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A UNIFIED THERMODYNAMIC/VORTICAL THEORY FOR THE AERODYNAMIC FORCE ANALYSIS

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A unified thermodynamic/vortical theory for the aerodynamic force analysis

ABSTRACT

Aerodynamic force prediction and decomposition into physical constituents is a fundamental research topic and a clear identification of lift-induced, viscous and wave drag in a generally compressible and dissipative flow is still an open issue.

The Lamb vector, cross-product of flow vorticity and velocity, is at the basis of different far-field methods developed in the last decades for the aerodynamic force analysis, as an alternative to the nowadays well-assessed thermodynamic methods. They offer far-field formulations and decompositions of the *whole* aerodynamic force (drag alone is instead addressed by thermodynamic methods), with identification of local flow structures associated with the force generation. Different *exact* Lamb-vector-based force formulations were proposed in the past and the decomposition of total drag into *lift-induced* and *parasite* contributions was subject of very recent studies, where new induced drag definitions were proposed for viscous and compressible flows. However, inconsistencies were also reported, raising questions on the physical role of a non-zero induced drag found in two-dimensional flows. Moreover, the separation of parasite drag into viscous and wave drag contributions by Lamb vector field analysis is still questioned. Additionally, the effect of the arbitrary pole (appearing in Lamb-vector-based formulas) on the accuracy of such methods in numerical applications was never sufficiently investigated.

In this dissertation, the steady flow regime is first analysed, investigating on the accuracy of different force formulas and of the drag decomposition in lift-induced and parasite contributions. Consistency between lift-induced/parasite drag definitions and the drag generated in isentropic flow conditions (*reversible* drag) or due to entropy production (*irreversible* drag) is analysed. A spurious contribution (related to the streamwise momentum perturbation) is identified, resulting in negative induced drag values in two-dimensional flows. The effect of a shift in the pole location is also analysed, with reference to its impact on the total force and on drag decomposi-

tion accuracy, then a new approach to parasite drag breakdown into physical constituents (viscous and wave contributions) is proposed and results are compared against predictions obtained using consolidated thermodynamic-based methods and against available results from recent literature.

Then investigations are extended to the unsteady regime. Exact vortical force formulas are derived extending, when necessary, recent steady compressible formulas to the unsteady regime and considering inertial or non-inertial frames, viscous or inviscid flows, fixed or moving bodies. For incompressible flows, a mixed inertial/non-inertial formulation, originally proposed for two-dimensional viscous cases, is extended to three-dimensional and to inviscid flows. Two aerodynamic force decompositions are proposed, allowing for Thrust/Drag Bookkeeping (TDB) in two-dimensional viscous and unsteady flows, addressing an open topic in moving wings aerodynamics where no TDB methodology is actually available for oscillating aerofoils in dissipative and compressible flow regime. An additional force decomposition is also formulated, which is independent of the arbitrary pole appearing in vortical formulas. Unsteady terms are re-formulated in an analytically equivalent expression, which will be shown to provide higher accuracy in numerical applications, eliminating large errors observed in computations using the standard unsteady terms expression.

Finally, a new unified thermodynamic/vortical method is developed, where exact Lamb-vector-based formulas are used in combination with a thermodynamic-based calculation of the Lamb vector through Crocco's equation. The resulting hybrid method, which does not require an explicit vorticity calculation, provides results in far better agreement with regards to near-field force data when compared to standard vorticity-based approaches, especially in the presence of shock waves, where inaccuracies of domain integrals involving the Lamb vector were systematically reported by different authors.

The proposed unified approach also led to the formulation of a new *exact* force formula, representing an innovative thermodynamic method, able to compute the total force in generally unsteady viscous and compressible flows (overcoming the limitations of previous thermodynamic methods, which only compute the drag force). The new formula is based on a far-field expression of pressure forces on the body (the only ones deserving a phenomenological decomposition) while friction stresses are retained in their standard *near-field* expression.

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THIS IS DEDICATED TO EDUARDO, MY *THIRD* GRANDFATHER. WITHOUT HAVING A CLEAR IDEA OF WHAT A PH.D. IS, HE MOTIVATED MOST OF MY WORK.

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Italian Aerospace Research Centre

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Nomenclature

List of symbols

$\alpha(t)$	$= \bar{\alpha} + \hat{\alpha} \sin [2\pi f(t - t_0)]$, angle of attack of the free-stream flow
$\alpha_e(t)$	$= \alpha(t) + \arctan \left[\left(\frac{1}{2} - a \right) \frac{b}{V_\infty} \dot{\alpha} \right] + \arctan \left(\frac{\dot{h}}{V_\infty} \right)$, effective inflow angle of attack, considering both the free-stream and the aerofoil motion
$\mathcal{AR}$	wing Aspect Ratio
$\bar{\alpha}$	mean angle of attack during pitching oscillations
$\bar{g}$	ensemble average of the random property g , or time-averaged g quantity, depending on the context of use
β	angle of side-slip of the free-stream flow
$\ddot{h}$	second time derivative of the aerofoil vertical position
$\delta \underline{V}$	$= \underline{V} - \underline{V}_\infty$, velocity perturbation
$\dot{\mathcal{A}}$	anergy production in $\mathcal{V}$
$\dot{\mathcal{E}}$	exergy flow rate through the boundaries of $\mathcal{V}$
$\underline{\dot{\Theta}}$	fluid particle angular velocity
$\dot{h}$	time derivative of the aerofoil vertical position
ε	fluid exergy, per unit mass, referred to asymptotic atmospheric conditions
$\frac{D}{Dt}$	material time derivative
Γ	velocity circulation or weight-specific aircraft energy altitude, depending on the context of use
γ	ratio of air specific heat capacities
γ_b	bound vorticity distribution (unsteady potential theories)
γ_w	free vorticity distribution (unsteady potential theories)

Nomenclature

$\hat{a}$	amplitude of the pitching oscillation
$\hat{h}$	amplitude of the plunging oscillation
$\mathcal{N}$	space dimension ($\mathcal{N} = 2, 3$)
$\mathcal{T}$	time period of the oscillatory motion
$\mathcal{V}$	control volume enclosed by Σ and ∂B
$\mathcal{V}_{sw}$	control volume enclosed by $\partial \mathcal{V}_{sw}$ (shock region)
μ	dynamic (molecular) viscosity of the fluid
μ_t	eddy viscosity
ν	kinematic viscosity of the fluid
ω	vorticity magnitude or specific turbulence dissipation rate, depending on the context of use
$\partial \mathcal{V}_{sw}$	selected closed surface around the shock wave
∂B	body surface
φ	scalar velocity potential
ρ	mass density
Σ	control volume far-field surface
$\tilde{g}$	<i>Favre</i> average of the random property g
$\tilde{h}$	$= \hat{h}/c$, non-dimensional amplitude of the plunging oscillation
$\underline{\underline{\tau}}$	stress tensor
$\underline{\underline{\tau}}_R$	Reynolds stress tensor
$\underline{\underline{\tau}}_{vt}$	$= \underline{\underline{\tau}}_v + \underline{\underline{\tau}}_R$, effective viscous stress tensor in turbulent flow regime
$\underline{\underline{\tau}}_v$	viscous stress tensor
$\underline{\ell}$	$= \underline{\omega} \times \underline{V}$, Lamb vector
$\hat{n}$	unit normal vector on ∂B , pointing outwards of the body
$\hat{r}$	position vector, referred to the pole
$\underline{\omega}$	flow vorticity
$\underline{\underline{\Omega}}(t)$	rotational velocity of the non-inertial reference frame

$\underline{\psi}$	vector velocity potential
$\underline{a}$	fluid particle acceleration
$\underline{e}_x$	unit vector aligned with and oriented as $\underline{V}_\infty$
$\underline{e}_y$	unit vector orthogonal to $\underline{e}_x$ and $\underline{e}_z$, such that $\underline{e}_x$, $\underline{e}_y$ and $\underline{e}_z$ form a right-handed system
$\underline{e}_z$	unit vector orthogonal to $\underline{V}_\infty$, in the plane of the resultant aerodynamic force
$\underline{F}$	aerodynamic force
$\underline{f}$	aerodynamic force per unit span
$\underline{g}$	gravitational acceleration
$\underline{I}_V^*$	density-weighted vortical impulse of the fluid in the control volume $\mathcal{V}$
$\underline{I}_V$	hydrodynamic impulse of the fluid in the control volume $\mathcal{V}$
$\underline{M}$	aerodynamic moment with regards to the pole P
$\underline{n}$	unit normal vector on Σ or ∂B , pointing outward of $\mathcal{V}$
$\underline{n}_{sw}$	unit normal vector on $\partial\mathcal{V}_{sw}$, pointing outward of the shock region
$\underline{P}_V$	momentum of the fluid in the control volume $\mathcal{V}$
$\underline{r}$	scaled position vector, referred to the pole
$\underline{U}(t)$	translational velocity of the non-inertial reference frame
$\underline{V}$	fluid particle velocity
$\underline{V}_{\partial B}$	velocity of the body wall boundary ∂B
$\underline{V}_{\partial\mathcal{V}}$	velocity of the whole control volume boundary
$\underline{V}_\Sigma$	velocity of the external control volume boundary Σ
$\underline{V}_T$	transport velocity in a non-inertial reference frame
$\underline{V}_\infty$	asymptotic flow velocity
$\underline{x}$	position vector
$\underline{x}_p$	Pole position vector
a	chordwise distance of the aerofoil centre of rotation downstream of the mid-chord point, relative to b
a_∞	asymptotic flow speed of sound

Nomenclature

b	$= c/2$, aerofoil half-chord
c	aerofoil chord
C_D^f	three-dimensional friction drag coefficient
C_D^p	three-dimensional pressure drag coefficient
c_0	wing section chord at mid-span
$C_{D,irr}$	$= 2D_{irr} / (\rho_\infty V_\infty^2 S)$, three-dimensional irreversible drag coefficient
$C_{d,irr}$	$= 2d_{irr} / (\rho_\infty V_\infty^2 c)$, two-dimensional irreversible drag coefficient
$C_{D,i}$	$= 2D_i / (\rho_\infty V_\infty^2 S)$, three-dimensional lift-induced drag coefficient
$C_{d,i}$	$= 2d_i / (\rho_\infty V_\infty^2 c)$, two-dimensional lift-induced drag coefficient
$C_{D,p}$	$= 2D_p / (\rho_\infty V_\infty^2 S)$, three-dimensional parasite drag coefficient
$C_{d,p}$	$= 2d_p / (\rho_\infty V_\infty^2 c)$, two-dimensional parasite drag coefficient
$C_{D,rev}$	$= 2D_{rev} / (\rho_\infty V_\infty^2 S)$, three-dimensional reversible drag coefficient
$C_{d,rev}$	$= 2d_{rev} / (\rho_\infty V_\infty^2 c)$, two-dimensional reversible drag coefficient
C_D	$= (2 D) / (\rho_\infty V_\infty^2 S)$, three-dimensional drag coefficient
C_d	$= (2 d) / (\rho_\infty V_\infty^2 c)$, two-dimensional drag coefficient
C_L	$= (2 L) / (\rho_\infty V_\infty^2 S)$, three-dimensional lift coefficient
C_l	$= (2 l) / (\rho_\infty V_\infty^2 c)$, two-dimensional lift coefficient
C_m	$= (2 m) / (\rho_\infty V_\infty^2 c^2)$, two-dimensional pitching moment coefficient, with regards to the aerofoil centre of rotation
C_p	pressure coefficient
D	$= \underline{F} \cdot \underline{e}_x$, drag force component
d	$= \underline{f} \cdot \underline{e}_x$, drag force component per unit span
$d.c.$	drag count ($1d.c. = 0.0001$)
D_{irr}	irreversible drag, i.e. drag due to physical phenomena associated with entropy production
d_{irr}	irreversible drag per unit span
D_i	lift-induced drag

d_i	lift-induced drag per unit span
D_m	drag (or thrust) due to the motion of the body, in inertial frames
D_{pa}	propagation and acoustics drag
D_p	parasite drag
d_p	parasite drag per unit span
D_{rev}	reversible drag, i.e. drag due to physical phenomena not associated with entropy production
d_{rev}	reversible drag per unit span
D_{sp}	spurious drag
D_v	viscous drag
D_w	wave drag
e	fluid internal energy, per unit mass
f	$= 1/\mathcal{T}$, frequency of the oscillatory motion
g''	turbulent fluctuation of the random property g with respect to its <i>Favre</i> average
g'	turbulent fluctuation of the random property g with respect to its ensemble average
H	fluid total enthalpy, per unit mass
h	fluid enthalpy, per unit mass
$h(t)$	$= \hat{h} \sin[2\pi f(t - t_0)]$, aerofoil vertical position (positive downwards)
k	$= \pi f c / V_\infty$, reduced frequency of the oscillatory motion
k_t	turbulent kinetic energy
L	$= \underline{F} \cdot \underline{e}_z$, lift force component
l	$= \underline{f} \cdot \underline{e}_z$, lift force component per unit span
m	aerodynamic pitching moment per unit span, with regards to the aerofoil centre of rotation
m^*	aerodynamic pitching moment per unit span, with regards to the aerofoil mid-chord point
M_∞	asymptotic flow Mach number

Nomenclature

n_x	normal vector component along $\underline{e}_x$
n_z	normal vector component along $\underline{e}_z$
p	static pressure
$p_{d,\infty}$	asymptotic flow dynamic pressure
p_d	dynamic pressure
$p_{t,\infty}$	asymptotic flow total pressure
p_t	total pressure
R	gas constant of the air mixture
Re_∞	asymptotic flow Reynolds number
S	reference surface
s	fluid entropy, per unit mass
T	fluid static temperature
t	physical time
t_0	initial time of aerofoil oscillations
T_∞	asymptotic flow static temperature
U	velocity component along $\underline{e}_x$
u	$= U - V_\infty$, velocity perturbation component along $\underline{e}_x$
v	velocity perturbation component along $\underline{e}_y$
V_∞	asymptotic flow velocity magnitude
w	velocity perturbation component along $\underline{e}_z$

Subscripts

π	transverse component lying on a Trefftz plane
a	computed using the vortical far-field formulation “ a ”
AM	added-mass contribution
b	computed using the vortical far-field formulation “ b ”
bc	computed using the vortical far-field formulation “ bc ”

C	circulatory contribution
c	computed using the vortical far-field formulation “ c ”
d	computed using the vortical far-field formulation “ d ”
ff	computed using the far-field formulation
NC	non-circulatory contribution
nf	computed using the near-field formulation
QS	quasi-steady contribution
st	steady formulation
uns	unsteady formulation
w	wake or wave contribution, depending on the context of use

Superscripts

$'$	quantity referred to a non-inertial frame of reference, or turbulent fluctuation with respect to the ensemble average, depending on the context of use
$''$	turbulent fluctuation with respect to the <i>Favre</i> average
a	computed using the vortical far-field formulation “ a ”
b	computed using the vortical far-field formulation “ b ”
bc	computed using the vortical far-field formulation “ bc ”
c	computed using the vortical far-field formulation “ c ”
d	computed using the vortical far-field formulation “ d ”
DV	computed using the thermodynamic method of Destarac and van der Vooren
NF	direct output of the flow solver
nf	computed using the near-field formulation
PT	computed using the thermodynamic method of Paparone and Tognaccini
st	steady formulation
T	tensor transpose or transport velocity term, depending on the context of use
uns	unsteady formulation

Introduction

CONTEXT AND MOTIVATION

Air transportation is undergoing a significant growth over time even after the pandemic crisis, with the demand for commercial flights increasing at an average annual rate of 3.4% and expected to double in 2040 (International Air Transport Association, Global Outlook for Air Transport, June 2023). In this perspective, aviation advisory committees from different world areas are pushing towards a strong reduction of the air traffic environmental footprint, including ambitious targets on noise and pollutants emissions. Concerning fuel consumption, the strategic research agenda is focusing on four different areas of improvement: propulsion systems (via the employment of high by-pass ratio or highly-integrated efficient power-plants, as well as through the use of sustainable aviation fuels or electric energy extracted by hydrogen in fuel cells); airframe structures (reducing weight by increasing the use of light composite materials and/or novel architectures); flight avionics (e.g. allowing for load alleviation and wise power management); then, last but not least, aircraft aerodynamics (pursuing drag reduction as its first aim).

A predictive capability on the aerodynamic force experienced by aircrafts during the different flight phases is then of paramount importance and represents a core topic in fluid mechanics. The aerodynamic force which opposes to the motion of a vehicle immersed and advancing in the surrounding fluid (*drag* force) can be ascribed to different physical phenomena and flow features (trailing vortices, viscous wakes and shock waves). An accurate identification of the different drag contributions is of fundamental relevance, for both theoretical and practical reasons.

In fact, drag reduction on modern aircrafts (targeting highly demanding requirements for a greener next-generation fleet) is typically addressed through numerical approaches (rather than via expensive wind tunnel campaigns, impractical in the product design phase), based on shape optimisation procedures, which can exploit information on total drag breakdown to define wiser objective functions [21] and/or to evaluate trade-offs in multi-objective problems [228], improving the design process effectiveness and computational efficiency.

Usually, different design solutions are selected to address the different forms of drag, for example: innovative configurations, box wings or winglets, to reduce the vortex drag [26, 37, 170]; supercritical aerofoils to limit shock wave strength and associated drag in transonic cruise flight [159]; suitable wing leading-edge design to enlarge the natural laminar flow region, delaying tran-

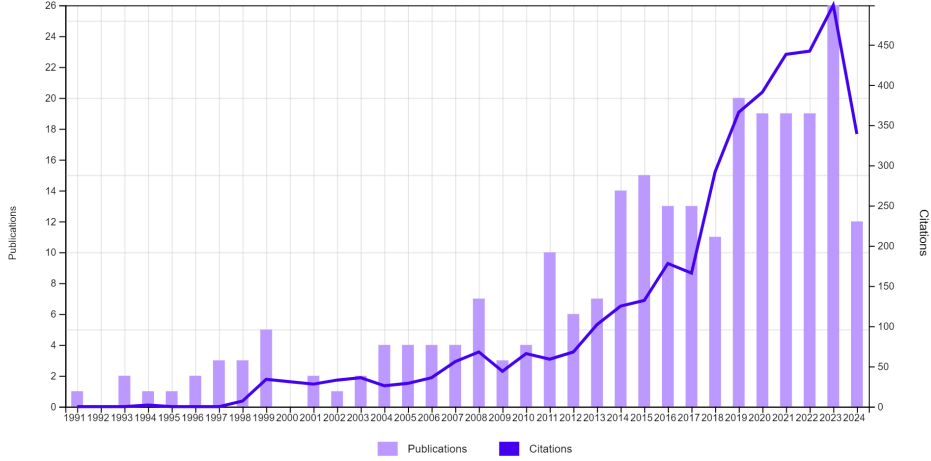


Figure 0.0.1: Results of a “drag decomposition” query in *Clarivate Web of Science* database: times cited and publications over time.

sition to the turbulent regime, associated to an increase in friction drag [27]; surface riblets, to reduce friction drag of turbulent boundary layers [25, 118]; flow control devices, to delay boundary layers separation, then limiting viscous pressure drag [158]. Indeed, all of these devices/solutions can be fine-tuned through an aerodynamic optimisation loop, where the capability to extract physically relevant information on the different drag sources can then be efficiently used.

Such methodologies can be extremely insightful also in the field of unsteady aerodynamics, playing a fundamental role in different engineering areas (from bio-inspired flight technologies to urban air mobility, from helicopter rotor design to fixed wing transonic buffeting). In particular, for propulsion systems like flapping wings, a phenomenological decomposition of the total axial force (i.e. the force component in the direction of flight) identifying the contributions responsible for thrust and the ones responsible for drag (Thrust/Drag Bookkeeping, TDB), in a generally viscous and unsteady flow, is still an open problem.

In aircrafts design, the analysis and decomposition of the aerodynamic force into its physical contributions, based on the inspection of the flow domain, is fundamental for the optimisation of aerodynamic shapes and propulsive systems.

On the other hand, advances in aerodynamic force prediction and decomposition theories also allow for a deeper understanding on the genesis of the different force contributions. This is why scientific interest in methods for an accurate force computation and its decomposition into physically distinct components is never fading, from the first era of fluid mechanics investigations to contemporary studies [48]. Fig. 0.0.1 illustrates the increase in indexed research in the field of drag decomposition, from 90’s to contemporary age.

Nowadays, the continuous development of modern High Performance Computing (HPC) capabilities and recent Digital Particle Image Velocimetry (DPIV) techniques led to the availability

of a big amount of data, coming from both Computational Fluid Dynamics (CFD) and Experimental Fluid Dynamics (EFD), which gave a fundamental role to data analysis tools, with the focus being on post-processing available results to extract physically relevant information. In this new context, Wu et al. [217] suggested to look back to classical aerodynamic theories and put a renewed interest in the theoretical analysis of the aerodynamic force, to develop modern far-field force theories and improved methods able to fruitfully exploit the available big data.

AIM AND OUTLINE

This dissertation deals with *Lamb-vector-based* methods for the analysis and decomposition of the aerodynamic force. They represent a class of *far-field* methods which revealed promising in the last decades, allowing for the computation of the total force (both lift and drag) as well as for the direct definition and calculation of lift-induced drag in generally viscous and compressible flows [217]. These methods were also identified to potentially provide *thrust/drag bookkeeping* in two-dimensional viscous flows for propulsive systems like flapping wings [169]. For such systems, in subsonic conditions, drag is typically estimated through steady-state calculations, which results are subsequently added to the unsteady thrust computed by inviscid flow simulations, of course neglecting combined effects [175].

However, different studies highlight that the accuracy of *Lamb-vector-based* force predictions did not yet reach a mature stage, for steady [56, 81] and especially for unsteady flows [140, 142], therefore these methods need to be sharpened and improved. Moreover, the research steps leading to their development are convoluted, with different formulations and decompositions that were proposed up to now, involving several degrees of freedom in the application of the methods and some aspects (such as, for example, wave drag identification) which are still questioned.

On the other hand, *thermodynamic* methods (a different class of far-field formulas developed at the beginning of the 21st century), well-assessed and nowadays routinely used in the aviation industry, offer a reliable approach to the identification of viscous and wave drag [200] and proved their superior accuracy. Nevertheless lift-induced drag can only be indirectly derived (by difference to the total drag) and the transverse (lift) component of the aerodynamic force is not computed.

A reasonable and desirable goal would be to ideally combine advantages offered by the two abovementioned approaches, to provide aerospace engineers with a new improved methodology for the analysis and decomposition of the aerodynamic force. This will also turn into a scientific achievement, providing unification of the available theories and offering new perspectives and insight on the generation of lift and drag.

The research activity, subject of this dissertation, started therefore with a preliminary effort focused on the identification of all the critical aspects and spurious effects influencing the robustness, accuracy and reliability of the force computation and breakdown in *Lamb-vector-based* methods [124]. Some of the highlighted problems affected the majority of numerical applications

published in the past, although very rarely discussed.

As a pre-requisite, the many different Lamb-vector-based force formulas available from literature were homogeneously organised in a general form, which is valid for either laminar or turbulent mean flows, viscous or inviscid regime, fixed or moving bodies, inertial or non-inertial reference frames. This also included the derivation of extended versions of some of the formulas, initially restricted to viscous and steady regimes, to deal with inviscid and/or unsteady flows [127].

Solutions to some of the open issues identified in the preliminary phase (such as the separation of viscous and wave drag from the total parasite contribution, or the accuracy of unsteady inertial formulas) were proposed and verified through numerical applications [124, 127].

Then, a new unified thermodynamic/vortical approach was conceived, developed and tested [122]. From the operative point of view, it allowed for solving most of the problems frequently encountered in the numerical application of Lamb-vector-based methods and highlighted in the early steps of this doctoral research, namely: the accuracy of force formulas in presence of shock waves, the non-physical dependence of total force coefficients on the control volume selection, a spurious local force production in non-rotational flow-field regions, etc..

From the theoretical point of view, the proposed unified approach also led to the formulation of a new *exact* force expression, representing an innovative thermodynamic method, able to compute the total force (not just drag alone) in generally unsteady viscous and compressible flows. The new formula is based on a far-field expression of pressure forces on the body (the only ones deserving a phenomenological decomposition) while friction stresses are retained in their standard *near-field* expression.

In what follows, an outline of the different chapter making up the structure of this dissertation is illustrated, highlighting dissemination activities and scientific publications associated with the research content included therein.

CHAPTER 1 provides an introduction to both *near-field* and *far-field* approaches for the computation and decomposition of the aerodynamic force, considering steady and unsteady regimes.

A comprehensive literature review on existing force analysis and breakdown methods is included, starting from an historical overview on pioneering studies and milestone formulas, which still today represent lighthouse references and guided the development of modern theories, extending and generalising first studies to complex flows, as well as providing innovative and insightful approaches and perspectives.

After a rigorous definition of the different forms of aerodynamic drag, identified on a physical and phenomenological basis, *operative* drag definitions (i.e. the available methods) used to predict each drag contribution will be illustrated. They consist in modern approaches developed for three-dimensional viscous flows, which have found implementations in both EFD and CFD contexts, based on generalisation of classical methods but also on the introduction of *thermodynamics*, *vorticity* (or the *Lamb vector*), *exergy* and *partial pressure fields*. Those methods will be thoroughly reviewed and discussed, using a homogeneous notation of symbols and formulas throughout all

chapter of this dissertation. Links between theories will also be identified and highlighted, in pursuit of the scientific goal of collocating the different methods into a unified picture.

Then the dissertation will address its core topic: Lamb vector methods.

CHAPTER 2 is dedicated to the *steady* regime. An extensive analysis on the accuracy of Lamb-vector-based total force formulas and of the drag decomposition in lift-induced and parasite contributions is first illustrated, considering both inviscid and viscous numerical flow solutions in nearly-incompressible, subsonic and transonic regime.

Some of the available formulas, originally developed for viscous cases, were extended to the analysis of inviscid flows and expressed using a general notation, which considers either laminar or turbulent mean flows.

Sensitivities to some degrees of freedom of Lamb-vector-based formulations (such as the control volume selection or the choice of an arbitrary pole) are analysed, the latter being rarely discussed in the literature (an exception is represented by the study included in the work by Kang et al. [81], although limited to one flow condition and one force formula).

This analysis was done for the total force (lift and drag) as well as for the drag decomposition in lift-induced and parasite contribution (in 2D and in 3D), clarifying for the first time the relationship between *reversible/irreversible* drag contributions and *lift-induced/parasite* drag definitions, removing the ambiguity that tended to identify the two in previous works. A spurious effect was identified, polluting the decomposition and resulting in negative induced drag values computed in two-dimensional flows. This will be shown to be related to the axial velocity perturbation, while thermodynamic methods will confirm the whole drag being of irreversible nature in all analysed two-dimensional cases, leaving no space to any non-zero reversible drag. Both high and low Reynolds number flow conditions are investigated, the latter being particularly interesting as Kang et al. [81] first reported negative lift-induced drag values post-processing two-dimensional laminar flow solutions, a feature that Minervino and Tognaccini [124, 125] recently observed also in transonic flows and in other particular conditions. In previous studies the decomposition was tacitly assumed to capture reversible and irreversible drag contributions, regardless of the control volume and pole position dependence, even in three-dimensional cases [116]. In this chapter, instead, the spurious effect, already identified on 2D cases and affecting the drag decomposition in induced and parasite contributions, will be highlighted also in 3D flows.

A new methodology to perform parasite drag breakdown, isolating viscous and wave contributions in two-dimensional transonic flow-fields, is then proposed and detailed numerical results are illustrated in case of high Reynolds number transonic flows around the RAE2822 and NACA0012 aerofoils (the latter considered in both symmetric and asymmetric flow conditions). A comparison against predictions obtained using consolidated thermodynamics-based methods [42, 148] is also discussed and benchmarking is provided with recent results [69, 117, 172], some of them obtained using completely different approaches.

The main results of the research illustrated in this chapter were presented at the 14th European

Fluid Mechanics Conference in 2022 [121], at the *AIAA SciTech Forum* in 2023 [125] and at the 57th *3AF International Conference on Applied Aerodynamics* [172], then published in the *Aerospace Science and Technology* journal [124].

CHAPTER 3, dedicated to the *unsteady* regime, illustrates a decomposition of the non-stationary vortical far-field formula, proposed to identify the unsteady parasite drag and the axial force induced by the unsteady lift.

For two-dimensional flows (no induced drag), this will be shown to allow for TDB in the total axial force on oscillating aerofoils in a viscous flow, at least at high Reynolds number, when the (zero) induced force can be accurately computed by Lamb-vector-based methods [81, 124]. This also gives a new insight on the physical phenomenon of thrust generation on flapping wings, complementary to that provided by Knoller [85] and Betz [16]: thrust is obtained as the axial force induced by the unsteady lift, i.e. by the time-variation of the aerofoil circulation and associated unsteady trailing vorticity. This aspect was quite interestingly observed by Saffman [173, p. 113] when analysing 2D accelerated inviscid flows. He highlighted that the velocity induced by vorticity may give a vortex force component in the free-stream direction. He therefore interpreted the birth of induced drag in 3D by analysing a 2D problem, observing that an induced force also exists in two-dimensional (yet unsteady) flows.

At the scope, different Lamb vector-based exact force formulas available in the literature for steady flows (a summary is provided in [124]) are extended to the unsteady regime and compared post-processing two-dimensional numerical flow-fields on moving grids.

A compact and general formulation will be introduced, which is valid for either laminar or turbulent mean flows, viscous or inviscid regime, fixed or moving bodies, inertial or non-inertial reference frames. Unsteady terms will also be re-formulated in an analytically equivalent expression, which will be shown to provide higher accuracy in numerical application, eliminating large errors observed in computations using the standard unsteady terms formula.

For incompressible flows, the mixed inertial/non-inertial formulation proposed by Ostieri et al. [140] and originally derived for two-dimensional viscous flow conditions, will be here extended to the general case of three-dimensional viscous/inviscid flow regimes.

Applications to a NACA0012 aerofoil oscillating in plunging or pitching motion are illustrated, considering subsonic and transonic flows and comparing far-field results to near-field estimates as well as, for incompressible flows, to classical linear theories by Theodorsen [194] and Garrick [61]. The effect of the integration domain size on the accuracy of the force formula is extensively analysed and selection criteria for the control volume and the arbitrary pole location to be used in TDB are illustrated. Finally, a new pole-independent force decomposition is also formulated and results are presented for the analysed unsteady viscous cases.

The main results of the research illustrated in this chapter were presented in 2023 at the *AIAA Aviation Forum* [188] and at the 57th *3AF International Conference on Applied Aerodynamics* [123], then published in the *International Journal of Numerical Methods for Heat & Fluid Flow* [127].

CHAPTER 4 is dedicated to illustrate the theoretical formulation, numerical application and detailed discussion on results/perspectives of the new *unified thermodynamic/Lamb-vector-based far-field force method*, developed in the framework of this doctoral research. It comes at the end of the thorough analysis of both steady and unsteady Lamb-vector methods illustrated in the following chapters, where spotlights were set on inconsistencies and inaccuracies previously left in the shadow, limiting the application and reliability of such force prediction and decomposition approaches. The proposed new methodology provides an effective solution to most of those identified accuracy issues.

The new mixed approach provides a unified formula where Lamb-vector-based and thermodynamic-based methods, so far considered as different alternatives, are combined in a new exact expression of the total aerodynamic force. This will be shown to provide a far-field formulation of the pressure force acting on the body, where different contributions can be identified. In a future research perspective, this is particularly interesting, as an alternative approach to near-field pressure force decomposition was recently provided by Partial Pressure Fields (PPFs) methods, first derived by Schmitz and Coder for the incompressible regime [178] and then extended to compressible flows [177].

In this chapter, the proposed unified method is applied to post-process two-dimensional and three-dimensional numerical flow-fields from both steady and unsteady CFD simulations. A NACA0012 aerofoil is considered, either fixed or oscillating with a pure-plunging motion, in a subsonic or transonic, viscous or inviscid, flow. Finally, an application to a 3D elliptic wing in transonic viscous and turbulent flow is also illustrated.

The main results of the research illustrated in this chapter were presented at the 1st *European Fluid Dynamics Conference* in 2024 [126] and published in the *Physics of Fluids* journal [122].

... the fluid leaves the body along a line called the line of separation, forming the generation of a surface of scission such that the speeds are not the same on both sides of the surface, thus forming a vortical layer.

Ludwig Von Prandtl

1

The aerodynamic force

IN THIS DISSERTATION, FOCUS IS GIVEN ON EXTERNAL AERODYNAMICS FOR AIR VEHICLES ENGINEERING. Therefore, the typical assumptions used in aeronautics will be introduced, starting from continuum mechanics governing equations restricted to non-micropolar fluids. A Newtonian model for the (symmetric) stress tensor will be used along with Poisson-Cauchy constitutive laws, where the bulk (longitudinal) viscosity coefficient is neglected for the air mixture. The preferential point of view will be from the side of compressible gas dynamics and the first (shear) viscosity coefficient will directly multiply the deviatoric part of the strain rate tensor. Nevertheless it is worth mentioning that the use of the triple stress tensor decomposition proposed by Wu et al. [222, pp. 52-55], highlighting the role of flow vorticity on internal fluid forces, represents an alternative and insightful approach especially in the case of incompressible flows with constant viscosity.

The system of external forces exerted on the body by the surrounding fluid is equivalently represented by resultant force (which will be at the centre of this dissertation) applied at a chosen pole, and by the aerodynamic moment around the selected pole. The latter, is usually discussed in less detail as, in the assumed hypotheses, the total and extrinsic angular momentum do coincide. Therefore the local contribution to the aerodynamic moment can be derived from that to the resultant force, multiplied by the position vector (referred to the selected pole), neglecting intrinsic contributions. Moreover, decomposition methods do not (yet) address the aerodynamic

moment^{*}, which limits the interest in the corresponding far-field formulations. Although this dissertation will mainly address the resultant force, formulas for the aerodynamic moment will also be derived in relevant cases, such as for Lamb-vector-based methods (see Appendix A.4.2), which set in the core of this doctoral research.

1.1 NEAR-FIELD VS. FAR-FIELD APPROACHES

The standard approach to calculate the aerodynamic force ($\underline{F}$) in CFD is based on the integration of fluid stresses ($\underline{\tau}$) on the body surface (∂B) exposed to the fluid interaction (*near-field* method):

$$\underline{F}_{nf} = \int_{\partial B} \underline{\tau} \cdot \underline{\hat{n}} \, dS = - \int_{\partial B} p \underline{\hat{n}} \, dS + \int_{\partial B} \underline{\tau}_v \cdot \underline{\hat{n}} \, dS = \int_{\partial B} p \underline{\hat{n}} \, dS - \int_{\partial B} \underline{\tau}_v \cdot \underline{\hat{n}} \, dS = \underline{F}_p + \underline{F}_\tau \quad (1.1)$$

where p is the static pressure, $\underline{\tau}_v$ is the viscous stress tensor (symmetric, under the assumed hypotheses) and $\underline{\hat{n}}$ is a unit normal vector on ∂B , pointing outwards of the body (opposed to $\underline{n}$, pointing outwards of the adjacent fluidic region)[†]. According to Newton's third law, $\underline{F}_{nf}$ is also opposed to the force exerted by the body on the surrounding fluid.

Eq. (1.1) only allows for a mechanical distinction between friction ($\underline{F}_\tau$) and pressure ($\underline{F}_p$) force contributions (with straightforward definition of the two symbols as the individual shear and normal stress integrals appearing in the formula). While friction stresses are purely related to viscous effects, wall pressure contributes to both *lift-induced* drag (see Sec. 1.2.1.4) and *parasite* drag (sum of viscous and wave drag). The entire computed flow-field does not *directly* come into play in Eq. (1.1): the near-field method uses field data at the wall surface only.

In EFD, instead, an *extrinsic* approach is usually adopted, where the aerodynamic force is derived by evaluating the *external* force needed to balance the former, obtained from measurements through a balance apparatus. However, accurate drag measurements require other methods, due to difference in magnitude between axial and transverse net force components loading the balance. The *intrinsic* way to calculate the aerodynamic force (near-field approach) is also not suitable for experimental applications, where wall friction is hard to be measured and normal stress data obtained using pressure taps or sensitive paints cannot be used to accurately estimate lift or drag.

For these reasons and to address the phenomenological drag breakdown into viscous, wave and induced components, alternative formulations (*far-field* methods) were developed, based on the integration of flow quantities in a control volume ($\mathcal{V}$) surrounding the body (Fig. 1.1.1) and on its external boundary (Σ) [200, 232]. An integral balance of linear momentum in $\mathcal{V}$ is used at the

^{*}an exception is represented by unsteady linear theories, which provide interesting decompositions of the pitching moment in two-dimensional inviscid and incompressible slightly perturbed flows (Sec. 1.2.2).

[†]Rigorously, Eq. (1.1) represents the whole force exerted on the body by the surrounding air and is not strictly the *aerodynamic* force, which is instead defined by difference with the *aerostatic* force. In this dissertation, however, the latter will always be neglected, for the cases of interest.

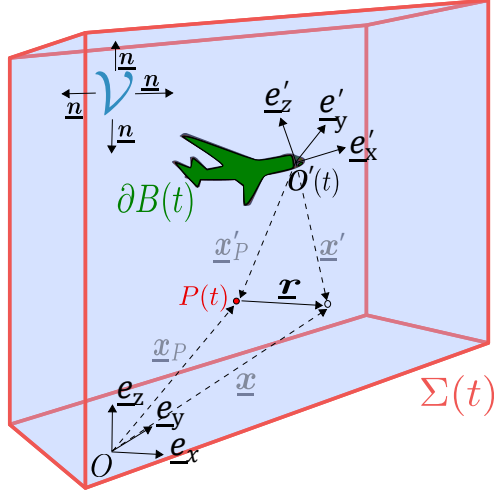


Figure 1.1.1: Generic control volume for far-field force calculation.

scope and can be concisely written as: $\underline{F}_{nf} = \underline{F}_{ff}$, where the following definition has been made:

$$\begin{aligned} \underline{F}_{ff} = & - \int_V \frac{\partial (\rho \underline{V})}{\partial t} dV - \int_{\partial B} \rho \underline{V} \underline{V} \cdot \underline{n} dS + \\ & - \int_{\Sigma} \left(p \underline{n} - \underline{\tau}_v \cdot \underline{n} + \rho \underline{V} \underline{V} \cdot \underline{n} \right) dS + \int_V \rho \underline{f}_{-m} dV \end{aligned} \quad (1.2)$$

In Eq. (1.2), $\underline{V}$ and ρ are respectively the fluid velocity and the mass density, t is the physical time and $\underline{f}_{-m}$ is a body force density per unit mass (for example due to gravitational or electromagnetic effects). It is worth highlighting that the advective momentum flux at the body skin has been attributed to $\underline{F}_{ff}$ instead of $\underline{F}_{nf}$ (as instead usually done in presence of propulsors or flow control systems) because this dissertation will mainly address aerodynamic forces excerpted on impermeable bodies, for which momentum advection at the body boundary is non-zero only in the case of moving walls. ‡

‡ It is worth highlighting that Eq. (1.2) is derived from momentum balance by applying the Gauss theorem, which relies on differentiable fields defined in V . This is the case of viscous flows governed by Navier-Stokes equations, but the expression of the far-field force is instead different for inviscid flows governed by the Euler equations. In this case, in fact, discontinuities might be present in the flow-field (e.g. shock waves and/or shear surfaces). Flow quantities will not be differentiable through these surfaces of discontinuity, which therefore need to be considered as additional boundaries of the control volume V where integral theorems are applied. In principle, the same derivation can be followed, replacing the body boundary ∂B with $\partial W \cup \partial B \cup \partial(SW)$, where W is the infinitely-thin vortical wake (shear discontinuity) and SW is the system of all shock waves in the flow-field. Branch-cuts need also be included as domain boundaries, to restore single connectivity in V , nevertheless providing zero overall contribution to the force. In the inviscid subsonic case (i.e. without shock waves) and for steady flows, the resulting far-field force expression is not modified with regards to Eq. (1.2). In fact,

As immediately observable, Eq. (1.2) (and any other exact force expression obtained by algebraic sum of different terms) provides a further decomposition of the aerodynamic force, alternative to the splitting in pressure and viscous contribution obtained from Eq. (1.1). The physical interpretation of such a breakdown is however less straightforward compared to the mechanical decomposition of forces on the body skin and, therefore, less appealing from an engineering point of view. In this perspective, a *phenomenological* force breakdown is instead desired.

Momentum balance also expresses the equivalence between the force exerted by the body on the surrounding fluid through normal and shear stresses ($-\underline{F}_{nf}$) and the resultant of the effects produced in the fluid by the moving body, generating vortices, boundary layers, shock waves and wakes in the flow, which are measured by $-\underline{F}_{ff}$. The latter interpretation of $\underline{F}_{ff}$ leads to the identification of three phenomenologically distinct drag sources: induced, viscous and wake drag, which will be extensively discussed in the following sections. Drag breakdown methods attempt to separate and to quantify the different contributions, however without providing an *exact* splitting of the total drag, which is theoretically not achievable due to the interplay of the different physical phenomena involved, their effects being therefore superimposed within the entire flow-field [202]. Nevertheless, good approximations of such an ideal splitting are feasible and discussed throughout this and the following chapters.

Instead of just using field data at the body skin, far-field approaches directly involve field data from the entire control domain surrounding the body and on its external boundary, through volume and surface integrals. In CFD applications, accuracy of far-field force prediction thus requires the flow-field to be sufficiently converged, reducing momentum imbalance to a negligible level.

When surface integrals can be converted into volume integrals – by applying Gauss’s theorem and/or Derivative Moment Transformations (DMTs, [138][222, pp. 698-705]) – far-field methods also offer the possibility to identify the source of the aerodynamic force in the flow domain, relating the force to local flow quantities [100, 217, 221, 226]. In this case, some authors [228] also use the “mid-field methods” appellation. In the following however, we can consider them as sub-class of far-field methods. The additional possibility, for high Reynolds number flows, to restrict both volume and surface integration to limited regions of the flow-field is also provided by some force formulations using integrated quantities that are negligible elsewhere (*mid-field property* [117]). The relevance of the mid-field property is twofold in CFD: it allows excluding, from the aerodynamic force computation, part of the so-called *spurious* drag (caused by numerical errors and artificial dissipation of advective fluxes schemes) and obtaining a weaker grid dependency of drag estimations [148, 198]; then it also reduces the numerical dependence of the computed far-field force on the control volume (CV) size, avoiding the integration of quantities in flow-field regions far from the body, where the spatial grid is usually not sufficiently refined.

In EFD, far-field force expressions are usually cast in forms which only use wake integration

the Kelvin theorem ensures that the vortical wake is a stream surface (no mass-flux through W). The advective momentum flux on ∂W is therefore zero, as well as the diffusive momentum flux since the wake supports no pressure load (consequence of the discontinuous momentum balance across the shear surface).

(i.e. surface integrals performed downstream of the body) and are historically widely employed, as they represent an effective and non-intrusive technique to extract drag measurements from a wind tunnel test [206]. In the last decades, the development of modern DPIV measurement techniques significantly contributed to a renewed research focus on far-field methods in EFD [58, 208].

Classical steady far-field methods are represented, for example, by the well-known Kutta-Joukowski theorem [78, 93] (linking lift to velocity circulation, in two-dimensional (2D) inviscid and incompressible flow regime), Maskell's formula [114] (relating lift-induced drag to the transverse velocity field on a Trefftz plane, in incompressible flows) and Betz's formula [17] (expressing parasite drag in terms of total pressure losses, in non-lifting incompressible flows). These methods were originally derived assuming simplified flow models, although their validity has been often verified to hold in more general flow regimes, through wind-tunnel wake survey measurements or post-processing CFD solutions [23, 29, 34, 64, 72, 73, 89, 203, 204]. Research efforts to extend classical far-field formulas to viscous and/or compressible flows were made by Schmitz and Coder [178], Schmitz [177] and Liu et al. [101, 104]. Recent studies on the vortical nature of the aerodynamic force also lead to a generalisation of the Kutta-Joukowski theorem in transonic and supersonic regime [101, 104, 231].

Here below, in Sec. 1.2, an excursus on the foundations of modern aerodynamic force analysis and decomposition theories is provided, starting from pioneering studies and classical methods, till the age of CFD (second half of the 20th century), highlighting the key milestones that still today represent consolidated references.

Modern methods developed for three-dimensional viscous flows, which have found implementations in both EFD and CFD contexts, are instead illustrated in following dedicated sections (1.4 and 1.5). Their validity, or extension, for compressible and unsteady flows will also be highlighted.

1.2 HISTORICAL OVERVIEW

1.2.1 STEADY REGIME AND UNIVERSAL THEORIES FROM ASYMPTOTICS

It is a common, yet provocative, choice to trace back the origin of aerodynamic force studies to 1752, when the French physician d'Alembert highlighted how, in an ideal incompressible fluid with zero viscosity, no force could be generated around bodies of any shape according to the potential flow theory, in clear contrast with experimental observations (*d'Alembert paradox*). Although historically referred to drag alone, the paradox leads to zero lift as well, a point recently remarked by Spalart [184] who also extended the result to elongated bodies.

Investigations then focused on viscosity effects, the only physical root justifying the observed forces. First investigations on the resistance of ships by Froude in 1872 analysed skin-friction forces and realised experiments measuring the resistance of a series of towed flat plates.

Experiments carried out by Reynolds in 1883 assessed transitional conditions between laminar and turbulent flow regimes (already observed by Da Vinci in 1510) and allowed quantifying the

relative importance of inertial over viscous effects through the definition of the non-dimensional *Reynolds number*. It was however in 1904 that studies reached a quantitatively mature stage, with the formulation of a *boundary layer* theory by Prandtl first [152] and later on by von Kármán [212]. They assumed the continuity of material particle velocity through the body skin and the effects of fluid viscosity, for high Reynolds number flows, were recognised to significantly act in a limited region of the flow domain only, i.e. around solid walls and inside wakes. *Form* drag added to *friction* drag as a result of the viscous/inviscid interaction between the boundary layer and the external flow, modifying wall pressure distributions.

Theories needed to be supported by evidence, at that time only available via tunnel experiments. The first far-field methods were then developed and tested, with the aim to measure the *profile* drag (sum of friction and form drag, in subsonic cases) from wake surveys performed using bent or Pitot tubes and confined to vortical wakes.

Some authors [53] account these methods in the category of thermodynamic formulations, as they are based on pressure, which is a thermodynamic variable. Nevertheless, in this dissertation, it was preferred to list them among *classical methods*, i.e. those far-field formulations directly derived from momentum equation without introducing constitutive relations of the fluid (as instead done for entropy-based methods, described in the following) or taking the curl to work with flow vorticity (as done in vortical approaches). Finally one should also consider that, for incompressible fluids, pressure is often attributed a non-thermodynamic meaning [67].

1.2.1.1 BETZ'S FORMULA

A classical far-field method is represented, for example, by the Betz's formula [17], expressing profile drag (D_p) in terms of total pressure losses, in non-lifting incompressible flows. Indeed the idea of calculating drag from energy losses (total pressure defect, in the incompressible case) was already attributed to Ackeret [2]. The method proposed by Betz (still widely used in tunnel measurements) used a control volume with lateral surfaces extending to infinity, then, combining integrals on planes ahead and behind the body, he obtained the formula:

$$D_p = \int_W (p_{t,\infty} - p_t) dS - \frac{\rho_\infty}{2} \int_W (u' - u) (2V_\infty - u' - u) dS \quad (1.3)$$

where p_t is the flow total pressure and u' is the local velocity of a potential flow with the same free-stream total pressure of the flow producing drag. In order to guarantee the continuity, this potential flow necessitates sources and generates thrust, which is accounted for in the momentum balance expressed by Eq. (1.3). Betz showed that, far downstream from the body, the second integral in Eq. (1.3) can be neglected, its value being estimated in the order of $1/20$ of the first integral by a rough quantitative analysis. Moreover, he also extended the method to lifting bodies, obtaining a formula identical to Eq. (1.3). The same expression was independently derived also by Taylor [191], using the same flow assumptions.

1.2.1.2 JONES' FORMULA

An improved method to compute profile drag by wake surveys using a Pitot tube was developed by Jones [76] in 1936. He assumed that the wake can be approximated as “stream tubes” (with constant total pressure) which do not mix as they move from the measurement plane (W) to a plane far downstream, where the static pressure is considered restored to its free-stream value. The final formula he derived avoids the introduction of the potential flow velocity u' and writes as:

$$D_p = p_{d,\infty} \int_W 2\sqrt{p_d^* - \Delta p^*} (1 - \sqrt{p_d^*}) dS \quad (1.4)$$

where p_d is the fluid dynamic pressure, $\Delta p = p - p_\infty$ and the superscript $*$ indicates non-dimensional values with regards to $p_{d,\infty}$. Bicknell [18] in 1939 derived a generalised Jones' formula in the case of compressible fluids, highlighting that Eq. (1.4) is obtained as a particular case neglecting compressibility effects. A compressible modification of Jones' formula was also derived by Lock et al. [106] in 1940 and successively simplified four years later in the work of Thompson [195]. An alternative profile drag calculation method was also proposed in the period by Squire and Young [186] based on the estimation of momentum thickness downstream of an aerofoil.

1.2.1.3 KUTTA-JOUKOWSKI THEOREM

In parallel to profile drag studies, lift generation was addressed by Kutta first (in 1902, unpublished thesis) and then by Joukowski [78] in 1906. For inviscid and incompressible two-dimensional flows, they identified the genesis of a lift force (l) in the existence of a velocity difference between suction and pressure aerofoil sides, better described by the *circulation*:

$$l = \rho_\infty V_\infty \Gamma \quad \text{with} \quad \Gamma = \oint_C \underline{V} \cdot \underline{t} dl \quad (1.5)$$

where Γ is the velocity circulation along a closed circuit C surrounding the aerofoil, with unit tangent vector $\underline{t}$ (clockwise oriented). A rigorous proof of Eq. (1.5) requires a very complex analysis of the asymptotic behaviour of far-field integrals as $\Sigma \rightarrow \infty$, although simplified derivations are often encountered. Nevertheless, it is worth stressing how the Kutta-Joukowski (KJ) theorem is proved for asymptotics, but then its validity holds also for finite control volumes, being the circulation Γ invariant on any closed contour surrounding the body [217].

Nevertheless, the scientific community initially rejected this result, based on Kelvin's theorem guaranteeing that no circulation could be generated from an initially irrotational ideal flow. It was again Prandtl who, based on the first intuitions of Lanchester, justified lift generation introducing the *effectively-inviscid* flow model [153]. This is an asymptotic regime where zero viscosity is approached by a real viscous fluid and differentiates from the ideal model of truly inviscid fluids where no circulation can indeed be generated. The underlying existence of a (vanishing) viscosity was appointed as the physical root of the Kutta condition [93], allowing to close the potential flow

problem (which otherwise admits infinite solutions) for sharp geometries, by selecting the unique flow-field with finite velocity at the aerofoil trailing-edge.

Indeed, very recently, Gonzalez and Taha [67] proved that the uniqueness can be guaranteed without involving fluid viscosity at all. They derived a variational formulation of flow governing equations for incompressible fluids and used Hertz's principle of least curvature to close the problem even in the case of smooth geometries, highlighting how the Kutta condition is obtained as a particular case for sharp-edged foils.

After Prandtl's work, Bryant and Williams [23] showed that Eq. (1.5) can also be applied to rotational flows, at condition that the downstream portion of the integration path is selected orthogonal to the velocity direction in the wake.

1.2.1.4 PRANDTL'S FORMULA

The effectively-inviscid flow assumption and Lanchester/Prandtl trailing vorticity model also led to the quantitative evaluation of *lift-induced* drag, for wings of finite span. Initially identified by the pioneering work of Prandtl at the beginning of the 20th century [154, pp. 185-225] and related to the trailing vorticity [153] (indeed, it is also referred to as *vortex-drag*), it represents the axial (drag) force associated to the transverse (lift) component, generating wake vortices in three-dimensional flows. It adds to profile drag in the build-up of the total axial force, for lifting bodies. Following the studies of Lighthill [98] (further discussed by Batchelor [12, pp. 277-293] years later), vorticity is produced at the wall, in the form of an infinitely-thin sheet, as a result of the tangent pressure gradient, an aspect recently discussed also by Wu et al. [223–225].

The famous Prandtl's formula:

$$C_{D,i} = \frac{C_L^2}{\pi \mathcal{R}e} \quad (1.6)$$

can be used to compute the induced drag coefficient ($C_{D,i}$) from the knowledge of lift coefficient (C_L) of lightly-loaded planar wings with high aspect ratio ($\mathcal{R}$), in incompressible flows. However, the Oswald's factor (e) is usually not known a priori, unless for very particular cases. Moreover, the formula returns zero lift-induced drag at zero global lift coefficient, which is not true in the case of a locally non-zero spanwise lift distribution, with associated trailing vorticity generating drag. For the above reasons and, generally, in viscous and compressible flows, other methods are therefore necessary to estimate lift-induced drag and to relate it to local flow structures, although this is still an open research topic, as highlighted, for example, by Spalart [183] with reference to a rigorous definition of induced drag.

1.2.1.5 MASKELL'S FORMULA

A first attempt to provide, for steady incompressible viscous flows, a total drag breakdown in profile and vortex drag on wings of finite span was proposed by Maskell [114], who also provided a formula to compute lift. He used quantities measured on a wake plane (W) orthogonal to the

free-stream velocity and assumed negligible variations of the stream-wise velocity (*Trefftz* plane). His results are also applicable to wind tunnel measurements, as the formulas can be restricted on the vortical portion of W . Maskell's expression for profile drag is a corrected version on Betz's formula, taking into account for the tunnel blockage (non-negligible in his reported experiments). Lift-induced drag is instead provided by:

$$D_i = \frac{\rho_\infty}{2} \int_W (v^2 + w^2) dS \quad (1.7)$$

where v and w are two components of the transverse velocity field ($\underline{V}_\pi$) on W . It is here explicitly observed that the stream-wise velocity perturbation (u) does not appear in the original Maskell's expression for induced drag.

The restriction of Eq. (1.7) to vortical wakes intercepting W is performed introducing the scalar potential (φ) and stream function (ψ) of the transverse velocity field, defined to satisfy:

$$\nabla_\pi^2 \psi = -\omega_x \quad \text{and} \quad \nabla_\pi^2 \varphi = \sigma \quad (1.8)$$

where ∇_π is the transverse nabla operator, ω_x is the stream-wise vorticity component and σ is a source term expressed by $\nabla_\pi \cdot \underline{V}_\pi$. They allow re-writing Maskell's formula for lift-induced drag as:

$$D_i = \frac{\rho_\infty}{2} \int_W (\psi \omega_x - \varphi \sigma) dS \quad (1.9)$$

where the second term is a small correction of the first and can be neglected, provided that W is a Trefftz plane. In this case, being the vorticity confined to wake portions of W , the integral in Eq. (1.7) can be restricted to that regions and the problem shifts on the computation of the stream function ψ . Kusunose [90] proposed two methods: i) a numerical computation of ψ through the iterative solution of the Poisson's equation; ii) an analytical solution of the problem using the Green's function. The relationship between Maskell's formula (still widely adopted in wind tunnel test campaigns) and Prandtl's lifting line theory has been analysed in [34, 64, 91] and a review of induced drag prediction and reduction methods is included in [87].

1.2.1.6 FILON THEOREM

The analogous of the KJ theorem for drag is represented by Filon theorem [51], enunciated in 1926 for 2D viscous and incompressible flows around cylindrical bodies of arbitrary cross-section. It expresses profile drag as:

$$d_p = \rho_\infty V_\infty E \quad (1.10)$$

where E is the “total outwards flow” (as defined by Filon). It is the flow, through the a control contour surrounding the aerofoil, of the velocity field obtained from the stream function solution of the problem, retaining leading terms only. It is counterbalanced by an equal and opposite inflow at the tail of the object [51] to preserve a zero net overall flux (steady incompressible flow around

the impermeable cylinder). Filon also provided the formula for lift, obtaining the same expression provided by the KJ theorem, then Imai derived the formulas from both linear and angular momentum balance, therefore addressing pitching moment as well [74].

The analogy with Eq. (1.5) comes after observing that E can be expressed as the inflow of the vortical (transverse) part of the velocity obtained through the Helmholtz decomposition [104], i.e.:

$$E = Q_{\underline{\psi}} = - \oint_C (\nabla \times \underline{\psi}) \cdot \underline{n} \, dl \quad (1.11)$$

with $\underline{\psi}$ being the divergence-free vector potential and $\underline{n}$ the normal unit vector along the closed circuit C surrounding the aerofoil.

It is again observed that the theorem is proved for asymptotics, but then its validity holds also for finite control volumes, being the transverse velocity field divergence-free and $Q_{\underline{\psi}}$ invariant on any closed contour surrounding the body.

It is also worth highlighting that the circulation Γ in the KJ theorem can also be expressed in terms of one component of the Helmholtz velocity field decomposition. In fact the potential flow assumed by Kutta and Joukowski allows identifying the velocity with its longitudinal component ($\underline{V} = \nabla \varphi$, where φ is the scalar potential), then:

$$\Gamma = \Gamma_{\varphi} = \oint_C \nabla \varphi \cdot \underline{t} \, dl \quad (\text{in potential flow}) \quad (1.12)$$

1.2.1.7 UNIFIED KUTTA-JOUKOWSKI/FILON THEOREM

A very important result, obtained recently by Liu et al. [101, 104] but mentioned here to keep proximity with previously enunciated theorems, is the so-called *unified Kutta-Joukowski/Filon theorem*, extending the validity of KJ and Filon formulas to compressible and three-dimensional (3D) flows. It is enunciated as:

“For an $\mathcal{N}$ -dimensional steady flow of viscous and compressible fluid over a rigid body, $\mathcal{N} = 2, 3$, the lift and drag exerted to the body are solely determined by the multi-valueness and/or singularities of the velocity potential φ in the circulation Γ_{φ} and the vortical stream function $\underline{\psi}$ in the inflow $Q_{\underline{\psi}}$, respectively”:

$$\underline{F} = (\mathcal{N} - 1) \rho_{\infty} \underline{V}_{\infty} \times \Gamma_{\varphi} + \rho_{\infty} \underline{V}_{\infty} Q_{\underline{\psi}} \quad (1.13)$$

where it is explicitly observed that the first term on the Right-Hand Side (RHS) of Eq. (1.13) gives a transverse force (lift) only, while the second one an axial force (drag) only. One should also note that the latter provides total drag but not its phenomenological decomposition. Nevertheless it is important to underline that the decomposition in lift and drag components provided by Eq. (1.13) is independent on the choice of the control volume (total lift and drag are intrinsic quantities) and this is obtained because Γ_{φ} and $Q_{\underline{\psi}}$ are invariant with respect to a change of Σ . In 3D, integrals defining the inflow and circulation are of course extended on the external control volume surface (Σ) and the (vectorial) circulation is generalised using cross-product with the unit normal vector

on Σ , i.e.:

$$\underline{\Gamma}_\varphi = \int_{\Sigma} \underline{n} \times \underline{\nabla} \varphi \, dS, \quad Q_{\underline{\psi}} = - \int_{\Sigma} \underline{n} \cdot (\underline{\nabla} \times \underline{\psi}) \, dS \quad (1.14)$$

The universality and exactness of Eq. (1.13) comes at the cost of φ and $\underline{\psi}$ being non-testable, either experimentally or numerically [217]. For this reason, the authors also proposed a *testable* version of the theorem, valid for Σ lying in the *linear far-field* (a zone, far from the body, where the governing equations can be linearized [102]) and formulated as:

$$\underline{F} \doteq \rho_{\infty} \underline{V}_{\infty} \times \underline{\Gamma} + \rho_{\infty} \underline{V}_{\infty} Q_W \quad (1.15)$$

where $\doteq$ denotes *asymptotic equality* (within the linear far-field) and the total circulation $\underline{\Gamma}$ is now used, along with a re-defined inflow Q_W evaluated on the downstream portion of Σ (wake plane, W):

$$\underline{\Gamma} = \int_{\Sigma} \underline{n} \times \underline{V} \, dS = \underline{\Gamma}_\varphi + \int_{\Sigma} \underline{n} \times (\underline{\nabla} \times \underline{\psi}) \, dS \quad (1.16)$$

$$Q_W = - \frac{\rho_{\infty}}{\mathcal{N} - 1} \int_W (\underline{x} \times \underline{\omega}) \cdot \underline{e}_x \, dS \quad (1.17)$$

where $\underline{x}$ is the position vector and $\underline{e}_x = \underline{V}_{\infty}/V_{\infty}$. In (1.16) and (1.17), only testable quantities appear (total flow velocity and first vorticity moment) and the formula was numerically tested and verified also on turbulent statistically-stationary flows solving Reynolds-Averaged Navier-Stokes (RANS) [231].

1.2.2 UNSTEADY REGIME AND TWO-DIMENSIONAL LINEAR THEORIES

Analytical theories of the aerodynamic force in unsteady conditions were historically developed for aero-elastic studies (a survey is provided by Garrick [62] and, more recently and extensively by Demasi [36]) and considered the analysis of thin oscillating aerofoils, producing small perturbations in a uniform free-stream flow in non-dissipative and incompressible regime. In these conditions, the kinematic problem for the perturbation velocity potential can be linearized, and the effects of aerofoil curvature, thickness and mean angle of attack (addressed by the steady thin aerofoil theory) can be superimposed to that due to the unsteady motion, which can therefore be studied for a flat plate aligned with the free-stream and undergoing pitching and/or plunging oscillations.

The latter problem was solved by different authors (a detailed discussion is provided in [20, Ch. 5]), following two alternative methods: i) Birnbaum's approach [19], based on singularities superposition over the aerofoil chord and wake line (*bound* and *free* vorticity), followed by Cicala [30] and Küssner [88]; ii) Wagner's method [213], based on conformal mapping to exploit the harmonic property of complex analytic functions, used by Glauert [66] and Theodorsen [194], leading to the famous formulas for the unsteady lift and pitching moment included in Sec. 1.2.2.2

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and valid for harmonic aerofoil oscillations.

An exhaustive illustration of the abovementioned theories goes beyond the scope of this introductory chapter. It is here instead underlined how, using bounded and free vorticity distributions ($\gamma_b(x, t)$ and $\gamma_w(x, t)$, respectively) together with the linearized unsteady Bernoulli's theorem (Birnbaum's approach), it is possible to derive the following 2D lift expressions, generally valid with no restriction on the type of aerofoil unsteady motion:

$$l = \rho_\infty V_\infty \Gamma - \rho_\infty \frac{d}{dt} \int_{-b}^b (x - b) \gamma_b dx \quad (1.18)$$

$$l = \rho_\infty V_\infty \Gamma - \rho_\infty \frac{d}{dt} \int_{-b}^b x \gamma_b dx - \rho_\infty V_\infty b \gamma_w(b, t) \quad (1.19)$$

where x is the axial coordinate, computed from the mid-chord aerofoil point and b is the half chord length.

Eq. (1.18) highlights that the unsteady lift can be computed from the knowledge of the bound vorticity only (the total circulation is in fact provided by: $\Gamma = \int_{-b}^b \gamma_b dx$). Both expressions include a Kutta-Joukowski lift term on the RHS, together with the time variation of the *hydrodynamic impulse*§ excluding the wake (first bounded vorticity moment, times mass density: $\rho_\infty \int_{-b}^b r \gamma_b dx$), where r is the lever arm computed from the aerofoil trailing edge (Eq. (1.18)) or from the mid-chord point (Eq. (1.19)). In the latter case, a contribution of the free vorticity at the body/wake interface (trailing edge) is identified. It is therefore useful to re-write Eq. (1.19) as:

$$l = l_\ell + l_\tau + l_s \quad (1.20)$$

with straightforward correspondence of the new symbols used in (1.20) with the three terms on the RHS of (1.19). The notation used herein will be justified in Appendix A.5, where links between these lift formulas and the general vortex force theory of Wu et al. [222, pp. 617-640] are highlighted. It is nevertheless observed since now that Eq. (1.18) and (1.19) represent the first *impulse theories*, involving the time derivative of the fluid impulse (down to the trailing edge).

With a similar procedure, it is also possible to derive an expression for the two-dimensional pitching moment with regards to the aerofoil centre of rotation (positive for pull-up), in terms of bounded and wake vorticity:

$$m = -\rho_\infty V_\infty \int_{-b}^b x \gamma_b dx + \frac{\rho_\infty}{2} \frac{d}{dt} \int_{-b}^b x^2 \gamma_b dx + \rho_\infty V_\infty \frac{b^2}{2} \gamma_w(b, t) + abl \quad (1.21)$$

The first three terms represent the aerodynamic moment with regards to the plate mid-chord point, which is summed to the pitching contribution of the overall lift force (l) applied therein (ab is the corresponding lever arm), to retrieve the moment around the centre of rotation.

Links between this pitching moment formula and the general vortex force/moment theory of

§The impulse concept was originally introduced by Thomson [196].

Wu et al. [222, pp. 617-640] are highlighted in Appendix A.5, which justifies re-writing Eq. (1.21) using the following notation:

$$m = m_{\underline{\ell}}^* + m_{\tau}^* + m_S^* + abl \quad (1.22)$$

with straightforward correspondence of the new symbols used in (1.22) with the first three terms on the RHS of (1.21). Recalling Eq. (1.20), the aerodynamic moment can be re-expressed as:

$$m = m_{\underline{\ell}} + m_{\tau} + m_S \quad (1.23)$$

where

$$m_{\underline{\ell}} = m_{\underline{\ell}}^* + abl_{\underline{\ell}} \quad (1.24)$$

$$m_{\tau} = m_{\tau}^* + abl_{\tau} \quad (1.25)$$

$$m_S = m_S^* + abl_S \quad (1.26)$$

Clearly, the quantities with a star superscript are analogous to those on the Left-Hand-Side (LHS), but refer to the aerodynamic moment computed with regards to the mid-chord point.

1.2.2.1 VON KÁRMÁN & SEARS' THEORY

In Birnbaum's approach, the bound vorticity distribution (γ_b) can be found considering the induced velocities of the whole vortical system (bound and free) over the flat plate and imposing boundary conditions at the moving wall. This method (followed for example by Ostieri et al. [139, 140] and Russo et al. [169]) solves Glauert's equation [65, p. 137] in closed integral form (see Bisplinghoff et al. [20, p. 217]) and is different from the approach originally followed by Kármán and Sears [211], although leading to the same results. The bound vorticity $\gamma_b(x, t)$ is considered composed of two parts: “the vorticity, $\gamma_0(x, t)$, which would be produced, according to the thin aerofoil theory, by the motion of the aerofoil... if the wake had no effect” and “the vorticity, $\gamma_1(x, t)$, which is induced by the wake”. The final Kármán & Sears's (K-S) unsteady lift formula is:

$$l = \rho_{\infty} V_{\infty} \Gamma_0 - \rho_{\infty} \frac{d}{dt} \int_{-b}^b x \gamma_0 dx + \rho_{\infty} V_{\infty} b \int_b^{\infty} \frac{\gamma_w}{\sqrt{x^2 - b^2}} dx \quad (1.27)$$

where $\Gamma_0 = \int_{-b}^b \gamma_0 dx$. The bound vorticity induced by the wake (γ_1) does not explicitly appear in Eq. (1.27).

The first term on the RHS of Eq. (1.27) is the *quasi-steady* lift: l_{QS} . Its interpretation is straightforward, representing the force produced in steady-state conditions at the aerofoil instantaneous velocity and angle of attack. The second one represents an *added-mass* contribution (l_{AM}). In fact, it is the only non-vanishing term in (1.27) in the ideal case where the aerofoil motion does not produce circulation (in this situation, the only forces exerted on the aerofoil are those related to

the reaction of the accelerated fluid masses, i.e. to its apparent mass) and the authors proved that it exactly returns the body integration of the “impulsive pressures” [211]. This contribution is also called *non-circulatory* lift (l_{NC}) as, following Wagner’s method and for harmonic oscillations, it can be obtained from the velocity potential of a distribution of sources and sinks over the body skin [194] (see also [20, Ch. 5]). The third term is explicitly depending on the wake vorticity and is therefore indicated with the symbol l_w . As explained by the authors, it takes into account transient effects on the aerodynamic force generation in an unsteady motion [211]. The K-S formula can then be re-arranged as[¶]:

$$l = l_{QS} + l_{AM} + l_w = l_{NC} + l_C \quad (1.28)$$

The two different decompositions (K-S’s and Theodorsen’s) of the lift force are related by:

$$l_{NC} = l_{AM} \quad \text{non-circulatory contribution} \quad (1.29)$$

$$l_C = l_{QS} + l_w \quad \text{circulatory contribution} \quad (1.30)$$

and the so-called *circulatory* lift l_C can be obtained, for harmonic oscillations, from a distribution of bound and free vortex singularities [194] (see also [20, Ch. 5]).

Kármán and Sears [211] also derived a similar expression for the aerodynamic pitching moment around the plate mid-chord point (this particular choice of the pole is denoted using a star superscript: m^*):

$$\begin{aligned} m^* = & -\rho_\infty V_\infty \int_{-b}^b x \gamma_0 dx + \frac{\rho_\infty}{2} \frac{d}{dt} \int_{-b}^b \left(x^2 - \frac{b^2}{2} \right) \gamma_0 dx + \\ & + \rho_\infty V_\infty \frac{b^2}{2} \int_b^\infty \frac{\gamma_w}{\sqrt{x^2 - b^2}} dx \end{aligned} \quad (1.31)$$

where the pitching contribution of the lift force (abl) does not explicitly appear as in Eq. (1.21) because of the different chosen pole. The sign in Eq. (1.31) is opposite to that reported in [211, p. 383] (Eq. 20) as a diving moment was considered positive by Kármán and Sears.

Similarly with the described interpretation of lift terms in Eq. (1.27), the K-S moment formula can then be physically explained by analogous considerations, leading to the following notation:

$$m^* = m_{QS}^* + m_{AM}^* + m_w^* = m_{NC}^* + m_C^* \quad (1.32)$$

Nevertheless, for the aerodynamic moment, the quasi-steady contribution is non-circulatory [194]

[¶]the second equality, introducing Theodorsen’s lift decomposition (Sec. 1.2.2.2) is indeed meaningful for harmonic oscillations only.

