,X_3) = \frac{\left(X_1^d + X_3^d\right)}{(X_1+X_3)} = \frac{\left(X_1^k + X_3^k\right)^{2^j}}{(X_1+X_3)},$$

the highest homogeneous term in $\frac{\sum_{0<i\le d} b_{i,0}\left(X_1^i+X_3^i\right)}{a_1^2(X_1+X_3)}$, is a square in $\overline{\mathbb{F}_q}[X_1,X_3]$.

- If $j > 0$ then $(X_1+X_3)^{2^j-1}$ is the highest power of $(X_1+X_3)$ dividing $L_d(X_1,X_3)$. Since $2^j - 1$ is odd, $L_d(X_1,X_3)$ is not a square in $\overline{\mathbb{F}_q}[X_1,X_3]$ and the surface $\mathcal{Y}$ is irreducible.
- If $j = 0$, following the same argument as above, if $\mathcal{Y}$ is reducible then $k = 1$, a contradiction to $d > 1$.

This shows that if $d = d_2 > d_1 = 1$ then $\mathcal{Y}$ and therefore $\mathcal{S}_{f_1,f_2}$ is absolutely irreducible or contains an absolutely irreducible component defined over $\mathbb{F}_q$. $\square$

**Proposition 6.5.3.** *Let $\mathcal{S}_{f_1,f_2}$ be as in (6.4), $d = d_2 > d_1 > 1$, $(j_1, j_2) = (1, 0)$. If $d_2 + 1 > 2d_1$ then $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.*

*Proof.* The polynomial $F(X_0, X_1, X_2, X_3, X_4)$ reads

$$(X_2 + X_4)^2 X_0^{d-1} + (X_2 + X_4) \sum_{i \le d_1} a_i(X_1^i + X_3^i) X_0^{d-i} + (X_1 + X_3) \sum_{0<i\le d} b_{i,0} \left(X_1^i + X_3^i\right) X_0^{d-i}.$$

The affine part of $\mathcal{S}_{f_1,f_2}$ can be written as

$$Z^2 + Z \frac{\sum_{i\le d_1} a_i(X_1^i + X_3^i)}{a_{d_1}(X_1^{d_1} + X_3^{d_1})} + (X_1 + X_3) \frac{\sum_{0<i\le d} b_{i,0}(X_1^i + X_3^i)}{a_{d_1}^2 (X_1^{d_1} + X_3^{d_1})^2}, \tag{6.7}$$

where $Z = (X_2 + X_4)/\left(a_{d_1}(X_1^{d_1} + X_3^{d_1})\right)$.

We will prove that the surface $\mathcal{Y}$ defined by the affine equation (6.7) is absolutely irreducible.

Now, the surface $Z^2 + Z \frac{\sum_{i\le d_1} a_i(X_1^i+X_3^i)}{a_{d_1}(X_1^{d_1}+X_3^{d_1})} + (X_1 + X_3) \frac{\sum_{0<i\le d} b_{i,0}(X_1^i+X_3^i)}{a_{d_1}^2(X_1^{d_1}+X_3^{d_1})^2}$ is reducible only if there exists $\overline{Z}(X_1, X_3) = \frac{\overline{Z}_1(X_1,X_3)}{\overline{Z}_2(X_1,X_3)}$, with $\overline{Z}_i(X_1, X_3) \in \overline{\mathbb{F}_q}[X_1, X_3]$, such that

$$\begin{aligned} &(\overline{Z}_1(X_1, X_3))^2 + \overline{Z}_1(X_1, X_3)\overline{Z}_2(X_1, X_3) \frac{\sum_{i\le d_1} a_i(X_1^i + X_3^i)}{a_{d_1}(X_1^{d_1} + X_3^{d_1})} + \\ &\quad +(\overline{Z}_2(X_1, X_3))^2 (X_1 + X_3) \frac{\sum_{0<i\le d} b_{i,0}(X_1^i + X_3^i)}{a_{d_1}^2 (X_1^{d_1} + X_3^{d_1})^2} = 0. \end{aligned} \tag{6.8}$$

Then Equation (6.8) implies that

$$\begin{aligned} &(\overline{Z}_1(X_1, X_3))^2 a_{d_1}^2 (X_1^{d_1} + X_3^{d_1})^2 + \overline{Z}_1(X_1, X_3)\overline{Z}_2(X_1, X_3) a_{d_1}(X_1^{d_1} + X_3^{d_1}) \sum_{i\le d_1} a_i(X_1^i + X_3^i) + \\ &\quad +(\overline{Z}_2(X_1, X_3))^2 (X_1 + X_3) \sum_{0<i\le d} b_{i,0}(X_1^i + X_3^i) = 0. \end{aligned} \tag{6.9}$$

Since $d_2 + 1 > 2d_1$, $\deg(\overline{Z}_1) > \deg(\overline{Z}_2)$, otherwise $2\deg(\overline{Z}_1) + 2d_1 \le \deg(\overline{Z}_1) + \deg(\overline{Z}_2) + 2d_1 < 2\deg(\overline{Z}_2) + d_2 + 1$ and (6.9) cannot be satisfied.

Also, the same argument shows that $2\deg(\overline{Z}_1) + 2d_1 = 2\deg(\overline{Z}_2) + d_2 + 1$ and therefore the highest homogeneous terms in $(\overline{Z}_1(X_1, X_3))^2 a_{d_1}^2 (X_1^{d_1} + X_3^{d_1})^2$ and in $(\overline{Z}_2(X_1, X_3))^2 (X_1 + X_3) \sum_{0<i\le d} b_{i,0}(X_1^i + X_3^i)$ must be equal.

Write $d = d_2 = 2^j k$, with $j \ge 0$ and $k$ odd. The above argument yields that, in particular,

$$L_d(X_1, X_3) = (X_1 + X_3)(X_1^d + X_3^d) = (X_1 + X_3)(X_1^k + X_3^k)^{2^j},$$

the highest homogeneous term in $(X_1 + X_3) \sum_{0<i\le d} b_{i,0}(X_1^i + X_3^i)$, is a square in $\overline{\mathbb{F}_q}[X_1, X_3]$.

- If $j > 0$ then $(X_1 + X_3)^{2^j+1}$ is the highest power of $(X_1 + X_3)$ dividing $L_d(X_1, X_3)$. Since $2^j + 1$ is odd, $L_d(X_1, X_3)$ is not a square in $\overline{\mathbb{F}_q}[X_1, X_3]$ and the surface $\mathcal{Y}$ is irreducible.

- If $j = 0$, following the same argument as above, if $\mathcal{Y}$ is reducible then $k = 1$, a contradiction to $d > 1$.

This shows that if $d = d_2 > d_1 > 1$ then $\mathcal{Y}$ and therefore $\mathcal{S}_{f_1,f_2}$ is absolutely irreducible or contains an absolutely irreducible component defined over $\mathbb{F}_q$. This completes the proof.

□

**Proposition 6.5.4.** *Let $\mathcal{S}_{f_1,f_2}$ be as in (6.4), $d = d_2 > d_1 > 1$, $(j_1, j_2) = (1, 0)$. If $d_2 + 1 \leq 2d_1$ and $d_2$ is even then $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.*

*Proof.* Consider the curve

$$\mathcal{C} : F(1, X_1, X_2, 0, 0) = X_2^2 + X_2 \sum_{i \leq d_1} a_i X_1^i + X_1 \sum_{0<i\leq d} b_{i,0} X_1^i = 0,$$

which is equivalent to

$$\mathcal{C}' : X_2^2 + X_2 + \frac{X_1 \sum_{0<i\leq d} b_{i,0} X_1^i}{\left(\sum_{i\leq d_1} a_i X_1^i\right)^2} = 0.$$

Such a curve is absolutely irreducible if there exists $\eta \in \mathbb{P}^1(\overline{\mathbb{F}_q}) = \overline{\mathbb{F}_q} \cup \{\infty\}$ such that

$$2 \nmid v_\eta \left( \frac{X_1 \sum_{0<i\leq d} b_{i,0} X_1^i}{\left(\sum_{i\leq d_1} a_i X_1^i\right)^2} \right) < 0.$$

Consider $\eta = \infty$. In this case

$$2 \nmid v_\eta \left( \frac{X_1 \sum_{0<i\leq d} b_{i,0} X_1^i}{\left(\sum_{i\leq d_1} a_i X_1^i\right)^2} \right) = d_2 + 1 - 2d_1 < 0$$

since $d_2$ is even.

This shows that $\mathcal{C}'$ and therefore $\mathcal{S}_{f_1,f_2}$ are both absolutely irreducible. □

The following theorem summarizes the results about the even characteristic case.

**Theorem 6.5.5.** *Let $\mathcal{S}_{f_1,f_2}$ be as in (6.4) contains an absolutely irreducible component defined over $\mathbb{F}_q$ unless $1 < d_1 < d_2 \leq 2d_1 - 1$, $d_2$ odd, $j_2 = 0$.*

## 6.6 $p > 2$

**Proposition 6.6.1.** *Let $\mathcal{S}_{f_1,f_2}$ be as in (6.4), and $p > 2$. If $d = d_2 > d_1$ and $(j_1, j_2) = (1,1)$ then $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.*

*Proof.* By Proposition 6.3.2, we can assume that $b_{d-1,1} = 0$. The surface $\mathcal{S}_{f_1,f_2}$ reads

$$(X_2 - X_4)^2 X_0^{d-1} + (X_2 - X_4)\left[\sum_i a_i (X_1^i - X_3^i) X_0^{d-i}\right]$$
$$+(X_1 - X_3)\left[\sum_{i>0} b_{i,0}(X_1^i - X_3^i)X_0^{d-i} + \sum_{0\le i\le d-2} b_{i,1}(X_1^i X_2 - X_3^i X_4)X_0^{d-i-1}\right] = 0$$

Let $\mathcal{W} = \mathcal{S}_{f_1,f_2} \cap (X_2 = X_4)$ be defined by

$$(X_1 - X_3)\left[\sum_{i>0} b_{i,0}(X_1^i - X_3^i)X_0^{d-i} + X_2 \sum_{0<i\le d-2} b_{i,1}(X_1^i - X_3^i)X_0^{d-i-1}\right] = 0.$$

If $b_{i,1} \neq 0$ for some $0 < i \le d-2$, then $\mathcal{W}$ is of degree 1 in $X_2$.
Thus $b_{i,1} = 0$ for $i > 0$ and Remark 6.3.1 yields the claim.

□

**Proposition 6.6.2.** *Let $\mathcal{S}_{f_1,f_2}$ be as in (6.4), $d = d_2 > d_1 = 1$, $(j_1, j_2) = (1, 0)$, and $p > 2$. If $d_2 \neq p^j$, with $j \ge 0$, or $b_{i,j} \neq 0$ for some $(i,j) \neq (d_2, 0)$, or $h = \infty$ and $a_{d_1} = 0$ then $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.*

*Proof.* The surface $\mathcal{S}_{f_1,f_2}$ reads

$$(X_2 - X_4)^2 X_0^{d-1} + a_1(X_1 - X_3)(X_2 - X_4)X_0^{d-1} + (X_1 - X_3)\left[\sum_{i\le d_2} b_{i,0}(X_1^i - X_3^i)X_0^{d-i}\right].$$

If $\mathcal{S}_{f_1,f_2}$ is reducible then by Remark 6.4.2, there exist $r(X_1, X_3)$, $s(X_1, X_3)$ such that

$$\begin{aligned}(X_1 - X_3)\left[\sum_{i\le d_2} b_{i,0}(X_1^i - X_3^i)\right] &= r(X_1, X_3)s(X_1, X_3),\\ a_1(X_1 - X_3) &= r(X_1, X_3) + s(X_1, X_3).\end{aligned}$$

Consider the affine part of $\mathcal{S}_{f_1,f_2}$ and note that the surface is reducible if and only if

$$\Delta(X_1, X_3) = a_1^2(X_1 - X_3)^2 - 4(X_1 - X_3)\sum_{i\le d_2} b_{i,0}(X_1^i - X_3^i)$$

is a square in $\overline{\mathbb{F}}_q[X_1, X_3]$.

Suppose that $\Delta(X_1, X_3)$ is a square in $\overline{\mathbb{F}_q}[X_1, X_3]$. Then, the highest homogeneous term $L_{d_2}(X_1, X_3) = -4b_{d_2,0}(X_1 - X_3)(X_1^{d_2} - X_3^{d_2})$ is a square in $\overline{\mathbb{F}_q}[X_1, X_3]$ too. Write $d_2 = kp^j$, with $p \nmid k$. Then $L_{d_2}(X_1, X_3) = -4b_{d_2,0}(X_1 - X_3)\left(X_1^k - X_3^k\right)^{p^j}$.

- Let $k > 1$. If $L_{d_2}(X_1, X_3)$ is a square in $\overline{\mathbb{F}_q}[X_1, X_3]$ then $R(X_1) = L_{d_2}(X_1, 1) = (X_1 - 1)(X_1^k - 1)^{p^j}$ is a square in $\overline{\mathbb{F}_q}[X_1]$.

  Every root of $R(X_1)$ different from 1 has algebraic multiplicity equal to $p^j$ and $R(X_1)$ cannot be a square in $\overline{\mathbb{F}}_q[X_1]$.

- Let now $k = 1$ and $d_2 = p^j$ for some $j \geq 0$. First, suppose $h < \infty$. Write

$$\begin{aligned} \Delta(X_1, X_3) &= L_{d_2}(X_1, X_3) + L_h(X_1, X_3) + \cdots \\ &= \left((X_1 - X_3)^{(p^j+1)/2} + M(X_1, X_3) + \cdots\right)^2, \end{aligned}$$

  where $M(X_1, X_3)$ has degree $(h + 1) - (p^j + 1)/2$. Now $L_h(X_1, X_3) = 2(X_1 - X_3)^{(p^j+1)/2}M(X_1, X_3)$ equals

$$\begin{cases} -4b_{h,0}(X_1 - X_3)(X_1^h - X_3^h), & \text{if } h > 1; \\ (a_1^2 - 4b_{1,0})(X_1 - X_3)^2, & \text{if } h = 1. \end{cases}$$

  If $h > 1$ then $(X_1 - X_3)^{(p^j-1)/2} \mid (X_1^h - X_3^h)$. Write $h = k_1p^{j_1}$, so that $(X_1^h - X_3^h) = (X_1^{k_1} - X_3^{k_1})^{p^{j_1}}$. Since $(X_1 - X_3)^{p^{j_1}}$ is the highest power of $(X_1 - X_3)$ dividing $(X_1^{k_1} - X_3^{k_1})^{p^{j_1}}$, we must have $(p^j - 1)/2 \leq p^{j_1}$. Since $j_1 \leq j - 1$,

$$p^j - 1 \leq 2p^{j_1} \leq 2p^j/p,$$

  and therefore $p^j \leq \frac{p}{p-2}$. This yields $p^j = 3$, that is $j = 1$ and $p = 3$. Now, $(X_1 - X_3)^4 + \alpha(X_1 - X_3)(X_1^2 - X_3^2) + \beta(X_1 - X_3)^2$, $\alpha, \beta \in \mathbb{F}_q$, is a square in $\overline{\mathbb{F}_q}[X_1, X_3]$ only if $(X_1 - X_3)^2 + \alpha(X_1 + X_3) + \beta$ is a square. By direct checking, this is possible only if $\alpha = \beta = 0$.

  If $h = 1$, in this case $M(X_1, X_3)$ has degree $2 - (p^j + 1)/2$ and this yields $p^j = 3$, a contradiction.

  This shows that if there exists $i$, $0 < i < d_2$, such that $b_{i,0} \neq 0$ then $\Delta(X_1, X_3)$ is not a square in $\overline{\mathbb{F}_q}[X_1, X_3]$.

  If $h = \infty$, $r(X_1, X_3)s(X_1, X_3) = b_{d_2,0}(X_1 - X_3)^{p^j+1}$ and $r(X_1, X_3) + s(X_1, X_3) = a_1(X_1 - X_3)$. Since $p^j + 1 > 2$, $r(X_1, X_3) = \sqrt{-b_{d_2,0}}(X_1 - X_3)^{(p^j+1)/2} = -s(X_1, X_3)$ and

$$a_1(X_1 - X_3) = r(X_1, X_3) + s(X_1, X_3) \equiv 0,$$

  yielding $a_1 = 0$.

$\square$

**Proposition 6.6.3.** *Let $\mathcal{S}_{f_1,f_2}$ be as in (6.4), and $p > 3$. If $d = d_2 > d_1 > 1$ and $(j_1, j_2) = (1, 0)$ then $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$, unless $(p, d_1, d_2, h) \in \{(5, 2, 5, 3), (7, 2, 7, 3), (7, 3, 7, 5)\}$ or $d = d_2 = 2d_1 - 1 \leq 11$.*

*Proof.* Note that $\mathcal{S}_{f_1,f_2}$ reads

$$(X_2 - X_4)^2 X_0^{d-1} + (X_2 - X_4)\left[\sum_{i \leq d_1} a_i(X_1^i - X_3^i)X_0^{d-i}\right] + (X_1 - X_3)\left[\sum_{i \leq d_2} b_{i,0}(X_1^i - X_3^i)X_0^{d-i}\right].$$

If $\mathcal{S}_{f_1,f_2}$ is reducible then by Remark 6.4.2, there exist $r(X_1, X_3)$, $s(X_1, X_3)$ such that

$$\begin{aligned}(X_1 - X_3)\left[\sum_{i \leq d_2} b_{i,0}(X_1^i - X_3^i)\right] &= r(X_1, X_3)s(X_1, X_3),\\ \left[\sum_{i \leq d_1} a_i(X_1^i - X_3^i)\right] &= r(X_1, X_3) + s(X_1, X_3).\end{aligned}$$

If $\deg(r) > \deg(s)$ then we can suppose that $\deg(r) = d_1$ and $(X_1^{d_1} - X_3^{d_1}) \mid (X_1 - X_3)(X_1^{d_2} - X_3^{d_2})$. Since $d_1 > 1$, this yields $d_1 \mid d_2$ and thus $d_1 \leq d_2/2$. Now,

$$1 + d_2 = \deg(r) + \deg(s) \leq d_1 + d_1 - 1 \leq d_2 - 1,$$

a contradiction.

From now on we consider the case $\deg(r) = \deg(s) = (d_2+1)/2$. Then $d_1 \leq (d_2+1)/2$. Recall that $\mathcal{S}_{f_1,f_2}$ is reducible if and only if

$$\Delta(X_1, X_3) = \left[\sum_i a_i(X_1^i - X_3^i)\right]^2 - 4(X_1 - X_3)\left[\sum_i b_{i,0}(X_1^i - X_3^i)\right]$$

is a square in $\overline{\mathbb{F}}_q[X_1, X_3]$. We distinguish the cases where $d_1 \leq d_2/2$ and $d_2 + 1 = 2d_1$.

1) If $d_1 \leq d_2/2$ the homogeneous term of the highest degree in $\Delta(X_1, X_3)$ is $L(X_1, X_3) = -4b_{d_2,0}(X_1 - X_3)(X_1^{d_2} - X_3^{d_2})$. If $\Delta(X_1, X_3)$ is a square in $\overline{\mathbb{F}_q}[X_1, X_3]$ so it is $L(X_1, X_3)$. Let $d_2 = kp^j$ for some non-negative $j$ and $p \nmid k$.

   If $k > 1$ then $L(X_1, X_3)$ cannot be a square in $\overline{\mathbb{F}_q}[X_1, X_3]$.

   From now on we can assume that $k = 1$ and then $d_2 = p^j$. In this case $d_1 \leq (d_2 - 1)/2$.

   Suppose that $b_{i,0} = 0$, for any $0 < i < d_2$, i.e. $h = \infty$. In this case $r(X_1, X_3)s(X_1, X_3) = b_{d_2,0}(X_1 - X_3)^{p^j+1}$ and $r(X_1, X_3) + s(X_1, X_3) = \sum_i a_i(X_1^i - X_3^i)$. Since $p^j + 1 > 2d_1$, $r(X_1, X_3) = \sqrt{-b_{d_2,0}}(X_1 - X_3)^{(p^j+1)/2} = -s(X_1, X_3)$ and

$$\sum_i a_i(X_1^i - X_3^i) = r(X_1, X_3) + s(X_1, X_3) \equiv 0,$$

a contradiction to $a_{d_1} \neq 0$.