(see also [20, Ch. 5]), therefore the decompositions in (1.32) are related by:

$$m_{NC}^* = m_{QS}^* + m_{AM}^* \quad (1.33)$$

$$m_C^* = m_w^* \quad (1.34)$$

It is observed that $m_w^* = (b/2)l_w$, therefore l_w contributes to m^* as it was applied at the plate quarter-chord point.

Since $m = m^* + abl$, and recalling Eq. (1.28), the following definitions can be done:

$$m_{QS} = m_{QS}^* + abl_{QS} \quad \text{quasi-steady moment} \quad (1.35)$$

$$m_{AM} = m_{AM}^* + abl_{AM} \quad \text{added-mass contribution} \quad (1.36)$$

$$m_w = m_w^* + abl_w \quad \text{wake contribution} \quad (1.37)$$

$$m_{NC} = m_{NC}^* + abl_{NC} \quad \text{non-circulatory part} \quad (1.38)$$

$$m_C = m_C^* + abl_C \quad \text{circulatory part} \quad (1.39)$$

1.2.2.2 THEODORSEN'S THEORY

Birnbaum's approach can also be used to derive the unsteady lift and pitching moment closed formulas originally obtained by Theodorsen [194], who used conformal mapping instead (for a detailed description, see Bisplinghoff et al. [20, pp. 251-281]). As anticipated in the previous section, this leads to a decomposition of the aerodynamic force and moment acting on the oscillating plate in *circulatory* and *non-circulatory* contributions, obtained from the velocity potential of, respectively, bound and free vortex singularities, or a distribution of sources and sinks over a circle in the complex plane (which is mapped to the body boundary).

Following the vorticity distribution approach [139, 140, 169], instead, the solution is achieved by analytically expressing γ_0 (and then l_{QS} , l_{AM} , m_{QS}^* and m_{AM}^*), in terms of the prescribed and known plate motion parameters, in the case of harmonic pitching and plunging oscillations:

$$l_{QS} = 2\pi\rho_\infty V_\infty b \left[V_\infty a + \dot{h} + \left(\frac{1}{2} - a \right) b\dot{a} \right] \quad (1.40)$$

$$l_{NC} = l_{AM} = \pi\rho_\infty b^2 \left(V_\infty \dot{a} - ab\ddot{a} + \ddot{h} \right) \quad (1.41)$$

$$m_{QS}^* = \pi\rho_\infty V_\infty b^2 \left(V_\infty a + \dot{h} - ab\dot{a} \right) \quad (1.42)$$

$$m_{AM}^* = -\frac{\pi}{8}\rho_\infty b^4 \ddot{a} \quad (1.43)$$

where h and a are, respectively, the aerofoil plunging displacement (positive downwards) and pitching rotation around a point located at $(ab, 0)$ (positive clockwise and measured between the

the second equality, introducing Theodorsen's decomposition (Sec. 1.2.2.2) is indeed meaningful for harmonic oscillations only.

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direction of $\underline{V}_\infty$ and the instantaneous position of the aerofoil). We explicitly observe that both pitching and plunging oscillations are assumed to vary with the same time frequency. Nevertheless, this is not a limitation of the theory as, due to the linearity of the unsteady flow problem (for small perturbations), oscillations with different frequencies can also be considered, superimposing solutions obtained for the individual motions.

Based on Eq. (1.35) and (1.36), closed form expressions can also be obtained for m_{QS} and m_{AM} :

$$m_{QS} = 2\pi\rho_\infty V_\infty b^2 \left(a + \frac{1}{2} \right) \left[V_\infty a + \dot{h} + b \left(\frac{1}{2} - a \right) \dot{a} \right] \quad (1.44)$$

$$m_{AM} = \pi\rho_\infty b^2 \left[ab\ddot{h} + abV_\infty\dot{a} - b^2 \left(\frac{1}{8} + a \right) \ddot{a} \right] \quad (1.45)$$

The need to compute γ_w appearing in l_w and m_w is then eliminated by deriving an analytical function for the ratio between l_C and l_{QS} (the latter is known). This is Theodorsen's function $C(k)$ (a rigorous mathematical definition is provided by Bisplinghoff et al. [20, p. 272]), which is a complex function only depending on the oscillation reduced frequency $k = \pi f c / V_\infty$:

$$C(k) = F(k) + iG(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)} \quad (1.46)$$

where $i = \sqrt{-1}$ and $H_0^{(2)}$ and $H_1^{(2)}$ are Hankel's function of the second kind, related to Bessel's functions of the first and second kind by: $H_\nu^{(2)} = J_\nu - iY_\nu$ (see also [1], Ch. 9). As mentioned above, this allows expressing lift as:

$$l = l_{NC} + C(k)l_{QS} \quad (1.47)$$

where one should consider the real or imaginary part of $C(k)$ to obtain the unsteady lift corresponding to co-sinusoidal or sinusoidal oscillation laws, respectively. The complex nature of $C(k)$, however, reveals that circulatory lift is delayed with regards to the oscillatory motion that produces the unsteady force.

With reference to the first lift decomposition expressed by Eq. (1.28) and being $l_C = l_{QS} + l_w$, it follows that the wake contribution to lift is given by:

$$l_w = [C(k) - 1] l_{QS} \quad (1.48)$$

It is possible to prove that, for the pitching moment, the homologous contributions are given by:

$$m_w^* = \frac{b}{2} [C(k) - 1] l_{QS} \quad (1.49)$$

$$m_w = \frac{b}{2} [C(k) - 1] l_{QS} + ab [C(k) - 1] l_{QS} \quad (1.50)$$

Then, circulatory and non-circulatory moment contributions are then expressed by:

$$m_C = m_w^* + abl_C = \frac{b}{2} [C(k) - 1] l_{QS} + abC(k)l_{QS} \quad (1.51)$$

$$m_{NC} = m_{QS}^* + m_{AM}^* + abl_{AM} \quad (1.52)$$

The same procedure can be adopted to derive, for harmonic oscillations, analytical functions relating $l_{\underline{\ell}}$, l_{τ} or l_S in Eq. (1.20) to the known terms l_{QS} and l_{AM} (decomposition (1.28)):

$$l_{\underline{\ell}} = D(k)l_{QS} \quad (1.53)$$

$$l_S = ikD(k)l_{QS} \quad (1.54)$$

$$l_{\tau} = l_{AM} + ikE(k)l_{QS} \quad (1.55)$$

with

$$D(k) = -\frac{2e^{-ik}}{\pi ik[H_1^{(2)}(k) + iH_0^{(2)}(k)]} \quad (1.56)$$

$$E(k) = \frac{\pi k^2[H_0^{(2)}(k) + H_2^{(2)}(k)] + 4(k-i)e^{-ik}}{2\pi ik^2[H_1^{(2)}(k) + iH_0^{(2)}(k)]} \quad (1.57)$$

By comparison with Eq. (1.47), Theodorsen's function can then be expressed as:

$$C(k) = D(k) + ikD(k) + ikE(k) = \tilde{C}(k) + ikE(k) \quad (1.58)$$

where $\tilde{C}(k) \rightarrow C(k)$ for $k \rightarrow 0$ [139, 140] and, in the practise, well approximates $C(k)$ up to $k \approx 0.1$.

Similarly, for the aerodynamic moment, the following relations can be derived:

$$m_{\underline{\ell}}^* = m_{QS}^* + \frac{b}{2} E(k)l_{QS} \quad (1.59)$$

$$m_S^* = -\frac{b}{2} ikD(k)l_{QS} \quad (1.60)$$

$$m_{\tau}^* = m_{AM}^* + \frac{b}{2} [C(k) + ikD(k) - E(k) - 1] l_{QS} \quad (1.61)$$

Then, since $m = m^* + abl$ and recalling Eq. (1.53)-(1.55):

$$m_{\underline{\ell}} = m_{QS}^* + \left[\frac{b}{2} E(k) + abD(k) \right] l_{QS} \quad (1.62)$$

$$m_S = \left[-\frac{b}{2} ikD(k) + abikD(k) \right] l_{QS} \quad (1.63)$$

$$m_{\tau} = m_{AM}^* + abl_{AM} + \frac{b}{2} [C(k) + ikD(k) - E(k) - 1 + 2aikE(k)] l_{QS} \quad (1.64)$$

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Finally, Theodorsen's formulas for lift and pitching moment coefficients (C_l and C_m , the latter referred to the aerofoil centre of rotation) are given by:

$$C_{l_{NC}} = \pi \left(\frac{b}{V_\infty} \dot{a} - \frac{ab^2}{V_\infty^2} \ddot{a} + \frac{b}{V_\infty^2} \ddot{h} \right) \quad (1.65)$$

$$C_{l_c} = 2\pi C(k) \left[a + \left(\frac{1}{2} - a \right) \frac{b}{V_\infty} \dot{a} + \frac{\dot{h}}{V_\infty} \right] \quad (1.66)$$

$$C_{m_{NC}} = \frac{\pi}{2} \left[a - \left(\frac{1}{8} + a^2 \right) \frac{b^2}{V_\infty^2} \ddot{a} + \frac{\dot{h}}{V_\infty} + a \frac{b}{V_\infty^2} \ddot{h} \right] \quad (1.67)$$

$$C_{m_c} = -\pi \left[\frac{1}{2} - \left(a + \frac{1}{2} \right) C(k) \right] \left[a + \left(\frac{1}{2} - a \right) \frac{b}{V_\infty} \dot{a} + \frac{\dot{h}}{V_\infty} \right] \quad (1.68)$$

Lift and moment coefficients corresponding to the decompositions in Eq. (1.20) and (1.23) are instead expressed by:

$$C_{l_\ell} = 2\pi D(k) \left[a + \left(\frac{1}{2} - a \right) \frac{b}{V_\infty} \dot{a} + \frac{\dot{h}}{V_\infty} \right] \quad (1.69)$$

$$C_{l_s} = 2\pi i k D(k) \left[a + \left(\frac{1}{2} - a \right) \frac{b}{V_\infty} \dot{a} + \frac{\dot{h}}{V_\infty} \right] \quad (1.70)$$

$$C_{l_\tau} = \pi \left(\frac{b}{V_\infty} \dot{a} - \frac{ab^2}{V_\infty^2} \ddot{a} + \frac{b}{V_\infty^2} \ddot{h} \right) + 2\pi i k E(k) \left[a + \left(\frac{1}{2} - a \right) \frac{b}{V_\infty} \dot{a} + \frac{\dot{h}}{V_\infty} \right] \quad (1.71)$$

$$\begin{aligned} C_{m_\ell} = & \frac{\pi}{2} \left(a + \frac{\dot{h}}{V_\infty} - \frac{ab}{V_\infty} \dot{a} \right) + \\ & + \pi \left[\frac{1}{2} E(k) + a D(k) \right] \left[a + \frac{\dot{h}}{V_\infty} + \left(\frac{1}{2} - a \right) \frac{b}{V_\infty} \dot{a} \right] \end{aligned} \quad (1.72)$$

$$C_{m_s} = \pi \left[-\frac{1}{2} i k D(k) + a i k D(k) \right] \left[a + \frac{\dot{h}}{V_\infty} + \left(\frac{1}{2} - a \right) \frac{b}{V_\infty} \dot{a} \right] \quad (1.73)$$

$$\begin{aligned} C_{m_\tau} = & -\frac{\pi}{16} \frac{b^2}{V_\infty^2} \ddot{a} + \frac{\pi}{2} \frac{ab}{V_\infty} \left(\dot{a} - \frac{ab}{V_\infty} \ddot{a} + \frac{\ddot{h}}{V_\infty} \right) + \\ & + \frac{\pi}{2} [C(k) + i k D(k) - E(k) - 1 + 2a i k E(k)] \left[a + \frac{\dot{h}}{V_\infty} + \left(\frac{1}{2} - a \right) \frac{b}{V_\infty} \dot{a} \right] \end{aligned} \quad (1.74)$$

An important quantity for oscillating aerofoils is the *aerodynamic damping*, defined for example for a pure pitching motion as:

$$\zeta = - \int_t^{t+\mathcal{T}} C_m \, d\alpha \quad (1.75)$$

and representing the work done by the aerodynamic forces arising from the aerofoil rotation of period $\mathcal{T}$. This will be stable if $\zeta > 0$ (the fluid absorbs external energy, needed to sustain the oscillation) or unstable if $\zeta < 0$ (energy is transferred from the fluid to the oscillating aerofoil). A geometrical representation of ζ is given by the area enclosed by the C_m vs. α loop curve, obtained during the aerofoil oscillation.

1.2.2.3 GARRICK'S AXIAL FORCE ANALYSIS

While previous studies focused on the transverse (lift) force and pitching moment only, von Kármán and Burgers [210] also addressed the force in the direction of flight (axial force, resulting in a thrust or drag action), an aspect later on treated in more details by Garrick [61].

From a physical point of view, thrust generation on flapping wings can be explained by the so-called *Knoller-Betz effect*. Independently both Knoller [85] and Betz [16] observed that the motion of the flapping wing creates an effective inflow angle of attack, generating an aerodynamic force with both transverse and axial components. This was later on experimentally verified by Katzmayer [82]. The generation of such a propulsive axial force is at the basis of birds' flight and fish locomotion [162]. Looking in more details at the flow-field, it can be observed that thrust generation is strongly related to the vortex shedding: when thrust is generated by an oscillating foil, the wake behind it has a jet-shaped average velocity profile [77]. This jet flow is associated with a staggered array of vortices moving downstream from the foil, closely resembling the von Kármán vortex street behind bluff objects, but with reverse rotational direction [210][151].

Garrick [61] made use of Theodorsen's compact expressions for lift and pitching moment and considered a flat plate with three independent degrees of freedom: plunging motion, pitching aerofoil rotation and the rotation of a trailing edge flap around its hinge point, located on the chord line. Indeed this third degree of freedom (occurring at the same time frequency as the other two) was already considered by Theodorsen [194], who also provided formulas for the hinge moment coefficient, not reported in Sec. 1.2.2.2.

He followed two different methods to compute the time-averaged axial force: i) an *energy formula*, which considers that the externally supplied power (needed to sustain the oscillatory motions) is used to increase the wake kinetic energy per unit time and, in case of power excess (defect), to generate a propulsive (resistance) power; ii) a *force formula* where forces normal to the plate surface are projected along the velocity direction and averaged in time, to obtain the mean axial component.

Garrick proved that both methods return mathematically equivalent expressions for the time-averaged axial force. It is worth mentioning that, in the force approach, considering normal forces projection along $\underline{V}_\infty$ is not enough to obtain the axial force but the effect of a *suction force* at the plate leading edge (arising from the locally infinite vorticity [210]) has to be included. Differently from the trailing edge (where Kutta's condition is imposed to guarantee a smooth velocity field), a leading edge singularity is still obtained in ideal unsteady flows around an infinitely thin plate, which indicates that this should only be considered as an asymptotic idealisation of non-dissipa-

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tive flows around shapes with rounded and smooth leading edges, in the limit of zero curvature radius.

The final expression for the averaged axial force obtained by Garrick is here reported for two particular cases, where either a pure plunging or pure pitching aerofoil motion is considered:

$$\bar{C}_T = 4\pi \left(k\tilde{h} \right)^2 \left[F^2(k) + G^2(k) \right] \quad \text{pure plunging} \quad (1.76)$$

$$\begin{aligned} \bar{C}_T = \pi (k\hat{a})^2 \left\{ \left[F^2(k) + G^2(k) \right] \left[\frac{1}{k^2} + \left(\frac{1}{2} - a \right)^2 \right] + \right. \\ \left. + \frac{1}{2} \left(\frac{1}{2} - a \right) - \frac{F(k)}{k^2} - \left(\frac{1}{2} - a \right) \frac{G(k)}{k} \right\} \quad \text{pure pitching} \quad (1.77) \end{aligned}$$

where $\bar{C}_T$ is the time-averaged thrust coefficient, $\hat{a}$ is the pitching amplitude and $\tilde{h} = \hat{h}/c$ is the non-dimensional plunging amplitude.

Garrick also observed that the averaged axial force provided by Eq. (1.76) is always of propulsive sign (thrust force). This is generally not true for different aerofoil motions and, even in the case of for pure pitching, a thrust or drag can be generated depending on the value of the reduced oscillation frequency k [61].

Recently, Fernandez-Feria [49], using impulse theory, corrected Garrick's axial force formulas to account for the suction force not just at the thin aerofoil leading edge, but all over the body. The formulas he derived differ, for increasing reduced frequency values, from Garrick's results, providing lower averaged thrust coefficients and leading to a propulsive efficiency of the oscillating aerofoil (mean thrust power over mean input power required to sustain the aerofoil oscillation) which tends to zero as $k \rightarrow \infty$ (instead of approaching 1/2, as in Garrick's theory). Later on, Fernandez-Feria and Alaminos-Quesada [50] also tried to take into account the leading-edge vortex in the analysis of forces exerted on oscillating aerofoils.

1.3 DRAG DEFINITIONS

Before describing modern far-methods (detailed in the following sections) it is worth starting with clarifying the employed definitions for the different forms of aerodynamic drag, identified on a physical and phenomenological basis. Then, in the following section, the *operative* definitions (i.e. the available methods) used herein to predict the different drag contributions will be illustrated and described.

Two different decompositions of the total far-field aerodynamic drag (D_{ff}) are obtained from the following definitions, summarised by the relation:

$$D_{ff} = D_{rev} + D_{irr} = D_i + D_p \quad (1.78)$$

where D_{rev} , D_{irr} , D_i and D_p are defined as:

1.3.1 REVERSIBLE DRAG (D_{rev})

It is here defined as the aerodynamic drag obtained in isentropic flow conditions.

It is therefore an *objective* quantity, in the sense that (for example, in steady conditions) its value only depends on the shape and lift of the body, as well as on the flow Mach number (it is worth observing that, for Newtonian fluids with negligible longitudinal viscosity coefficient, compressing processes are isentropic as long as the flow regime remains subsonic).

However, when a far-field method is used to (directly or indirectly) compute it from a numerical flow solution, a unique reversible drag value cannot be obtained and results depend on the downstream extension of the control volume used by the method. As a matter of fact, in CFD computations another (spurious) form of drag is introduced (caused by numerical errors and artificial dissipation of advective fluxes schemes) and the reversible drag content is gradually converted into spurious drag (detected through entropy raising), keeping a constant sum of the two. Indeed, for numerical applications, a spurious drag contribution (D_{sp}) adds to the right-hand-side decomposition in Eq. (1.78), while on the LHS it is usually included in D_{irr} as it contributes to entropy production.

1.3.2 IRREVERSIBLE DRAG (D_{irr})

It is here defined as the drag due to flow phenomena related to entropy production.

In non-isentropic flow conditions, it therefore represents the additional contribution to total drag, adding-on to the reversible drag. Once again, in non-isentropic flow conditions (now even in absence of numerical effects) reversible and irreversible drag are *non-objective* quantities, as the physical dissipation also converts reversible into irreversible drag, with a continuous entropy generation as the flow progresses downstream of the body. Therefore, far-field methods aiming at separating reversible from irreversible drag, in a viscous flowfield, are expected to predict different breakdowns as the control volume is progressively extended downstream of the body and, usually, the decomposition is performed immediately before a significant conversion from reversible into irreversible drag is established.

1.3.3 LIFT-INDUCED DRAG (D_i)

First identified by Prandtl for inviscid and incompressible flows [154, pp. 185-225], is the drag (axial aerodynamic force component) directly related to lift (transverse, or *deviating*, component).

As discussed in Sec. 1.2.1.4, Prandtl introduced the concept of *effectively-inviscid* flow ($\mu \rightarrow 0$, but $\mu \neq 0$) [153], to justify an aerodynamic force generation not possible for truly inviscid fluids ($\mu = 0$, d'Alembert paradox). This fluid model is characterised by an infinitely thin bounded and free vortical sheet which is necessary to guarantee Kutta's condition, but which induces vertical velocities responsible of a downstream axial force on upwards lifting surfaces [154, pp. 185-225]. Although Prandtl's theory was formulated for incompressible flows and for lightly-loaded wings of

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high aspect ratio, the concepts at the basis of lift-induced drag generation remain valid for general isentropic flows, with trailing vortices identified as the flow feature this form of drag is related to.

1.3.3.1 CONSISTENCY WITH REVERSIBLE DRAG

As lift-induced drag genesis is inherently based on an inviscid flow model, its definition in the framework of dissipative flows assumes that the overall lift is regarded as obtained in isentropic conditions. It therefore downfalls into the definition of reversible drag.

On the other hand, since no aerodynamic force is generated for truly inviscid fluids (according to the d'Alembert's paradox), reversible drag is only defined for effectively-inviscid flows and, therefore, only in presence of circulation. The latter, in turns, implies the presence of a deviating force, then reversible drag can be identified with the drag due to lift (lift-induced drag).

1.3.4 PARASITE DRAG (D_p)

It is the drag not directly related to lift, then it represents a drag force with no lift counterpart.

Also referred to as “profile drag”, it accounts for losses due to shock waves, boundary layers and viscous wakes. However it is worth recalling that, without viscosity, no aerodynamic force could be generated, therefore parasite drag is indirectly related to lift as well (vortex sheets in inviscid flows are models for boundary layers and viscous wakes in the vanishing viscosity limit**).

1.3.4.1 CONSISTENCY WITH IRREVERSIBLE DRAG

Based on Eq. (1.78), and because of consistency between D_{rev} and D_i , this follows as a consequence. Drag decompositions in reversible/irreversible or lift-induced/parasite contributions therefore theoretically match each other. Nevertheless, in will be shown that the *operative* drag definitions proposed in literature do not always ensure such a matching.

1.4 FAR-FIELD FORMULAS AND OPERATIVE DRAG DEFINITIONS

Hereafter, advanced methods and formulas to compute the aerodynamic force with a far-field approach are introduced, together with proposed formulas decompositions to calculate the different forms of drag previously defined, which will be used in the following chapters of this dissertation.

The starting far-field force equation (Eq. (1.2)) will be here considered valid for both laminar and turbulent flow regimes. In the last case, mass density (ρ) and fluid particle velocity ($\underline{V}$) are intended as their corresponding *ensemble* and *Favre* averages respectively, then a turbulent (or “effective”) viscous stress tensor is adopted, including Reynolds stresses ($\underline{\tau}_{vt} = \underline{\tau}_v + \underline{\tau}_R$).

**The classical physical justification of the Kutta condition for sharp-edged geometries in inviscid flows is still assumed here as a reference, although it was mentioned in the previous sections that a new approach, based on variational mechanics principles, was very recently proposed [67], avoiding the involvement of fluid viscosity to justify closure conditions in Euler flow problems.

1.4. Far-field formulas and operative drag definitions

1.4.1 THERMODYNAMIC METHODS

1.4.1.1 OSWATITSCH'S FORMULA AND ENTROPY DRAG CONCEPT

Following the work of Betz and Jones on profile (or parasite) drag estimation, after the second world war in 1945, Oswatitsch first introduced thermodynamics in the momentum balance to compute parasite drag in compressible gases using a wake survey of flow entropy variations [143] (an English-translated version by H. Sockel of the original German text is available at [144]). He assumed no variations of the flow specific total enthalpy and considered a slightly perturbed parallel flow in the body wake, retaining only first order terms in the momentum and energy balance equations. The final formula derived by Oswatitsch directly relates parasite drag to entropy variations in the fluid caused by the presence of the body (*entropy drag* concept), clearly linking parasite drag to irreversible processes:

$$D_p = \frac{T_\infty}{V_\infty} \int_\Sigma \rho \Delta s (\underline{V} \cdot \underline{n}) dS \quad (1.79)$$

Due to the aforementioned assumptions, this formula is however only accurate in the case of weak shocks. Nevertheless, it has been widely adopted in experimental tests, as for example in the work by Kusunose et al. [90, 92], who computed and decomposed profile drag using wake surveys and a vorticity magnitude cut-off to distinguish viscous and shock wakes. This decomposition is however sensitive to the position of the wake plane, being influenced by the outer flow entrainment in the viscous wave.

Steger and Baldwin [187] were among the first to implement drag extraction methods in CFD using Oswatitsch entropy drag formula. However, Yu et al. [230], in their study aiming at separating induced and wave drag in three-dimensional inviscid flows, highlighted that the far-field drag (which parasite contribution was calculated by Steger and Baldwin's method) was not in good agreement with the near-field drag. This problem was explained by Lock [107], who introduced the concept of spurious drag, later on also addressed by Destarac [41]. Lock [108] also used Oswatitsch's formula to compute wave drag in inviscid flows, in terms of entropy jumps across shock waves.

1.4.1.2 METHODS BASED ON VELOCITY MAGNITUDE DECOMPOSITION

A decomposition of parasite drag into viscous and wave contributions in transonic flows was proposed for the first time in 1990 by Van der Vooren and Slooff [207], based on thermodynamic considerations. However, it was only in the 21st century that thermodynamic-based far-field methods developed to a mature stage [42, 148], nowadays representing a consolidated approach, widely used in numerical applications [22, 48, 110].

These methods, routinely adopted also in the aviation industry, only predict drag (lift is usually not addressed) and, in particular, that part of the total drag which is due to flow phenomena

increasing fluid entropy (i.e. the irreversible drag), such as viscosity or supersonic compression. Drag due to other, reversible, phenomena (e.g. lift-induced drag) is usually obtained by difference from the total drag, although these methods can also be formulated the other way around, i.e. to directly estimate the reversible drag and indirectly obtaining (by difference) the irreversible part [59, 201, 202]. Nevertheless, this is not the standard approach in thermodynamic methods, as an explicit dependence on flow static pressure appears in such formulations of the reversible drag, preventing from identifying a relationship with local flow structures, as pressure variations are not confined to flow-field regions where physical phenomena occur (boundary layers, viscous and shock wakes).

These methods revealed quite robust and accurate also in transonic flows, where they allow for a decomposition of the irreversible drag into viscous and wave contributions [83, 96, 118, 203]. Later on, such approaches have been successfully extended to deal with unsteady flows [59, 201] and applied to complex and three-dimensional flow-fields [202].

They rely on the possibility to express, for a perfect gas, the velocity magnitude in terms of variations (with regards to asymptotic free-stream conditions) of specific total enthalpy (ΔH), entropy (Δs) and static pressure (Δp) [207] as:

$$\begin{aligned} \frac{V}{V_\infty} &= \sqrt{1 + 2 \left(\frac{\Delta H}{V_\infty^2} \right) - \frac{2}{(\gamma - 1)M_\infty^2} \left[\left(\frac{\Delta p}{p_\infty} + 1 \right)^{\frac{\gamma-1}{\gamma}} e^{\frac{\Delta s}{R} \frac{\gamma-1}{\gamma}} - 1 \right]} \\ &= f(\Delta p/p_\infty, \Delta s/R, \Delta H/V_\infty^2) \end{aligned} \quad (1.80)$$

where M_∞ is the free-stream Mach number, R is the gas constant for the air mixture and γ is the ratio of specific heats.

Focusing on steady flows without body forces (in which case, only the third integral in Eq. (1.2) is non-zero, for impermeable bodies), Eq. (1.80) can be used in the axial projection of Eq. (1.2) to obtain a thermodynamic expression of the total drag in terms of Δp , Δs and ΔH (neglecting viscous stresses on Σ , supposing it lays outside of the boundary layer and orthogonal to the free-stream velocity in the body wake).

In this expression, total entropy variations are due to dissipative effects, as well as total enthalpy variations in the case of steady flows and un-powered configurations. Therefore, in the hypothesis of no impact of flow dissipation on static pressure variations on Σ , the irreversible drag contribution is defined as:

$$D_{irr} \stackrel{\text{def}}{=} \underline{F}_{ff}^{\Delta p=0} \cdot \underline{e}_x = -V_\infty \int_V \underline{\nabla} \cdot \left[\rho \hat{v}_x f \left(\frac{\Delta s}{R}, \frac{\Delta H}{V_\infty^2} \right) \underline{V} \right] dV \quad (1.81)$$

where $\underline{e}_x$ is a unit vector aligned with $\underline{V}_\infty$ and the divergence form has been used applying Gauss' theorem*. The local axial velocity has been expressed as $\underline{V} \cdot \underline{e}_x = V \hat{v}_x = V \hat{v}_x$ where $\hat{v}_x$ is related to

*In this as well as in other document sections, Gauss's divergence theorem (or different integral identities relying on regular fields defined in simply-connected regions) was applied without restric-

the angle between local and free-stream velocity vectors (non-negligible in the wake downstream of lifting bodies).

From this point on, two methods were developed independently by Paparone and Tognaccini in 2003 and by Destarac and van der Vooren in 2004. They differentiate for the the approximations used in the computation of D_{irr} .

It is worth observing since now that the divergence form of Eq. (1.81) allows identifying the local generation of irreversible drag in the flow-field (examples will be provided in Section 2.2.3.7), with possibility to isolate, in CFD solutions, at least part of the so-called *spurious drag* in high-Reynolds number flows, considering any drag generation identified in the external regions of the flow-field assumed homo-entropic and homo-enthalpic [148]. Then, for the dissipative region, the contributions of viscous drag (D_v) and wave drag (D_w) can be further separated from D_{irr} , using appropriate solution sensors for the identification of the viscous and shock parts of the flow domain $\mathcal{V}$ [45, 109, 148, 197]. Therefore, thermodynamic methods based on the velocity magnitude decomposition, allow expressing D_{irr} in Eq. (1.78) as:

$$D_{irr} = D_v + D_w + D_{sp} \quad (1.82)$$

This further decomposition is based on a zonal decomposition of the flow domain $\mathcal{V}$ (instead of on a formula decomposition, as for the reversible and irreversible contributions) and the last term (D_{sp}) is non-zero in numerical computations only†.

It is additionally observed that the underlying hypothesis on pressure variations, which neglects any impact from dissipation at the external CV surface, complicates the application of thermodynamic methods based on such a velocity decomposition to supersonic flows. In such cases, the shock-waves system extends considerably far from the body, then the use of either very large CVs or other artefacts (like the use of under-resolved grids, accounting for the associated numerical dissipation in the wave drag calculation [38]) is necessary.

Both methods mentioned above have been applied in this dissertation to isolate and compute the irreversible drag in generally viscous and compressible flow-fields, obtained from numerical

tions on the space dimensionality (2D or 3D) and on the flow regime. In fact, despite the double-connectivity topology of the control volume in 2D, one can adopt a branch cut of the domain to restore the single connectivity, then making the two cut-edges collapse onto each other with no effect on the final formula, as the two edges are opposed in orientation, cancelling-out their combined contribution. Moreover, no singularities will be considered for the integrated flow-fields. As a matter of fact, the different far-field formulations illustrated in this document will only deal with Navier-Stokes or with *weak* Euler solutions, which do not admit discontinuities. This is the reason why, even in inviscid and unsteady flow regime, no shear discontinuity was considered in the body wake and, for the same reason, shock waves in transonic flows were not considered as CV domain boundaries.

†It is explicitly observed that the zonal decomposition of $\mathcal{V}$ is ambiguous in regions of interaction between different drag sources: for example, at the foot of a shock wave, due to its interaction with the boundary layer. In such cases, those local drag contributions are arbitrarily assigned to either the viscous or the wave drag component, assumed that a unique criterion is consistently adopted in all method's applications. Additionally, as mentioned before, the spurious drag is not fully contained in the flow domain external to the viscous and wave regions.

solutions. Calculated drag values will be denoted as D_{irr}^{PT} (Paparone and Tognaccini) and D_{irr}^{DV} (Destarac and van der Vooren).

METHOD OF PAPARONE AND TOGNACCINI The method of Paparone and Tognaccini [147, 148] neglects total enthalpy variations (for a unitary Prandtl number gas) and approximates the effect of entropy variations using a Taylor series expansion of $f(\Delta s/R)$, halted at second order terms. The resulting formula for the irreversible drag is:

$$D_{irr}^{PT} \stackrel{\text{def}}{=} -V_\infty \int_V \underline{\nabla} \cdot \left\{ \rho \hat{v}_x \left[-\frac{1}{\gamma M_\infty^2} \frac{\Delta s}{R} - \frac{1 + (\gamma - 1)M_\infty^2}{2\gamma^2 M_\infty^4} \left(\frac{\Delta s}{R} \right)^2 \right] \underline{V} \right\} dV \quad (1.83)$$

and it can be easily proved that, at first order in $\Delta s/R$, it returns Oswatitsch's entropy drag expression.

Finally, the method was extended in 2005 to deal with power-on configurations by Tognaccini [198], who proposed a thrust/drag bookkeeping formula for jet propelled aircrafts.

METHOD OF DESTARAC AND VAN DER VOOREN A parallel flow in the body wake is assumed in the method of Destarac and van der Vooren ($\hat{v}_x \approx 1$, [42]), but specific total enthalpy variations are allowed and no Taylor expansion is performed for f .

The method is usually formulated using the axial velocity perturbation ($u = U - V_\infty$, with $U = \underline{V} \cdot \underline{e}_x \approx V$) instead of the relative velocity magnitude (V/V_∞) [38], although formally the two approaches are equivalent as V_∞ can be subtracted in the axial momentum expression with no effect on D_{irr} because of the flow continuity. Therefore its typical formulation is written as:

$$D_{irr}^{DV} \stackrel{\text{def}}{=} - \int_V \underline{\nabla} \cdot \left\{ \rho \left[V_\infty f \left(\frac{\Delta s}{R}, \frac{\Delta H}{V_\infty^2} \right) - V_\infty \right] \underline{V} \right\} dV \quad (1.84)$$

with the term included in square brackets representing the *irreversible* axial velocity perturbation (u_{irr}) in the hypotheses of Destarac and van der Vooren's method. Verley [209] and then Méheut [115] also developed a formulation able to compute the thrust and torque generated by helicopter rotors and propellers.

Bailly [10] very recently proposed a methodology to extend Destarac and van der Vooren's approach to accurately compute and decompose the far-field drag in supersonic flows, using control volumes which not necessarily extend to fully include the entire shock-wave system.

It has to be mentioned that the function (obtained through Eq. (1.80)) expressing the irreversible axial velocity perturbation decomposition may be discontinuous within the flow-field, as pointed out by Méheut [133]. This occurs whenever the flow total pressure is lower than the free-stream static pressure (p_∞). For steady flows, this situation is generally achieved when a boundary layer separates from the wall, in the core of strong vortices, in boundary layers when the local Mach number exceeds the free-stream Mach number, or downstream of strong shocks [201] (i.e. inside any region of drag source), nevertheless without affecting the computed irreversible drag in most

cases. Méheut and Bailly [134] also proposed a continuous function alternative, for steady flows, later on also used by Gariépy and Trépanier [60]. This however implies losses in accuracy in some cases, compared to the use of the standard (yet potentially ill-defined) formulation, as described in [201, 202], due to the explicit dependence on the static pressure.

GENERALISATION TO UNSTEADY FLOWS BY GARIÉPY ET AL. The abovementioned continuous formulation is based on approaching the velocity decomposition the other way around, i.e. expressing the reversible part of the axial velocity perturbation explicitly and the irreversible part by difference with regards to the total perturbation.

This reveals useful in unsteady flows, where total enthalpy variations, even for un-powered configurations, cannot be fully ascribed to dissipative effects. Instead, unsteady pressure variations need to be taken into account and Gariépy et al. [59] generalised the approach of Destarac and van der Vooren by assuming that, far downstream from the body, they are the only cause of total enthalpy variations and are solely due to the body trailing vortices (for lifting configurations).

Then they further decomposed the reversible axial velocity perturbation isolating a contribution that would also be active in the steady regime ($\frac{\partial}{\partial t} = 0$, u_{rev}^{st}) and the difference, only non-zero for unsteady flows (u_{rev}^{uns}). The velocity decomposition used by Gariépy et al. is therefore expressed by:

$$u_{rev} = V_{\infty} \left\{ \sqrt{1 + 2 \left(\frac{\Delta H}{V_{\infty}^2} \right) - \frac{2}{(\gamma - 1)M_{\infty}^2} \left[\left(\frac{\Delta p}{p_{\infty}} + 1 \right)^{\frac{\gamma-1}{\gamma}} - 1 \right] - \frac{v^2 + w^2}{V_{\infty}^2} - 1} \right\} \quad (1.85)$$

$$u_{rev}^{st} = V_{\infty} \left\{ \sqrt{1 - \frac{2}{(\gamma - 1)M_{\infty}^2} \left[\left(\frac{\Delta p}{p_{\infty}} + 1 \right)^{\frac{\gamma-1}{\gamma}} - 1 \right] - \frac{v^2 + w^2}{V_{\infty}^2} - 1} \right\} \quad (1.86)$$

$$u_{rev}^{uns} = u_{rev} - u_{rev}^{st} \quad (1.87)$$

$$u_{irr} = u - u_{rev} \quad (1.88)$$

The far-field drag expression was derived by Gariépy et al. [59] in a generally non-inertial frame of reference, therefore based on Eq. (1.2) which includes both the unsteady volume integral and the contribution of apparent forces due to the relative motion in non-inertial frames (through the body forces term).

The irreversible drag (defined through u_{irr}) can be split into viscous, wave and spurious (for numerical cases) contributions, as discussed for the steady thermodynamic methods in the previous sections. Then, the reversible drag component (D_{rev}), was also split based on the u_{rev} decomposition illustrated above, isolating an unsteady contribution (D_{uns} , which not only includes the effect of u_{rev}^{uns} but also the unsteady and apparent forces terms in the far-field drag expression) and a steady

part, which the authors appointed as the induced drag contribution (here denoted by the symbol: D_i^{st}). Finally, the drag decomposition provided by the method is therefore expressed by:

$$D_{ff} = D_{rev} + D_{irr} = D_{uns} + D_i^{st} + D_v + D_w + D_{sp} \quad (1.89)$$

NEW UNSTEADY DRAG DECOMPOSITION BY TOUBIN ET AL. As described in the previous paragraph, the approach of Gariépy et al. [59] gathers together all unsteady and apparent forces contributions in the so-called *unsteady drag* term (D_{uns}), therefore lacking a phenomenological interpretation for this term and, at the same time, hiding any unsteady effect directly related to the viscous and wave drag sources.

These considerations led Toubin and Bailly [201] to derive a different unsteady thermodynamic formulation for the far-field drag, which retains the usual three-components phenomenological splitting of drag (induced, viscous and wave contributions) and incorporates unsteady effects in the three different components (namely, D_i^{uns} , D_v^{uns} , D_w^{uns}), rather than in a standalone term difficult to interpret.

The general formulation of Toubin and Bailly [201] far-field drag decomposition is not included herein for the sake of conciseness. Nevertheless, the main assumptions and differences with the previous approach (Sec. 1.4.1.2) are highlighted hereafter.

They worked in an inertial frame of reference, therefore, compared to the approach of Gariépy et al. [59], apparent forces are here replaced by a skin momentum flux term (denoted by the symbol D_m), representing the drag/thrust directly associated to the movement of the body (if any), which adds to the three-components splitting and vanishes for steady flows and impermeable walls. The decomposition is then achieved in the form:

$$D_{ff} = D_i^{uns} + D_m + D_v^{uns} + D_w^{uns} + D_{sp} \quad (1.90)$$

where, of course, a spurious contribution is also introduced in numerical cases.

Another remarkable difference with the approach of the previous paragraph is in the splitting of the axial velocity perturbation, which is done isolating the irreversible contribution (subject to the discussed ill-conditioning issues) rather than the reversible part. This is done through Eq. (1.80) and under the same hypotheses of Destarac and Van der Vooren [42] ($p \approx p_\infty$, $\hat{v}_x \approx 1$) but, for unsteady flows, specific total enthalpy variations are not only attributed to dissipative effects. Nevertheless the contribution from unsteady pressure variation is still accounted for by Toubin and Bailly [201] as an irreversible effect if it is obtained in a flow-field region where dissipation is occurring (i.e. in the viscous or shock wakes). Similarly, the authors also attributed the unsteady volume integrand in Eq. (1.2) to either the viscous or wave drag component, when integrated in their corresponding flow-field regions.

However, in their formulation, these regions were not identified through a flow feature decomposition but using *stream-tubes*, therefore assuming that the separation between the viscous and shock wakes is a stream surface (which is not generally true due to the entrainment of the outer

flow in the boundary layer).

In their approach, the viscous stress tensor is not neglected in the definition of the viscous drag and the unsteady induced drag is obtained by difference from the total far-field drag, once the viscous, wave and motion contributions have been calculated.

Moreover, the spurious drag is here obtained as the imbalance between near-field and far-field total drag ($D_{sp} = D_{nf} - D_{ff}$). In CFD analyses, it includes numerical effects from round-off errors and artificial dissipation.

Finally, a year later, Toubin et al. [202] refined the method by further isolating, in their unsteady induced drag definition, a term (D_{pa}) which they directly related to the propagation in the flow domain of acoustic waves and other reversible effects, which globally affect the flow-field. This allowed to sharpen the unsteady induced drag calculation, leading to the final form of their decomposition:

$$D_{ff} = D_i^{uns} + D_{pa} + D_m + D_v^{uns} + D_w^{uns} + D_{sp} \quad (1.91)$$

where D_i^{uns} is of course redefined with regards to the homologous term in Eq. (1.90).

1.4.2 CLASSICAL METHODS (GENERALISED)

In this section a derivation of generalised expressions (valid for 3D viscous and compressible flows) of the Kutta-Joukowski theorem, Maskell's and Betz's formulas is included. Their correlation with force decomposition provided by Lamb vector methods (Sec. 1.4.3.2) has been recently discussed by Fournis et al. [57] and will also be addressed and examined in more detail in Ch. 2. It is therefore important to highlight some key properties of such generalised expressions even before introducing Lamb-vector-based methods. Moreover, their explicit derivation from momentum balance equation is not available from the above reference and is provided hereafter for further discussion*.

Starting from Eq. (1.2)†, taking into account that $\int_{\Sigma} \underline{n} \, dS = \underline{0}$ for any closed surface Σ , then considering that:

$$\int_V \frac{\partial \rho}{\partial t} dV + \int_{\Sigma \cup \partial B} \rho \underline{V} \cdot \underline{n} \, dS = 0 \quad (\text{continuity equation, integral form})$$

*Fournis et al. [57] obtained these generalised expressions, for steady flows and without body forces, starting from Lamb-vector-based force formulas. The derivation included herein, extended to unsteady flows and considering body forces as well, straightforwardly moves from momentum balance, instead of introducing the vortex force thorough DMTs and then reverting to flow momentum again (as done in [57]).

†considered valid for either laminar or turbulent flow regime, as discussed in Sec. 1.4.1

it is possible to write:

$$\begin{aligned} \underline{F}_{ff} = & - \int_{\mathcal{V}} \frac{\partial (\rho \delta \underline{V})}{\partial t} d\mathcal{V} - \int_{\partial B} \rho \delta \underline{V} \underline{V} \cdot \underline{n} dS + \\ & - \int_{\Sigma} \left[(p - p_{\infty}) \underline{n} - \underline{\tau}_{vt} \cdot \underline{n} + \rho \delta \underline{V} \underline{V} \cdot \underline{n} \right] dS + \int_{\mathcal{V}} \rho \underline{f}_m d\mathcal{V} \end{aligned}$$

where the velocity perturbation $\delta \underline{V} = \underline{V} - \underline{V}_{\infty}$ is introduced, with Cartesian components (u, v, w) along coordinate axes.

The derivation will now proceed assuming steady flows and no body forces. In these hypotheses, volumetric force densities are zero and advective fluxes at the body skin are also zero for impermeable walls (this is generally not true, for unsteady flows, even if assuming impermeable bodies, in the case of moving walls)[‡]. The total force is therefore expressed in terms of surface integrals only, over the external control volume boundary.

Adding and subtracting $\int_{\Sigma} \rho (\delta \underline{V}) \underline{V}_{\infty} \cdot \underline{n} dS$, we obtain:

$$\underline{F}_{ff} = \int_{\Sigma} - \left[(p - p_{\infty}) \underline{n} - \underline{\tau}_{vt} \cdot \underline{n} + \rho \delta \underline{V} \delta \underline{V} \cdot \underline{n} + \rho (\delta \underline{V}) \underline{V}_{\infty} \cdot \underline{n} \right] dS$$

Then, considering the double cross-product rule:

$$\underline{V}_{\infty} \times \underline{n} \times \rho \delta \underline{V} = (\underline{V}_{\infty} \cdot \rho \delta \underline{V}) \underline{n} - (\underline{V}_{\infty} \cdot \underline{n}) \rho \delta \underline{V}$$

the force formula can be rewritten as:

$$\begin{aligned} \underline{F}_{ff} = & \int_{\Sigma} - \left[(p - p_{\infty}) \underline{n} - \underline{\tau}_{vt} \cdot \underline{n} + \rho \delta \underline{V} \delta \underline{V} \cdot \underline{n} + (\underline{V}_{\infty} \cdot \rho \delta \underline{V}) \underline{n} \right] dS + \\ & - \int_{\Sigma} (\underline{V}_{\infty} \times \underline{n} \times \rho \delta \underline{V}) dS \end{aligned}$$

Finally, considering that:

$$\frac{1}{2} \delta \underline{V} \cdot \delta \underline{V} = \frac{1}{2} (V^2 - V_{\infty}^2) - \underline{V}_{\infty} \cdot \delta \underline{V}$$

after some algebra, the far-field aerodynamic force expression can be written as:

$$\begin{aligned} \underline{F}_{ff} = & \int_{\Sigma} \left[(p_{t,\infty} - p_t) \underline{n} + \frac{1}{2} V_{\infty}^2 (\rho - \rho_{\infty}) \underline{n} + \underline{\tau}_{vt} \cdot \underline{n} \right] dS + \\ & + \int_{\Sigma} \left[-\rho \delta \underline{V} \delta \underline{V} \cdot \underline{n} + \frac{1}{2} \rho (\delta V^2) \underline{n} + \underline{V}_{\infty} \times \underline{n} \times \rho \delta \underline{V} \right] dS \stackrel{\text{def}}{=} \\ & \stackrel{\text{def}}{=} \underline{F}_{BETZ} + \underline{F}_{\delta V} \end{aligned} \quad (1.92)$$

[‡]These assumptions will be removed at the end of the section and are only temporarily introduced to simplify the notation.

1.4. Far-field formulas and operative drag definitions

where p_t is the fluid total pressure. The first and second integrals in Eq. (1.92) have been denoted with the symbols $\underline{F}_{BETZ}$ and $\underline{F}_{\delta V}$ respectively (the nomenclature of the first will be justified in the following) and the last term in $\underline{F}_{\delta V}$ has no drag component, being orthogonal to $\underline{V}_\infty$. §

In non-dissipative flows without shock waves, entropy production is zero and the aerodynamic force exerted in these conditions on the body, if any (effectively ideal flows), has been appellated to as *reversible force* (i.e. force not related to any entropy production). Its component orthogonal and parallel to $\underline{V}_\infty$ are respectively lift and lift-induced drag.

If the flow also originates from uniform free-stream conditions (homo-enthalpic and homo-entropic flow) and the regime is incompressible, then total pressure variations (first term in $\underline{F}_{BETZ}$) are zero, as well as density variations (second term) and viscous fluxes (third term).

Then $\underline{F}_{BETZ} = \underline{0}$ in Eq. (1.92). In these conditions, exact lift and lift-induced drag formula can be written. Assuming the x -axis aligned with free-stream velocity and z -axis orthogonal to $\underline{V}_\infty$ and in the direction of the transverse (lift) force ($\underline{e}_x$ and $\underline{e}_z$ denote the corresponding unit vectors), it is:

$$\begin{aligned} \underline{L} &= (\underline{F}_{ff} \cdot \underline{e}_z) \underline{e}_z = (\underline{F}_{\delta V} \cdot \underline{e}_z) \underline{e}_z = \\ &= \rho_\infty \underline{V}_\infty \times \int_\Sigma \underline{n} \times \delta \underline{V} dS + \rho_\infty \int_\Sigma \left[\frac{1}{2} (\delta V^2) n_z - w \delta V \cdot \underline{n} \right] dS \underline{e}_z \end{aligned} \quad (1.93)$$

The first term on the RHS of Eq. (1.93) is a three-dimensional Kutta-Joukowski theorem and the second term is a finite domain viscous correction, identified for the first time by Schmitz and Coder [178] and further analysed by Schmitz [176] in 2D, where the Kutta-Joukowski theorem is formulated. Schmitz and Coder [178] also proved that its contribution is zero in a purely irrotational flow while, in the viscous case, it approaches zero in the far-field wake of 2D flows and remains non-zero in 3D. Then, for the axial component:

$$\begin{aligned} D &= D_i = \underline{F}_{ff} \cdot \underline{e}_x = \underline{F}_{\delta V} \cdot \underline{e}_x = - \int_\Sigma \rho u \delta V \cdot \underline{n} dS + \int_\Sigma \frac{1}{2} \rho \delta V^2 n_x dS \\ &= \rho_\infty \int_\Sigma \left(\frac{1}{2} u^2 n_x - u \delta V \cdot \underline{n} \right) dS + \rho_\infty \int_\Sigma \frac{1}{2} (v^2 + w^2) n_x dS \end{aligned} \quad (1.94)$$

The RHS of Eq. (1.94) is a widely adopted lift-induced drag expression, written for a general external control volume boundary (Σ).

It reduces to the more familiar wake integral $\frac{1}{2} \int_W \rho (v^2 + w^2 - u^2) dS \underline{e}_x$ when the lateral portion of Σ is sufficiently far from the body and the wake plane W is orthogonal to the freestream velocity. As discussed in [89], Maskell's formula can be obtained from this equation.

An extension of these definitions to generally viscous and compressible flows, however, requires some assumptions. In particular, supposing that:

- the upstream and lateral portions of Σ are not too close to the body, the flow is not supersonic and characterised by a high Reynolds number (so that velocity perturbations can be

§Eq. (1.92), is also valid for inviscid subsonic flows governed by the (strong) Euler equations (see footnote on page 11).

assumed to algebraically decay far from the body and wake [12, pp. 114-124]).

- the downstream portion of Σ is orthogonal to the free-stream velocity $\underline{V}_\infty$.
- no large viscous wakes intercept the Σ surface (e.g. far from stall conditions).

then the sum of the first two integrals in $\underline{F}_{BETZ}$ can be restricted on the wake plane, where $\underline{n} = \underline{e}_x$ (proof is provided in Appendix A). Moreover, the turbulent viscous stress tensor flux (third term in $\underline{F}_{BETZ}$) on the wake plane can be neglected (in absence of large separations) and therefore the lift component of $\underline{F}_{BETZ}$ can be neglected:

$$\underline{F}_{ff} \approx F_{BETZ} \underline{e}_x + \underline{F}_{\delta V} \quad (1.95)$$

Then:

$$\begin{aligned} \underline{L} &\approx (\underline{F}_{ff} \cdot \underline{e}_z) \underline{e}_z = (\underline{F}_{\delta V} \cdot \underline{e}_z) \underline{e}_z = \\ &= \underline{V}_\infty \times \int_{\Sigma} \underline{n} \times \rho \delta \underline{V} dS + \int_{\Sigma} \left[\frac{1}{2} \rho (\delta V^2) n_z - \rho w \delta \underline{V} \cdot \underline{n} \right] dS \underline{e}_z \end{aligned} \quad (1.96)$$

where the RHS of Eq. (1.96) is a generalised Kutta-Joukowski theorem (for three-dimensional viscous and compressible flows, on finite domains).

Under the specified hypotheses, $\underline{F}_{BETZ}$ represents a drag force with no lift counterpart, i.e. a parasite drag. If $\underline{F}_{\delta V} \cdot \underline{e}_x$ is still assumed as lift-induced drag definition, i.e.:

$$D_i \stackrel{\text{def}}{=} \underline{F}_{\delta V} \cdot \underline{e}_x \quad (1.97)$$

this leads to:

$$\begin{aligned} D &= \underline{F}_{ff} \cdot \underline{e}_x = (\underline{F}_{BETZ} + \underline{F}_{\delta V}) \cdot \underline{e}_x = \\ &= \int_{\Sigma} \left[(p_{t,\infty} - p_t) \underline{n} + \frac{1}{2} V_\infty^2 (\rho - \rho_\infty) \underline{n} + \underline{\tau}_{vt} \cdot \underline{n} \right] dS \cdot \underline{e}_x + \\ &+ \int_{\Sigma} \left(\frac{1}{2} \rho u^2 n_x - \rho u \delta \underline{V} \cdot \underline{n} \right) dS + \int_{\Sigma} \frac{1}{2} \rho (v^2 + w^2) n_x dS \stackrel{\text{def}}{=} D_p + D_i \end{aligned} \quad (1.98)$$

where the first integral in Eq. (1.98) is a compressible Betz's formula (justifying the subscript notation used in $\underline{F}_{BETZ}$), assumed as parasite drag definition:

$$D_p \stackrel{\text{def}}{=} \underline{F}_{BETZ} \cdot \underline{e}_x \quad (1.99)$$

and the second and third integrals represent an extended Maskell's formula, defining lift-induced drag.

The following definitions will be therefore used for the axial and transverse component vectors

1.4. Far-field formulas and operative drag definitions

of $\underline{F}_{\delta V}$:

$$\underline{F}_{MSK} \stackrel{\text{def}}{=} \left(\underline{F}_{\delta V} \cdot \underline{e}_x \right) \underline{e}_x = D_i \underline{e}_x \quad (1.100)$$

$$\underline{F}_{KJ} \stackrel{\text{def}}{=} \left(\underline{F}_{\delta V} \cdot \underline{e}_z \right) \underline{e}_z \quad (1.101)$$

which allow re-writing the classical far-field steady force formula as:

$$\underline{F}_{ff} = \underline{F}_{KJ} + \underline{F}_{MSK} + \underline{F}_{BETZ} \quad (1.102)$$

Note that, unlike for Eq. (1.95), no hypotheses on the CV and-or wake flow were made for the validity of Eq. (1.102), which is exact and does not neglect any lift component potentially coming from $\underline{F}_{BETZ}$.

Removing the other assumptions initially made at the beginning of this section, Eq. (1.102) can be further generalised to unsteady flows and non-inertial reference frames as:

$$\underline{F}_{ff} = \underline{F}_{KJ} + \underline{F}_{MSK} + \underline{F}_{BETZ} + \underline{F}_{TIME} + \underline{F}_{APP} + \underline{F}_{BODY} \quad (1.103)$$

where $\underline{F}_{TIME}$, $\underline{F}_{APP}$ and $\underline{F}_{BODY}$ are defined respectively as the unsteady term, apparent forces term and skin advective flux term included in Eq. (1.2) and temporarily neglected to simplify the derivation of other terms. Therefore:

$$\underline{F}_{TIME} = - \int_V \frac{\partial (\rho \delta V)}{\partial t} dV \quad (1.104)$$

$$\underline{F}_{APP} = \int_V \rho \underline{f}_m dV \quad (1.105)$$

$$\underline{F}_{BODY} = - \int_{\partial B} \rho \delta \underline{V} \underline{V} \cdot \underline{n} dS \quad (1.106)$$

When dealing with rigid impermeable bodies, where typically the non-inertial reference frame is chosen solidal to the moving body, only one term between $\underline{F}_{APP}$ and $\underline{F}_{BODY}$ is non-zero in the formula: $\underline{F}_{APP}$ in non-inertial frames and $\underline{F}_{BODY}$ in inertial frames.

1.4.2.1 CONSISTENCY OF LIFT-INDUCED/PARASITE DRAG FORMULAS WITH REVERSIBLE/IRREVERSIBLE DRAG CONTRIBUTIONS

As highlighted before, in the steady inviscid and incompressible regime, the induced drag defined by Eq. (1.94) provides the reversible drag. On the other hand, however, for generally viscous and compressible flows, the total drag decomposition in lift-induced drag and parasite contributions defined by Eq. (1.98) is not consistent with the total drag breakdown in reversible and irreversible components.

Consider, for example, a steady non-dissipative subsonic flow regime. In this case, there's

no entropy production, then the whole drag is *reversible* drag. Nevertheless, D_p as defined in Eq. (1.98) is non-zero in compressible flows, therefore D_i does not coincide with the total reversible drag.

The relationship between lift-induced/parasite definitions and the reversible/irreversible drag breakdown will be analysed in more details in the following chapter. At this time, it is worth under-lying that lift-induced and parasite drag definitions provided by Eqs. (1.97) and (1.99) are subject to the hypotheses discussed above for the validity of Eq. (1.95). A general choice of the control volume would in fact result in a non-zero lift component of $\underline{F}_{BETZ}$, which would modify the general lift expression in Eq. (1.96) and, above all, would not justify the appellation of *parasite* drag attributed to $\underline{F}_{BETZ} \cdot \underline{e}_x$.

1.4.3 VORTICAL METHODS

Vortical methods provide *exact* formulas to compute the total aerodynamic force (both lift and drag) in unsteady and viscous flows from Eq. (1.2), where the vorticity is introduced and pressure is eliminated.

They were initially developed for EFD applications to extract forces from DPIV measurements, which produce two-dimensional velocity fields but no pressure data. But, above all, the importance of eliminating pressure lies in the fact that its presence in the force formula prevents from identifying a relationship between the force and local flow structures. Pressure variations are in fact not confined to flow-field regions where physical phenomena occur (boundary layers, viscous and shock wakes), therefore some information on the causal mechanisms generating the force remains hidden under pressure terms. For incompressible flows, vortical methods provides formulas which only require the knowledge of the time-dependent velocity field and its derivatives*.

In theory, for incompressible flows, pressure on the body surface can be related to the (known) body movement and to the flow vorticity at the wall by the relation [220]:

$$\underline{n} \times \underline{\nabla} p = -\underline{n} \times \underline{a} + \nu \underline{n} \cdot \underline{\nabla} \underline{\omega} - \nu (\underline{n} \times \underline{\nabla}) \cdot [(\underline{\omega} - 2\underline{\Omega}) \times \underline{nn}] \quad (1.107)$$

where ν is the fluid kinematic viscosity, while $\underline{a}$ and $\underline{\Omega}$ are, respectively, the wall acceleration and angular body velocity. This could be integrated for the pressure, nevertheless defined up to an arbitrary additive constant, which is not easy to be inferred [136]. Moreover, for experimental purposes, the evaluation of vorticity gradients on the body surface is also prohibitive. However, in principle, the Σ surface in Eq. (1.2) could be taken to coincide with the body wall (∂B), allowing pressure to be evaluated through Eq. (1.107), unless for the abovementioned difficulties.

Another option (also applicable to incompressible flows only) would be to keep the Σ surface distant from the body surface ∂B and to use the Pressure Poisson Equation (PPE) to derive flow

*body forces in Eq. (1.2) are also supposed to be only depending on the velocity field, as for example in the case of apparent forces due to the relative motion, in non-inertial reference frames.

static pressure from the knowledge of the velocity field. Nevertheless, PPE boundary conditions on Σ are not known a priori [136].

An elegant way to eliminate pressure from momentum equation was proposed by Quartapelle and Napolitano in 1983 [160], still for incompressible flows. Their method achieves the goal, however at the cost of introducing another (harmonic) function in the final force formula. This additional function is flow-field-independent and can be determined from the knowledge of the control volume geometry only (i.e. from the body and Σ shapes). Neumann boundary conditions are used on $\partial\mathcal{V}$ to close the problem, nevertheless analytical solutions can only be obtained for very simple geometries of both ∂B and Σ . Numerical approaches are otherwise necessary, as implemented for example in the calculations by Protas et al. [157].

Other approaches to pressure elimination, initially developed for incompressible flows and later on extended to compressible gases as well, are based on the introduction of derivative moment transformations [138][222, pp. 698-705], which in this case represent the integral counterpart of the differential operation of taking the curl of flow momentum equation [217]. This time, the cost is represented by the introduction of an arbitrary reference pole in the formulations, as it will be described into details in the following sections.

Vortical methods, i.e. those methods derived from momentum equation where the vorticity appears and pressure is eliminated, can be distinguished in two main categories, depending on the unsteady term in Eq. (1.2) being formulated with the time derivative inside (Lamb vector methods) or outside (impulse theories) the integral sign†.

1.4.3.1 IMPULSE THEORIES

In the case of impulse theories, Eq. (1.2) is therefore preliminarily modified as:

$$\begin{aligned} \underline{F}_{ff} = & -\frac{d}{dt} \int_{\mathcal{V}} \rho \underline{V} d\mathcal{V} - \int_{\partial B} \rho \underline{V} (\underline{V} - \underline{V}_{\partial B}) \cdot \underline{n} dS + \\ & - \int_{\Sigma} \left[p \underline{n} - \underline{\tau}_v \cdot \underline{n} + \rho \underline{V} (\underline{V} - \underline{V}_{\Sigma}) \cdot \underline{n} \right] dS + \int_{\mathcal{V}} \rho \underline{f}_m d\mathcal{V} \end{aligned} \quad (1.108)$$

where $\underline{V}_{\partial B}$ and $\underline{V}_{\Sigma}$ are the velocities of the body boundary and of the external CV boundary, respectively.

The introduction of flow vorticity in incompressible impulse theories is inherently related to the Derivative Moment Transformation provided in Appendix A.1 by Eq. (A1). In fact, with $\underline{f} = \rho \underline{V}$, Eq. (A1) directly relates the flow momentum in $\mathcal{V}$ (appearing in the first term on the RHS of Eq. (1.108)) to the first moment of flow vorticity, times the constant mass density (hydrodynamic impulse in $\mathcal{V}$). It is important to observe that the vorticity appears linearly in the impulse (the same is not true in Lamb vector methods), since this allows superimposing the effects of different parts of the vorticity field [217].

†An application of the Reynolds transport theorem is necessary to shift the time derivative and domain integral signs (Appendix A, Sec. A.3.1).

Chapter 1. The aerodynamic force

As mentioned in sec. 1.2.2, a first example of incompressible impulse theory (2D inviscid and linearized) is represented by Eq. (1.18), (1.19) and by von Kármán & Sears' theory [211]. A generalisation to non-linear viscous and three-dimensional flows was formulated, among others[‡], by Wu [218] and goes under the name of *vorticity-moment theory*.

VORTICITY-MOMENT THEORY Wu's formulation for incompressible flows [218] considers no body forces and a control volume $\mathcal{V}_\infty$ which includes the *entire vorticity field*, with its external surface Σ therefore extending to an infinite distance far from the body. All the vorticity is produced at the body wall and, when it decays exponentially at large distance from the body, using Biot-Savart law it is possible to show that the asymptotic behaviour of the velocity field is of order $r^{-\mathcal{N}}$, where r is the distance from the body and $\mathcal{N}$ is the dimensionality of the problem ($\mathcal{N} = 2, 3$). This allows neglecting momentum and shear stress terms in the Σ -integral of Eq. (1.2) when $\Sigma \rightarrow \infty$, then the pressure integral is tacitly assumed to approach zero [136], although it can be rigorously proved that this assumption is not necessary for the validity of the theory's final results [79, 138].

The final formula of Wu's vorticity-moment theory, derived in general for N impermeable bodies, is here reported for $N = 1$:

$$\underline{F}_{ff} = -\frac{d}{dt} (\underline{I}_{\mathcal{V}_\infty} + \underline{I}_B - \underline{P}_B) \quad (1.109)$$

where:

$$\underline{I}_{\mathcal{V}_\infty} = \rho_\infty \int_{\mathcal{V}_\infty} \underline{r} \times \underline{\omega} d\mathcal{V} \quad \text{hydrodynamic impulse of the fluid in } \mathcal{V}_\infty \quad (1.110)$$

$$\underline{I}_B = \rho_\infty \int_B \underline{r} \times \underline{\omega} d\mathcal{V} \quad \text{hydrodynamic impulse of the body } B \quad (1.111)$$

$$\underline{P}_B = \rho_\infty \int_B \underline{V} d\mathcal{V} \quad \text{momentum of the body } B \quad (1.112)$$

and $\underline{r}$ denotes a scaled position vector, computed from an arbitrary pole:

$$\underline{r} = \frac{\underline{x} - \underline{x}_p}{\mathcal{N} - 1} \quad (1.113)$$

Eq. (1.109) was derived by Wu [218] in the case of solid bodies animated by rigid motions, but the force method can be generalised to *fluidic bodies*§.

Using identity (A1) (Appendix A.1) it is easy to prove that body terms in Eq. (1.109) can be

[‡]see also the work of Batchelor [12, Ch. 7] and Lamb [95, Ch. 6].

§The fluidic body concept, extensively used by Saffman [173, p. 77], replaces the body by a mass of fluid of the same (constant) density and arbitrary viscosity. When the same viscosity of the surrounding fluid is considered, the body boundary is effectively replaced by a streamline of the entire flow-field. When an infinite viscosity is assumed, an impermeable solid body is modelled, which can be either rigid or deformable [136].

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cast in the form:

$$\frac{d}{dt} (P_B - I_B) = \rho_\infty \frac{d}{dt} \int_{\partial B} \underline{r} \times (\underline{n} \times \underline{V}) dS \quad (1.114)$$

where $\underline{n}$ points to the body inside. It is sometimes useful (for example to link impulse and Lamb vector theories) to shift the time derivative and the integral signs in the RHS of Eq. (1.114). This can be done using an identity provided by Aris [7, p. 63] for vector fields (and generalised by Noca et al. [138] to tensor fields of higher rank) which is basically the counterpart of the Reynolds transport theorem when surface integrals (instead of volume integrals) are involved (Appendix A, Sec. A.3.2)¶.

For irrotational flows ($\underline{V} = \nabla \varphi$ where φ is the scalar velocity potential) $I_{V_\infty} = \underline{0}$) and the RHS is exactly the *added* (or *virtual* or *apparent*) mass definition provided by Saffman [173, pp. 81-84], as Eq. (A2) can be applied to derive:

$$\underline{F}_{ff} = \frac{d}{dt} (P_B - I_B) = -\rho_\infty \frac{d}{dt} \int_{\partial B} \varphi \underline{n} dS \quad (1.115)$$

It should be observed that Eq. (1.109) is inherently unsteady, providing no force in the case of steady impulses and momentum. As a matter of fact, being the control volume extending to infinity, the flow-field is always unsteady whenever there is a relative motion of the body with regards to the fluid (even in the case of a steady body motion) [102] and, therefore, whenever an aerodynamic force arises. Nevertheless, it was rigorously proved that Wu's theory involving the global vorticity field is not applicable to steady flows due to the slow decay of the velocity and vorticity of the fluid as $\Sigma \rightarrow \infty$ in this case [156].

FINITE-DOMAIN IMPULSE THEORIES The generalisation of impulse theories to finite control volumes is due to Saffman [173, Ch. 3] for inviscid flows and to Moreau [129] for viscous and rotational flows. Differently from what done in these previous works, Noca [136] and Noca et al. [137, 138] provided a mathematical derivation which does not use the fluidic body concept and directly applies derivative moment transformations instead of integration by parts. The impulse equation they derived is written in the form:

$$\underline{F}_{ff} = -\frac{d}{dt} I_V + \int_{\Sigma_{imp}} \underline{\gamma} \cdot \underline{n} dS + \rho_\infty \frac{d}{dt} \int_{\partial B} \underline{r} \times (\underline{n} \times \underline{V}) dS - \rho_\infty \int_{\partial B} \underline{n} \cdot (\underline{V} - \underline{V}_{\partial B}) \underline{V} dS \quad (1.116)$$

with:

$$\begin{aligned} \underline{\gamma}_{imp} = & \rho_\infty \left[\frac{1}{2} V^2 \underline{I} - \underline{V} \underline{V} - (\underline{V} - \underline{V}_\Sigma) (\underline{r} \times \underline{\omega}) + \underline{\omega} (\underline{r} \times \underline{V}) \right] + \\ & + \left[\underline{r} \cdot \left(\underline{\nabla} \cdot \underline{\tau}_v \right) \underline{I} - \underline{r} \left(\underline{\nabla} \cdot \underline{\tau}_v \right) \right] + \underline{\tau}_v \end{aligned} \quad (1.117)$$

where $\underline{I}$ is the 2^{nd} -rank unit tensor.

¶The integrand of Eq. (1.114) can be cast in the form of Aris' identity by writing: $\underline{r} \times (\underline{n} \times \underline{V}) = \underline{n} \cdot \left[(\underline{r} \cdot \underline{V}) \underline{I} - \underline{r} \underline{V} \right]$.

The last integral in Eq. (1.116) is null in the case of impermeable bodies, while the time derivative of the other skin integral can be replaced by $-\frac{d}{dt}(I_B - P_B)$, in the fluidic body approach. Therefore, Eq. (1.116) reduces to Wu's formula (Eq. (1.109)) in the case of $\Sigma \rightarrow \infty$, under the abovementioned assumption for the asymptotic behaviour of both vorticity and velocity (which makes the Σ -integral vanish). This also proves that the hypothesis assuming pressure integral on Σ vanishing for $\Sigma \rightarrow \infty$ is not necessary for the validity of Wu's formula.

The generality of Eq. (1.116) comes at the cost of loosing the neatness of Wu's formulation for unbounded domains (Eq. (1.109)). This was recovered by Kang et al. [79] who derived their *minimum-domain impulse theory* on the hypothesis that the external CV surface does not cut any vortical structure (i.e. $\underline{\omega} = \underline{0}$ and $\frac{\partial \underline{\omega}}{\partial n} = \underline{0}$ on Σ). In this case the far-field force expression is given by:

$$\underline{F}_{ff} = -\frac{d}{dt}(I_V + I_B - P_B) + \underline{F}_{\underline{\ell}} + \underline{F}_{\underline{\ell},B} \quad (1.118)$$

where $\underline{F}_{\underline{\ell}}$ is the so-called *vortex-force* (a concept which will be extensively discussed in Sec. 1.4.3.2 and in Ch. 2 and 3), defined for incompressible flows as:

$$\underline{F}_{\underline{\ell}} = -\rho_{\infty} \int_V \underline{\omega} \times \underline{V} dV \quad (1.119)$$

and $\underline{F}_{\underline{\ell},B}$ is the same quantity, integrated in the fluidic body domain. In Eq. (1.118), the volume V can be limited to a minimum domain including only the vortical structures attached to the body, as initially proposed in the work of Li and Lu [97].

Noca et al. [138] also provided two other far-field force expressions, which are not impulse formulas. Nevertheless, they are reported hereafter to preserve continuity of their work. The first one is based on momentum (instead of impulse) and can be written as:

$$\underline{F}_{ff} = -\frac{d}{dt}P_V + \int_{\Sigma} \underline{\gamma}_{mom} \cdot \underline{n} dS - \rho_{\infty} \int_{\partial B} \underline{n} \cdot (\underline{V} - \underline{V}_{\partial B}) \underline{V} dS \quad (1.120)$$

with:

$$\begin{aligned} \underline{\gamma}_{mom} = & \rho_{\infty} \left[\frac{1}{2} V^2 \underline{I} - (\underline{V}_{\Sigma} - \underline{V}) \underline{V} - \underline{V} (\underline{r} \times \underline{\omega}) + \underline{\omega} (\underline{r} \times \underline{V}) \right] + \\ & - \rho_{\infty} \left[\left(\underline{r} \cdot \frac{\partial \underline{V}}{\partial t} \right) \underline{I} - \underline{r} \left(\frac{\partial \underline{V}}{\partial t} \right) \right] + \\ & + \left[\underline{r} \cdot \left(\underline{\nabla} \cdot \underline{\tau}_v \right) \underline{I} - \underline{r} \left(\underline{\nabla} \cdot \underline{\tau}_v \right) \right] + \underline{\tau}_v \end{aligned} \quad (1.121)$$

Eq. (1.120) is interesting because it allows direct comparison against Eq. (1.108), allowing a closed-form evaluation of the pressure term in terms of the velocity field and its derivatives only. Noca et al. [138], through this expression, also highlighted that the pressure integral on Σ in Eq. (1.120) indeed does not vanish as $\Sigma \rightarrow \infty$, but is only conditionally convergent in 3D [173, p. 51][12, pp. 408, 518-519].

The other formula provided by Noca et al. [138] is a far-field force expression based on surface

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integrals only, which is a convenient form for experimental applications. It can be written as:

$$\underline{F}_{ff} = \int_{\Sigma} \underline{\gamma}_{flux} \cdot \underline{n} \, dS - \rho_{\infty} \int_{\partial B} \underline{n} \cdot (\underline{V} - \underline{V}_{\partial B}) \underline{V} \, dS - \frac{d}{dt} \rho_{\infty} \int_{\partial B} \underline{n} \cdot [\underline{V}(\underline{x} - \underline{x}_p)] \, dS \quad (1.122)$$

with:

$$\begin{aligned} \underline{\gamma}_{flux} = & \rho_{\infty} \left[\frac{1}{2} \underline{V}^2 \underline{I} - \underline{V} \underline{V} - \underline{V}(\underline{r} \times \underline{\omega}) + \underline{\omega}(\underline{r} \times \underline{V}) \right] + \\ & - \rho_{\infty} \left[\left(\underline{r} \cdot \frac{\partial \underline{V}}{\partial t} \right) \underline{I} - \underline{r} \left(\frac{\partial \underline{V}}{\partial t} \right) + \frac{\partial \underline{V}}{\partial t} (\underline{x} - \underline{x}_p) \right] + \\ & + \left[\underline{r} \cdot (\underline{\nabla} \cdot \underline{\tau}_v) \underline{I} - \underline{r} (\underline{\nabla} \cdot \underline{\tau}_v) \right] + \underline{\tau}_v \end{aligned} \quad (1.123)$$

It is worth mentioning that, before the introduction of Eq. (1.122), wake surveys could only be used to measure time-averaged forces in stochastically periodic flows [136] and far downstream of the body, where further assumptions on pressure can be adopted.

Recent studies on impulse theories and applications to flapping aerofoils can also be found in [63, 68, 128].

COMPRESSIBLE EXTENSION OF IMPULSE THEORIES An impulse theory for the compressible regime was first introduced by Huang [71]. He still applied the DMT (Eq. (A1)) to the flow momentum on the RHS of Eq. (1.108), obtaining:

$$\underline{F}_{ff} = -\frac{d\underline{I}_{\mathcal{V}_{\infty}}^*}{dt} + \frac{d}{dt} \int_{\partial B} \underline{r} \times (\underline{n} \times \rho \underline{V}) \, dS \quad (1.124)$$

which is very similar to Eq. (1.109) and (1.114) from Wu's theory [218]. $\underline{I}_{\mathcal{V}_{\infty}}^*$ is a density-weighted vortical impulse in $\mathcal{V}_{\infty}$, defined as:

$$\underline{I}_{\mathcal{V}_{\infty}}^* = \int_{\mathcal{V}_{\infty}} \underline{r} \times \underline{\omega}^* \, d\mathcal{V} \quad \text{with} \quad \underline{\omega}^* = \underline{\nabla} \times (\rho \underline{V}) \quad (1.125)$$

An exact formula for finite CVs ($\mathcal{V}$ instead of $\mathcal{V}_{\infty}$) can also be obtained, but no minimum-domain formulation is instead possible, due to the nature of $\underline{\omega}^* = \rho \underline{\omega} + \underline{\nabla} \rho \times \underline{V}$, which involves density variations (not confined to compact structures).

Kang et al. [80] recently developed a minimum-domain vorticity-moment theory for compressible flows, which uses the integral:

$$\underline{I}_{\rho, \mathcal{V}_{\infty}} = \int_{\mathcal{V}_{\infty}} \underline{r} \times \rho \underline{\omega} \, d\mathcal{V} \quad (1.126)$$

which cannot however be directly obtained from a DMT application to the total flow momentum (as done by Wu [218] and Huang [71]) and therefore does not represent an impulse, leading to a differentiation of impulse and vorticity-moment theories in the compressible regime [217].

1.4.3.2 LAMB-VECTOR METHODS

Another class of far-field vortical methods was developed in the last fifteen years on the basis of the new vortical force theory by Wu et al. [222, pp. 617-640][221], initially formulated for incompressible (but viscous) and unsteady flows and later on generalised to the compressible regime as well [222, p. 621][100, 103]. The theory provides *exact* formulations of the aerodynamic force (both lift and drag) and moves from Eq. (1.2) in the form of Newton's second law (i.e. in terms of the flow acceleration instead of momentum time derivative, Eq. (A16)), which directly introduces flow vorticity in the equation involving the *Lamb vector* ($\underline{\ell}$).

First introduced by Lamb [94], $\underline{\ell}$ is defined as the cross product of flow vorticity and velocity ($\underline{\ell} = \underline{\omega} \times \underline{V}$). For steady high-Reynolds number and incompressible flows, it is responsible for the generation of the whole aerodynamic force. It is non negligible in a limited region of the flow-field only (mid-field property): boundary layers, viscous and shock-wave wakes.

The Lamb vector ($\underline{\ell} = \underline{\omega} \times \underline{V} = 2\dot{\underline{\Theta}} \times \underline{V}$) represents the centripetal acceleration of a fluid particle which is rotating with angular velocity $\dot{\underline{\Theta}}$ (combination of trajectory rotation and spinning). Considering momentum balance equation in the form of Crocco's theorem, this expresses the local equilibrium of all elementary forces acting on the fluid particles (inertial and external forces) as:

$$\rho \frac{\partial \underline{V}}{\partial t} + \rho \underline{\ell} = \rho (T \underline{\nabla} s - \underline{\nabla} H) + \underline{\nabla} \cdot \underline{\tau}_v + \rho \underline{f}_m \quad (1.127)$$

Therefore, in absence of body forces ($\rho \underline{f}_m = \underline{0}$), in a two-dimensional steady non-dissipative and incompressible flow (homoentropic and homoentropic), $-\rho \underline{\ell}$ represents the volumetric Kutta-Joukowski force density acting on the fluid particle (an application of Stokes theorem to the elementary vortex tube of infinite length normal to the plane proves the result).

The different formulas and methods based on Lamb vector field integration developed in the last decades will be subject of detailed analysis and new extensions in Ch. 2 (dedicated to the steady regime) and in Ch. 3 (dedicated to the unsteady regime). Hereafter, an introductory review of Lamb-vector-based methods is included, leaving to the following two chapters their thorough analysis, as it constitutes part of the core of this dissertation.

First intuitions of a link between the aerodynamic force and the Lamb vector date back to Prandtl, then the concept of vortex force (i.e. the volume integral of $-\rho \underline{\ell}$) was defined by von Kármán and Burgers [210]. Saffman identified and discussed its relation with the aerodynamic force in the case of incompressible flows with bounded vorticity [173, p. 46], which however represents a very special case in aerodynamics, usually characterised by wakes with vorticity convected infinitely downstream. He also envisaged [173, p. 113] the relationship between the vortex force and lift-induced drag.

The first incompressible Lamb-vector-based force formulation, valid also in presence of unbounded vorticity, is due to Wu [222, pp. 617-640][221] and has been extended to turbulent flows governed by Reynolds-Averaged Navier Stokes (RANS) equations by Marongiu and Tognaccini [112]. The relationship between the vortex force component in the flow direction and a

widely used expression for lift-induced drag [181] was also identified by Marongiu et al. [113].

An extension of Lamb-vector-based far-field force formula to the compressible flow regime has also been proposed by Wu et al. [222, p. 621] and further developed by Liu et al. [100, 103]. In, this formulation, however, the mid-field property of the original incompressible formula is lost. Mele and Tognaccini [119] derived an alternative exact formula, valid in steady compressible flow regimes, which preserves the mid-field property. In the same paper, the authors also extended their formulation to the analysis of numerically computed statistically stationary flows governed by the RANS equations, although they observed numerical inaccuracies on drag predictions in the transonic flow regime, post-processing two-dimensional (2D) RANS solutions. Indeed it was observed that, as the control volume includes (also partially) any strong shock, the computed drag is considerably higher than the reference near-field value, with unacceptable differences that can reach hundreds of drag counts for a 2D aerofoil in transonic flow. This was attributed to the lower accuracy of numerical flow solutions in the shock wave region, where usually non-linear schemes (e.g. those based on limiters) are used to guarantee Total Variation Diminishing solutions, although locally compromising with numerical errors. Mele et al. [117] then derived a modified mid-field exact formula which revealed more accurate in transonic flow conditions overcoming the numerical difficulties encountered when performing volume integration in the shock wave region.

A definition of lift-induced drag in viscous compressible flows was also proposed by Mele et al. [116] and tested on three-dimensional (3D) flow solutions for an elliptic wing in subsonic and transonic flight conditions. Nevertheless, using this definition of induced drag in steady compressible flow, Kang et al. [81] post-processed two-dimensional subsonic flow-fields and highlighted non-zero (negative) induced drag values at low Reynolds numbers, vanishing in the limit of effectively inviscid flow regime ($Re_\infty \rightarrow \infty$). This unexpected, yet interesting, result was then no longer investigated.

Concerning drag decomposition in lift-induced and parasite contributions, instead, an additional contribution was given by Fournis et al. [57] who updated Mele et al. definitions [116] and extended the results obtained by Marongiu et al. [113] to the compressible flow regime, establishing links between the vortex-force theory and classical far-field analyses (generalised formulations of the Kutta–Joukowski theorem, Maskell’s and Betz’s formulas). Moreover, Fournis et al. also proved that their decomposition could be independent from the choice of the arbitrary pole used for moment transformations in Lamb-vector-based formulations.

However, the same authors [56] highlighted how the total drag breakdown in lift-induced and parasite drag contributions is still a fundamental issue under investigation. As a matter of fact, after the initial observations by Kang et al. on 2D flow-fields at low Reynolds numbers [81], different authors reported inconsistencies using the proposed definitions of lift-induced drag for steady viscous and compressible flows. Negative induced drag values were predicted for example post-processing two-dimensional flow-fields in transonic regime [124], raising doubts on the physical or non-physical role of this non-zero induced drag contribution. Indeed the available far-field defi-

nitions of lift-induced drag (e.g. *Maskell's* formula [114]) do not mathematically ensure zero drag in 2D flows. Moreover, in two-dimensional but *unsteady* flows, the velocity induced by vorticity may give a vortex force component in the free-stream direction, an aspect quite interestingly observed by Saffman [173, p. 113] when analysing 2D accelerated inviscid flows. A classical example is the generation of a thrust force on flapping wings and its relation to the Lamb vector fields [140]. Saffman therefore interpreted the birth of induced drag in 3D by analysing a 2D problem, observing that induced drag also exists in 2D (yet unsteady) flows. Nonetheless, in two-dimensional steady conditions, where no trailing vorticity is generated, one usually expects zero induced drag, so the question on whether or not the computed negative values are legitimated was worth a clarification.

Nevertheless, accepting this bias on parasite drag calculation (previously ignored and identified during the work reported in this dissertation, Ch. 2), its breakdown in viscous and wave contributions using Lamb-vector field analysis was first addressed by Mele et al. [117], who proposed a decomposition method using a wake plane positioned just downstream of the shock wave. Ostieri and Tognaccini [141], however, verified that this method overestimates wave drag when compared to results obtained using thermodynamic methods and concluded that the wake plane cannot be placed just downstream of the shock, as the contribution of at least its near wake has to be taken into account. Nevertheless, Mele et al. [117] already observed that moving the wake plane downstream leads to a drag exchange from wave to viscous contributions due to the flow entrainment in the boundary layer. Then, Ostieri and Tognaccini [141] derived a parasite drag formulation based on volume integrals (as in thermodynamic-based methods [42, 148]) instead of surface integrals, therefore allowing for a drag decomposition in analogy with the procedure used in methods [42, 148] and based on the sub-division of the control volume in viscous, wave and eventually spurious drag regions. This method was tested by the authors on a two-dimensional symmetric transonic flow-field around the NACA0012 aerofoil (at zero incidence) and not in different conditions. Recently, a Lamb-vector-based parasite drag decomposition approach was also discussed by Fournis [54]. He related parasite drag definitions by Mele et al. [116] and Fournis et al. [57] respectively to Oswatitsch's entropy drag formula [143] and to Destarac and Van der Vooren irreversible drag definition [42]. Then he performed the decomposition on a wake plane (as in the initial investigations by Mele et al. [117]), concluding that the method can correctly capture wave drag, provided that the wake of the shock wave has not yet been entrained into the wake of the boundary layer.

The possibilities, offered by this class of methods, to analyse the aerodynamic force also in unsteady conditions were recently investigated, starting from incompressible flows and considering either inertial or non-inertial reference frames. An example of application of this kind of methods to flapping aerofoils can be found in [105]. The authors proposed an approximated simple lift formula and evidenced that the von Kármán and Sears [211] linear theory is able to also take into account for nonlinear effects, if the required bound vortex sheet over the plate is found by integrating the actual vorticity field, obtained by numerical simulations of the viscous flow around

a pitching and plunging flat plate. Wang et al. [215] also applied the simple lift formula to the analyses of numerical solutions around pitching finite wings.

Additionally, incompressible studies revealed promising in identifying non-stationary formulations of the axial force decomposition [105, 112, 139], including TDB [169], once links with classical linear theories have been established for the transverse (lift) component [139, 140].

Recently, Ostieri et al. [140] proposed a new mixed inertial/non-inertial Lamb-vector-based formula for two-dimensional viscous and incompressible flows, which revealed more accurate than inertial formulations when dealing with moving grid problems. The authors also showed that this formula allowed to correctly compute the dynamic force derivatives even in strongly non-linear flows. In addition, they provided a connection between the original Lamb vector formula of Wu [222, pp. 619-620] and Theodorsen linear theory [194]. Russo et al. [169] used this formula to compute thrust in both steady and unsteady flows and proposed a thrust-drag bookkeeping methodology separating the total axial force in a reversible and an irreversible contribution.

First compressible analyses of unsteady Lamb vector fields were also approached [142], concluding that further investigations are worth being considered to improve knowledge of fundamental phenomena such as flapping wing aerodynamics.

1.4.4 EXERGY METHODS

Far-field formulations previously described fall under the category of *force methods*, as they all move from momentum balance vectorial equation. In different situations, one may be interested in a single force component (axial force in the direction of flight, for example) and a scalar equation could be used in this case to derive a far-field method. In particular, for propelled configurations, engineers are interested in identifying the local sources of thrust and that of drag in the flow domain (*Thrust/Drag Bookkeeping*: TDB), to define improvement areas to be addressed by airframe and engine designers respectively [189]. Unless for standard *tube-and-wing* configurations, this is however not always feasible, especially for highly integrated designs (like those employing distributed propulsion or boundary layer ingestion) and still represents an open research topic. The problem has then been addressed in the last decades from a different point of view, by deriving far-field methods based on the scalar kinetic energy or *exergy* equations.

These methods analyse how the energy/exergy introduced in a fluidic control volume surrounding the aircraft by propulsion, flow control, heat transfer etc. is used for flight, wasted away (i.e. lost outside of the CV) or destroyed within the CV, allowing comparison of different powered configurations on a new basis, alternative to the classical TDB. The underlying principles of energy/exergy approaches will be briefly discussed in the following subsections, without describing all formulations in the details, being this dissertation mainly focused on force methods.

A pioneering work to derive an energy method for aerospace application was performed by Drela in 2009 [43], who focused on the integral balance of kinetic energy, in a control volume $\mathcal{V}$ surrounding the powered aircraft. He considered both jet propulsion (where the engine is typi-

cally not modelled and hidden inside inflow/outflow CV surfaces*) and propeller-driven aircrafts (where rotating blades are simulated and the flow-field is periodically unsteady). His approach was then generalised by Arntz et al. [8], considering not only mechanical but also thermal energy†.

In doing so, they considered that, differently from the mechanical part, thermal energy is not completely reversible, but only a part of it can theoretically be converted into useful work‡. For this reason, the method is not formulated for the total enthalpy (sum of kinetic, internal and flow energy§) but for the system *exergy*, which only considers reversible energy and is therefore the quantity worth being managed carefully.

Exergy is defined as the maximum amount of theoretically recoverable work that can be extracted from a system when it evolves, following a reversible transformation, from its thermo-fluid-dynamic state towards equilibrium (*dead state*) with an ideal reservoir, typically the surrounding environment¶. Exergy is therefore a property of the system together with its environment and, in the aeronautical context, the system is constituted by the fluid (perturbed by the aircraft) while the environment is Earth's atmosphere, at the considered flight altitude. Along with exergy, *anergy* is also defined as that part of the total energy which cannot be converted into useful work. Denoting with a subscript ∞ the asymptotic atmospheric conditions, flow exergy (per unit mass) can be expressed by:

$$\varepsilon = (H - H_\infty) - T_\infty (s - s_\infty) = \Delta H - T_\infty \Delta s \quad (1.128)$$

where H and s are mass-specific total enthalpy and flow entropy, respectively, while T_∞ is the reference (atmospheric) static temperature.

At this point, assuming local thermodynamic equilibrium, a balance equation for exergy can be derived from those of total enthalpy and entropy:

$$\rho \frac{D\varepsilon}{Dt} = \rho \frac{D\Delta H}{Dt} - \rho T_\infty \frac{D\Delta s}{Dt} \quad (1.129)$$

once the latter have been formulated for the *variations* of their balanced quantities with regards to atmospheric conditions⌘. This also allows identifying diffusive fluxes and production terms for flow specific exergy.

The final form of the integral exergy balance for a fluid control volume $\mathcal{V}$ surrounding the aircraft is given by:

$$\dot{\mathcal{E}}_{prop} + \dot{\mathcal{E}}_q = W\dot{T} + \dot{\mathcal{E}}_m + \dot{\mathcal{E}}_{th} + \dot{\mathcal{A}}_{tot} \quad (1.130)$$

*a similar approach can be used for flow control, where actuators are typically not simulated.

†The method of Arntz et al. was however formulated for steady flows only, without considering periodically unsteady regimes and, therefore, in absence of moving surfaces as propeller blades.

‡through an ideal Carnot machine.

§potential energy is neglected.

¶see also the work of Rant [161] and, recently, of Berhouni et al. [14].

⌘this can be easily done introducing mass balance in the equations and neglecting the fuel mass addition into the CV by the propulsors [8].

1.4. Far-field formulas and operative drag definitions

where the different terms and symbols used in Eq. (1.130) are described hereafter. Advective and diffusive exergy fluxes through the CV internal boundary** appear on the LHS, while exergy advection through the CV external surface (Σ) is left on the RHS ††, together with exergy destruction in $\mathcal{V}$, denoted by $\dot{\mathcal{A}}_{tot}$ (it represents an anergy generation).

Advection at the CV interior boundary is due to mass flow at permeable surface delimiting jet propulsors or flow control devices and the corresponding integral exergy flow rate is therefore denoted by $\dot{\mathcal{E}}_{prop}$. Exergy diffusion at the body (stationary) walls is essentially due to heat conduction‡‡, with integral flow rate denoted by $\dot{\mathcal{E}}_q$. The sum of the two ($\dot{\mathcal{E}}_{prop} + \dot{\mathcal{E}}_q$) constitute the total exergy supplied to the fluidic CV.

This is managed as detailed by the RHS of 1.130. First it is recognised that, from the advective exergy fluxes on Σ , a contribution can be isolated which directly represents the rate of work done by the gross axial aerodynamic force on the aircraft (sum of engine thrust and airframe drag), which is moving with velocity $-\underline{V}_\infty$ *. Any thrust/drag imbalance will therefore result in an aerodynamic force rate of work which, for an aircraft of weight W , is denoted by $W\dot{\Gamma}$, with Γ being the aircraft *energy height* (accounting for both potential and kinetic energy†). This is a convenient way to express that this power can be used to either increase aircraft altitude, or its flight speed (or both). For un-powered configurations, $W\dot{\Gamma} = -DV_\infty$, which provides a link between far-field exergy and drag methods (often formulated in absence of propulsion effects).

The remaining exergy advection terms through Σ are related to the flow rate of mechanical exergy ($\dot{\mathcal{E}}_m$, sum of kinetic perturbation energy and pressure perturbation work) and thermo-compressible exergy ($\dot{\mathcal{E}}_{th}$, sum internal energy perturbation and isobaric pressure work, minus the anergy related to entropy variations). The sum of the two ($\dot{\mathcal{E}}_m + \dot{\mathcal{E}}_{th}$), representing the total exergy flow rate through Σ , is a measure of the provided exergy which is wasted outside of the CV and represents a potential for work recovery.

Finally, another part of the exergy externally provided to the control volume, is destroyed within the CV through the action of $\dot{\mathcal{A}}_{tot}$, which includes effects of viscous dissipation, thermal mixing and anergy generation across shock waves.

Exergy methods offer the capability of identifying room for aerodynamic design improvement: exergy waste is in fact quantified and, above all, drag reduction can be driven by an intuitive approach based on exergy recovery.

Arntz and Hue [9] applied the far-field exergy method derived in [8] to the analysis of the

**including the body skin as well as any permeable surface delimiting jet propulsors or flow control devices.

††diffusion fluxes at the Σ surface are neglected in the formulation of Arntz et al. [8].

‡‡no moving walls are considered by Arntz et al. [8].

*the far-field aerodynamic force expression obtained from momentum balance is used at the scope, and identified in the exergy balance. For un-powered configurations, the rate of work is simply given by $-DV_\infty$, where D is the aerodynamic drag.

†It represents the altitude where the aircraft, considered at rest, would have a potential energy equivalent to the sum of the real potential and kinetic energy it has, when flying at a given speed and altitude.

wing-body NASA Common Research Model, then Berhouni et al. [13] extended the method to the analysis in rotating reference frames and also refined the decomposition proposing an approach to separate thermal and compressible contributions in the thermo-compressible exergy flow rate [15]. The drag breakdown, for steady flows, has been analysed by Aguirre and Duplaa [4] and a velocity decomposition allowing identifying isentropic and non-isentropic contributions to the exergetic balance has been proposed by Aguirre et al. [6]. The exergy analysis in unsteady conditions has been studied by Ruscio et al. [166–168] and, very recently, applications to aircraft engines at industrial level of complexity have been documented by Paladini et al. [146], Petropoulos et al. [150] and Morelli et al. [130]. Mutangara et al. [132], following the work of Drela [43], also applied a far-field mechanical energy balance to the analysis of aircraft configurations with boundary layer ingestion, simulated using actuator disks.

1.5 NEAR-FIELD FORCE DECOMPOSITION: A NOVEL APPROACH

Till the last decade, a phenomenological decomposition of the aerodynamic force was only possible via far-field methods, described in the previous sections. Near-field approaches only allowed distinguishing pressure and viscous contributions to the force (Sec. 1.1).

Indeed a phenomenological decomposition of pressure forces (friction stresses are entirely due to flow viscosity) was proposed in 2015 by Schmitz and Coder [178], initially for the incompressible regime. Their near-field approach, complementary to far-field methods, moved from the observation that, in the momentum equation, the pressure gradient balances both viscous and inviscid terms (in the sense that the latter also appear in the Euler equations):

$$\underline{\nabla} p = - \left[\rho_\infty \underline{\nabla} \left(\frac{V^2}{2} \right) + \rho_\infty \underline{\omega} \times \underline{V} \right] + \underline{\nabla} \cdot \underline{\tau}_v \quad (1.131)$$

The linearity of the gradient operator, allowed them to express pressure effects on the fluid particle equilibrium as the combination of two *Partial Pressure Fields* (PPFs): an Euler contribution p_E (or circulatory contribution, for incompressible flows) and a viscous contribution p_μ , i.e. $\underline{\nabla} p = \underline{\nabla}(p_E + p_\mu) = \underline{\nabla} p_E + \underline{\nabla} p_\mu$. The decomposition method then relies on the assumption to decouple viscous and inviscid effects by *independently* satisfying the two equations:

$$\underline{\nabla} p_E = - \left[\rho_\infty \underline{\nabla} \left(\frac{V^2}{2} \right) + \rho_\infty \underline{\omega} \times \underline{V} \right] \quad (1.132)$$

$$\underline{\nabla} p_\mu = \underline{\nabla} \cdot \underline{\tau}_v \quad (1.133)$$

since Eq. (1.132) would be satisfied in an Euler flow.

The numerical technique used by Schmitz and Coder to solve the above equations and to compute the PPFs goes through the transformation of (1.132) and (1.133) in Poisson's equations, which can be iteratively solved even at runtime, integrating the decomposition method in the CFD solver (appropriate boundary conditions were also formulated to guarantee a well-posed

problem).

The method and theory were extended a few years later to compressible flows by Schmitz [177]. The author also identified links between the lift and drag provided by surface integration of the inviscid pressure only and the classical Kutta-Joukowski theorem and Maskell's formulas, respectively. The drag contribution deriving from the viscous pressure field (p_μ) and friction stresses was instead related to the Oswatitsch's formula, while its lift contribution was shown to be negligible unless in cases with large wakes (e.g. in stall conditions).

In Sec. 1.4.2 the steady aerodynamic force was decomposed as: $\underline{F}_{ff} = \underline{F}_{\delta V} + \underline{F}_{BETZ}$, where the axial and transverse components of $\underline{F}_{\delta V}$ provide, respectively, the generalised Kutta-Joukowski and Maskell force ($\underline{F}_{KJ}$ and $\underline{F}_{MSK}$). Using PPFs, the total force is expressed by the sum of the aerodynamic force due to the Euler pressure ($\underline{F}_{pE}$) and that due to the dissipative pressure and shear stress ($\underline{F}_{p\mu, \tau_{\mu t}}$): $\underline{F}_{ff} = \underline{F}_{pE} + \underline{F}_{p\mu, \tau_{\mu t}}$. It is important to highlight the relationship between the two decompositions. The Euler pressure force includes $\underline{F}_{\delta V}$ but also the density variation term in $\underline{F}_{BETZ}$ (second integral in Eq. (A39)) and part of total pressure losses term (first integral in Eq. (A39)), namely the one *related to the Euler PPF*. The last two contributions to $\underline{F}_{pE}$ represent a *viscous-inviscid interactional aerodynamics* term, as defined by Schmitz [177]. He also proved that $\underline{F}_{p\mu, \tau_{\mu t}}$ is related to $\underline{F}_{BETZ}$ (which, in turn, can be related to the Oswatitsch's formula [54]) plus exactly the same interactional aerodynamics term appearing in $\underline{F}_{pE}$, but taken with opposite sign.

In compressible flows and in presence of shock waves, the peculiarities of the method make the inviscid PPF include both induced and wave drag, differently from other far-field approaches, where wave drag is usually grouped to viscous drag.

To further isolate wave drag, Coder and Schmitz [31] proposed a thermodynamic approach based on the density field splitting, successfully de-coupling wave and induced drag in inviscid transonic flows. The problem was then very recently addressed for viscous and compressible flows by Hart and Schmitz [69], in 2D. The method in this case requires the solution of another, now incompressible, flow-field around the same geometry under investigation, to allow a velocity splitting (in compressible and incompressible contributions) required to isolate wave drag. Applications of the proposed approach were provided to the NACA0012 and RAE2822 aerofoils, as well as to a flapped S207 wing section. Comparison is provided in Ch. 2 (Sec. 2.2.3.7) between results provided by Hart and Schmitz's method versus those obtained using similar-purpose far-field approaches, including a new Lamb-vector-based decomposition method developed in the framework of this dissertation.

An ambition which will have to wait is a rigorous definition of induced drag in viscous flows.

Philippe Roland Spalart

2

Steady regime

2.1 LAMB-VECTOR METHODS

THE FAR-FIELD FORCE FORMULA (EQ. (1.2)), OBTAINED THROUGH AN INTEGRAL BALANCE OF LINEAR MOMENTUM, CAN BE EXPRESSED IN TERMS OF FLOW VORTICITY ($\underline{\omega}$) via the Lamb vector ($\underline{\ell} = \underline{\omega} \times \underline{V}$) and pressure can be eliminated by applying Derivative Moment Transformations (DMTs [138][222, pp. 698-705], Appendix A.1), which can be regarded as a multi-dimensional version of integration by parts [222, pp. 593, 700]. The proof is given in Appendix A.4 for the original vortical formulation of the far-field force, first derived by Wu et al. [222, p. 621]. In the same appendix, the Lamb-vector-based aerodynamic moment formula is also derived.

Different *exact* formulations involving the Lamb vector, developed in the last two decades [200, 217] and expressing the aerodynamic force exerted on a rigid body in *steady* and viscous flow regime, are herein briefly recalled, summarising the key properties of each formulation and highlighting individual constituents of the total force. The following formula are written in a general form which is valid for both steady laminar and statistically-steady turbulent flow regimes. In the first case, *Reynolds* stresses and variables fluctuations are zero. In the second case, mass density and fluid particle velocity are intended as their corresponding *ensemble* and *Favre* averages respectively. The definition of all symbols used herein is included in the nomenclature section. Nevertheless, for reader's convenience, it is quickly resumed that, in what follows, $\underline{F}$ is the aerody-

dynamic force exerted by the fluid on the body B immersed in it, while $\mathcal{V}$ is the control volume used for the integral balance of linear momentum, bounded by the aerofoil surface ∂B and the far-field surface Σ (Fig. 1.1.1).

All proposed equations are exact in the sense that they can be directly derived, via the Navier-Stokes equations, from the standard definition of the aerodynamic force as integration of wall stresses on the body surface, the so-called *near-field* formula (Eq. (1.1)), which will be adopted as a reference in all calculations.

2.1.1 ORIGINAL FORMULA

The original incompressible formulation derived by Wu et al. [221] for the laminar flow regime in inertial reference frames and neglecting gravitational effects is expressed as:

$$\underline{F}_{ff} = -\rho \int_{\mathcal{V}} \underline{\ell} d\mathcal{V} - \rho \int_{\Sigma} \underline{r} \times (\underline{n} \times \underline{\ell}) dS + \int_{\Sigma} \underline{r} \times (\underline{n} \times \underline{\nabla} \cdot \underline{\tau}_v) dS + \int_{\Sigma} \underline{\tau}_v \cdot \underline{n} dS \quad (2.1)$$

The symbol $\underline{r}$ denotes a scaled position vector, computed from an arbitrary pole (Fig. 1.1.1):

$$\underline{r} = \frac{\underline{x} - \underline{x}_p}{\mathcal{N} - 1} \quad (2.2)$$

with $\mathcal{N}$ being the space dimension (2 or 3). As shown by Saffman [173, pp. 46-48], the first integral in Eq. (2.1) gives a fundamental contribution to the aerodynamic force in many flow problems characterised by bounded vorticity. The second integral, instead, allows extending Saffman's theory to the case of unconfined vorticity fields, representing a break-through contribution of the exact vortical force theory by Wu et al. The last two integrals, involving the viscous stress tensor, give a negligible contribution to the total force in high-Reynolds number flows when the control surface Σ lies sufficiently outside of the viscous boundary layer [112]. Therefore, for steady high-Reynolds number flows, the aerodynamic force is linked by Eq. (2.1) to the Lamb vector field only, which then represents the local source of the force. In the same hypotheses, the second integral is a wake integral, i.e. it can be restricted to the Σ portion intercepted by the body wake, being the Lamb vector negligible elsewhere on Σ . In this case it represents a finite Reynolds number effect, as the Lamb vector in the wake will also vanish for steady effectively-inviscid flows ($Re_{\infty} \rightarrow \infty$), since Kelvin's theorem ensures that flow vorticity is aligned with the fluid velocity in the free vortex system*. Finally, it can be easily proved that, even when this wake integral is non-zero, its lift contribution can be cancelled if the wake portion of Σ is a planar surface orthogonal to $\underline{V}_{\infty}$ and the DTM pole is located on this plane†. The same formula (then extended to turbulent

*This argumentation, although intuitive, is not rigorous as, for inviscid flows, the control volume frontier ($\partial\mathcal{V}$) must bound the shear discontinuity represented by the vortical wake (see footnote on page 11).

†In fact, in this case, $\underline{n} = \underline{e}_x$, then $\underline{F}_S = \ell_z r_z \underline{e}_x - r_x \underline{l}_z$. Choosing the DMT pole on the wake plane ($r_x = 0$) then cancels the transverse $\underline{F}_S$ component.

flows by Marongiu and Tognaccini [112]) is valid also in the compressible regime [222, p. 621] at condition of replacing the Lamb vector (or, better, $\rho \underline{\ell}$) with the “compressible Lamb vector” $\left(\rho \underline{\ell} - \frac{V^2}{2} \underline{\nabla} \rho\right)$ as defined by Liu et al. [100, 103]. This formula is hereafter denoted with the subscript “a” (to distinguish it from the different formulations introduced in the following sections) and the *far-field* (ff) subscript will be omitted from here on, in vortical force formulas, to simplify the notation.

$$\underline{F}_a = - \int_{\mathcal{V}} \left(\rho \underline{\ell} - \frac{V^2}{2} \underline{\nabla} \rho \right) d\mathcal{V} - \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\rho \underline{\ell} - \frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS + \underline{F}_\mu \quad (2.3)$$

where:

$$\begin{aligned} \underline{F}_\mu = & \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \underline{\nabla} \cdot \left(\underline{\tau}_v + \underline{\tau}_R \right) \right] dS + \int_{\Sigma} \left(\underline{\tau}_v + \underline{\tau}_R \right) \cdot \underline{n} dS + \\ & + \int_{\mathcal{V}} \underline{r} \times \left(\underline{\nabla} \times \tilde{\delta}_{\rho V} \right) d\mathcal{V} \end{aligned} \quad (2.4)$$

$\underline{F}_\mu$ includes terms which are explicitly depending on fluid viscosity and on turbulent fluctuations (through the laminar viscous stress tensors $\underline{\tau}_v$ and Reynolds stresses $\underline{\tau}_R$). The last integral in Eq. (2.4) is a turbulent term, introduced by Mele and Tognaccini for compressible flows [119], with:

$$\tilde{\delta}_{\rho V} = \overline{V'' \underline{\nabla} \cdot (\rho V'')} \quad (2.5)$$

where the ensemble average is indicated by $\bar{\cdot}$, the *Favre* average by $\tilde{\cdot}$, while $''$ denotes random velocity fluctuations with respect to the *Favre* average.

Eq. (2.3) can be written in compact form as:

$$\boxed{\underline{F}_a = \underline{F}_\ell + \underline{F}_\rho + \underline{F}_S + \underline{F}_\mu} \quad (2.6)$$

defining:

$$\underline{F}_\ell = - \int_{\mathcal{V}} \rho \underline{\ell} d\mathcal{V} \quad \text{vortex force} \quad (2.7)$$

$$\underline{F}_\rho = \int_{\mathcal{V}} \frac{V^2}{2} \underline{\nabla} \rho d\mathcal{V} + \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS \quad \text{compressibility correction} \quad (2.8)$$

$$\underline{F}_S = - \int_{\Sigma} \underline{r} \times [\underline{n} \times (\rho \underline{\ell})] dS \quad \text{elementary vortex forces circulation moment} \quad (2.9)$$

$\underline{F}_\ell$ is the *vortex force* (namely, the volume integral of the opposite Lamb vector, times mass density, in $\mathcal{V}$). $\underline{F}_\rho$ is a compressibility correction, taking into account density variations in the flow-field. It can be included in the total force formula as an alternative to the replacement of the Lamb vector with its compressible form. $\underline{F}_S$ represents the contribution of flow vorticity from outside of the selected control volume $\mathcal{V}$.

Differently from the classical far-field force expression (Eq. (1.92) or Eq. (1.102)) which, for steady flows, is only based on surface integrals, vortical force formulas as Eq. (2.6) also include

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domain integrals in $\mathcal{V}$, a feature which allows linking the aerodynamic force to its local generation in the flow-field (examples will be provided in Section 2.2.3.7 and 4.2.5). Eq. (2.6), however, might lead to the conclusion that, even for high-Reynolds number flows, the aerodynamic force is generated in the entire flow-field, due to the presence of the density gradient in $\underline{F}_\rho$.

2.1.2 MODIFIED FORMULA

A modification of formula “a” was then proposed by Mele and Tognaccini [119] using a re-formulated compressibility correction term. Similarly to the Lamb vector, it has the property of being non-zero only in the rotational part of the flow-field (*mid-field* property), for flows originating from uniform free-stream conditions. The total force formula is written as:

$$\underline{F}_b = - \int_{\mathcal{V}} (\rho \underline{\ell} + \underline{m}_\rho) d\mathcal{V} - \int_{\Sigma} \underline{r} \times (\underline{n} \times \rho \underline{\ell}) dS + \underline{F}_\mu \quad (2.10)$$

where:

$$\begin{aligned} \underline{m}_\rho &= \underline{r} \times \left\{ \underline{\nabla} \times \left[\rho \underline{\nabla} \left(\frac{V^2}{2} \right) \right] \right\} = -\underline{r} \times \left[\underline{\nabla} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] = \\ &= \underline{r} \times \left[\underline{\nabla} \rho \times \underline{\nabla} \left(\frac{V^2}{2} \right) \right] \end{aligned} \quad (2.11)$$

In this form, the formula now evidences that the aerodynamic force is only generated in the rotational regions of the flow-field.

Eq. (2.10) can be written in compact form as:

$$\boxed{\underline{F}_b = \underline{F}_\ell + \underline{F}_{m_\rho} + \underline{F}_S + \underline{F}_\mu} \quad (2.12)$$

defining:

$$\underline{F}_{m_\rho} = - \int_{\mathcal{V}} \underline{m}_\rho d\mathcal{V} \quad \text{mid-field compressibility correction} \quad (2.13)$$

2.1.3 ALTERNATIVE FORMULA

Starting from Eq. (2.12), Mele et al. [117] derived an alternative formula converting volume integrals $\underline{F}_\ell$ and $\underline{F}_{m_\rho}$ into surface integrals, leading to a new expression for the steady aerodynamic force (denoted with subscript “c”) containing surface integrals only (as for classical methods), with exception for the last term in Eq. (2.4):

$$\underline{F}_c = - \int_{\Sigma} \rho \underline{V} \underline{V} \cdot \underline{n} dS + \int_{\Sigma} \rho \frac{V^2}{2} \underline{n} dS + \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS + \underline{F}_S + \underline{F}_\mu \quad (2.14)$$

Eq. (2.14) can be written in compact form as:

$$\underline{F}_c = \underline{F}_{\underline{\ell}+m_p} + \underline{F}_S + \underline{F}_\mu \quad (2.15)$$

defining:

$$\underline{F}_{\underline{\ell}+m_p} = - \int_{\Sigma} \rho \underline{V} \underline{V} \cdot \underline{n} \, dS + \int_{\Sigma} \rho \frac{V^2}{2} \underline{n} \, dS + \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS \quad (2.16)$$

It has to be observed that $\underline{F}_c$ is partly based on flow velocity instead of on the Lamb vector, as in Eq. (2.16) the vortex force does not explicitly appear anymore and the Lamb vector is only involved in the total force formula through the outer vorticity contribution ($\underline{F}_S$ term). Indeed, it should not be accounted among Lamb-vector-based far-field formulas, standing mid-way between classical and vortical methods.

2.1.4 TRANSONIC FORMULA

In the same article [117], a hybrid formulation to be used in transonic flows was also proposed (“c” formulation in the shock wave region $\mathcal{V}_{SW}$, Fig. 2.1.1, combined with “b” formulation in the remaining part of the fluid domain) to avoid computing volume integrals in the shock wave region, where numerical inaccuracies were observed. It can be written in compact form, using the already defined force components in Eq. (2.7) and Eq. (2.9), by adopting a modified expression for the mid-field compressibility correction term:

$$\begin{aligned} \underline{F}_{m_{pc}} = & - \int_{\mathcal{V} \setminus \mathcal{V}_{SW}} \underline{m}_p \, d\mathcal{V} - \int_{\partial \mathcal{V}_{SW}} \rho \underline{V} \underline{V} \cdot \underline{n}_{SW} \, dS + \int_{\partial \mathcal{V}_{SW}} \rho \frac{V^2}{2} \underline{n}_{SW} \, dS + \\ & + \int_{\partial \mathcal{V}_{SW}} \underline{r} \times \left[\underline{n}_{SW} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS + \int_{\mathcal{V}_{SW}} \rho \underline{\ell} \, d\mathcal{V} \end{aligned} \quad (2.17)$$

leading to:

$$\underline{F}_{bc} = \underline{F}_{\underline{\ell}} + \underline{F}_{m_{pc}} + \underline{F}_S + \underline{F}_\mu \quad (2.18)$$

2.1.5 MODIFIED TRANSONIC FORMULA

Ostieri and Tognaccini [141] converted the vortex forces circulation moment ($\underline{F}_S$, Eq. (2.9)) from surface to volume integrals. The corresponding modified total force formulation is denoted with subscript “d” and written in compact form as:

$$\underline{F}_d = \underline{F}_{\underline{\ell}} + \underline{F}_{m_{pc}} + \underline{F}_{S\mathcal{V}} + \underline{F}_\mu \quad (2.19)$$

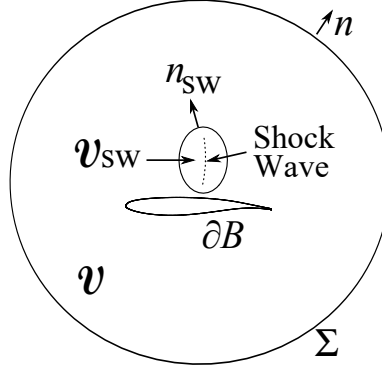


Figure 2.1.1: Sample (two-dimensional) control volume for the application of the transonic formula, with selected no-integration region, $\mathcal{V}_{sw}$, around the shock wave.

where:

$$\underline{F}_{SV} = - \int_{\mathcal{V}} \underline{\nabla}(\underline{r} \cdot \underline{\rho \ell}) d\mathcal{V} + \int_{\mathcal{V}} \underline{\nabla} \cdot (\underline{r} \underline{\rho \ell}) d\mathcal{V} \quad (2.20)$$

This alternative expression for the vortex forces circulation moment can be used also in the other force formulas introduced above and, in particular, choosing formulation “b”, it allows identifying the local force production in steady high-Reynolds number flows ($\underline{F}_\mu \approx \underline{0}$, see Ch. 4, Sec. 4.2.5), being in this case the total force formula based on volume integrals only.

2.1.6 THE INVISCID FLOW CASE: GENERALISATION OF THE FORCE FORMULAS

To extend the above listed force formulas to the case of an effectively inviscid flow, corrections to vortex force, modified compressibility corrections and vortex forces circulation moment are necessary. Indeed, additional integrals over the body boundary (∂B) need to be included as highlighted below (terms in blue) to account for a non-zero slip velocity at the wall on the fluid side, or,

alternatively, for the volume integration in the infinitely-thin bounded vortex sheet region [81]‡.

$$\underline{F}_\ell = - \int_{\mathcal{V}} \rho \underline{\ell} dV - \int_{\partial B} \frac{\rho V^2}{2} \underline{n} dS \quad (2.21)$$

$$\underline{F}_{m_\rho} = - \int_{\mathcal{V}} \underline{m}_\rho dV - \int_{\partial B} \underline{r} \times \left[\underline{n} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS \quad (2.22)$$

$$\underline{F}_{m_{\rho c}} = - \int_{\mathcal{V} \setminus \mathcal{V}_{sw}} \underline{m}_\rho dV - \int_{\partial \mathcal{V}_{sw}} \rho \underline{V} \underline{V} \cdot \underline{n}_{sw} dS + \int_{\partial \mathcal{V}_{sw}} \rho \frac{V^2}{2} \underline{n}_{sw} dS + \quad (2.23)$$

$$+ \int_{\partial \mathcal{V}_{sw}} \underline{r} \times \left[\underline{n}_{sw} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS + \int_{\mathcal{V}_{sw}} \rho \underline{\ell} dV + \quad (2.24)$$

$$- \int_{\partial B} \underline{r} \times \left[\underline{n} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS \quad (2.25)$$

$$\underline{F}_{sv} = - \int_{\mathcal{V}} \underline{\nabla} (\underline{r} \cdot \rho \underline{\ell}) dV + \int_{\mathcal{V}} \underline{\nabla} \cdot (r \rho \underline{\ell}) dV + \int_{\partial B} \underline{r} \times (\underline{n} \times \rho \underline{\ell}) dS \quad (2.26)$$

Kang et al. derived the additional terms for the vortex force and for the modified compressibility correction [81], while here the inviscid flow correction for the vortex forces circulation moment (volume integrals formulation, Eq. (2.26)) is introduced for the first time. The other force constituents introduced above ($\underline{F}_\rho$, $\underline{F}_{\ell+m_\rho}$ and $\underline{F}_s$) do not need any correction, while obviously $\underline{F}_\mu$ is zero in the case of effectively inviscid flow. It is also explicitly observed that the additional surface integrals on the body boundary cancel to zero in the case of viscous flow fields, due to the no-slip boundary condition at the wall. It is straightforward to verify that, in inviscid, incompressible and irrotational flows, the near-field formula is recovered from all far-field expressions introduced in this section, via the Bernoulli's theorem.

Once again, it is remarked that this section deals with the inviscid extension for *weak* Euler solutions (the ones actually obtained from CFD, and used in inviscid numerical applications reported in this dissertation). As long as *strong* solutions are instead considered (for theoretical argumentation), discontinuities need to be included among CV boundaries. In this case and for subsonic flows (i.e. in absence of shock waves), the exact steady formulation can be obtained by replacing integrals over the body boundary (∂B) with integrals over the body and wake boundaries ($\partial B \cup \partial W$). In steady conditions, the shear discontinuity due to the free vortical system is only present in 3D flows, while shear singularities may also arise in steady transonic 2D flows as a consequence of total pressure variations across shock waves.

‡For steady flows, these skin terms are not conceptually different from the accompanying domain integral terms, as they represent the volume contribution of the bound vortex sheet modelling the effectively-inviscid fluid in the domain region between the body wall and the external irrotational flow. In this perspective, force formulas valid for inviscid flows (i.e. including skin terms) can be regarded standard formulas written for viscous flows and no-slip body boundaries, where domain integrals are split between the two volume regions (outer irrotational zone and vortical sheet).

2.1.7 DRAG DECOMPOSITION IN LIFT-INDUCED/PARASITE CONTRIBUTIONS AND LINKS TO CLASSICAL FORMULAS

Mele et al. [117] proposed a decomposition of the total drag provided by the axial component of Eq. (2.18) in lift-induced and parasite contributions. Later on, Fournis et al. [55] sharpened the original decomposition to the form:

$$D_i^{bc} = \underline{e}_x \cdot \left(\underline{F}_\ell + \underline{F}_{m_{pc}} - \underline{F}_{\nabla\rho} \right) \quad (2.27)$$

$$D_p^{bc} = \underline{e}_x \cdot \left(\underline{F}_S + \underline{F}_\mu + \underline{F}_{\nabla\rho} \right) \quad (2.28)$$

where

$$\underline{F}_{\nabla\rho} = \frac{V_\infty^2}{2} \int_\Sigma (\rho - \rho_\infty) \underline{n} \, dS + \int_\Sigma \underline{r} \times \left(\underline{n} \times \frac{V_\infty^2}{2} \nabla\rho \right) dS \quad (2.29)$$

It is an additional compressibility correction, that gradually vanishes as the control volume external surface (Σ) extends away from the body [56]. Its introduction has no impact on the total expression (it is both added and subtracted in $\underline{F}_{bc}$) and, in the case of effectively inviscid flow, no correction is required to Eq. (2.29) as its derivation did not assume no-slip flow condition at the wall.

Fournis et al. [55] also identified the links between the vortex-force theory and classical far-field analyses (the Kutta-Joukowski theorem, Maskell's formula and Betz's formula) in compressible flow regime. This decomposition is in fact based on two notable identities, derived by the authors and hereafter reported:

$$\underline{F}_\ell + \underline{F}_{m_{pc}} - \underline{F}_{\nabla\rho} = \underline{F}_{KJ} + \underline{F}_{MSK} \quad (2.30)$$

$$\underline{F}_S + \underline{F}_\mu + \underline{F}_{\nabla\rho} = \underline{F}_{BETZ} \quad (2.31)$$

They numerically verified that $\underline{F}_{BETZ}$ did not provide a lift contribution in their computations, then lift was only attributed to $\underline{F}_\ell + \underline{F}_{m_{pc}} - \underline{F}_{\nabla\rho}$, whose drag component is therefore due to lift. This justified the proposed decomposition, summarised by Eq. (2.27), (2.28). It should be mentioned however that, from a theoretical point of view, $\underline{F}_{BETZ}$ does not provide a lift contribution only in the case of Σ sufficiently distant from the body surface (so that integration can be restricted on a wake plane), no large separations (e.g. far from stall) and assuming the wake plane orthogonal to the free-stream velocity, as discussed in Sec. 1.4.2.1.

In the incompressible flow regime, neglecting the effects of viscous terms on Σ , Eq. (2.31) provides an identification of $\underline{F}_S$ with the classical far-field total pressure loss integration, routinely used in wind tunnel practise to estimate drag.

It is worth observing that, based on identities (2.30) and (2.31), the decomposition provided by Eq. (2.27) and Eq. (2.28) is independent from the choice of the pole used in the definition of $\underline{r}$ as $\underline{F}_{KJ}$, $\underline{F}_{MSK}$ and $\underline{F}_{BETZ}$ do not include any position vector in their defining equations (1.101),

(1.100) and (1.92).

Finally, as the various (mathematically equivalent) vortical force formulations (i.e. $\underline{F}_a$, $\underline{F}_b$, $\underline{F}_c$, $\underline{F}_{bc}$ and $\underline{F}_d$) only differ by the adopted expression of the compressibility correction and of the vortex forces circulation moment, the drag breakdown expressed by Eq. (2.27) and Eq. (2.28) for the bc formulation, can be easily extended to the other formulations as well. For the sake of completeness, this is explicitly done hereafter:

$$D_i^a = \underline{e}_x \cdot \left(\underline{F}_\ell + \underline{F}_p - \underline{F}_{\nabla\rho} \right) \quad (2.32)$$

$$D_p^a = \underline{e}_x \cdot \left(\underline{F}_S + \underline{F}_\mu + \underline{F}_{\nabla\rho} \right) = D_p^{bc} \quad (2.33)$$

$$D_i^b = \underline{e}_x \cdot \left(\underline{F}_\ell + \underline{F}_{\underline{m}_p} - \underline{F}_{\nabla\rho} \right) \quad (2.34)$$

$$D_p^b = \underline{e}_x \cdot \left(\underline{F}_S + \underline{F}_\mu + \underline{F}_{\nabla\rho} \right) = D_p^{bc} \quad (2.35)$$

$$D_i^c = \underline{e}_x \cdot \left(\underline{F}_{\ell+\underline{m}_p} - \underline{F}_{\nabla\rho} \right) \quad (2.36)$$

$$D_p^c = \underline{e}_x \cdot \left(\underline{F}_S + \underline{F}_\mu + \underline{F}_{\nabla\rho} \right) = D_p^{bc} \quad (2.37)$$

$$D_i^d = \underline{e}_x \cdot \left(\underline{F}_\ell + \underline{F}_{\underline{m}_{pc}} - \underline{F}_{\nabla\rho} \right) = D_i^{bc} \quad (2.38)$$

$$D_p^d = \underline{e}_x \cdot \left(\underline{F}_{SV} + \underline{F}_\mu + \underline{F}_{\nabla\rho} \right) \quad (2.39)$$

where it can be observed that parasite drag expressions are identical for all but the “ d ” formulation (the symbol $D_p^{a,b,c,bc}$ will be therefore used) and induced drag expressions from formulations “ bc ” and “ d ” coincide (the symbol $D_i^{bc,d}$ will be therefore used).

2.1.8 CONSISTENCY OF LIFT-INDUCED/PARASITE DRAG FORMULAS WITH REVERSIBLE/IRREVERSIBLE DRAG CONTRIBUTIONS

Because of the links with classical methods discussed above, considerations from Sec. 1.4.2.1 similarly apply to vortical methods. Nevertheless, the capability of lift-induced and parasite drag definitions by Eqs. (2.27) and (2.28) to respectively represent reversible and irreversible drag contributions is further discussed herein, considering two particular conditions:

- subsonic inviscid flow: the total drag (D) is in this case due to reversible flow phenomena associated with the generation of trailing vorticity in three-dimensional lifting cases:

$$D = D_{rev}$$

However, decomposing total drag in lift-induced and parasite contributions:

$$D = D_i + D_p$$

and using definitions from Eq. (2.27) and (2.28), $\underline{F}_\mu = \underline{0}$ (the flow is inviscid) and $\underline{F}_S = \underline{0}$

(the steady inviscid flow is irrotational on Σ , the free vortical wake being an additional CV boundary, treated as ∂B – see footnote on page 11) but $\underline{F}_{\nabla p}$ is non-zero, unless Σ is put sufficiently far from the body skin. Therefore, parasite drag as defined by Eq. (2.28) is generally non-zero and lift-induced drag (Eq. (2.27)) does not coincide with the total reversible drag, unless Σ retreats to infinity.

- no trailing vorticity: the total drag (D) is in this case due to irreversible flow phenomena associated with the presence of boundary layers, shock waves and their wakes. The irreversible drag can be expressed as the sum of Destarac and van der Vooren (DV) definition [42] (which accounts for entropy and total enthalpy variations of the streamwise velocity field) plus an additional term (ΔD_{irr}) accounting for the irreversible drag not included in DV definition (associated to static pressure variations and transverse flow velocities). The last term reduces to zero as Σ retreats to infinity as, in the absence of trailing vorticity, zero transverse flow disturbances and free-stream static pressure are realised in the body far wake. Then:

$$D = D_{irr} = D_{irr}^{DV} + \Delta D_{irr} \quad ; \quad \lim_{\Sigma \rightarrow \infty} \Delta D_{irr} = 0$$

However, in a very recent work [54] Fournis highlighted how, in transonic inviscid flows, parasite drag defined by Eq. (2.28) does not coincide with D_{irr}^{DV} and the difference (ΔD_p) approaches zero only when Σ retreats to infinity:

$$D_p = D_{irr}^{DV} + \Delta D_p \quad ; \quad \lim_{\Sigma \rightarrow \infty} \Delta D_p = 0$$

Therefore, decomposing the total (irreversible) drag in lift-induced and parasite contributions as proposed by Eq. (2.27) and (2.28), using the above relations:

$$D = D_{irr}^{DV} + \Delta D_{irr} = D_i + D_p = D_i + D_{irr}^{DV} + \Delta D_p$$

which leads to non-zero lift-induced drag in this case ($D_i = \Delta D_{irr} - \Delta D_p$), unless Σ is chosen sufficiently far from the body skin.

Moreover, it has to be mentioned that definitions provided in Eq. (2.27) and (2.28) are linked respectively to an extended Maskell's formula ($\underline{F}_{MSK}$) and to a compressible Betz's formula ($\underline{F}_{BETZ}$, Section 1.4.2) where the contribution of $\int_{\Sigma} \rho \left(\frac{u^2}{2} n_x - u(\delta \underline{V} \cdot \underline{n}) \right) dS \underline{e}_x$ has been included in $\underline{F}_{MSK}$ instead of being attributed to the parasite drag ($\underline{F}_{BETZ}$), as suggested by Maskell [114].

This contribution (which reduces to the more familiar wake integral $-\frac{1}{2} \int_W \rho u^2 dS \underline{e}_x$ when the lateral portion of Σ is sufficiently far from the body and the wake plane W is orthogonal to the free-stream velocity) is questioned and has been analysed by Spalart [183] and Drela [43]. The former also discussed the possibility of the $-\rho u^2$ term to provide negative lift-induced drag (as allowed by the formula), a feasible circumstance at least in (idealised) two-dimensional flows. The

latter, justified its attribution to the induced drag in non-rotational flow regions of the Trefftz plane, based on kinetic energy considerations and analysing incompressible flow regimes. However, in rotational Trefftz plane regions (where vortex wakes are mixed with viscous and shock wakes) and/or in presence of compressibility effects, the role of this term is still ambiguous, and so is drag decomposition. This ambiguity can be eliminated if Σ retreats to infinity in the linear far-field, as the abovementioned drag contribution is in this case negligible.

2.2 NUMERICAL ANALYSES

Both two-dimensional and three-dimensional analyses were performed to deeply investigate on the Lamb-vector-based steady force formulas and associated decompositions, theoretically described in the previous sections.

It is explicitly remarked that forces are obtained, through either far-field or near-field formulas, from the computed CFD solutions, which represent the available flow-field data to validate the applied methodologies. In this perspective, a comparison against experimental force values is therefore not required.

2.2.1 REFERENCE NUMERICAL FLOW FIELDS

CFD solutions were obtained for the NACA0012 and RAE2822 aerofoils using the *ZEN* flow solver, developed at the Italian Aerospace Research Centre (CIRA) [111]. CIRA *ZEN* code is a multi-block structured flow solver, based on a cell-centred finite volumes formulation with central schemes and *Jameson*-like scalar artificial dissipation model [75]. Used in steady-state mode, the code adopts a solution procedure based on the pseudo-time marching concept, the relaxation operator consisting of an explicit multi-stage *Runge–Kutta* integration scheme. Convergence is accelerated by implicit residual averaging, local time stepping and multi-grid techniques.

Weak Euler solutions were computed in the case of effectively-inviscid flow simulations ($Re_\infty \rightarrow \infty$), while, in viscous flow simulations at low Reynolds number ($Re_\infty = 1 \times 10^4$), two-dimensional Navier-Stokes equations (2D NS) were directly solved assuming laminar flow. High Reynolds number analyses, instead, used a RANS formulation of flow governing equations, with turbulence closure based on $k - \omega$ turbulent/non-turbulent (TNT) model [86]. In this case, fully turbulent flow was assumed and pre-conditioning was used for the nearly-incompressible case simulations.

A slightly altered definition of the NACA0012 aerofoil providing closed trailing edge is used, according to the procedure illustrated in [165]. The reference geometry of the RAE2822 aerofoil is instead available in discrete points coordinates from [32] (Table 6.1). For both aerofoils, the computational mesh is a C-type grid (Fig. 2.2.1a), generated using NASA *HYPGEN* hyperbolic grid generator [28]. An overall number of $828 \times 200 = 165600$ hexahedral cells (428 around the aerofoil) was used for the NACA0012 aerofoil, while approximately the same dimension (432 cells, instead of 428, around the aerofoil) was employed for the RAE2822 aerofoil. This will be

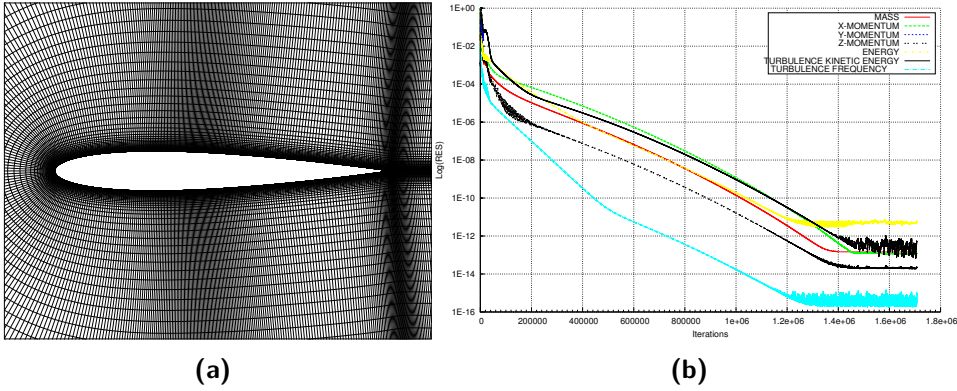


Figure 2.2.1: Visualisation of the NACA0012 computational mesh (828×200 cells) used for RANS simulations in the aerofoil region (Fig. 2.2.1a) and sample convergence history of CFD simulations at $M_\infty = 0.8$, $Re_\infty = 9 \times 10^6$, $\alpha = 3^\circ$ (Fig. 2.2.1b).

referred to, in the following text, as the “fine mesh”. In viscous flow simulations, cells are clustered around the aerofoil surface to allow computing numerical solutions down to the wall without using any wall function. The computational domain extends up to 300 chord lengths around the aerofoil.

Numerical solutions were computed at different angles of attack and for different flow regimes according to Table 2.2.1 and Table 2.2.2. In particular, for the NACA0012 aerofoil at $M_\infty = 0.70$ and $M_\infty = 0.90$, the angle of attack was selected to achieve a target $C_l = 0.5$, to allow comparison with drag breakdown data available in [117], [69] and [172]. For the RAE2822 aerofoil, flow conditions corresponding to cases 6 ($M_\infty = 0.725$) and 9 ($M_\infty = 0.730$) in [32] were used, with target $C_l = 0.796$ and $C_l = 0.803$ respectively, to benchmark with computations data reported in [172] and [69]. Convergence of flow solution to machine accuracy was achieved (Fig. 2.2.1b), although, for viscous NACA0012 simulations at zero angle of attack, computations revealed particularly stiff, requiring a much larger number of solver iterations, compared to other cases, to achieve a similar convergence level. Contour plots of local Mach number are illustrated in Fig. 2.2.2a for the RAE2822 aerofoil in viscous and transonic flow (case 9 conditions) and in Fig. 2.2.2b for the NACA0012 at the highest simulated Mach number (fish-tail shock configuration).

A steady three-dimensional test-case was also considered, analysing an elliptic-planform wing with no twist, straight trailing edge and $\mathcal{R} = 7$. Wing sections are shaped using a NACA0012 aerofoil with closed t.e. and the chord length at mid-span of the wing (c_0) is unitary. Calculations were performed in viscous transonic flow conditions ($M_\infty = 0.75$, $Re_\infty = 3 \times 10^6$) and at zero side-slip angle ($\beta = 0$).

Being the flow-field symmetrical with regards to the mid-plane at zero side-slip angle, a half of the wing configuration was analysed. The grid topology is of C-C type (Fig. 2.2.3), structured in two symmetrical blocks (upper-side and lower-side) with opposite node coordinates in the di-

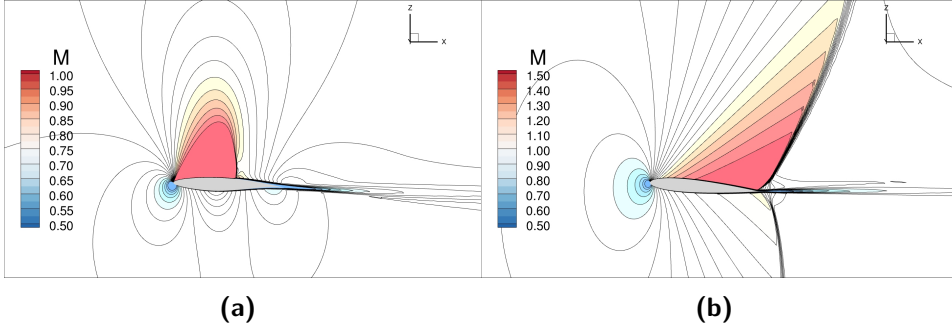


Figure 2.2.2: Contour plot of local Mach number in viscous transonic flow-fields: RAE2822 at $M_\infty = 0.73$, $Re_\infty = 6.5 \times 10^6$, $\alpha = 2.65^\circ$ (Fig. 2.2.2a) and NACA0012 at $M_\infty = 0.90$, $Re_\infty = 9 \times 10^6$, $\alpha = 4.70^\circ$ (Fig. 2.2.2b).

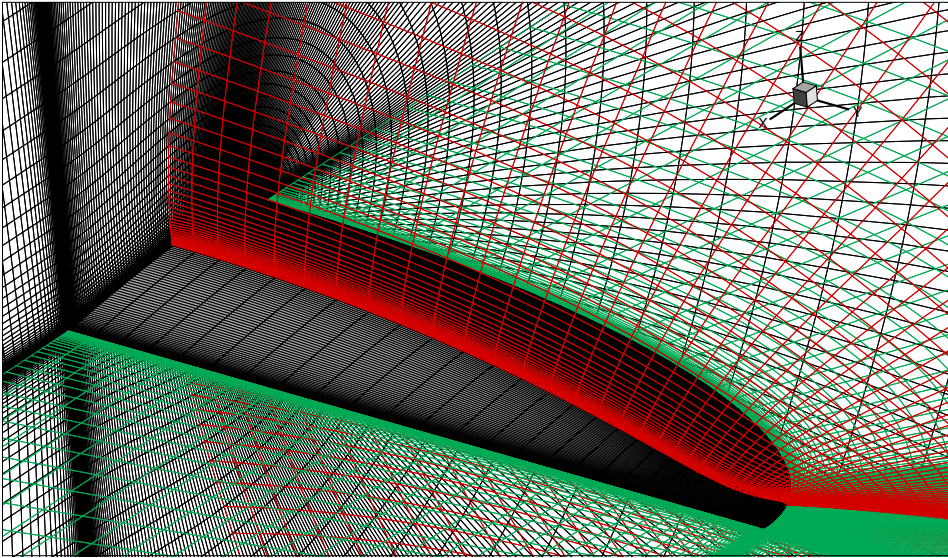


Figure 2.2.3: Computational grid used for the 3D elliptic wing simulations: surface mesh visualisation at the body wall and on orthogonal grid planes.

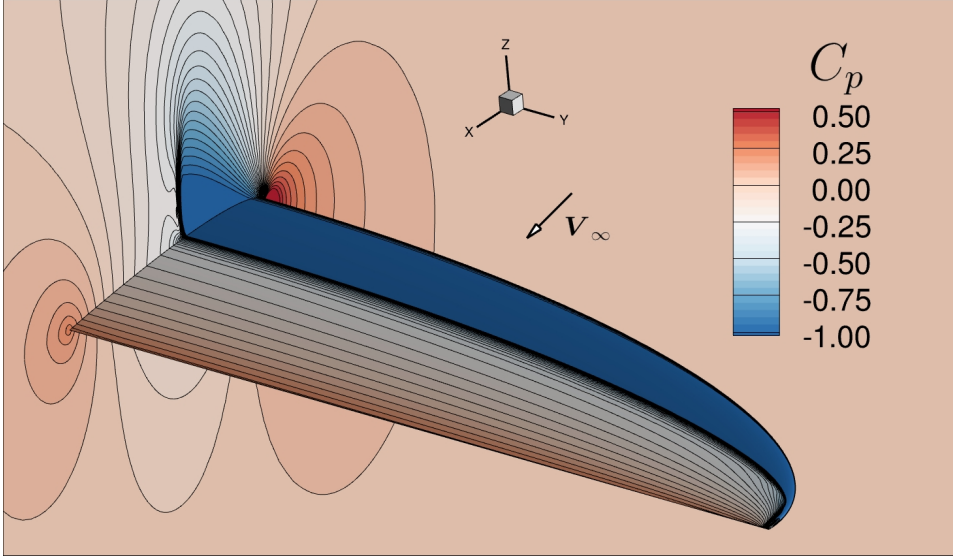


Figure 2.2.4: Contour plot of local pressure coefficient (on the wing surface and on the symmetry plane) for the 3D steady RANS simulation in viscous transonic conditions ($M_\infty = 0.75$; $Re_\infty = 3 \times 10^6$; $\alpha = 4^\circ$).

rection normal to the wing planform. The far field distance from the body is set at 30 root chord lengths. The volume mesh, generated using CIRA *ENGRID* grid generator [185], contains more than 3×10^6 hexahedral cells (384×64 quadrilateral elements over the wing surface, 64 nodes downstream in the wake and 96 interior nodes in the direction normal to the wall/wake). For the same computational mesh, a grid convergence study is already available in [116]. Also in this case, convergence to machine accuracy was achieved (12-orders of magnitude drop, for the average mass-flux residual).

The TNT model was preferred against other options available in the ZEN solver because of its convergence behaviour on the analysed test-cases, as a compromise between robustness and the lower computational cost offered by one-equation models. Compared to the widely adopted SST version of the $k - \omega$ model, the TNT option provided negligible difference on the computed drag, as evidenced for example by the 3D elliptic wing results (illustrated in Section 2.2.4), compared to the available data reported in [116] for the same flow conditions and obtained using a $k - \omega$ SST closure of RANS equations system. Free-stream conditions included a Turbulent Intensity Level (TIL) set at 1% of the mean flow velocity and a turbulent-to-laminar viscosity ratio equal to 0.1.

2.2.2.2 POST-PROCESSING METHOD

Two-dimensional numerical solutions described in Sec. 2.2.1 were post-processed using an in-house software, *UBF2D*, implementing far-field methods illustrated in Sec. 1.4.1 and Sec. 1.4.3.2 and developed in the framework of this doctoral research in co-operation between CIRA (Italian

Table 2.2.1: Simulated flow conditions, NACA0012 aerofoil.