From now on we suppose that $h < \infty$. We distinguish the cases where $h+1 > 2d_1$, $h+1 < 2d_1$ and $h+1 = 2d_1$:

a) $h+1 > 2d_1$. Write

$$\Delta(X_1, X_3) = \Big((X_1 - X_3)^{(p^j+1)/2} + M(X_1, X_3) + \dots\Big)^2,$$

where $M(X_1, X_3)$ has degree $(h+1) - (p^j+1)/2$. Now

$$2(X_1 - X_3)^{(p^j+1)/2} M(X_1, X_3) = -4b_{h,0}(X_1 - X_3)(X_1^h - X_3^h)$$

and then $(X_1 - X_3)^{(p^j-1)/2} \mid (X_1^h - X_3^h)$. Write $h = k_1 p^{j_1}$, so that $(X_1^h - X_3^h) = (X_1^{k_1} - X_3^{k_1})^{p^{j_1}}$. Since $(X_1 - X_3)^{p^{j_1}}$ is the highest power of $(X_1 - X_3)$ dividing $(X_1^{k_1} - X_3^{k_1})^{p^{j_1}}$, we must have $(p^j - 1)/2 \leq p^{j_1}$. Since $j_1 \leq j-1$,

$$p^j - 1 \leq 2p^{j_1} \leq 2p^j/p,$$

and therefore $p^j \leq \frac{p}{p-2}$. This yields $p^j = 3$, a contradiction to $p > 3$.

b) $h+1 < 2d_1$. Write

$$\Delta(X_1, X_3) = \Big((X_1 - X_3)^{(p^j+1)/2} + M(X_1, X_3) + \dots\Big)^2,$$

where $M(X_1, X_3)$ has degree $2d_1 - (p^j+1)/2$. Now

$$2(X_1 - X_3)^{(p^j+1)/2} M(X_1, X_3) = a_{d_1}^2 (X_1^{d_1} - X_3^{d_1})^2$$

and then $(X_1 - X_3)^{(p^j+1)/2} \mid (X_1^{d_1} - X_3^{d_1})^2$. Write $d_1 = k_1 p^{j_1}$, so that $(X_1^{d_1} - X_3^{d_1})^2 = (X_1^{k_1} - X_3^{k_1})^{2p^{j_1}}$. We must have $(p^j+1)/2 \leq 2p^{j_1}$. Since $j_1 \leq j-1$,

$$p^j + 1 \leq 4p^{j_1} \leq 4p^j/p,$$

and therefore $(p-4)p^j \leq -p$, a contradiction to $p > 3$.

c) $h+1 = 2d_1$. Write

$$\begin{aligned} r(X_1, X_3) &= r_{(d_2+1)/2}(X_1, X_3) + \dots + r_{d_1}(X_1, X_3) + \cdots \\ s(X_1, X_3) &= s_{(d_2+1)/2}(X_1, X_3) + \dots + s_{d_1}(X_1, X_3) + \cdots, \end{aligned}$$

where $s_i, r_i$ homogeneous of degree $i$ or the zero polynomial. Since $r(X_1, X_3) + s(X_1, X_3) = \Big[\sum_{i \leq d_1} a_i (X_1^i - X_3^i)\Big]$, $s_{(d_2+1)/2}(X_1, X_3) = -r_{(d_2+1)/2}(X_1, X_3)$, $\dots$, $s_{d_1+1}(X_1, X_3) = -r_{d_1+1}(X_1, X_3)$.
The term of degree $d_2 > 2d_1$ in $r(X_1, X_3)s(X_1, X_3)$ is

$$r_{(d_2+1)/2} s_{(d_2-1)/2} + r_{(d_2-1)/2} s_{(d_2+1)/2} = -2r_{(d_2+1)/2} r_{(d_2-1)/2} \equiv 0,$$

and then $s_{(d_2-1)/2} = r_{(d_2-1)/2} \equiv 0$.
By induction, the term of degree $i > (d_2+1)/2 + d_1 \geq 2d_1 = h+1$ in $r(X_1, X_3)s(X_1, X_3)$ is

$$r_{(d_2+1)/2}s_{i-(d_2+1)/2} + r_{i-(d_2+1)/2}s_{(d_2+1)/2} = -2r_{(d_2+1)/2}r_{i-(d_2+1)/2} \equiv 0,$$

and then $s_{i-(d_2+1)/2} = r_{i-(d_2+1)/2} \equiv 0$.
Consider $i = (d_2+1)/2 + d_1 > 2d_1 = h+1$ in $r(X_1, X_3)s(X_1, X_3)$ is

$$\begin{aligned} r_{(d_2+1)/2}s_{d_1} + r_{d_1}s_{(d_2+1)/2} &= r_{(d_2+1)/2}(s_{d_1} - r_{d_1}) \\ &= \sqrt{-b_{d_2,0}}(X_1 - X_3)^{(d_2+1)/2}(s_{d_1} - r_{d_1}), \\ &\equiv 0 \end{aligned}$$

Then $r_{d_1} = s_{d_1} = a_{d_1}(X_1^{d_1} - X_3^{d_1})/2$.
Suppose that $2d_1 < (d_2+1)/2$. The term of degree $2d_1 = h+1$ in $r(X_1, X_3)s(X_1, X_3)$ is

$$r_{d_1}s_{d_1} = b_{h,0}(X_1 - X_3)(X_1^h - X_3^h) = a_{d_1}^2(X_1^{d_1} - X_3^{d_1})^2/4.$$

If $p \nmid d_1$ and $p \mid h$, then $(X_1 - X_3)^2 \parallel (X_1^{d_1} - X_3^{d_1})^2$ but $(X_1 - X_3)^{p+1} \mid (X_1 - X_3)(X_1^h - X_3^h)$.
If $p \nmid d_1$ and $p \nmid h$, then $(X_1^h - X_3^h)/(X_1 - X_3)$ is separable, whereas $(X_1^{d_1} - X_3^{d_1})^2/(X_1 - X_3)^2$ is not.
If $p \mid d_1$ then $p \nmid h$ and so $(X_1 - X_3)^{2p} \mid (X_1^{d_1} - X_3^{d_1})^2$ and $(X_1 - X_3)^2 \parallel (X_1 - X_3)(X_1^h - X_3^h)$. In all the cases a contradiction arises.
Suppose that $2d_1 \geq (d_2+1)/2$. The term of degree $2d_1 = h+1$ in $r(X_1, X_3)s(X_1, X_3)$ is

$$\begin{aligned} &r_{d_1}s_{d_1} + r_{(d_2+1)/2}s_{2d_1-(d_2+1)/2} + r_{2d_1-(d_2+1)/2}s_{(d_2+1)/2} = b_{h,0}(X_1 - X_3)(X_1^h - X_3^h) \\ &= a_{d_1}^2(X_1^{d_1} - X_3^{d_1})^2/4 + \sqrt{-b_{d_2,0}}(X_1 - X_3)^{(d_2+1)/2}(s_{2d_1-(d_2+1)/2} - r_{2d_1-(d_2+1)/2}) \end{aligned}$$

If $p \nmid d_1$ and $p \mid h$, then $(X_1 - X_3)^2 \parallel (X_1^{d_1} - X_3^{d_1})^2$ but $(X_1 - X_3)^{p+1} \mid (X_1 - X_3)(X_1^h - X_3^h)$ and $(X_1 - X_3)^{(p+1)/2} \mid (X_1 - X_3)^{(d_2+1)/2}$.
If $p \nmid d_1$ and $p \nmid h$, then $(X_1 - X_3)^2 \parallel (X_1^{d_1} - X_3^{d_1})^2$ and $(X_1 - X_3)^2 \parallel (X_1 - X_3)(X_1^h - X_3^h)$ and $(X_1 - X_3)^5 \mid (X_1 - X_3)^{(d_2+1)/2}$, unless $d_2 = p^j \leq 7$.
So, suppose that $d_2 > 7$. If $s_{2d_1-(d_2+1)/2} = r_{2d_1-(d_2+1)/2}$ the same argument as for the case $2d_1 < (d_2+1)/2$ applies.
If $s_{2d_1-(d_2+1)/2} \neq r_{2d_1-(d_2+1)/2}$, we have

$$(X_1 - X_3)^{(d_2-3)/2} \Big| \frac{b_{h,0}(X_1^h - X_3^h)}{X_1 - X_3} - \frac{a_{d_1}^2(X_1^{d_1} - X_3^{d_1})^2}{4(X_1 - X_3)^2}.$$

This is equivalent to

$$X_1^{(d_2-3)/2} \Big| H(X_1, X_3) = \frac{b_{h,0}((X_1+X_3)^h - X_3^h)}{X_1} - \frac{a_{d_1}^2((X_1+X_3)^{d_1} - X_3^{d_1})^2}{4X_1^2}$$
$$= (b_{h,0}h - a_{d_1}^2 d_1^2/4)X_3^{h-1} + \left(b_{h,0}\binom{h}{2} - \frac{a_{d_1}^2}{2}d_1\binom{d_1}{2}\right) X_1 X_3^{h-2} +$$
$$+ \left(b_{h,0}\binom{h}{3} - \frac{a_{d_1}^2}{2}d_1\binom{d_1}{3} - \frac{a_{d_1}^2}{4}\binom{d_1}{2}^2\right) X_1^2 X_3^{h-3} + \cdots$$

which yields $b_{h,0}h - a_{d_1}^2 d_1^2/4 = 0$. Since we are supposing that $d_2 > 7$, so that $(d_2-3)/2 > 2$. If $h \not\equiv 1 \pmod p$ then $d_1 \not\equiv 1 \pmod p$ and

$$b_{h,0}\binom{h}{3} - \frac{a_{d_1}^2}{2}d_1\binom{d_1}{3} - \frac{a_{d_1}^2}{4}\binom{d_1}{2}^2 \equiv 0 \pmod p,$$

that is

$$\frac{a_{d_1}^2 d_1^2(h-1)(h-2)}{24} - \frac{a_{d_1}^2 d_1^2(d_1-1)(d_1-2)}{12} - \frac{a_{d_1}^2 d_1^2(d_1-1)^2}{16} \equiv 0 \pmod p.$$

Since $h = 2d_1 - 1$, one gets $d_1 \equiv 1 \pmod p$, a contradiction.
If $h \equiv 1 \pmod p$ then $d_1 \equiv 1 \pmod p$. Consider $p^{i_1} \mid\mid (h-1)$ and $p^{i_2} \mid\mid (d_1-1)$, since $d_1 - 1 = (h-1)/2$, $i_1 = i_2$. Now,

$$\begin{aligned} H(X_1, X_3) &= (b_{h,0}h - a_{d_1}^2 d_1^2/4)X_3^{h-1} + \\ &\left(b_{h,0}\binom{h}{p^{i_1}} - \frac{a_{d_1}^2}{2}d_1\binom{d_1}{p^{i_1}}\right) X_1^{p^{i_1}-1} X_3^{h-p^{i_1}} + \\ &+ \left(b_{h,0}\binom{h}{p^{i_1}+1} - \frac{a_{d_1}^2}{2}d_1\binom{d_1}{p^{i_1}+1}\right) X_1^{p^{i_1}} X_3^{h-p^{i_1}-1} \\ &+ \left(b_{h,0}\binom{h}{2p^{i_1}-1} - \frac{a_{d_1}^2}{4}\binom{d_1}{p^{i_1}}^2\right) X_1^{2p^{i_1}-2} X_3^{h-2p^{i_1}+1} + \cdots . \end{aligned}$$

Since $b_{h,0}h - a_{d_1}^2 d_1^2/4 = 0$,

$$H(X_1, X_3) = \left(b_{h,0}\binom{h}{2p^{i_1}-1} - \frac{a_{d_1}^2}{4}\binom{d_1}{p^{i_1}}^2\right) X_1^{2p^{i_1}-2} X_3^{h-2p^{i_1}+1} + \cdots$$

and therefore $(p^j - 3)/2 \le 2p^{i_1} - 2$, a contradiction to $p > 3$.
If $p \mid d_1$ then $p \nmid h$ and so $(X_1 - X_3)^2 \mid\mid (X_1 - X_3)(X_1^h - X_3^h)$ and $(X_1 - X_3)^{2p} \mid (X_1^{d_1} - X_3^{d_1})^2$ and $(X_1 - X_3)^3 \mid (X_1 - X_3)^{(d_2+1)/2}$, since $d_2 \ge p > 3$.

2) If $d_2 + 1 = 2d_1$ then the homogeneous term of the highest degree $L(X_1, X_3)$ in $\Delta(X_1, X_3)$ is

$$a_{d_1}^2 (X_1^{d_1} - X_3^{d_1})^2 - 4b_{2d_1-1,0}(X_1 - X_3)(X_1^{2d_1-1} - X_3^{2d_1-1}).$$

In the following we will prove that, if $d_1 > 6$ , then

$$L(X_1, 1) = (a_{d_1}^2 - 4b_{2d_1-1,0})X_1^{2d_1} + 4b_{2d_1-1,0}X_1^{2d_1-1} - 2a_{d_1}^2 X_1^{d_1} + 4b_{2d_1-1,0}X_1 + (a_{d_1}^2 - 4b_{2d_1-1,0})$$

is not a square in $\overline{\mathbb{F}_q}[X_1]$.
First, if $a_{d_1}^2 - 4b_{2d_1-1,0} = 0$ then $X_1 \mid\mid L(X_1, 1)$ and therefore $L(X_1, 1)$ is not a square in $\overline{\mathbb{F}_q}[X_1]$. So we can suppose $a_{d_1}^2 - 4b_{2d_1-1,0} \neq 0$.
In the following we will consider the derivatives of $L(X_1, 1)$

$$\begin{aligned}
L^{(1)}(X_1, 1) &= 2d_1(a_{d_1}^2 - 4b_{2d_1-1,0})X_1^{2d_1-1} + 4(2d_1 - 1)b_{2d_1-1,0}X_1^{2d_1-2} \\
&\quad -2d_1 a_{d_1}^2 X_1^{d_1-1} + 4b_{2d_1-1,0}, \\
L^{(2)}(X_1, 1) &= 2d_1(2d_1 - 1)(a_{d_1}^2 - 4b_{2d_1-1,0})X_1^{2d_1-2} \\
&\quad +4(2d_1 - 1)(2d_1 - 2)b_{2d_1-1,0}X_1^{2d_1-3} - 2d_1(d_1 - 1)a_{d_1}^2 X_1^{d_1-2}, \\
L^{(3)}(X_1, 1) &= 2d_1(2d_1 - 1)(2d_1 - 2)(a_{d_1}^2 - 4b_{2d_1-1,0})X_1^{2d_1-3} \\
&\quad +4(2d_1 - 1)(2d_1 - 2)(2d_1 - 3)b_{2d_1-1,0}X_1^{2d_1-4} \\
&\quad -2d_1(d_1 - 1)(d_1 - 2)a_{d_1}^2 X_1^{d_1-3}, \\
L^{(4)}(X_1, 1) &= 2d_1(2d_1 - 1)(2d_1 - 2)(2d_1 - 3)(a_{d_1}^2 - 4b_{2d_1-1,0})X_1^{2d_1-4} \\
&\quad +4(2d_1 - 1)(2d_1 - 2)(2d_1 - 3)(2d_1 - 4)b_{2d_1-1,0}X_1^{2d_1-5} \\
&\quad -2d_1(d_1 - 1)(d_1 - 2)(d_1 - 3)a_{d_1}^2 X_1^{d_1-4}.
\end{aligned}$$

Clearly $X_1 = 0$ cannot be a repeated root of $L(X_1, 1)$.
• If $d_1 \equiv 0 \pmod p$, then $L^{(1)}(X_1, 1) = -4b_{2d_1-1,0}(X_1^{2d_1-2} - 1)$. Now, $X_1^{d_1-1} = 1$ or $X_1^{d_1-1} = -1$ yield, together with $L(X_1, 1) = 0$, $X_1 = 1$ or $(a_{d_1}^2 - 4b_{2d_1-1,0})X_1^2 + (8b_{2d_1-1,0} + 2a_{d_1}^2)X_1 + (a_{d_1}^2 - 4b_{2d_1-1,0}) = 0$. Since $L^{(2)}(X_1, 1) = 8b_{2d_1-1,0}X_1^{2d_1-3}$, all the roots have at most multiplicity 2. So, $L(X_1, 1)$ cannot be a square in $\overline{\mathbb{F}_q}[X_1]$ since its degree is larger than 6.
• If $d_1 \equiv 1 \pmod p$, since

$$X_1 L'(X_1, 1) - L(X_1, 1) = (a_{d_1}^2 - 4b_{2d_1-1,0})(X_1^{2d_1} - 1),$$

the only repeated roots of $L(X_1, 1)$ satisfy $X_1^{2d_1} = 1$. If $X_1^{d_1} = 1$ or $X_1^{d_1} = -1$, then $L(X_1, 1) = 0$ yields $X_1 = 1$ or $4b_{2d_1-1,0}X_1^2 + 4(a_{d_1}^2 - b_{2d_1-1,0})X_1 + 4b_{2d_1-1,0} = 0$, respectively. This shows that there are at most three repeated roots of $L(X_1, 1)$. Since in this case $L^{(2)}(X_1, 1)$ reads $2d_1(2d_1-1)(a_{d_1}^2 - 4b_{2d_1-1,0})X_1^{2d_1-2}$, there are not roots of $L(X_1, 1)$ of multiplicity larger than 2. Our assumption $d_1 > 6$ is sufficient to prove that $L(X_1, 1)$ cannot be a square in $\overline{\mathbb{F}_q}[X_1]$, since its degree is larger than 6.

• If $2d_1 \equiv 1 \pmod p$, then

$$\begin{aligned} L^{(1)}(X_1, 1) &= (a_{d_1}^2 - 4b_{2d_1-1,0})X_1^{2d_1-1} - a_{d_1}^2 X_1^{d_1-1} + 4b_{2d_1-1,0} \\ L(X_1, 1) - (X_1 - 1)L^{(1)}(X_1, 1) &= a_{d_1}^2 (X_1^{d_1} - 1)(X_1^{d_1-1} - 1). \end{aligned}$$

Both $X_1^{d_1} = 1$ and $X_1^{d_1-1} = 1$ combined with $L(X_1, 1) = 0$ give $X_1 = 1$. So, the only repeated root of $L(X_1, 1)$ is $X_1 = 1$. Since $L^{(2)}(X_1, 1) = -(d_1 - 1)a_{d_1}^2 X_1^{d_1-2}$, $X_1 = 1$ is a double root of $L(X_1, 1)$, which proves that $L(X_1, 1)$ cannot be a square in $\overline{\mathbb{F}_q}[X_1]$ (its degree is larger than 12).

From now on we can suppose that $d_1(d_1 - 1)(2d_1 - 1) \not\equiv 0 \pmod p$. First, note that $X_1 = 1$ is a root of multiplicity at most 4 for $L(X_1, 1)$. In fact,

$$\begin{aligned} L(X_1 + 1, 1) &= (a_{d_1}^2 - 4b_{2d_1-1,0})(X_1 + 1)^{2d_1} + 4b_{2d_1-1,0}(X_1 + 1)^{2d_1-1} \\ &\quad -2a_{d_1}^2 (X_1 + 1)^{d_1} + 4b_{2d_1-1,0}X_1 + a_{d_1}^2 \\ &= \left( a_{d_1}^2 \left( \binom{2d_1}{2} - 2\binom{d_1}{2} \right) - 4b_{2d_1-1,0} \left( \binom{2d_1}{2} - \binom{2d_1 - 1}{2} \right) \right) X_1^2 + \\ &\quad \left( a_{d_1}^2 \left( \binom{2d_1}{3} - 2\binom{d_1}{3} \right) - 4b_{2d_1-1,0} \left( \binom{2d_1}{3} - \binom{2d_1 - 1}{3} \right) \right) X_1^3 + \\ &\quad \left( a_{d_1}^2 \left( \binom{2d_1}{4} - 2\binom{d_1}{4} \right) - 4b_{2d_1-1,0} \left( \binom{2d_1}{4} - \binom{2d_1 - 1}{4} \right) \right) X_1^4 + \cdots \\ &= \left( a_{d_1}^2 d_1^2 - 4b_{2d_1-1,0}(2d_1 - 1) \right) X_1^2 + \\ &\quad \left( a_{d_1}^2 d_1^2 (d_1 - 1) - 4b_{2d_1-1,0}(2d_1 - 1)(d_1 - 1) \right) X_1^3 + \\ &\quad \frac{\left( a_{d_1}^2 \left( 7d_1^4 - 17d_1^3 + 8d_1^2 + 2d_1 \right) - 4b_{2d_1-1,0} \left( 16d_1^3 - 48d_1^2 + 44d_1 - 12 \right) \right)}{12} X_1^4 + \cdots \end{aligned}$$

If both the coefficients of $X_1^2$ and $X_1^4$ vanish then $d_1^2(d_1 - 1)(2d_1 - 1)a_{d_1}^2 = 0$, a contradiction.

Denote $Y = X_1^{d_1}$, so that

$$\begin{aligned}
L(X_1,1) &= (a_{d_1}^2 - 4b_{2d_1-1,0})Y^2 + 4b_{2d_1-1,0}\frac{Y^2}{X_1} - 2a_{d_1}^2 Y \\
&\quad +4b_{2d_1-1,0}X_1 + (a_{d_1}^2 - 4b_{2d_1-1,0}) \\
L^{(1)}(X_1,1) &= 2d_1(a_{d_1}^2 - 4b_{2d_1-1,0})\frac{Y^2}{X_1} + 4(2d_1-1)b_{2d_1-1,0}\frac{Y^2}{X_1^2} \\
&\quad -2d_1 a_{d_1}^2\frac{Y}{X_1} + 4b_{2d_1-1,0}, \\
L^{(2)}(X_1,1) &= 2d_1(2d_1-1)(a_{d_1}^2 - 4b_{2d_1-1,0})\frac{Y^2}{X_1^2} + 4(2d_1-1)(2d_1-2)b_{2d_1-1,0}\frac{Y^2}{X_1^3} \\
&\quad -2d_1(d_1-1)a_{d_1}^2\frac{Y}{X_1^2}, \\
L^{(3)}(X_1,1) &= 2d_1(2d_1-1)(2d_1-2)(a_{d_1}^2 - 4b_{2d_1-1,0})\frac{Y^2}{X_1^3} \\
&\quad +4(2d_1-1)(2d_1-2)(2d_1-3)b_{2d_1-1,0}\frac{Y^2}{X_1^4} \\
&\quad -2d_1(d_1-1)(d_1-2)a_{d_1}^2\frac{Y}{X_1^3},
\end{aligned}$$

$$\begin{aligned}
L^{(4)}(X_1,1) &= 2d_1(2d_1-1)(2d_1-2)(2d_1-3)(a_{d_1}^2 - 4b_{2d_1-1,0})\frac{Y^2}{X_1^4} \\
&\quad +4(2d_1-1)(2d_1-2)(2d_1-3)(2d_1-4)b_{2d_1-1,0}\frac{Y^2}{X_1^5} \\
&\quad -2d_1(d_1-1)(d_1-2)(d_1-3)a_{d_1}^2\frac{Y}{X_1^4}.
\end{aligned}$$

The resultant of $L(X_1,1)$ and $L^{(1)}(X_1,1)$ with respect to $Y$ reads

$$16b_{2d_1-1,0}(a_{d_1}^2 - 4b_{2d_1-1,0})X_1^2(X_1-1)^2 Z(X_1),$$

where

$$\begin{aligned}
Z(X_1) &= b_{2d_1-1,0}(2d_1-1)^2(a_{d_1}^2 - 4b_{2d_1-1,0})X_1^2 + (a_{d_1}^4 d_1^2 - 8a_{d_1}^2 b_{2d_1-1,0}d_1^2 \\
&\quad +2a_{d_1}^2 b_{2d_1-1,0} + 32b_{2d_1-1,0}^2 d_1^2 - 32b_{2d_1-1,0}^2 d_1 + 8b_{2d_1-1,0}^2)X_1 \\
&\quad +b_{2d_1-1,0}(2d_1-1)^2(a_{d_1}^2 - 4b_{2d_1-1,0}).
\end{aligned}$$

Multiple roots of $L(X_1,1)$ distinct from $X_1 = 1$ must satisfy $Z(X_1) = 0$.

• If $d_1 \equiv 2 \pmod p$, then $L^{(4)}(X_1,1) = 24(a_{d_1}^2 - 4b_{2d_1-1,0})X_1^{2d_1-4}$ and all the roots have multiplicity at most 4. This shows that $L(X_1,1)$ cannot be a square in $\overline{\mathbb{F}_q}[X_1]$ (its degree is larger than 12).

• If $d_1 \not\equiv 2 \pmod{p}$, then the resultant of $L^{(2)}(X_1,1)/Y$ and $L^{(3)}(X_1,1)/Y$ with respect to $Y$ reads

$$4(2d_1-1)a_{d_1}^2 d_1(d_1-1)X_1(X_1a_{d_1}^2d_1^2-4X_1b_{2d_1-1,0}d_1^2+4b_{2d_1-1,0}d_1^2-8b_{2d_1-1,0}d_1+4b_{2d_1-1,0}),$$

where as the resultant of $L^{(3)}(X_1,1)/Y$ and $L^{(4)}(X_1,1)/Y$ with respect to $Y$ reads

$$8(d_1-2)(d_1-1)^2d_1a_{d_1}^2(2d_1-1)X_1 \cdot$$
$$\cdot(X_1a_{d_1}^2d_1^2-4X_1b_{2d_1-1,0}d_1^2+4b_{2d_1-1,0}d_1^2-10b_{2d_1-1,0}d_1+6b_{2d_1-1,0}).$$

The unique possible root of multiplicity larger than 4 must satisfy

$$X_1 = -\frac{4b_{2d_1-1,0}(d_1-1)^2}{d_1^2(a_{d_1}^2-4b_{2d_1-1,0})} = -\frac{2b_{2d_1-1,0}(2d_1^2-5d_1+3)}{d_1^2(a_{d_1}^2-4b_{2d_1-1,0})}$$

which implies $d_1(d_1-1)b_{2d_1-1,0}(a_{d_1}^2-4b_{2d_1-1,0}) = 0$, a contradiction. Therefore, all the (at most three) multiple roots of $L(X_1,1)$ have multiplicity at most 4 and so $L(X_1,1)$ cannot be a square in $\overline{\mathbb{F}_q}[X_1]$ (its degree is larger than 12).

□

**Proposition 6.6.4.** *Let $\mathcal{S}_{f_1,f_2}$ be as in (6.4), and $p=3$. If $d=d_2>d_1>1$ and $(j_1,j_2)=(1,0)$ then $\mathcal{S}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$, unless one of the following holds*

1. $d=d_2=2d_1-1\leq 11$;
2. $h+1<2d_1<d_2=3^j$ *and* $3 \mid d_1$;
3. $h+1=2d_1<d_2=3^j$ *and* $3 \mid (d_1-1)$.

*Proof.* The proof is similar to that of Proposition 6.6.3 and we use the same notations.

If $d_2+1=2d_1$ the claim follows as in Proposition 6.6.3. From now on we assume $d_1 \leq d_2/2$. The homogeneous term of the highest degree in $\Delta(X_1,X_3)$ is $L(X_1,X_3) = -4b_{d_2,0}(X_1-X_3)(X_1^{d_2}-X_3^{d_2})$. If $\Delta(X_1,X_3)$ is a square in $\overline{\mathbb{F}_q}[X_1,X_3]$ so it is $L(X_1,X_3)$. Let $d_2=k\cdot 3^j$ for some non-negative $j$ and $3 \nmid k$.

If $k>1$ then $L(X_1,X_3)$ cannot be a square in $\overline{\mathbb{F}_q}[X_1,X_3]$.

From now on we can assume that $k=1$ and then $d_2=3^j$. In this case $d_1 \leq (d_2-1)/2$ and $j>1$.

If $h=\infty$ the claim follows as in Proposition 6.6.3.

When $h<\infty$, we distinguish the cases where $h+1>2d_1$, $h+1<2d_1$ and $h+1=2d_1$:

a) $h+1 > 2d_1$. Following the same argument as in the proof of Proposition 6.6.3, we get $d = d_2 = 3$ and therefore $1 < d_1 \leq 1$.

b) $h+1 < 2d_1$. Write

$$\begin{array}{rcl} r(X_1,X_3) & = & r_{(d_2+1)/2}(X_1,X_3) + \cdots + r_{d_1}(X_1,X_3) + \cdots \\ s(X_1,X_3) & = & s_{(d_2+1)/2}(X_1,X_3) + \cdots + s_{d_1}(X_1,X_3) + \cdots , \end{array}$$

where $s_i, r_i$ homogeneous of degree $i$ or the zero polynomial. As in Proposition 6.6.3 it follows that $s_{(d_2+1)/2} = -r_{(d_2+1)/2} = -\sqrt{-b_{d_2,0}}(X_1 - X_3)^{(d_2+1)/2}$, $s_i(X_1,X_3) = r_i(X_1,X_3) \equiv 0$, for any $d_1 < i < (d_2+1)/2$, and $r_{d_1}(X_1,X_3) = s_{d_1}(X_1,X_3) = a_{d_1}(X_1^{d_1} - X_3^{d_1})/2$.

Suppose that $2d_1 < (d_2+1)/2$. The term of degree $2d_1 > h+1$ in $r(X_1,X_3)s(X_1,X_3)$ is

$$r_{d_1}s_{d_1} = a_{d_1}^2(X_1^{d_1} - X_3^{d_1})^2/4 \equiv 0,$$

a contradiction to $a_{d_1} \neq 0$.

Suppose that $2d_1 \geq (d_2+1)/2$. The term of degree $2d_1 > h+1$ in $r(X_1,X_3)s(X_1,X_3)$ is

$$\begin{array}{rcl} r_{d_1}s_{d_1} + r_{(d_2+1)/2}s_{2d_1-(d_2+1)/2} + r_{2d_1-(d_2+1)/2}s_{(d_2+1)/2} & = & \\ a_{d_1}^2(X_1^{d_1} - X_3^{d_1})^2/4 + \sqrt{-b_{d_2,0}}(X_1 - X_3)^{(d_2+1)/2}(s_{2d_1-(d_2+1)/2} - r_{2d_1-(d_2+1)/2}) & \equiv & 0 \end{array}$$

Then

$$a_{d_1}^2(X_1^{d_1} - X_3^{d_1})^2/4 = \sqrt{-b_{d_2,0}}(X_1 - X_3)^{(d_2+1)/2}(r_{2d_1-(d_2+1)/2} - s_{2d_1-(d_2+1)/2}).$$

If $s_{2d_1-(d_2+1)/2} = r_{2d_1-(d_2+1)/2}$ then $a_{d_1}$ must be zero.

If $s_{2d_1-(d_2+1)/2} \neq r_{2d_1-(d_2+1)/2}$ and $3 \nmid d_1$, then $(X_1 - X_3)^2 || (X_1^{d_1} - X_3^{d_1})^2$ but $(d_2+1)/2 > 2$. In both cases a contradiction arises.

c) $h+1 = 2d_1$. If $d_1 \not\equiv 1 \pmod 3$, the claim follows as Proposition 6.6.3.

$\square$

We are now in position to prove the main result of this chapter.

*Proof of Theorem 6.1.2.* By Remark 6.4.1, Theorem 6.5.5, Propositions 6.6.1, 6.6.2, 6.6.3, 6.6.4 if there exists $b_{i,j} \neq 0$ with $j \neq 0$ and none of the conditions listed above holds, then $\mathcal{S}_{f_1,f_2}$ defined as in (6.4) contains an absolutely irreducible $\mathbb{F}_q$-rational component. By Theorem 6.2.4 $\mathcal{O}_5(f_1,f_2)$ is not an ovoid.

# Chapter 7

# Classification of low-degree ovoids of $Q(6,q)$

It is well known that to any ovoid of $Q(6,q)$ two polynomials $f_1(x,y,z)$, $f_2(x,y,z)$ can be associated. In this chapter we classify ovoids of $Q(6,q)$ with $\max\{\deg(f_1),\deg(f_2)\} < (\frac{1}{6.3}q)^{\frac{3}{13}} - 1$. The results of this chapter can also be found in [8].

## 7.1 Introduction

Regarding ovoids of the parabolic quadric $Q(m,q)$ of $\mathrm{PG}(m,q)$, $m$ even, it is known that there are no ovoids if $m > 6$ (see [40, 79]). Hence for parabolic quadrics the only relevant cases are $m = 4$ and $m = 6$.

Let $(X_0, X_1, X_2, X_3, X_4, X_5, X_6)$ denote the homogeneous coordinates of $\mathrm{PG}(6,q)$ and by $H_\infty$ the hyperplane at infinity $X_0 = 0$. Up to collineations, in $\mathrm{PG}(6,q)$ there is a unique non-degenerate quadric, the parabolic quadric $Q(6,q)$ and in the sequel we fix the following parabolic quadric:

$$Q(6,q) : X_0X_6 + X_1X_5 + X_2X_4 + X_3^2 = 0.$$

It is well known that there are no ovoids of $Q(6,q)$ for $q$ even [79], and thus from now on, we will consider only $p > 2$. We recall that an ovoid of $Q(6,q)$ is a set of $q^3+1$ pairwise non-collinear points on the quadric. We can always assume that an ovoid of $Q(6,q)$ contains the points $(1,0,0,0,0,0,0)$ and $(0,0,0,0,0,0,1)$ and hence it can be written in the following form:

$$\begin{aligned}\mathcal{O}_6(f_1,f_2) &= \{(1,x,y,z,f_1(x,y,z),f_2(x,y,z),-z^2-yf_1(x,y,z)-xf_2(x,y,z))\}_{x,y,z\in\mathbb{F}_q}\\ &\quad \cup\{(0,0,0,0,0,0,1)\},\end{aligned}$$

for some functions $f_i : \mathbb{F}_q^3 \longrightarrow \mathbb{F}_q$ with $f_i(0,0,0) = 0$ for $i \in \{1,2\}$; see [85, Chapter 3]. The set $\mathcal{O}_6(f_1,f_2)$ is an ovoid if and only if

$$\begin{aligned}&\langle(1,x_1,y_1,z_1,f_1(x_1,y_1,z_1),f_2(x_1,y_1,z_1),-z_1^2-y_1f_1(x_1,y_1,z_1)-x_1f_2(x_1,y_1,z_1)), \quad (7.1)\\ &(1,x_2,y_2,z_2,f_1(x_2,y_2,z_2),f_2(x_2,y_2,z_2),-z_2^2-y_2f_1(x_2,y_2,z_2)-x_2f_2(x_2,y_2,z_2))\rangle \neq 0,\end{aligned}$$

for every $(x_1,y_1,z_1) \neq (x_2,y_2,z_2)$ in $\mathbb{F}_q^3$, where $\langle\, , \,\rangle$ is the symmetric form associated to the quadratic form of the quadric $Q(6,q)$.
The only known ovoids of $Q(6,q)$ are the two classes of ovoids listed in the following table (see [45, 71]).

Table 7.1: Known ovoids of $Q(6,q)$.

| Name | $f_1(x,y,z)$ | $f_2(x,y,z)$ | Restrictions |
|---|---|---|---|
| *Thas-Kantor* | $-ny^3+x^2y-xz$ | $-1/nx^3+xy^2+yz$ | $q=3^h$, $h \geq 1$<br>$n$ a non-square |
| *Ree-Tits* | $-x^{\sigma+3}+y^\sigma+x^2y-xz$ | $-x^{2\sigma+3}+x^\sigma y^\sigma-z^\sigma+xy^2+yz$ | $q=3^{2h+1}, h \geq 0$<br>$\sigma=\sqrt{3q}$ |

Note that Condition (7.1) reads

$$(z_1-z_2)^2+(x_2-x_1)(f_2(x_2,y_2,z_2)-f_2(x_1,y_1,z_1))+(y_2-y_1)(f_1(x_2,y_2,z_2)-f_1(x_1,y_1,z_1)) \neq 0, \tag{7.2}$$

for every $(x_1,y_1,z_1) \neq (x_2,y_2,z_2)$ in $\mathbb{F}_q^3$.

In order to obtain non-existence results and partial classifications for the ovoids $\mathcal{O}_6(f_1,f_2)$ of $Q(6,q)$, we will consider the hypersurface $\mathcal{W}_{f_1,f_2} \subset \mathrm{PG}(6,q)$ defined by

$$F_{f_1,f_2}(X_0,X_1,X_2,X_3,X_4,X_5,X_6)=0,$$

where $F_{f_1,f_2}(1,X_1,X_2,X_3,X_4,X_5,X_6)$ equals

$$\begin{aligned}(X_3-X_6)^2+(X_4-X_1)(f_2(X_4,X_5,X_6)-f_2(X_1,X_2,X_3))\\ +(X_5-X_2)(f_1(X_4,X_5,X_6)-f_1(X_1,X_2,X_3)).\end{aligned} \tag{7.3}$$

**Lemma 7.1.1.** *If $f_1, f_2 \not\equiv 0$, then $\mathcal{O}_6(f_1,f_2)$ is an ovoid of $Q(6,q)$ if and only if $\mathcal{W}_{f_1,f_2}$ contains no affine $\mathbb{F}_q$-rational points off the solid $X_1-X_4=X_2-X_5=X_3-X_6=0$.*

The main result of this chapter is the following.

**Theorem 7.1.2.** *Let $f_1(X,Y,Z)=\sum_{ijk} a_{i,j,k}X^iY^jZ^k$ and $f_2(X,Y,Z)=\sum_{ijk} b_{i,j,k}X^iY^jZ^k$. Suppose that $q > 6.3(d+1)^{13/3}$, where $\max\{\deg(f_1),\deg(f_2)\}=d$. If $\mathcal{O}_6(f_1,f_2)$ is an ovoid of $Q(6,q)$ then*

$$q=3^h, \quad f_1(X,Y,Z)=X^2Y-nY^3-XZ, \quad f_2(X,Y,Z)=-1/nX^3+XY^2+YZ.$$

*Hence $\mathcal{O}_6(f_1,f_2)$ is a Thas-Kantor ovoid.*

It is worth pointing out that the degree of the functions $f_1(X,Y,Z)$ and $f_2(X,Y,Z)$ associated with the Ree-Tits ovoid does not satisfy the bound $q > 6.3(d+1)^{13/3}$ in Theorem 7.1.2.

## 7.2 Classification of low degree ovoids

For our purposes, the existence of an absolutely irreducible $\mathbb{F}_q$-rational component in $\mathcal{W}_{f_1,f_2}$ is enough to provide asymptotic non-existence results for ovoids of $Q(6,q)$.

**Theorem 7.2.1.** *Let $\mathcal{W}_{f_1,f_2} : F_{f_1,f_2}(X_0,X_1,X_2,X_3,X_4,X_5,X_6) = 0$, where $F_{f_1,f_2}$ is defined as in (7.3). Suppose that $q > 6.3(d+1)^{13/3}$, $\max\{\deg(f_1),\deg(f_2)\} = d$, and $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component $\mathcal{V}$ defined over $\mathbb{F}_q$. Then $\mathcal{O}_6(f_1,f_2)$ is not an ovoid of $Q(6,q)$.*

*Proof.* Since $q > 6.3(d+1)^{13/3}$, by Theorem 6.2.3, with $\delta = d+1$ and $r = 5$, it is readily seen that $\mathcal{V}$ contains more than $q^3$ affine $\mathbb{F}_q$-rational points. This yields the existence of at least one affine $\mathbb{F}_q$-rational point $(a_1,b_1,c_1,a_2,b_2,c_2)$ in $\mathcal{V} \subset \mathcal{W}_{f_1,f_2}$ such that $a_1 \neq a_2$ or $b_1 \neq b_2$ or $c_1 \neq c_2$ and therefore $\mathcal{O}_6(f_1,f_2)$ is not an ovoid of $Q(6,q)$. □

### 7.2.1 Ovoids in $Q(4,q)$

A starting point in our investigation is the main result of [7].

**Theorem 7.2.2** ([7, Main Theorem])**.** *Let $f(X,Y) = \sum_{ij} a_{i,j}X^iY^j$, with $a_{0,1} = 0$ if $p > 2$. Suppose that $q > 6.3(d+1)^{13/3}$, where $\deg(f) = d$. If $\mathcal{O}_4(f)$ is an ovoid of $Q(4,q)$ then one of the following holds:*

1. *$p = 2$ and $f(X,Y) = a_{1,0}X + a_{0,1}Y$. Hence $\mathcal{O}_4(f)$ is an elliptic quadric.*
2. *$p > 2$ and $f(X,Y) = a_{p^j,0}X^{p^j}$, $j \geq 0$. Hence $\mathcal{O}_4(f)$ is either an elliptic quadric or a Kantor ovoid.*

**Corollary 7.2.3.** *Let $f(X,Y) = \sum_{ij} a_{i,j}X^iY^j$, with $p > 2$. Suppose that $q > 6.3(d+1)^{13/3}$, where $\deg(f) = d$. If*

$$f(X,Y) \neq a_{p^j,0}X^{p^j} + \frac{a_{0,1}^2}{4}X + a_{0,1}Y,$$

*for some $j \geq 0$, then $\mathcal{S}_f$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.*

*Proof.* Let $\mathcal{S}_f : F(X_1,X_2,X_3,X_4) = 0$, where

$$F(X_1,X_2,X_3,X_4) = (X_2 - X_4)^2 + (X_1 - X_3)\Big(f(X_1,X_2) - f(X_3,X_4)\Big)$$

be the hypersurface associated to an ovoid of $Q(4,q)$; see [7]. As already observed in [7], up to a change of variables we can always suppose that $a_{0,1} = 0$. Suppose that this is not the case, then consider

$$\mathcal{S}_{f'} : F(X_1, X_2 - a_{0,1}X_1/2, X_3, X_4 - a_{0,1}X_3/2) = 0$$

and so the coefficient of $(X_1-X_3)(X_2-X_4)$ in $F(X_1,X_2-a_{0,1}X_1/2,X_3,X_4-a_{0,1}X_3/2)$ vanishes, i.e. the coefficient $a'_{0,1}$ in $f'$ is 0. Therefore, by Theorem 7.2.2,

$$F(X_1,X_2-a_{0,1}X_1/2,X_3,X_4-a_{0,1}X_3/2)=(X_2-X_4)^2+(X_1-X_3)^{p^h+1},$$

otherwise $\mathcal{S}_{f'}$ (and so $\mathcal{S}_f$) contains an absolutely irreducible component defined over $\mathbb{F}_q$.

So, $F(X_1,X_2,X_3,X_4)$ equals

$$(X_2-X_4)^2+(X_1-X_3)\left((X_1-X_3)^{p^h}+a_{0,1}(X_2-X_4)+\frac{a_{0,1}^2}{4}(X_1-X_3)\right). \tag{7.4}$$

The claim follows. □

### 7.2.2 Ovoids in $Q(6,q)$

Corollary 7.2.3 above provides constraints on the shape of the functions $f_1,f_2$ associated with putative ovoids $\mathcal{O}_6(f_1,f_2)$ of $Q(6,q)$.

**Proposition 7.2.4.** *Suppose that*

$$(f_1(X,Y,Z),f_2(X,Y,Z))\neq\left(a_1(X)Y^{p^{h_1}}+\frac{c_1(X)^2Y}{4}+c_1(X)Z,a_2(Y)X^{p^{h_2}}+\frac{c_2(Y)^2X}{4}+c_2(Y)Z\right), \tag{7.5}$$

*for some $a_1,c_1\in\mathbb{F}_q[X]$, $a_2,c_2\in\mathbb{F}_q[Y]$, $h_1,h_2\geq 0$. If $q>6.3(d+1)^{13/3}$, $\max\{\deg(f_1),\deg(f_2)\}=d$, then $\mathcal{O}_6(f_1,f_2)$ is not an ovoid.*

*Proof.* Consider the variety $\mathcal{W}'_{f_1,f_2}=\mathcal{W}_{f_1,f_2}\cap(X_5=X_2=\lambda)$, $\lambda\in\mathbb{F}_q$, defined by

$$G'_{f_1,f_2,\lambda}(X_0,X_1,X_3,X_4,X_6)=F_{f_1,f_2}(X_0,X_1,\lambda,X_3,X_4,\lambda,X_6)=0,$$

where

$$G'_{f_1,f_2,\lambda}(1,X_1,X_3,X_4,X_6)\ =\ (X_3-X_6)^2+(X_4-X_1)(f_2(X_4,\lambda,X_6)-f_2(X_1,\lambda,X_3)),$$

where $f_2$ satisfies (7.5). Suppose that

$$f_2(X,\lambda,Z)\neq\alpha_\lambda X^{p^j}+\frac{\beta_\lambda^2}{4}X+\beta_\lambda Z, \tag{7.6}$$

for any $\alpha_\lambda,\beta_\lambda\in\mathbb{F}_q$. Then, by Corollary 7.2.3, $\mathcal{W}'_{f_1,f_2}$ possesses an absolutely irreducible component defined over $\mathbb{F}_q$. Such a component, since $q>6.3(d+1)^{13/3}$, contains at least an $\mathbb{F}_q$-rational point $(1,\overline{x}_1,\overline{x}_3,\overline{x}_4,\overline{x}_6)$, with $\overline{x}_1\neq\overline{x}_4$ or $\overline{x}_3\neq\overline{x}_6$. Therefore there exists a six-tuple $(1,\overline{x}_1,\lambda,\overline{x}_3,\overline{x}_4,\lambda,\overline{x}_6)$, with $(\overline{x}_1,\lambda,\overline{x}_3)\neq(\overline{x}_4,\lambda,\overline{x}_6)$ satisfying $F_{f_1,f_2}(X_0,X_1,X_2,X_3,X_4,X_5,X_6)=0$ and therefore $\mathcal{O}_6(f_1,f_2)$ is not an ovoid.