	Nearly Incompressible $M_\infty = 0.05$	Subsonic $M_\infty = 0.50$	Transonic $M_\infty = 0.70$ $M_\infty = 0.80$ $M_\infty = 0.90$		
Weak Euler: $Re_\infty \rightarrow \infty$	$\alpha = 0.00^\circ$ $\alpha = 4.00^\circ$	$\alpha = 0.00^\circ$ $\alpha = 4.00^\circ$	$\alpha = 0.00^\circ$ $\alpha = 2.00^\circ$		
RANS: $Re_\infty = 9 \times 10^6$	$\alpha = 0.00^\circ$ $\alpha = 4.00^\circ$	$\alpha = 0.00^\circ$ $\alpha = 4.00^\circ$	$\alpha = 3.04^\circ$	$\alpha = 0.00^\circ$ $\alpha = 3.00^\circ$	$\alpha = 4.70^\circ$
NS: $Re_\infty = 1 \times 10^4$		$\alpha = 2.00^\circ$ $\alpha = 4.00^\circ$			

Table 2.2.2: Simulated flow conditions, RAE2822 aerofoil.

	Transonic $M_\infty = 0.725$ $M_\infty = 0.730$	
RANS: $Re_\infty = 6.5 \times 10^6$	$\alpha = 2.65^\circ$	$\alpha = 2.65^\circ$

Aerospace Research Centre) and the *University of Naples Federico II*. The code is further described in Appendix F. Three-dimensional solutions were instead post-processed using a dedicated scripting routine implemented and executed in the *Tecplot*®360 commercial environment (see App. F for further details).

When dealing with turbulent flows governed by the RANS equations, the isotropic Reynolds stress tensor (related to the turbulent kinetic energy) was neglected and Boussinesq's approximation for the deviatoric component was assumed: $\underline{\tau}_R = 2\mu_t \left[\frac{1}{2} (\nabla \underline{V} + \nabla \underline{V}^T) - \frac{1}{3} \nabla \cdot \underline{V} \right]$.

Unless otherwise specified, in the post-processing of 2D CFD solutions, flow vorticity required in Lamb vector calculation has been computed using a nodes-based Green-Gauss formula, while, for 3D structured flow-fields, a 2^{nd} -order finite differences scheme was applied in the computational space. Volume integrals are numerically computed by simply summing-up cell centre values, multiplied by the cell volume, over the selected integration domain. Similarly, surface integrals are obtained computing the state in the cell surface centre obtained by centred averaging.

The adopted domain of integration has always been selected as a subset of the computational grid. The identification of the shock-wave region, required in formulations “bc” and “d”, was performed using the shock sensor defined by Lovely and Haimes [109] and selecting an appropriate cut-off value.

2.2.3 AERODYNAMIC FORCE ANALYSIS, IN 2D

2.2.3.1 ACCURACY OF THE TOTAL FORCE FORMULA AND OF THE DRAG DECOMPOSITION IN LIFT-INDUCED AND PARASITE CONTRIBUTIONS

The aerodynamic force (lift and drag components) was computed, in the post-processing phase, using both near-field surface stress integration and far-field Lamb-vector-based formulations summarised in Section 2.1. In particular, Table 2.2.3 lists the different quantities that were computed and compared in this section.

Table 2.2.3: Summary of computed and compared quantities.

Quantity	Definition
C_l^{NF} (or C_d^{NF})	Lift (or drag) coefficient, output of the flow solver
C_l^{nf} (or C_d^{nf})	Lift (or drag) coefficient, computed from Eq. (1.1)
C_l^a (or C_d^a)	Lift (or drag) coefficient, computed from Eq. (2.6)
C_l^b (or C_d^b)	Lift (or drag) coefficient, computed from Eq. (2.12)
C_l^c (or C_d^c)	Lift (or drag) coefficient, computed from Eq. (2.15)
C_l^{bc} (or C_d^{bc})	Lift (or drag) coefficient, computed from Eq. (2.18)
C_l^d (or C_d^d)	Lift (or drag) coefficient, computed from Eq. (2.19)
$C_{d,i}^a$	Lift-induced drag coefficient, computed from Eq. (2.32)
$C_{d,i}^b$	Lift-induced drag coefficient, computed from Eq. (2.34)
$C_{d,i}^c$	Lift-induced drag coefficient, computed from Eq. (2.36)
$C_{d,i}^{bc,d}$	Lift-induced drag coefficient, computed from Eq. (2.27)
$C_{d,p}^{a,b,c,bc}$	Parasite drag coefficient, computed from Eq. (2.28)
$C_{d,p}^d$	Parasite drag coefficient, computed from Eq. (2.39)
$C_{d,irr}^{DV}$	Irreversible drag coefficient, computed from Eq. (1.84)
$C_{d,irr}^{PT}$	Irreversible drag coefficient, computed from Eq. (1.83)

Each of the terms constituting the total force has been analysed individually (Appendix B.1) and the drag decomposition in lift-induced and parasite contributions has also been addressed. The only term that was not included in the computations is the last integral of Eq. (2.4), which is related to turbulent fluctuations and represents an unclosed term. This was neglected assuming *Morkovin's* hypothesis, stating that compressibility has a weak influence on the Reynolds stress [131].

Then the irreversible drag was also calculated using the method of Destarac and van der Vooren [42], as well as that of Paparone and Tognaccini [148] (Ch. 1, Sec. 1.4.1.2 and 1.4.1.2). Lift and drag coefficients are referred to the free-stream dynamic pressure and to either the aerofoil chord length (in 2D) or the wing surface (in 3D).

While, for given free-stream conditions and aerofoil angle of attack (α), the near-field method provides a unique value for the aerodynamic force, the theoretically equivalent far-field calculation might exhibit, for numerical reasons, a dependence on some of the method's degrees of freedom, such as the control volume choice.

In the analysed cases, the computed total force revealed very weakly influenced by the adopted integration domain as discussed in detail in the following sub-section and in Appendix B.1. Table 2.2.4 provides comparison of far-field lift coefficients obtained by the different force formula (Section 2.1) against near-field values.

Table 2.2.4: Total lift coefficient values (Σ distance: $\approx 15c$; fine mesh; pole at aerofoil l.e.).

	M_∞	Re_∞	α [$^\circ$]	C_l^{NF}	C_l^{nf}	C_l^a	C_l^b	C_l^c	C_l^{bc}	C_l^d
NACA0012	0.050	$\rightarrow \infty$	0.00	0.000	0.000	0.000	0.000	0.000	0.000	-0.000
			4.00	0.482	0.483	0.482	0.482	0.482	0.482	0.482
		9.0×10^6	0.00	0.000	-0.000	0.000	0.000	0.000	0.000	0.000
			4.00	0.443	0.443	0.443	0.443	0.443	0.443	0.443
	0.500	$\rightarrow \infty$	0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000
			4.00	0.574	0.574	0.574	0.574	0.574	0.574	0.574
		9.0×10^6	0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000
			4.00	0.516	0.516	0.516	0.516	0.516	0.516	0.515
		1.0×10^4	2.00	-0.039	-0.039	-0.039	-0.039	-0.039	-0.039	-0.039
			4.00	-0.032	-0.032	-0.032	-0.032	-0.032	-0.032	-0.033
		9.0×10^6	3.04	0.500	0.500	0.501	0.500	0.501	0.500	0.500
			0.00	0.000	0.000	0.000	0.000	0.000	-0.000	0.000
	0.800	$\rightarrow \infty$	2.00	0.557	0.557	0.557	0.549	0.558	0.551	0.551
			0.00	0.000	0.000	0.000	0.000	0.000	-0.000	0.000
		9.0×10^6	3.00	0.451	0.451	0.450	0.449	0.452	0.451	0.451
			4.70	0.500	0.500	0.479	0.481	0.501	0.499	0.498
	0.900	9.0×10^6	0.00	0.000	0.000	0.000	0.000	0.000	-0.000	0.000
			2.00	0.557	0.557	0.557	0.549	0.558	0.551	0.551
RAE 2822	0.725	6.5×10^6	2.65	0.796	0.796	0.797	0.794	0.796	0.794	0.792
	0.730	6.5×10^6	2.65	0.803	0.803	0.803	0.801	0.803	0.801	0.797

The latter are either directly obtained from the flow solver (C_l^{NF}) or re-computed by the post-processor based on the input numerical flow-field data (C_l^{nf}). Differences are negligible in almost all cases, with exception for the flow conditions with the strongest shocks (inviscid transonic case at non-zero incidence and high-transonic flow at $M_\infty = 0.9$), where formulation “c” provided more accurate results, although differences with other formulations still fall within two lift counts (0.02) accuracy, at most. The analysis at low Reynolds number showed that a laminar flow separation on the aerofoil upper surface (occurring at approximately 45% of the chord length) is responsible for the slightly negative lift coefficient, at positive incidence, observed in Table 2.2.4.

Total drag coefficients are instead compared in Table 2.2.5. Differences in near-field drag coefficients computed by the flow solver (C_d^{NF}) and by the post-processor (C_d^{nf}) mainly come from the friction contribution, and are due to the different location of the working variables (cells-centroids or grid-nodes) and gradient schemes used by the two codes. In the following, to maintain coherence on numerical approaches, comparison of far-field vs. near-field results will always be illustrated with reference to post-processor-computed values. It is however observed that the difference between C_d^{NF} and C_d^{nf} becomes relevant in laminar flow simulations at low Reynolds number (where wall velocity gradients are smaller, compared to turbulent solutions).

As long as no viscous or wave drag is considered (2D inviscid subsonic cases), zero drag is obtained with one count (0.0001) accuracy using all Lamb-vector-based formulation except “d” (mostly based on volume integrals). In all other cases, formulation “c” (mostly based on surface integrals) provided drag coefficients in better agreement with near-field values. A significant differ-

Table 2.2.5: Total drag counts (Σ distance: $\approx 15c$; fine mesh; pole at aerofoil l.e.).

	M_∞	Re_∞	$\alpha [^\circ]$	$10^4 \times$							$C_{d,irr}^{DV}$	$C_{d,irr}^{PT}$
				C_d^{NF}	C_d^{nf}	C_d^a	C_d^b	C_d^c	C_d^{bc}	C_d^d		
NACA0012	0.050	$\rightarrow \infty$	0.00	0.5	0.0	0.1	0.1	0.4	0.1	1.1		
			4.00	0.7	0.3	0.3	0.3	0.7	0.3	2.0		
		9.0×10^6	0.00	79.0	80.5	78.3	78.3	80.2	78.3	78.4		
			4.00	84.7	86.2	82.2	82.2	84.9	82.2	82.2		
	0.500	$\rightarrow \infty$	0.00	0.0	-0.1	-0.4	-0.3	0.0	-0.3	0.7	0.2	0.1
			4.00	0.4	0.3	-0.1	0.2	0.4	0.2	1.2	0.3	0.1
		9.0×10^6	0.00	80.0	81.8	79.0	79.0	81.2	79.0	79.1	81.3	80.3
			4.00	87.6	89.4	84.7	84.6	87.8	84.6	84.5	87.7	87.3
		1.0×10^4	2.00	414.1	405.9	392.7	392.4	416.5	392.4	392.5	416.2	413.0
			4.00	468.7	460.2	419.0	418.7	472.0	418.7	418.6	471.4	468.5
		9.0×10^6	3.04	147.9	149.3	161.0	160.6	148.5	142.6	142.3	148.4	147.9
	0.800	$\rightarrow \infty$	0.00	82.9	82.7	115.3	114.9	83.2	79.3	80.0	83.0	82.9
			2.00	416.7	416.6	574.9	570.0	417.9	414.0	414.6	417.3	416.3
		9.0×10^6	0.00	160.2	161.5	184.2	182.6	162.3	157.3	157.3	162.2	160.5
			3.00	506.4	507.6	606.3	599.9	508.8	501.8	501.6	508.1	506.1
	0.900	9.0×10^6	4.70	1556.0	1557.3	2109.5	2054.7	1563.0	1555.4	1554.3	1558.0	1554.5
RAE 2822	0.725	6.5×10^6	2.65	141.1	142.0	145.7	144.0	141.5	131.1	130.1	141.7	140.8
	0.730	6.5×10^6	2.65	160.6	161.4	170.0	168.0	161.1	150.1	148.6	161.2	159.6

ence is also observed, in transonic cases, between prediction of force formula based on volume integrals extended to the whole control volume (“a” and “b”) or corrected to avoid integrating in the shock wave region (“c”, “bc” and “d”). In the same table, drag coefficient values obtained using far-field thermodynamic methods [42, 148] are also included for the analysed cases in compressible flow regime. These methods (inaccurate in nearly-incompressible flow cases, when $M_\infty \rightarrow 0$) compute the drag associated with variations of flow entropy and total enthalpy (Destarac and van der Vooren approach: “DV” in Table 2.2.5) or with entropy variations only (Paparone and Tognaccini approach: “PT”). In both methods, for stationary and un-powered cases, the computed drag is totally due to dissipative effects and therefore represents an *irreversible* drag (i.e. drag associated with entropy production). The agreement of total drag (C_d) with computed irreversible drag ($C_{d,irr}$) reveals that the entire drag is due to dissipative effects, as expected in two-dimensional flows.

Finally, results of the total drag decomposition in lift-induced and parasite contributions, as defined in Section 2.1.7, are included in Table 2.2.6. Again, in absence of viscous and wave drag, zero lift-induced and zero parasite drag (in the limit of one drag count accuracy) are provided by all formulations except “d”.

When viscous drag is considered (subsonic cases at finite Reynolds number), negative induced drag coefficients are observed, with formulation “c” providing values closer to zero. Using the definitions proposed in [55] and illustrated in Section 2.1.7, *lift-induced* drag in these conditions is therefore not coincident with (zero) *reversible* drag and *parasite* drag is not coincident with (total)

Table 2.2.6: Lift-induced and parasite drag counts (Σ distance: $\approx 15c$; fine mesh; pole at aerofoil l.e.).

M_∞ Re_∞ α [°]			$10^4 \times$							
			$C^a_{d,i}$	$C^b_{d,i}$	$C^c_{d,i}$	$C^{bc,d}_{d,i}$	$C^{a,b,c,bc}_{d,p}$	$C^d_{d,p}$		
NACA0012	0.050	$\rightarrow \infty$	0.00	-0.3	-0.3	0.0	-0.3	0.4	1.5	Nearly Incompr.
			4.00	-0.1	-0.1	0.3	-0.1	0.3	2.1	
		9.0×10^6	0.00	-4.0	-4.0	-2.1	-4.0	82.3	82.4	
			4.00	-2.9	-2.9	-0.2	-2.9	85.1	85.0	
	0.500	$\rightarrow \infty$	0.00	-0.4	-0.3	-0.0	-0.3	0.0	1.0	Subsonic
			4.00	0.1	0.4	0.6	0.4	-0.2	0.8	
		9.0×10^6	0.00	-4.2	-4.3	-2.0	-4.3	83.3	83.3	
			4.00	-3.0	-3.1	0.1	-3.1	87.7	87.6	
		1.0×10^4	2.00	-39.0	-39.2	-15.2	-39.2	431.7	431.7	
			4.00	-66.9	-67.2	-13.8	-67.2	485.9	485.8	
		9.0×10^6	3.04	11.6	11.1	-0.9	-6.9	149.5	149.2	
	0.800	$\rightarrow \infty$	0.00	31.8	31.4	-0.3	-4.2	83.5	84.2	Transonic
			2.00	150.7	145.8	-6.4	-10.3	424.3	424.9	
		9.0×10^6	0.00	18.4	16.8	-3.5	-8.5	165.8	165.9	
			3.00	87.2	80.8	-10.3	-17.3	519.1	518.8	
	0.900	9.0×10^6	4.70	527.1	472.3	-19.4	-27.0	1582.4	1579.9	
RAE2822	0.725	6.5×10^6	2.65	5.2	3.5	1.0	-9.4	140.5	139.6	
	0.730	6.5×10^6	2.65	9.8	7.7	0.9	-10.1	160.2	158.8	

irreversible drag. The difference (represented by the non-zero induced drag) is particularly visible in low Reynolds number simulations, as already reported by Kang et al. [81], then here it is also observed at high Reynolds number in symmetric flow conditions ($\alpha = 0^\circ$), which revealed very stiff for the convergence of the numerical solution method, requiring a significantly larger number of solver iterations compared to other cases.

In transonic flow, as already observed on total drag results, formulations “a” and “b” are largely inaccurate. On the other hand, formulations corrected for shock region inaccuracies (“c”, “bc” and “d”) provide lower lift-induced drag coefficients, with nearly zero induced drag obtained using formulation “c” but only in the inviscid symmetric case ($\alpha = 0^\circ$). For the other transonic cases, formulation “c” still provides the smaller induced drag coefficients, but a non-zero induced drag is still computed (negative in sign for the symmetric NACA0012 aerofoil and positive for the cambered RAE2822). It can be neglected at Mach number close to 0.7 (where it remains in the limit of one *d.c.*), but it becomes significant in magnitude at higher free-stream Mach numbers. Consequently, parasite drag is higher than total drag for the other analysed transonic flow-fields even when using formulation “c”, which generally provided the best results as discussed above.

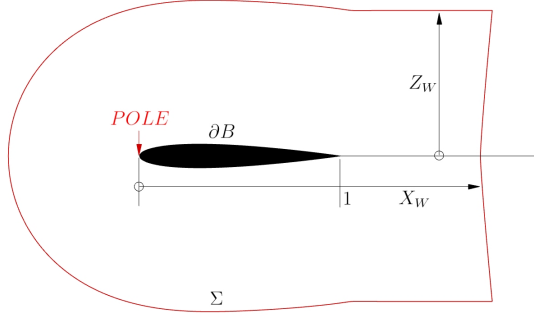


Figure 2.2.5: Integration domain size schematic.

2.2.3.2 SENSITIVITY TO THE INTEGRATION DOMAIN SIZE

The aerodynamic force computation using far-field methods relies on the selection of a fluidic control volume for the application of the integral momentum balance. Therefore, its definition represents a degree of freedom which, however, has theoretically no impact on the total force definition, supposed it fully surrounds the body. In the previous paragraph, force coefficients were reported using a control volume external boundary (Σ) approximately distant 15 chord lengths from the aerofoil surface. This choice is justified by the sensitivity analysis included in Appendix B.1, where results of total force far-field computations (lift and drag components) using different Lamb-vector-based formulations are illustrated for a variable-dimension control volume, identified by three grid lines of the computational mesh, as indicated in Figure 2.2.5 (where the two characteristic lengths used to define its size are also represented).

The downstream portion of Σ (wake plane) has chordwise distance from the aerofoil leading edge X_W , while the lateral distance of the C-shaped Σ edge from the aerofoil will be denoted as Z_W . Both X_W and Z_W will be expressed in chord units and, for the 3D wing, are measured on the symmetry plane. The DMT pole (Fig. 1.1.1) is located at the aerofoil leading edge (Fig. 2.2.5). It's worth observing that for $0 < X_W < 1$ the integration domain does not fully surrounds the aerofoil and the resulting computed force represents, in this case, the force exerted on the body portion effectively laying on the frontier of the selected CV§. In Appendix B.1, variations of the individual force constituents (vortex force, compressibility corrections, etc.) as a function of the integration domain size are also illustrated. Moreover, domain size sensitivity is also reported for the lift-induced and parasite drag coefficients, computed from far-field vortical methods based on the decompositions summarised in Section 2.1.7.

As an example of the complete sensitivity analysis performed during the present study (involv-

§In such situations, however, this does not represent the aerodynamic force because, even assuming negligible gravitational effects, the aerostatic force from the uniform pressure field still has to be subtracted and does not vanish as for a closed body surface.

ing the different simulated flow regimes listed in Table 2.2.1), Figure 2.2.6 and Figure 2.2.7 illustrate the variation of the total force (computed using formulation “c”) with the control volume size. The computed force difference (far-field minus near-field) is plotted against the integration domain size, which dimension is modified in this case varying both X_W and Z_W simultaneously (i.e. $X_W = Z_W$) and $X_W - 1$ (i.e. the relative Σ distance from trailing edge) is represented on the abscissa, using a logarithmic scale.

In all analysed cases, the total force convergence is achieved a few chord lengths downstream of the aerofoil, with non-negligible differences with regards to the near-field drag only observed at low Reynolds number (Fig. 2.2.7), as highlighted also in Section 2.2.3.1. It is anticipated that these, as well as other observed discrepancies, will be explained and corrected by the new unified theory illustrated and discussed in Ch. 4.

Results reported above, together with those included in Appendix B.1, illustrate that the control volume should extend at least from 1 to 5 chord lengths (depending on the analysed flow-field) around the aerofoil in order to reach convergence of computed far-field total lift and drag on constant values. Larger integration domains are usually avoided to prevent spurious effects from polluting drag decomposition results. As a matter of fact, moving the wake plane downstream of the aerofoil near wake, a conversion of reversible into irreversible drag is usually observed (especially in three-dimensional flows), mainly due to the the artificial viscosity of the numerical scheme, dissipating kinetic energy of trailing vortices into internal energy and resulting in a spurious entropy production, which is also active in inviscid flow simulations. Indeed, to allow for feasible computational costs of numerical simulations, the grid has typically inadequate resolution downstream of the near body wake and the mesh size rapidly grows together with the local artificial dissipation.

2.2.3.3 GRID CONVERGENCE

At the scope of investigating numerical errors impact on the observed behaviour of lift-induced and parasite drag coefficients (Table 2.2.6), a grid convergence analysis is performed on the NACA 0012 aerofoil for the inviscid transonic case at non-zero incidence angle ($\alpha = 2^\circ$). Two additional grids were generated, starting from the reference (fine) mesh and agglomerating two-by-two adjacent cells along each block direction (medium mesh: 41400 cells) or halving original grid cells by placing intermediate nodes along each quadrilateral edge (super-fine mesh: 662400 cells). Figure 2.2.8 illustrates grid convergence of lift-induced and parasite drag coefficients, where the generic mesh size on the abscissa is referred to the fine grid. Parasite drag is converging towards a value higher than total (wave) drag and lift-induced drag coefficient is apparently approaching a negative value as the grid is refined. This behaviour is different from that investigated by Destarac [39] in his grid convergence studies for two-dimensional Euler solutions. It cannot be explained by the same hypothesis formulated in [39] and related to near-wall grid cells with unitary aspect ratio because, in this case, the mesh is characterised by the usual local grid refinement normal to the wall, with highly stretched cells.

Moreover, the negligible difference in near-field total drag coefficients obtained on the fine and

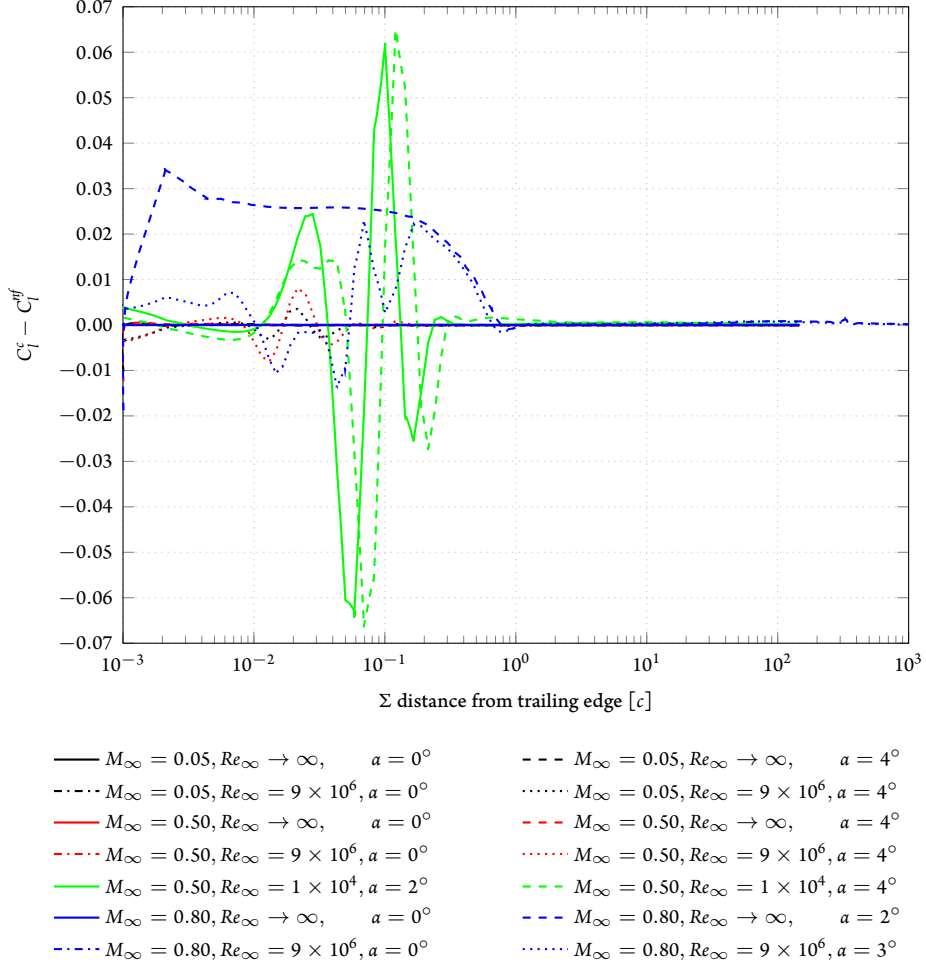


Figure 2.2.6: Far-field vs. near-field lift coefficient difference, as a function of the integration domain size, for different simulated flow conditions (NACA0012 aerofoil).

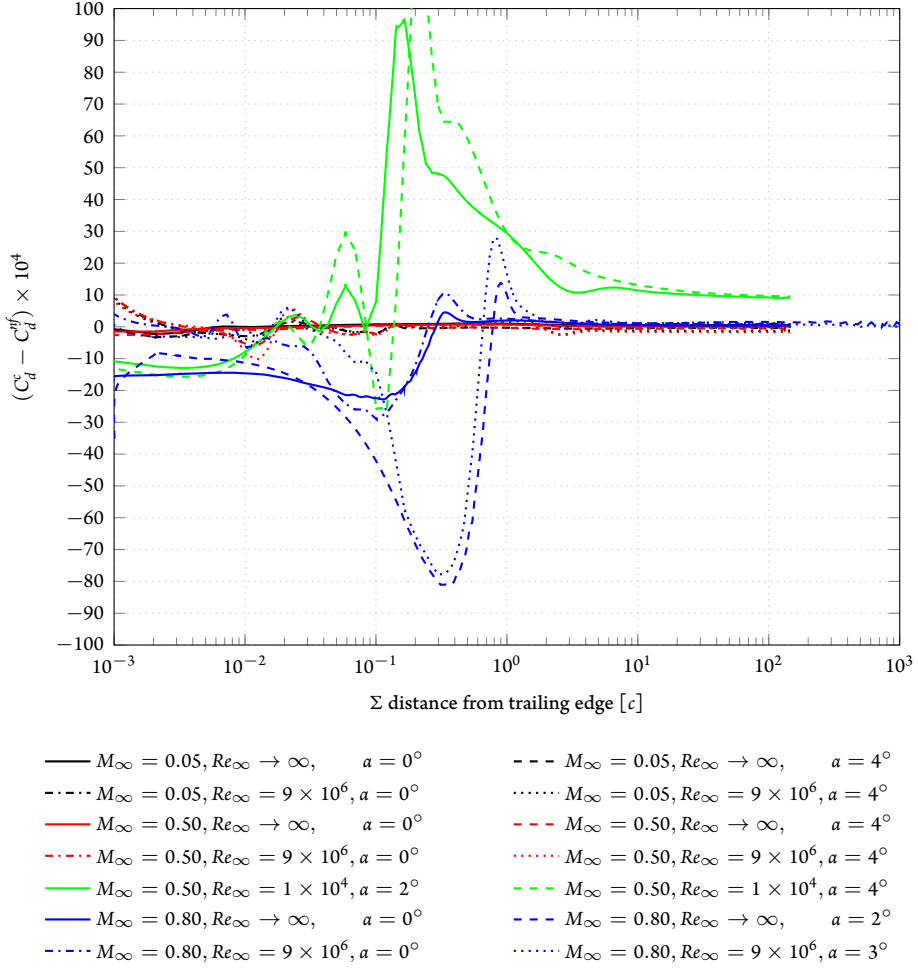
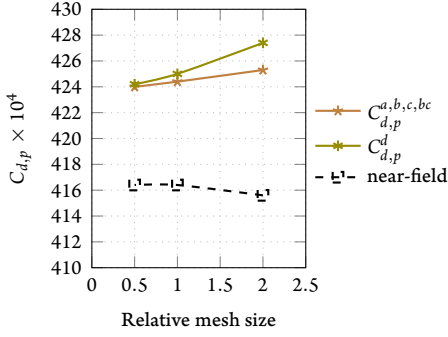
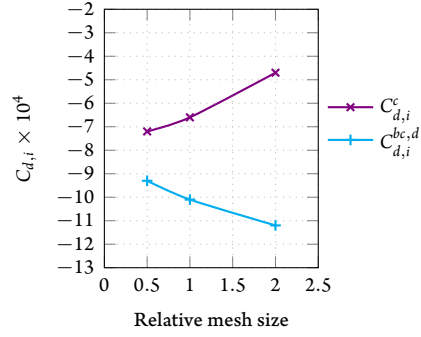


Figure 2.2.7: Far-field vs. near-field drag coefficient difference, as a function of the integration domain size, for different simulated flow conditions (NACA0012 aerofoil).

NACA0012 aerofoil - Weak Euler solution
($M_\infty = 0.8$; $\alpha = 2^\circ$)

(a)

NACA0012 aerofoil - Weak Euler solution
($M_\infty = 0.8$; $\alpha = 2^\circ$)

(b)

Figure 2.2.8: Grid convergence of parasite (Fig. 2.2.8a) and lift-induced (Fig. 2.2.8b) aerodynamic force coefficients, for different Lamb-vector-based formulations ($X_W \approx 16c$; $Z_W \approx 15c$; pole at aerofoil l.e.).

Table 2.2.7: Effect of far-field distance on total lift coefficient values, NACA0012 aerofoil ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

M_∞	Re_∞	α [$^\circ$]	$d^{FF}[c]$	C_l^{nf}	C_l^a	C_l^b	C_l^c	C_l^{bc}	C_l^d
0.80	$\rightarrow \infty$	2.0	300	0.557	0.557	0.549	0.558	0.551	0.551
0.80	$\rightarrow \infty$	2.0	3000	0.558	0.558	0.551	0.559	0.552	0.552
0.80	9×10^6	3.0	300	0.451	0.450	0.449	0.452	0.451	0.451
0.80	9×10^6	3.0	3000	0.451	0.449	0.449	0.452	0.451	0.450

super-fine meshes reveals that the spurious drag in the reference grid solution is negligible as well (a similar situation was observed for all computed 2D solutions).

The impact of grid far-field boundary distance on the observed behaviour has also been investigated. Indeed, a spurious far-field-boundary induced drag (independent from the grid refinement but strongly dependent on the size of the computational domain) was also identified by Destarac [40] analysing two-dimensional Euler solutions, then its potential impact on the computed results has to be assessed. At the scope, the reference fine mesh was modified enlarging the computational domain size by a factor 10. Additional cell layers were included in the original mesh to cover the region between the initial far-field boundary location ($\approx 300c$ away from the aerofoil surface) up to the extended domain boundary ($\approx 3000c$). The analysis was done on the NACA0012 aerofoil for both inviscid and viscous transonic cases (at $M_\infty = 0.8$ and non-zero incidence) where non-zero lift-induced drag coefficients were observed and results are included in Table 2.2.7, 2.2.8 and 2.2.9 (the approximate distance of the domain far-field boundary is indicated as d^{FF}).

No direct impact can be observed on negative induced drag coefficients, which indeed get slightly larger in magnitude as the computational domain size is increased, due to the larger lift

Table 2.2.8: Effect of far-field distance on total drag counts, NACA0012 aerofoil ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

M_∞	Re_∞	$\alpha [^\circ]$	$d^{FF}[c]$	$10^4 \times$					
				C_d^{nf}	C_d^a	C_d^b	C_d^c	C_d^{bc}	C_d^d
0.80	$\rightarrow \infty$	2.0	300	416.6	574.9	570.0	417.9	414.0	414.6
0.80	$\rightarrow \infty$	2.0	3000	417.8	576.3	571.8	418.9	414.9	415.6
0.80	9×10^6	3.0	300	507.6	606.3	599.9	508.8	501.8	501.6
0.80	9×10^6	3.0	3000	507.8	606.8	600.4	508.9	501.8	501.5

Table 2.2.9: Effect of far-field distance on lift-induced and parasite drag counts, NACA0012 aerofoil ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

M_∞	Re_∞	$\alpha [^\circ]$	$d^{FF}[c]$	$10^4 \times$					
				$C_{d,i}^a$	$C_{d,i}^b$	$C_{d,i}^c$	$C_{d,i}^{bc,d}$	$C_{d,p}^{a,b,c,bc}$	$C_{d,p}^d$
0.80	$\rightarrow \infty$	2.0	300	150.7	145.8	-6.4	-10.3	424.3	424.9
0.80	$\rightarrow \infty$	2.0	3000	150.7	146.1	-6.7	-10.7	425.6	426.3
0.80	9×10^6	3.0	300	87.2	80.8	-10.3	-17.3	519.1	518.8
0.80	9×10^6	3.0	3000	87.4	81.0	-10.5	-17.6	519.4	519.1

obtained as free-stream conditions are imposed far-away from the aerofoil surface (the flow solver has no *point-vortex* correction, i.e. it is not imposed, at the far-field, the same circulation as obtained on the aerofoil).

2.2.3.4 SENSITIVITY TO THE POLE LOCATION

The aerodynamic force (both lift and drag) is independent from the choice of the pole used in Lamb-vector-based formulations. Moreover, as highlighted in Sec. 2.1.7, the drag decomposition provided by Eq. (2.27), (2.28) is also independent from the selected pole. However, due to numerical errors, different Lamb-vector-based formulations may exhibit a sensitivity of both the total force coefficients and of the drag decomposition to the pole location. Hereafter, these effects are highlighted and discussed with reference to computed cases, illustrated in Table 2.2.1.

At zero incidence angle (symmetric flow fields), no variation with the pole location was observed, therefore the sensitivity analysis will only report results for non-zero angles of attack.

It is worth observing that, in all Lamb-vector-based formulations, the (scaled) position vector only appears in double cross products and is measured from the chosen pole. Therefore, if position vectors are measured from the origin of a selected reference frame (different, in general, from the chosen pole), any integral involving position vectors can be written as:

$$\int (\underline{r} - \underline{r}_p) \times (\underline{V}_1 \times \underline{V}_2) = \int \underline{r} \times (\underline{V}_1 \times \underline{V}_2) - \underline{r}_p \times \int \underline{V}_1 \times \underline{V}_2 \quad (2.40)$$

where $\underline{r}_p$ is the position of the chosen pole in the selected reference frame and $\underline{V}_1$ and $\underline{V}_2$ are generic

vector fields.

In two dimensions, both $\underline{V}_1$ and $\underline{V}_2$ vectors appearing in Lamb-vector-based formula are contained in the plane of the flow, therefore their cross product (and its integral) is orthogonal to the plane. As a consequence, its momentum around the Z-axis (lift force) is only influenced by the X-coordinate of the pole and, analogously, drag is only influenced by pole Z-coordinate.

The complete diagrams analysing sensitivities to pole location are included in Appendix B.2. In all analysed cases, formulations “c” and “a” (mostly based on surface integrals) exhibited negligible sensitivity of the total force (lift and drag) with regards to the pole location, while unacceptable variations were observed for the other formulations. In particular, formulation “d” (using mainly volume integrals) provided the worst results with large inaccuracies on drag even for a pole shift of a few chord lengths ($\approx 200d.c.$ drag variation for a pole shift of 3 chord lengths in the Z-direction).

Pole independence of the drag decomposition in lift-induced and parasite contributions was also numerically verified for the most accurate formulations: “c” and “a”. Concerning lift-induced drag only, it has to be mentioned that this result was expected for the two highlighted formulations, being in this case the D_i expressions (Eq. (2.32) and (2.36)) mathematically independent of the DMT pole (identical terms containing $\underline{r}$ in $\underline{F}_{\underline{\nabla}_p}$ and $\underline{F}_p$ or $\underline{F}_{\underline{\ell}+m_p}$ cancel-out).

It is also observed that choosing a pole position near the body surface ($0 \leq X_p \leq 1$ and $Z_p \approx 0$) allows in any case to obtain values of the total force and of the drag decomposition in good agreement to those computed using pole-independent formulas.

At the scope of investigating numerical errors impact on the illustrated sensitivities to pole position variations, diagrams of total lift and drag variations for the NACA0012 inviscid transonic case at $AoA = 2^\circ$ are proposed in Figure 2.2.9 for two different grids: the reference fine mesh (identified with “L3” in the figure) and the refined grid already used for the grid convergence analysis in Section 2.2.3.3 (“L4” in the figure). A consistent reduction of the highlighted sensitivities is observed for values computed using all formulations, suggesting an amplification of local truncation errors by the local $\underline{r}$ vector magnitude.

Another numerical investigation was performed on the NACA0012 viscous transonic flow-field at $M_\infty = 0.8$ and $AoA = 3^\circ$, changing the gradient reconstruction method from the baseline nodes-based Green-Gauss formula (G-G), used in all reported applications, to a Weighted Least Squares method (WLS) [182, 190]. Formulations “c” and “a” exhibited negligible total force sensitivities to pole variations with both gradient methods, while no clear trend with regards to the gradient scheme modification was identified for the observed sensitivities of other force formulations.

2.2.3.5 REVERSIBLE DRAG, IN 2D

For two-dimensional flows, where no trailing vortices are realised, the expected reversible drag is zero and the total drag is entirely due to irreversible flow phenomena (viscosity or supersonic compression) increasing fluid entropy. Figure 2.2.10 illustrates the difference between the total

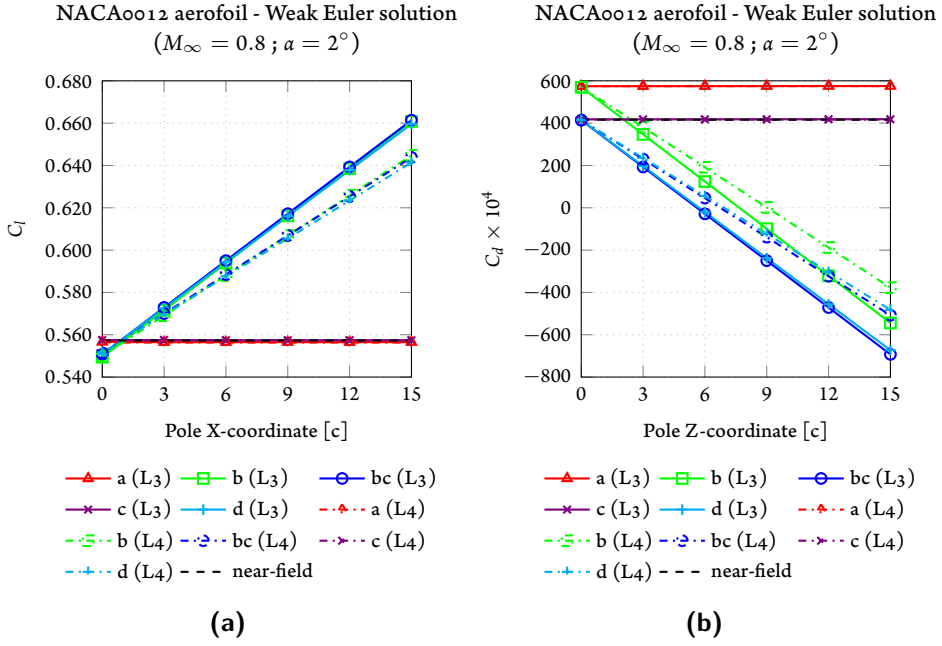


Figure 2.2.9: Effect of mesh refinement on sensitivity of total aerodynamic force coefficients to pole location, for different Lamb-vector-based formulations ($X_W \approx 16c$; $Z_W \approx 15c$; L3: fine mesh; L4: super-fine mesh).

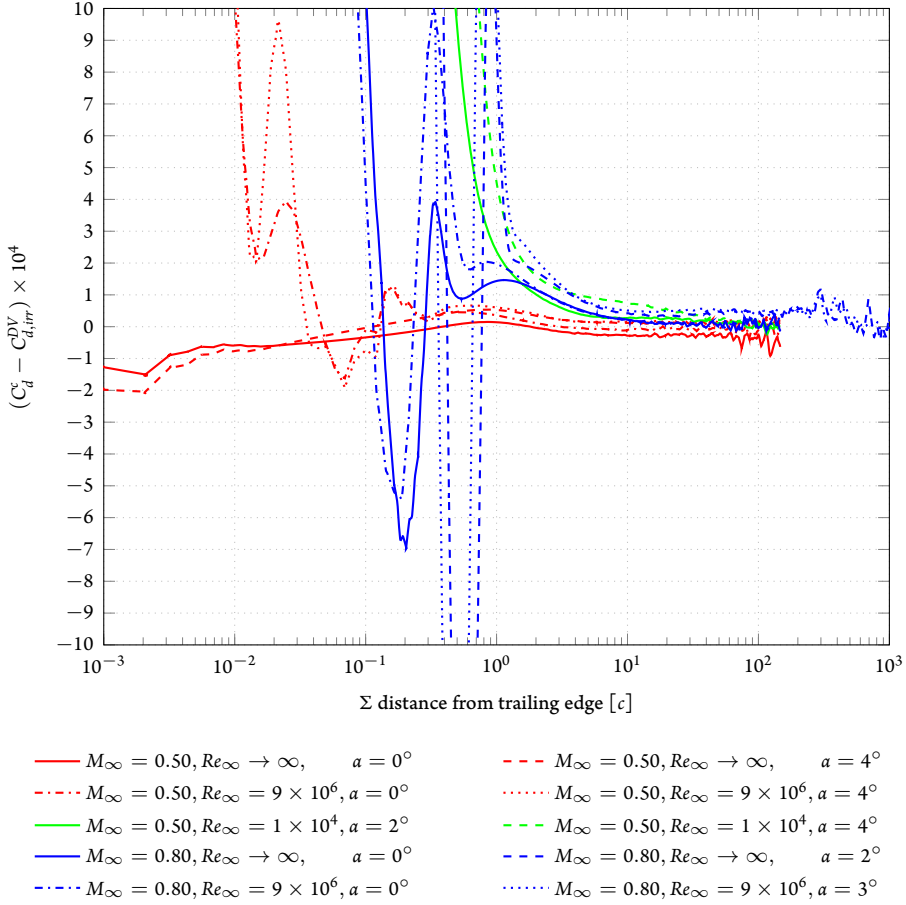


Figure 2.2.10: Irreversible vs. total far-field drag counts difference as a function of the integration domain size, for different simulated flow conditions (NACA0012 aerofoil).

far-field drag (computed from the vortical formula in Eq. (2.15)) and the irreversible drag (computed using the method of Destarac and van der Vooren [42]): an indirect reversible drag measure.

It is clear how they converge onto each other within one *d.c.* accuracy in the same range where far-field drag was observed to stabilise (after three or four chord lengths). This numerically confirms the irreversible nature of total drag for the analysed two-dimensional flows.

2.2.3.6 IDENTIFICATION OF NON-ZERO LIFT-INDUCED DRAG, IN 2D

The consistency of total drag decomposition in lift-induced and parasite contributions with the same drag breakdown in reversible and irreversible parts, theoretically discussed in Section 2.1.8, has a numerical evidence in the results illustrated in Fig. 2.2.11, where lift-induced drag coefficients (computed using Eq. (2.36)) are plotted as a function of the integration domain size. It is

clearly visible that, in most of simulated cases (black curves), lift-induced drag rapidly approaches zero and is negligible ($-1d.c. < C_{d,i}^c < +1d.c.$) for a Σ distance ≥ 5 aerofoil chords. On the other hand, in stiff numerical simulations (blue curves), transonic asymmetric cases (red curves) or low Reynolds number conditions (green curves), the convergence of lift-induced drag to the (null) reversible drag is much slower, and significant negative values of such a *spurious* lift-induced drag are still obtained for a control volume size in the order of 10 or 100 chords. In particular, for the two transonic cases with non-zero incidence, data included in Fig. 2.2.11 are related to an extended computational domain with far-field boundary at $\approx 3000c$ instead of $\approx 300c$, to highlight how zero lift-induced drag is indeed achieved in these cases on very large control volumes (Σ distance $\approx 1000c$). At low Reynolds number conditions, results are in line with what was already observed by Kang et al. [81]. They used formulation “b” of lift-induced drag (without the additional compressibility correction $\underline{E}_{\nabla p}$, which is however negligible in this case, although here not illustrated) with a wake plane location between 2 and 3 chord lengths downstream of the aerofoil trailing edge and observed negative induced drag coefficients in the order of approximately $-100d.c.$ at $Re_\infty = 1 \times 10^4$ ([81], Fig. 21, pag. 4636). This value however provides a rough estimate only, as they simulated subsonic flow conditions at $M_\infty = 0.3$ (instead of $M_\infty = 0.5$) and $Re_\infty = 1 \times 10^3$, $Re_\infty = 1 \times 10^6$ and $Re_\infty = 1 \times 10^7$ (i.e. value at $Re_\infty = 1 \times 10^4$ is interpolated).

The decomposition in lift-induced and parasite contributions is therefore not independent on the selected control volume in the highlighted cases, even in absence of trailing vorticity (2D flows) and consistency with total drag breakdown in reversible and irreversible parts (which, instead, stabilise quite quickly downstream of the aerofoil near wake) is only achieved on very large integration domains.

RELATIONSHIP WITH THE AXIAL VELOCITY PERTURBATION Figure 2.2.12 also illustrates, for some of the analysed cases, how the different rates of convergence for lift-induced drag observed in Fig. 2.2.11 correspond to different decay rates of the streamwise velocity perturbation (u).

The above argumentation suggests that a drag breakdown in lift-induced and parasite contributions capturing respectively reversible and irreversible drag components cannot be obtained with an arbitrary choice of the external integration domain surface (Σ), that should be placed far away from the body surface in order to effectively decompose total drag in reversible and irreversible parts. Unfortunately, in numerical applications, the integration domain cannot be extended too far downstream of the body, where a spurious conversion of reversible into irreversible drag also occurs, as discussed in Sec. 2.2.3.2. This limits the accuracy of drag decompositions in Lamb-vector-based methods.

2.2.3.7 NEW METHODOLOGY FOR PARASITE DRAG DECOMPOSITION

Accepting the uncertainty mentioned above, a decomposition of parasite drag in transonic flow, isolating the viscous contribution, is hereafter proposed and described.

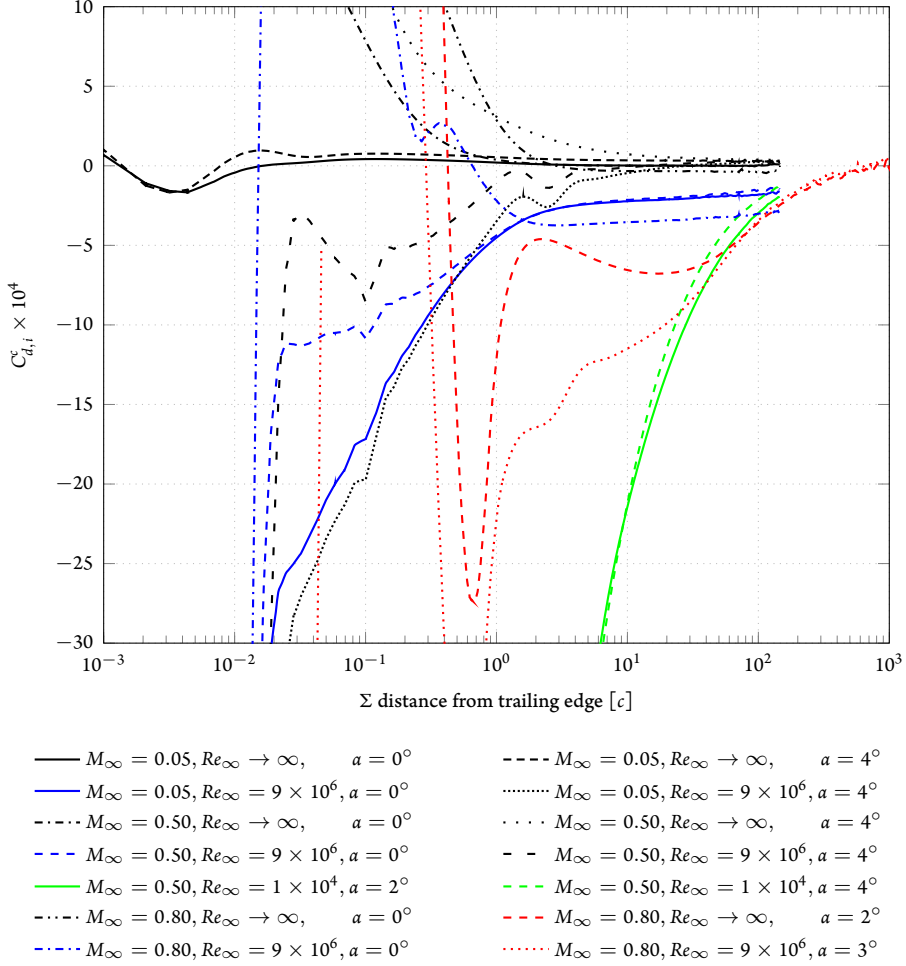


Figure 2.2.11: Computed lift-induced drag coefficient as a function of the integration domain size, for all simulated 2D flow conditions.

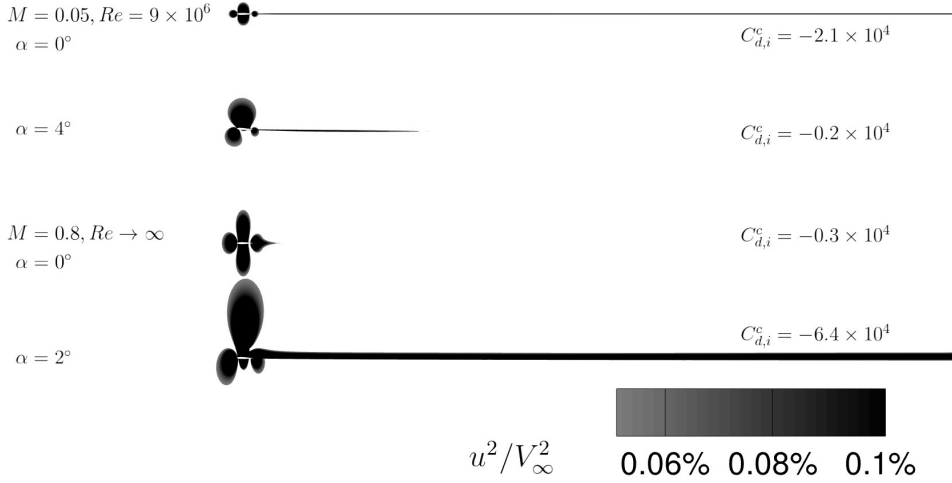


Figure 2.2.12: Contour plot of non-dimensional streamwise velocity perturbation (squared) in nearly-incompressible viscous flow solutions and inviscid transonic flow solutions ($C_{d,i}^c$ computed with Σ distance ≈ 15 aerofoil chords).

As described in Sec. 1.4.3.2, such a breakdown is not straightforward, due to the entrainment in the viscous boundary layer of the flow past shock waves, and previous attempts to correctly separate the two drag contributions [54, 141] revealed not truly successful. In the present approach, flow entrainment is explicitly taken into account and, once the viscous contribution has been isolated, its complement to the total parasite drag can then be attributed to wave drag, in the hypothesis of neglecting spurious drag (expected to be limited to a few drag counts, on sufficiently refined grids). The proposed method provides therefore an *indirect* definition of wave drag. As reported in Section 2.1.7, the definition of parasite drag in far-field Lamb-vector-based force formulations proposed in [55] is made up of three different terms and is the same in formulations “a”, “b”, “c” and “bc”:

$$D_p^{a,b,c,bc} = \underline{e}_x \cdot \left(\underline{F}_S + \underline{F}_\mu + \underline{F}_{\nabla\rho} \right) \quad (2.41)$$

The second term ($\underline{F}_\mu$), being explicitly dependent on the fluid viscosity, is totally attributed to the viscous drag, while for the other two terms a decomposition in viscous and wave contributions is required (eventually, the $\underline{F}_{\nabla\rho}$ term could also be ignored in Eq. (2.41), because it vanishes quite quickly as the control volume surrounding the body grows in size, as evidenced in Appendix B.1).

After having identified, in a computed flow-field, the boundary layer and viscous wake regions (using, for example, appropriate solution sensors [141, 148]), the decomposition can be accomplished splitting either surface or volume integrals, depending on the selected formulation for the individual terms to be decomposed. For example, in Eq. (2.41), vortex forces circulation moment ($\underline{F}_S$) and the compressibility correction term ($\underline{F}_{\nabla\rho}$) are both expressed as surface integrals over the control volume far-field boundary. This boundary can be decomposed in two parts, as illustrated in Figure 2.2.13, isolating the external surface of the viscous domain ($\mathcal{V}_{\text{viscous}}$) and the frontier of

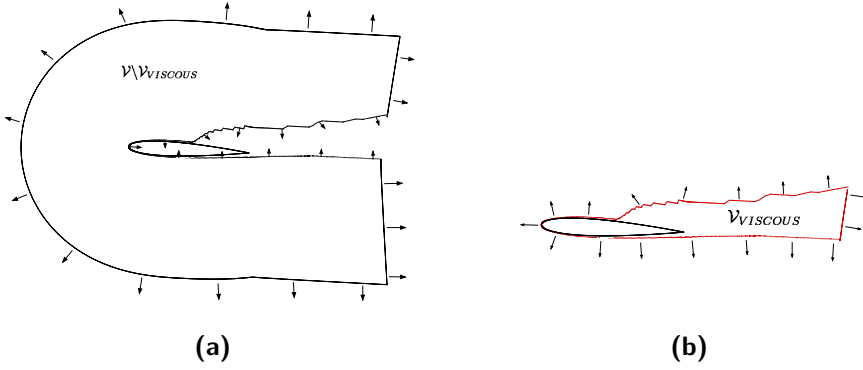


Figure 2.2.13: Example of domain decomposition and outward-pointing normal vectors: viscous domain (Fig. 2.2.13b) and complement to viscous domain (wave drag region, Fig. 2.2.13a).

the remaining part of the control volume ($\mathcal{V} \setminus \mathcal{V}_{VISCOUS}$), being surface integral contributions on the two common interface surfaces exactly equal and opposite in sign, using outward-pointing normal vectors.

The proposed decomposition in viscous and wave drag contributions can be therefore initially defined by:

$$D_{p,VISCOUS}^{a,b,c,bc} = \underline{e}_x \cdot \left[\underline{F}_\mu + \left(\underline{F}_S + \underline{F}_{\nabla p} \right) \Big|_{\partial \mathcal{V}_{VISCOUS}} \right] \quad (2.42)$$

$$D_{p,WAVE}^{a,b,c,bc} = D_p^{a,b,c,bc} - D_{p,VISCOUS}^{a,b,c,bc} \quad (2.43)$$

where $\Big|_{\partial \mathcal{V}_{VISCOUS}}$ indicates that surface integration is restricted to the frontier of the volume portion $\mathcal{V}_{VISCOUS}$, instead of the complete control volume $\mathcal{V}$.

Another option is to use volume integral formulations and splitting integrals in the two volumetric partitions of the integration domain ($\mathcal{V}_{VISCOUS}$ and $\mathcal{V} \setminus \mathcal{V}_{VISCOUS}$). A formula based on volume integrals for the vortex forces circulation moment has already been proposed in [141] (Eq. (2.20)), while a similar formula for the compressibility correction $\underline{F}_{\nabla p}$ is hereafter derived. For the first term on the RHS of Eq. (2.29), an application of Gauss's theorem gives:

$$\frac{V_\infty^2}{2} \int_\Sigma (\rho - \rho_\infty) \underline{n} \, dS = \frac{V_\infty^2}{2} \int_{\mathcal{V}} \nabla \rho \, d\mathcal{V} - \frac{V_\infty^2}{2} \int_{\partial B} (\rho - \rho_\infty) \underline{n} \, dS \quad (2.44)$$

while, for the second term of Eq. (2.29) RHS, using the double cross product identity:

$$\underline{r} \times \left[\underline{n} \times \left(\frac{V^2}{2} \nabla \rho \right) \right] = \left(\frac{V^2}{2} \nabla \rho \cdot \underline{r} \right) \underline{n} - (\underline{n} \cdot \underline{r}) \frac{V^2}{2} \nabla \rho \quad (2.45)$$

Gauss's theorem can again be applied, leading to the equivalent formula for the compressibility

correction $\underline{F}_{\underline{\nabla}\rho}$:

$$\begin{aligned} \underline{F}_{\underline{\nabla}\rho} = & -\frac{V_\infty^2}{2} \int_{\partial B} (\rho - \rho_\infty) \underline{n} \, dS + \\ & + \frac{V_\infty^2}{2} \int_{\mathcal{V}} \underline{\nabla}\rho \, d\mathcal{V} + \int_{\mathcal{V}} \underline{\nabla} \left(\frac{V^2}{2} \underline{\nabla}\rho \cdot \underline{r} \right) d\mathcal{V} - \int_{\mathcal{V}} \underline{\nabla} \cdot \left(\underline{r} \frac{V^2}{2} \underline{\nabla}\rho \right) d\mathcal{V} = \underline{F}_{\underline{\nabla}\rho}^{\partial B} + \underline{F}_{\underline{\nabla}\rho}^{\mathcal{V}} \end{aligned} \quad (2.46)$$

where the sum of last three volume integrals in Eq. (2.46) has been indicated with $\underline{F}_{\underline{\nabla}\rho}^{\mathcal{V}}$ and the integral over body boundary ($\underline{F}_{\underline{\nabla}\rho}^{\partial B}$) has to be attributed to viscous drag. A parasite drag formulation based on volume integrals can therefore be written as (the superscript “e” will be used in this case):

$$D_p^e = \epsilon_x \cdot \left(\underline{F}_{S\mathcal{V}} + \underline{F}_\mu + \underline{F}_{\underline{\nabla}\rho}^{\mathcal{V}} + \underline{F}_{\underline{\nabla}\rho}^{\partial B} \right) \quad (2.47)$$

and hybrid formulations (using $\underline{F}_{S\mathcal{V}} + \underline{F}_{\underline{\nabla}\rho}$ or $\underline{F}_S + \underline{F}_{\underline{\nabla}\rho}^{\mathcal{V}} + \underline{F}_{\underline{\nabla}\rho}^{\partial B}$) can also be obtained (the first one has already been used in Section 2.2.3.2 and appointed as D_p^d). Using the expression in Eq. (2.47) to define parasite drag, the initial decomposition in viscous and wave drag contributions is therefore given by:

$$D_{p,VISCOUS}^e = \epsilon_x \cdot \left[\underline{F}_\mu + \left(\underline{F}_{S\mathcal{V}} + \underline{F}_{\underline{\nabla}\rho}^{\mathcal{V}} \right) \Big|_{\mathcal{V}_{VISCOUS}} + \underline{F}_{\underline{\nabla}\rho}^{\partial B} \right] \quad (2.48)$$

$$D_{p,WAVE}^e = D_p^e - D_{p,VISCOUS}^e \quad (2.49)$$

where $\Big|_{\mathcal{V}_{VISCOUS}}$ indicates that volume integration is restricted to the portion $\mathcal{V}_{VISCOUS}$ of the complete control volume $\mathcal{V}$.

Now, the parasite drag formulation based on volume integrals (Eq. (2.47)) can be used to visualise local contributions of individual grid cells to parasite drag. Eventually, the same can be done for the induced drag, using formulations of the compressibility corrections based on volume integrals ($\underline{F}_{\underline{m}_p}$ and $\underline{F}_{\underline{\nabla}\rho}^{\mathcal{V}}$):

$$D_i^e = \epsilon_x \cdot \left(\underline{F}_\ell + \underline{F}_{\underline{m}_p} - \underline{F}_{\underline{\nabla}\rho}^{\mathcal{V}} - \underline{F}_{\underline{\nabla}\rho}^{\partial B} \right) \quad (2.50)$$

while Eq. (2.19)(x-component) can be used to visualise total drag.

Figure 2.2.14a provides a visualisation of the quantities under volume integral sign in (2.47), multiplied by the cell volume and in non-dimensional form. Therefore, the visualised quantity represents the elementary parasite drag coefficient of individual grid cells in the fluid domain, apart from the contribution of terms $\underline{F}_{\underline{\nabla}\rho}^{\partial B}$ and $\underline{F}_\mu$. A similar visualisation is provided in Figure 2.2.15 with reference to the elementary irreversible drag coefficient as computed using Destarac and van der Vooren thermodynamic method [42]. While the local irreversible drag is well confined to the viscous and shock wave regions, parasite drag computed using Lamb-vector-based formulation in Eq. (2.47) is not confined to rotational flow regions, mainly due to the compressibility correction term $\underline{F}_{\underline{\nabla}\rho}^{\mathcal{V}}$, as evidenced in Figure 2.2.14b where the $\underline{F}_{\underline{\nabla}\rho}^{\mathcal{V}}$ term was not considered. In the same

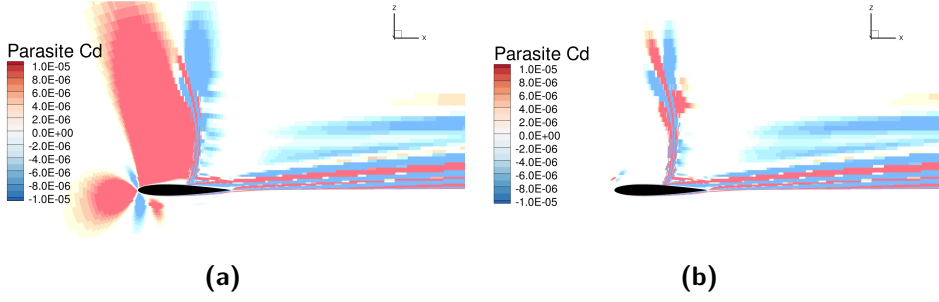


Figure 2.2.14: Volumetric contribution to parasite drag of individual grid cells, with the $\underline{F}_{\nabla\rho}^V$ term included (Fig. 2.2.14a) or not (Fig. 2.2.14b), $M_\infty = 0.8$, $Re_\infty = 9 \times 10^6$, $\alpha = 3^\circ$.

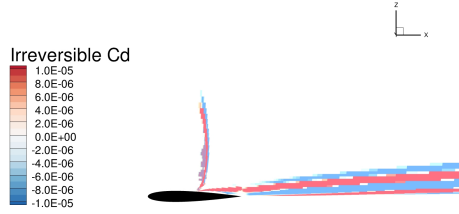


Figure 2.2.15: Volumetric contribution to the irreversible drag of individual grid cells ($M_\infty = 0.8$, $Re_\infty = 9 \times 10^6$, $\alpha = 3^\circ$).

figure it can also be appreciated how the shock-wave wake contributes to the drag production, which is not observed in thermodynamic methods (Figure 2.2.15).

The lack of the mid-field property highlighted for the $\underline{F}_{\nabla\rho}^V$ term means that, for example, in a viscous and subsonic flow-field, the drag production would not be confined to the boundary layer and viscous wake regions, therefore the (total) viscous drag could not be obtained limiting domain integrals to the viscous region, as defined by Eq. (2.48). For these reasons and because a decomposition of the $\underline{F}_{\nabla\rho}$ term is indeed not necessary (as recalled at the beginning of this paragraph), the proposed drag breakdown method can be sharpened and finally formulated as:

$$D_{p,VISCOUS}^{a,b,c,bc} = \underline{e}_x \cdot \left(\underline{F}_\mu + \underline{F}_S | \partial \mathcal{V}_{VISCOUS} \right) \quad (2.51)$$

$$D_{p,WAVE}^{a,b,c,bc} = \underline{F}_S - \underline{F}_S | \partial (\mathcal{V}_{VISCOUS}) \quad (2.52)$$

or, in volume integrals form:

$$D_{p,VISCOUS}^d = \underline{e}_x \cdot \left(\underline{F}_\mu + \underline{F}_{SV} | \mathcal{V}_{VISCOUS} \right) \quad (2.53)$$

$$D_{p,WAVE}^d = \underline{F}_{SV} - \underline{F}_{SV} | \mathcal{V}_{VISCOUS} \quad (2.54)$$

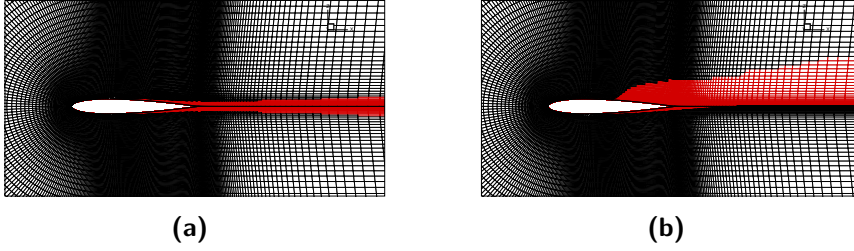


Figure 2.2.16: Domain decomposition in viscous transonic flow (NACA0012, $M_\infty = 0.8$; $Re_\infty = 9 \times 10^6$) at $\alpha = 0^\circ$ (Fig. 2.2.16a) and $\alpha = 3^\circ$ (Fig. 2.2.16b): grid cells identified in the viscous domain highlighted in red.

Hereafter, results of parasite drag decomposition are presented for the different analysed viscous transonic flow-fields (for both the NACA0012 and the RAE2822 aerofoils) using Eq. (2.51) and (2.52) based on surface integrals for $\underline{F}_S$. Formulations based on volume integrals (Eq. (2.53) and (2.54)) provided similar results.

Two different sensors, designed for fully turbulent flows, are used to select grid cells in the viscous drag region. A first one ($F_{bl} = (\mu + \mu_t) / \mu$) is based on eddy viscosity and assumes values significantly higher than 1 in the boundary layer and in the viscous wake. The other adopted sensor is simply turbulence specific dissipation rate (ω), which assumes very high values near the body surface and, in particular, in the viscous sub-layer, where the eddy viscosity vanishes. Cut-off values on both sensors are used to identify grid cells in $\mathcal{V}_{VISCOUS}$, which are highlighted in red in Figure 2.2.16. Due to the implementation of turbulent production terms in the adopted flow solver, the turbulent wake is however dissipated very soon past the body and, for this reason, the parasite drag decomposition was limited to 4 chord lengths downstream of the aerofoil. This implies a higher impact on the drag breakdown of the spurious non-zero induced drag highlighted in Section 2.2.3.2.

Results of *parasite* drag decomposition using the described Lamb-vector-based methodology are compared to those obtained using the well-assessed thermodynamic method by Paparone and Tognaccini [148] (Figure 2.2.17, 2.2.18 and 2.2.19), decomposing the *irreversible* drag, using the same control volume subdivision.

Figure 2.2.17, 2.2.18 and 2.2.19 show a good agreement of the Lamb-vector-based drag breakdown against thermodynamic-based estimates, with differences compatible with the highlighted parasite drag over-prediction, due to the spurious non-zero induced contribution. In the same figures, in fact, total drag values computed with the two methodologies are also compared against near-field drag. As observed also in Section 2.2.3.1 (Table 2.2.5 and 2.2.6), parasite drag as defined by Eq. (2.28) is higher than total aerofoil drag (negative induced drag cannot be ignored), while the thermodynamic method provides total irreversible drag values in very good agreement with near-field data. Compared to the thermodynamic method, wave drag computed using Lamb-vector-based formula takes a few chord lengths downstream the shock before reaching a constant

value, as expected being the Lamb vector field non-zero in the shock wake.

Finally, Tables from 2.2.10 to 2.2.15 provide a similar comparison, where results from the thermodynamic method by Destarac and van der Vooren [42] are also included, as well as available breakdown data from different literature articles. In particular, Table 2.2.11 includes data from Ostieri and Tognaccini [141] for the symmetric flow-field around the NACA0012 aerofoil at $M0.8$ and $Re_\infty = 9 \times 10^6$ in absence of flow incidence. The comparison provides reasonable results and it is interesting to note that a negative induced drag contribution (very similar in magnitude) was already observed and reported by the authors in [141], although using a different grid, flow solver and far-field force formulation. As mentioned in Ch. 1, this was the only viscous test-case where the methodology by Ostieri and Tognaccini was tested, while now the proposed drag breakdown method is also applied to an asymmetric flow-field at non-zero incidence angle (Table 2.2.12). In this case, the spurious induced-drag contribution is much higher and results in an over-prediction of both viscous and wave drag components.

For the other two transonic conditions ($M_\infty = 0.7$ and $M_\infty = 0.9$), breakdown data for the NACA0012 drag are also available from Mele et al. [117] (Table 2.2.10, 2.2.13). They used the formulation “bc” and the parasite drag breakdown was performed using a wake plane positioned just downstream of the shock wave. This was verified to result in an over-estimation of the wave drag by Ostieri and Tognaccini [141], which is here also confirmed from the plots included in Figure 2.2.18 (where formulation “c” was used).

At $M_\infty = 0.7$, drag breakdown data from two very recent studies are also available (Table 2.2.10). In particular, Saetta and Tognaccini [172] used a thermodynamic method [148], although the separation of viscous and wave drag was achieved using a different domain decomposition, based on a machine-learning approach instead of on deterministic solution sensors and associated cut-off values. Wave drag results are in acceptable agreement with the data obtained by the present method, although total and viscous drag values reported by the authors are higher, while wave drag is slightly lower (a coarser grid was used: 428 cells on the aerofoil surface for the present study, 256 cells in [172]).

The comparison with data from Hart and Schmitz [69] is particularly interesting, as they obtained the breakdown using a completely different approach, based on Partial Pressure Fields (PPFs) [178][177]. Nevertheless drag breakdown values are very close to those obtained using the proposed Lamb-vector-based approach. The same consideration can be done for the comparison of PPFs data [69] against drag breakdown values obtained by the present Lamb-vector method and by the reference thermodynamic methods for the RAE2822 aerofoil, at case 9 flow conditions [32] and $C_l = 0.803$ (Table 2.2.15).