Suppose now that $f_2(X,Y,Z)$ contains a term $a_{i,j,k}X^iY^jZ^k\neq 0$, with $(i,k)\notin\{(p^{h_2},0),(1,0),(0,1)\}$. The coefficient of $X^iZ^k$ in $f_2(X,\lambda,Z)$ is determined by $A_{i,k}(\lambda)=\sum_j a_{i,j,k}\lambda^j$ and we have that $A_{i,k}(\lambda)=0$ for each $\lambda\in\mathbb{F}_q$. Since $\deg A_{i,k}(Y)\leq d<q$,

$A_{i,k}(Y) \equiv 0$ for each $(i,k) \notin \{(p^{h_2},0),(1,0),(0,1)\}$. Also, $(A_{0,1}(\lambda))^2 = 4A_{1,0}(\lambda)$ for each $\lambda \in \mathbb{F}_q$. Since $\deg(A_{0,1}(Y))^2 \leq 2d < q$, we have that $(A_{0,1}(Y))^2 = 4A_{1,0}(Y)$. Summing up,

$$f_2(X,Y,Z) = a_2(Y)X^{p^{h_2}} + \frac{c_2(Y)^2}{4}X + c_2(Y)Z,$$

for some $a_2, c_2 \in \mathbb{F}_q[Y]$.

A similar argument, for any $\lambda \in \mathbb{F}_q$, holds for $\mathcal{W}''_{f_1,f_2} = \mathcal{W}_{f_1,f_2} \cap (X_4 = X_1 = \lambda)$ defined by

$$G''_{f_1,f_2,\lambda}(X_0,X_2,X_3,X_5,X_6) = F_{f_1,f_2}(X_0,\lambda,X_2,X_3,\lambda,X_5,X_6) = 0,$$

that is

$$G''_{f_1,f_2,\lambda}(1,X_2,X_3,X_5,X_6) \quad = \quad (X_3-X_6)^2 + (X_5-X_2)(f_1(\lambda,X_5,X_6) - f_1(\lambda,X_2,X_3)).$$

So, if $\mathcal{O}_6(f_1,f_2)$ is an ovoid then

$$f_1(X,Y,Z) = a_1(X)Y^{p^{h_1}} + \frac{c_1(X)^2}{4}Y + c_1(X)Z,$$

for some $a_1, c_1 \in \mathbb{F}_q[X]$. □

By Proposition 7.2.4, if $\mathcal{O}_6(f_1,f_2)$ is an ovoid then the polynomials $F(1,X_1,X_2,X_3,X_4,X_5,X_6)$ defining $\mathcal{W}_{f_1,f_2}$ read

$$\begin{aligned}(X_3-X_6)^2 + (X_4-X_1)\Bigg(& a_2(X_5)X_4^{p^{h_2}} - a_2(X_2)X_1^{p^{h_2}} + \frac{c_2^2(X_5)}{4}X_4 \\ & -\frac{c_2^2(X_2)}{4}X_1 + c_2(X_5)X_6 - c_2(X_2)X_3\Bigg) \\ +(X_5-X_2)\Bigg(& a_1(X_4)X_5^{p^{h_1}} - a_1(X_1)X_2^{p^{h_1}} + \frac{c_1^2(X_4)}{4}X_5 \\ & -\frac{c_1^2(X_1)}{4}X_2 + c_1(X_4)X_6 - c_1(X_1)X_3\Bigg).\end{aligned} \tag{7.7}$$

**Proposition 7.2.5.** *Suppose that* $\deg(\mathcal{W}_{f_1,f_2}) = 2$. *Then* $\mathcal{O}_6(f_1,f_2)$ *is not an ovoid.*

*Proof.* In this case (7.7) defines a quadric in $\mathrm{PG}(6,q)$, and by Theorem 6.2.3, it contains at least $q^4$ affine $\mathbb{F}_q$-rational points. Thus, there exists at least an affine $\mathbb{F}_q$-rational point off the solid $X_1 - X_4 = X_2 - X_5 = X_3 - X_6 = 0$ and, by Theorem 7.2.1, $\mathcal{O}_6(f_1,f_2)$ is not an ovoid. □

In the sequel, we will make use of the following notations.

$$\deg(a_1) = A_1, \quad \deg(a_2) = A_2, \quad \deg(c_1) = C_1, \quad \deg(c_2) = C_2. \tag{7.8}$$

When one of the above polynomials vanishes, we use $-\infty$ to indicate its degree.

**Proposition 7.2.6.** *Let $p^{h_1} = p^{h_2} = 1$. Then $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.*

*Proof.* By way of contradiction, suppose that $\mathcal{O}_6(f_1, f_2)$ is an ovoid. By Proposition 7.2.5, we can suppose that $\deg(\mathcal{W}_{f_1,f_2}) \geq 3$.

Now, $F(1, X_1, X_2, X_3, X_4, X_5, X_6)$ reads

$$(X_3 - X_6)^2 + (X_4 - X_1)\Bigg(\alpha_1(X_5)X_4 - \alpha_1(X_2)X_1 + \beta_1(X_5)X_6 - \beta_1(X_2)X_3\Bigg)$$
$$+(X_5 - X_2)\Bigg(\alpha_2(X_4)X_5 - \alpha_2(X_1)X_2 + \beta_2(X_4)X_6 - \beta_2(X_1)X_3\Bigg), \quad (7.9)$$

for some polynomials $\alpha_1, \alpha_2, \beta_1, \beta_2$ of degrees $A_1, A_2, B_1, B_2$ (possibly $-\infty$). Note that at least one among $A_1, A_2, B_1, B_2$ is positive, since $\deg(\mathcal{W}_{f_1,f_2}) \geq 3$.

We distinguish a number of cases, analyzing the homogeneous component $M(X_1, X_2, X_3, X_4, X_5, X_6)$ of highest degree, say $\ell$, in $F(1, X_1, X_2, X_3, X_4, X_5, X_6)$.

C.1. $\ell = A_1 + 2 \geq 3$.
Then $A_1 > 0$. The component of highest degree $N(X_1, X_4, X_5, X_6)$ in $M(X_1, 1, 1, X_4, X_5, X_6)$ is

$$u(X_4 - X_1)X_5^{A_1}X_4 + v(X_4 - X_1)X_5^{B_1}X_6 + wX_4^{A_2}X_5^2 + zX_4^{B_2}X_5X_6,$$

for some $u, v, w, z \in \mathbb{F}_q$, $u \neq 0$. It is already seen that $\deg_{X_1}(N(X_1, X_4, X_5, X_6)) = 1$ and therefore $N(X_1, X_4, X_5, X_6) = 0$ contains a nonrepeated absolutely irreducible component defined over $\mathbb{F}_q$. Such a component extends to an absolutely irreducible component defined over $\mathbb{F}_q$ in $\mathcal{W}_{f_1,f_2}$ by applying twice Lemma 6.2.5.

C.2. $\ell = B_1 + 2 \geq 3$ and $B_1 > A_1$.
Then $B_1 > 0$. The component of highest degree $N(X_1, X_4, X_5, X_6)$ in $M(X_1, 1, 1, X_4, X_5, X_6)$ is

$$v(X_4 - X_1)X_5^{B_1}X_6 + wX_4^{A_2}X_5^2 + zX_4^{B_2}X_5X_6,$$

for some $v, w, z \in \mathbb{F}_q$, $v \neq 0$, and $\deg_{X_1}(N(X_1, X_4, X_5, X_6)) = 1$. The claim follows similar then in Case C.1.

C.3. $\ell = B_2 + 2 \geq 3$, and $B_2 > A_1$, $B_2 > B_1$ .
Then $B_2 > 0$. The component of highest degree $N(X_2, X_4, X_5, X_6)$ in $M(1, X_2, 1, X_4, X_5, X_6)$ is

$$(X_5 - X_2)(wX_4^{A_2}X_5 + zX_4^{B_2}X_6),$$

for some $w, z \in \mathbb{F}_q$, $z \neq 0$, and $\deg_{X_6}(N(X_2, X_4, X_5, X_6)) = 1$. The claim follows similar then in Case C.1.

C.4. $\ell = A_2 + 2 \geq 3$ and $A_2 > A_1$, $A_2 > B_1$, $A_2 > B_2$.
Then $A_2 > 0$. The component of highest degree $N(X_2, X_4, X_5, X_6)$ in $M(1, X_2, 1, X_4, X_5, X_6)$ is $w(X_5 - X_2)X_4^{A_2}X_5$, with $w \in \mathbb{F}_q^*$. The claim follows similar then in Case C.1.

In all the cases above we obtain that $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$. $\square$

**Proposition 7.2.7.** *Let*

$$\begin{aligned}\Delta(X_1,X_2,X_4,X_5) = &\Big((X_4-X_1)c_2(X_5)+(X_5-X_2)c_1(X_4)\Big)^2 \\ &-4(X_4-X_1)\left(a_2(X_5)X_4^{p^{h_2}}-a_2(X_2)X_1^{p^{h_2}}+\frac{c_2^2(X_5)}{4}X_4-\frac{c_2^2(X_2)}{4}X_1\right)\\ &-4(X_5-X_2)\left(a_1(X_4)X_5^{p^{h_1}}-a_1(X_1)X_2^{p^{h_1}}+\frac{c_1^2(X_4)}{4}X_5-\frac{c_1^2(X_1)}{4}X_2\right)\end{aligned} \tag{7.10}$$

*If $\Delta$ is not a square in $\overline{\mathbb{F}_q}[X_1,X_2,X_4,X_5]$ then $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.*

*Proof.* Consider the hypersurface $\mathcal{W}'_{f_1,f_2}=\mathcal{W}_{f_1,f_2}\cap(X_3=0)$. Note that, since the polynomial $F(1,X_1,X_2,0,X_4,X_5,X_6)$ is of degree 2 in $X_6$, $\mathcal{W}'_{f_1,f_2}$ is absolutely irreducible if and only if the discriminant $\Delta(X_1,X_2,X_4,X_5)$ with respect to $X_6$ is a non-square in $\overline{\mathbb{F}_q}[X_1,X_2,X_4,X_5]$. Therefore, if $\Delta(X_1,X_2,X_4,X_5)$ is not a square in $\overline{\mathbb{F}_q}[X_1,X_2,X_4,X_5]$ then $\mathcal{W}'_{f_1,f_2}$ is absolutely irreducible and by Lemma 6.2.5 the hypersurface $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$. $\square$

**Proposition 7.2.8.** *Let $p^{h_1}=1$ and $p^{h_2}\geq 3$ or $p^{h_1}\geq 3$ and $p^{h_2}=1$. Then $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.*

*Proof.* Suppose by way of contradiction that $\mathcal{W}_{f_1,f_2}$ does not contain an absolutely irreducible component defined over $\mathbb{F}_q$.

Without loss of generality, we can suppose that $p^{h_1}\geq 3$ and $p^{h_2}=1$. Let $\widetilde{a_2}(X)=a_2(X)+\frac{c_2^2(X)}{4}$, with $\deg(\widetilde{a_2})=\widetilde{A_2}$. Now, the polynomial $F(1,X_1,X_2,X_3,X_4,X_5,X_6)$ reads

$$(X_3-X_6)^2+(X_4-X_1)\Big(\widetilde{a_2}(X_5)X_4-\widetilde{a_2}(X_2)X_1+c_2(X_5)X_6-c_2(X_2)X_3\Big)$$
$$+(X_5-X_2)\left(a_1(X_4)X_5^{p^{h_1}}-a_1(X_1)X_2^{p^{h_1}}+\frac{c_1^2(X_4)}{4}X_5-\frac{c_1^2(X_1)}{4}X_2+c_1(X_4)X_6-c_1(X_1)X_3\right).$$

First, we claim that the homogeneous term $M(X_1,X_2,X_3,X_4,X_5,X_6)$ of highest degree in the polynomial $F(1,X_1,X_2,X_3,X_4,X_5,X_6)$ is a linear combination (with not all the coefficients zero) of

$$\begin{aligned} t_1 &= (X_4-X_1)(X_5^{\widetilde{A_2}}X_4-X_2^{\widetilde{A_2}}X_1),\\ t_2 &= (X_4-X_1)(X_5^{C_2}X_6-X_2^{C_2}X_3),\\ t_3 &= (X_5-X_2)(X_4^{A_1}X_5^{p^{h_1}}-X_1^{A_1}X_2^{p^{h_1}}),\\ t_4 &= (X_5-X_2)(X_4^{2C_1}X_5-X_1^{2C_1}X_2).\end{aligned}$$

To see this, it is enough to observe that any such linear combination cannot vanish since each of the following monomials

$$X_1X_4X_5^{\widetilde{A_2}}, X_1X_6X_5^{C_2}, X_1^{A_1}X_2^{p^{h_1}}X_5, X_1^{2C_1}X_2X_5$$

is contained in a unique $t_i$. Also, if $t_1$, $t_2$ or $t_4$ are nonzero then $\widetilde{A_2}$, $C_2$, or $C_1$ (respectively) are nonnegative, since $\deg(\mathcal{W}_{f_1,f_2}) > 3$.

Thus $M(X_1,X_2,X_3,X_4,X_5,X_6) = u_1t_1 + u_2t_2 + u_3t_3 + u_4t_4$ for some $u_i \in \mathbb{F}_q$, not all vanishing.

If $u_2 \neq 0$, then $M(X_1,X_2,X_3,X_4,X_5,X_6)$ is of degree one in $X_6$ and therefore the hypersurface $M(X_1,X_2,X_3,X_4,X_5,X_6) = 0$ contains a nonrepeated absolutely irreducible component defined over $\mathbb{F}_q$. Such a component extends to an absolutely irreducible component defined over $\mathbb{F}_q$ in $\mathcal{W}_{f_1,f_2}$ by applying Lemma 6.2.5, a contradiction.

From now on we can suppose that $u_2 = 0$. If $u_1 \neq 0$, then the part of the highest degree in $M(X_1,X_2,X_3,X_4,X_5,X_6)$ is of degree one in $X_1$ and it contains a nonrepeated absolutely irreducible component defined over $\mathbb{F}_q$. Such a component extends to an absolutely irreducible component defined over $\mathbb{F}_q$ in $\mathcal{W}_{f_1,f_2}$ by applying twice Lemma 6.2.5, a contradiction.

Thus, only $u_3$ and $u_4$ can be nonzero. If $u_4 \neq 0$ or $u_4 = 0$ and $A_1 \neq 0$ then (since $C_1 > 0$) $X_5 - X_2$ is a nonrepeated absolutely irreducible factor defined over $\mathbb{F}_q$ of $M(X_1,X_2,X_3,X_4,X_5,X_6)$. Arguing as above we get a contradiction.

So $u_1 = u_2 = u_4 = 0$, $u_3 \neq 0$, $A_1 = 0$, and $M(X_1,X_2,X_3,X_4,X_5,X_6) = u_3(X_5 - X_2)^{p^{h_1}+1}$. Note that $p^{h_1} + 1 > \max\{2C_1 + 2, C_2 + 2\}$.

We now investigate the discriminant $\Delta(X_1,X_2,X_4,X_5)$ of $F(1,X_1,X_2,0,X_4,X_5,X_6)$ with respect to $X_6$. Recall that, since $\mathcal{W}_{f_1,f_2}$ does not contain an absolutely irreducible component defined over $\mathbb{F}_q$, $\Delta(X_1,X_2,X_4,X_5)$ must be a square.

By direct computations, $\Delta(X_1,X_2,X_4,X_5)$ reads

$$\begin{array}{rl} & ((X_4-X_1)c_2(X_5)+(X_5-X_2)c_1(X_4))^2 - 4(X_4-X_1)(a_2(X_5)X_4 - a_2(X_2)X_1) \\ & -(X_4-X_1)(c_2^2(X_5)X_4 - c_2^2(X_2)X_1) - 4\alpha(X_5-X_2)^{p^{h_1}+1} - (X_5-X_2)(c_1^2(X_4)X_5 - c_1^2(X_1)X_2) \\ = & -4\alpha(X_5-X_2)^{p^{h_1}+1} + (X_4-X_1)X_1(c_2^2(X_2)-c_2^2(X_5)) + (X_5-X_2)X_2(c_1^2(X_1)-c_1^2(X_4)) \\ & +2(X_4-X_1)(X_5-X_2)c_2(X_5)c_1(X_4) - 4(X_4-X_1)(a_2(X_5)X_4 - a_2(X_2)X_1), \end{array}$$

for some $\alpha \neq 0$.

Clearly, the homogeneous component of highest degree in $\Delta$ is $-4\alpha(X_5 - X_2)^{p^{h_1}+1}$, which is a square. Now we claim that the homogeneous term $N(X_1,X_2,X_4,X_5)$ of the second highest degree in $\Delta(X_1,X_2,X_4,X_5)$ is a linear combination of

$$\begin{array}{rcl} z_1 & = & (X_4-X_1)X_1(X_2^{2C_2} - X_5^{2C_2}), \\ z_2 & = & (X_5-X_2)X_2(X_1^{2C_1} - X_4^{2C_1}), \\ z_3 & = & (X_4-X_1)(X_5^{A_2}X_4 - X_2^{A_2}X_1), \\ z_4 & = & (X_4-X_1)(X_5-X_2)X_5^{C_2}X_4^{C_1}. \end{array}$$

We can suppose that $C_1, C_2 > 0$, otherwise we just exclude $z_1$ and $z_2$.

If $C_1 = 0$ and $C_2 > 0$, then $z_2 \equiv 0$, $\deg(z_1) > \deg(z_4)$, and any linear combination of $z_1$ and $z_3$ does not vanish. A similar argument applies to the case $C_1 > 0$ and $C_2 = 0$. If $C_1 = C_2 = 0$ then $z_1 \equiv z_2 \equiv 0$ and any linear combination of $z_3$ and $z_4$ does not vanish.

So we consider only $C_1, C_2 > 0$.

1. $(C_1, A_2) \neq (1, C_2+1)$. The four monomials $X_1^2 X_5^{2C_2}, X_2^2 X_4^{2C_1}, X_4^2 X_5^{A_2}, X_1 X_2 X_4^{C_1} X_5^{C_2}$ belong to a unique $z_i$ and therefore a nontrivial linear combination of $z_i$ cannot vanish.

2. $(C_1, A_2) = (1, C_2 + 1)$. If $C_2 > 1$ the degree of $z_1$ is the highest among the $z_i$'s.

3. $(C_1, C_2, A_2) = (1, 1, 2)$. If $\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3 + \lambda_4 z_4 = 0$ then $\lambda_3 = 0$ since $(X_5 - X_2)$ divides $z_1, z_2, z_4$ but not $z_3$. Now, it can be readily seen that $\lambda_1 z_1 + \lambda_2 z_2 + \lambda_4 z_4$ cannot vanish for any $(\lambda_1, \lambda_2, \lambda_4) \neq (0, 0, 0)$.

Thus we know that

$$N(X_1, X_2, X_4, X_5) = \delta_1 w_1^2 z_1 + \delta_2 w_2^2 z_2 + 2 w_1 w_2 \delta_4 z_4 + \delta_3 w_3 z_3,$$

for some $w_1, w_2, w_3 \in \mathbb{F}_q^*$, and $\delta_1, \delta_2, \delta_3, \delta_4 \in \{0, 1\}$, where $\delta_i = 1$ only if the corresponding $z_i$ has the highest degree.

Recall that since $\Delta(X_1, X_2, X_4, X_5)$ is a square, $(X_5 - X_2)^{(p^{h_1}+1)/2} \mid N(X_1, X_2, X_4, X_5)$ and $(p^{h_1} + 1)/2 \geq 2$; see Lemma 6.2.7.

If $C_1 = C_2 = 0$ then $N(X_1, X_2, X_4, X_5) = \delta_3 w_3 (X_4 - X_1)(X_5^{A_2} X_4 - X_2^{A_2} X_1) + 2\delta_4 w_1 w_2 (X_4 - X_1)(X_5 - X_1)$ which is never divisible by $(X_5 - X_2)^2$.

If $C_1 = 0$ and $C_2 > 0$ then $N(X_1, X_2, X_4, X_5)$ is

$$\delta_1 w_1^2 (X_4 - X_1) X_1 (X_2^{2C_2} - X_5^{2C_2}) + \delta_3 w_3 (X_4 - X_1)(X_5^{A_2} X_4 - X_2^{A_2} X_1) + 2\delta_4 w_1 w_2 (X_4 - X_1)(X_5 - X_1) X_5^{C_2},$$

which is divisible by $(X_5 - X_2)$ only if $\delta_3 = 0$. Also $C_2 > 0$ yields $\delta_4 = 0$ since $\deg(z_1) > \deg(z_4)$. This forces $p \mid 2C_2$.

Analogously, $C_1 > 0$ and $C_2 = 0$ yields a contradiction.

Suppose now that $C_1, C_2 > 0$. Since $(X_5 - X_2)$ must divide $N(X_1, X_2, X_4, X_5)$, $\delta_3 = 0$. Now $(X_5 - X_2)^2$ divides $N(X_1, X_2, X_4, X_5)$ only if

$$\phi(X_1, X_2, X_4) = 2C_2 \delta_1 w_1^2 (X_4 - X_1) X_1 X_2^{2C_2 - 1} + \delta_2 w_2^2 X_2 (X_1^{2C_1} - X_4^{2C_1}) + 2\delta_4 w_1 w_2 (X_4 - X_1) X_2^{C_2} X_4^{C_1}$$

is the zero-polynomial.

If $\delta_2 \neq 0$, then, considering $X_4 = 0$, we get

$$\phi(X_1, X_2, X_4) = -2C_2 \delta_1 w_1^2 X_1^2 X_2^{2C_2 - 1} + \delta_2 w_2^2 X_2 X_1^{2C_1},$$

which does not vanish unless $C_1 = C_2 = 1$. In this case $\delta_1 = \delta_2 = \delta_4 = 1$ and

$$\phi(X_1, X_2, X_4) = 2w_1^2 (X_4 - X_1) X_1 X_2 + w_2^2 X_2 (X_1^2 - X_4^2) + 2 w_1 w_2 (X_4 - X_1) X_2 X_4$$

vanishes only if $2w_1^2 = 2w_1w_2$, $2w_1^2 = w_2^2$, which yields $w_1 = w_2 = 0$, a contradiction.

So we can suppose that $\delta_2 = 0$ and

$$\phi(X_1, X_2, X_4) = 2C_2\delta_1 w_1^2(X_4 - X_1)X_1X_2^{2C_2-1} + 2\delta_4 w_1w_2(X_4 - X_1)X_2^{C_2}X_4^{C_1}.$$

It is readily seen that $\phi$ vanishes only if $\delta_4 = 0$ and $p \mid 2C_2$.

Summing up, $N(X_1, X_2, X_4, X_5)$ is divisible by $(X_2-X_5)^{(p^{h_1}+1)/2\geq 2}$ only if $N(X_1, X_2, X_4, X_5) = (X_4 - X_1)X_1(X_2^{2C_2} - X_5^{2C_2})$ with $p \mid 2C_2$. Let $\ell$ be the highest power of $p$ dividing $2C_2$. From $p^{h_1} + 1 > \max\{2C_1 + 2, C_2 + 2\}$, $\ell \leq h_1 - 1$ and $(X_5 - X_2)^{p^\ell+1}$ does not divide $N(X_1, X_2, X_4, X_5)$. A final contradiction arises from $(p^{h_1}+1)/2 \leq p^\ell \leq p^{h_1-1}$. The claim follows from Proposition 7.2.7. □

By Proposition 7.2.8, if $\mathcal{W}_{f_1,f_2}$ does not contain an absolutely irreducible component defined over $\mathbb{F}_q$ then both $p^{h_1} > 1$ and $p^{h_2} > 1$ and $\deg(\mathcal{W}_{f_1,f_2}) \geq 4$.

**Proposition 7.2.9.** *Let $\mathcal{W}_{f_1,f_2}$ be the hypersurface defined as in (7.7). Suppose that $\deg(\mathcal{W}_{f_1,f_2}) \geq 4$ and $p^{h_1}, p^{h_2} > 1$. If $A_1A_2 > 0$ then $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.*

*Proof.* We proceed as in the proof of Proposition 7.2.8. Consider the variety $\mathcal{W}_{f_1,f_2}^\infty = \mathcal{W}_{f_1,f_2} \cap (X_0 = 0)$. It is defined by the homogeneous part $L(X_1, X_2, X_3, X_4, X_5, X_6)$ of highest degree in (7.7) and it arises from a linear combination (with not all the coefficients zero) of

$$\begin{aligned}
t_1 &= (X_4 - X_1)(X_5^{A_2}X_4^{p^{h_2}} - X_2^{A_2}X_1^{p^{h_2}}),\\
t_2 &= (X_4 - X_1)(X_5^{2C_2}X_4 - X_2^{2C_2}X_1),\\
t_3 &= (X_5 - X_2)(X_4^{A_1}X_5^{p^{h_1}} - X_1^{A_1}X_2^{p^{h_1}}),\\
t_4 &= (X_5 - X_2)(X_4^{2C_1}X_5 - X_1^{2C_1}X_2).
\end{aligned}$$

To see this, it is enough to observe any such linear combination cannot vanish since each of the following monomials

$$X_1^{p^{h_2}}X_2^{A_2}X_4,\ X_1X_2^{2C_2}X_4,\ X_1^{A_1}X_2^{p^{h_1}}X_5,\ X_1^{2C_1}X_2X_5$$

is contained in a unique $t_i$.

Thus

$$L(X_1, X_2, X_3, X_4, X_5, X_6) = \delta_1u_1t_1 + \delta_2u_2t_2 + \delta_3u_3t_3 + \delta_4u_4t_4,$$

with $\delta_i \in \{0, 1\}$, $u_i \in \mathbb{F}_q^*$, and not all the $\delta_i$'s vanish. We distinguish two cases.

1. $(\delta_1, \delta_2) \neq (0, 0)$. The homogeneous term in $L(X_1, 1, X_3, X_4, X_5, X_6)$ of highest degree is

$$M = \delta_1u_1(X_4-X_1)X_5^{A_2}X_4^{p^{h_2}} + \delta_2u_2(X_4-X_1)X_5^{2C_2}X_4 + \delta_3u_3X_4^{A_1}X_5^{p^{h_1}+1} + \delta_4u_4X_5^2X_4^{2C_1},$$

which is of degree 1 in $X_1$. Thus there exists a non-repeated $\mathbb{F}_q$-rational factor of $M$ (depending on $X_1$) which is absolutely irreducible. Therefore $L(X_1, 1, X_3, X_4, X_5, X_6)$ contains a non-repeated absolutely irreducible factor defined over $\mathbb{F}_q$, by Lemma 6.2.5. Again by Lemma 6.2.5 the corresponding variety extends to a non-repeated $\mathbb{F}_q$-rational absolutely irreducible component in $\mathcal{W}^{\infty}_{f_1,f_2}$.

2. $(\delta_1, \delta_2) = (0, 0)$. So, $(\delta_3, \delta_4) \neq (0, 0)$. The homogeneous term in $L(1, X_2, X_3, X_4, X_5, X_6)$ of highest degree is

$$M = \delta_3 u_3 (X_5 - X_2) X_4^{A_1} X_5^{p^{h_1}} + \delta_4 u_4 (X_5 - X_2) X_4^{2C_1} X_5,$$

which is of degree 1 in $X_2$. Thus we obtain the same conclusion as above.

□

**Proposition 7.2.10.** *Let $\mathcal{W}_{f_1,f_2}$ be the hypersurface defined as in (7.7). Suppose that* $\deg(\mathcal{W}_{f_1,f_2}) \geq 4$ *and* $p^{h_1}, p^{h_2} > 1$. *If* $(A_1, A_2) \notin \{(0, -\infty), (-\infty, 0), (-\infty, -\infty), (0, 0)\}$ *then the hypersurface $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over* $\mathbb{F}_q$.

*Proof.* Suppose that $A_1 > 0$, the case $A_2 > 0$ follows completely similar. In view of Proposition 7.2.9, we only have to consider $A_2 \leq 0$.

Following the same notation as in Proposition 7.2.9, let $L(X_1, X_2, X_3, X_4, X_5, X_6)$ be the part of the highest degree in (7.7). It arises from a linear combination (with not all the coefficients zero) $\delta_1 u_1 t_1 + \delta_2 u_2 t_2 + \delta_3 u_3 t_3 + \delta_4 u_4 t_4$, where

$$\begin{aligned} t_1 &= (X_4 - X_1)^{p^{h_2}+1}, \\ t_2 &= (X_4 - X_1)(X_5^{2C_2} X_4 - X_2^{2C_2} X_1), \\ t_3 &= (X_5 - X_2)(X_4^{A_1} X_5^{p^{h_1}} - X_1^{A_1} X_2^{p^{h_1}}), \\ t_4 &= (X_5 - X_2)(X_4^{2C_1} X_5 - X_1^{2C_1} X_2). \end{aligned}$$

Note that if $A_2 = -\infty$, that is $a_2 \equiv 0$, we may assume $\delta_1 = 0$.

To see this, it is enough to observe any such linear combination cannot vanish since each of the following monomials

$$X_1^{p^{h_2}+1}, X_1 X_2^{2C_2} X_4, X_1^{A_1} X_2^{p^{h_1}} X_5, X_1^{2C_1} X_2 X_5$$

is contained in a unique $t_i$.

Suppose that $(\delta_3, \delta_4) \neq (0, 0)$. The homogeneous term in $L(1, X_2, X_3, X_4, X_5, X_6)$ of highest degree is

$$M = \delta_1 u_1 X_4^{p^{h_2}+1} + \delta_2 u_2 X_4^2 X_5^{2C_2} + \delta_3 u_3 (X_5 - X_2) X_4^{A_1} X_5^{p^{h_1}} + \delta_4 u_4 (X_5 - X_2) X_4^{2C_1} X_5,$$

which is of degree 1 in $X_2$. Thus there exists a non-repeated $\mathbb{F}_q$-rational factor of $M$ (depending on $X_1$) which is absolutely irreducible. Therefore $L(1, X_2, X_3, X_4, X_5, X_6)$

contains a non-repeated absolutely irreducible factor defined over $\mathbb{F}_q$, by Lemma 6.2.5. Again by Lemma 6.2.5 the corresponding variety extends to a non-repeated $\mathbb{F}_q$-rational absolutely irreducible component in $\mathcal{W}^{\infty}_{f_1,f_2}$. Thus $(\delta_3, \delta_4) = (0, 0)$.

If $\delta_1 = 0$ or $\delta_1\delta_2 \neq 0$ then $C_2 > 0$ and $(X_4 - X_1)$ is a non-repeated $\mathbb{F}_q$-rational absolutely irreducible factor of $L(X_1, X_2, X_3, X_4, X_5, X_6)$ and, arguing as above, $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.

Thus, we have only to deal with the case $\delta_1 \neq 0$ (which implies $A_2 = 0$) and $\delta_2 = \delta_3 = \delta_4 = 0$. Note that $p^{h_2} + 1 > \max\{2C_2 + 2, A_1 + p^{h_1} + 1, 2C_1 + 2\}$.

Consider now $\Delta(X_1, X_2, X_4, X_5)$ defined as in (7.10) and we will prove that it is not a square in $\overline{\mathbb{F}_q}[X_1, X_2, X_4, X_5]$ using Lemma 6.2.7.

Suppose by way of contradiction that $\Delta(X_1, X_2, X_4, X_5)$ is a square in $\overline{\mathbb{F}_q}[X_1, X_2, X_4, X_5]$.

Hence, $\Delta(X_1, X_2, X_4, 0)$ is a square in $\overline{\mathbb{F}_q}[X_1, X_2, X_4]$. Recalling that $p^{h_1} \geq 3$, it is readily seen that its homogeneous component of the second largest degree is

$$N(X_1, X_2, X_4) \;=\; \delta_1 u_1 X_2^2 X_4^{2C_1} + \delta_2 u_2 (X_4 - X_1) X_1 X_2^{2C_2} + \delta_3 u_3 X_2^{p^{h_1}+1} X_1^{A_1} - \delta_1 u_1 X_2^2 X_1^{2C_1},$$

for some $\delta_1, \delta_2, \delta_3 \in \{0, 1\}$ not all vanishing, and $u_1, u_2, u_3 \in \mathbb{F}_q$. By Lemma 6.2.7, $(X_1 - X_4)^{(p^{h_2}+1)/2}$ must divide $N(X_1, X_2, X_4)$. This immediately forces $\delta_3 = 0$ and thus $p^{h_1} + 1 + A_1 < \min\{2 + 2C_1, 2 + 2C_2\}$.

Also, $N'(X_1, X_2, X_4) = N(X_1, X_2, X_4)/(X_4 - X_1)$ reads

$$\delta_1 u_1 X_2^2 \left( \sum_{i=0}^{2C_1 - 1} X_4^i X_1^{2C_1 - 1 - i} \right) + \delta_2 u_2 X_1 X_2^{2C_2}.$$

Now, $(X_4 - X_1)^2 \mid N(X_1, X_2, X_4)$ if and only if $N'(X_1, X_2, X_1) \equiv 0$, that is

$$\delta_1 u_1 (2C_1) X_2^2 X_1^{2C_1 - 1} + \delta_2 u_2 X_1 X_2^{2C_2} \equiv 0.$$

The case $C_2 = 1 = C_1$ must be excluded, since $4 \leq p^{h_1+1} + A_1 < \min\{2 + 2C_1, 2 + 2C_2\} = 4$ yields a contradiction. The only possibility is then $\delta_2 = 0$, $\delta_1 = 1$, $p \mid C_1$. In this case $N(X_1, X_2, X_4) = u_1 X_2^2 (X_4^{2C_1} - X_1^{2C_1}) = u_1 X_2^2 (X_4^{C_1} - X_1^{C_1})(X_4^{C_1} + X_1^{C_1})$, which is not divisible by $(X_1 - X_4)^{(p^{h_2}+1)/2}$, since $2C_1 < p^{h_2} - 1$. The claim follows from Proposition 7.2.7. □

**Proposition 7.2.11.** *Let $\mathcal{W}_{f_1,f_2}$ be the hypersurface defined as in (7.7). Suppose that* $\deg(\mathcal{W}_{f_1,f_2}) \geq 4$ *and $p^{h_1}, p^{h_2} > 1$. If $(A_1, A_2) = (0, -\infty)$ or $(A_1, A_2) = (-\infty, 0)$ then the hypersurface $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$ or $\mathcal{O}_6(f_1, f_2)$ is not an ovoid.*

*Proof.* We consider only the case $(A_1, A_2) = (0, -\infty)$, the case $(A_1, A_2) = (-\infty, 0)$ being similar.

Note that $C_2 \neq -\infty$, otherwise $f_2 \equiv 0$.

If $C_1 = -\infty$ and $C_2 = 0$ then $\mathcal{W}_{f_1,f_2}$ reads

$$(X_3 - X_6)^2 + (X_4 - X_1)\left( \frac{c_2^2}{4}(X_4 - X_1) + c_2(X_6 - X_3) \right) + a_1 (X_5 - X_2)^{p^{h_1}+1} = 0$$

and contains all the points on $X_5 = X_2$, $2(X_6 - X_3) + c_2(X_4 - X_1) = 0$ and $\mathcal{O}_6(f_1, f_2)$ is not an ovoid.

Following the same notation as in Proposition 7.2.9, let $L(X_1, X_2, X_3, X_4, X_5, X_6)$ be the part of the highest degree in (7.7). It arises from a linear combination (with not all the coefficients zero) $\delta_2 u_2 t_2 + \delta_3 u_3 t_3 + \delta_4 u_4 t_4$, where

$$\begin{aligned} t_2 &= (X_4 - X_1)(X_5^{2C_2} X_4 - X_2^{2C_2} X_1), \\ t_3 &= (X_5 - X_2)^{p^{h_1}+1}, \\ t_4 &= (X_5 - X_2)(X_4^{2C_1} X_5 - X_1^{2C_1} X_2). \end{aligned}$$

To see this, it is enough to observe any such linear combination cannot vanish since each of the following monomials

$$X_1 X_2^{2C_2} X_4, X_2^{p^{h_1}+1}, X_1^{2C_1} X_2 X_5$$

is contained in a unique $t_i$.

If $\delta_2 \neq 0$ then $C_2 > 0$ since $\deg(\mathcal{W}_{f_1,f_2}) > 3$ and the homogeneous term in $L(X_1, 1, X_3, X_4, X_5, X_6)$ of highest degree is of degree 1 in $X_1$. Arguing as in the previous propositions $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$. So $\delta_2 = 0$. If $\delta_4 \neq 0$ then $C_1 > 0$ and $(X_5 - X_2)$ is a nonrepeated factor of $L(X_1, X_2, X_3, X_4, X_5, X_6)$. Arguing as above, $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.

Thus $\delta_3 \neq 0$ and $\delta_2 = \delta_4 = 0$, which yields $p^{h_1} + 1 > \max\{2C_2 + 2, 2C_1 + 2\}$.

We consider now $\Delta(X_1, X_2, X_4, X_5)$ as in (7.10) and we show that it is not a square in $\overline{\mathbb{F}_q}[X_1, X_2, X_4, X_5]$ by means of Lemma 6.2.7. We have that

$$\begin{aligned} \Delta(X_1, X_2, X_4, X_5) &= \Big((X_4 - X_1)c_2(X_5) + (X_5 - X_2)c_1(X_4)\Big)^2 \\ &\quad -4(X_4 - X_1)\left(\frac{c_2^2(X_5)}{4}X_4 - \frac{c_2^2(X_2)}{4}X_1\right) \\ &\quad -4(X_5 - X_2)\left(a_1 X_5^{p^{h_1}} - a_1 X_2^{p^{h_1}} + \frac{c_1^2(X_4)}{4}X_5 - \frac{c_1^2(X_1)}{4}X_2\right) \\ &= X_1(X_4 - X_1)(c_2^2(X_2) - c_2^2(X_5)) + X_2(X_5 - X_2)(c_1^2(X_1) - c_1^2(X_4)) + \\ &\quad \alpha(X_5 - X_2)^{p^{h_1}+1} + 2(X_5 - X_2)(X_4 - X_1)c_1(X_4)c_2(X_5). \end{aligned}$$

If $C_1 = -\infty$ and $C_2 > 0$, $\Delta(X_1, X_2, X_4, X_5)$ is not a square and, by Proposition 7.2.7, $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.

So we can suppose that $C_1, C_2 \geq 0$.

If $\Delta(X_1, X_2, X_4, X_5)$ is a square in $\overline{\mathbb{F}_q}[X_1, X_2, X_4, X_5]$, then the homogeneous term of the highest degree in $\Delta(X_1, X_2, X_4, X_5)$ is clearly $(X_5 - X_2)^{p^{h_1}+1}$ and $\Delta(0, X_2, X_4, X_5)$ is a square in $\overline{\mathbb{F}_q}[X_2, X_4, X_5]$. Recalling that $p^{h_1} \geq 3$, it is readily seen that its homogeneous component of the second largest degree is

$$N(X_2, X_4, X_5) = \delta_1 u_1 X_2 (X_5 - X_2) X_4^{2C_1} + \delta_2 u_2 (X_5 - X_2) X_4^{C_1+1} X_5^{C_2},$$

for some $\delta_1, \delta_2 \in \{0,1\}$ not all vanishing, and $u_1, u_2 \in \mathbb{F}_q$. By Lemma 6.2.7, $(X_5 - X_2)^{(p^{h_1}+1)/2}$ must divide $N(X_2, X_4, X_5)$. This immediately forces $p^{h_1} = 3$, $C_1 = C_2 = 1$, a contradiction to $4 = p^{h_1} + 1 > \max\{2C_2 + 2, 2C_1 + 2\}$. Thus $\Delta(X_1, X_2, X_4, X_5)$ is not a square in $\overline{\mathbb{F}_q}[X_1, X_2, X_4, X_5]$ and, by Proposition 7.2.7, $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$. $\square$

**Proposition 7.2.12.** *Let $\mathcal{W}_{f_1,f_2}$ be the hypersurface defined as in (7.7). Suppose that $\deg(\mathcal{W}_{f_1,f_2}) \geq 4$, $(A_1, A_2) = (-\infty, -\infty)$, and $p^{h_1}, p^{h_2} > 1$. Then the hypersurface $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.*

*Proof.* Since $(A_1, A_2) = (-\infty, -\infty)$, the polynomial $F(1, X_1, X_2, X_3, X_4, X_5, X_6)$ defining $\mathcal{W}_{f_1,f_2}$ reads

$$(X_3 - X_6)^2 + (X_4 - X_1)\left(\frac{c_2^2(X_5)}{4}X_4 - \frac{c_2^2(X_2)}{4}X_1 + c_2(X_5)X_6 - c_2(X_2)X_3\right)$$
$$+(X_5 - X_2)\left(\frac{c_1^2(X_4)}{4}X_5 - \frac{c_1^2(X_1)}{4}X_2 + c_1(X_4)X_6 - c_1(X_1)X_3\right). \qquad (7.11)$$

We can consider $C_1, C_2 \geq 0$ otherwise $f_1$ or $f_2$ vanishes. Also, either $C_1$ or $C_2$ is positive otherwise $\mathcal{W}_{f_1,f_2}$ has degree 2.

Thus

$$\begin{aligned}
\Delta(X_1, X_2, X_4, X_5) &= \Big((X_4 - X_1)c_2(X_5) + (X_5 - X_2)c_1(X_4)\Big)^2 \\
&\quad -4(X_4 - X_1)\left(\frac{c_2^2(X_5)}{4}X_4 - \frac{c_2^2(X_2)}{4}X_1\right) \\
&\quad -4(X_5 - X_2)\left(\frac{c_1^2(X_4)}{4}X_5 - \frac{c_1^2(X_1)}{4}X_2\right) \\
&= X_1(X_4 - X_1)(c_2^2(X_2) - c_2^2(X_5)) + X_2(X_5 - X_2)(c_1^2(X_1) - c_1^2(X_4)) \\
&\quad +2(X_5 - X_2)(X_4 - X_1)c_2(X_5)c_1(X_4).
\end{aligned}$$

If $C_2 > C_1$ (resp. $C_1 > C_2$) then the highest-degree homogeneous term in $\Delta(X_1, X_2, X_4, X_5)$ is $\alpha^2 X_1(X_4 - X_1)(X_2^{2C_2} - X_5^{2C_2})$ (resp. $\alpha^2 X_2(X_5 - X_2)(X_1^{2C_1} - X_4^{2C_1})$) which is not a square. If $C_2 = C_1 > 1$ then the highest-degree homogeneous term in $\Delta(X_1, X_2, 0, 0)$

$$-\alpha^2 X_1^2 X_2^{2C_2} - \beta^2 X_2^2 X_1^{2C_1} = X_1^2 X_2^2(-\alpha^2 X_2^{2C_1-2} - \beta^2 X_1^{2C_1-2})$$

is not a square.

If $C_2 = C_1 = 1$ then the highest-degree homogeneous term in $\Delta(X_1, X_2, X_4, X_5)$

$$\alpha^2 X_1(X_4 - X_1)(X_2^2 - X_5^2) + \beta^2 X_2(X_5 - X_2)(X_1^2 - X_4^2) + 2\alpha\beta(X_5 - X_2)(X_4 - X_1)X_4X_5$$
$$= (X_4 - X_1)(X_5 - X_2)(-\alpha^2 X_1(X_2 + X_5) - \beta^2 X_2(X_1 + X_4) + 2\alpha\beta X_4X_5)$$

is not a square, since $(X_5 - X_2)$ does not divide $(-\alpha^2 X_1(X_2 + X_5) - \beta^2 X_2(X_1 + X_4) + 2\alpha\beta X_4X_5)$ unless $\alpha = \beta = 0$, which is not possible. In all these cases, by Lemma 6.2.7,

$\Delta(X_1, X_2, X_4, X_5)$ is not a square and, by Proposition 7.2.7, $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$. $\square$

**Proposition 7.2.13.** *Let $\mathcal{W}_{f_1,f_2}$ be the hypersurface defined as in (7.7). Suppose that $\deg(\mathcal{W}_{f_1,f_2}) \geq 4$, $(A_1, A_2) = (0, 0)$, and $p^{h_1}, p^{h_2} > 1$. Then the hypersurface $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$ unless*

$$p^{h_1} = p^{h_2} = 3, \text{ and } C_1, C_2 \leq 1.$$

*Proof.* Similar as in Proposition 7.2.12, we have that

$$\begin{aligned}
\Delta(X_1, X_2, X_4, X_5) &= \Big((X_4 - X_1)c_2(X_5) + (X_5 - X_2)c_1(X_4)\Big)^2 \\
&\quad -4(X_4 - X_1)\left(a_2X_4^{p^{h_2}} - a_2X_1^{p^{h_2}} + \frac{c_2^2(X_5)}{4}X_4 - \frac{c_2^2(X_2)}{4}X_1\right) \\
&\quad -4(X_5 - X_2)\left(a_1X_5^{p^{h_1}} - a_1X_2^{p^{h_1}} + \frac{c_1^2(X_4)}{4}X_5 - \frac{c_1^2(X_1)}{4}X_2\right) \\
&= X_1(X_4 - X_1)(c_2^2(X_2) - c_2^2(X_5)) + X_2(X_5 - X_2)(c_1^2(X_1) - c_1^2(X_4)) \\
&\quad +2(X_5 - X_2)(X_4 - X_1)c_2(X_5)c_1(X_4) - 4a_2(X_4 - X_1)^{p^{h_2}+1} \\
&\quad -4a_1(X_5 - X_2)^{p^{h_1}+1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta(0, X_2, 1, X_5) &= X_2(X_5 - X_2)(c_1^2(0) - c_1^2(1)) + 2(X_5 - X_2)c_2(X_5)c_1(1) \\
&\quad -4a_2 - 4a_1(X_5 - X_2)^{p^{h_1}+1},
\end{aligned}$$

where $a_1a_2 \neq 0$.