A drag decomposition for the RAE2822 aerofoil was also obtained by Saetta and Tognaccini [172] for the flow conditions corresponding to case 6 [32] and $C_l = 0.796$. In these conditions, however, the wave drag reported in [172] is higher than the corresponding values obtained from the present simulations (Table 2.2.14), although the size of the two different grids is similar in this case. The observed discrepancies might be due to the different separation of the viscous and wave

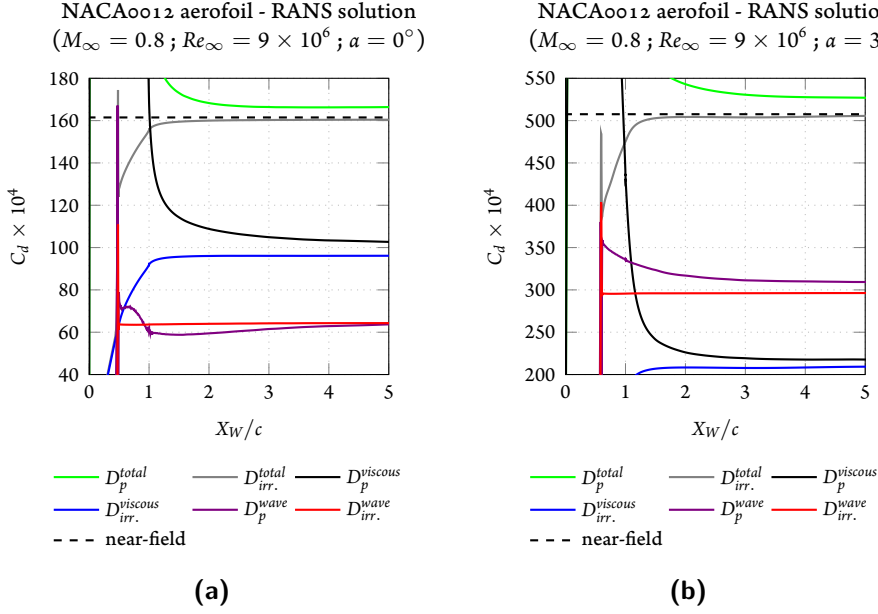


Figure 2.2.17: Decomposition in viscous and wave contributions for the parasite drag (Lamb-vector-based) and for the irreversible drag (thermodynamic-based), using the same integration domain subdivision ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

regions in the computed flow-field, achieved using either deterministic sensors (as in the present study) or a machine learning approach (as in [172]).

One may wonder why surface integrals decomposition has not been performed using the external control volume boundary only, i.e. according to Figure 2.2.20, as this would represent the natural way to split far-field integrals. However, this approach leads to a decomposition which highly depends on the wake plane longitudinal position (X_W , Figure 2.2.21), because part of the air flow crossing the shock wave is downstream entrained in the viscous region, as already observed by Mele et al. [117].

The integral contribution on the two (coincident) interior surfaces at the interface between

Table 2.2.10: Drag breakdown for the NACA0012 aerofoil at $M_\infty = 0.7$, $Re_\infty = 9 \times 10^6$, $C_l = 0.5$ ($Z_W \approx 15c$; $X_W \approx 5c$; fine mesh; pole at aerofoil l.e.).

	$C_d \times 10^4$			
	total	induced	viscous	wave
Lamb vector (formulation “c”)	148	-4	98	54
Thermodynamic (DV)	148		96	52
Thermodynamic (PT)	148		94	54
Mele et al. [117]	153	6	78	69
Hart and Schmitz [69]	148		96	52
Saetta and Tognaccini [172]	161		111	50

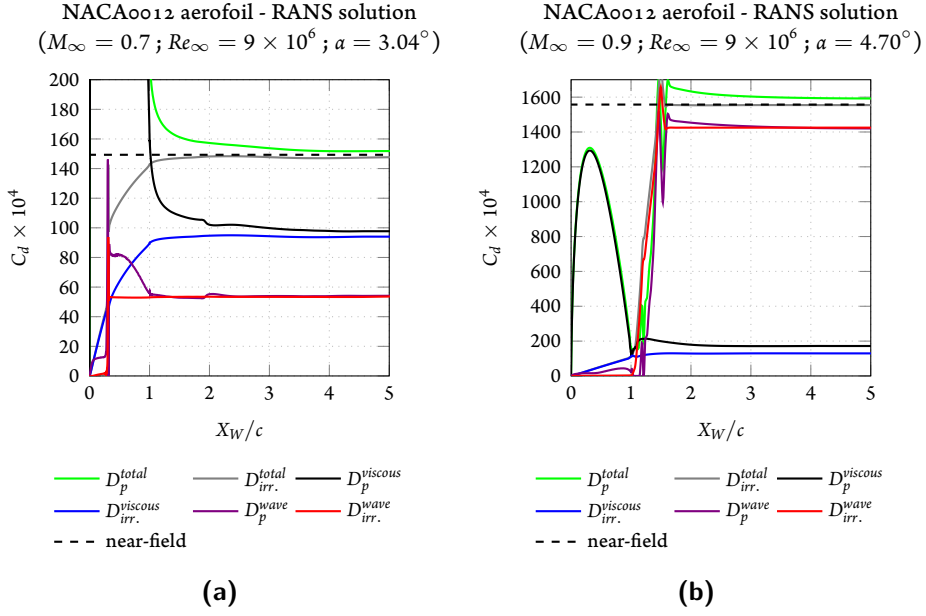


Figure 2.2.18: Decomposition in viscous and wave contributions for the parasite drag (Lamb-vector-based) and for the irreversible drag (thermodynamic-based), using the same integration domain subdivision ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

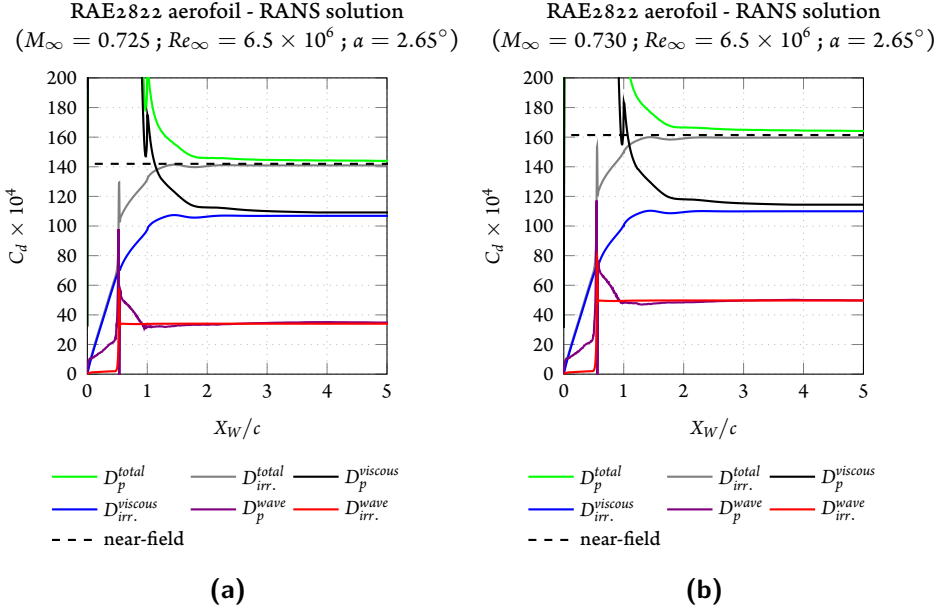


Figure 2.2.19: Decomposition in viscous and wave contributions for the parasite drag (Lamb-vector-based) and for the irreversible drag (thermodynamic-based), using the same integration domain subdivision ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

Table 2.2.11: Drag breakdown for the NACA0012 aerofoil at $M_\infty = 0.8$, $Re_\infty = 9 \times 10^6$, $\alpha = 0.0^\circ$ ($Z_W \approx 15c$; $X_W \approx 5c$; fine mesh; pole at aerofoil l.e.).

	$C_d \times 10^4$			
	total	induced	viscous	wave
Lamb vector (formulation “c”)	162	-5	103	64
Thermodynamic (DV)	162		99	63
Thermodynamic (PT)	160		96	64
Ostieri and Tognaccini [141]	157	-4	101	60

Table 2.2.12: Drag breakdown for the NACA0012 aerofoil at $M_\infty = 0.8$, $Re_\infty = 9 \times 10^6$, $\alpha = 3.0^\circ$ ($Z_W \approx 15c$; $X_W \approx 5c$; fine mesh; pole at aerofoil l.e.).

	$C_d \times 10^4$			
	total	induced	viscous	wave
Lamb vector (formulation “c”)	508	-19	218	309
Thermodynamic (DV)	508		215	293
Thermodynamic (PT)	505		209	296

Table 2.2.13: Drag breakdown for the NACA0012 aerofoil at $M_\infty = 0.9$, $Re_\infty = 9 \times 10^6$, $C_l = 0.5$ ($Z_W \approx 15c$; $X_W \approx 5c$; fine mesh; pole at aerofoil l.e.).

	$C_d \times 10^4$			
	total	induced	viscous	wave
Lamb vector (formulation “c”)	1561	-32	172	1421
Thermodynamic (DV)	1558		145	1413
Thermodynamic (PT)	1554		129	1425
Mele et al. [117]	1550	-29	153	1426

Table 2.2.14: Drag breakdown for the RAE2822 aerofoil at $M_\infty = 0.725$, $Re_\infty = 6.5 \times 10^6$, $C_l = 0.796$ ($Z_W \approx 15c$; $X_W \approx 5c$; fine mesh; pole at aerofoil l.e.).

	$C_d \times 10^4$			
	total	induced	viscous	wave
Lamb vector (formulation “c”)	142	-2	109	35
Thermodynamic (DV)	142		110	32
Thermodynamic (PT)	141		107	34
Saetta and Tognaccini [172]	145		92	53

Table 2.2.15: Drag breakdown for the RAE2822 aerofoil at $M_\infty = 0.730$, $Re_\infty = 6.5 \times 10^6$, $C_l = 0.803$ ($Z_W \approx 15c$; $X_W \approx 5c$; fine mesh; pole at aerofoil l.e.).

	$C_d \times 10^4$			
	total	induced	viscous	wave
Lamb vector (formulation “c”)	162	-2	114	50
Thermodynamic (DV)	161		115	46
Thermodynamic (PT)	160		111	49
Hart and Schmitz [69]	163		113	50

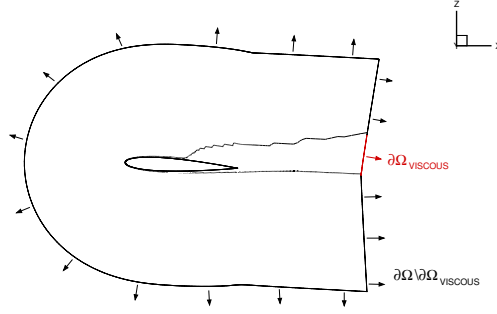


Figure 2.2.20: Example of decomposition using the far-field surface only.

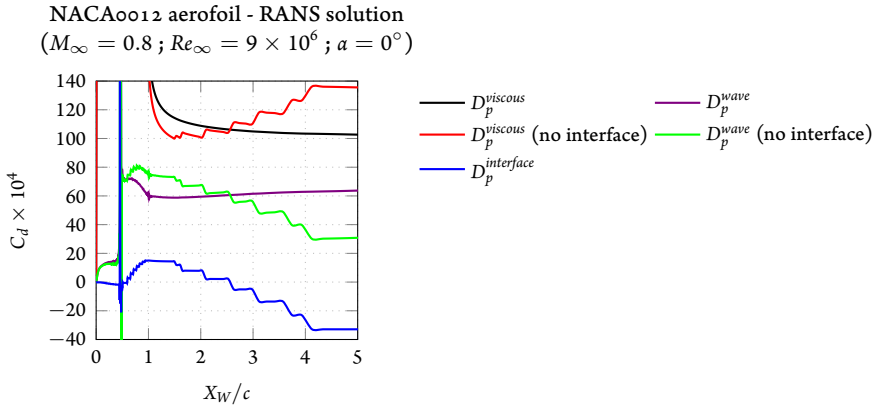


Figure 2.2.21: Parasite drag decomposition in viscous and wave contributions using surface integrals, with or without considering the viscous/wave regions interface boundary ($Z_W \approx 15c$; fine mesh; pole at aerofoil i.e.; interface integral computed on the viscous domain frontier).

$\mathcal{V}_{viscous}$ and $\mathcal{V} \setminus \mathcal{V}_{viscous}$ (Figure 2.2.13, dashed line in Figure 2.2.20) accounts for the flow entrainment. In fact, if the boundary layer edge was a streamline (i.e. no flow entrainment) this integral would be zero, being the Lamb vector orthogonal to flow streamlines, therefore with no circulation on such paths. Downstream of the aerofoil near wake ($X_W > 2.5$), flow entrainment is non-negligible, viscous drag estimated at the wake plane increases, taking over part of the wave drag, which in turns decreases by the same amount. This is quantified by the surface integral at the interface (Figure 2.2.21) which, if taken into account in parasite drag decomposition, can therefore eliminate the breakdown dependence on the wake plane position.

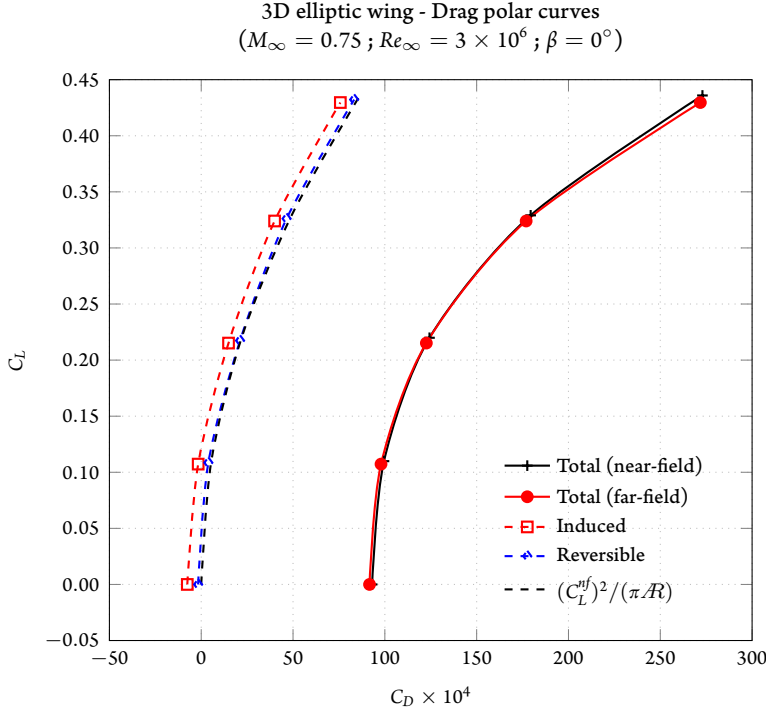


Figure 2.2.22: Drag polar breakdown isolating lift-induced contribution, for the 3D elliptic-planform wing, computed using $\underline{F}_a$ Lamb-vector far-field force formula ($X_W/c_0 = 4.5$). Reversible drag by Destarac and van der Vooren method ($X_W/c_0 = 1.0$).

2.2.4 AERODYNAMIC FORCE ANALYSIS, IN 3D

In the previous subsections, it was observed how an accurate drag breakdown in lift-induced and parasite contribution requires control volumes extending far away downstream of the body. Unfortunately, in numerical applications and especially in 3D, the integration domain cannot be extended too far downstream of the body, because a spurious conversion of reversible into irreversible drag also occurs, as discussed in Section 1.3.

In Fig. 2.2.22, the drag polar curve for the elliptic wing case described in Section 2.2.1 was computed by direct surface stress integration (near-field method) and compared with the same data obtained by the Lamb-vector-based far-field formula $\underline{F}_a$ (Eq. (2.6))[¶].

In the same figure, the total far-field drag is decomposed isolating the lift-induced contribution. As observed in Sec. 2.2.3.4 (and verified in Appendix B.2), this contribution is analytically independent on the pole location, when computed using $\underline{F}_a$. Results provided by the incompressible Prandtl formula are also included in the plot, being the Oswald factor known and unitary (elliptic

[¶]In this case, despite the presence of shock waves in the flow-field, the accuracy of Eq. (2.6) was recovered using the new thermodynamic Lamb-vector definition [122] illustrated in Ch. 4, instead of using the alternative formula $\underline{F}_c$ (Eq. (2.15)) as done for two-dimensional test cases.

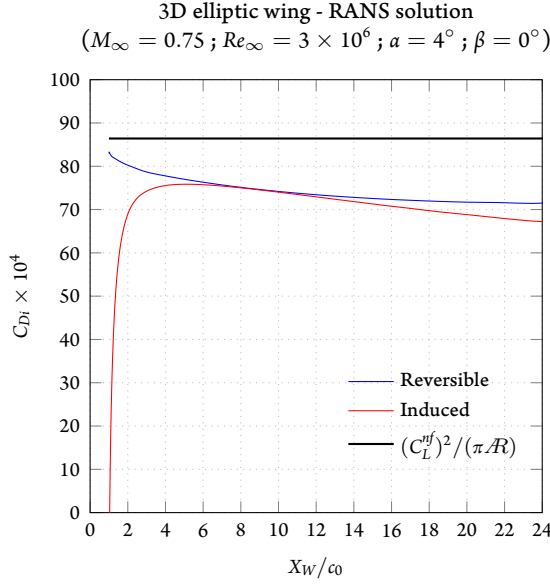


Figure 2.2.23: Lift-induced and reversible drag of the 3D elliptic-planform wing at non-zero incidence (lifting case), compared to Prandtl formula value ($Z_W/c_0 = 10.1$).

planform) and considering that the wing has no aerodynamic twist. Computed induced drag values are very close to the reference incompressible Prandtl curve, although smaller by a few drag counts.

The control volume selected for the drag decomposition illustrated in Fig. 2.2.22 extends approximately 4.5 root chord lengths downstream of the wing trailing edge. As a matter of fact, the computed induced drag is underestimated and affected by the spurious negative contribution already observed in two-dimensional flows. This effect is stronger in the near wing's wake (where the axial velocity defect is higher) and slowly vanishes downstream, reducing the underestimation. On the other hand, the reversible drag content of the flow-field is also gradually reducing, due to the physical and numerical dissipation, then the computed lift-induced drag has a maximum when this effect overcomes the former (Fig. 2.2.23). A typical choice, in Lamb-vector methods, is to assume the peak value as best estimate for the lift-induced drag. The reversible drag contribution was also calculated through the method of Destarac and van der Vooren [42], Eq. (1.81), by difference to the total near-field drag and plotted in Fig. 2.2.23. At the trailing edge of the wing, the computed reversible drag is surprisingly close to the value predicted by Prandtl's incompressible formula, then it gradually decreases as the wake plane is moved downstream of the wing, due to the reasons discussed above. By the way, it is worth remarking that the validity of the Prandtl formula in the transonic regime (and even in supersonic conditions) is not at all a straightforward result, as already evidenced by Mele et al. [116] and by Kang et al. [81].

A very similar result was obtained, for the same elliptic-planform wing, by Kang et al. [81]

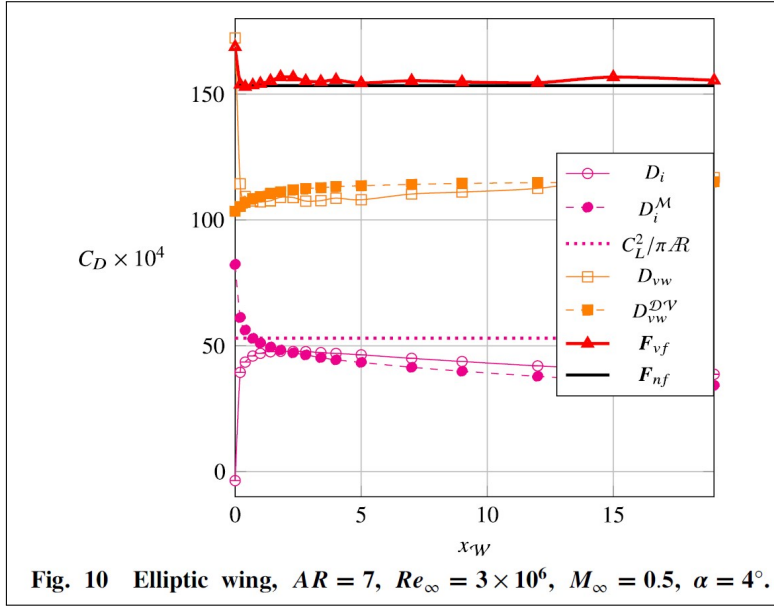


Fig. 10 Elliptic wing, $AR = 7$, $Re_\infty = 3 \times 10^6$, $M_\infty = 0.5$, $\alpha = 4^\circ$.

Figure 2.2.24: Drag variation with the wake plane position for the 3D elliptic-planform wing, figure extracted from [81], pag 4630.

(Fig. 2.2.24 is extracted from the cited reference: pag. 4630, Fig. 10). They analysed a subsonic case ($M_\infty = 0.5$) and observed an underestimation of lift-induced drag by the Lamb vector method (solid line with empty circular marks), compared to the Prandtl formula (purple dotted line). The reversible drag contribution was not reported and the spurious drag contribution was non-negligible despite the absence of shock-waves, as the vortical formula E_b was used instead of E_c (see Tab. 2.2.6 for an estimation of the difference on 2D flows) using the standard vorticity-based definition of the Lamb vector ($\underline{\ell} = \underline{\omega} \times \underline{V}$). As mentioned above, here the application to a transonic flow condition ($M_\infty = 0.75$) was possible due to the alternative thermodynamic Lamb-vector definition [122], proposed and described in Ch. 4.

Another computation, again using the same elliptic wing (although with a different and finer mesh), was analysed by Zou et al. [232] in incompressible flow conditions and at $Re_\infty = 1 \times 10^5$ ($\alpha = 4^\circ$). In Fig. 2.2.25, the underestimation of lift-induced drag computed using a formula derived from Lamb vector methods (“exact induced drag”, black solid line with filled squares), with regards to the Prandtl formula (horizontal green line) is visible. Comparison was provided against values computed using Maskell’s formula (instead of reversible drag), which also start very close to the Prandtl line near the wing trailing edge.

The simulation at zero incidence (and zero lift) is instead analysed in Fig. 2.2.26. A similar behaviour is observed for the lift-induced and reversible drag variations with the wake plane position, the main difference consisting in the absence of reversible drag ($C_{D,i} = 0$ is also predicted by Prandtl’s formula) and, therefore, in the lack of conversion into irreversible drag. Neverthe-

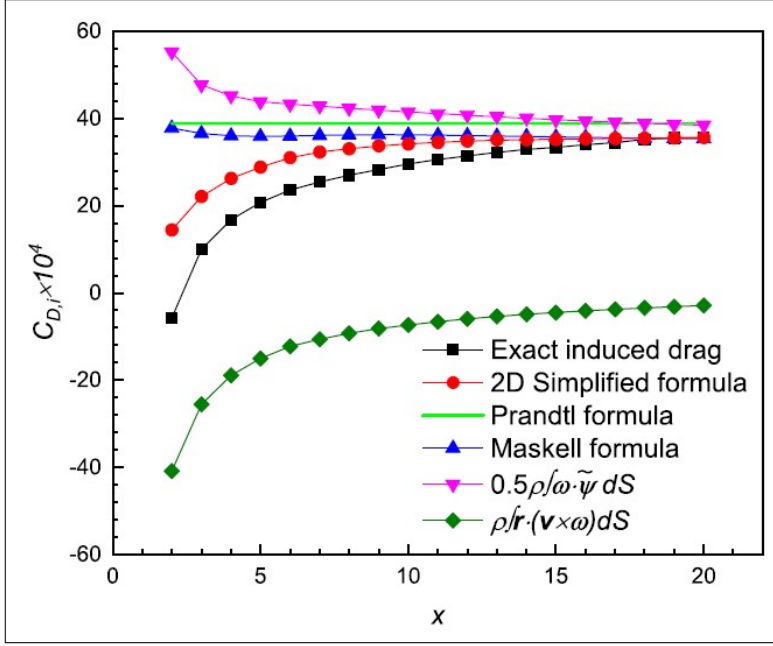


Figure 2.2.25: Induced drag variation with the wake plane position for the 3D elliptic-planform wing, figure extracted from [232], pag 12.

less, the computed lift-induced drag is not able to approach the zero value as the spurious negative contribution vanishes very slowly (a logarithmic scale was used in Fig. 2.2.11) and in this case the computational domain extends up to 30 root chord lengths only, downstream of the wing.

2.3 DISCUSSION

The Lamb-vector-based drag decomposition in lift-induced and parasite contributions proposed by Fournis et al. [57] (Eq. (2.27), (2.28)) differs from the original compressible drag decomposition of Mele et al. [117] by the term $\underline{F}_{\nabla\rho}$.

However, the introduction of $\underline{F}_{\nabla\rho}$ in the formulas makes the original decomposition loose its mid-field property. As a matter of fact, density variations are non-negligible outside of flow rotational regions, an aspect which has already been evidenced in Sec. 2.2.3.7 with regards to the local parasite drag generation.

The main benefit of using the $\underline{F}_{\nabla\rho}$ term in Fournis et al. decomposition consists in the possibility to establish links with the classical Maskell's and Betz's formulas (Eq. (2.30), (2.31)), allowing to achieve the drag breakdown independence on the DMT pole.

Nevertheless, these are *generalised* Maskell's and Betz's formulas (discussed in Ch. 1, §1.4.2.1)

Indeed, Mele et al. formulation also neglects the $\underline{F}_\mu$ term in Eq. (2.28), a valid assumption for high Reynolds number flows.

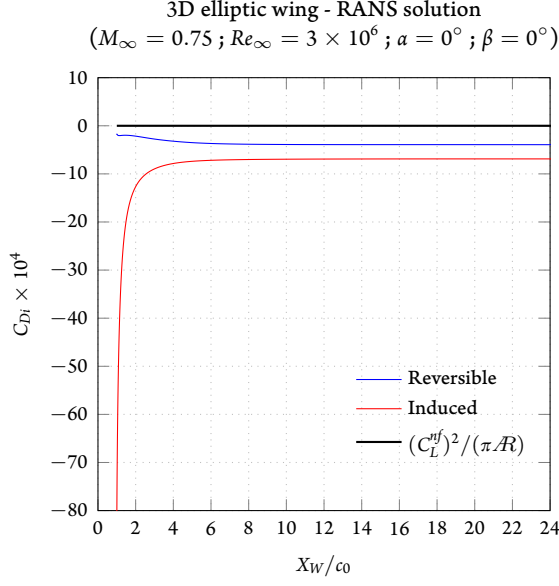


Figure 2.2.26: Lift-induced and reversible drag of the 3D elliptic-planform wing at zero incidence (non-lifting case) compared to Prandtl formula value ($Z_W/c_0 = 2.4$).

and inconsistencies were highlighted in the identification of reversible and irreversible drag contributions using such definitions in compressible flows.

On the other hand, the pole dependency of Mele et al. decomposition could be handled in a different way, i.e. by identifying appropriate criteria to *fix* a pole location. Indeed, first it was highlighted that the DMT pole needs also being selected as the pole where the aerodynamic moment is required to be computed (Appendix A). Then, it was observed in Sec. 2.2.3.4 that the pole should be located near the body, in order to reduce the weight in far-field formulas of quantities computed in the near-wall grid cells, where the local flux residuals are typically higher than average. Additionally, considering for example the two-dimensional cases analysed in Sec. B.2, a pole position near the body surface ($0 \leq X_p \leq 1$ and $Z_p \approx 0$) allows in any case to obtain values of the total force and of the drag decomposition in good agreement to those computed using pole-independent formulas.

For the above reasons, Mele et al. decomposition is still to be preferred in most cases, especially when a following parasite drag breakdown has to be performed. Concerning the identified spurious induced drag contribution (Sec. 2.2.3.6), using either Mele et al. or Fournis et al. breakdown definitions leads to no different conclusions, since the $\underline{F}_{\nabla p}$ term vanishes at large distance from the body.

2.4 SUMMARY AND REMARKS

- In present chapter, the accuracy of different Lamb-vector-based force formulations was assessed in nearly-incompressible, subsonic and transonic flow regimes, analysing both inviscid and viscous numerical solutions at low and high Reynolds number [124]. A loss of accuracy was observed in association with the Lamb vector integration in the control volume interior, with the best results obtained using formulation “c” (involving the Lamb vector only through surface integrals) and the worst obtained from formulation “d” (mostly based on volume integrals). It is anticipated that the observed inaccuracies can now be significantly reduced through a thermodynamic-based calculation of the Lamb vector, representing the core of the unified methodology proposed in this dissertation (Ch. 4).
- The author also analysed, for different steady Lamb-vector-based force formulas, sensitivities of the computed force and of the drag decomposition to the control volume selection and to the choice of the arbitrary pole used in DMTs, the latter being rarely addressed in the literature [81]. Formulations “c” and “a” provided the most accurate results (nearly zero sensitivity in a reasonable range of pole locations) while unacceptable variations with the DMT pole were observed using other formulations, at least with the adopted numerical method. Results of a grid convergence analysis revealed an impact of discretization errors on the highlighted sensitivities, which represent pure numerical effects.
- Inconsistencies experienced in previous works and related to the drag decomposition in lift-induced and parasite contributions (with ambiguities on the physical or non-physical role of a non-zero induced drag found in two-dimensional flows) have been definitively clarified. Indeed, for the first time, the relationship between reversible/irreversible drag definitions and lift-induced/parasite drag contributions has been clearly addressed, removing the ambiguity that tended to identify the two in previous works. Using Lamb-vector-based lift-induced drag definitions, a non-zero spurious induced drag was clearly identified post-processing two-dimensional numerical solutions, increasing in magnitude as the Reynolds number is reduced or the Mach number increased, and vanishing as the distance of the outer control volume boundary from the aerofoil surface is significantly increased.
- Generalised forms of the classical Kutta-Joukowski theorem, Maskell and Betz formulas were derived (in Ch. 1) starting from the integral momentum balance, for steady flows, as an alternative to the originally proposed derivation starting from the vortical force formulas [57]. The capability of Lamb-vector-based lift-induced and parasite drag definitions (directly linked to generalised classical formulas) to respectively capture reversible and irreversible drag contributions was theoretically discussed, then numerical support to the theoretical argumentation was provided post-processing three-dimensional and two-dimensional CFD solutions. The latter ones were analysed in nearly-incompressible, subsonic and transonic regime, considering both inviscid and viscous flows (at high or low

Reynolds number). In 2D, consistency of lift-induced drag predictions with the expected zero reversible drag (confirmed by thermodynamic methods) was shown to hold on very large control volumes only. A spurious induced drag contribution, related to the axial velocity perturbation, is otherwise computed, resulting in negative induced drag values.

- The above argumentation reveals the domain dependence of the drag breakdown in lift-induced and parasite contributions, and suggests that a decomposition capturing respectively reversible and irreversible drag components cannot be obtained with an arbitrary choice of the external control surface, that should be placed far away from the body skin in order to effectively decompose the total drag in reversible and irreversible parts. Unfortunately, in numerical applications (and especially in 3D), the integration domain cannot be extended too far downstream of the body, where a spurious conversion of reversible into irreversible drag also occurs, polluting drag decomposition results. This limits the accuracy of the drag breakdown achievable through classical or Lamb-vector-based methods, and highlights that the proposed direct definitions of lift-induced drag in viscous and compressible flows still need to be sharpened to remove the bias represented by the identified spurious effect. In high Reynolds number flows without strong shocks, Lamb-vector-based methods can instead already provided a reliable drag breakdown.
- It was evidenced that, to decompose parasite drag in viscous and wave contributions, Lamb-vector integrals cannot be limited to the wake plane only, but flow entrainment in the viscous wake needs to be taken into account, and its effect was quantified by integrals extended at the interface between the viscous and the outer flow regions.
- A new methodology to perform parasite drag decomposition in viscous and wave contributions accounting for the flow entrainment in the viscous region was then proposed and tested on transonic flow-fields around aerofoils. Results were compared to viscous and wave drag estimates obtained using consolidated thermodynamic-based methods, and a benchmarking was provided including recent results also obtained using different decomposition approaches [69, 117, 172]. A reasonable agreement was observed, at condition to account for the spurious induced drag observed on control volumes of limited size.
- A modified formulation (based on volume instead of surface integrals) was also derived for lift-induced and parasite drag contributions, allowing for a visualisation of the drag sources in the flow domain using Lamb-vector methods.

The contribution to the vortex force from the velocity induced by the vorticity in the trailing sheet gives a component parallel to the free-stream. This is the 'induced drag' associated with the generation of kinetic energy in the trailing vortex system.

Philip Geoffrey Saffman

3

Unsteady regime

THIS CHAPTER WILL MAINLY FOCUS ON MOVING BODIES, although the methods illustrated herein can also be applied to different unsteady flows around fixed objects (e.g. low Reynolds vortex shedding, transonic buffeting, etc.). In this context, working in non-inertial frames of reference could be a convenient choice, in particular when a rigid body motion is considered and a solidal frame can be used. At the scope, it is worth recalling the expression of Navier-Stokes equations in a generally non-inertial reference frame, together with some useful transformations involving time derivatives in inertial and non-inertial frames. This is done in Appendix A.2.

Differently from the steady case, the aerodynamic force exerted on moving bodies like flapping wings, for example, is also characterised by the presence of a *thrust* component. In this Chapter, a general thrust force theory using Lamb-vector formulas is also proposed, which encompasses Garrick's linear theory [61] as a particular case. This allows, for example, the direct thrust calculation on oscillating aerofoils in viscous flows, not possible before.

3.1 LAMB-VECTOR METHODS

Different *exact* Lamb-vector-based force formulas developed in the last two decades are herein briefly recalled, summarising the key properties of each formulation and extending, where necessary, their validity to the unsteady flow regime. In what follows, similarly to what introduced for steady flows (Chapter 2), $\underline{F}$ is the aerodynamic force exerted by the fluid on the body B immersed

in it, while $\mathcal{V}$ is the control volume used for the integral balance of linear momentum, bounded by the aerofoil surface ∂B and the far-field surface Σ , where $\underline{n}$ denotes the unit normal vector pointing outwards of $\mathcal{V}$ (Fig. 1.1.1).

Here a neat formulation is introduced, where the exact force formulas are written for a generally non-inertial reference frame, including apparent body forces due to the relative motion (a derivation is provided in [140], although restricted to incompressible flows). In this form, formulas are valid for both inertial and non-inertial reference frames (respectively $(O, \underline{e}_x, \underline{e}_y, \underline{e}_z)$ and $(O', \underline{e}'_x, \underline{e}'_y, \underline{e}'_z)$ in Fig. 1.1.1), at condition of considering $\underline{V}$ as, respectively, the absolute or the relative fluid velocity and interpreting partial time derivatives as appropriate for the selected reference frame (i.e. respectively at constant absolute, or relative, position). Apparent body forces due to the relative frame motion will of course be zero in the case of inertial reference frames. The general case of non-zero fluid velocity at the body wall is also considered. This allows applying the force formula to viscous and to effectively inviscid flows, as well as to fixed or moving bodies. Similarly, as already introduced for steady flows in Chapter 2, the adopted generalised notation is also valid for both laminar and turbulent flow regimes. In the first case, *Reynolds* stresses and variables fluctuations are zero. In the second case, mass density (ρ) and fluid particle velocity ($\underline{V}$) are intended as their corresponding *ensemble* and *Favre* averages respectively.

3.1.1 ORIGINAL FORMULA

The original unsteady compressible formulation, derived by Wu et al. [222, p. 621] for the laminar flow regime in inertial reference frames (further extended to the turbulent regime by Mele and Tognaccini [119]), is hereafter written for a generally non-inertial reference frame, including apparent body forces due to the relative motion (the general derivation is included in Appendix A.4, for both the aerodynamic force and moment formulas). This is not necessarily solidal to the body and, at this stage, no hypothesis is even made on the body being rigid or deformable.

This formula is denoted with subscript “a” and can be expressed in compact form as:

$$\underline{F}_a = \underline{F}_\ell + \underline{F}_\rho + \underline{F}_t + \underline{F}_S + \underline{F}_\mu + \underline{F}_m \quad (3.1)$$

where:

$$\underline{F}_\ell = - \int_{\mathcal{V}} \rho \underline{\ell} d\mathcal{V} - \int_{\partial B} \frac{\rho V^2}{2} \underline{n} dS \quad (3.2)$$

$$\underline{F}_t = - \int_{\mathcal{V}} \underline{r} \times \left[\underline{\nabla} \times \left(\rho \frac{\partial \underline{V}}{\partial t} \right) \right] d\mathcal{V} + \int_{\partial B} \underline{r} \times \left[\underline{n} \times \left(\rho \frac{\partial \underline{V}}{\partial t} \right) \right] dS \quad (3.3)$$

$$\underline{F}_m = \int_{\mathcal{V}} \rho \underline{f}_m d\mathcal{V} + \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\rho \underline{f}_m \right) \right] dS \quad (3.4)$$

In Eq. (3.1), the symbols $\underline{F}_\rho$, $\underline{F}_S$ and $\underline{F}_\mu$ are defined as already introduced for steady flows (Eq. (2.8), (2.9) and (2.4)) and the vortex force term ($\underline{F}_\ell$) here is explicitly corrected by the body

surface integral, which is non-zero for inviscid flows (Sec. 2.1.6) or moving bodies*. $\underline{F}_t$ collects terms containing time derivatives of the fluid velocity and $\underline{F}_m$ is the contribution of body forces per unit mass ($\underline{f}_m$), including gravitational acceleration ($\underline{g}$) and apparent forces, if any, due to non-inertial effects:

$$\underline{f}_m = \underline{g} - \frac{d\underline{U}}{dt} - \frac{d\underline{\Omega}}{dt} \times \underline{x} - \underline{\Omega} \times (\underline{\Omega} \times \underline{x}) - 2\underline{\Omega} \times \underline{V} \quad (3.5)$$

where $\underline{x}$ is the position vector referred to the frame origin, which is moving with linear velocity $\underline{U}(t)$ and angular velocity $\underline{\Omega}(t)$ with regards to an inertial reference. In the following, gravitational effects will be neglected (i.e. $\underline{g} = \underline{0}$).

3.1.2 MODIFIED FORMULA

The mid-field modification of formula “a” proposed by Mele and Tognaccini [119] in the steady regime (Ch. 2, Sec. 2.1.2) is also applicable to the unsteady formula. It will be here considered for comparison against other formulas. Nevertheless, the mid-field property is lost in this case even in inertial reference frames ($\underline{F}_m = \underline{0}$) because, in unsteady conditions, non-dissipative flow regions are not homo-enthalpic (an assumption used to prove the mid-field property for steady flows) and, moreover, the unsteady volume integrand in $\underline{F}_t$ is also non-zero everywhere in the flow-field.

The modified force formula can be written in compact form as:

$$\underline{F}_b = \underline{F}_\ell + \underline{F}_{m_p} + \underline{F}_t + \underline{F}_s + \underline{F}_\mu + \underline{F}_m \quad (3.6)$$

where $\underline{F}_{m_p}$ is defined as in Ch. 2 (Eq. (2.22)).

3.1.3 ALTERNATIVE FORMULA AND ITS EXTENSION TO UNSTEADY FLOWS

The alternative formula derived, for steady flows, by Mele et al. [117] and containing surface integrals only (with exception for the last term in Eq. (2.4)) proved to be numerically more accurate than the ones previously described [124], also in presence of shock waves, where numerical issues were reported with formulations using volume integrals in the shock wave region [117, 124].

This formulation is here extended to unsteady flows (and denoted with subscript “c”) by introducing two additional terms (which will be highlighted in the following) one of which is still a volume integral. Moreover, for unsteady flows $\underline{F}_t$ and $\underline{F}_m$ also contain volume integrals. Therefore the accuracy of this unsteady alternative formula in presence of shock waves has to be assessed, which is one of the objectives of this chapter.

*The velocity at the body wall, involved in skin integrals terms, is composed by the fluid velocity relative to the wall, plus the moving wall velocity. For viscous flows and impermeable bodies (zero relative fluid velocity) skin terms are therefore still non-zero for moving walls, except in the particular case of a zero-thickness body (e.g. an ideal plate), where the contribution of wall velocities on opposite sides of the body cancel-out.

Chapter 3. Unsteady regime

In compact form, the formula can be expressed as:

$$\boxed{\underline{F}_c = \underline{F}_{\ell+m_p} + \underline{F}_t + \underline{F}_s + \underline{F}_\mu + \underline{F}_m} \quad (3.7)$$

defining:

$$\begin{aligned} \underline{F}_{\ell+m_p} = & - \int_{\Sigma} \rho \underline{V} \underline{V} \cdot \underline{n} \, dS + \int_{\Sigma} \rho \frac{V^2}{2} \underline{n} \, dS + \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] \, dS + \\ & - \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \underline{V} \, d\mathcal{V} - \int_{\partial B} \rho \underline{V} \underline{V} \cdot \underline{n} \, dS \end{aligned} \quad (3.8)$$

where the last two terms in Eq. (3.8) are new with regards to the steady formula derived in [117] (see Eq. (2.16)). In particular, the last body integral is non-zero for impermeable but moving walls. Finally, the RHS expression in Eq. (3.8) can also be simplified combining the first and last integrals into: $-\int_{\partial \mathcal{V}} \rho \underline{V} \underline{V} \cdot \underline{n} \, dS$.

3.1.4 TRANSONIC FORMULA AND ITS EXTENSION TO UNSTEADY FLOWS

The steady hybrid formulation proposed by Mele et al. for transonic flows [117] (“c” formulation in the shock wave region, combined with “b” formulation in the remaining of the fluid domain, Ch. 2, Sec. 2.1.4) can also be extended to unsteady flows. This is done here, in analogy with what derived in Sec. 3.1.3. The formula can be written in compact form as:

$$\boxed{\underline{F}_{bc} = \underline{F}_{\ell} + \underline{F}_{m_{pc}} + \underline{F}_t + \underline{F}_s + \underline{F}_\mu + \underline{F}_m} \quad (3.9)$$

with:

$$\begin{aligned} \underline{F}_{m_{pc}} = & - \int_{\mathcal{V} \setminus \mathcal{V}_{sw}} \underline{m}_\rho \, d\mathcal{V} - \int_{\partial \mathcal{V}_{sw}} \rho \underline{V} \underline{V} \cdot \underline{n}_{sw} \, dS + \int_{\partial \mathcal{V}_{sw}} \rho \frac{V^2}{2} \underline{n}_{sw} \, dS + \\ & + \int_{\partial \mathcal{V}_{sw}} \underline{r} \times \left[\underline{n}_{sw} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] \, dS + \int_{\mathcal{V}_{sw}} \left(\rho \underline{\ell} - \frac{\partial \rho}{\partial t} \underline{V} \right) \, d\mathcal{V} + \\ & - \int_{\partial B} \underline{r} \times \left[\underline{n} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] \, dS \end{aligned} \quad (3.10)$$

where $\mathcal{V}_{sw}$ is the selected shock wave domain (with outward-pointing normal vector $\underline{n}_{sw}$) and the time derivative term in the fifth integral of Eq. (3.10) is new with regards to the steady formula derived in [117].

The original intention of the authors to avoid volume integration in $\mathcal{V}_{sw}$ is preserved as far as the elementary vortex forces ($-\rho \underline{\ell}$) are concerned, nevertheless a new volume integral in $\mathcal{V}_{sw}$ now arises for unsteady compressible flows, due to the presence of mass density time derivative in the fifth integral of Eq. (3.10). This however does not involve spatial derivatives, as instead occurring for term containing the Lamb vector.

It should also be pointed out that the identification of the shock wake region (generally non-connected, in case of multiple shocks in the flow-field) is now required as a function of time (i.e. $\mathcal{V}_{sw} = \mathcal{V}_{sw}(t)$ and $\partial\mathcal{V}_{sw} = \partial\mathcal{V}_{sw}(t)$), following the evolution of the shock waves system. This introduces an additional complexity in the application of Eq. (3.10) to numerically-computed flow-fields, where the force sensitivity to the choice of $\mathcal{V}_{sw}$ might behave differently depending on the selected time step, resulting in a variable accuracy of the far-field method.

3.1.5 MODIFIED TRANSONIC FORMULA AND ITS EXTENSION TO UNSTEADY FLOWS

Having extended the transonic formula to unsteady flows in the previous section, it is straightforward to derive its modification where the outer vorticity contribution ($\underline{E}_S$) is obtained using domain integrals in $\mathcal{V}$ instead of surface integrals on Σ . The conversion proposed by Ostieri and Tognaccini [141] (Ch. 2, Sec. 2.1.5) is of course applicable also to the unsteady transonic formula and the corresponding modified total force expression, denoted with subscript “d”, can be written in compact form as:

$$\underline{E}_d = \underline{E}_\ell + \underline{E}_{m_{pc}} + \underline{E}_t + \underline{E}_{S\mathcal{V}} + \underline{E}_\mu + \underline{E}_m \quad (3.11)$$

where:

$$\underline{E}_{S\mathcal{V}} = - \int_{\mathcal{V}} \nabla(\underline{r} \cdot \rho \underline{\ell}) d\mathcal{V} + \int_{\mathcal{V}} \nabla \cdot (r \rho \underline{\ell}) d\mathcal{V} + \int_{\partial B} \underline{r} \times (\underline{n} \times \rho \underline{\ell}) dS \quad (3.12)$$

3.1.6 NEW UNSTEADY TERMS FORMULATION

Eventually, unsteady terms included in $\underline{E}_t$ can be expressed, for all compressible force formulas previously illustrated, in an alternative way by adding and subtracting in $\underline{E}_t$ the integral $\int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\rho \frac{\partial \underline{V}}{\partial t} \right) \right] dS$ and applying a Derivative Moment Transformation (DMT) [222, pp. 700-701] to the integrals in $\mathcal{V}$ and on $\partial\mathcal{V} = \Sigma \cup \partial B$. The new $\underline{E}_t$ expression is:

$$\underline{E}_t^* = - \int_{\mathcal{V}} \rho \frac{\partial \underline{V}}{\partial t} d\mathcal{V} - \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\rho \frac{\partial \underline{V}}{\partial t} \right) \right] dS \quad (3.13)$$

and has the advantage of not containing spatial gradients of flow variables, which could bring numerical advantages in post-processing CFD solutions. Moreover, it is worth observing that the new formulation also contains no pole dependence in the volume integral, highlighting that there is a force contribution from the unsteady velocity field variation which does not depend on the selected DMT pole. This was not evident in the original $\underline{E}_t$ expression, with the reference pole appearing in both integrals of Eq. (3.3). Additionally, in Eq. (3.13) the surface integral represents now a far-field contribution (being computed on Σ) instead of a near-field one (as in Eq. (3.3), where it is computed on ∂B). Then, this alternative expression of unsteady terms is also fundamental for the derivation of the new exact aerodynamic force and moment formulas, which will be illustrated in Ch. 4.

In the following, $\underline{F}_a$, $\underline{F}_b$ and $\underline{F}_c$ symbols will be added a superscript * to denote that, in this case, the corresponding force formulas are defined using unsteady terms as expressed in Eq. (3.13) instead of Eq. (3.3).

3.1.7 THE INCOMPRESSIBLE MIXED INERTIAL/NON-INERTIAL FORMULATION AND ITS EXTENSION TO 3D AND TO INVISCID FLOWS

For incompressible fluids (constant density) or ideal gases in the incompressible flow regime (negligible effects of density variations), the vortical force formula (no matter the original or modified expressions, as compressibility corrections vanish) can be written in a mixed inertial/non-inertial form, following the approach of Ostieri et al. [140]. This is a technique also exploited in geophysical flows and is based on a mixed Lamb vector definition which uses the absolute flow vorticity ($\underline{\omega}$) and the relative fluid velocity ($\underline{V}'$). As discussed in Appendix A.2 and differently from the simplified notation used in previous sections, here it is necessary to explicitly use prime superscripts to distinguish relative quantities from the absolute ones, as they co-exist in the same formula: $\underline{\ell}^{mix} = \underline{\omega} \times \underline{V}'$.

The mixed inertial/non-inertial formulation, initially derived in [140] for two-dimensional viscous flows, will be here extended to 3D and to inviscid flows.

The mixed formulation is introduced at the scope of taking advantages from both sides. In fact, some authors [139, 140] reported that using relative flow velocities computed in a non-inertial reference frame can provide higher accuracy in numerical applications when dealing with moving grids. Nevertheless, for incompressible flows, the use of an inertial frame of reference guarantees the mid-field property of the far-field force formula, for high-Reynolds number flows. In fact, in this case, $\underline{F}_p = \underline{0}$, $\underline{F}_m = \underline{0}$, $\underline{F}_\mu \approx \underline{0}$ and the unsteady term ($\underline{F}_t$) can be expressed as:

$$\underline{F}_t = -\rho \int_V \underline{r} \times \frac{\partial \underline{\omega}}{\partial t} dV + \rho \int_{\partial B} \underline{r} \times \left(\underline{n} \times \frac{\partial \underline{V}}{\partial t} \right) dS \quad (3.14)$$

where the Schwarz theorem has been applied to the volume integrand. This is then negligible in non-rotational regions of the flow-field, providing the mid-field property. It is worth observing that the relative flow vorticity

$$\underline{\omega}' = \underline{\nabla} \times \underline{V}' = \underline{\omega} - 2\underline{\Omega} \quad (3.15)$$

is instead generally non-zero in the entire flow-field due to the frame rotational velocity $\underline{\Omega}^\dagger$. The idea is then to keep using the absolute flow vorticity, together with the relative fluid velocity, defining the Lamb vector as: $\underline{\ell}^{mix} = \underline{\omega} \times \underline{V}'$.

Let us explicitly write the inertial and non-inertial forms of the incompressible formula. Start-

[†]the last equality in Eq. (3.15) follows as the curl of the velocity equals twice the fluid particle rotation appearing in the vorticity tensor.

ing with the inertial one, i.e. Eq. (3.1) with $\underline{F}_m = \underline{0}$ and $\underline{F}_\rho = \underline{0}$:

$$\underline{F}_{ff} = \underline{F}_\ell + \underline{F}_t + \underline{F}_S + \underline{F}_\mu \quad (3.16)$$

It is convenient, for the following derivation, to re-define the symbols on the RHS, isolating body skin integrals in an additional term, i.e.:

$$\underline{F}_\ell = -\rho \int_{\mathcal{V}} \underline{\ell} \, d\mathcal{V} \quad (3.17)$$

$$\underline{F}_t = -\rho \int_{\mathcal{V}} \underline{r} \times \frac{\partial \underline{\omega}}{\partial t} \, d\mathcal{V} \quad (3.18)$$

$$\underline{F}_S = -\rho \int_{\Sigma} \underline{r} \times (\underline{n} \times \underline{\ell}) \, dS \quad (3.19)$$

$$\underline{F}_B = \rho \int_{\partial B} \underline{r} \times \left(\underline{n} \times \frac{\partial \underline{V}}{\partial t} \right) \, dS - \rho \int_{\partial B} \frac{V^2}{2} \underline{n} \, dS \quad (3.20)$$

$$\underline{F}_\mu = \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \underline{\nabla} \cdot \left(\underline{\tau}_v + \underline{\tau}_R \right) \right] \, dS + \int_{\Sigma} \left(\underline{\tau}_v + \underline{\tau}_R \right) \cdot \underline{n} \, dS \quad (3.21)$$

leading to:

$$\underline{F}_{ff} = \underline{F}_\ell + \underline{F}_t + \underline{F}_S + \underline{F}_B + \underline{F}_\mu \quad (\text{inertial form}) \quad (3.22)$$

Of course this is purely a matter of convention and, in the following, it will be assumed that, when the $\underline{F}_B$ term is not explicitly included in the force formula, body skin integrals are individually included in the other contributing terms, as done in the previous sections.

The same can be done in non-inertial reference frames, where the absolute velocity ($\underline{V}$) is replaced by the corresponding relative quantity ($\underline{V}'$), partial time derivatives are evaluated at constant relative (instead of absolute) position ($\frac{\partial}{\partial t'}$ instead of $\frac{\partial}{\partial t}$) and apparent forces are of course included through the $\underline{F}_m$ term:

$$\underline{F}_{ff} = \underline{F}'_\ell + \underline{F}'_t + \underline{F}'_S + \underline{F}'_B + \underline{F}_\mu + \underline{F}_m \quad (\text{non-inertial form}) \quad (3.23)$$

where

$$\underline{F}'_\ell = -\rho \int_{\mathcal{V}} \underline{\ell}' \, d\mathcal{V} \quad (3.24)$$

$$\underline{F}'_t = -\rho \int_{\mathcal{V}} \underline{r} \times \frac{\partial \underline{\omega}'}{\partial t'} \, d\mathcal{V} \quad (3.25)$$

$$\underline{F}'_S = -\rho \int_{\Sigma} \underline{r} \times (\underline{n} \times \underline{\ell}') \, dS \quad (3.26)$$

$$\underline{F}'_B = \rho \int_{\partial B} \underline{r} \times \left(\underline{n} \times \frac{\partial \underline{V}'}{\partial t'} \right) \, dS - \rho \int_{\partial B} \frac{V'^2}{2} \underline{n} \, dS \quad (3.27)$$

$$\underline{F}_m = \rho \int_{\mathcal{V}} \underline{f}_m \, d\mathcal{V} + \rho \int_{\Sigma} \underline{r} \times (\underline{n} \times \underline{f}_m) \, dS \quad (3.28)$$

and the relative Lamb vector has been defined as $\underline{\ell}' = \underline{\omega}' \times \underline{V}'$.

It is worth observing that prime superscripts were omitted for scaled position vectors ($\underline{r}$), as they are always referred to the Derivative Moment Transformation (DMT) pole (Fig. 1.1.1). The relative position of a generic point in space, with regards to the DMT pole, is in fact independent of the reference system used to compute both the point and pole locations (i.e.: $\underline{x} - \underline{x}_p = \underline{x}' - \underline{x}'_p$). Similarly, unit normal vectors at the CV boundaries ($\underline{n}$) are intrinsic and do not transform between frames. Finally, based on the considerations included in Appendix A.2, the viscous term $\underline{F}_\mu$ is also frame invariant.

Now, the derivation of the mixed inertial/non-inertial formulation starts from the inertial formula (Eq. (3.22)). Using Eq. (A17), the body term ($\underline{F}_B$) can also be expressed as:

$$\underline{F}_B = \rho \int_{\partial B} \underline{r} \times (\underline{n} \times \underline{a}) dS - \rho \int_{\partial B} \underline{r} \times (\underline{n} \times \underline{\ell}) dS \stackrel{\text{def}}{=} \underline{F}_{t_B} + \underline{F}_{S_B} \quad (3.29)$$

where the definition of the last two terms in Eq. (3.29) is straightforward. The inertial formula is then re-written as:

$$\underline{F}_{ff} = \underline{F}_\ell + \underline{F}_t + \underline{F}_S + \underline{F}_{t_B} + \underline{F}_{S_B} + \underline{F}_\mu \quad (3.30)$$

and it is also observed that $\underline{F}_t$ and $\underline{F}_{S_B}$, for two-dimensional flows around impermeable bodies and for Σ fixed in time, are related to the time rate of the hydrodynamic impulse in the control volume ($\underline{I}_V = \rho \int_V \underline{r} \times \underline{\omega} dV$):

$$\frac{d\underline{I}_V}{dt} = -(\underline{F}_t + \underline{F}_{S_B}) \quad (3.31)$$

In fact, in these hypotheses, $\underline{F}_{S_B}$ can be written as:

$$\begin{aligned} \underline{F}_{S_B} &= -\rho \int_{\partial B} \underline{r} \times [\underline{n} \times (\underline{\omega} \times \underline{V})] dS = -\rho \int_{\partial B} \underline{r} \times [(\underline{n} \cdot \underline{V}) \underline{\omega} - (\underline{n} \cdot \underline{\omega}) \underline{V}] dS = \\ &= -\rho \int_{\partial B} \underline{r} \times \underline{\omega} [(\underline{V}_{\partial B} + \underline{V} - \underline{V}_{\partial B}) \cdot \underline{n}] dS = -\rho \int_{\partial B} \underline{r} \times \underline{\omega} (\underline{V}_{\partial B} \cdot \underline{n}) dS = \\ &= -\rho \int_{\partial V} \underline{r} \times \underline{\omega} (\underline{V}_{\partial V} \cdot \underline{n}) dS \end{aligned} \quad (3.32)$$

where it was considered that, for 2D flows, the vorticity is orthogonal to the planar vector $\underline{n}$ and that, for impermeable bodies, the relative velocity at the body wall ($\underline{V} - \underline{V}_{\partial B}$) is always tangent to the skin. Therefore, $\underline{F}_{S_B}$ reduces to the body integral of the first vorticity moment, times the flux of the moving wall velocity. In the case of Σ fixed in time ($\underline{V}_\Sigma = \underline{0}$), the integral can also be extended to ∂V , and then Eq. (3.32) exactly provides the boundary term needed to apply Reynolds transport theorem (Appendix A, Sec. A.3) to $\underline{F}_t$, proving Eq. (3.31).

This equation represents a link, for 2D flows, between incompressible Lamb-vector methods and impulse theories [138, 218]. It can be extended to 3D flows, provided that the non-zero integral $\rho \int_{\partial B} \underline{r} \times \underline{V} \omega_n dS$ is subtracted on the RHS of Eq. (3.31). Its validity also stands for moving Σ , as long as it does not intersect any vortical structure (i.e. $\underline{\omega} = \underline{0}$ on Σ). This was used for example

by Kang et al. [79]. Moreover, in the case of Σ intersecting non-zero vorticity and for generally non-impermeable bodies, Eq. (3.31) can be further extended to the form:

$$\frac{dI_{\mathcal{V}}}{dt} = - \left\{ \underline{F}_t + \underline{F}_S + \underline{F}_{S_B} - \rho \int_{\partial\mathcal{V}} \underline{r} \times \underline{V} \omega_n dS + \rho \int_{\partial\mathcal{V}} \underline{r} \times \underline{\omega} [(\underline{V} - \underline{V}_{\partial\mathcal{V}}) \cdot \underline{n}] dS \right\} \quad (3.33)$$

In previous studies on incompressible Lamb-vector-based methods [139, 140, 169], body skin terms were always formulated as $\underline{F}_{t_B} + \underline{F}_{S_B}$ (RHS of Eq. (3.29)) instead of as $\underline{F}_B$ (Eq. (3.20)). The latter expression, derived and used in the present work, has two advantages in numerical applications compared to the former: i) it does not involve spatial gradients of flow variables, which is beneficial for the accuracy of the compute results; ii) in the steady case, it is also independent of the arbitrary pole used for Derivative Moment Transformations. For these reasons, it was selected as the reference formulation of body skin terms in all calculations included and reported herein.

Following on the derivation of the mixed formula and considering that $\underline{\ell} = \underline{\ell}^{mix} + \underline{\omega} \times \underline{V}_T$, the vortex force term re-writes as follows:

$$\underline{F}_{\underline{\ell}} = -\rho \int_{\mathcal{V}} \underline{\ell}^{mix} dV - \rho \int_{\mathcal{V}} \underline{\omega} \times \underline{V}_T dV = \underline{F}_{\underline{\ell}}^{mix} + \underline{F}_{\underline{\ell}}^T \quad (3.34)$$

In the same manner, the other terms in Eq. (3.30) that contain the Lamb vector can be re-written as the sum of two quantities: one depending on $\underline{\ell}^{mix}$ (namely, $\underline{F}_{\underline{\ell}}^{mix}$) and another one depending on the transport Lamb vector $\underline{\omega} \times \underline{V}_T$ (namely, $\underline{F}_{\underline{\ell}}^T$):

$$\underline{F}_{S_B} = \underline{F}_{S_B}^{mix} + \underline{F}_{S_B}^T \quad (3.35)$$

$$\underline{F}_S = \underline{F}_S^{mix} + \underline{F}_S^T \quad (3.36)$$

while, by means of Eq. (A9), $\underline{F}_t$ is modified as follow:

$$\begin{aligned} \underline{F}_t &= -\rho \int_{\mathcal{V}} \underline{r} \times \frac{\partial \underline{\omega}}{\partial t'} dV + \rho \int_{\mathcal{V}} \underline{r} \times (\underline{V}_T \cdot \underline{\nabla} \underline{\omega}) dV - \rho \int_{\mathcal{V}} \underline{r} \times (\underline{\Omega} \times \underline{\omega}) dV = \\ &= \underline{F}_t^{mix} + \underline{F}_t^T - \rho \int_{\mathcal{V}} \underline{r} \times (\underline{\Omega} \times \underline{\omega}) dV \end{aligned} \quad (3.37)$$

where the last term is generally non-zero for 3D flows, while it vanishes in the two-dimensional case, when both the flow vorticity ($\underline{\omega}$) and the frame angular velocity ($\underline{\Omega}$) are orthogonal to the planar space. By means of Eq. (A1):

$$\underline{F}_{\underline{\ell}}^T = -\rho \int_{\mathcal{V}} \underline{r} \times [\underline{\nabla} \times (\underline{\omega} \times \underline{V}_T)] dV + \rho \int_{\partial B \cup \Sigma} \underline{r} \times [\underline{n} \times (\underline{\omega} \times \underline{V}_T)] dS \quad (3.38)$$

Considering that:

$$\underline{\nabla} \times (\underline{\omega} \times \underline{V}_T) = \underline{\omega} (\underline{\nabla} \cdot \underline{V}_T) \overset{0}{\rightarrow} + \underline{V}_T \cdot \underline{\nabla} \underline{\omega} - \underline{\omega} \cdot \underline{\nabla} \underline{V}_T - \underline{V}_T (\underline{\nabla} \cdot \underline{\omega}) \overset{0}{\rightarrow} \quad (3.39)$$

where the first and last terms are null, respectively, because all diagonal terms in $\underline{\nabla}V_T$ are identically zero, and because a rotor field is divergence-free. Also note that the third term vanishes in the case of 2D flows.

Therefore, concerning the transport term in the vortex force, we can write:

$$\begin{aligned}\underline{F}_{\underline{\ell}}^T &= -\rho \int_{\mathcal{V}} \underline{r} \times (\underline{V}_T \cdot \underline{\nabla} \underline{\omega} - \underline{\omega} \cdot \underline{\nabla} V_T) d\mathcal{V} + \rho \int_{\partial B \cup \Sigma} \underline{r} \times [\underline{n} \times (\underline{\omega} \times \underline{V}_T)] dS = \\ &= -\underline{F}_t^T + \rho \int_{\mathcal{V}} \underline{r} \times (\underline{\omega} \cdot \underline{\nabla} V_T) d\mathcal{V} - \underline{F}_{S_B}^T - \underline{F}_S^T\end{aligned}\quad (3.40)$$

Eventually, all the transport terms ($\underline{F}^T$) in the total force cancel-out and the aerodynamic force formula becomes:

$$\underline{F}_{ff}^{mix} = \underline{F}_{\underline{\ell}}^{mix} + \underline{F}_t^{mix} + \underline{F}_S^{mix} + \underline{F}_{S_B}^{mix} + \underline{F}_{t_B} + \underline{F}_{\mu} + \underline{F}_{3D} \quad (3.41)$$

with:

$$\underline{F}_{3D} = -\rho \int_{\mathcal{V}} \underline{r} \times [(\underline{\Omega} \times \underline{\omega}) - \underline{\omega} \cdot \underline{\nabla} V_T] d\mathcal{V} \quad (3.42)$$

This term is new and was not present in the two-dimensional formulation derived by Ostieri et al. [140]. Similarly, the body surface integral $\underline{F}_{S_B}^{mix}$ was not considered in [140] because no-slip wall conditions in the relative reference frame were assumed in the case of viscous flows.

Finally, considering that $\underline{\ell}^{mix} = \underline{\omega}' \times \underline{V}' + 2\underline{\Omega} \times \underline{V}' = \underline{\ell}' + 2\underline{\Omega} \times \underline{V}'$ and recalling Eq. (A10) and (A2), we can re-write body terms avoiding the integration of spatial derivatives at the wall:

$$\begin{aligned}\underline{F}_B^{mix} &= \underline{F}_{S_B}^{mix} + \underline{F}_{t_B} = \rho \int_{\partial B} \underline{r} \times [\underline{n} \times (\underline{a} - \underline{\ell}^{mix})] dS = \\ &= \rho \int_{\partial B} \underline{r} \times \underline{n} \times \left[\frac{\partial \underline{V}'}{\partial t'} + \frac{d\underline{U}}{dt} + \left(\frac{d\underline{\Omega}}{dt} \times \underline{x}' \right) + \underline{\Omega} \times (\underline{\Omega} \times \underline{x}') \right] dS - \rho \int_{\partial B} \frac{V'^2}{2} \underline{n} dS\end{aligned}\quad (3.43)$$

and re-arrange the total force formula as:

$$\boxed{\underline{F}_{ff}^{mix} = \underline{F}_{\underline{\ell}}^{mix} + \underline{F}_t^{mix} + \underline{F}_S^{mix} + \underline{F}_B^{mix} + \underline{F}_{\mu} + \underline{F}_{3D}} \quad (3.44)$$

For viscous flows around rigid and impermeable bodies, assuming the non-inertial reference frame solidal to the moving body, the no-slip wall boundary condition ($\underline{V}' = \underline{0}$) makes this term reducing to:

$$\underline{F}_B^{mix} = \rho \int_{\partial B} \underline{r} \times \underline{n} \times \left[\frac{d\underline{U}}{dt} + \left(\frac{d\underline{\Omega}}{dt} \times \underline{x}' \right) + \underline{\Omega} \times (\underline{\Omega} \times \underline{x}') \right] dS \quad (3.45)$$

in which only the velocities of the non-inertial reference frame are present. Therefore this term is in this case independent of the flow-field and represents the direct contribution of the body

motion to the aerodynamic force.

3.1.7.1 THE CASE OF PURE PLUNGING MOTION

It is worth observing that, in the case of pure plunging motion ($\underline{\Omega} = \underline{0}$), absolute and relative vorticity coincide, and so is for the non-inertial and mixed Lamb vector definitions ($\underline{\ell}' = \underline{\ell}^{mix}$). Therefore, the vortex force, as well as unsteady and outer vorticity terms in Eq. (3.23) and Eq. (3.44) also coincide ($\underline{F}'_{\ell} = \underline{F}^{mix}_{\ell}$; $\underline{F}'_t = \underline{F}^{mix}_t$; $\underline{F}'_S = \underline{F}^{mix}_S$). Moreover, in the expression of Eq. (3.43) with $\underline{\Omega} = \underline{0}$, a DMT can be applied using Eq. (A1) to the space-independent vector field $\frac{d\underline{U}}{dt}$, obtaining:

$$\underline{F}^{mix}_B = \underline{F}'_B + \underline{F}_m \quad (\text{for pure plunging}) \quad (3.46)$$

The transport speed $\underline{V}_T$ is also space-independent in absence of frame rotation ($\underline{\Omega} = \underline{0}$), then the three-dimensional effects term is also zero ($\underline{F}_{3D} = \underline{0}$).

Therefore, in the case of pure plunging motion, the mixed and non-inertial formulations do coincide.

3.1.8 LAMB-VECTOR-BASED FORCE DECOMPOSITION

Different decompositions of the aerodynamic force are herein proposed and illustrated using the original compressible vortical force formulation ($\underline{F}_a$), although similar considerations apply to all other previously illustrated formulations.

Two-dimensional flows around oscillating aerofoils are considered and the integration domain is selected with a planar downstream portion of Σ (*wake plane*) orthogonal to the flight direction and intercepting the aerofoil trailing edge, where the DMT pole is also located. Note that this CV choice, for high-Reynolds number flows, tends to separate the bound vorticity (which is kept in the interior of $\mathcal{V}$) from the free vorticity in the unsteady wake (which is left outside of $\mathcal{V}$). This distinction is theoretically achieved in an exact way only in the asymptotic case of effectively-inviscid flows ($Re_{\infty} \rightarrow \infty$) since, in the case of finite Reynolds number, in the very near wake a vorticity directly generated by the body boundary layer is still present.

When the DMT pole is on the wake plane, provided the remaining portion of Σ is not too close to the body, $\underline{F}_S$ and $\underline{F}_{\mu}$ have no lift component. For pitching oscillations ($\underline{\Omega} \neq \underline{0}$), this is only obtained in inertial reference frames, as the $\underline{F}_S$ integral cannot otherwise be restricted to the wake plane. In the case of pure plunging motion, instead, it can be generalised to non-inertial reference frames as well. In such cases, $\underline{F}_S + \underline{F}_{\mu}$ provides the *parasite* drag and lift is then only given by $\underline{F}_{\ell} + \underline{F}_p + \underline{F}_t + \underline{F}_m$.

Links between this vortical lift expression and the classical linear theory by Theodorsen (Ch. 1, Sec. 1.2.2.2) have been evidenced in [139, 142, 169] and a rigorous derivation and discussion is provided in Appendix A.5.

The drag (or thrust) force, D_i , induced by the unsteady lift is therefore given by:

$$D_i = \left(\underline{F}_\ell + \underline{F}_p + \underline{F}_t + \underline{F}_m \right) \cdot \underline{e}_x \quad (3.47)$$

where $\underline{e}_x$ is a unit vector aligned with the free-stream velocity V_∞ . Its time-average in the oscillation period is expected to be of negative sign (thrust) in the inviscid linear incompressible case and for pure plunging motion, as proved by Garrick [61].

Here an hypothesis is formulated (subject to following verification in Section 3.2.7), stating that thrust will still be provided by D_i for plunging oscillations in a generally viscous and compressible flow, while a positive drag will instead be provided by the parasite contribution:

$$D_p = \left(\underline{F}_s + \underline{F}_\mu \right) \cdot \underline{e}_x \quad (3.48)$$

The above decomposition corresponds to the unsteady version of Mele et al. [117] decomposition, proposed by the authors for steady flows (Ch. 2, Sec. 2.1.7). Compared to the last unsteady extension of this decomposition by Ostieri et al. [142], Eq. (3.47) and (3.48) are now generalised to non-inertial frames of reference and to inviscid flows.