We distinguish a few cases:

1. $c_1(1) = 0$.
   Then $\Delta(0, X_2, 1, X_5)$ reads $-4a_2 - 4a_1(X_5 - X_2)^{p^{h_1}+1} + c_1^2(0)(X_5 - X_2)X_2$ and it cannot be a square.

2. $c_1(1) \neq 0$.

   - If $C_2 = p^{h_1} > 1$ then $\Delta(0, X_2, 1, X_5)$ cannot be a square since its highest-degree homogeneous term is $-4(X_5 - X_2)^{p^{h_1}+1} + 2c_1(1)c_2(X_5)(X_5 - X_2)$, a nonsquare.
   - If $C_2 > p^{h_1}$, then $2c_1(1)c_2(X_5)(X_5 - X_2)$, the highest-degree homogeneous term in $\Delta(0, X_2, 1, X_5)$ is not a square.
   - If $C_2 < p^{h_1}$, the highest-degree homogeneous term in $\Delta(0, X_2, 1, X_5)$ is $-4a_1(X_5 - X_2)^{p^{h_1}+1}$. Since no other homogeneous term in $\Delta(0, X_2, 1, X_5)$ is divisible by $(X_5 - X_2)^{(p^{h_1}+1)/2}$ if $p^{h_1} + 1 > 4$, $\Delta(0, X_2, 1, X_5)$ is not a square if $p^{h_1} > 3$.

Also, when $p^{h_1} = 3$, and $C_2 < p^{h_1}$, we must have $C_2 \leq 1$, otherwise the second highest-degree homogeneous term in $\Delta(0, X_2, 1, X_5)$ is $X_5^2(X_5 - X_2)$ which is not divisible by $(X_5 - X_2)^{(p^{h_1}+1)/2} = (X_5 - X_2)^2$. By Lemma 6.2.7 $\Delta(0, X_2, 1, X_5)$ is not a square.

This shows that $\Delta(0, X_2, 1, X_5)$ is not a square unless $p^{h_1} = 3$ and $C_2 \leq 1$.

A similar argument applied to $\Delta(X_1, 0, X_4, 1)$ shows that is not a square unless $p^{h_2} = 3$ and $C_1 \leq 1$.

So, unless $p^{h_2} = 3 = p^{h_1}$ and $C_1, C_2 \leq 1$, by Proposition 7.2.7, $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$.

□

In view of the previous propositions, if we assume that $\mathcal{O}_6(f_1, f_2)$ is an ovoid and thus $\mathcal{W}_{f_1,f_2}$ does not contain any $\mathbb{F}_q$-rational absolutely irreducible components, then

1. $p^{h_1} = p^{h_2} = 3$;
2. $C_1, C_2 \leq 1$ (possibly $-\infty$),
3. $a_1(X) = a_1$, $a_2(X) = a_2$, $a_1a_2 \neq 0$.

Secondly, from now we assume that

$$c_1(X) = CX + D, \quad c_2(X) = EX + F,$$

for $C, D, F, E \in \mathbb{F}_q$.

**Proposition 7.2.14.** *If $p^{h_1} = p^{h_2} = 3$, $C_1, C_2 \leq 1$ (possibly $-\infty$), and $A_1 = A_2 = 0$, then $\mathcal{W}_{f_1,f_2}$ contains an absolutely irreducible component defined over $\mathbb{F}_q$, unless $(D, F) = (0, 0)$, $C = -E$, and $a_2 = E^4/a_1$. In this case, $\mathcal{O}_6(f_1, f_2)$ is an ovoid if and only if $-a_1$ is not a square in $\mathbb{F}_q$.*

*Proof.* We want to determine whether $\Delta(X_1, X_2, X_4, X_5)$ is a square by applying Lemma 6.2.7. We have

$$\begin{aligned}
\Delta(0, X_2, X_4, 0) &= -(4a_1X_2^4 - C^2X_2^2X_4^2 + 4a_2X_4^4) + 2CX_2X_4(DX_2 - FX_4) - 2FDX_2X_4 \\
&= -(a_1X_2^4 + 2C^2X_2^2X_4^2 + a_2X_4^4) - CX_2X_4(DX_2 - FX_4) + FDX_2X_4.
\end{aligned}$$

First note that $C \neq 0$, otherwise $(4a_1X_2^4 + 2C^2X_2^2X_4^2 + 4a_2X_4^4)$ cannot be a square. Since $FDX_2X_4$ must be a square, $FD = 0$. Now, in order to have that $\Delta(0, X_2, X_4, 0)$ is a square, $-CX_2X_4(DX_2 - FX_4)$ must be a square and this can happen only if it vanishes, i.e. $F = D = 0$. Also, if $(a_1X_2^4 + 2C^2X_2^2X_4^2 + a_2X_4^4)$ is a square, then $C^4 = a_1a_2$.

Now, $\Delta(X_1, X_2, X_4, X_5)$ reads

$$\begin{aligned}
&(E(X_4 - X_1)X_5 + C(X_5 - X_2)X_4)^2 \\
&-(X_4 - X_1)\left(a_2(X_4 - X_1)^3 + E^2X_5^2X_4 - E^2X_2^2X_1\right) \\
&-(X_5 - X_2)\left(a_1(X_5 - X_2)^3 + C^2X_4^2X_5 - C^2X_1^2X_2\right).
\end{aligned}$$

In particular

$$a_1\Delta(X_1,0,X_4,1) = -a_2(X_1-X_4)^4 + Ea_1(EX_1+CX_4)(X_1-X_4) - a_1,$$

which is a square only if $C=-E$. In this case

$$-a_1F(1,X_1,X_2,X_3,X_4,X_5,X_6) = F_1(1,X_1,X_2,X_3,X_4,X_5,X_6)F_2(1,X_1,X_2,X_3,X_4,X_5,X_6),$$

where

$$\begin{aligned} F_1(1,X_1,X_2,X_3,X_4,X_5,X_6) &= E^2X_1^2+E^2X_1X_4+\eta^2\sqrt{a_1}EX_1X_5+a_1X_2^2-\eta^2\sqrt{a_1}EX_2X_4 \\ &\quad +a_1X_2X_5-\eta^2\sqrt{a_1}X_3+E^2X_4^2+a_1X_5^2+\eta^2\sqrt{a_1}X_6; \\ F_2(1,X_1,X_2,X_3,X_4,X_5,X_6) &= E^2X_1^2+E^2X_1X_4-\eta^2\sqrt{a_1}EX_1X_5+a_1X_2^2+\eta^2\sqrt{a_1}EX_2X_4 \\ &\quad +a_1X_2X_5+\eta^2\sqrt{a_1}X_3+E^2X_4^2+a_1X_5^2-\eta^2\sqrt{a_1}X_6, \end{aligned}$$

with $\eta\in\mathbb{F}_9$ satisfying $\eta^2-\eta-1=0$.

If $q=3^{2h}$ and $a_1\in\Box_q$ or $q=3^{2h+1}$ and $a_1\notin\Box_q$ then both the factors $F_1$ and $F_2$ are defined over $\mathbb{F}_q$ and both $F_1=0$ and $F_2=0$ are absolutely irreducible quadrics. Therefore, $\mathcal{O}_6(f_1,f_2)$ is not an ovoid.

If $q=3^{2h}$ and $a_1\notin\Box_q$ or $q=3^{2h+1}$ and $a_1\in\Box_q$ then both the factors $F_1$ and $F_2$ are not defined over $\mathbb{F}_q$ and both $F_1=0$ and $F_2=0$ are absolutely irreducible quadrics. Therefore the $\mathbb{F}_q$-rational solutions satisfy $F_1+F_2=0$ and $F_1-F_2=0$, that is

$$\begin{cases} EX_1X_5+2EX_2X_4+2X_3+X_6 & =0, \\ (EX_1+\eta^2\sqrt{a_1}X_2+2EX_4-\eta^2\sqrt{a_1}X_5)(EX_1-\eta^2\sqrt{a_1}X_2+2EX_4+\eta^2\sqrt{a_1}X_5) & =0. \end{cases}$$

Since both the factors $G_1$ and $G_2$ of the second equation in the system above are not defined over $\mathbb{F}_q$, the $\mathbb{F}_q$-rational solutions also satisfy $G_1+G_2=0$ and $G_1-G_2=0$, and thus

$$\begin{cases} EX_1X_5+2EX_2X_4+2X_3+X_6 & =0, \\ X_1-X_4 & =0, \\ X_2-X_5 & =0. \end{cases}$$

It is readily seen that the system above has only the solutions $X_1-X_4=X_2-X_5=X_3-X_6=0$ and therefore $\mathcal{O}_6(f_1,f_2)$ is an ovoid. The claim follows by observing that the two conditions $q=3^{2h}$ and $a_1\notin\Box_q$ or $q=3^{2h+1}$ and $a_1\in\Box_q$ are equivalent to $-a_1$ not being a square in $\mathbb{F}_q$. $\Box$

### 7.2.3 Conclusion

Summing up, the only choice of $f_1$ and $f_2$ for which $\mathcal{O}_6(f_1,f_2)$ is an ovoid is

$$\begin{aligned} f_1(x,y,z) &= a_1y^3+E^2x^2y-Exz; \\ f_2(x,y,z) &= \frac{E^4}{a_1}x^3+E^2y^2x+Eyz. \end{aligned}$$

In this case the ovoid $\mathcal{O}_6(f_1, f_2)$ reads

$$\left\{\left(1, x, y, z, a_1 y^3 + E^2 x^2 y - Exz, \frac{E^4}{a_1} x^3 + E^2 y^2 x + Eyz, -z^2 - a_1 \left(y^2 + \frac{E^2}{a_1} x^2\right)^2\right)\right\}_{x,y,z \in \mathbb{F}_q}$$
$$\cup\{(0,0,0,0,0,0,1)\},$$

which can be written as

$$\left\{\left(1, x/E, y, z, a_1 y^3 + x^2 y - xz, E\left(x^3/a_1 + y^2 x + yz\right), -z^2 - a_1\left(y^2 + x^2/a_1\right)^2\right)\right\}_{x,y,z \in \mathbb{F}_q}$$
$$\cup\{(0,0,0,0,0,0,1)\}.$$

For any $E \in \mathbb{F}_q^*$ the above example is equivalent to the Thas-Kantor ovoid via the projectivity

$$(X_0, X_1, X_2, X_3, X_4, X_5, X_6) \mapsto (X_0, EX_1, X_2, X_3, X_4, X_5/E, X_6)$$

that fixes the quadric $Q(6, q)$.

Combining the previous propositions and with Theorem 7.2.1, we have proved the following

**Theorem 7.2.15.** *Let* $f_1(X, Y, Z) = \sum_{ijk} a_{i,j,k} X^i Y^j Z^k$ *and* $f_2(X, Y, Z) = \sum_{ijk} b_{i,j,k} X^i Y^j Z^k$. *Suppose that* $q > 6.3(d+1)^{13/3}$, *where* $\max\{\deg(f_1), \deg(f_2)\} = d$. *If* $\mathcal{O}_6(f_1, f_2)$ *is an ovoid of* $Q(6, q)$ *then*

$$q = 3^h, \quad f_1(X, Y, Z) = X^2 Y - nY^3 - XZ, \quad f_2(X, Y, Z) = -1/nX^3 + XY^2 + YZ.$$

*Hence* $\mathcal{O}_6(f_1, f_2)$ *is a Thas-Kantor ovoid.*

# Chapter 8

# $m$-Ovoids of $Q^+(7, q)$, $q$ odd

The results of this chapter have appeared in [1].

## 8.1 Introduction

In 1965, Segre [75] proved that if the elliptic quadric $Q^-(5, q)$ has a non-trivial $m$-ovoid then $q$ is odd and $m = (q+1)/2$. Secondly, Segre constructed an example of a 2-ovoid of $Q^-(5, 3)$. In 2005, Cossidente and Penttila (see [18]) constructed examples of $(q + 1)/2$-ovoids of $Q^-(5, q)$ for any $q$ odd, generalizing Segre's example.

In this chapter, we will construct $m$-ovoids of the hyperbolic quadric $Q^+(7, q)$, $q$ odd, as the disjoint union of $m$-ovoids of different elliptic quadrics $Q^-(5, q)$ contained in $Q^+(7, q)$. This chapter is organized as follows.

In Section 8.2, we determine intersection patterns for $m$-ovoids of the elliptic quadric $Q^-(2n+1, q)$ with embedded elliptic quadrics $Q^-(2n-1, q)$. For the sake of completeness, we also present similar results for $m$-ovoids of the symplectic space $W(2n + 1, q)$. In Section 8.3, we describe how to construct $(q + 1)$-ovoids of $Q^+(7, q)$ as disjoint union of two $(q + 1)/2$-ovoids of two different embedded elliptic quadrics $Q^-(5, q)$. In the final section, we investigate spreads of $\mathrm{PG}(3, q)$ with as many 2-secants as possible to an elliptic quadric. This is used eventually to construct $m$-ovoids of $Q^+(7, 3)$ for $m \in \{2, 4, 6, 8, 10\}$ by using 2-ovoids of $Q^-(5, 3)$ and the particular line spread.

## 8.2 Intersection patterns for $m$-ovoids

In this section, we will assume that $n \geq 2$.

**Lemma 8.2.1.** *Suppose that $\mathcal{O}$ is an $m$-ovoid of $Q^-(2n + 1, q)$, then it induces an $m$-ovoid in any $Q(2n, q) \subset Q^-(2n + 1, q)$.*

*Proof.* Let $\pi$ be an hyperplane of $\mathrm{PG}(2n + 1, q)$ such that $\pi \cap Q^-(2n + 1, q) = Q(2n, q)$. Since any generator of $Q(2n, q)$ is also a generator of $Q^-(2n + 1, q)$, it is easy to see that $\mathcal{O}' = \pi \cap \mathcal{O}$ is an $m$-ovoid of $Q(2n, q)$. $\square$

**Proposition 8.2.2.** *Suppose that $\mathcal{O}$ is an $m$-ovoid of $Q^-(2n+1,q)$, with $m \geq 1$. Then any elliptic quadric $Q^-(2n-1,q) \subset Q^-(2n+1,q)$ meets $\mathcal{O}$ in exactly $(m-2)q^{n-1}+m$, $(m-1)q^{n-1}+m$ or $mq^{n-1}+m$ points, and all of these cases occur.*

*Proof.* Consider an elliptic quadric $Q^-(2n-1,q)$ in an $(2n-1)$-dimensional subspace $\pi$, then $\pi^\perp$ is a line $\ell$ of $\mathrm{PG}(2n+1,q)$, meeting $Q^-(2n+1,q)$ in 2 points. Suppose that $|\ell \cap \mathcal{O}| = c$, then $c \leq 2$. Consider now any point $P \in \ell$. If $P \notin Q^-(2n+1,q)$, then $P^\perp \cap Q^-(2n+1,q) = Q(2n,q)$ and hence $|P^\perp \cap \mathcal{O}| = m(q^n+1)$, since, by Lemma 8.2.1, $\mathcal{O}$ induces an $m$-ovoid in any $Q(2n,q) \subset Q^-(2n+1,q)$. If $P \in Q^-(2n+1,q) \setminus \mathcal{O}$, then $|P^\perp \cap \mathcal{O}| = m(q^n+1)$. For any point $P \in \mathcal{O}$, we get $|P^\perp \cap \mathcal{O}| = (m-1)q^n + m$.

By counting the pairs $\{(P,R) : P \in \ell, R \in \mathcal{O} \setminus \ell, P \in R^\perp\}$, we find that

$$m(q+1-c)(q^n+1) + c(m-1)(q^n+1) = m(q^{n+1}+1) - c + xq,$$

This simplifies to

$$m(q+1)(q^n+1) - cm(q^n+1) + cm(q^n+1) - c(q^n+1) = m(q^{n+1}+1) - c + xq,$$

which yields $xq = mq(q^{n-1}+1) - cq^n$, with $x$ the number of points of $\mathcal{O} \cap \pi$ and $c \in \{0,1,2\}$. Thus the assertion follows. □

**Proposition 8.2.3.** *Suppose that $\mathcal{O}$ is an $m$-ovoid of $W(2n+1,q)$, $m \geq 1$. Then any symplectic space $W(2n-1,q) \subset W(2n+1,q)$ meets $\mathcal{O}$ in exactly $(m-i)q^{n-1}+m$ points, $i \in \{0,1,\ldots,q+1\}$.*

*Proof.* Consider a symplectic space $W(2n-1,q)$ in an $(2n-1)$-dimensional subspace $\pi$, then $\pi^\perp$ is a line $\ell$ of $\mathrm{PG}(2n+1,q)$ contained in $W(2n+1,q)$. Suppose that $|\ell \cap \mathcal{O}| = c$, then $c \leq q+1$.

If $P \in W(2n+1,q) \setminus \mathcal{O}$, then $|P^\perp \cap \mathcal{O}| = m(q^n+1)$. For any point $P \in \mathcal{O}$, we get $|P^\perp \cap \mathcal{O}| = (m-1)q^n + m$.

Hence, we can count the pairs $\{(P,R) : P \in \ell, R \in \mathcal{O} \setminus \ell, P \in R^\perp\}$ in order to find that

$$m(q+1-c)(q^n+1) + c(m-1)(q^n+1) = m(q^{n+1}+1) - c + xq.$$

This results in $x = m(q^{n-1}+1) - cq^{n-1}$, where $x$ is the number of points of $\mathcal{O} \cap \pi$ and $c \in \{0,1,\ldots,q+1\}$. Thus the assertion follows. □

**Lemma 8.2.4.** *Let $n, m, i$ be three integers such that $n \geq 2$, $m \geq 1$ and $i \in \{0,1,\ldots,q+1\}$. Then $(m-i)q^{n-1}+m = 0$ if and only if $(n,i,m) = (2,q+1,q)$.*

*Proof.* We have that $(m-i)q^{n-1}+m = 0$ if and only if $m = iq^{n-1}/(q^{n-1}+1)$. Note that $q^{n-1}+1 \mid iq^{n-1}$ if and only if $q^{n-1}+1 \mid i$. This is only possible whenever $(n,i) = (2,q+1)$. □

This immediately implies the following corollary.

**Corollary 8.2.5.** *The followings hold:*

- *an $m$-ovoid $\mathcal{O}$ of $Q^-(2n+1,q)$, $m \geq 1$, has non-empty intersection with any $Q^-(2n-1,q) \subset Q^-(2n+1,q)$;*
- *if an $m$-ovoid $\mathcal{O}$ of $W(2n+1,q)$, $m \geq 1$, has empty intersection with some $W(2n-1,q) \subset W(2n+1,q)$ then $(n,m) = (2,q)$.*

**Proposition 8.2.6.** *Let $\mathcal{O}$ be an $m$-ovoid of $Q^-(2n+1,q)$, $m \geq 1$. If $\mathcal{O}$ contains an elliptic quadric $Q^-(2n-1,q)$, then $\mathcal{O} = Q^-(2n+1,q)$.*

*Proof.* By contradiction, assume that $\mathcal{O} \neq Q^-(2n+1,q)$ and define $\tilde{\mathcal{O}}$ to be the complement of $\mathcal{O}$. Then $\tilde{\mathcal{O}}$ is a $\left(\frac{q^n-1}{q-1} - m\right)$-ovoid that has empty intersection with $Q^-(2n-1,q)$, a contradiction. □

**Proposition 8.2.7.** *Let $\mathcal{O}$ be an $m$-ovoid of $W(2n+1,q)$, $m \geq 1$. If $\mathcal{O}$ contains a symplectic space $W(2n-1,q)$ and $(n,m) \neq (2,q^2+1)$, then $\mathcal{O} = W(2n+1,q)$.*

*Proof.* Let $\tilde{\mathcal{O}}$ be the complement of $\mathcal{O}$. By contradiction, assume that $\mathcal{O} \neq W(2n+1,q)$. Then $\tilde{\mathcal{O}}$ is a $\left(\frac{q^{n+1}-1}{q-1} - m\right)$-ovoid and has empty intersection with $W(2n-1,q)$, a contradiction. □

## 8.3 Constructing $(q+1)$-ovoids of $Q^+(7,q)$

We recall the following theorem obtained by B. Segre [75] in 1965.

**Theorem 8.3.1** (Segre's Theorem)**.** *Let $\mathcal{O}$ be a non-trivial $m$-ovoid of $Q^-(5,q)$, then $q$ is odd and $m = (q+1)/2$.*

We recall that the Klein correspondence is a bijective map between lines of $\mathrm{PG}(3,q^2)$ and points of $Q^+(5,q^2)$. In particular, the lines of a hermitian surface $H(3,q^2)$ are mapped onto points of an elliptic quadric $Q^-(5,q)$ obtained by intersecting $Q^+(5,q^2)$ with a subgeometry isomorphic to $\mathrm{PG}(5,q)$. Hence to any $m$-ovoid $\mathcal{O}$ of $Q^-(5,q)$ there corresponds a set of lines $\mathcal{L}$ of $H(3,q^2)$ with the property that every point of $H(3,q^2)$ is incident with exactly $m$ lines of $\mathcal{L}$. From Segre's Theorem it then follows that $q$ odd and $m = (q+1)/2$, thus the lineset $\mathcal{L}$ is called a *hemisystem*. In [75], Segre constructed an example of such a hemisystem of $H(3,9)$. Later, in [18], Cossidente and Penttila constructed a family of hemisystems of $H(3,q^2)$, for $q$ odd. Hence generalizing the construction of Segre. From this we obtain that there exist examples of $(q+1)/2$-ovoids in $Q^-(5,q)$ for $q$ odd.

From now on, we will assume that $q$ is odd. We describe a construction of a $(q+1)$-ovoid of $Q^+(7,q)$. Before doing this, we prove the following proposition.

**Proposition 8.3.2.** *Consider $Q^+(7,q)$, and suppose that $\pi_1$ and $\pi_2$ are 5-dimensional subspaces in the ambient space $\mathrm{PG}(7,q)$ such that $\dim(\pi_1 \cap \pi_2) = 3$. Suppose both $\pi_1$ and $\pi_2$ intersect $Q^+(7,q)$ in an elliptic quadric $Q^-(5,q)$, that will be called $\mathcal{Q}_1$ and $\mathcal{Q}_2$ respectively. Then there exists a projectivity $\Phi \in \mathrm{PGO}^+(8,q)$ such that $\mathcal{Q}_1$ is mapped on $\mathcal{Q}_2$ and the set $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is fixed pointwised.*

*Proof.* Suppose that a 3-dimensional subspace $\pi$ intersects $Q^+(7,q)$ in an elliptic quadric $\mathcal{Q}_1$. Then $\pi^\perp$, which is also of dimension 3, intersects $Q^+(7,q)$ in another elliptic quadric $\mathcal{Q}_2$. Hence both quadrics are an elliptic quadric $Q^-(3,q)$. Since $\pi$ and $\pi^\perp$ are disjoint, by abuse of notation, $\mathbb{F}_q^8 = \pi \oplus \pi^\perp$.

Now suppose that $\varphi \in \mathrm{PGO}^-(4,q)$, then we can extend $\varphi$ to an element $\tilde{\varphi}$ of $\mathrm{PGO}^+(8,q)$ as follows

$$\tilde{\varphi}(x+y) = x + \varphi(y), \qquad x \in \pi, y \in \pi^\perp .$$

The notation is explained as follows. The map $\varphi$ is represented by a $4\times 4$ matrix $M$, and $\tilde{\varphi}(x+y)$ is represented by the $8 \times 8$ matrix

$$M' = \begin{pmatrix} M & O \\ O & I_4 \end{pmatrix},$$

where $O$ and $I_4$ are the zero-matrix and identity-matrix, respectively. As such, $M'$ will represent an element of $\mathrm{PGO}^+(8,q)$.

Now suppose that $\sigma$ is a 5-dimensional subspace through $\pi$, intersecting $Q^+(7,q)$ in an elliptic quadric $Q^-(5,q)$. The polarity corresponding to this $Q^-(5,q)$ sends $\pi$ to $\ell = \sigma \cap \pi^\perp$. Since this polarity is elliptic and $\pi$ intersects $Q^-(5,q)$ in an elliptic quadric $Q^-(3,q)$, $\ell$ will be a hyperbolic line with respect to $Q^-(3,q)$. In general the span of a line $\ell \subset \pi^\perp$ and $\pi$ is hyperbolic, elliptic, or singular respectively if $\ell$ is elliptic, hyperbolic or singular. Since $\mathrm{PGO}^-(4,q)$ acts transitively on the hyperbolic lines in $\pi^\perp$, we now know that the pointwise stabiliser of $\pi \cap Q^+(7,q)$ in $\mathrm{PGO}^+(8,q)$ works transitively on the elliptic 5-spaces through $\pi$. □

**Construction 8.3.3.** First, we note that any $m$-ovoid of an elliptic quadric $Q^-(5,q)$ contained in $Q^+(7,q)$ is also an $m$-ovoid of $Q^+(7,q)$. Indeed, it is easy to see that any generator of $Q^+(7,q)$ meets $Q^-(5,q)$ in a generator of $Q^-(5,q)$. Let $\sigma$ be a generator of $Q^+(7,q)$ and let $\pi$ be a 5-dimensional subspace such that $Q^+(7,q) \cap \pi = Q^-(5,q)$. Now,

$$\sigma \cap Q^-(5,q) = \sigma \cap \pi \cap Q^+(7,q) = \sigma \cap \pi.$$

Since $Q^-(5,q)$ has rank 2, then $\dim(\sigma \cap \pi) = 1$. Moreover, $m$-ovoids of $Q^-(5,q)$ and $Q^+(7,q)$ have the same size $m(q^3+1)$.

Let $\pi_1$ and $\pi_2$ be two distinct 5-dimensional spaces such that $\pi_1 \cap Q^+(7,q) = \mathcal{Q}_1$ and $\pi_2 \cap Q^+(7,q) = \mathcal{Q}_2$ are two elliptic quadrics $Q^-(5,q)$ and $\dim(\pi_1 \cap \pi_2) = 3$. It is possible to choose $\pi_1$ and $\pi_2$ in such as way that $\pi_1 \cap \pi_2 \cap Q^+(7,q) = \mathcal{Q}_1 \cap \mathcal{Q}_2$ is an elliptic quadric $Q^-(3,q)$. By Proposition 8.3.2, there exists a collineation $\Phi \in \mathrm{PGO}^+(8,q)$ that maps $\mathcal{Q}_1$ on $\mathcal{Q}_2$ and fixes $\mathcal{Q}_1 \cap \mathcal{Q}_2$ pointwise.

Now, let $\mathcal{O}_1$ be a $(\frac{q+1}{2})$-ovoid of $\mathcal{Q}_1$. Then, $\Phi(\mathcal{O}_1)$ is a $(q+1)/2$-ovoid of $\mathcal{Q}_2$, hence its complement $\mathcal{O}_2$ in $\mathcal{Q}_2$ is again a $(q+1)/2$-ovoid. Since $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint in $Q^+(7,q)$, then $\mathcal{O}_1 \cup \mathcal{O}_2$ is a $(q+1)$-ovoid of $Q^+(7,q)$.

The following Theorem summarizes the result of this section.

**Theorem 8.3.4.** *There exist $(q+1)$-ovoids in $Q^+(7,q)$ which are the union of two $\frac{q+1}{2}$-ovoids in distinct $Q^-(5,q)$'s.*

*Proof.* See construction 8.3.3. □

**Remark 8.3.5.** There are known examples of 1-systems of $Q^+(7,q)$, see e.g. [60, 61], hence by Theorem 1.7.6 $(q+1)$-ovoids of $Q^+(7,q)$ exist. This construction yields eventually $(q+1)$-ovoids of $Q^+(7,q)$ not coming from a 1-system.

## 8.4 Spreads without tangents to $Q^-(3,q)$

Consider the elliptic quadric $Q^-(3,q)$ with ambient projective space $\mathrm{PG}(3,q)$. Our first goal is to construct a line spread of $\mathrm{PG}(3,q)$ containing as many 2-secants to $Q^-(3,q)$ as possible. The maximum number clearly equals $|Q^-(3,q)|/2 = (q^2+1)/2$, which is evidently only attainable if $q$ is odd. We will show that this number can be reached if $q = 3^h$, $h$ odd, using the Klein correspondence, and when $q \equiv 1 \pmod 4$.

Case 1: $q = 3^h$, $h$ odd.
Then $q \equiv -1 \pmod 4$ and so $-1$ is a non-square.

Let $f_1(x,y) = x+y$, $f_2(x,y) = x+2y$. Then, Equation (6.1) reads

$$(x_1-x_2)^2 + (y_1-y_2)^2 \neq 0.$$

Since $-1$ is not a square in $\mathbb{F}_q$, the equation above is satisfied for any $(x_1,y_1) \neq (x_2,y_2)$ in $\mathbb{F}_q^2$. Hence, we get the following.

**Theorem 8.4.1.** *Let $q = 3^h$, $h$ odd, and suppose that $f_1(x,y) = x+y$, $f_2(x,y) = x+2y$. Then $\mathcal{O}(f_1,f_2)$ is an ovoid of $Q^+(5,q)$ and the set $\mathcal{S}(f_1,f_2) = \{\ell_{x,y} : (x,y) \in \mathbb{F}_q^2\} \cup \{\ell_\infty\}$ is a line spread of $\mathrm{PG}(3,q)$, where $\ell_\infty : X_0 = X_1 = 0$ and*

$$\ell_{x,y} : \begin{cases} X_2 = -(x+y)X_0 + (x+2y)X_1 \\ X_3 = xX_0 + yX_1. \end{cases}$$

The quadratic form $F(X_0,X_1,X_2,X_3) = X_0X_1 + X_2^2 + X_3^2$ defines an elliptic quadric. Indeed, since $-1$ is not a square, $X_2^2 + X_3^2$ is irreducible. Now, consider the line spread $\mathcal{S}(f_1,f_2)$ defined in the previous theorem. Note that the line $\ell_\infty$ is a 0-secant to $Q^-(3,q)$ and the line $\ell_{0,0}$ is a 2-secant to $Q^-(3,q)$. Let $(x,y) \neq (0,0)$, then the line $\ell_{x,y}$ contains the points $(1,0,-x-y,x)$ and $(0,1,x+2y,y)$, which do not lie on $Q^-(3,q)$. Hence, this line is tangent if and only if

$$\begin{gathered} F((1,0,-x-y,x) + T(0,1,x+2y,y)) = \\ (x^2+xy+2y^2)T^2 + (x^2+2xy+2y^2+1)T + 2x^2+2xy+y^2 \end{gathered}$$

has a unique root, which is equivalent to

$$x^4 + 2x^2y^2 + y^4 + x^2 + 2xy + 2y^2 + 2 = 0.$$

So finding tangent lines to $Q^-(3,q)$ is equivalent to finding $\mathbb{F}_q$-rational points of the curve $\mathcal{C} : H(x,y) = 0$, where $H(x,y) = x^4 + 2x^2y^2 + y^4 + x^2 + 2xy + 2y^2 + 2$.

Let $i \in \overline{\mathbb{F}}_q$ be such that $i^2 = -1$. Let $t := x + iy$ and $z := x - iy$. Now, the equation of $\mathcal{C}$ reads

$$t^2z^2 + (i+2)t^2 + (2i+2)z^2 + 2 = 0,$$

or equivalently

$$(z^2 - (1-i))(t^2 - (1+i)) = 0. \tag{8.1}$$

Note that the smallest field containing the square roots of $1-i$ and $1+i$ is $\mathbb{F}_{3^4}$. Since $q = 3^h$, with $h$ odd, $\mathbb{F}_{3^4}$ is not a subfield of $\mathbb{F}_{q^2}$, thus Equation (8.1) has no solutions over $\mathbb{F}_{q^2}$. It follows that the curve $\mathcal{C}$ has no $\mathbb{F}_q$-rational points. Therefore, the spread $\mathcal{S}(f_1, f_2)$ of $\mathrm{PG}(3,q)$ has no tangent lines to $Q^-(3,q)$.

Case 2: $q \equiv 1 \pmod 4$.

We construct an example of Desarguesian spread, and show that it does not contain any tangent lines to $Q^-(3,q)$. Choose a non-square $\alpha \in \mathbb{F}_q$. Then the quadratic form $F(X_0,X_1,X_2,X_3) = X_0X_1 + X_2^2 - \alpha X_3^2$ defines an elliptic quadric.

Every line in $\mathrm{PG}(3,q)$ can be represented as the column space of a full rank $4 \times 2$ matrix. Write $A = \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix}$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then

$$\mathcal{S} = \left\{ \begin{pmatrix} I_2 \\ xI_2 + yA \end{pmatrix} : x, y \in \mathbb{F}_q \right\} \cup \left\{ \begin{pmatrix} O \\ I_2 \end{pmatrix} \right\}$$

is a Desarguesian spread. Note that the line $\begin{pmatrix} O \\ I_2 \end{pmatrix}$ is 0-secant and the line $\begin{pmatrix} I_2 \\ 0 \end{pmatrix}$ is 2-secant.

So take another line $\begin{pmatrix} I_2 \\ xI_2 + yA \end{pmatrix} \in \mathcal{S}$, $(x,y) \neq (0,0)$. This line goes through the points $(1,0,x,y)$ and $(0,1,\alpha x,y)$ which do not lie on $Q^-(3,q)$ due to $(x,y) \neq (0,0)$. Hence this line is tangent to $Q^-(3,q)$ if and only if

$$F((1,0,x,y) + T(0,1,\alpha x,y)) = \alpha(\alpha y^2 - x^2)T^2 + T + (x^2 - \alpha y^2)$$

has a unique root, which is equivalent to

$$1 - 4\alpha(\alpha y^2 - x^2)(x^2 - \alpha y^2) = 1 + 4\alpha(x^2 - \alpha y^2)^2 = 0,$$

since $\alpha(\alpha y^2 - x^2) \neq 0$ if $(x,y) \neq (0,0)$. Note that since $\alpha$ is a non-square, $4\alpha(x^2 - \alpha y^2)^2$ is a non-square. If $q \equiv 1 \pmod 4$, $-1$ is a square and the above equation has no solutions. Therefore, the Desarguesian spread $\mathcal{S}$ contains no tangent lines to $Q^-(3,q)$.

The results above are summarized in the following proposition.

**Proposition 8.4.2.** *Consider $Q^-(3,q)$ in the ambient space $\mathrm{PG}(3,q)$. If $q = 3^h$ or $q \equiv 1 \pmod 4$ (this covers the case $q = 3^h$ with $h$ even), then there exists a line spread of $\mathrm{PG}(3,q)$ containing $|Q^-(3,q)|/2 = (q^2+1)/2$ 2-secant lines to $Q^-(3,q)$.*

The reason we are interested in these spreads is the construction of $m$-ovoids (see Theorem 8.4.4). First let us prove one more lemma.

**Lemma 8.4.3.** *Consider the elliptic quadric $Q^-(5, 3)$ and let $\sigma$ be an elliptic 3-space. Then for any 2 distinct points $P, R \in \sigma$, there exists a 2-ovoid $\mathcal{O}$ of $Q^-(5, 3)$ with $\mathcal{O} \cap \sigma = \{P, R\}$.*

*Proof.* There exists a 2-ovoid $\mathcal{O}_1$ of $Q^-(5, 3)$. Let $\pi$ be the ambient 5-space of $Q^-(5, 3)$. By Proposition 8.2.2 there is a 3-space $\sigma' \subset \pi$ which intersects $\mathcal{O}_1$ in 2 points. Since $\mathrm{PGO}^-(6, 3)$ acts transitively on the elliptic 3-spaces, there also exists a 2-ovoid $\mathcal{O}_2$ intersecting $\sigma$ in 2 points $\{P', R'\}$. The group $\mathrm{PGO}^-(4, 3)$ acts 2-transitively on the points of $Q^-(3, 3)$. Thus there exists a projectivity $\varphi$ of $\sigma$ which fixes $Q^-(5, 3) \cap \sigma$ and maps $\{P', R'\}$ to $\{P, R\}$. As in Proposition 8.3.2, we can choose an appropriate element of the stabilizer of $\sigma^\perp \cap Q^-(5, 3)$, which is isomorphic to $\mathrm{PGO}^+(2, 3)$, to extend $\varphi$ to an element of $\mathrm{PGO}^-(6, 3)$. This extended projectivity maps $\mathcal{O}_2$ to a 2-ovoid of $Q^-(5, 3)$ intersecting $\sigma$ exactly in $\{P, R\}$. □

**Theorem 8.4.4.** *The hyperbolic quadric $Q^+(7, 3)$ contains $m$-ovoids for $m \in \{2, 4, 6, 8, 10\}$.*

*Proof.* We will prove that $Q^+(7, 3)$ has 5 pairwise disjoint 2-ovoids. The theorem then immediately follows by taking unions of some of these 2-ovoids.

Take an elliptic 3-space $\sigma$. Then $\sigma^\perp$ is also an elliptic 3-space. By Proposition 8.4.2, there exists a line spread of $\sigma^\perp$ containing 5 2-secant lines $\ell_1, \ldots, \ell_5$ to $Q^+(7, 3) \cap \sigma^\perp \cong Q^-(3, 3)$. Let $\pi_i$ denote the span of $\ell_i$ and $\sigma$ for $i = 1, \ldots, 5$. Then $\pi_1, \ldots, \pi_5$ are elliptic 5-spaces pairwise intersecting in $\sigma$. Partition the 10 points of $\sigma \cap Q^+(7, 3)$ into 5 pairs $\{P_i, R_i\}$, $i = 1, \ldots, 5$. For every $i$ choose a 2-ovoid $\mathcal{O}_i$ of $\pi_i$ which intersects $\sigma$ precisely in $\{P_i, R_i\}$. Such a 2-ovoid exists by Lemma 8.4.3. Then $\mathcal{O}_1, \ldots, \mathcal{O}_5$ are pairwise disjoint 2-ovoids of $Q^+(7, 3)$. □

**Remark 8.4.5.** Notice that the same technique cannot be used for $q > 5$, since, by Proposition 8.2.2, every $\frac{q+1}{2}$-ovoid of $Q^-(5, q)$ intersects every elliptic 3-space in more than $|Q^-(3, q)|/3$ points. Finally, we were also not able to find 3 pairwise disjoint 3-ovoids of $Q^+(7, 5)$.

# Chapter 9

# Non-linear MRD codes from cones over exterior sets

In this chapter, we provide a geometric construction for a class of non-linear $(n, n, q; d)$-MRD $\mathcal{C}_{\sigma,T}$, $1 \in T \subset \mathbb{F}_q^*$, $2 \leq d \leq n-1$, and puncturing properly a code in this relevant class, one gets a code described in [28]. We shall show that this class is effectively new, i.e. any code is not equivalent to a non-linear code constructed by Otal and Özbudak. The results of this chapter can also be found in [32].

## 9.1 Embeddings and cones over exterior sets

Let $\Sigma = \Sigma_{n,n} \cong \mathrm{PG}(n-1, q)$ be a canonical subgeometry of $\mathrm{PG}(n-1, q^n)$ and consider a subspace $\Lambda^\star$ of rank $k$ disjoint from $\Sigma$ and $\Lambda$ a subspace of $\mathrm{PG}(n-1, q^n)$ of rank $n-k$ disjoint from $\Lambda^\star$. Let $\Gamma$ be the projection of $\Sigma$ from $\Lambda^\star$ to $\Lambda$, i.e.

$$\Gamma = \mathrm{p}_{\Lambda^\star,\Lambda}(\Sigma) = \{\langle \Lambda^\star, P\rangle \cap \Lambda : P \in \Sigma\}.$$

We recall the following definition given in [63].

**Definition 9.1.1.** Let $\Gamma = \mathrm{p}_{\Lambda^\star,\Lambda}(\Sigma)$ be an $\mathbb{F}_q$-linear set of rank $n$. The set $\Gamma$ is called an $(n-k-1)$-*embedding of* $\Sigma$ if any subspace of $\Sigma$ of rank $n-k$ is disjoint from $\Lambda^\star$.

Note that $\Gamma = \mathrm{p}_{\Lambda^\star,\Lambda}(\Sigma)$ is an $(n-k-1)$-embedding if and only if for any choice of $n-k$ independent points $R_1, R_2 \ldots, R_{n-k}$ of $\Sigma$,

$$\Lambda = \langle R_1', R_2' \ldots, R_{n-k}'\rangle$$

where $R_i' = \mathrm{p}_{\Lambda^\star,\Lambda}(R_i)$, $i = 1, 2, \ldots, n-k$, also this is equivalent to say that

$$\mathrm{p}_{\Lambda^\star,\Lambda} : P \in \Sigma \longrightarrow \langle P, \Lambda^\star\rangle \cap \Lambda \in \Lambda$$

induces an injective map from the set of all subspaces of $\Sigma$ of rank $h \leq n-k-1$ to the set of all subspaces of $\Lambda$ of the same rank $h$.

**Lemma 9.1.2** ([63, Theorem 6])**.** *The $\mathbb{F}_q$-linear set $\Gamma = p_{\Lambda^\star,\Lambda}(\Sigma)$ is an $(n-k-1)$-embedding of $\Sigma$ if and only if any point $P \in \Lambda^\star$ is of type $\geq n-k+1$.*

*Proof.* By definition $\Gamma$ is an $(n-k-1)$-embedding if and only if $\Lambda^\star$ is disjoint from any subspace of $\Sigma$ of rank $n-k$. This is equivalent to saying that for any point $P$ of $\Lambda^\star$ the subspace $L(P)$ has rank at least $n-k+1$, i.e. the point $P$ is of type at least $n-k+1$. □

In [63], it has been proved a geometric property for the cone having vertex $\Lambda^\star$ and base a linear set of rank $n$ to produce a linear MRD-code. More precisely,

**Theorem 9.1.3** ([63, Theorem 5])**.** *Let $\Sigma \cong \mathrm{PG}(n-1,q)$ be a canonical subgeometry of $\mathrm{PG}(n-1,q^n)$ and let $\Lambda^\star$ and $\Lambda$ be subspaces of $\mathrm{PG}(n-1,q^n)$ of rank $k$ and $n-k$, respectively, such that $\Lambda^\star \cap \Sigma = \emptyset = \Lambda^\star \cap \Lambda$.*
*Let $\Gamma = \mathrm{p}_{\Lambda^\star,\Lambda}(\Sigma)$ be an $(n-k-1)$-embedding of $\Sigma$ and let $\Omega \subset \Lambda$ be an $\mathbb{F}_q$-linear set of rank $n$. If no points of $\Omega$ belong to a subspace spanned by at most $n-k$ points of $\Gamma$, then any point of $\mathcal{K} = \mathcal{K}(\Lambda^\star,\Omega)$ is of type $\geq n-k+1$.*
*Moreover, the set $\mathcal{C} = \left\{\sum_{i=0}^{n-1} \alpha_i x^{\sigma^i} : (\alpha_0,\alpha_1,\ldots,\alpha_{n-1}) \in \mathcal{K}\right\}$ is a linear MRD-code with minimum distance $d = n-k+1$.*

Here, one shall adapt the result above to the case of cone such that its base is not necessarily a linear set.