As illustrated in Ch. 2, in the steady regime Fournis et al. [57] proposed an update of the decomposition which is mathematically pole-independent. This was achieved through the proof of two vectorial identities, linking vortical to classical force formulas, recalled hereafter for the sake of clarity:

$$\underline{F}_\ell + \underline{F}_p - \underline{F}_{\nabla\rho} = \underline{F}_{KJ} + \underline{F}_{MSK} \quad (3.49)$$

$$\underline{F}_s + \underline{F}_\mu + \underline{F}_{\nabla\rho} = \underline{F}_{BETZ} \quad (3.50)$$

A similar effort is done here for the unsteady regime. Indeed, Eq. (3.47) and (3.48) are pole-dependent (as steady version by Mele et al. [117]) and the above vectorial identities are no more valid in their original (steady) form. Their extension to the unsteady regime is instead formulated in this context (and for the first time) as:

$$\begin{aligned} \underline{F}_\ell + \underline{F}_p + \underline{F}_t + \underline{F}_m - \underline{F}_{\nabla\rho} = & \underline{F}_{KJ} + \underline{F}_{MSK} + \underline{F}_{TIME} + \underline{F}_{APP} + \underline{F}_{BODY} + \\ & - \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\rho \frac{\partial \underline{V}}{\partial t} - \rho \underline{f}_m \right) \right] dS \end{aligned} \quad (3.51)$$

$$\underline{F}_s + \underline{F}_\mu + \underline{F}_{\nabla\rho} = \underline{F}_{BETZ} + \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\rho \frac{\partial \underline{V}}{\partial t} - \rho \underline{f}_m \right) \right] dS \quad (3.52)$$

A pole independent decomposition can therefore still be achieved by incorporating the highlighted integrals in Eq. (3.51) and Eq. (3.52) in the $\underline{F}_{\nabla\rho}$ term, leading to an unsteady expression

of the additional compressibility correction:

$$\underline{F}_{\nabla\rho}^{uns} = \underline{F}_{\nabla\rho} - \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\rho \frac{\partial V}{\partial t} - \rho \underline{f}_m \right) \right] dS \quad (3.53)$$

which leads to revised unsteady identities, where the RHSs now contain no pole dependency:

$$\underline{F}_{\ell} + \underline{F}_{\rho} + \underline{F}_t + \underline{F}_m - \underline{F}_{\nabla\rho}^{uns} = \underline{F}_{KJ} + \underline{F}_{MSK} + \underline{F}_{TIME} + \underline{F}_{APP} + \underline{F}_{BODY} \quad (3.54)$$

$$\underline{F}_S + \underline{F}_{\mu} + \underline{F}_{\nabla\rho}^{uns} = \underline{F}_{BETZ} \quad (3.55)$$

The corresponding decomposition of the axial vortical force (LHSs, projected along $\underline{e}_x$), representing the unsteady version of Fournis et al. [57] drag decomposition, cannot however be interpreted as a breakdown in induced and parasite contributions. In fact, the lift component of $\underline{F}_S + \underline{F}_{\mu} + \underline{F}_{\nabla\rho}^{uns}$ is now non-zero (for example, the integral in Eq. (3.53) cannot be restricted on the wake plane), therefore it does not represent a *parasite* drag. Its interpretation is instead provided by Eq. (3.54) and (3.55), and it is worth observing that, for the two proposed decompositions, all unsteady and apparent force terms are entirely attributed to the induced drag component, following an approach similar to that used by Gariépy et al. [59] in their unsteady thermodynamic-based decomposition.

3.2 NUMERICAL ANALYSES

3.2.1 COMPUTED FLOW FIELDS

CFD solutions were obtained for an oscillating NACA0012 aerofoil (unitary chord length) in pitching or plunging motion, at different values of the free-stream Mach number (M_{∞}) and Reynolds number (Re_{∞}), using *Stanford University Unstructured (SU2)* compressible and incompressible flow solvers [46, 145] (version 7.3.1 “*Blackbird*”). *SU2* solves flow governing equations using a nodes-centred Finite Volume Method (FVM) based on dual grids. Unsteady simulations were performed using a dual time stepping approach, with moving mesh technique and central *Jameson-Schmidt-Turkel* scheme [75]. Pseudo-time marching is based on an implicit Euler algorithm and no multi-grid procedure was used in this case to accelerate convergence. Weak (entropic) Euler solutions were computed in the case of effectively-inviscid flow simulations ($Re_{\infty} \rightarrow \infty$), while viscous analyses used an Unsteady Reynolds-Averaged Navier-Stokes (URANS) formulation of flow governing equations, with turbulence closure based on $k - \omega$ SST *Menter’s* model [120], assuming fully turbulent flow. Incompressible simulations assumed a constant molecular flow viscosity (independent of the static temperature) and the energy equation was not solved†.

†Indeed, in absence of heat flux through the body walls, the variation of static temperature due to viscous dissipation of kinetic into internal energy is negligible in the considered flow conditions (high Reynolds number and low Eckert number). This was also numerically verified in Sec. 3.2.4.

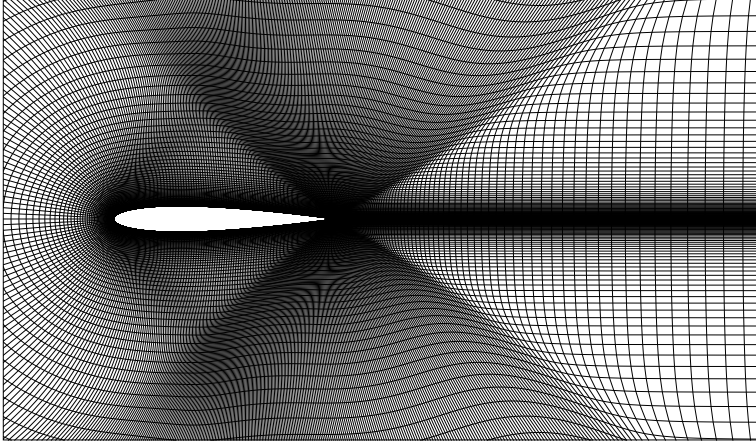


Figure 3.2.1: NACA0012 grid used in incompressible plunging URANS simulations.

The *SU2* flow solver was validated for the simulation of pitching and plunging oscillations in very recent studies, dealing with propulsive performance of flapping aerofoils [99] or with fluid-structure interaction and aeroelastic flutter [52]. Nevertheless, a validation study is also included in Appendix C, considering plunging oscillations in known flow conditions ($M_\infty \rightarrow 0$ and $Re_\infty = 2 \times 10^4$), where both experimental and numerical results are available from literature data.

As for steady simulations (Ch. 2), a slightly altered definition of NACA0012 aerofoil providing closed trailing edge (t.e.) is used [165]. Two different C-type grid were used for incompressible and compressible plunging simulations respectively. They were generated using the open-source software *Construct2D* (COMputational fluid dyNAMics STRUctured grid CreaTor for 2D aerofoils) [155] and the computational domain extends up to 300 chord lengths around the aerofoil, where free-stream boundary conditions are applied.

The mesh used in incompressible plunging simulations has 429 nodes on the aerofoil and 200 along the wake, resulting in an overall number of hexahedral cells equal to $828 \times 200 = 165600$ (viscous grid, Fig. 3.2.1). For compressible (and for pitching) simulations, instead, a refined mesh was used, with the same number of nodes on the aerofoil (429) but 1000 (instead of 200) along the wake, resulting in an overall number of hexahedral cells equal to $2428 \times 200 = 485600$ (viscous grid, Fig. 3.2.2) or $2428 \times 150 = 364200$ (inviscid grid). Grid lines were also forced to keep almost aligned with Cartesian axes downstream of the aerofoil, overriding default mesh-generator settings which tended to relax the grid refinement at the trailing edge off the body (as in Fig. 3.2.1). In viscous flow simulations, cells are clustered around the aerofoil surface to allow computing numerical solutions down to the wall, without using any wall function.

Grid convergence data justifying the adopted spatial and time discretization are reported in Appendix D.2.

The plunging motion was simulated at different values of the reduced frequency $k = \pi fc / V_\infty$ and of the non-dimensional plunging amplitude $\hat{h} = \hat{h} / c$, at right angles to the direction of flight

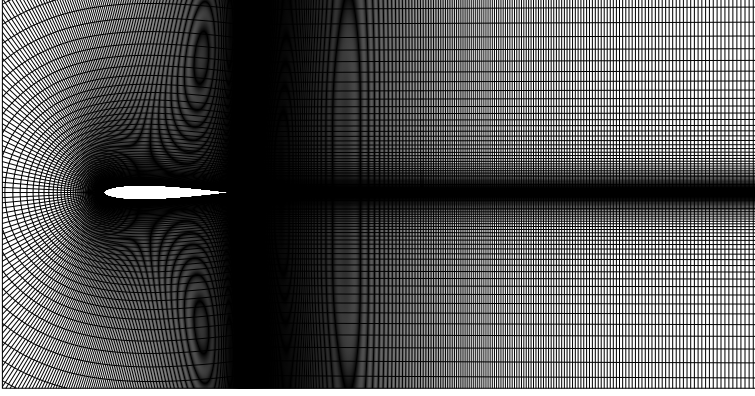


Figure 3.2.2: NACA0012 grid used in compressible or pitching URANS simulations.

(zero mean effective angle of attack: $\bar{\alpha}_e = 0^\circ$). Simulations were initialised from uniform free-stream conditions at time $t_0 = 0s$, when the harmonic plunging oscillation of time period $\mathcal{T}$ starts with the aerofoil moving downwards. Pitching oscillations, instead, started with the aerofoil increasing its angle of attack with regards to the free-stream flow, from an initially zero incidence setting. The centre of rotation was assumed at the quarter-chord point and the mean and amplitude values of the aerofoil angle of attack are denoted with $\bar{\alpha}$ and $\hat{\alpha}$ respectively, i.e.: $\alpha(t) = \bar{\alpha} + \hat{\alpha} \sin [2\pi f(t - t_0)]$. A summary of the analysed cases is reported in Tab. 3.2.1 and Tab. 3.2.2, where the physical time step (Δt) and the number of Dual Time Iterations (DTIs) used are also reported§.

In compressible simulations, the physical time step was selected as $\mathcal{T}/250$ for inviscid analyses and as $\mathcal{T}/500$ in the viscous case, while the number of Dual Time Iterations (DTIs) was set respectively at 2500 (inviscid case) or at 5000 (viscous case). This choice is conservative with regards to literature results where larger time steps were used for plunging aerofoils and time convergence was shown [99, 140]. As viscous simulations need more solver iterations to achieve similar convergence and residuals drop, instead of just increasing the number of dual time iterations, a combined approach was preferred and the physical time step was also halved to reduce peak values of flux residuals between time steps.

A contour plot of flow vorticity magnitude is illustrated in Fig. 3.2.3 for the subsonic viscous

§It is worth highlighting that, when the incompressible solver is used (as in unsteady simulations included herein for $M_\infty \rightarrow 0$), a divergence-free velocity field is always imposed, regardless of the free-stream and oscillatory motion conditions. This means that, even if $M_\infty = V_\infty/a_\infty \ll 1$, the fluid is considered incompressible whichever the reduced frequency value, when compressibility effects might instead arise in ideal gases (the Mach number based on the motion characteristic speed, $M_{motion} = (\pi f c)/a_\infty$, should also be considered). An infinite speed of sound is assumed ($a_\infty \rightarrow \infty$) and the ideal gas model is replaced with a constant-density thermodynamic model, with specific internal energy (and entropy variations) only depending on the fluid static temperature. Specific flow enthalpy, instead, is a function of both temperature and pressure ($h = e + p/\rho$), the latter being no more a thermodynamic variable.

Table 3.2.1: Summary of the analysed plunging cases.

M_∞	Re_∞	k	$\tilde{h}$	$\bar{\alpha}_e$	Δt	DTIs
$\rightarrow 0$	$\rightarrow \infty$	0.05	0.1	0.0°	$\mathcal{T}/100$	300
$\rightarrow 0$	9×10^6	0.05	0.1	0.0°	$\mathcal{T}/500$	1000
$\rightarrow 0$	$\rightarrow \infty$	0.50	0.1	0.0°	$\mathcal{T}/100$	300
$\rightarrow 0$	$\rightarrow \infty$	0.50	0.2	0.0°	$\mathcal{T}/100$	300
$\rightarrow 0$	9×10^6	0.50	0.1	0.0°	$\mathcal{T}/500$	1000
$\rightarrow 0$	$\rightarrow \infty$	1.00	0.1	0.0°	$\mathcal{T}/100$	500
$\rightarrow 0$	$\rightarrow \infty$	1.00	0.2	0.0°	$\mathcal{T}/100$	500
$\rightarrow 0$	9×10^6	1.00	0.1	0.0°	$\mathcal{T}/500$	1000
$\rightarrow 0$	$\rightarrow \infty$	2.00	0.1	0.0°	$\mathcal{T}/500$	500
$\rightarrow 0$	9×10^6	2.00	0.1	0.0°	$\mathcal{T}/500$	1000
0.5	$\rightarrow \infty$	1.00	0.1	0.0°	$\mathcal{T}/250$	2500
0.5	9×10^6	1.00	0.1	0.0°	$\mathcal{T}/500$	5000
0.7	$\rightarrow \infty$	1.00	0.1	0.0°	$\mathcal{T}/250$	2500
0.7	9×10^6	1.00	0.1	0.0°	$\mathcal{T}/500$	5000

Table 3.2.2: Summary of the analysed pitching cases.

M_∞	Re_∞	k	$\hat{\alpha}$	$\bar{\alpha}$	Δt	DTIs
$\rightarrow 0$	$\rightarrow \infty$	1.00	5.0°	0.0°	$\mathcal{T}/250$	2500
$\rightarrow 0$	9×10^6	1.00	5.0°	0.0°	$\mathcal{T}/500$	5000

flow-field ($M_\infty = 0.5$; $Re_\infty = 9 \times 10^6$) after 10 computed oscillation cycles, using two different zero-centred vorticity scales. The larger scale ($-5/s < \omega < 5/s$, Fig. 3.2.3a) is enough to visualise the unsteady wake vorticity produced by the plunging oscillations. However, with a smaller scale ($-0.05/s < \omega < 0.05/s$, Fig. 3.2.3b), spurious vorticity waves are observed, travelling with flow dilatation waves, revealing that vorticity is numerically not confined to dissipative flow regions as it should be in theory. Indeed, in high Reynolds number flows, the initially irrotational flow-field, should remain irrotational outside of boundary layers, shock waves and associated wakes. A very similar picture as Fig. 3.2.3b was also obtained for the inviscid subsonic case and a visualisation of flow entropy variation field shows that it is also affected by spurious waves. Considering the vorticity balance equation (either absolute or relative vorticity, as they coincide in a 2D pure plunging motion):

$$\frac{D\omega}{Dt} = \underline{\omega} \cdot \underline{\nabla} V - \underline{\omega} (\underline{\nabla} \cdot \underline{V}) + \frac{1}{\rho^2} \underline{\nabla} \rho \times \underline{\nabla} p + \underline{\nabla} \times \left[\frac{\underline{\nabla} \cdot \left(\underline{\tau}_v + \underline{\tau}_R \right)}{\rho} \right] + \underline{\nabla} \times \underline{f}_m \quad (3.56)$$

it is clear how, from the five terms on the RHS, the first (stretching/tilting) term is zero in a two-dimensional flow-field, the fifth (apparent forces) term is zero even in non-inertial reference frames (where $\underline{f}_m \neq \underline{0}$) as apparent forces are space-independent (and thus irrotational) in pure plunging motion, the fourth (viscous) term is also basically zero for high-Reynolds number flows in regions

where dissipative effects are negligible and, for the same reasons, the third (baroclinic) term is also zero as the unsteady flow would be isentropic outside of boundary layers, shock waves and associated wakes. The only non-zero term on the RHS of Eq. (3.56) is the second one (dilatation term). However, integrating vorticity equation along path-lines, when starting from freestream irrotational conditions ($\omega_\infty = 0$) there's no possibility for flow dilatation to produce fluid particles vorticity variations, as the velocity divergence is multiplied by a locally-null vorticity vector. However, in a numerical calculation, the artificial dissipation is also active, causing spurious vorticity to arise also in inviscid subsonic conditions and the flow dilatation term to become then effective.

Contour plots of relative Mach number around the plunging aerofoil in viscous transonic conditions are instead included in Fig. 3.2.4, at different time fractions of the oscillation period, where shock waves on both upper and lower aerofoil sides are visualised.

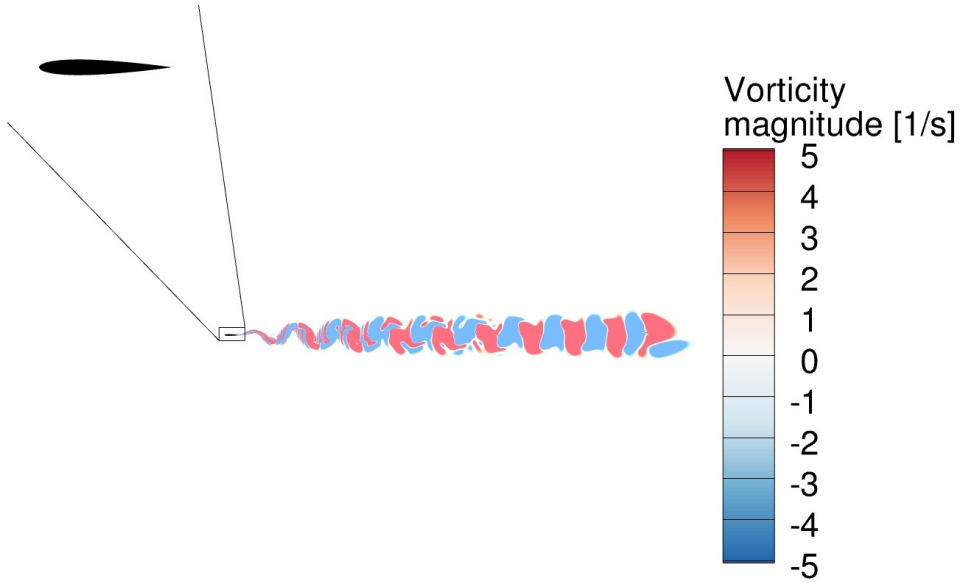
3.2.2 POST-PROCESSING METHOD

Numerical solutions described in Sec. 3.2.1 were post-processed using *UBF2D*, which also implements unsteady far-field methods illustrated in Sec. 3.1 (Appendix F). Results obtained with compressible far-field formulations “bc” and “d” are not included hereafter for two reasons: first, it will be shown (Sec. 3.2.6) that relatively small integration domains are often required for an accurate force calculation, which limits the applicability of Eq. (3.9) and (3.11) as they require the shock wave region to be fully included in the selected control volume; then, in Ch. 4, the need to modify the compressibility correction (E_{m_p}) in transonic flows to avoid domain integration around strong shocks will be overcome by the new approach proposed therein.

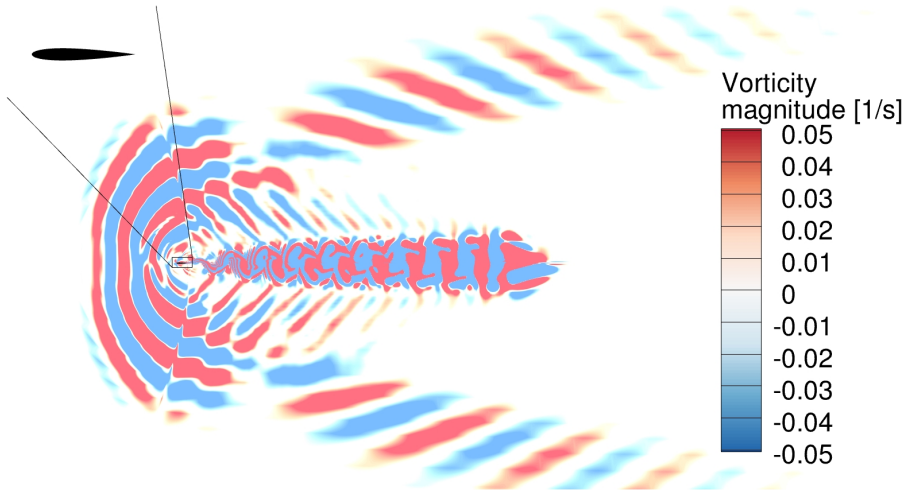
When dealing with turbulent flows governed by the RANS equations, the isotropic Reynolds stress tensor (related to the turbulent kinetic energy) was neglected and Boussinesq's approximation for the deviatoric component was assumed: $\underline{\tau}_R = 2\mu_t \left[\frac{1}{2} (\underline{\nabla} V + \underline{\nabla} V^T) - \frac{1}{3} \underline{\nabla} \cdot \underline{V} \right]$.

In the post-processing of 2D CFD solutions, spatial gradients of flow variables were computed using finite differences in the computational (non-curvilinear) space for incompressible flow-fields, while a nodes-based Green-Gauss formula has been used in compressible cases. Time derivatives were always computed using a 2^{nd} -order central finite differences scheme, while asymmetric stencils were adopted at time domain boundaries to preserve accuracy. Volume and surface integrals are numerically computed as described in Ch. 2, Sec. 2.2.2.

One oscillation cycle was analysed, after adequate convergence level of unsteady forces was reached (closed loops of instantaneous lift and drag). This corresponded to start post-processing after n complete computed oscillations, with $n = 10$ for the subsonic cases ($M_\infty = 0.5$) and $n = 4$ for transonic simulations ($M_\infty = 0.7$). The pole used in DMTs is located at the aerofoil trailing edge and therefore moves with it in time, as well as the employed CV which has always been selected as a subset of the (moving) computational grid and is identified by three orthogonal grid lines of the computational mesh, as illustrated in Fig. 3.2.5 (where the characteristic lengths used to define its size are also represented). In particular, X_W measures the chordwise position of the



(a) Larger scale.



(b) Smaller scale.

Figure 3.2.3: Flow vorticity around a plunging NACA0012 ($k = 1$; $\tilde{h} = 0.1$; $M_\infty = 0.5$; $Re_\infty = 9 \times 10^6$; $\bar{\alpha}_e = 0^\circ$). Snapshot at the end of 10 complete oscillations, using different vorticity scales.

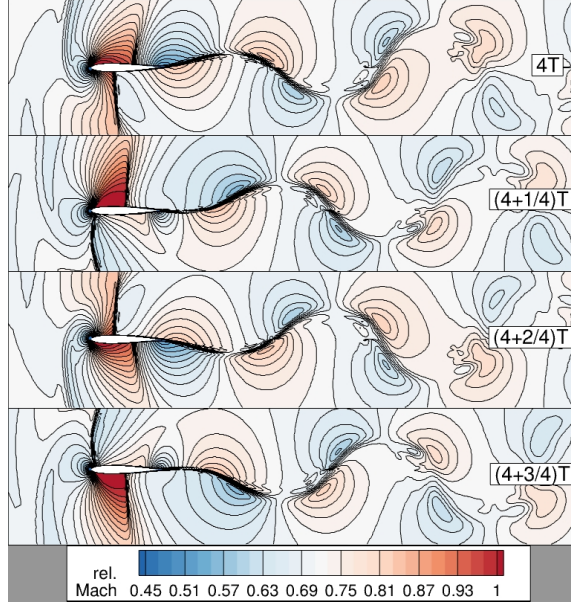


Figure 3.2.4: Relative Mach number contour plots, at different time period fractions - plunging NACA0012 ($k = 1$; $\tilde{h} = 0.1$; $M_\infty = 0.7$; $Re_\infty = 9 \times 10^6$; $\bar{\alpha}_e = 0^\circ$).

“wake plane”, delimiting the integration domain downstream of the aerofoil. The lateral distance of the C-shaped Σ edge from the aerofoil will be denoted as Z_W . Both X_W and Z_W will be expressed in chord units.

In what follows, a non-inertial frame of reference solidal with the rigid oscillating aerofoil (origin at leading edge, x-axis in chord-wise direction and z-axis pointing upwards) has been considered. This facilitates the computation of relative time derivatives in the case of moving grid simulations. Another, inertial, frame of reference is also considered, initially coincident in origin and axes orientation with the non-inertial one and fixed in time. The computation of absolute time derivatives (to be used in the inertial frame of reference) can move from the corresponding relative time derivatives using the transformations in Appendix A.2.

3.2.3 PLUNGING OSCILLATIONS IN INCOMPRESSIBLE FLOW

A comparison between near-field and far-field calculations of the instantaneous aerodynamic coefficients (the latter obtained using the mixed inertial/non-inertial formula) is reported in Fig. 3.2.6 and 3.2.7, for the simulations characterised by a plunging amplitude $\hat{h} = 0.1c$. Generally, far-field and near-field predictions are in satisfactory agreement, for both viscous and inviscid simulations. Results from Theodorsen’s linear theory [194] are also included in lift-coefficient plots. The reduction of the mean slope of the lift curve at moderately and highly unsteady regime ($k = 0.5, k = 1.0, k = 2.0$), already evidenced in the calculations by Ostieri et al. [140], reveals

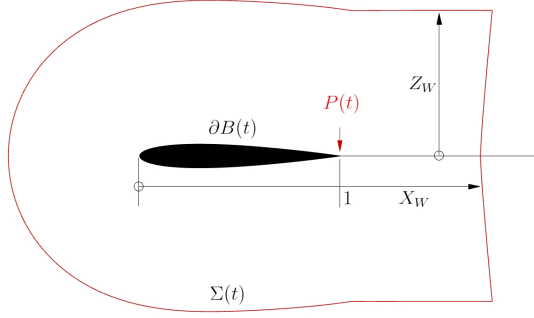


Figure 3.2.5: Control volume shape and characteristic dimensions.

an opposite trend compared to what observed in the quasi-steady regime (up to $k = 0.05$), where the aerofoil thickness results in a slightly higher average lift slope compared to Theodorsen's thin aerofoil theory. Indeed, the latter is an expected behaviour, as it approaches what experienced in steady-state conditions, where the thickness effect increases lift of thin aerofoils in the linear range. With the introduction of viscosity, the average lift curve slope is slightly reduced and the drag coefficient increases, leading to a reduction of the mean thrust coefficient as observed in Fig. 3.2.8b for both near-field and far-field results.

In the same figure, mean (time-averaged) thrust coefficients (both near-field and far-field estimates) computed for inviscid flow are compared against values obtained using Garrick's formula (Ch. 1, Eq. (1.76)), valid in the linear case (small perturbations assumption).

The agreement with reference linear theory is good even if, at the higher values of the reduced frequency, mean thrust coefficients are slightly overestimated by Garrick's formula. Such a discrepancy can be associated to non-linear effects, since the small perturbation hypothesis is lost as the oscillation frequency is increased and larger values of the effective incidence angle are experienced during the plunging oscillation. Additionally, even for small perturbations, Fernandez-Feria [49] highlighted that Garrick's theory needs to be corrected to account for suction effects not just at the aerofoil leading edge (Ch. 1, Sec. 1.2.2.3) but all over the body, which results in lower $\bar{C}_T$ values as the reduced frequency is increased.

The free-stream flow conditions used for inviscid flow simulations are: $V_\infty = 34\text{m/s}$, $\rho_\infty = 1.225\text{kg/m}^3$ and $T_\infty = 288.15^\circ\text{K}$, corresponding to $M_\infty = 0.10$. It was verified that, if a compressible (instead of incompressible) simulation is performed for the same flow conditions, the computed (near-field) mean thrust coefficient is closer to Garrick's linear estimate (Fig.3.2.8b). Such a simulation was performed using the compressible *SU2* solver, at the highest tested value of the reduced frequency ($k = 2$). However, this behaviour is misleading as non-linear discrepancies are hidden by compressibility effects, that tend to accentuate flow expansions at the leading-edge and to increase the thrust coefficient. This was the case of flow simulations reported in

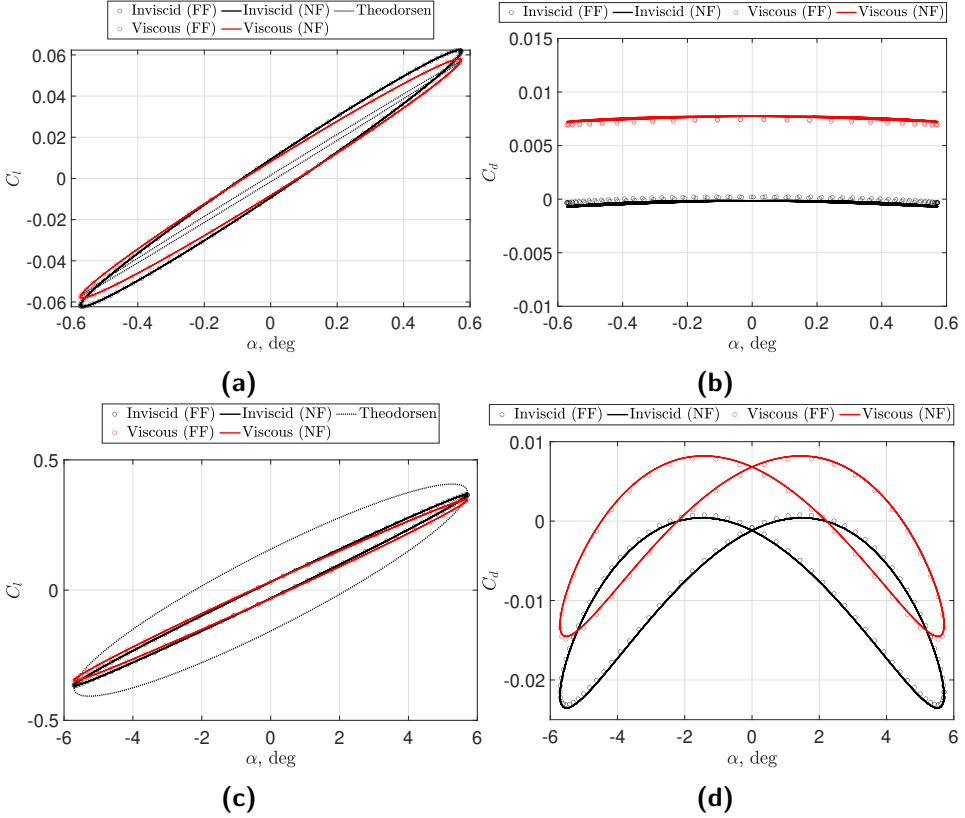


Figure 3.2.6: Lift and drag coefficients versus effective angle of attack $\alpha_e = \arctan(\dot{h}/V_\infty)$, comparison of near-field (NF), far-field (FF) and Theodorsen's estimations. Plunging NACA0012 ($\tilde{h} = 0.1$; $\bar{\alpha}_e = 0.0^\circ$; $M_\infty \rightarrow 0$; $Re_\infty = 9 \times 10^6$). DMT pole at the aerofoil trailing edge; $X_W/c = 1$, $Z_W/c = 1$. $k = 0.05$ in (a) and (b), $k = 0.50$ in (c) and (d).

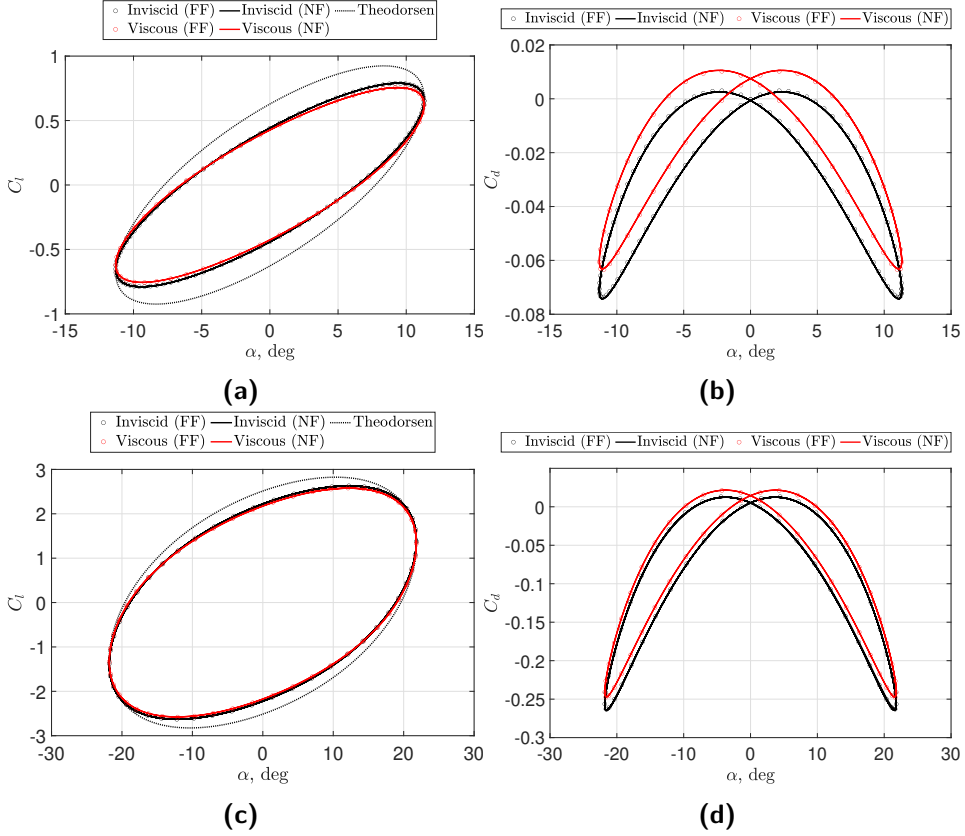


Figure 3.2.7: Lift and drag coefficients versus effective angle of attack $\alpha_e = \arctan(\dot{h}/V_\infty)$, comparison of near-field (NF), far-field (FF) and Theodorsen's estimations. Plunging NACA0012 ($\tilde{h} = 0.1$; $\bar{\alpha}_e = 0^\circ$; $M_\infty \rightarrow 0$; $Re_\infty = 9 \times 10^6$). DMT pole at the aerofoil trailing edge; $X_W/c = 1$, $Z_W/c = 1$. $k = 1.00$ in (a) and (b), $k = 2.00$ in (c) and (d).

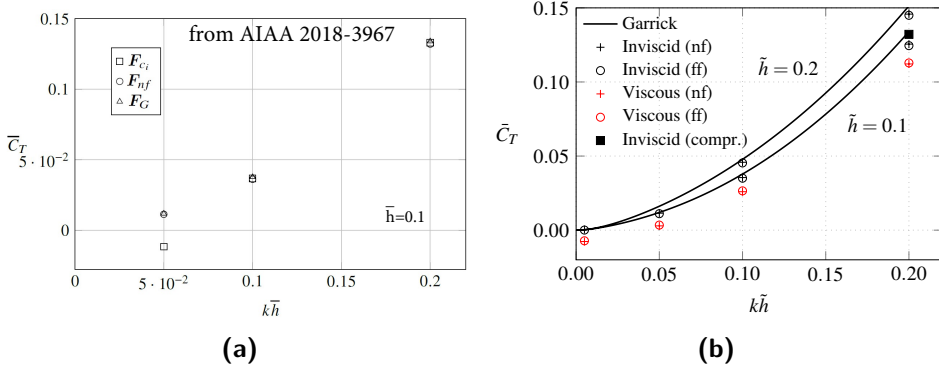


Figure 3.2.8: Comparison between far-field (ff), near-field (nf) and Garrick's mean thrust coefficients: 3.2.8a is extracted from [169] (Pag. 13, Fig. 9); 3.2.8b refers to the present study. Plunging NACA0012 ($\bar{\alpha}_e = 0^\circ$; $M_\infty \rightarrow 0$). DMT pole at the aerofoil trailing edge; $X_W/c = 1$, $Z_W/c = 1$.

[169] (Pag. 13, Fig. 9), where a compressible solver was used to compute unsteady flow solutions at $M_\infty = 0.10$, showing an apparently better agreement at high k values. For reader's convenience, the referenced plot extracted from [169] is also included in Fig. 3.2.8a. In the legend of the extracted graph, E_G indicates values computed using Garrick's formula, E_{nf} represent near-field data obtained by the compressible flow solver and E_{ci} denote far-field values computed using the mixed inertial/non-inertial vortical force formula on the inviscid flow solutions. Moreover, it should also be noted that the present comparison between near-field and far-field estimations (Fig. 3.2.8b) shows a good agreement also in the case at $k\tilde{h} = 0.05$, where Russo et al. did not report a satisfactory matching (Fig. 3.2.8a).

Results from viscous simulations are also included in Fig. 3.2.8b. Results computed with the mixed inertial/non-inertial formula are in satisfactory agreement with near-field data also in these fully turbulent high-Reynolds-number flow conditions, in the full range of analysed reduced frequency values.

In Fig. 3.2.9 an analysis of the (numerical) variation of far-field total drag coefficient with the integration domain dimension is illustrated, for the plunging case with unitary reduced frequency value and $\tilde{h} = 0.1$ (a similar behaviour was observed in all other simulated cases). In particular, time histories of the aerodynamic drag coefficients (both far-field and near-field estimates) are included for different control volumes, which are also shown in the figure together with vorticity contour plots.

It can be observed that, as the wake edge of Σ is moved downstream of the aerofoil trailing edge (in Fig. 3.2.9 $X_s = X_W - 1$ and varies from 0 to 300 chord lengths), the far-field aerodynamic force computation loses accuracy, with strong deviations from the near-field data observed up to $X_s \approx 50$. An improvement is then obtained on larger control volumes and no significant variations are observed for $50 < X_sc < 300$. A similar behaviour was also observed on lift coefficients,

although not reported in Fig. 3.2.9.

This can be explained by analysing the different terms involved in the mixed inertial/non-inertial far-field force formula. It contains domain integrals (in $\mathcal{V}$), external surface integrals (on Σ) and body skin integrals (on ∂B). The last terms are dominant over all others, especially for inviscid flows. Based on what observed in Fig. 3.2.9, domain and external surface integrals are for sure affected by spurious numerical effects in the region where $0 < X_W - 1 < 50$. Nevertheless, integrals on Σ vanish on large control volumes, as the flow-field is still undisturbed at large distances from the aerofoil, not yet reached by flow advection at the simulated time. For the same reason, the pollution of domain integrals observed in the wake region has a relatively lower impact on larger control volumes, where the majority of the integration domain falls outside of the computed unsteady wake. The dominant skin terms therefore tend to mask pollution effects on large control volumes and far-field force inaccuracy is then reduced. It is anticipated that the observed inaccuracies can now be significantly reduced through the unified thermodynamic/vortical methodology proposed in this dissertation, which will be discussed and illustrated in Ch. 4.

3.2.4 PITCHING OSCILLATIONS IN INCOMPRESSIBLE FLOW

Since in the case of pure plunging motion the mixed and non-inertial formulations coincide, to demonstrate the advantages of using the mixed formulation for unsteady incompressible flows, pitching oscillations were additionally considered, as detailed in Tab. 3.2.2. In particular, the weak Euler simulation ($Re_\infty \rightarrow \infty$) is also used as a test-case for the extension of the mixed formula to the analysis of inviscid flow-field derived in Sec. 3.1.7.

CFD simulations used the refined grid (Fig. 3.2.2) providing higher resolution especially in the unsteady wake region, to better analyse the far-field force variation with the control volume size. The energy equation was now solved (in the viscous case) and a Sutherland's dynamic viscosity model was used. Nevertheless, negligible temperature variations in the entire flow-field were observed, confirming the assumption used in Sec. 3.2.3. The post-processing covered one oscillation period, starting after 10 complete pitching cycles were computed.

Figure 3.2.10 and 3.2.11 provide comparison of lift and drag coefficients respectively, computed using the inertial (Eq. (3.22)), non-inertial (Eq. (3.23)) and mixed (Eq. (3.44)) far-field formulas, against reference data computed using near-field integration on the body wall (Eq. (1.1)), for the inviscid pitching simulation. A red arrow along curve loops indicates the marching direction during the analysed pitch oscillation. The selected control volume is relatively small in size ($X_W = 1.2, Z_W = 0.2$) and satisfactory results are provided by all far-field formulas. Similar plots are provided in Figure 3.2.12 and 3.2.13 for the viscous pitching simulation.

To better appreciate differences in compared formulations, which cannot be easily deduced from the included plots because of the larger unsteady variation scale, the time series standard deviation of a generic far-field force coefficient ($C_{l,d}^{ff}$), with regards to the corresponding near-field

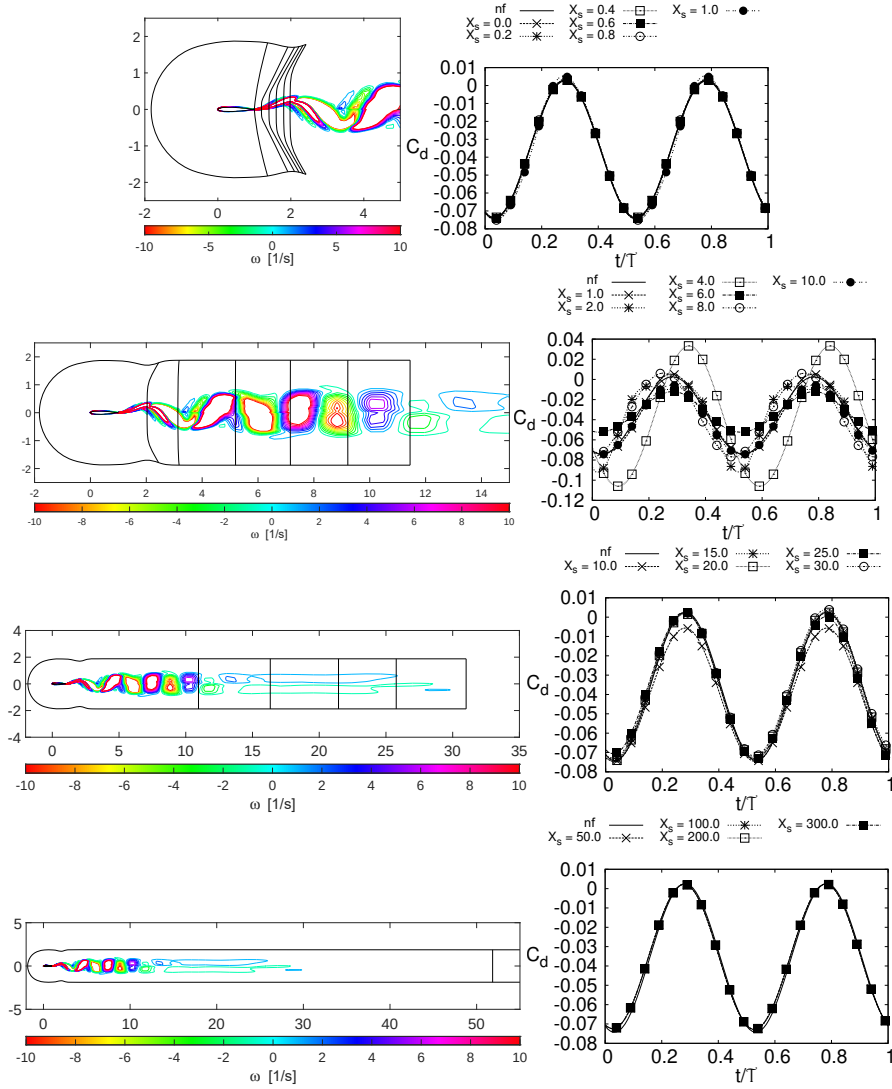


Figure 3.2.9: Far-field drag variation changing the wake plane position, for the plunging NACA0012 in incompressible inviscid flow ($\tilde{h} = 0.1$; $k = 1.0$; $\bar{\alpha}_e = 0^\circ$; $M_\infty \rightarrow 0$; $Re_\infty \rightarrow \infty$). DMT pole at the aerofoil trailing edge; $Z_W \approx 2$, $X_s = X_W - 1$.

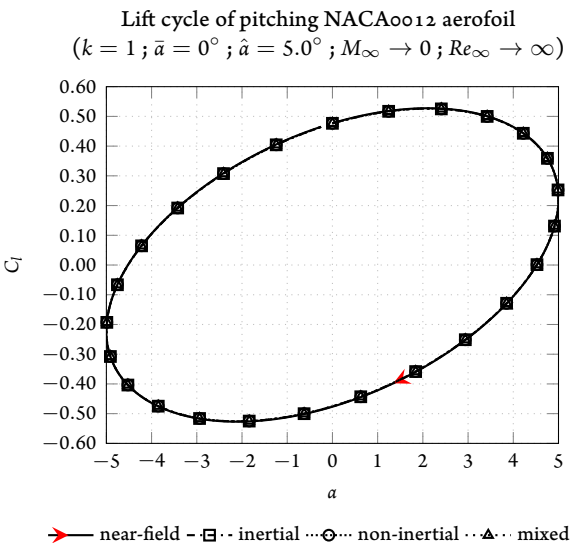


Figure 3.2.10: Near-field vs. far-field lift coefficient using incompressible vortical force formulas in inviscid flow ($X_W = 1.2$, $Z_W = 0.2$).

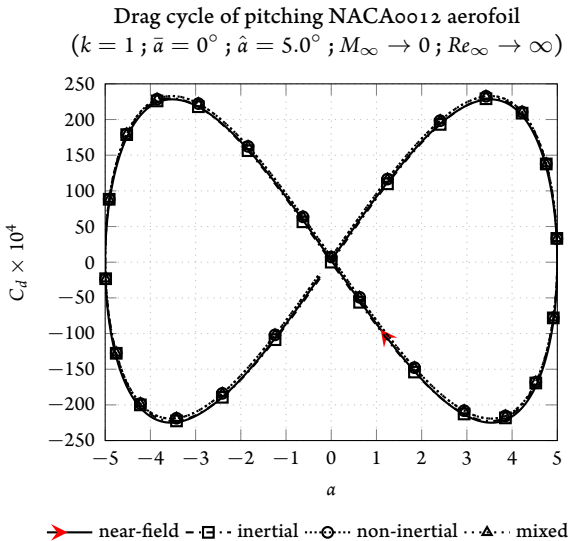


Figure 3.2.11: Near-field vs. far-field drag coefficient using incompressible vortical force formulas in inviscid flow ($X_W = 1.2$, $Z_W = 0.2$).

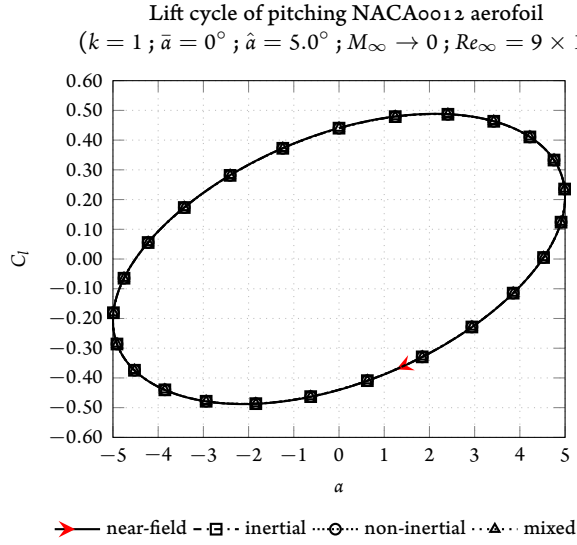


Figure 3.2.12: Near-field vs. far-field lift coefficient using incompressible vortical force formulas in viscous flow ($X_W = 1.2$, $Z_W = 0.2$).

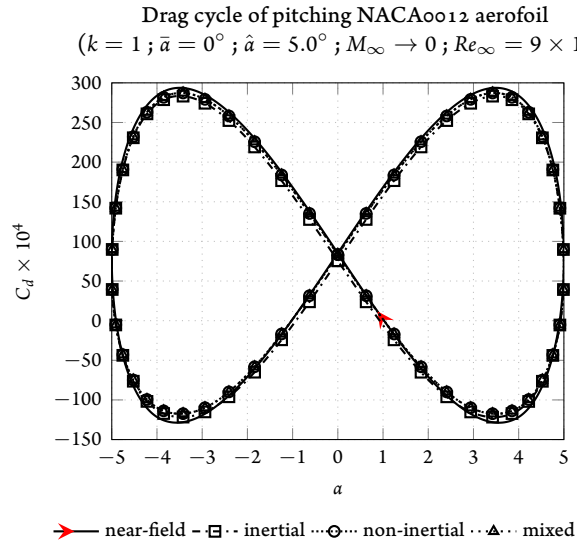


Figure 3.2.13: Near-field vs. far-field drag coefficient using incompressible vortical force formulas in viscous flow ($X_W = 1.2$, $Z_W = 0.2$).

Table 3.2.3: Standard deviation of far-field force coefficient time series ($\underline{F}_{ff}$), with regards to near-field data ($\underline{F}_{nf}$), in inviscid incompressible flow.

Formulation	σ_{C_l}	σ_{C_d}
inertial	8.94e-04	3.45e-04
non-inertial	9.20e-04	5.14e-04
mixed	8.47e-04	5.24e-04

Table 3.2.4: Standard deviation of far-field force coefficient time series ($\underline{F}_{ff}$), with regards to near-field data ($\underline{F}_{nf}$), in viscous incompressible flow.

Formulation	σ_{C_l}	σ_{C_d}
inertial	4.80e-04	9.55e-04
non-inertial	5.25e-04	8.16e-04
mixed	4.27e-04	8.13e-04

data ($C_{l|d}^{nf}$), is defined as follows (t_i is the discrete time):

$$\sigma_{C_{l|d}} = \sqrt{\frac{\sum_{i=1}^N \left[C_{l|d}^{ff}(t_i) - C_{l|d}^{nf}(t_i) \right]^2}{N}} \quad (3.57)$$

where N is the number of total time steps of the computed periodic series.

Table 3.2.3 and 3.2.4 include standard deviation data for the three compared formulations, highlighting that lift is accurate in all cases, while a loss in drag accuracy is observed in presence of viscosity. This will be improved by using the unified thermodynamic/vortical approach discussed in Ch. 4 and results included in Section 4.2.6 will show that accuracy is recovered in most cases. Previously, other authors [139, 140] reported much larger inaccuracies using the inertial formula, which were however not encountered in this work.

Nevertheless, as the integration domain is extended downstream of the pitching aerofoil, all formulas loose accuracy on drag (Fig. 3.2.14) and on lift (similar plot, not included herein), as already observed in Sec. 3.2.3. However, this behaviour is divergent for the non-inertial formulation, due to the lack of the mid-field property in this case. As the control volume size is enlarged, the integration is extended to flow-field regions where the computational mesh is not sufficiently refined, accumulating errors in the computed far-field force as their contribution is non-negligible in absence of the mid-field property. In particular, the diverging behaviour is due to the presence of position vectors $\underline{x}$ in the expression of apparent forces ($\underline{f}_m$), which are themselves multiplied by (scaled) position vectors in the surface integral of the $\underline{E}_m$ term, resulting in a non-linear dependence on $\underline{x}$ vectors. Their magnitude increases far from the centre of rotation, but the grid is therein quite coarse and does not provide adequate resolution for apparent forces.

This effect is even stronger when the lateral CV dimension Z_W is also increased (from 0.2 to 10.0, Fig. 3.2.15), resulting in large inaccuracies of the non-inertial formula at any X_W position.

The new unsteady terms formulation ($\underline{F}_t^*$, applicable to both inertial and non-inertial formulas) allows reducing inaccuracies in the non-inertial frame of reference (although not at a satisfactory level), while almost no effect was observed from its application in inertial frames for this pitching case¶. Two different scales for the drag coefficients are used in Fig. 3.2.14 and 3.2.15 to highlight the computed force divergence in non-inertial reference frames (observed for both standard and new unsteady terms formulations). The (mid-field) inertial and mixed formulations provide reasonable accuracy as long as the unsteady wake is captured with sufficient accuracy ($X_W < 2$). Further downstream the flow vorticity is gradually dissipated and the far-field methods suffer of similar problems as those already observed and discussed in Sec. 3.2.3. The mixed formula however behaves best among others, with smaller errors in the region of lower accuracy (σ_{C_d} peak). Very similar plots (not included herein) were obtained for the post-processing of the viscous pitching simulation⌘.

It is worth mentioning that the accuracy recovery downstream of the convected wake ($X_W > 50$, Fig. 3.2.15), already observed in plunging simulations, is not realised in the case with $Z_W = 0.2$ (Fig. 3.2.14). Surface integrals on Σ do not vanish in this case even if the wake plane is in a region of undisturbed flow, as the lateral CV dimension is too small, causing the C-edge crossing the unsteady wake (Fig. 3.2.16). A similar phenomenon was also observed and discussed in steady simulations (Appendix B, Fig. B.1.17).

3.2.5 PLUNGING OSCILLATIONS IN SUBSONIC FLOW

Fig. 3.2.17 and Fig. 3.2.18 illustrate, for plunging oscillations in viscous subsonic flow, a comparison of near-field lift and drag coefficients versus their far-field counterparts, computed using the $\underline{F}_a$ formula in either inertial or non-inertial reference frame and with the unsteady terms computed by Eq. (3.3) or Eq. (3.13). The wake plane is located at the aerofoil trailing edge and the Z_W coordinate (Fig. 3.2.5) is equal to 0.2. Time marching along lift and drag loops is indicated highlighting points at the beginning ($n\mathcal{T}$) of the time period $\mathcal{T}$ and after a fraction of it ($n\mathcal{T} + \mathcal{T}/25$). The aerofoil non-dimensional upwards displacement from its initial reference position is plotted on the ordinates axis, taking the opposite of the relative vertical position (h/c), which is conventionally positive in the downwards direction for a plunging motion.

All compared formulations provided results in close agreement with near-field data (three of the four symbols are superimposed, altogether looking like a filled circle), except when the unsteady terms were computed in the standard way (Eq. (3.3)) and based on absolute velocities. This inaccuracy can be related to the anticipated computation of spatial gradients (involved in Eq. (3.3) and not in Eq. (3.13)) using absolute velocities, also considering that in pure plunging

¶Its advantage will be much more clear on plunging cases, illustrated in the next sections. Here it was highlighted that the use of $\underline{F}_t^*$ can bring benefits also on different cases, such as the pitching oscillations considered herein.

⌘It is anticipated that the observed inaccuracies can now be significantly reduced through the unified thermodynamic/vortical methodology proposed in this dissertation, which will be discussed and illustrated in Ch. 4.

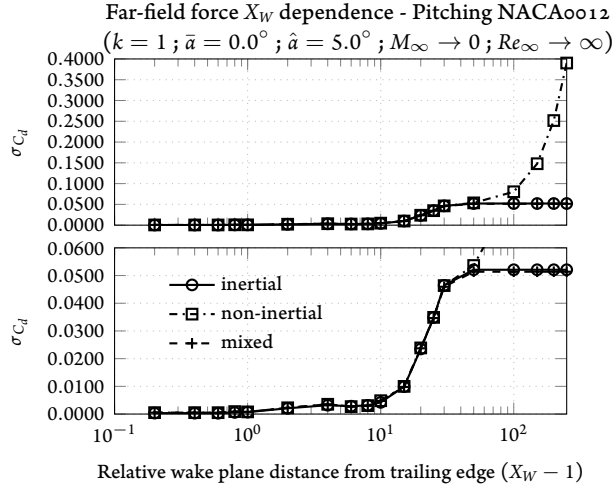


Figure 3.2.14: Total far-field drag variation with wake plane position, for $Z_W = 0.2$: large (top) and small (bottom) drag scales.

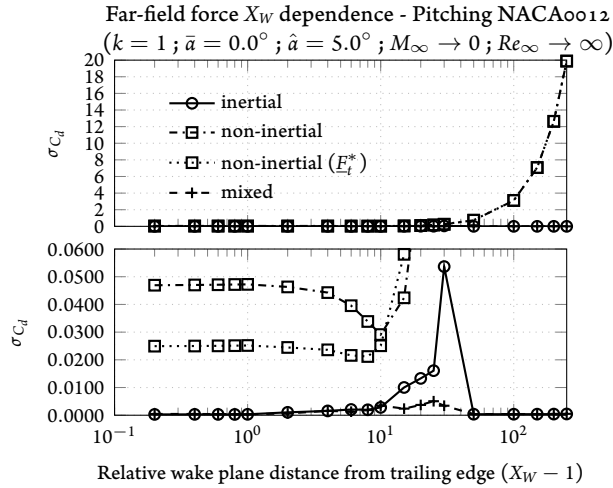


Figure 3.2.15: Total far-field drag variation with wake plane position, for $Z_W = 10.0$: large (top) and small (bottom) drag scales.

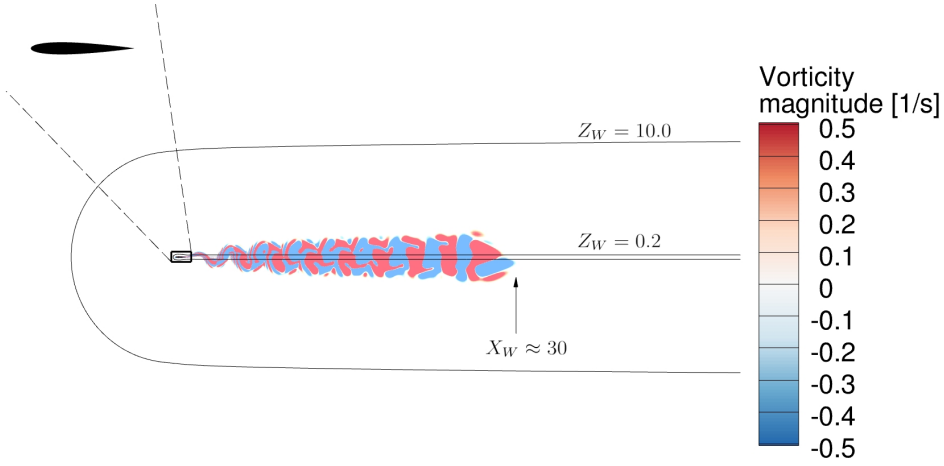


Figure 3.2.16: Flow vorticity around a pitching NACA0012 ($k = 1$; $M_\infty \rightarrow 0$; $Re_\infty \rightarrow \infty$; $\bar{\alpha} = 0^\circ$; $\hat{\alpha} = 5^\circ$). Snapshot at the end of 9 complete oscillations and CV reference dimensions.

Table 3.2.5: Standard deviation of far-field force coefficient time series, with regards to near-field data, in viscous subsonic flow.

Formulation		σ_{C_l}	σ_{C_d}
U.V.E. ¹	$\underline{F}_a$ inertial	1.09e-01	1.22e-03
	$\underline{F}_a^*$ inertial	4.58e-04	4.02e-04
	$\underline{F}_a$ non-inertial	4.50e-04	3.99e-04
	$\underline{F}_a^*$ non-inertial	4.75e-04	3.99e-04
	$\underline{F}_b^*$ non-inertial	5.11e-04	4.28e-04
	$\underline{F}_c^*$ non-inertial	4.74e-04	2.61e-04
		S.V.E. ²	

¹ Unsteady Variant Effect. ² Steady Variant Effect.

motion grid velocities only affect the vertical speed component. This inaccuracy is avoided using the alternative formulation of unsteady terms: $\underline{F}_t^*$.

Drag coefficient has two minima per oscillation cycle: one in the up-stroke and one in the down-stroke, the average drag being of negative sign (thrust) at the simulated Reynolds number (Knoller-Betz or Katzmayr effect [16, 82, 85]).

Tab. 3.2.5 includes data of the calculated standard deviation (as defined in Sec. 3.2.4) for the different $\underline{F}_a$ formulations illustrated in Fig. 3.2.17 and Fig. 3.2.18 (Unsteady Variant Effect: U.V.E.).

Selecting the non-inertial reference frame along with the alternative unsteady terms expression ($\underline{F}_t^*$), a further comparison is presented in Tab. 3.2.5 for the different force formulas ($\underline{F}_a^*$, $\underline{F}_b^*$ and $\underline{F}_c^*$) as defined in Sec. 3.1 (Steady Variant Effect: S.V.E.). Formulation “c” provided the most accurate results, a feature that the author already observed post-processing steady RANS solutions [124].

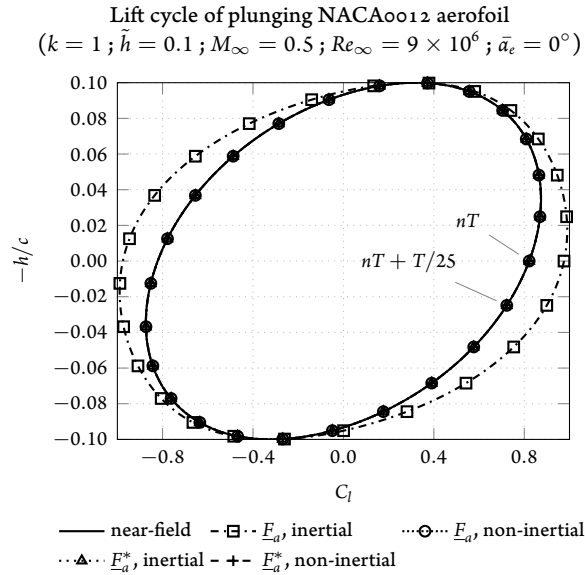


Figure 3.2.17: Near-field vs. far-field lift coefficient using force formula “a”, in viscous subsonic flow.

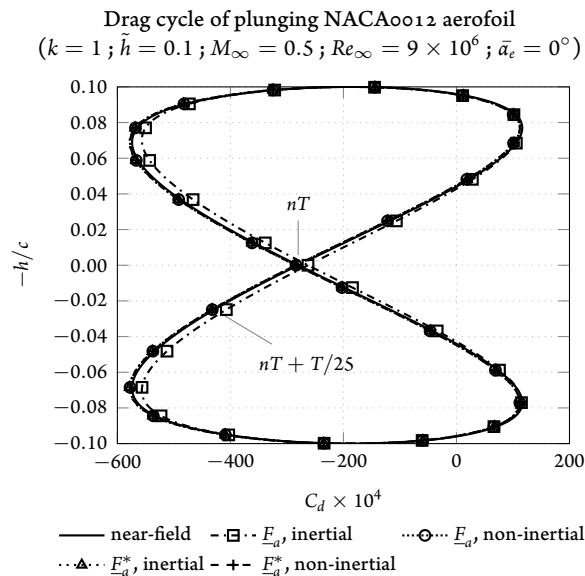


Figure 3.2.18: Near-field vs. far-field drag coefficient using force formula “a”, in viscous subsonic flow.

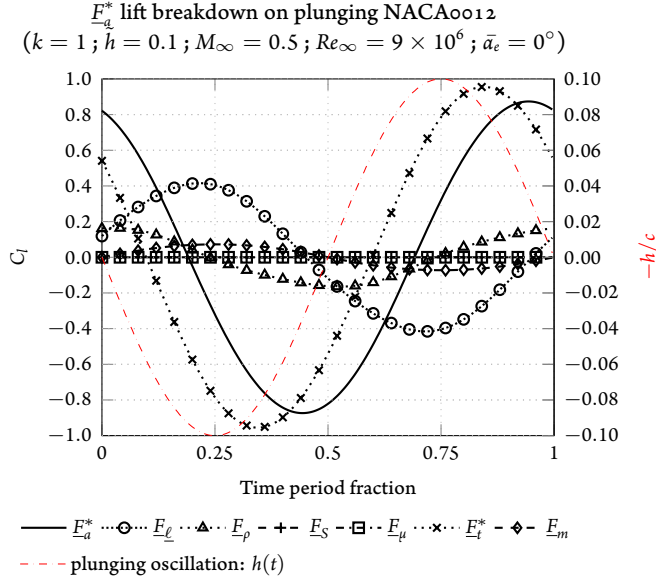


Figure 3.2.19: Lift breakdown into individual $\underline{F}_a^*$ constituents, in the non-inertial reference frame and for viscous subsonic flow.

A breakdown of total lift and drag (computed using $\underline{F}_a^*$ formula in the non-inertial reference frame) into individual components of the vortical force expression is provided in Fig. 3.2.19 and Fig. 3.2.20. A strong temporal delay in lift generation, with regards to the plunging motion, is visible (Fig. 3.2.19), as expected being the flow regime highly unsteady at the simulated reduced frequency value ($k = 1$). The unsteady term ($\underline{F}_t^*$) is dominant on lift and oscillating around zero on drag, while compressibility corrections are comparable in magnitude with the vortex force components. It is observed that the $\underline{F}_S$ term only provides a drag contribution, almost constant in time, with the adopted choice of the CV.

Also in this case, a sensitivity analysis on the variation of total lift and drag (computed using the non-inertial $\underline{F}_c^*$ formula) with the CV selection, has been performed modifying the wake plane position for different values of the Z_W dimension. Fig. 3.2.21 and Fig. 3.2.22 illustrate the variation of respectively lift and drag cycles standard deviations from near-field data. The parameter $\sigma_{C_{l,d}}$ is a measure of the absolute error between far-field and near-field force coefficients over one oscillation period and, for drag, a measure of the relative error is also provided in Fig. 3.2.22 considering the ratio of σ_{C_d} over the modulus of the time-averaged near-field drag coefficient C_d^{nf} (for lift, the time average is zero in this case). Although the mid-field property is not verified by any of the compressible Lamb-vector formulas, errors do not diverge in this case as the CV size is progressively increased, differently from what observed in presence of pitching oscillations for the non-inertial formulation. For pure plunging, apparent forces are in fact spatially independent.

It is clear that, again, a loss of accuracy occurs as the CV extends in the region were large vortic-

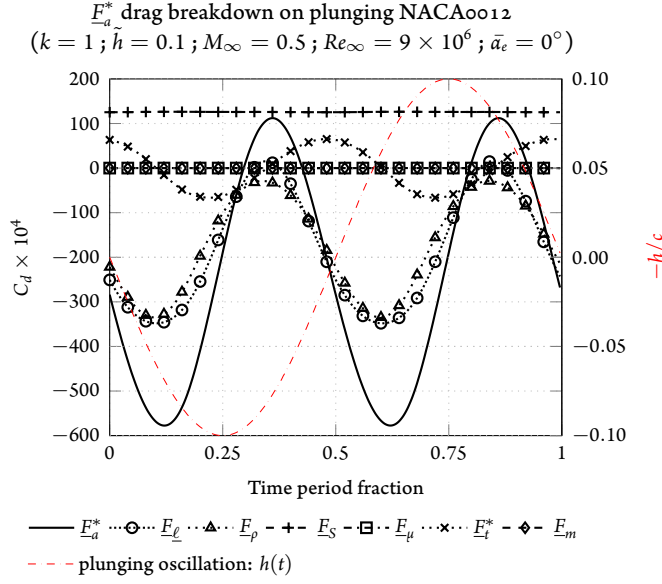


Figure 3.2.20: Drag breakdown into individual F_a^* constituents, in the non-inertial reference frame and for viscous subsonic flow.

ity variations were observed (Fig. 3.2.23), with a peak on CVs with size between 20 and 30 aerofoil chords. This region corresponds to the location up to which the vortical structures highlighted in Fig. 3.2.3 have been advected downstream of the aerofoil at the simulated time used in post-processing ($10T \leq t < 11T$). When the wake plane is located a few chord lengths downstream of the aerofoil, total lift is correctly computed for both small and high values of the Z_W distance, while small Z_W values are required to achieve an accurate matching between far-field vs. near-field total drag. Selecting an X_W position downstream of the computed unsteady wake, total drag is correctly computed in the inviscid flow case for both small and high values of the Z_W distance, while large inaccuracies are still observed in the viscous case for all analysed Z_W values.

Different reasons can be at the basis of the observed inaccuracies, causing a pollution of computed results which decreases (but does not vanish) for CVs encompassing the vorticity waves region. Indeed, a first source of inaccuracy is envisaged in the presence of a shear surface in the aerofoil wake during the unsteady oscillation. Like shock waves, this shear surface is a flow-field discontinuity for inviscid flows (and, as such, should be considered as a CV boundary**) as far as *strong* solutions of the Euler equations are concerned. This is useful, for example, in establishing links between the vortical force theory and classical unsteady potential theories (Appendix A, Sec. A.5). Nevertheless, when dealing with Navier-Stokes equations (which do not admit discontinuities) or looking for weak Euler solutions (as in numerical applications) shock waves and shear

**see, for example, Fig. A.5.1a in Appendix A, in the case of small perturbations (rectilinear wake discontinuity).

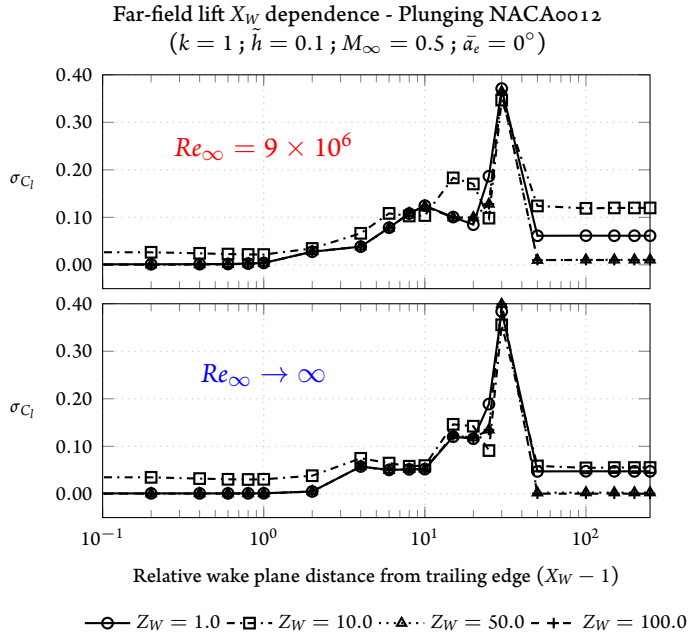


Figure 3.2.21: Total far-field lift variation with wake plane position in viscous (top) and inviscid (bottom) subsonic flow (Γ_c^* non-inertial).

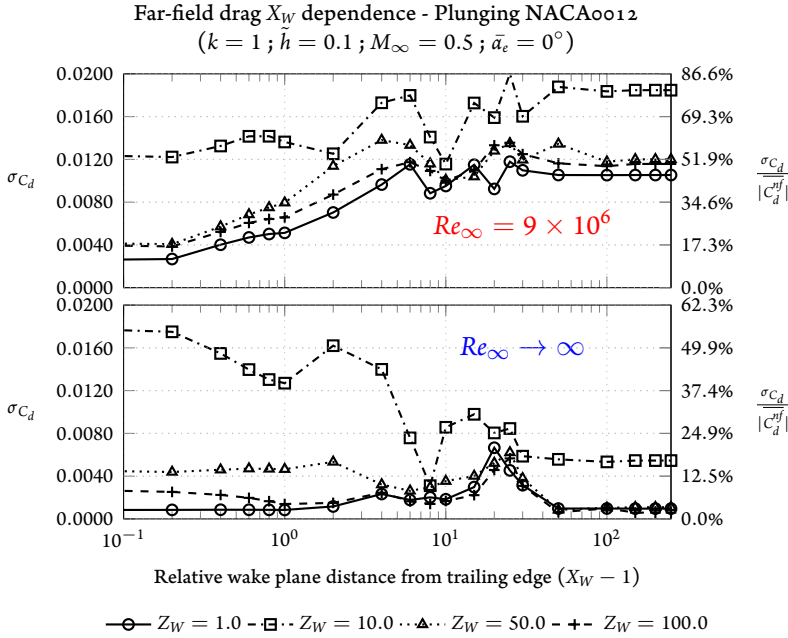


Figure 3.2.22: Total far-field drag variation with wake plane position in viscous (top) and inviscid (bottom) subsonic flow (Γ_c^* non-inertial).

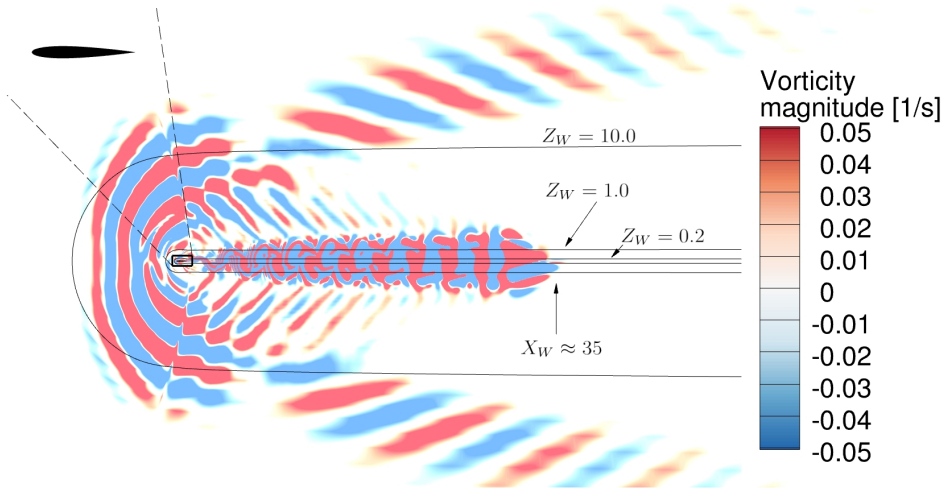


Figure 3.2.23: Flow vorticity around a pitching NACA0012 ($k = 1$; $\tilde{h} = 0.1$; $M_\infty = 0.5$; $Re_\infty = 9 \times 10^6$; $\bar{\alpha}_e = 0^\circ$). Snapshot at the end of 10 complete oscillations and CV reference dimensions.

layers can be included in the interior of the CV (the flow-field still being differentiable therein), nevertheless a loss in accuracy is expected in those regions due to the steep gradients of flow variables involved.

Eventually, the improvement observed for the inviscid case in far-field predictions on CVs fully including the vorticity waves, can be explained by the presence of non-zero body surface integrals in the vortical force expression. These (near-field) terms are dominant over volume (mid-field) integrals and mask their inaccuracy, instead emerging in the viscous case. Surface integrals over the external CV boundary (Σ) provide no contribution on such *large* CVs (for either viscous or inviscid cases) as the far-field is still undisturbed in the simulated time interval.

A sensitivity study of the analysed far-field force methods to the spatial grid resolution is also included in Appendix D.3.

The inaccuracies observed herein will be largely reduced by the introduction of the unified thermodynamic/vortical approach discussed in Ch. 4, representing part of the advantages of the innovative far-field method proposed in this dissertation. In particular, the impact of the Lamb vector calculation approach on the highlighted force variations with the CV size will be investigated and discussed in Section 4.2.6.

3.2.6 PLUNGING OSCILLATIONS IN TRANSONIC FLOW

Starting from inviscid simulations, having observed that sufficiently small values of σ_{C_l} and σ_{C_d} are obtained in inviscid conditions also on large CVs, the post-processing of Euler transonic solutions was performed locating the outer integration domain boundary 50 chord-lengths off from the aerofoil surface, which of course includes any shock wave in the flow-field.

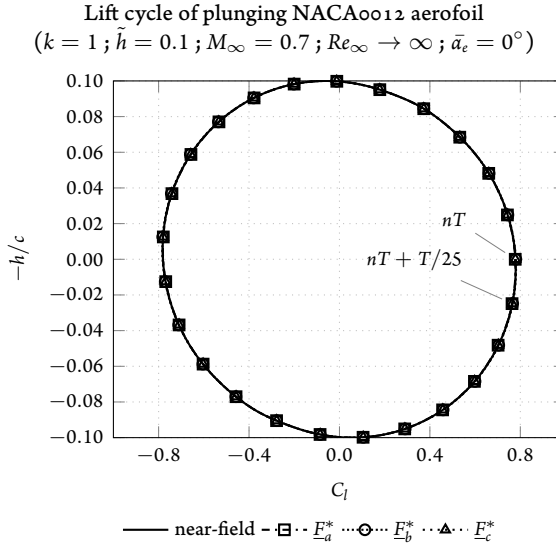


Figure 3.2.24: Near-field vs. far-field lift coefficient using non-inertial force formulas, in inviscid transonic flow.

A comparison of the different Lamb-vector-based drag formulas in the non-inertial reference frame is illustrated in Fig. 3.2.24 and Fig. 3.2.25 (unsteady terms are always computed according to Eq. (3.13)). As expected, formulations “a” and “b” are largely inaccurate on drag in presence of shock waves [116, 124], while far-field drag computed by formulation “c” is in satisfactory agreement with near-field data, despite the presence of the new volume integral term highlighted in Sec. 3.1.3.

The inaccuracy highlighted in the viscous subsonic case when using the inertial formulation with standard expression of unsteady terms, is not observed here for this inviscid case (Tab. 3.2.6), while it is clearly visible on lift computed for the viscous transonic solutions (Tab. 3.2.7, Fig. 3.2.26 and Fig. 3.2.27). Finally, although not reported herein, similar results as those illustrated for the

Table 3.2.6: Standard deviation of far-field force coefficient time series, with regards to near-field data, in inviscid transonic flow.

Formulation		σ_{C_l}	σ_{C_d}
S.V.E. ¹	$\underline{F}_a^*$ non-inertial	7.52e-04	8.40e-03
	$\underline{F}_b^*$ non-inertial	1.15e-03	7.70e-03
	$\underline{F}_c^*$ non-inertial	2.81e-04	2.10e-04
	$\underline{F}_c$ inertial	1.43e-03	4.68e-04
U.V.E. ²	$\underline{F}_c^*$ inertial	1.51e-03	5.07e-04
	$\underline{F}_c$ non-inertial	1.33e-04	1.84e-04

¹ Steady Variant Effect. ² Unsteady Variant Effect.

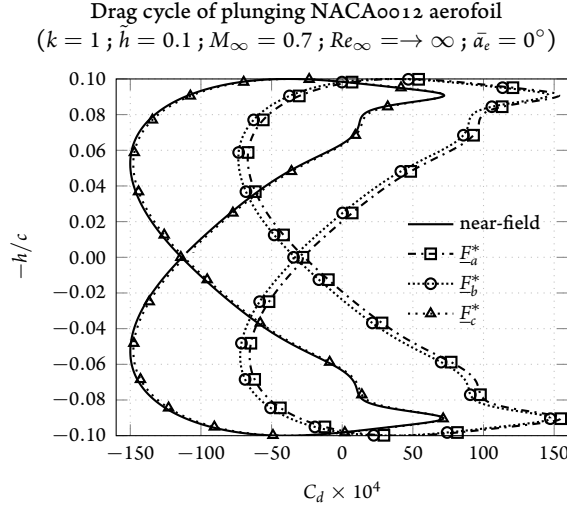


Figure 3.2.25: Near-field vs. far-field drag coefficient using non-inertial force formulas, in inviscid transonic flow.

Table 3.2.7: Standard deviation of far-field force coefficient time series ($\underline{F}_c$), with regards to near-field data, in viscous transonic flow.

Formulation	σ_{C_l}	σ_{C_d}
$\underline{F}_c$ inertial	3.02e-01	1.54e-02
$\underline{F}_c^*$ inertial	1.61e-03	6.98e-04
$\underline{F}_c$ non-inertial	1.50e-03	7.43e-04
$\underline{F}_c^*$ non-inertial	1.58e-03	7.78e-04

subsonic case, were obtained for the sensitivity of far-field total lift and drag coefficients to the integration domain selection.

3.2.7 THRUST/DRAG BOOKKEEPING

The proposed decompositions of the aerodynamic force, described in Section 3.1.8, are herein applied to the plunging NACA0012 aerofoil in viscous subsonic and transonic conditions. The hypothesis formulated in Section 3.1.8 and related to the identification of *thrust* and *drag* on plunging aerofoils immersed in a generally viscous and compressible flow will be verified. In all analysed cases, the wake plane intercepts the aerofoil trailing edge ($X_W = 1$), where the DMT pole is also located.

Fig. 3.2.28-3.2.33 illustrate results of total lift and drag decomposition respectively for the viscous incompressible, subsonic and transonic plunging simulations, using definitions from Eq. (3.47)/(3.48). Force coefficients are plotted as a function of the effective angle of attack: $\alpha_e = \arctan(\dot{h}/V_\infty)$.

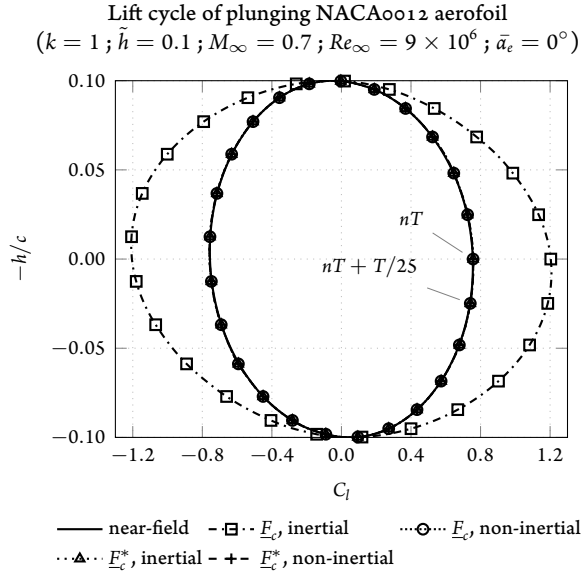


Figure 3.2.26: Near-field vs. far-field lift coefficient using force formula “c”, in viscous transonic flow.

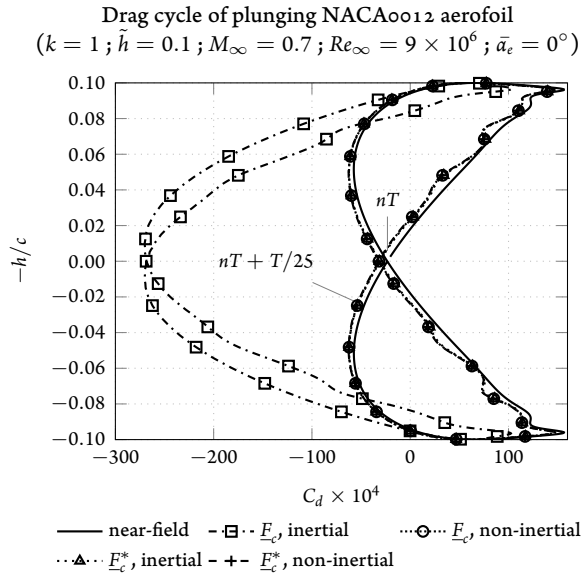


Figure 3.2.27: Near-field vs. far-field drag coefficient using force formula “c”, in viscous transonic flow.

Numerical results confirm that, with the selected integration domain and pole position, no lift is obtained by $\underline{F}_s + \underline{F}_\mu$ (Fig. 3.2.28, 3.2.30 and 3.2.32), then the unsteady axial force is effectively decomposed in lift-induced and parasite contributions (Fig. 3.2.29, 3.2.31 and 3.2.33).

In the same figures, drag coefficients from *steady* viscous simulations are included at incidence angles equal to $-10^\circ, -5^\circ, 0^\circ, 5^\circ$ and 5° (to compare against unsteady parasite drag), together with total axial force coefficients from the unsteady inviscid simulation at the same Mach number (to compare against unsteady induced drag). It is interesting to note the remarkable agreement between the identified unsteady thrust in viscous flow and the unsteady *inviscid* total axial force at $M_\infty \rightarrow 0$ and $M_\infty = 0.5$. Discrepancies are observed in transonic conditions, but in this case such a comparison is unfair, as the inviscid axial force does not purely represent a thrust contribution: it also includes wave drag. Moreover, the strength of viscous and inviscid shock waves is quite different. It is also highlighted that the unsteady parasite drag exhibits very slight variations with the effective angle of attack experienced by the aerofoil during its plunging oscillation, despite the strong incidence effects instead observed in steady conditions. For example, it should be mentioned that the steady viscous flow at $M_\infty = 0.5$ and $\alpha = \pm 10^\circ$ separates from the aerofoil (the corresponding drag coefficients are oscillating and the average value is out of scale in Fig. 3.2.31), while fully attached flow is instead realised at the same effective angle of attack during the unsteady motion, where separation has not enough time to develop. This also happens in viscous transonic conditions, where a shock-induced separation was observed at $\alpha = \pm 5^\circ$ ($C_d = 381d.c.$, out of scale in Fig. 3.2.33) which does not occur under unsteady conditions.

Finally, Fig. 3.2.34-3.2.37 illustrate results of the pole-independent total force decomposition. As anticipated in Sec. 3.1.8, and with the current CV selection ($X_W = 1.0$; $Z_W = 0.2$), this cannot however be interpreted as a drag breakdown in induced and parasite contributions, as the lift component of $\underline{F}_s + \underline{F}_\mu + \underline{F}_{\nabla\rho}^{uns}$ is non-zero. Like for steady flows, a sensitivity analysis of this (and of the former) unsteady force decomposition with the control volume selection should be performed, to investigate, among other things, on the possibility to obtain a pole-independent drag decomposition identifying the unsteady lift-induced and parasite drag. This however was not yet possible at the time of the reported research because of the numerical variation of the total force with the control volume size (extensively evidenced and discussed throughout this chapter), preventing from analysing different breakdowns of the *same* unsteady drag.

3.3 SUMMARY AND REMARKS

- The validity of three existing Lamb-vector-based force formulas (“c”, “bc” and “d”) was extended to the unsteady flow regime, introducing additional terms. Formulation “c” provided the most accurate results on all analysed flow-fields (viscous or inviscid, subsonic or transonic), a feature already observed in the steady regime and now also confirmed for unsteady flows and including the new additional terms. This was not straightforward since, differently from the steady version, the additional terms in the unsteady formula contain

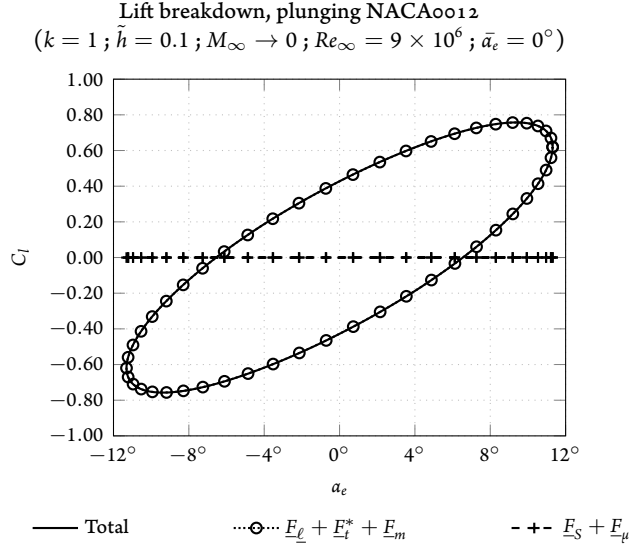


Figure 3.2.28: Lift decomposition isolating zero lift contribution for the viscous incompressible flow case at $k = 1$, using $\underline{F}_a$ in the non-inertial reference frame.

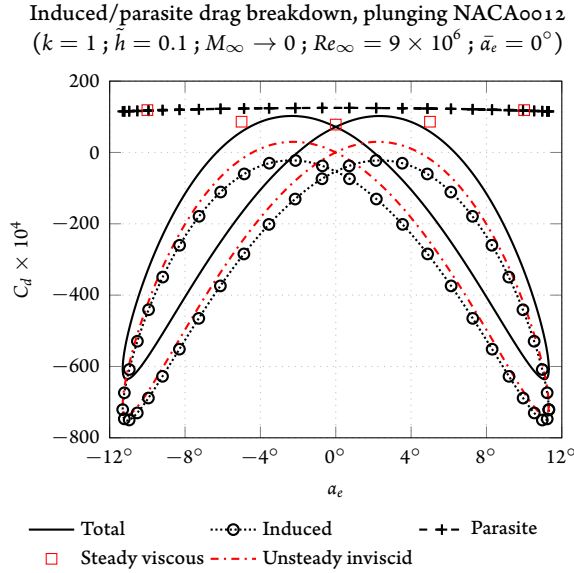


Figure 3.2.29: Drag decomposition in induced and parasite contributions for the viscous incompressible flow case at $k = 1$, using $\underline{F}_a$ in the non-inertial reference frame.

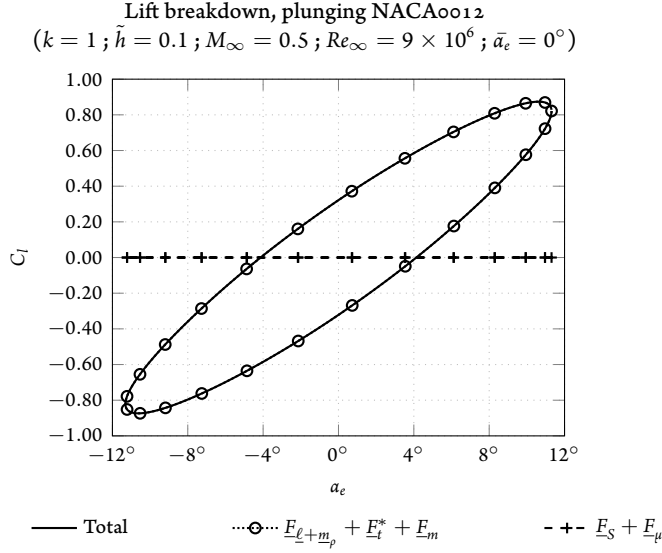


Figure 3.2.30: Lift decomposition isolating zero lift contribution for the viscous subsonic flow case, using $\underline{F}_c^*$ in the non-inertial reference frame.

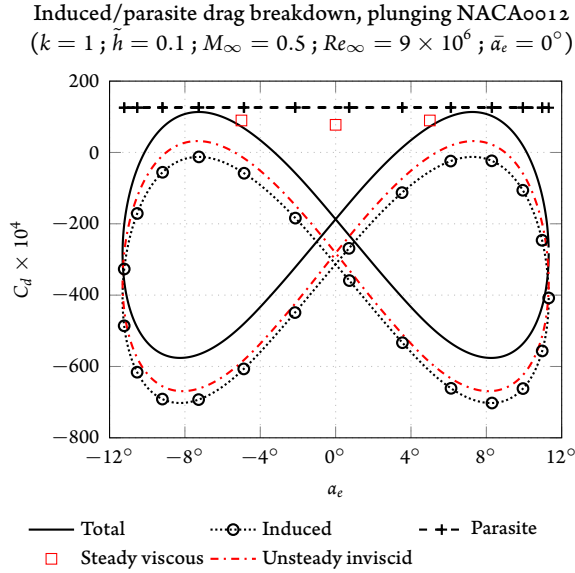


Figure 3.2.31: Drag decomposition in induced and parasite contributions for the viscous subsonic flow case, using $\underline{F}_c^*$ in the non-inertial reference frame.

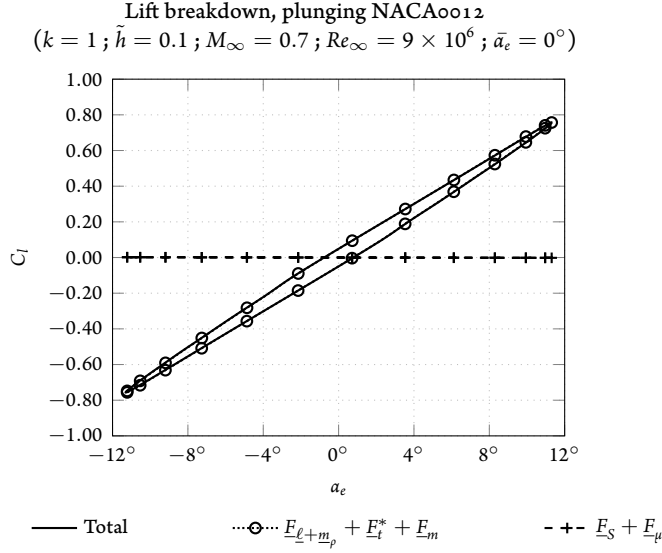


Figure 3.2.32: Lift decomposition isolating zero lift contribution for the viscous transonic flow case, using $\underline{F}_c^*$ in the non-inertial reference frame.

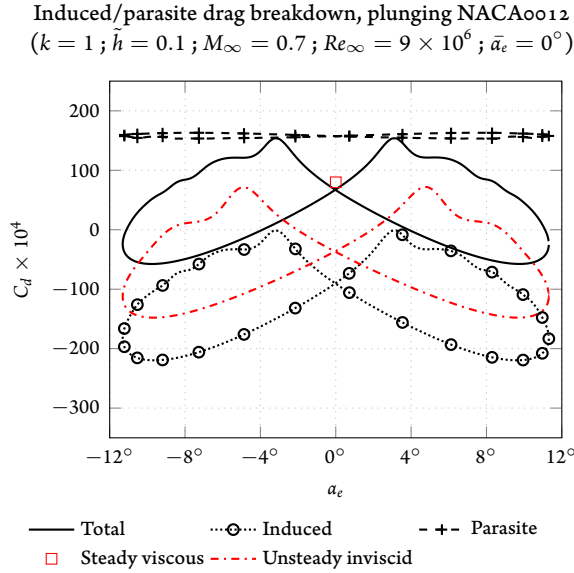


Figure 3.2.33: Drag decomposition in induced and parasite contributions for the viscous transonic flow case, using $\underline{F}_c^*$ in the non-inertial reference frame.

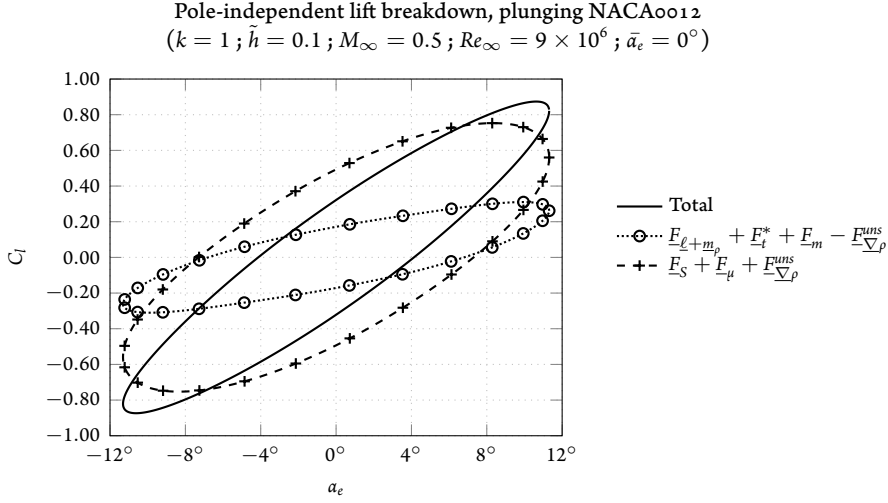


Figure 3.234: Pole-independent lift decomposition for the viscous subsonic flow case ($\underline{F}_c^*$ non-inertial).

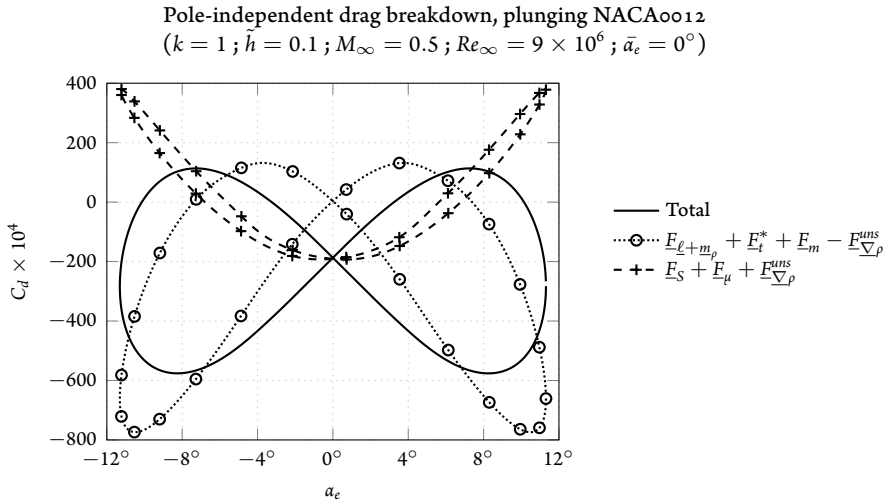


Figure 3.235: Pole-independent drag decomposition for the viscous subsonic flow case ($\underline{F}_c^*$ non-inertial).

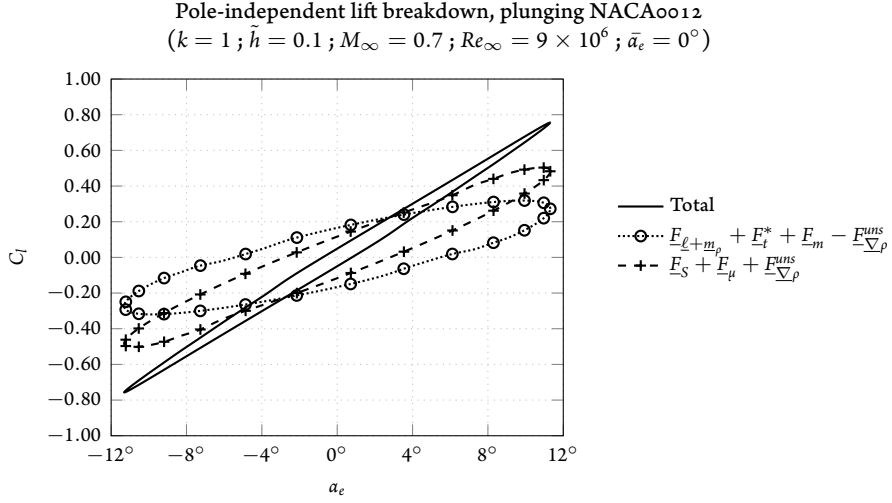


Figure 3.2.36: Pole-independent lift decomposition for the viscous transonic flow case ($\underline{F}_c^*$ non-inertial).

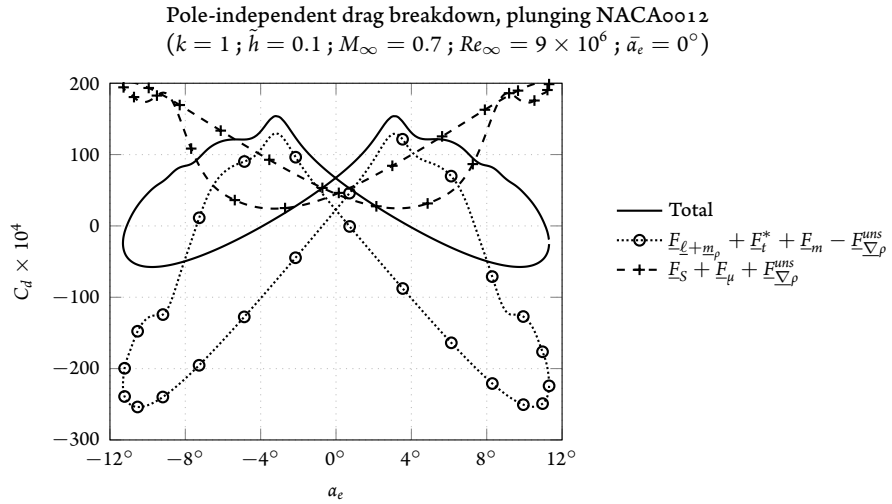


Figure 3.2.37: Pole-independent drag decomposition for the viscous transonic flow case ($\underline{F}_c^*$ non-inertial).

volume integrals, which were appointed to cause accuracy issues, especially in presence of shock waves, by different authors working on steady Lamb-vector methods.

- A recently proposed mixed inertial/non-inertial unsteady incompressible vortical force formulation, originally derived for two-dimensional viscous flows, has been also extended to three-dimensional and to inviscid flow conditions. Furthermore, a new expression of body integral terms, appearing in unsteady vortical force formulas, has been derived and successfully applied to post-process viscous and inviscid numerical flow solutions around an oscillating NACA0012 aerofoil in pure plunging motion. The new expression of body skin terms has the advantage of being numerically more accurate, as no computation of spatial derivatives on the body surface is required. Moreover, for steady inviscid cases, this expression is also independent of the arbitrary pole used for derivative moment transformations.
- Different unsteady Lamb-vector-based force formulas were compared and their accuracy assessed in inertial and non-inertial reference frames, for oscillating aerofoils simulated using moving grids.
- For a pure plunging motion, the best results were obtained working in non-inertial reference frames. In these cases, inertial formulations revealed inaccurate on both lift and drag, for viscous flows. A new expression was derived for the integrals containing time derivative of the velocity field. This does not involve the computation of spatial gradients and its use eliminates the inaccuracy observed for Lamb-vector-based force formula in inertial reference frames. In non-inertial formulations, its use resulted in a negligible impact.
- For a pure pitching motion, the best results were obtained using inertial or mixed formulations, while for the non-inertial formulation errors diverged as the CV size was increased, due to the lack of the mid-field property combined with space-dependent apparent forces (poorly resolved on coarse meshes, far from the body).
- For plunging oscillations in an incompressible flow, the discrepancy observed in a literature paper [169] between near-field and far-field estimates of mean thrust coefficients, at low values of the reduced frequency, is not observed in the reported results, which exhibit a satisfactory agreement with linear predictions of Garrick's theory in the inviscid flow regime. In addition, it was proved that, using a compressible flow solver, may lead to an apparently better agreement of inviscid mean thrust calculations with Garrick's results, at high values of the reduced frequency. This is however an effect of the higher leading-edge flow expansion during aerofoil plunging motion (increasing the mean thrust), which masks the discrepancies instead observed in incompressible flow simulations and justified by the loss of the small perturbations hypothesis.
- The numerical dependence of unsteady far-field force predictions on the integration domain has been broadly and systematically analysed (an aspect almost never addressed in

literature). A significant loss of accuracy was identified in all force formulas when extending the selected control volume far from the body, in the region of computed vortical structures (aerofoil unsteady wake and, for compressible flows, spurious vorticity waves). On small control volumes, the observed inaccuracies were not so large, although not negligible on drag, especially in viscous cases. This revealed a pollution of computed integrals, and it was anticipated that the observed inaccuracies can now be significantly reduced through the unified thermodynamic/vortical methodology proposed in this dissertation.

- Two different unsteady force decompositions were discussed. The proposed axial force breakdown in unsteady induced force and parasite drag allows for TDB in two-dimensional flows, and a pole-independent force decomposition was also derived. Indeed, a methodology was proposed for oscillating aerofoils identifying, in the far-field Lamb-vector-based axial force expression, the individual force terms providing thrust and those providing drag. It is interestingly highlighted that the decomposition is achieved using a control volume that tends to separate the bound vorticity on the body surface from the free vorticity in the unsteady wake. The proposed vortical approach to the TDB problem also gives a new insight on the physical phenomenon of thrust generation on flapping wings, complementary to that provided by Knoller and Betz (related to non-zero effective angles of attack): thrust is obtained as the axial force induced by the unsteady lift, i.e. by the time-variation of the aerofoil circulation and associated unsteady trailing vorticity.
- In subsonic flow, the decomposition provides satisfactory results when thrust and drag contributions are compared respectively to the inviscid force (thrust only) and to the steady force (drag only). Nonetheless, it is also highlighted that the computed unsteady parasite drag exhibits very slight variations with the effective angle of attack experienced by the aerofoil during its plunging oscillation, while strong incidence effects are instead observed on the steady drag. Decomposition results in transonic flow were also analysed, although in this case the inviscid force does not purely represent a thrust contribution, but also includes wave drag.
- It has to be mentioned that the proposed TDB method and the associated separation of bound and free vorticity are difficult to extend to three-dimensional configurations, due to the discussed limitations on the selection of the wake plane and DMT pole locations.

... a fundamental theory will be considered better than others if it involves fewer unknown variables in more localised integration domains (a property to be said “neater”), and it can reveal more clearly the key physics underlying the force.

Jie-Zhi Wu

4

Unified thermodynamic/Lamb-vector-based far-field force method

IN CFD, LAMB-VECTOR-BASED METHODS ARE MUCH TAILORED FOR NUMERICAL SOLUTIONS WHERE FLOW VORTICITY IS DIRECTLY COMPUTED (e.g. by incompressible flow solvers using vorticity-based algorithms) [138, 179, 216], while they might exhibit lower accuracy when flow momentum is instead computed (e.g. in compressible solvers using primitive flow variables) [124], then vorticity is obtained through a numerical differentiation of the velocity field ($\underline{\omega} = \underline{\nabla} \times \underline{V}$), to finally compute the Lamb vector. Indeed the non-linear differential vector identity $\underline{V} \cdot \underline{\nabla} \underline{V} = \underline{\nabla} (V^2/2) + (\underline{\nabla} \times \underline{V}) \times \underline{V}$, which is used in the theoretical derivation of Lamb-vector-based methods, might not numerically hold, depending on the selected numerical differential operators [11]. Therefore the discrete momentum equation, which is numerically satisfied at convergence according to the solver’s order of accuracy (e.g. 2^{nd} -order) and with the selected advective flux formulation, might not be satisfied (at least not with the same accuracy) when the discrete expressions of $\underline{\nabla} (V^2/2)$ and $\underline{\ell} = (\underline{\nabla} \times \underline{V}) \times \underline{V}$ are instead considered, as in a numerical post-processing procedure using Lamb-vector-based momentum methods. Moreover, the expected solution accuracy might not be locally achieved in flow-field regions around shock waves (where high-resolution schemes may blend towards lower order accuracy) or in areas of the computational domain where the local flux residuals are hard to drop below a selected threshold.

In this chapter, a unified approach to far-field aerodynamic force prediction and decomposition is proposed, where the exact Lamb-vector-based formula derived by Wu et al. [222, p. 621] and later on modified by Mele and Tognaccini [119] is used in combination with a thermodynamic-based calculation of the Lamb vector through Crocco's equation [33] (page 48, Eq. (1.127), recalled hereafter for the sake of completeness).

$$\rho \underline{\ell} = \rho (T \underline{\nabla} s - \underline{\nabla} H) + \underline{\nabla} \cdot \underline{\tau}_v + \rho \underline{f}_m - \rho \frac{\partial V}{\partial t}$$

This equation is an exact alternative expression of momentum equation, highlighting the presence of conservative forces. It relates the Lamb vector to thermodynamic quantities, and therefore represents a local bridge between Lamb-vector and thermodynamic methods. Its introduction is then the chain link to build the unified far-field force method illustrated herein.

As anticipated in Ch. 1 (Sec. 1.4.3.2), an interesting interpretation of Crocco's equation can be given as a *local expression of the Kutta-Joukowski theorem applied to the fluid particle*, in 2D steady inviscid and incompressible flows, without body forces.

Generally, Crocco's equation states that the elementary vortex force, $-\rho \underline{\ell}$, is balanced by the resultant of all other external and inertial forces, and this resultant (per unit mass) can be replaced to the Lamb vector in far-field force formulas.

In post-processing CFD solutions, this way of computing the Lamb vector (which does not require an explicit vorticity calculation) is therefore based on the numerical form of flow momentum equation and discretely satisfies the local (and integral) momentum balance which far-field methods rely on.

As detailed in Ch. 2, inaccuracies of Lamb-vector-based force formulas using flow vorticity were often reported [117, 124], with best results provided by formulation “c”, which however does not involve the Lamb vector (and flow vorticity) unless in one term, computed on the external control volume surface (Σ). Formulations integrating flow vorticity in the CV interior are nevertheless valuable, as they allow identifying the local sources of the aerodynamic force in the flow-field. However, in their application, the computed force, theoretically independent on the selected control volume, was observed to vary as the wake plane is moved downstream of the body in the near wake, stabilising past several characteristic lengths [124] (see Appendix B). Then, above all, in presence of shock waves, large inaccuracies of computed domain integrals were systematically observed (Ch. 2, Sec. 2.2.3.1) and reported [117, 124], polluting far-field force results. Indeed it was observed that, as the control volume includes (also partially) any strong shock, the computed drag is considerably higher than the reference near-field value, with unacceptable differences that can reach hundreds of drag counts for a 2D aerofoil in transonic flow [117]. This was attributed to the lower accuracy of numerical flow solutions in the shock wave region, where usually non-linear schemes (e.g. based on limiters) are used to guarantee Total Variation Diminishing solutions, although locally compromising with numerical errors.

To overcome shock-related inaccuracy issues, the transonic variant of Mele and Tognaccini's

formula [117] ($\underline{F}_{bc}$, Ch. 2, Sec. 2.1.4) was based on the exclusion of the shock wave region from domain integrals. The selection of the shock wave region, however, introduces other degrees of freedom in the force formula and results are numerically dependent on the particular choice, with higher accuracy only obtained excluding the entire control volume from domain integration ($\underline{F}_c$, Sec. 2.1.3), retaining surface integrals only [124]. Moreover, the proposed variant is not applicable to the original compressible vortical force formula as derived by Wu et al. [222, p. 621] ($\underline{F}_a$, Sec. 2.1.1).

The new mixed approach is applicable to both Lamb-vector-based formulations ($\underline{F}_a$ and $\underline{F}_b$) and provides a unified formula where Lamb-vector-based and thermodynamic-based methods, so far considered as different alternatives*, are combined in a new exact expression of the total aerodynamic force. This will be shown to provide a far-field formulation of the pressure force acting on the body, where different contributions can be identified. In a future research perspective, this is particularly interesting, as an alternative approach to near-field pressure force decomposition was recently provided by Partial Pressure Fields (PPFs) methods, first derived by Schmitz and Coder for the incompressible regime [178] and then extended to compressible flows [177].

In this work, the proposed unified method is applied to post-process two-dimensional and three-dimensional numerical flow-fields from both steady and unsteady CFD simulations. A NACA0012 aerofoil is considered, either fixed or oscillating with a pure-plunging motion, in a subsonic or transonic, viscous or inviscid, flow. Finally, an application to a 3D elliptic wing in transonic viscous and turbulent flow is also illustrated.

4.1 DEVELOPMENT OF THE UNIFIED FORCE METHOD

4.1.1 THE BASELINE FAR-FIELD FORCE FORMULA

To develop and illustrate the method, the *exact* unsteady compressible formula derived by Mele and Tognaccini [119] ($\underline{F}_b$, Ch. 3, Sec. 3.1.2) is here adopted as a baseline. Nonetheless, as discussed above, the method can be similarly applied to the other illustrated unsteady vortical formulas ($\underline{F}_a$ and $\underline{F}_c$, Sec. 3.1.1 and Sec. 3.1.3).

In Sec. 3.1.2, the force formula was introduced using a neat formulation which provides generalisation to turbulent flows, non-inertial reference frames, inviscid flows and/or moving bodies. Here its particularisation to laminar flows will be assumed as the baseline form to derive the unified force method. In fact, it will be shown that the extension to the turbulent regime depends on the selected Lamb vector expression. Moreover, the alternative expression of unsteady terms (derived in Sec. 3.1.6) will also be incorporated in the adopted formula, in view of its following numerical application which already revealed beneficial as evidenced in Sec. 3.2.

Therefore, the baseline far-field formula (which is hereafter recalled for the sake of complete-

*although first attempts to links the two approaches were made by Kang et al. [81] and by Fournis [54].

ness) can be expressed in compact form as:

$$\underline{F}_b = \underline{F}_\ell + \underline{F}_{m_p} + \underline{F}_t^* + \underline{F}_S + \underline{F}_\mu + \underline{F}_m \quad (4.1)$$

where:

$$\underline{F}_\ell = - \int_{\mathcal{V}} \rho \underline{\ell} d\mathcal{V} - \int_{\partial B} \frac{\rho V^2}{2} \underline{n} dS \quad (4.2)$$

$$\underline{F}_{m_p} = - \int_{\mathcal{V}} \underline{r} \times \left[\underline{\nabla} \rho \times \underline{\nabla} \left(\frac{V^2}{2} \right) \right] d\mathcal{V} - \int_{\partial B} \underline{r} \times \left[\underline{n} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS \quad (4.3)$$

$$\underline{F}_t^* = - \int_{\mathcal{V}} \rho \frac{\partial \underline{V}}{\partial t} d\mathcal{V} - \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\rho \frac{\partial \underline{V}}{\partial t} \right) \right] dS \quad (4.4)$$

$$\underline{F}_S = - \int_{\Sigma} \underline{r} \times [\underline{n} \times (\rho \underline{\ell})] dS \quad (4.5)$$

$$\underline{F}_\mu = \int_{\Sigma} \underline{r} \times \left(\underline{n} \times \underline{\nabla} \cdot \underline{\tau}_v \right) dS + \int_{\Sigma} \underline{\tau}_v \cdot \underline{n} dS \quad (4.6)$$

$$\underline{F}_m = \int_{\mathcal{V}} \rho f_m d\mathcal{V} + \int_{\Sigma} \underline{r} \times \left[\underline{n} \times (\rho f_m) \right] dS \quad (4.7)$$

4.1.2 STANDARD VORTICITY-BASED LAMB VECTOR METHOD

In Eq. (4.1) the Lamb vector appears through $\underline{F}_\ell$ and $\underline{F}_S$ and can be computed using its definition $\underline{\ell} = \underline{\omega} \times \underline{V}$ i.e.:

$$\underline{F}_\ell^{vrt} = - \int_{\mathcal{V}} \rho (\underline{\omega} \times \underline{V}) d\mathcal{V} - \int_{\partial B} \frac{\rho V^2}{2} \underline{n} dS \quad (4.8)$$

$$\underline{F}_S^{vrt} = - \int_{\Sigma} \underline{r} \times \{ \underline{n} \times [\rho (\underline{\omega} \times \underline{V})] \} dS \quad (4.9)$$

The resulting *vorticity-based* force formula can be expressed in compact form as:

$$\boxed{\underline{F}_b^{vrt} = \underline{F}_\ell^{vrt} + \underline{F}_{m_p} + \underline{F}_t^* + \underline{F}_S^{vrt} + \underline{F}_\mu + \underline{F}_m} \quad (4.10)$$

The extension of Eq. (4.10) to turbulent flows governed by the Reynolds-Averaged Navier-Stokes (RANS) equations is discussed in Appendix E.1.

4.1.3 NEW THERMODYNAMIC-BASED APPROACH TO THE LAMB VECTOR METHOD

Alternatively, the Lamb vector appearing in $\underline{F}_\ell$ and $\underline{F}_S$ can be computed from momentum equation (for a generally non-inertial reference frame) in the form of Crocco's theorem, obtaining:

$$\underline{F}_\ell^{thd} = - \int_{\mathcal{V}} \left[\rho (T \underline{\nabla} s - \underline{\nabla} H) + \underline{\nabla} \cdot \underline{\tau}_v + \rho f_m - \rho \frac{\partial V}{\partial t} \right] d\mathcal{V} - \int_{\partial B} \frac{\rho V^2}{2} \underline{n} dS \quad (4.11)$$

$$\underline{F}_S^{thd} = - \int_{\Sigma} \underline{r} \times \left\{ \underline{n} \times \left[\rho (T \underline{\nabla} s - \underline{\nabla} H) + \underline{\nabla} \cdot \underline{\tau}_v + \rho f_m - \rho \frac{\partial V}{\partial t} \right] \right\} dS \quad (4.12)$$

The resulting *thermodynamic-based* force formula can therefore be expressed in compact form as:

$$\underline{F}_b^{thd} = \underline{F}_\ell^{thd} + \underline{F}_m + \underline{F}_t^* + \underline{F}_S^{thd} + \underline{F}_\mu + \underline{F}_m \quad (4.13)$$

The extension of Eq. (4.13) to turbulent flows governed by the RANS equations is derived in Appendix E.2.

4.1.4 NEW EXACT FORCE FORMULA

It is worth observing that, when the Lamb vector in $\underline{F}_\ell$ and $\underline{F}_S$ is computed using Crocco's equation, apparent forces terms and time-derivative terms in Eq. (4.11) and Eq. (4.12) cancel-out with the analogous (and opposite) terms in $\underline{F}_m$ and $\underline{F}_t^*$. The same happens for the viscous stress term in Eq. (4.12), which is exactly equal and opposite in sign to that already included in $\underline{F}_\mu$. Moreover, applying Gauss' divergence theorem to the analogous term in Eq. (4.11), viscous stress fluxes on Σ cancel-out with the opposite term in $\underline{F}_\mu$, then fluxes on body skin only remain in the force formula.

Therefore, defining the vector field $\underline{G} = \underline{\nabla} H - T \underline{\nabla} s$, the unified method formula reduces to:

$$\begin{aligned} \underline{F}_b^{THD} = & \int_{\mathcal{V}} \rho \underline{G} d\mathcal{V} - \int_{\partial B} \frac{\rho V^2}{2} \underline{n} dS + \\ & - \int_{\mathcal{V}} \underline{r} \times \left[\underline{\nabla} \rho \times \underline{\nabla} \left(\frac{V^2}{2} \right) \right] d\mathcal{V} - \int_{\partial B} \underline{r} \times \left[\underline{n} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS + \\ & + \int_{\Sigma} \underline{r} \times (\underline{n} \times \rho \underline{G}) dS - \int_{\partial B} \underline{n} \cdot \underline{\tau}_v dS \end{aligned} \quad (4.14)$$

which represents a new thermodynamic method, now expressing the total force (both lift and drag) instead of the irreversible drag alone as in previous thermodynamic based algorithms. For isothermal flows, $\underline{G}$ represents the gradient of total Gibbs free energy. It is related to the gradient of flow exergy ε (referred to the free-stream state, Eq. (1.128)) by: $\underline{G} = \underline{\nabla} \varepsilon - \Delta T \underline{\nabla} s$, where $\Delta T = T - T_\infty$. A notation with uppercase superscripts ($\underline{F}_b^{THD}$) is used to differentiate this new exact expression with regards to its mathematically equivalent formulation represented by Eq. (4.13), where lowercase superscripts were instead adopted. Both formulas were in fact considered to calculate the aerodynamic force in numerical applications described in the following

section. The simplified exact formula of the unified method can be expressed in compact form as:

$$\boxed{\underline{F}_b^{THD} = \underline{F}_G + \underline{F}_{m_p} + \underline{F}_S^G + \underline{F}_\tau} \quad (4.15)$$

where:

$$\underline{F}_G = \int_V \rho \underline{G} dV - \int_{\partial B} \frac{\rho V^2}{2} \underline{n} dS \quad (4.16)$$

$$\underline{F}_S^G = \int_\Sigma \underline{r} \times [\underline{n} \times (\rho \underline{G})] dS \quad (4.17)$$

$$\underline{F}_\tau = - \int_{\partial B} \underline{n} \cdot \underline{\tau}_v dS \quad (4.18)$$

Indeed, $\underline{F}_\tau$ was already defined in Ch. 1, Eq. (1.1) and represents a near-field contribution.

The extension of Eq. (4.15) to turbulent flows governed by the RANS equations is also discussed in Appendix E.2.

It is worth highlighting that no explicit dependence on time derivatives and apparent forces (due to relative motion, in non-inertial reference frames) appears in Eq. (4.14) recovering a property of the near field formula. Flow unsteadiness is implicitly accounted for in the variations of total enthalpy, which also includes (together with $\underline{F}_{m_p}$) the absolute or relative velocity magnitude, depending on the selected inertial or non-inertial reference frame.

4.1.4.1 LINK WITH OSWATITSCH'S FORMULA

In Eq. (4.15)), the integral term over the external control-volume surface ($\underline{F}_S^G$) is linked to Oswatitsch's entropy drag formula [143] (Ch. 1, Sec. 1.4.1.1). In fact (a similar derivation is also provided in [54]), introducing total enthalpy and entropy perturbations with regards to free-stream conditions ($\Delta H = H - H_\infty$, $\Delta s = s - s_\infty$), it is:

$$\begin{aligned} \underline{F}_S^G &= \int_\Sigma \underline{r} \times [\underline{n} \times (\rho \underline{G})] dS = \int_\Sigma \underline{r} \times \{ \underline{n} \times [\rho (\underline{\nabla} H - T \underline{\nabla} s)] \} dS \\ &= \int_\Sigma \underline{r} \times \{ \underline{n} \times [\rho (\underline{\nabla}(\Delta H) - T \underline{\nabla}(\Delta s))] \} dS \end{aligned}$$

For steady un-powered high-Reynolds number flows around adiabatic bodies, assuming a unitary Prandtl number ($Pr \approx 1$) and uniform free-stream conditions, variations of specific total enthalpy in the entire flow-field are negligible ($\Delta H \approx 0$). Then, considering also the perfect gas equation $p = \rho RT$ (with R being the gas constant):

$$\begin{aligned} \underline{F}_S^G &\approx - \int_\Sigma \underline{r} \times \{ \underline{n} \times [\rho T \underline{\nabla}(\Delta s)] \} dS \\ &= - \int_\Sigma \underline{r} \times \{ \underline{n} \times [\underline{\nabla}(\rho T \Delta s)] \} dS + \int_\Sigma \underline{r} \times \{ \underline{n} \times [\Delta s \underline{\nabla}(\Delta p/R)] \} dS \end{aligned}$$

4.1. Development of the unified force method

with $\Delta p = p - p_\infty$. The first integral on the RHS can be transformed using identity Eq. (A2), then the $\underline{F}_S^G$ drag component is obtained by projection along the free-stream direction $\underline{e}_x = \underline{V}_\infty / V_\infty$:

$$D^G = \underline{e}_x \cdot \underline{F}_S^G \approx \frac{1}{V_\infty} \int_\Sigma \rho T \Delta s (\underline{V}_\infty \cdot \underline{n}) dS + \underline{e}_x \cdot \int_\Sigma \underline{r} \times \{ \underline{n} \times [\Delta s \underline{\nabla}(\Delta p/R)] \} dS$$

Introducing static temperature and velocity perturbations ($T = T_\infty + \Delta T$, $\underline{V} = \underline{V}_\infty + \Delta \underline{V}$), we can write:

$$\begin{aligned} D^G &\approx \frac{T_\infty}{V_\infty} \int_\Sigma \rho \Delta s (\underline{V}_\infty \cdot \underline{n}) dS + \frac{1}{V_\infty} \int_\Sigma \rho \Delta T \Delta s (\underline{V}_\infty \cdot \underline{n}) dS + \\ &\quad + \underline{e}_x \cdot \int_\Sigma \underline{r} \times \{ \underline{n} \times [\Delta s \underline{\nabla}(\Delta p/R)] \} dS = \\ &= \frac{T_\infty}{V_\infty} \int_\Sigma \rho \Delta s (\underline{V} \cdot \underline{n}) dS - \frac{T_\infty}{V_\infty} \int_\Sigma \rho \Delta s (\Delta \underline{V} \cdot \underline{n}) dS + \\ &\quad + \frac{1}{V_\infty} \int_\Sigma \rho \Delta T \Delta s (\underline{V}_\infty \cdot \underline{n}) dS + \underline{e}_x \cdot \int_\Sigma \underline{r} \times \{ \underline{n} \times [\Delta s \underline{\nabla}(\Delta p/R)] \} dS \end{aligned}$$

Therefore, under Oswatitsch's assumption of a parallel flow with small perturbations on a sufficiently distant Σ surface from the aerofoil [143], only the first-order perturbation term can be retained, leading to:

$$D^G \approx \frac{T_\infty}{V_\infty} \int_\Sigma \rho \Delta s (\underline{V} \cdot \underline{n}) dS \quad (4.19)$$

which is Oswatitsch's entropy drag definition Eq. (1.79).

4.1.5 NEW EXACT MOMENT FORMULA

The unified formulation is complete once the aerodynamic moment around a selected pole is also addressed, closing the total force definition as an applied vector.

The baseline moment formula is represented by Eq. A28, which is here reported for the reader's convenience and denoted with the subscript “b”:

$$\underline{M}_b = \underline{M}_\ell + \underline{M}_{m_p} + \underline{M}_t^* + \underline{M}_S + \underline{M}_\mu + \underline{M}_m \quad (4.20)$$

Provided that, similarly to what illustrated in Sec. 4.1.3, a thermodynamic-based calculation of the Lamb vector can also be introduced in $\underline{M}_\ell$ and $\underline{M}_S$ instead of using standard vorticity-based formulas, a new exact expression for the aerodynamic moment can be derived following the same procedure described in Sec. 4.1.4 for the resultant force.

Simplifications of unsteady and apparent forces terms are straightforward, as well as cancellations involving the viscous stress term in $\underline{M}_S$. For the viscous term included in $\underline{M}_\ell$, instead, preliminarily to the application of Gauss' divergence theorem (in line with the derivation included in

Sec. 4.1.4) the following differential vector identity has to be taken into account:

$$-\hat{r} \times \left(\underline{\nabla} \cdot \underline{\tau}_{\underline{v}} \right) = \underline{\nabla} \cdot \left(\underline{\tau}_{\underline{v}} \times \hat{r} \right) + \cancel{\underline{\tau}_{\underline{v}} \times \hat{r}}^0 \quad (4.21)$$

where $\underline{\tau}_{\underline{v}}$ is the vector associated to the skew-symmetric part of the tensor $\underline{\tau}_{\underline{v}}$, which is null under the assumed hypothesis of a non-micropolar fluid, with symmetric stress tensor.

Finally, the new *exact* aerodynamic moment formula is expressed by:

$$\boxed{\underline{M}_b^{THD} = \underline{M}_G + \underline{M}_{m_p} + \underline{M}_S^G + \underline{M}_\tau} \quad (4.22)$$

where:

$$\underline{M}_G = \int_{\mathcal{V}} \hat{r} \times \rho \underline{G} d\mathcal{V} - \int_{\partial B} \hat{r} \times \rho \frac{V^2}{2} \underline{n} dS \quad (4.23)$$

$$\underline{M}_S^G = -\frac{1}{2} \int_{\Sigma} \hat{r}^2 [\underline{n} \times (\rho \underline{G})] dS \quad (4.24)$$

$$\underline{M}_\tau = - \int_{\partial B} \hat{r} \times \left(\underline{\tau}_{\underline{v}} \cdot \underline{n} \right) dS \quad (4.25)$$

and $\underline{M}_{m_p}$ is given by Eq. (A26) in Appendix A.

4.1.6 SOME REMARKS

Most of the observations and remarks included hereafter are discussed with reference to the force formulas, although similar considerations may be done for the aerodynamic moment as well.

- Comparing Eq. (4.14) with Eq. (1.1), it is highlighted that all terms but the last one on the RHS of Eq. (4.14) can be interpreted as a far-field expression of the pressure force acting on the body, while the viscous force (last integral) appears in its near-field formulation.

Indeed pressure force ($\underline{F}_p$, Eq. (1.1)) is the only one which deserves a decomposition, being attributed to different physical phenomena (viscosity, compressibility and trailing vorticity). In standard Lamb-vector-based far-field methods, the direct presence of pressure (varying in the entire flow-field) in the force formula is eliminated in favour of fluid velocity and its spatial derivatives. Here it is eliminated introducing entropy and total enthalpy.

- A similar structure can be observed between the new exact formula (Eq. (4.15)) and the steady inertial version of the baseline Lamb-vector-based formula (Eq. (4.1) with $\underline{F}_t^* = \underline{0}$ and $\underline{F}_m = \underline{0}$). The latter represents a far-field expression of the whole aerodynamic force. By comparison, if only the pressure force ($\underline{F}_p$) is expressed by far-field integrals, the exact formula is obtained simply replacing $\underline{F}_\mu$ with $\underline{F}_\tau$ and $-\underline{\ell}$ with $\underline{G}$. Moreover, the validity of the hybrid far-field/near-field formula obtained in this way is not limited to steady flows in

inertial frames of references, but it holds exactly for unsteady flows in generally non-inertial frames.

- The lack of an explicit dependence on time derivatives represents an advantage for numerical applications of the simplified far-field formula. In fact, post-processors do not need to use data at different physical times other than the working time, then the instantaneous far-field force can be computed post-processing one flow-field at the time (saving memory and reducing the computational effort).
- The unified approach also allows deriving a far-field expression independent of time derivatives and apparent forces for the *whole* aerodynamic force. It is obtained from $\underline{F}_b^{thd}$ simply avoiding the last simplification involving viscous terms:

$$\boxed{\underline{F}_b^{ThD} = \underline{F}_\ell^{ThD} + \underline{F}_{\underline{m}_p} + \underline{F}_S^{ThD} + \underline{F}_\mu} \quad (4.26)$$

with

$$\underline{F}_\ell^{ThD} = \int_{\mathcal{V}} [\rho \underline{G} - \underline{\nabla} \cdot \underline{\tau}_v] d\mathcal{V} - \int_{\partial B} \frac{\rho V^2}{2} \underline{n} dS \quad (4.27)$$

$$\underline{F}_S^{ThD} = \int_{\Sigma} \underline{r} \times \left\{ \underline{n} \times [\rho \underline{G} - \underline{\nabla} \cdot \underline{\tau}_v] \right\} dS \quad (4.28)$$

It represents a mid-way expression between $\underline{F}_b^{thd}$ and $\underline{F}_b^{THD}$ (hence, the superscript ThD), where the viscous force is still computed by far-field integrals.

The independence on time derivatives and apparent forces is a peculiarity of the near-field formula (Eq. (1.1)), while the possibility to get rid of such terms in far-field expressions (Eq. (1.2)) is not a trivial result, as the momentum balance naturally brings them into play. From the point of view of numerical applications, computing the total force with either Eq. (4.27) or (4.11) gives identical results (therefore, only $\underline{F}_b^{thd}$ was computed in the next section), as unsteady and apparent forces terms are not computed in the first formula, or computed twice (with opposite signs) in the second formula, of course with zero net contribution in this case.

- Retaining the original force formula structure as in Eq. (4.13) (i.e. ignoring all simplifications leading to Eq. (4.15)) one can still compute and analyse the classical Lamb-vector-based total force decompositions into individual vortical constituents (i.e. $\underline{F}_\ell^{thd}$, $\underline{F}_{\underline{m}_p}$, $\underline{F}_t^*$, $\underline{F}_S^{thd}$, $\underline{F}_\mu$ and $\underline{F}_m$) or into lift, lift-induced drag and parasite drag [57], but now using thermodynamic-based expressions to calculate $\underline{F}_\ell$ and $\underline{F}_S$, which can result in higher accuracy for the overall integral momentum balance. Hybrid formulations are also possible, using a thermodynamic-based expression to calculate only one between $\underline{F}_\ell$ and $\underline{F}_S$, the other term still being computed with a vorticity-based expression. This can be interesting for example to keep linking the vortex force to flow vorticity, while expressing the wake integral ($\underline{F}_S$, re-

lated to parasite drag) in terms of thermodynamic quantities, which results in a more compact drag production field (Section 4.2.5) allowing for an easier decomposition of parasite drag into viscous and wave contributions.

- Compared to classical thermodynamic methods, the new formulas introduced in this chapter do not suffer of any ill-definition in their integral expressions. This is an issue of all drag formulas based on velocity decompositions (Ch. 1, Sec. 1.4.1.2), which involve square roots of quantities that can drop below zero. No satisfactory solution to this problem has been achieved yet, especially for unsteady flows around rotating bodies [135], while the new thermodynamic methods derived from the proposed unified approach overcome this limitation.
- For steady high-Reynolds number flows, Lamb-vector-based force formulas were often identified as confined expressions of the aerodynamic force involving rotational regions of the flow-field only (*mid-field* property) [113, 116]. This argumentation was indeed based on Crocco's theorem, which associates zero Lamb vector to homoentropic and homoentropic flow regions, allowing a restriction of both domain and surface integrals in $\underline{F}_\ell$, $\underline{F}_m$ and $\underline{F}_s$ to rotational flow regions only. From a numerical point of view, the new mixed approach now does ensure that the computed Lamb vector is indeed zero in flow-field regions where dissipative effects are negligible and outside shock waves and associated wakes (homoentropic and homoentropic regions), which is evidently not guaranteed by the standard vorticity-based calculation in numerical applications, given the difference in Lamb vector fields computed by the two approaches (Section 4.2.4).

4.2 NUMERICAL ANALYSES

4.2.1 SELECTED TEST-CASES

CFD solutions using Finite Volume Methods (FVMs) were obtained for a NACA0012 aerofoil (unitary chord length) either fixed or oscillating in pure plunging motion and immersed in an effectively-inviscid or viscous flow (at low and high Reynolds number), with different values of the free-stream Mach number (M_∞). All simulated conditions assumed dry air and no variation of specific heat capacities with the static temperature was considered (ratio of specific heat capacities fixed at 1.4). The ideal gas constant for the air mixture was set at the value of 287.058 and the molecular viscosity was computed based on flow static temperature, using Sutherland's viscosity law. As for previous flow simulations (Ch. 2, Ch. 3), a slightly altered definition of NACA0012 aerofoil providing closed trailing edge (t.e.) is used [165] and the computational domain extends up to 300 chord lengths around the aerofoil. Impermeable and adiabatic wall boundary conditions (b.c.) were used at solid surfaces, assuming no-slip for viscous cases and zero shear stress (free-slip) for inviscid simulations. Characteristics-based free-stream b.c. were instead adopted at the far-field boundary of the entire computational domain.

Table 4.2.1: Summary of the analysed 2D test-cases.

M_∞	Re_∞	k	$\tilde{h}$	$\bar{\alpha}_e$
0.5	1×10^4	0.00	0.0	2.0°
0.5	9×10^6	0.00	0.0	4.0°
0.8	$\rightarrow \infty$	0.00	0.0	2.0°
0.8	9×10^6	0.00	0.0	3.0°
$\rightarrow 0$	9×10^6	0.05	0.1	0.0°
$\rightarrow 0$	9×10^6	0.50	0.1	0.0°
$\rightarrow 0$	1×10^4	1.00	0.1	0.0°
0.7	$\rightarrow \infty$	1.00	0.1	0.0°

Denoting with c the aerofoil chord and with V_∞ the free-stream velocity magnitude, the plunging motion is characterised by the reduced oscillation frequency ($k = \pi fc/V_\infty$), the non-dimensional plunging amplitude ($\tilde{h} = \hat{h}/c$) and the mean effective angle of attack ($\bar{\alpha}_e$, where $\alpha_e = \alpha + \arctan(\dot{h}/V_\infty)$). Plunging oscillations were simulated at different values of the reduced frequency (from quasi-steady to highly-unsteady regime) and at right angles to the direction of flight ($\bar{\alpha}_e=0$). Table 4.2.1 summaries the selected two-dimensional test-cases, where steady-state solutions for the fixed aerofoil can be regarded as flow solutions for aerofoil oscillations at zero frequency and/or zero amplitude. Except for the unsteady incompressible flow simulation at low Reynolds number, all test cases were selected from the data-set of steady and unsteady flow-fields already analysed in Ch. 2 and Ch. 3.

The elliptic-planform wing three-dimensional flow solution, illustrated and analysed in Ch. 2, was also selected as a test-case for the developed unified method. The adopted geometry and computational grid are the same used by Marongiu et al. [113] and Mele et al. [116], which motivates the selection of this 3D test-case.

Details on the procedure and tools used to generate the selected CFD solutions are available from Sec. 2.2.1 and Sec. 3.2.1.

A grid convergence analysis (additional to that provided in Appendix D.2) justifying the employed spatial discretization for plunging simulations in incompressible laminar flow is reported in Appendix D.1.

A visualisation of the unsteady flow-field after 4 computed oscillation periods is provided in Fig. 4.2.1 and Fig. 4.2.2, respectively, for the low Reynolds viscous incompressible and for the inviscid transonic cases. The amplitude of the oscillatory plunging motion is set at one tenth of the aerofoil chord length and snapshots are provided at time frames corresponding to: midway down-stroke ($4T$), lower dead-point ($(4 + 1/4)T$), midway up-stroke ($(4 + 2/4)T$) and upper dead-point ($(4 + 3/4)T$). Bound and free vorticity are visible in Fig. 4.2.1 for the viscous case, as well as in the inviscid flow (through the unsteady wake shear layer visible in Fig. 4.2.2). Being far from a linear slightly-perturbed flow regime, the wake pattern is of non-straight and oscillatory shape, with upwards (downwards) wake deflection near the aerofoil trailing edge in correspondence of midway down-stroke (up-stroke) points, while nearly-straight initial wake is observed at

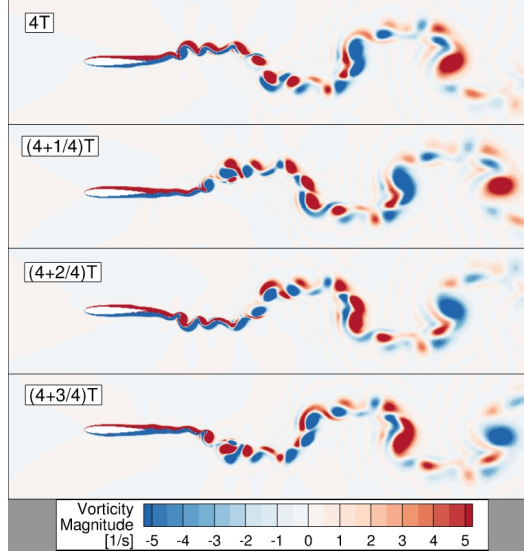


Figure 4.2.1: Vorticity magnitude contour plots, at different time period fractions - plunging NACA0012 ($k = 1$; $\tilde{h} = 0.1$; $M_\infty \rightarrow 0$; $Re_\infty = 1 \times 10^4$; $\bar{\alpha}_e = 0^\circ$).

saddle points (Fig. 4.2.1). During the unsteady motion at $k = 1$, the effective angle of attack experienced by the aerofoil resulting from the actual inflow conditions (combination of freestream and vertical velocities) oscillates between $\pm 11.31^\circ$, values reached at midway down-stroke (positive incidence) and midway up-stroke (negative incidence). The flow field response to the actual inflow is however characterised by a strong delay and the shock waves on the upper and lower sides are stronger at dead-points, which are reached later on during the oscillation, when the instantaneous angle of attack has changed to zero again (Fig. 4.2.2).

4.2.2 POST-PROCESSING METHOD

The aerodynamic force (lift and drag) was computed, in the post-processing phase, using near-field surface stress integration (Eq. (1.1)), far-field Lamb-vector-based formulations summarised in Section 4.1 (Eq. (4.10) and Eq. (4.13)) and the new simplified exact force formula (Eq. (4.15)). In the application of far-field methods to RANS solutions (Appendix E), the isotropic Reynolds stress tensor (related to the turbulent kinetic energy) was neglected and Boussinesq's approximation for the deviatoric part was assumed: $\underline{\underline{\tau}}_R = 2\mu_t \left[\frac{1}{2} (\underline{\nabla} \underline{V} + \underline{\nabla} \underline{V}^T) - \frac{1}{3} \underline{\nabla} \cdot \underline{V} \right]$.

Moreover, the last integral in Eq. (E2) and the last integral in Eq. (E12), which are related to turbulent fluctuations and represent unclosed terms, were not computed. In particular, the first one was neglected assuming *Morkovin's* hypothesis, stating that compressibility has a weak influence on the Reynolds stress [131]. Interestingly, results will show that turbulent force formulas well predict the aerodynamic force from RANS solutions even when the highlighted unclosed terms are neglected.

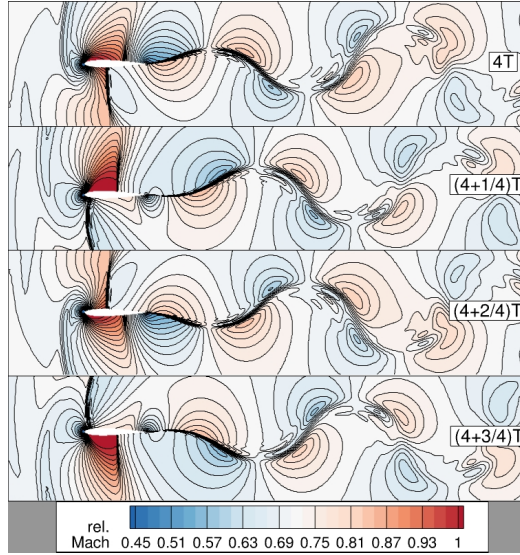


Figure 4.2.2: Relative Mach number contour plots, at different time period fractions - plunging NACA0012 ($k = 1$; $\tilde{h} = 0.1$; $M_\infty = 0.7$; $Re_\infty \rightarrow \infty$; $\bar{\alpha}_e = 0^\circ$).

For unsteady simulations, one plunging cycle was analysed starting after 4 complete oscillations were computed. The DMT pole (Fig. 1.1.1) is located at the aerofoil leading edge (i.e.) for steady cases (for the 3D wing, the mid-span wing section is considered) or at the trailing edge for unsteady simulations. In the latter case, it therefore moves in time with the aerofoil, as well as the employed control volume, which has always been selected as a subset of the (moving) computational grid. It is identified by three orthogonal grid lines of the computational mesh, as illustrated in Fig. 4.2.3, where the characteristic lengths used to define its size are also represented. In particular, X_W measures the chordwise position of the “wake plane”, delimiting the integration domain downstream of the aerofoil. The lateral distance of the C-shaped Σ edge from the aerofoil will be denoted as Z_W . Both X_W and Z_W will be expressed in chord units and, for the 3D wing, are measured on the symmetry plane.