**Theorem 9.1.4.** *Let $\Sigma = \mathrm{PG}(n-1,q)$ be a canonical subgeometry of $\mathrm{PG}(n-1,q^n)$ and let $\Lambda^\star$ and $\Lambda$ be subspaces of $\mathrm{PG}(n-1,q^n)$ of rank $k-2$ and $n-k+2$, respectively, such that $\Lambda^\star \cap \Sigma = \emptyset = \Lambda^\star \cap \Lambda$. Let $\Gamma = \mathrm{p}_{\Lambda^\star,\Lambda}(\Sigma)$ be an $(n-k+1)$-embedding of $\Sigma$ and let $\mathcal{E} \subset \Lambda$ be a (maximum) exterior set with respect to $\Omega_{n-k-1}(\Gamma)$.*
*Then, $\mathcal{K} = \mathcal{K}(\Lambda^\star,\mathcal{E})$ is a (maximum) exterior set with respect to $\Omega_{n-k-1}(\Sigma)$.*

*Proof.* Let $P, Q$ be two points in $\mathcal{K} = \mathcal{K}(\Lambda^\star,\mathcal{E})$ and one shall distinguish some cases:

- $P, Q \in \Lambda^\star$. Since $\Gamma$ is an $(n-k+1)$-embedding the line $PQ$ is disjoint from the subspaces of rank $(n-k)$ of $\Sigma$ and so from $\Omega_{n-k-1}(\Sigma)$.

- $P, Q \in \mathcal{E}$. Suppose that the line $PQ$ meets a subspace $S$ of $\Sigma$ of rank $(n-k)$ and consider
$$\emptyset \neq \langle \Lambda^\star, S \cap PQ\rangle \cap \Lambda \subset \langle \Lambda^\star, S\rangle \cap \Lambda \cap PQ. \tag{9.1}$$
This is a contradiction by the hypothesis on $\mathcal{E}$ and since $\langle \Lambda^\star, S\rangle \cap \Lambda$ is a subspace of rank $n-k$.

- the line $PQ \subset \mathcal{K} \setminus (\Lambda^* \cup \mathcal{E})$. Without loss of generality, one may suppose that $P \in \Lambda^\star$ and $Q \in \mathcal{E}$. If the line $PQ$ meets a subspace $S$ of $\Sigma$ of rank $(n-k)$, then
$$Q \in \langle \Lambda^\star, S \cap PQ\rangle \cap \Lambda \subset \langle \Lambda^\star, S\rangle \cap \Lambda. \tag{9.2}$$
Then $Q$ belongs to a space spanned by points of $\Gamma$ with rank $(n-k)$, a contradiction.

- the line $PQ \not\subset \mathcal{K}$. If $PQ$ meets a subspace $S$ of $\Sigma$ of rank $(n-k)$, then

$$\emptyset \neq \langle \Lambda^\star, S \cap PQ \rangle \cap \Lambda \subset \langle \Lambda^\star, S \rangle \cap \langle \Lambda^\star, PQ \rangle \cap \Lambda. \tag{9.3}$$

Since the projection from $\Lambda^\star$ to $\Lambda$ of the line $PQ$ is a line through two points $P', Q'$ of $\mathcal{E}$, one gets that this line meets the space $\langle \Lambda^\star, S \rangle \cap \Lambda$ spanned by $(n-k)$ points of $\Gamma$, a contradiction.

Then one has showed that any line through two points of $\mathcal{K}$ is disjoint from $\Omega_{n-k-1}(\Sigma)$. Now, since $\langle \Gamma \rangle = \Lambda$, if $\mathcal{E}$ is a maximum exterior set with respect to $\Omega_{n-k-1}(\Gamma)$, one gets

$$\begin{aligned} |\mathcal{K}(\Lambda^\star, \mathcal{E})| &= |\Lambda^\star| + |\mathcal{E}| + |\Lambda^\star||\mathcal{E}|(q^n-1) \\ &= \frac{q^{n(k-2)}-1}{q^n-1} + \frac{q^{2n}-1}{q^n-1} + \frac{q^{n(k-2)}-1}{q^n-1} \cdot \frac{q^{2n}-1}{q^n-1}(q^n-1) \\ &= \frac{q^{2n}-1}{q^n-1} + \frac{q^{n(k-2)}-1}{q^n-1} q^{2n} \\ &= \frac{q^{nk}-1}{q^n-1}. \end{aligned}$$

□

## 9.2 The class of non-linear MRD codes $\mathcal{C}_{\sigma,T}$

In this section, as application of Theorem 9.1.4, by an appropriate choice of $\Lambda^\star$, $\Lambda$, an $(n-k+1)$-embedding $\Gamma$ of $\Sigma$, $k \leq n-1$, and a maximum exterior set with respect to $\Omega_{n-k-1}(\Gamma)$ in $\mathrm{PG}(n-1, q^n)$, one will get a class of non-linear MRD codes in $\mathcal{L}_{n,q,\sigma}$. Precisely, let

$$\Lambda^\star : X_0 = X_1 = \ldots = X_{n-k+1} = 0$$

and

$$\Lambda : X_{n-k+2} = X_{n-k+3} = \ldots = X_{n-1} = 0$$

be disjoint subspaces of rank $(k-2)$ and $(n-k+2)$ in $\mathrm{PG}(n-1, q^n)$, respectively. Consider the linear set of rank $n$

$$\Gamma = \mathrm{p}_{\Lambda^\star,\Lambda}(\Sigma) = \{(\alpha, \alpha^\sigma, \ldots, \alpha^{\sigma^{n-k+1}}, 0, \ldots, 0) : \alpha \in \mathbb{F}_{q^n}^*\} \tag{9.4}$$

where $\Sigma$ is the canonical geometry as in (2.3) and $P = (0, \ldots, 0, \beta_{n-k+2}, \ldots, \beta_{n-1}) \in \Lambda^\star$. This is a point of type at least $n-k+3$. Indeed, the linearized polynomial

$$f(x) = \beta_{n-k+2} x^{\sigma^{n-k+2}} + \ldots + \beta_{n-1} x^{\sigma^{n-1}}$$

has rank at least $n-k+3$ because the polynomial $f(x) \circ x^{\sigma^{k-2}} = \beta_{n-k} x + \ldots + \beta_{n-1} x^{\sigma^{k-3}}$ has at most $q^{k-3}$ roots and $\mathrm{rk}(f(x) \circ x^{\sigma^{k-2}}) = \mathrm{rk} f(x)$. So, by Lemma 9.1.2, one gets that $\Gamma$ is an $(n-k+1)$-embedding of $\Sigma$. Finally, let

$$A = (\underbrace{0, \ldots, 0}_{n-k+1}, 1, 0, \ldots, 0) \text{ and } B = (1, 0, \ldots, 0)$$

be points in $\Lambda$ and consider

$$\mathcal{X} = \bigcup_{a \in \mathbb{F}_q^*} \mathcal{X}_a \cup \{A, B\},$$

where

$$\mathcal{X}_a = \left\{(1, t, t^{\sigma+1}, \ldots, t^{\sigma^{n-k}+\ldots+\sigma+1}, 0, \ldots, 0) : t \in N_a\right\}.$$

The set $\mathcal{X}$ is the $C_F^\sigma$-set with vertices $A$ and $B$ generated by a $\sigma$-collineation $\Phi$ between the star of lines through $A$ and $B$ contained in $\Lambda$. Let $T \subset \mathbb{F}_q^*$, $1 \in T$, then the set

$$\mathcal{E} = \left(\mathcal{X} \setminus \bigcup_{a \in T} \mathcal{X}_a\right) \cup \bigcup_{a \in T} J_a \tag{9.5}$$

with

$$J_a = \left\{(1, \underbrace{0, \ldots, 0}_{n-k}, (-1)^{n-k} t, 0, \ldots, 0) : t \in N_a\right\}, \quad a \in \mathbb{F}_q^*.$$

is a maximum exterior set with respect to $\Omega_{n-k-1}(\Gamma)$ as proved in [28, Theorem 5.1] of size $q^n + 1$. Therefore, the hypothesis of Theorem 9.1.4 are satisfied and the cone $\mathcal{K}(\Lambda^\star, \mathcal{E})$ is a maximum exterior set with respect to $\Omega_{n-k-1}(\Sigma)$ and by Theorem 2.2.3, the set $\mathcal{C}_{\sigma,T} \subset \mathbb{F}_q^{n\times n}$, as in (2.4), is a non linear-MRD with minimum distance $d = n-k+1$ which cannot be a translated version of an additive code.
Note that the punctured code $\mathcal{C}'_{\sigma,T} \subset \mathbb{F}_q^{(n-k+2)\times n}$ obtained by $\mathcal{C}_{\sigma,T}$ deleting the last $(k-2)$ rows is exactly the code constructed in [28, Theorem 5.1] or for $k = 2$ the code appeared in [33] and for $n = 3, k = 2$ in [17].

By Theorem 9.1.4 and Theorem 2.2.3, the $\mathcal{C}_{\sigma,T}$ is a subset of square matrices of order $n$ and so, it can be seen as a subset of $\mathcal{L}_{n,q,\sigma}$. Indeed, fixing $T \subset \mathbb{F}_q^*$, $1 \in T$, the homogeneous coordinates of the points belonging to $\mathcal{E}$ have one of the following shapes

$$\begin{gathered}
(\alpha, \alpha^\sigma \xi, \alpha^{\sigma^2} \xi^{\sigma+1}, \ldots, \alpha^{\sigma^{n-k+1}} \xi^{\sigma^{n-k}+\ldots+1}, 0, \ldots, 0) \\
(\alpha, \underbrace{0, \ldots, 0}_{n-k}, (-1)^{n-k} \alpha^\sigma \eta, 0, \ldots, 0) \\
A = (\underbrace{0, \ldots, 0}_{n-k+1}, \alpha, 0, \ldots, 0) \text{ and } B = (\alpha, 0, \ldots, 0)
\end{gathered} \tag{9.6}$$

with $\mathrm{N}_{q^n/q}(\xi) \in \mathbb{F}_q^* \setminus T$ and $\mathrm{N}_{q^n/q}(\eta) \in T$. So the non linear $(n, n, q; d)$, $d = n - k + 1$, MRD-code $\mathcal{C}_{\sigma,T}$ is the set of $\sigma$-linearized polynomials with $\sigma$-degree at most $n - 1$ whose coefficients are the homogeneous coordinates of a point in $\mathcal{K}(\Lambda^\star, \mathcal{E})$ with the zero map, i.e.

$$\begin{aligned}\mathcal{C}_{\sigma,T} &= \left\{\sum_{i=0}^{d} \lambda\alpha^{\sigma^i}\xi^{\frac{\sigma^i-1}{\sigma-1}}x^{\sigma^i} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \lambda,\alpha,\beta_i \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\xi) \in \mathbb{F}_q^* \setminus T\right\}\\ &\cup \left\{\lambda\alpha x + (-1)^{d+1}\lambda\alpha^{\sigma}\eta x^{\sigma^d} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \lambda,\alpha,\beta_i \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\eta) \in T\right\}\\ &\cup \left\{\alpha x^{\sigma^d} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \alpha,\beta_i \in \mathbb{F}_{q^n}\right\} \cup \left\{\alpha x + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \alpha,\beta_i \in \mathbb{F}_{q^n}\right\}.\end{aligned}$$

## 9.3 The equivalence issue for $\mathcal{C}_{\sigma,T}$

Finally, in order to state the novelty of the class of non-linear codes obtained, we have to compare a code of type $\mathcal{C}_{\sigma,T}$, $1 \in T \subset \mathbb{F}_q^*$, with the only known non-linear MRD codes made up of square matrices: the codes of type $\mathcal{C}_{n,k,\sigma,I}$ with $I \subset \mathbb{F}_q$, cf. (2.2).
Firstly, note that the sets

$$\mathcal{U} = \left\{\alpha x^{\sigma^d} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \alpha,\beta_i \in \mathbb{F}_{q^n}\right\} = \left\{f \circ x^{\sigma^d} : f \in \mathcal{G}_{k-1,\sigma}\right\} \tag{9.7}$$

and

$$\mathcal{V} = \left\{\alpha x + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \alpha,\beta_i \in \mathbb{F}_{q^n}\right\} = \left\{f \circ x^{\sigma^{d+1}} : f \in \mathcal{G}_{k-1,\sigma}\right\} \tag{9.8}$$

contained in $\mathcal{C}_{\sigma,T}$ are equivalent to a generalized Gabidulin code $\mathcal{G}_{k-1,\sigma}$ and their intersection

$$\left\{\sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \beta_i \in \mathbb{F}_{q^n}\right\} = \left\{f \circ x^{\sigma^{d+1}} : f \in \mathcal{G}_{k-2,\sigma}\right\}$$

is equivalent to a generalized Gabidulin code $\mathcal{G}_{k-2,\sigma}$. While the non-linear $(n,n,q;n-k+1)$-MRD code $\mathcal{C}_{n,k,\sigma,I} = \mathcal{C}^{(1)}_{n,k,\sigma,I} \cup \mathcal{C}^{(2)}_{n,k,\sigma,I}$, introduced in Subsection 2.3., contains the set

$$\left\{\sum_{i=1}^{k-1} \gamma_i x^{\sigma^i} : \gamma_i \in \mathbb{F}_{q^n}\right\} = \left\{f \circ x^{\sigma} : f \in \mathcal{G}_{k-1,\sigma}\right\},$$

which is equivalent to a generalized Gabidulin code $\mathcal{G}_{k-1,\sigma}$ and it is straightforward to see that $\mathcal{C}_{n,k,\sigma,I}$ is contained in a generalized Gabidulin code $\mathcal{G}_{k+1,\sigma}$.

Clearly, if $q = 2$ or $T = \mathbb{F}_q^*$, $\mathcal{C}_{\sigma,T}$ is equivalent to the generalized Gabidulin code $\mathcal{G}_{k,\sigma}$. Then, by [69, Corollary 2.1], it follows

**Proposition 9.3.1.** *Let $I \subset \mathbb{F}_q$, $1 \in T \subset \mathbb{F}_q^*$ and $\sigma$ be the Frobenius automorphism of $\mathbb{F}_{q^n}$ defined by $\sigma : x \mapsto x^{q^s}$ with $(n,s) = 1$. If $q = 2$ or $T = \mathbb{F}_q^*$, and $I \in \{\{0\}, \mathbb{F}_q, \mathbb{F}_q^*, \emptyset\}$, the codes $\mathcal{C}_{n,k,\sigma,I}$ and $\mathcal{C}_{\sigma,T}$ are both equivalent to $\mathcal{G}_{k,\sigma}$.*

In the remainder of this section, we shall show that apart from these cases the codes $\mathcal{C}_{\sigma,T}$ and $\mathcal{C}_{n,k,\sigma,I}$ may not be equivalent. In order to do this, we recall the following result that holds for the $\sigma$-linearized polynomials as well.

**Theorem 9.3.2** ([76, Theorem 3]). *A subspace $\mathcal{W}$ of $\mathcal{G}_k$, $k \leq n-1$, is equivalent to $\mathcal{G}_r$ if and only if there exist invertible linearized polynomials $f, g$ such that*

$$\mathcal{W} = \mathcal{G}_r^{(f,g)} = \{f \circ \alpha \circ g : \alpha \in \mathcal{G}_r\},$$

*where $f_0 = 1$, and $\deg_q(f) + \deg_q(g) \leq k - r$.*

By a similar argument used in [76, Lemma 4] for the generalized twisted Gabidulin codes, we prove the following

**Theorem 9.3.3.** *Let $I \notin \{\{0\}, \mathbb{F}_q, \mathbb{F}_q^*, \emptyset\}$, the non linear $(n, n, q; n-k+1)$-MRD code $\mathcal{C}_{n,k,\sigma,I}$ contains a unique subspace equivalent to $\mathcal{G}_{k-1,\sigma}$.*

*Proof.* We have already noted that $\mathcal{C}_{n,k,\sigma,I}$ is contained in $\mathcal{G}_{k+1,\sigma}$. Now, let $\mathcal{W}$ be a subspace of $\mathcal{C}_{n,k,\sigma,I}$ equivalent to $\mathcal{G}_{k-1,\sigma}$. By Theorem 9.3.2, there exist invertible $\sigma$-linearized polynomials $f, g$ such that

$$\mathcal{W} = \mathcal{G}_{k-1,\sigma}^{(f,g)} = \{f \circ \alpha \circ g : \alpha \in \mathcal{G}_{k-1,\sigma}\}$$

with $f_0 \neq 0$ and $\deg_\sigma(f) + \deg_\sigma(g) \leq 2$.
If $\deg_\sigma(f) = t$ then $\deg_\sigma(g) \leq 2 - t$. Let $\alpha(x) = \sum_{i=0}^{k-2} \alpha_i x^{\sigma^i}$ be an element of $\mathcal{G}_{k-1,\sigma}$. Then the coefficient of $x$ in the polynomial $f \circ \alpha \circ g$ is $f_0\alpha_0 g_0$, while the coefficient of $x^{\sigma^k}$ is $f_t \alpha_{k-2}^{\sigma^t} g_{2-t}^{\sigma^{k+t-2}}$. Since $\mathcal{C}^{(1)}_{n,k,\sigma,I}$ and $\mathcal{C}^{(2)}_{n,k,\sigma,I}$ are disjoint, first suppose that $\mathcal{W} \subset \mathcal{C}^{(1)}_{n,k,\sigma,I}$. Then, $\mathrm{N}_{q^n/q}(f_0\alpha_0 g_0) \in I$ and $f_t \alpha_{k-2}^{\sigma^t} g_{2-t}^{\sigma^{k+t-2}} = 0$ for any $\alpha_0, \alpha_{k-2} \in \mathbb{F}_{q^n}$. So, we distinguish three cases:

1. Suppose $t = 2$, then $\deg_\sigma(g) = 0$. Since $f_0, f_2 \neq 0$ this is possible if and only if $0 \in I$ and $g_0 = 0$. This yields $g$ is the zero polynomial, a contradiction.

2. Suppose $t = 1$, then $\deg_\sigma(g) \leq 1$. Since $f_0, f_1 \neq 0$ this is possible if and only if $0 \in I$ and $g_1 = 0$. If $g_0 \neq 0$, since the map $x \in \mathbb{F}_{q^n} \mapsto N_{q^n/q}(f_0 x g_0) \in \mathbb{F}_q$ is surjective, $I = \mathbb{F}_q$ a contradiction. This yields $g$ is the zero polynomial, leading to a contradiction again.

3. Suppose $t = 0$, then $\deg_\sigma(g) \leq 2$. Since $f_0 \neq 0$ this is possible if and only if $0 \in I$ and $g_2 = 0$ and $g_0 = 0$ as in the previous case. It follows $g(x) = g_1 x^\sigma$, implying $\mathcal{W} = \{\alpha \circ x^\sigma : \alpha \in \mathcal{G}_{k-1,\sigma}\}$.

A similar argument can be applied when $\mathcal{W} \subset \mathcal{C}^{(2)}_{n,k,s,I}$. The claim follows. □

**Theorem 9.3.4.** *Let $I \notin \{\{0\}, \mathbb{F}_q, \mathbb{F}_q^*, \emptyset\}$ and $1 \in T \subset \mathbb{F}_q^*$. Then the codes of type $\mathcal{C}_{n,k,\sigma,I}$ and $\mathcal{C}_{\sigma,T}$ are neither equivalent nor adjointly equivalent.*

*Proof.* By contradiction, assume that $\mathcal{C}_{n,k,\sigma,I}$ and $\mathcal{C}_{\sigma,T}$ are equivalent or adjointly equivalent. Then, there exist $f, g, h \in \mathcal{L}_{n,q,\sigma}$ with $f, g$ invertible and $\rho \in \mathrm{Aut}(\mathbb{F}_q)$ such that

$$\mathcal{C}_{n,k,\sigma,I} = f \circ \mathcal{C}^{\rho}_{\sigma,T} \circ g + h \quad \text{or} \quad \mathcal{C}_{n,k,\sigma,I} = f \circ (\mathcal{C}^{\top}_{\sigma,T})^{\rho} \circ g + h.$$

Let $\mathcal{U}, \mathcal{V}$ be the subspaces in $\mathcal{C}_{\sigma,T}$ equivalent to $\mathcal{G}_{k-1,\sigma}$ as defined in (9.7) and (9.8). If $\mathcal{C}_{n,k,\sigma,I}$ and $\mathcal{C}_{\sigma,T}$ are equivalent, then $f \circ \mathcal{U}^{\rho} \circ g + h$ and $f \circ \mathcal{V}^{\rho} \circ g + h$ are two different subspaces of $\mathcal{C}_{n,k,\sigma,I}$ equivalent to $\mathcal{G}_{k-1,\sigma}$. By Theorem 9.3.3, a contradiction follows.
Next, suppose that $\mathcal{C}_{n,k,\sigma,I} = f \circ (\mathcal{C}^{\top}_{\sigma,T})^{\rho} \circ g + h$. Then $f \circ (\mathcal{U}^{\top})^{\rho} \circ g + h$ and $f \circ (\mathcal{V}^{\top})^{\rho} \circ g + h$ are two different subspaces of $\mathcal{C}_{n,k,\sigma,I}$ equivalent to $\mathcal{G}_{k-1,\sigma}$, see [76, Lemma 1], a contradiction again. □

**Remark 9.3.5.** For linear codes, left and right idealisers are invariants under equivalence [65, Proposition 4.1] and they are subalgebras of $\mathcal{L}_{n,q,\sigma}$ isomorphic to subfields of $\mathbb{F}_{q^n}$ in the case of square linear MRD-codes [65, Theorem 5.4]. This, as already stated in [65], does not hold for non-linear codes. In the following, we shall show an example of two equivalent codes such that their left idealisers have different size.
Let $\mathbb{F}_5$ be the field with five elements and let

$$\mathcal{C}_1 = \left\{ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix} \right\} \subset \mathbb{F}_5^{2\times 2}. \tag{9.9}$$

Consider the matrix $H = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}$ and the code

$$\mathcal{C}_2 = \mathcal{C}_1 + H = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \right\}. \tag{9.10}$$

It is straightforward to see that $I_L(\mathcal{C}_1)$ contains only the identity matrix, whereas any matrix $I_2$ of the shape

$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}$$

with $a, b \in \mathbb{F}_5$ is an element of $I_L(\mathcal{C}_2)$.

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