A non-inertial frame of reference solidal with the rigid oscillating aerofoil (origin at the leading edge, x-axis in chord-wise direction and z-axis pointing upwards) has been considered. This facilitates the computation of relative time derivatives in the case of moving grid simulations. Another, inertial, frame of reference is also considered, initially coincident in origin and axes orientation with the non-inertial one, then fixed in time. Of course, in the case of steady solutions ($k = 0$), the inertial and non-inertial frames of reference do coincide.

Two-dimensional numerical solutions described in Sec. 2.2.1 were post-processed using the in-house software, *UBF2D*, implementing far-field methods illustrated in Sec. 1.4.1 and Sec. 1.4.3.2. Three-dimensional solutions were instead post-processed using a dedicated scripting routine implemented and executed in the *Tecplot*®360 commercial environment (see App. F for further de-

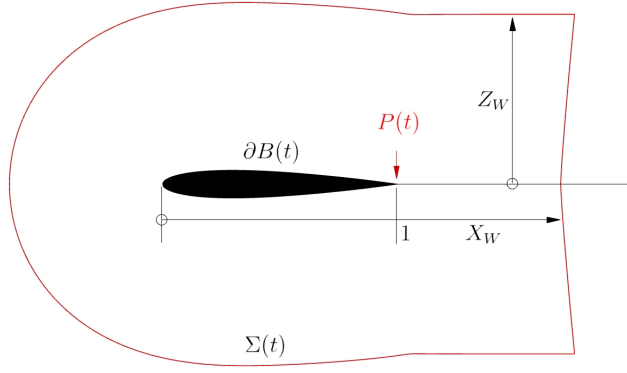


Figure 4.2.3: Control volume shape and characteristic dimensions.

tails).

The post-processing moves on from an input flow-field, where variables are provided at grid nodes. Spatial gradients of flow variables are computed using a nodes-based Green-Gauss formula [182] while time derivatives (in the non-inertial reference frame) are calculated using a 2nd-order-accurate finite differences formula (an asymmetric stencil is used at the boundaries of the analysed time period to preserve accuracy). Volume integrals are numerically computed by simply summing up cell centre values, multiplied by the cell volume, over the selected integration domain. Similarly, surface integrals are obtained computing the state in the cell surface centre, obtained by centred averaging. An exception occurs when the integration domain boundaries coincide with grid boundaries (e.g. at the body wall). In this case, wall spatial gradients required to evaluate the skin friction term ($\underline{F}_\tau$) and the body integral of the compressibility correction ($\underline{F}_{m_p}$) are obtained by extrapolation of the corresponding values computed at the first adjacent cell (for the velocity gradient), or at the first two adjacent cells (for the density gradient).

4.2.3 FAR-FIELD FORCE ACCURACY AND SENSITIVITY TO THE WAKE PLANE POSITION

Post-processing a steady subsonic 2D NS solution at low Reynolds number ($Re_\infty = 1 \times 10^4$), the aerodynamic drag computed through the far-field Lamb-vector-based method using flow vorticity ($\underline{F}_b^{vrt}$) needs a control volume extending approximately 4 chords downstream of the aerofoil to converge (Fig. 4.2.5), while, using Crocco's equation to compute the Lamb vector, the two formulas ($\underline{F}_b^{thd}$ and $\underline{F}_b^{THD}$) suddenly stabilise as soon as the whole aerofoil surface is enclosed in the selected control volume ($X_W \geq 1$, Fig. 4.2.5). Of course the total force (both lift and drag) is intrinsically independent of any control volume selection. The latter is a degree of freedom in the far-field formulas (the near-field method is instead independent of it, as it does not involve any control volume), but because of the equivalence between the two, and because they both calcu-

late the intrinsic force value, far-field results should be theoretically independent on both X_W and Z_W . Nevertheless, in numerical applications, variations of the computed far-field force with the control volume size are generally observed, and Fig. 4.2.5 illustrates in this case the much lower sensitivity of the unified method to the wake plane location.

Moreover, the converged drag values obtained using vorticity-based formulas ($\underline{F}_b^{vrt}$) differ by $\approx 12d.c.$ from the corresponding values computed using the near-field formula ($\underline{F}_{nf}$, Eq. (1.1)), while the new unified thermodynamic/Lamb-vector-based methods ($\underline{F}_b^{thd}$ and $\underline{F}_b^{THD}$) provide results basically coincident with the near-field formula, in the accuracy limit of $1d.c.$. No significant differences are observed on lift (Fig. 4.2.4), although it is highlighted that, also in this case, computations using the unified methods are closer to near-field data. A laminar flow separation on the aerofoil upper surface (occurring at approximately 45% of the chord length) is responsible for the slightly negative lift coefficient values, at positive incidence.

Similar results were obtained also post-processing a RANS solution for the same Mach number, but at high-Reynolds number ($Re_\infty = 9 \times 10^6$, Fig. 4.2.6, 4.2.7). The difference (on both lift and drag) between the two thermodynamic-based formulas is negligible and their results are basically coincident with the near-field reference data. The standard vorticity-based method, in this case, converges to a value which is $4d.c.$ lower (Fig. 4.2.7).

In presence of shock waves ($M_\infty = 0.8$), the vorticity-based formulation is largely inaccurate on drag (Fig. 4.2.9), as already observed by Mele et al. [117] and Minervino and Tognaccini [125]. To overcome this issue on the $\underline{F}_b^{vrt}$ formula, Mele et al. [117] also proposed a transonic variant of the method, based on a modified expression of the compressibility correction term ($\underline{F}_{m_p}$), which excludes the shock wave region from domain integration. The selection of the shock wave region, however, introduces other degrees of freedom in the force formula (apart those related to the control volume selection and to the DMT pole location) and the obtained results are numerically dependent on the particular size and shape of the excluded region surrounding the shock wave (or even multiple shock waves), with higher accuracy only obtained excluding the entire control volume from domain integration, retaining surface integrals only [125]. The modification proposed in [117] is moreover not applicable to the original compressible vortical force formula as derived by Wu et al. [222, p. 621], while the new unified approach is applicable to both Lamb-vector-based formulations and computed drag values are in very close agreement with the reference data obtained by near-field integration ($\underline{F}_{nf}$, Eq. (1.1)). Again results from the two thermodynamic-based formulas coincide and the difference with regards to the reference near-field drag is in this case limited to $3d.c.$, well below the discrepancy obtained using the vorticity-based formula. A slight reduction of the (acceptable) discrepancy observed on lift for the vorticity-based method ($\underline{F}_b^{vrt}$), with regards to the reference near-field data, is also obtained using the new approaches ($\underline{F}_b^{thd}$ and $\underline{F}_b^{THD}$, Fig. 4.2.8).

Similar results were obtained also post-processing a turbulent RANS solution at the same Mach number ($Re_\infty = 9 \times 10^6$, Fig. 4.2.10, 4.2.11). In this case, despite the higher angle of attack, flow viscosity attenuates the shock wave and the differences observed between the vortical formula and

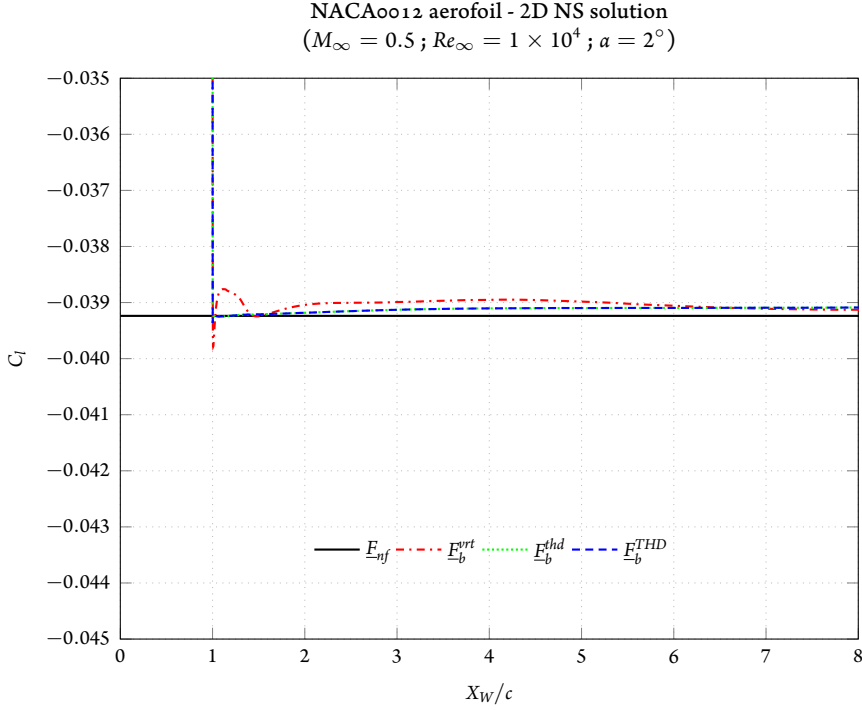


Figure 4.2.4: Far-field lift as a function of the wake plane position for the vorticity-based and the thermodynamic-based Lamb-vector methods, in low-Reynolds number viscous subsonic flow ($Z_W = 15c$; DMT pole at aerofoil l.e.).

the reference data are slightly reduced, although still unacceptable on drag (Fig. 4.2.11). Results obtained using the new unified approach are instead coincident with those provided by the near-field method (less than one count on drag).

Although the formulas derived in Section 4.1 are exact and generally valid also in 3D, the accuracy and sensitivity to the wake plane position of the three compared far-field formulas were also assessed on the 3D elliptic wing, in transonic viscous and turbulent flow. Results illustrated in Fig. 4.2.12 and Fig. 4.2.13, for lift and drag coefficients respectively, further confirm the higher accuracy of the new proposed mixed approach.

The improved accuracy and reduced domain sensitivity, observed for the unified method formulas, have important practical implications. Smaller integration domains can be used, compared to the standard vorticity-based approach, to obtain accurate force results, which is a computational advantage especially for the post-processing of unsteady flow solutions. Moreover, this is beneficial in the perspective of a following force decomposition analysis. In fact, the use of larger integration domains usually pollutes the results of a drag breakdown, as lift-induced drag (linked to trailing vorticity) is dissipated downstream of the body due to the artificial viscosity, an effect which is generally predominant with regards to that produced by the physical viscosity [113, 116].

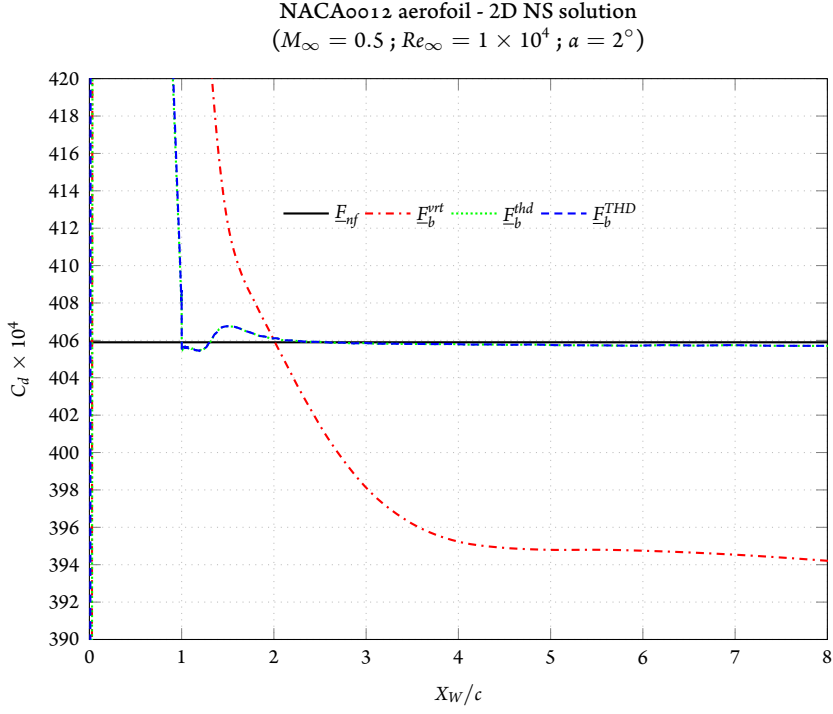


Figure 4.2.5: Far-field drag as a function of the wake plane position for the vorticity-based and the thermodynamic-based Lamb-vector methods, in low-Reynolds number viscous subsonic flow ($Z_W = 15c$; DMT pole at aerofoil l.e.).

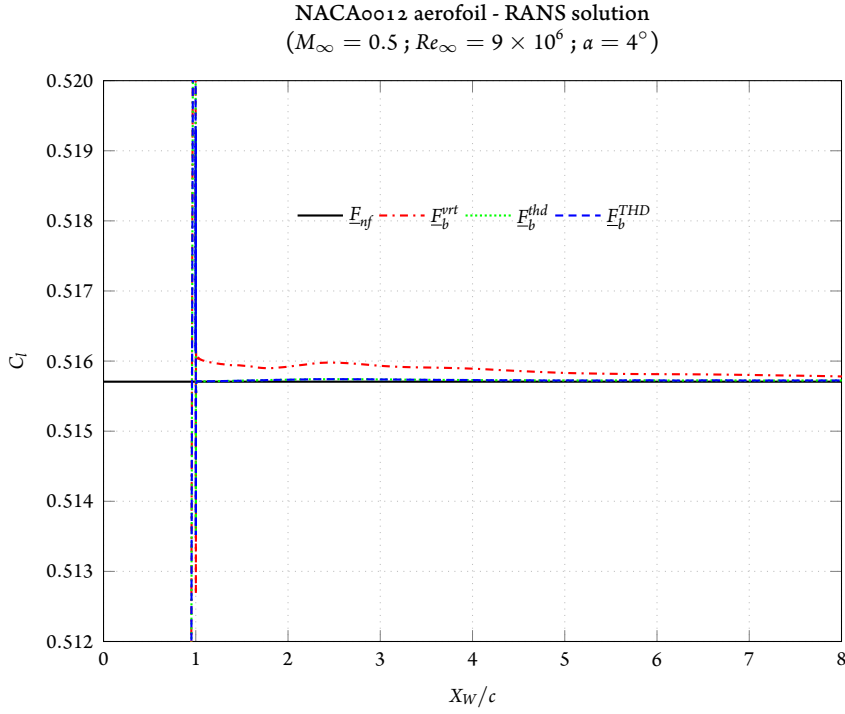


Figure 4.2.6: Far-field lift as a function of the wake plane position for the vorticity-based and the thermodynamic-based Lamb-vector methods, in high-Reynolds number viscous subsonic flow ($Z_W = 15c$; DMT pole at aerofoil l.e.).

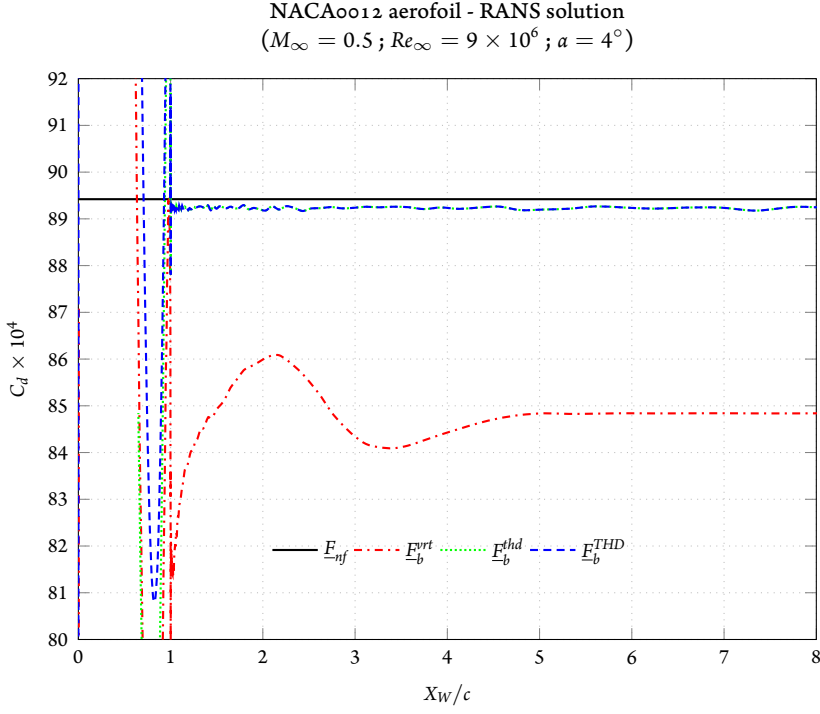


Figure 4.2.7: Far-field drag as a function of the wake plane position for the vorticity-based and the thermodynamic-based Lamb-vector methods, in high-Reynolds number viscous subsonic flow ($Z_W = 15c$; DMT pole at aerofoil l.e.).

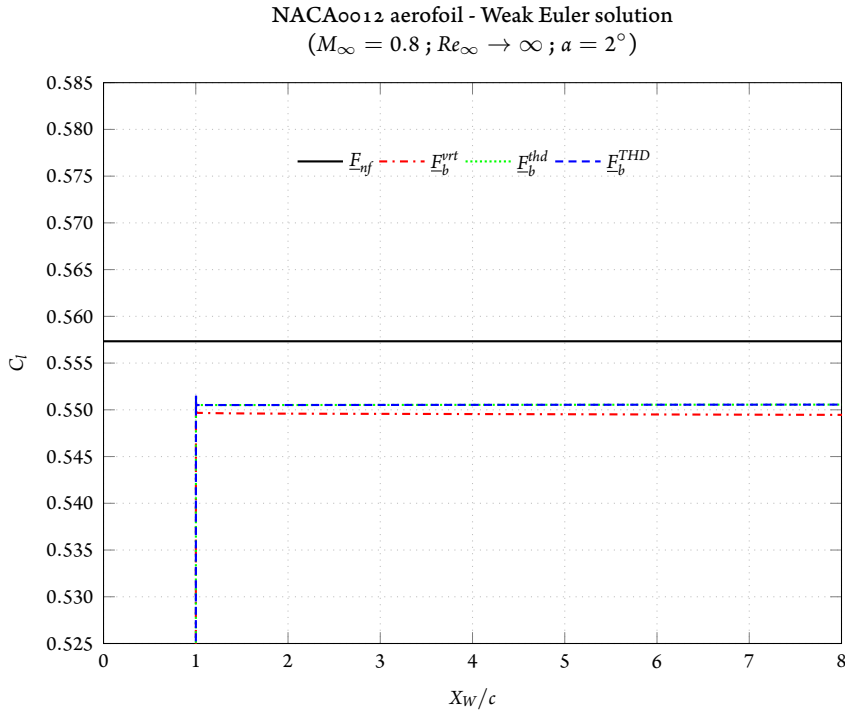


Figure 4.2.8: Far-field lift as a function of the wake plane position for the vorticity-based and the thermodynamic-based Lamb-vector methods, in inviscid transonic flow ($Z_W = 15c$; DMT pole at aerofoil l.e.).

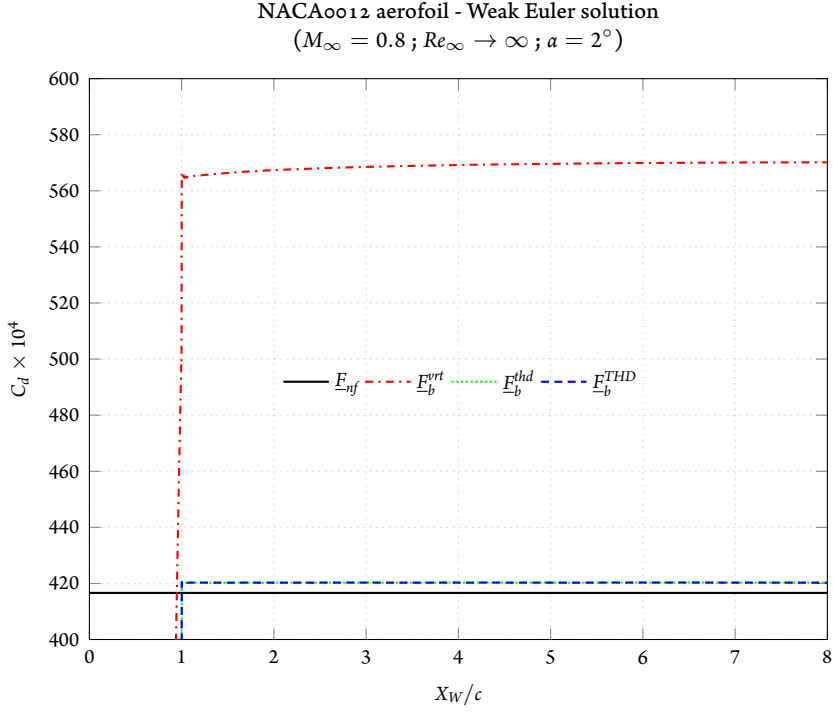


Figure 4.2.9: Far-field drag as a function of the wake plane position for the vorticity-based and the thermodynamic-based Lamb-vector methods, in inviscid transonic flow ($Z_W = 15c$; DMT pole at aerofoil l.e.).

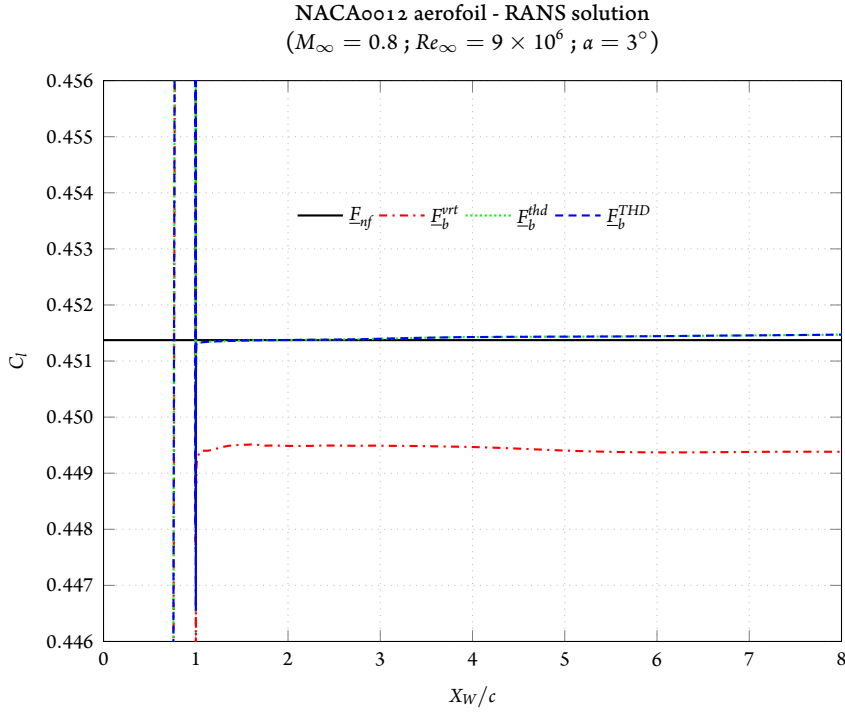


Figure 4.2.10: Far-field lift as a function of the wake plane position for the vorticity-based and the thermodynamic-based Lamb-vector methods, in high-Reynolds number viscous transonic flow ($Z_W = 15c$; DMT pole at aerofoil i.e.).

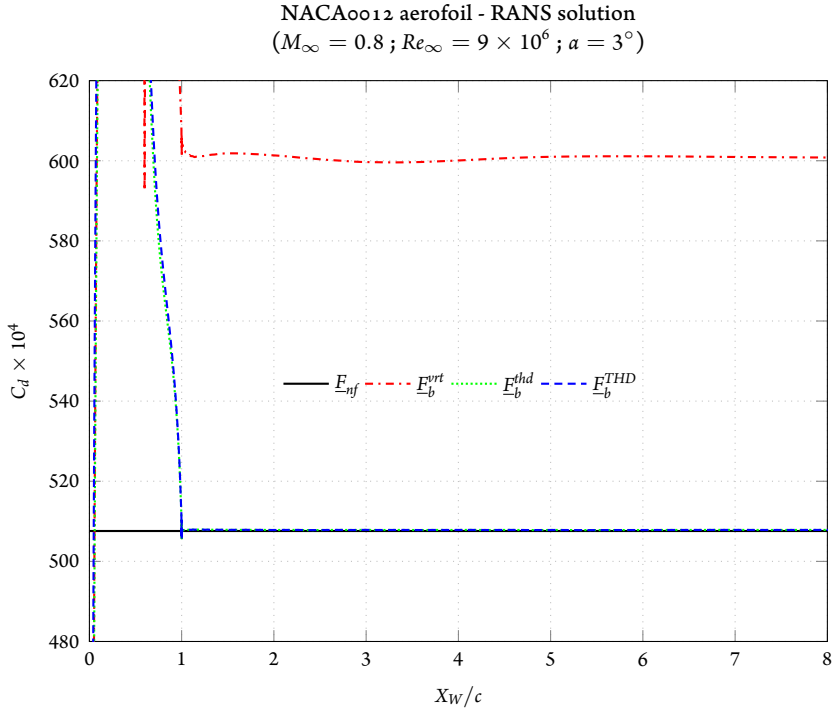


Figure 4.2.11: Far-field drag as a function of the wake plane position for the vorticity-based and the thermodynamic-based Lamb-vector methods, in high-Reynolds number viscous transonic flow ($Z_W = 15c$; DMT pole at aerofoil l.e.).

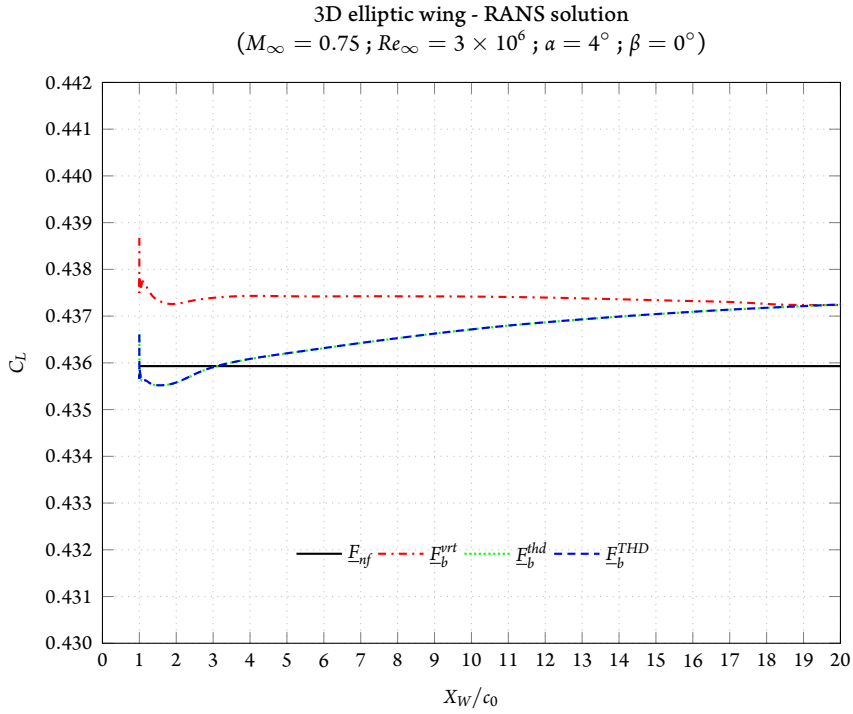


Figure 4.2.12: Far-field lift as a function of the wake plane position for the vorticity-based and the thermodynamic-based Lamb-vector methods, in 3D high-Reynolds number viscous transonic flow ($Z_W/c_0 = 10.1$; DMT pole at l.e. of mid-span wing section).

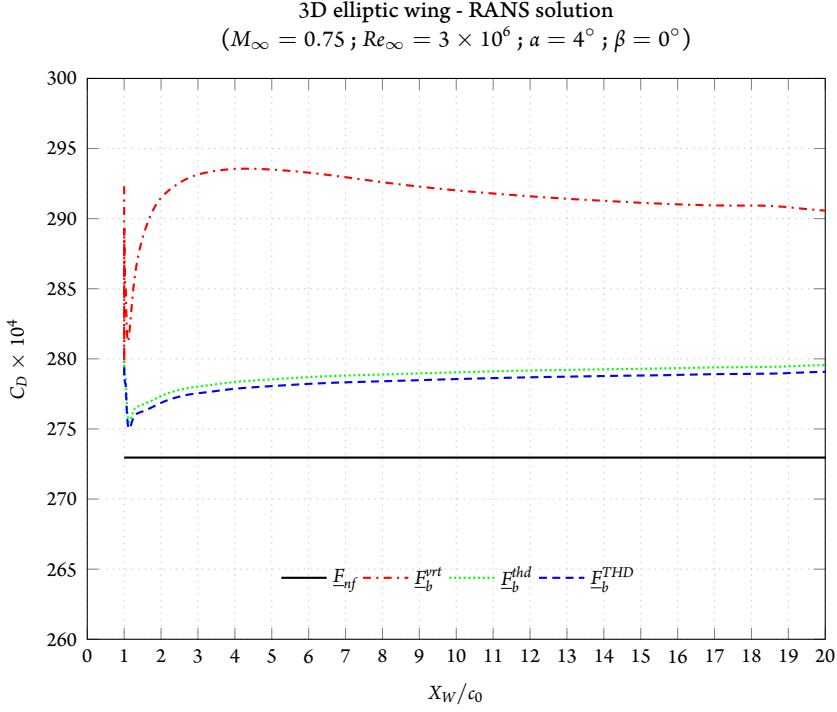


Figure 4.2.13: Far-field drag as a function of the wake plane position for the vorticity-based and the thermodynamic-based Lamb-vector methods, in 3D high-Reynolds number viscous transonic flow ($Z_W/c_0 = 10.1$; DMT pole at l.e. of mid-span wing section).

Additionally, the reduced domain sensibility also implies a reduced uncertainty in the integration domain selection, which is an issue in Lamb vector methods more than in other far-field approaches [125, 188]. Finally, as mentioned above, the higher accuracy is crucial in presence of strong shocks, where vorticity-based methods fail in computing drag and different approaches were proposed, although at the cost of introducing additional degrees of freedom in the formula (then, additional uncertainties in the practical application of the method).

A similar improvement of far-field drag computations in presence of shock waves using the new thermodynamic-based, instead of the standard vorticity-based Lamb vector calculation, is also observed in the unsteady regime, where Fig. 4.2.15 illustrates results of a flow simulation for the plunging NACA0012 aerofoil in inviscid transonic flow. Time marching along lift and drag loops in Fig. 4.2.14 and Fig. 4.2.15 is indicated highlighting points at the beginning ($4\mathcal{T}$) of the time period $\mathcal{T}$ and after a fraction of it. Slight differences on far-field ($\underline{F}_b^{thd}$ and $\underline{F}_b^{THD}$) versus near-field ($\underline{F}_{nf}$) drag values are still visible, probably related to the accuracy of CFD solutions in the unsteady wake and shock waves regions. It is interestingly observed that, applying the same methods using formulation “a” as the baseline, the accuracy of $\underline{F}_a^{THD}$ is almost perfect (Fig. 4.2.16). Indeed, the difference between formulations “a” and “b” is the expression of the compressibility correction, which in the last case depends on position vectors in the entire control volume, amplifying errors in the unsteady wake region where the computational mesh is not sufficiently refined.

In the unsteady case (whichever the baseline formula), the two thermodynamic-based methods did not provide matching results and a slight difference between the two is observed, with the simplified formula providing closer results compared to the reference near-field data. This has to be attributed to the different treating of the skin friction force, which is expressed by far-field integrals in $\underline{F}_b^{thd}$ while its reference near-field formulation is used in $\underline{F}_b^{THD}$. The average drag is of negative sign (thrust) at the simulated Reynolds number (*Knoller-Betz* or *Katzmayr* effect [16, 82, 85]), a bio-inspired propulsive mechanism which is still subject of recent studies [149, 214]. All tested formulas exhibit negligible differences on lift (Fig. 4.2.14).

Results illustrated in steady viscous and subsonic flow are also confirmed in the unsteady regime, with figures from 4.2.17 to 4.2.22 providing comparison of lift and drag calculations for the plunging NACA0012 aerofoil in viscous and incompressible flow at different values of the reduced frequency k . While the three far-field formulas exhibit negligible differences on lift (Fig. 4.2.17, 4.2.19 and 4.2.21), the accuracy of drag calculation is improved using the unified thermodynamic/Lamb-vector-based method. This result is confirmed at the three different analysed values of the reduced frequency, as well in laminar (Fig. 4.2.22) or turbulent flow condition (Fig. 4.2.18 and 4.2.20). At the upper-bound of the quasi-steady flow regime ($k = 0.05$), where drag variations are confined to less than 6 *d.c.* over the whole aerofoil oscillation cycle (Fig. 4.2.18), far-field drag calculations obtained using the unified method formulas differ from the reference near-field data by one *d.c.* only, compared to drag results of the vorticity-based formula, which are $\approx 5d.c.$ smaller. This result is in very good agreement with the analogous observations discussed for the steady subsonic and turbulent case (Fig. 4.2.7).

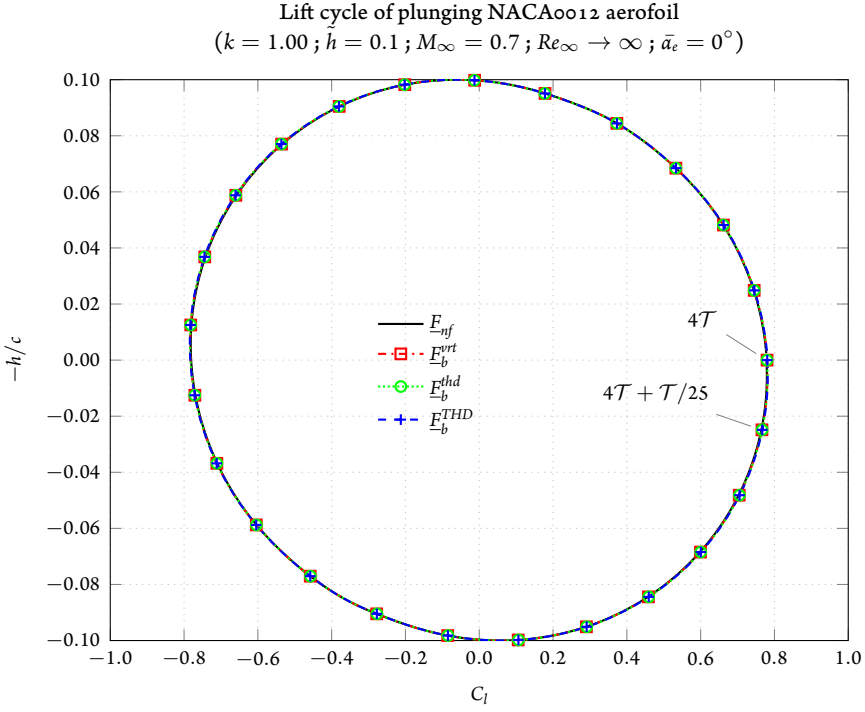


Figure 4.2.14: Far-field lift from vorticity-based and thermodynamic-based Lamb-vector methods, compared to near-field force in transonic inviscid flow ($X_W = 50c$; $Z_W = 50c$; DMT pole at aerofoil t.e.; non-inertial).

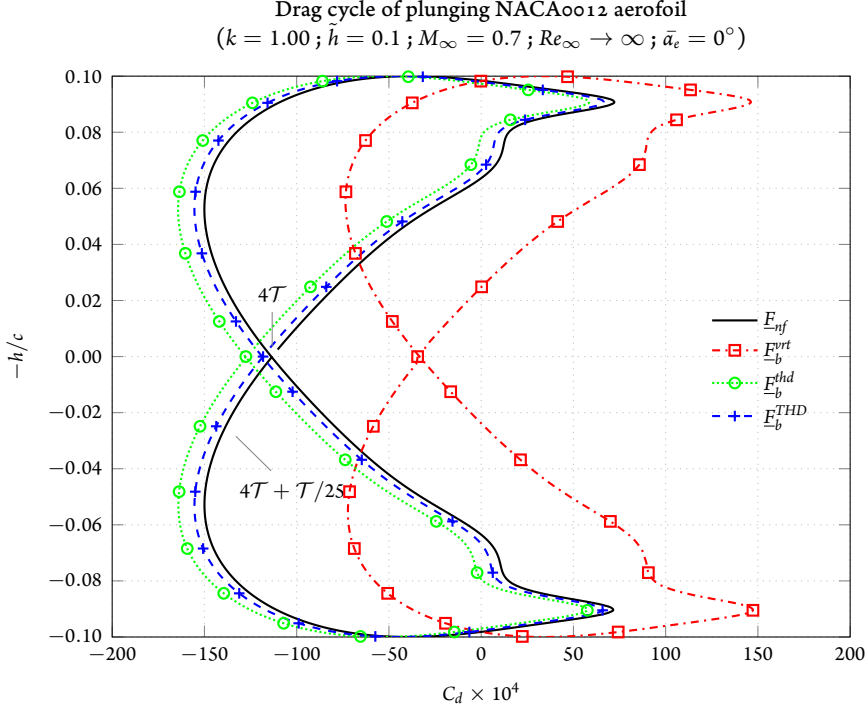


Figure 4.2.15: Far-field drag from vorticity-based and thermodynamic-based Lamb-vector methods (“ b ” formulation), compared to near-field force in transonic inviscid flow ($X_W = 50c$; $Z_W = 50c$; DMT pole at aerofoil t.e.; non-inertial).

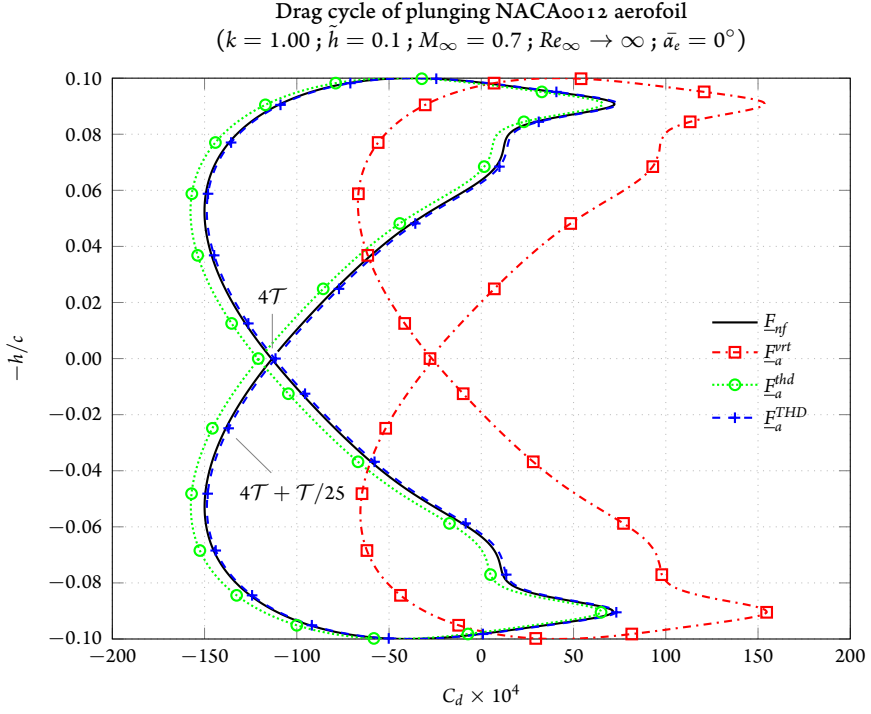


Figure 4.2.16: Far-field drag from vorticity-based and thermodynamic-based Lamb-vector methods ("a" formulation), compared to near-field force in transonic inviscid flow ($X_W = 50c$; $Z_W = 50c$; DMT pole at aerofoil t.e.; non-inertial).

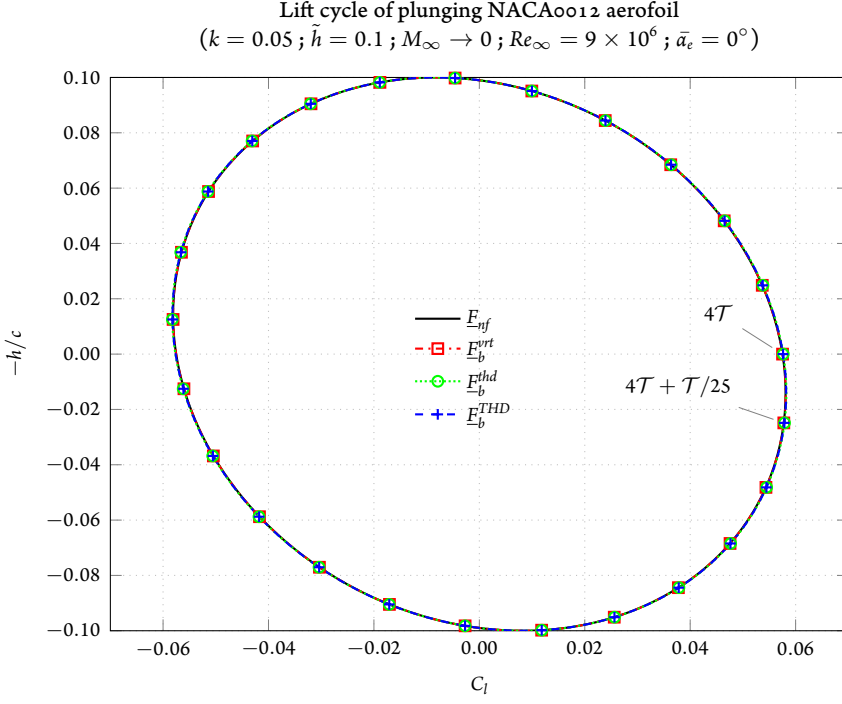


Figure 4.2.17: Far-field lift from vorticity-based and thermodynamic-based Lamb-vector methods, compared to near-field force in incompressible viscous and turbulent flow ($X_W = 1.0c$; $Z_W = 0.2c$; DMT pole at aerofoil t.e.; non-inertial).

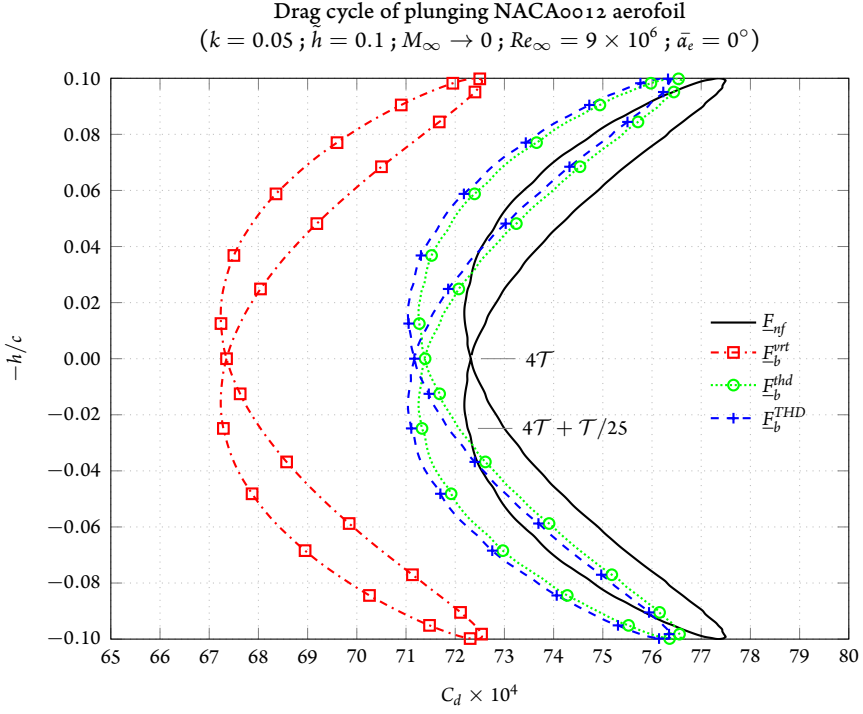


Figure 4.2.18: Far-field drag from vorticity-based and thermodynamic-based Lamb-vector methods, compared to near-field force in incompressible viscous and turbulent flow ($X_W = 1.0c$; $Z_W = 0.2c$; DMT pole at aerofoil t.e.; non-inertial).

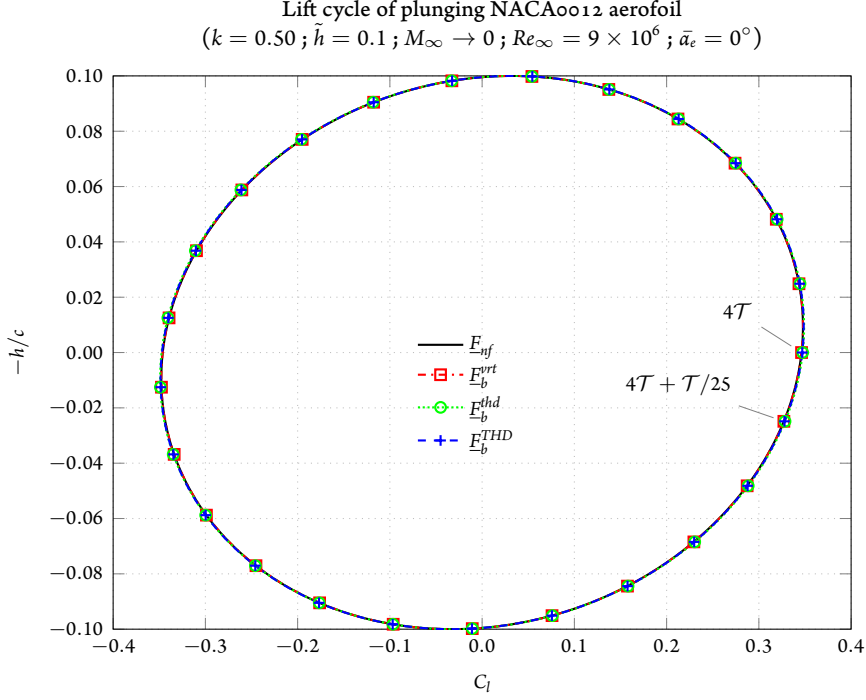


Figure 4.2.19: Far-field lift from vorticity-based and thermodynamic-based Lamb-vector methods, compared to near-field force in incompressible viscous and turbulent flow ($X_W = 1.0c$; $Z_W = 0.2c$; DMT pole at aerofoil t.e.; non-inertial).

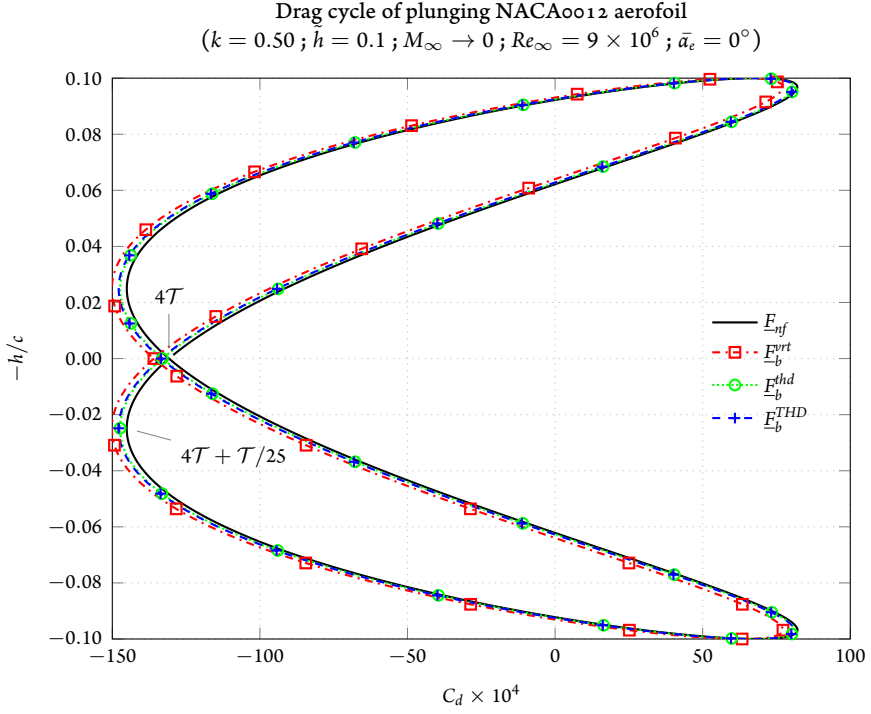


Figure 4.2.20: Far-field drag from vorticity-based and thermodynamic-based Lamb-vector methods, compared to near-field force in incompressible viscous and turbulent flow ($X_W = 1.0c$; $Z_W = 0.2c$; DMT pole at aerofoil t.e.; non-inertial).

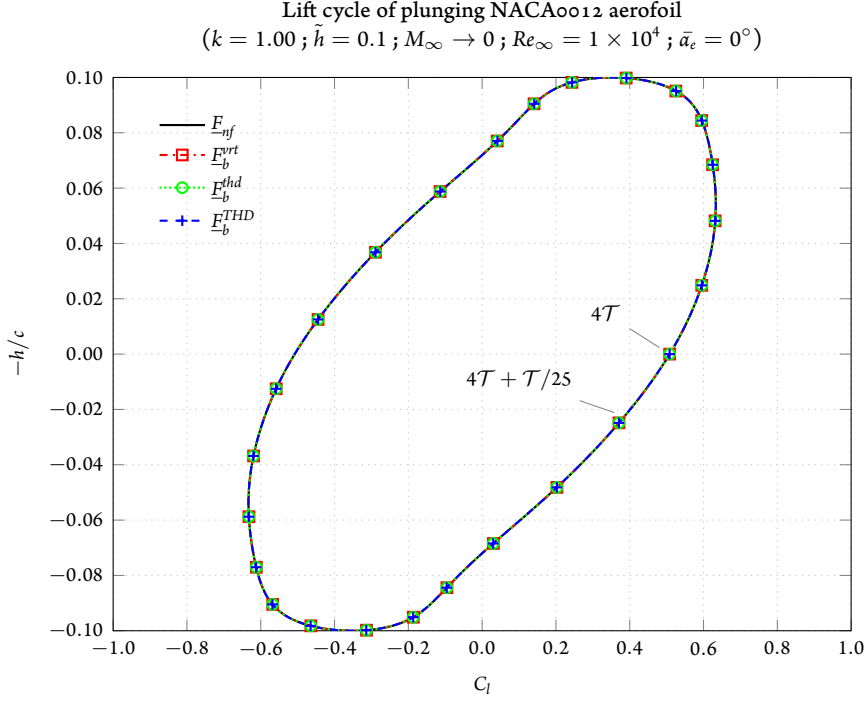


Figure 4.2.21: Far-field lift from vorticity-based and thermodynamic-based Lamb-vector methods, compared to near-field force in incompressible viscous and laminar flow ($X_W = 1.0c$; $Z_W = 0.2c$; DMT pole at aerofoil t.e.; non-inertial).

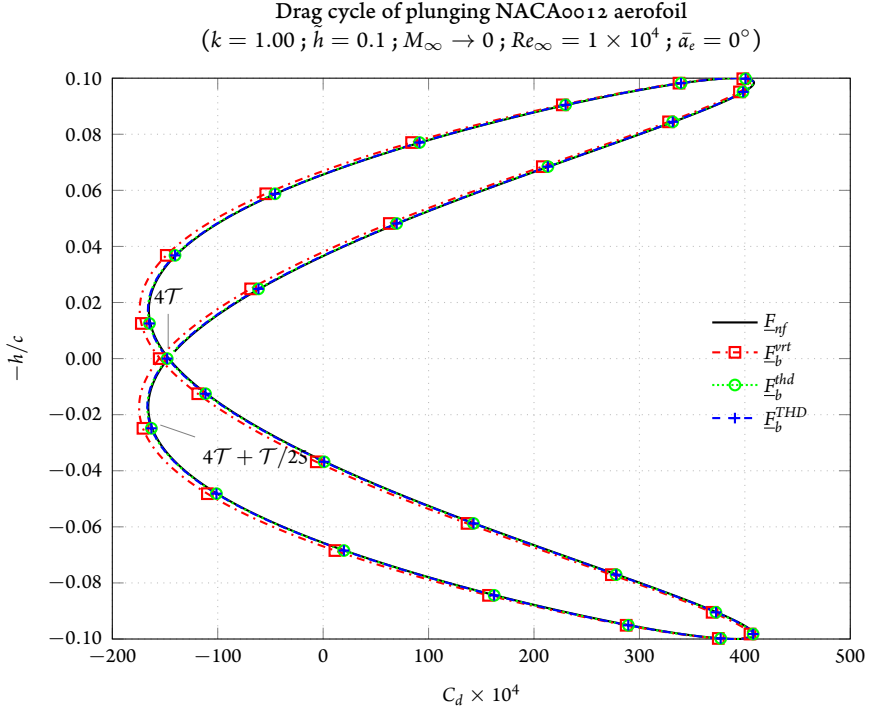


Figure 4.2.22: Far-field drag from vorticity-based and thermodynamic-based Lamb-vector methods, compared to near-field force in incompressible viscous and laminar flow ($X_W = 1.0c$; $Z_W = 0.2c$; DMT pole at aerofoil t.e.; non-inertial).

4.2.4 LAMB VECTOR AND VORTICITY FIELDS

A visualisation of the (negative) Lamb vector field around the NACA0012 aerofoil in steady viscous and transonic flow ($M_\infty = 0.8$, $Re_\infty = 9 \times 10^6$, $\alpha = 3^\circ$) is presented in Fig. 4.2.23; it represents the local generation of the Kutta-Joukowski force on the aerofoil. As already evidenced by [113] for incompressible flows, the Lamb vector is essentially generated in the proximity of the body wall (boundary layer and viscous wake) and across shock waves (the contribution in the wake of the shock is small and not visible in Fig. 4.2.23). This is the flow-field region where the aerodynamic force is generated and all vectors illustrated in Fig. 4.2.23 have their tail points in such a small portion of the domain. It is observed that the largest magnitude of the Lamb vector is obtained near the aerofoil leading edge and the magnitude difference between upper-side and lower-side i.e. lobes is the major contribution to the aerofoil lift.

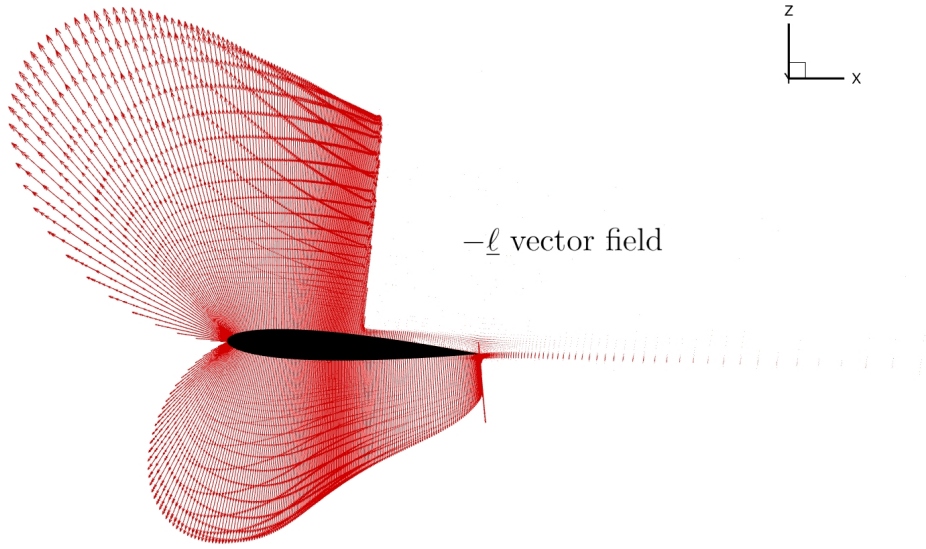
The Lamb vector is computed, from the analysed steady CFD solution, directly from flow vorticity using its definition ($\underline{\ell} = \underline{\omega} \times \underline{V}$, Fig. 4.2.23a), or indirectly from Crocco's equation (Fig. 4.2.23b). In the first case, the Lamb vector field is of course orthogonal to flow streamlines. Vectors directions therefore appear perfectly aligned and normally oriented with regards to the aerofoil surface in Fig. 4.2.24a, as flow streamlines in the boundary layer are almost tangent to the wall. In the second case, deviations in the Lamb vector directions are observed, particularly in proximity of the aerofoil surface and at its trailing edge (Fig. 4.2.24b), where a spurious streamwise Lamb vector component is highlighted. This evidences the local imbalance between the computed Lamb vector and the resultant of all other forces (per unit mass) acting on the fluid particle. To better appreciate the local difference in Lamb vector fields, they are also illustrated superimposed one onto each other in Fig. 4.2.25, where a closer visualisation is provided near the wall upstream of the strong shock (Fig. 4.2.25a, showing the large deviations in the first cells adjacent to the surface) and near the trailing edge (Fig. 4.2.25b, where the largest deviations occur, although confined to the proximity of the sharp t.e.).

It is also interesting to plot the Lamb vector field from an unsteady simulation of a plunging aerofoil (Fig. 4.2.26). Once again, it is clearly observed that the major contribution to thrust generation (maximum at mid strokes, Fig. 4.2.22) is provided by the vortex force.

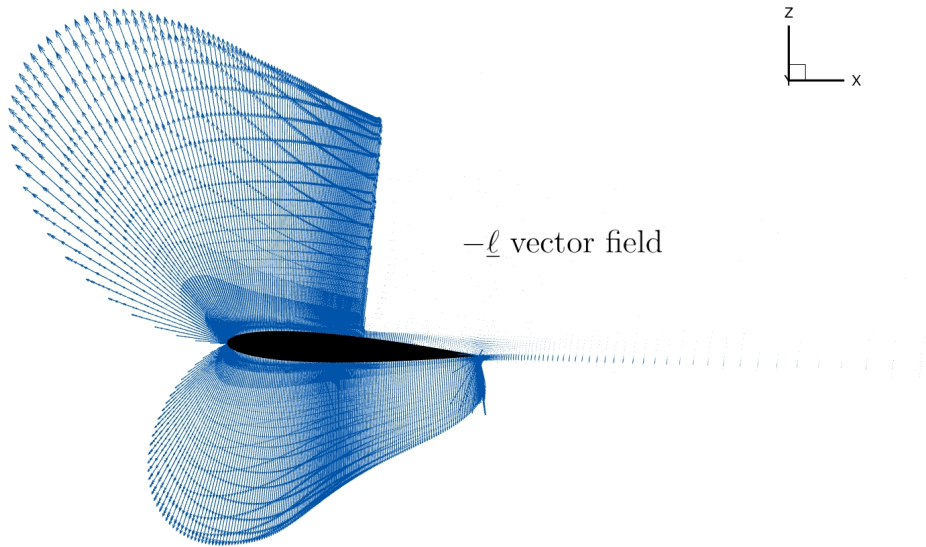
In the case the Lamb vector is computed from Crocco's equation, an indirect calculation of the flow vorticity can also be obtained, as an alternative to the direct differentiation the velocity field. Indeed, the cross-product *inverse* to derive $\underline{\omega}$ from $\underline{\ell} = \underline{\omega} \times \underline{V}$ is not uniquely defined. If $\underline{\ell}$ and $\underline{V}$ are orthogonal, then the solutions are:

$$\underline{\omega} = \frac{\underline{\ell} \times \underline{V}}{\underline{V} \cdot \underline{V}} + q\underline{V} \quad (4.29)$$

for any arbitrary q scalar field. Therefore, this indirect vorticity calculation (a technique also proposed in geophysical applications [35]) ignores the spurious streamwise Lamb vector component and only allows for computing the *transverse* vorticity field (by setting $q = 0$), which in two-dimensional flows represents the entire vorticity. The Lamb vector knowledge would not suffice

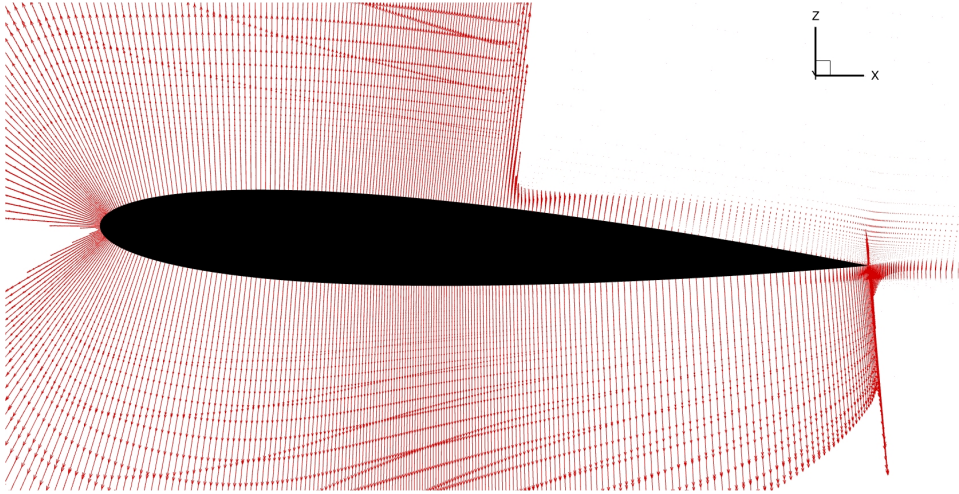


(a)

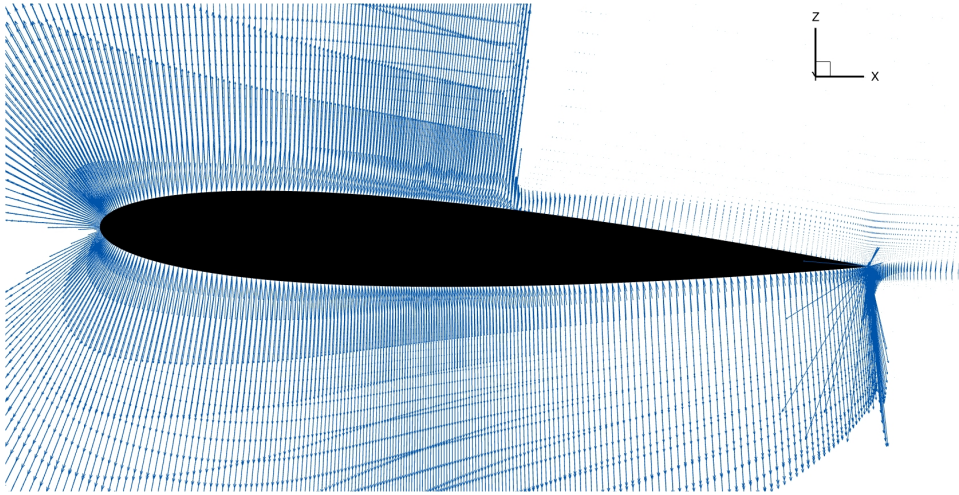


(b)

Figure 4.2.23: Visualisation of the negative Lamb vector fields computed from flow vorticity (4.2.23a) or from Crocco's equation (4.2.23b), for the steady viscous transonic flow-field ($M_\infty = 0.8$; $Re_\infty = 9 \times 10^6$; $\alpha = 3^\circ$).

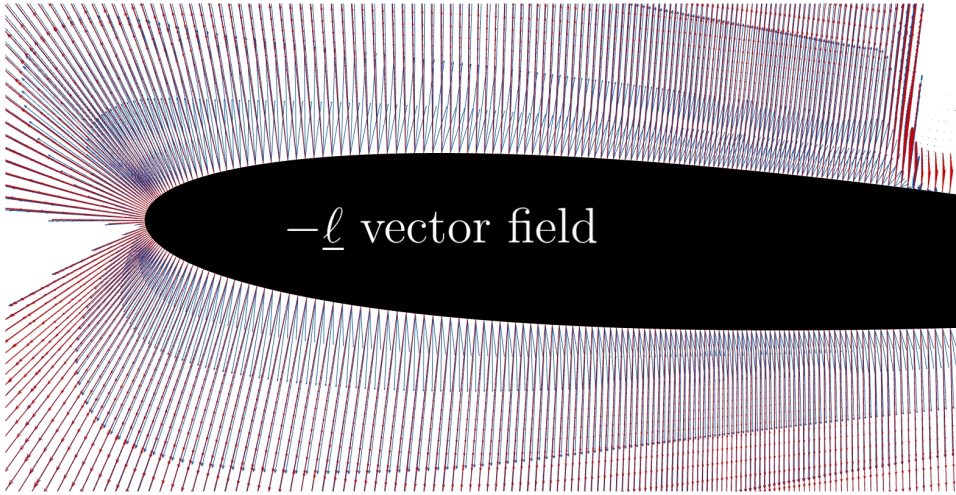


(a)

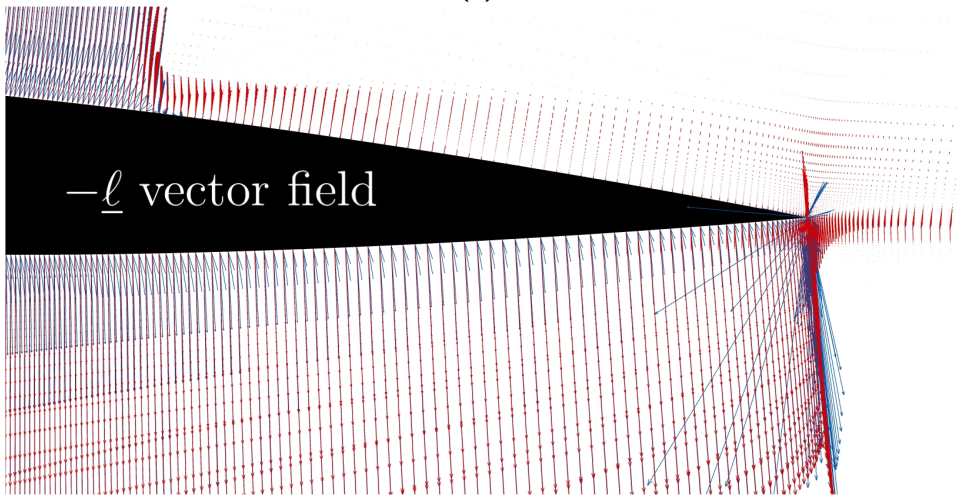


(b)

Figure 4.2.24: Zoomed visualisation of the negative Lamb vector fields computed from flow vorticity (4.2.24a) or from Crocco's equation (4.2.24b), for the steady viscous transonic flow-field ($M_\infty = 0.8$; $Re_\infty = 9 \times 10^6$; $\alpha = 3^\circ$).



(a)



(b)

Figure 4.2.25: Comparison of the negative Lamb vector fields computed from flow vorticity (red) or from Crocco's equation (blue) in the region of aerofoil leading-edge (4.2.25a) and trailing-edge (4.2.25b), for the steady viscous transonic flow-field ($M_\infty = 0.8$; $Re_\infty = 9 \times 10^6$; $\alpha = 3^\circ$).

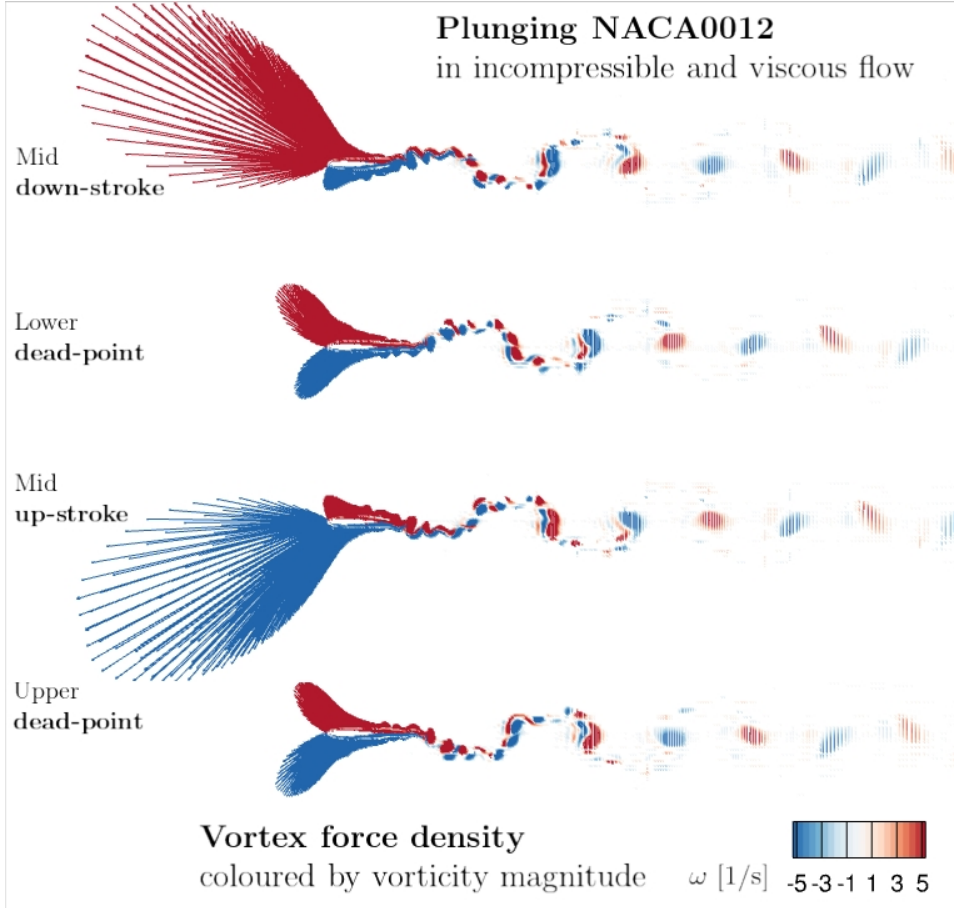


Figure 4.2.26: Visualisation of the unsteady negative Lamb vector field (computed from flow vorticity) for the laminar and incompressible flow-field around a plunging NACA0012 ($k = 1$; $\tilde{h} = 0.1$; $M_\infty \rightarrow 0$; $Re_\infty = 1 \times 10^4$; $\tilde{\alpha}_e = 0^\circ$).

however to indirectly compute the vorticity in three-dimensional flows, where a physical stream-wise vorticity might instead be present [205]. In this case, besides the Lamb vector, the additional knowledge of flow *helicity density* ($\underline{\omega} \cdot \underline{V}$) would be necessary to indirectly calculate the whole vorticity.

Fig. 4.2.27 illustrates, for the steady viscous transonic solution, a comparison between the vorticity magnitude computed from a direct differentiation of the velocity field (nodes-based Green-Gauss formula, Fig. 4.2.27a) or indirectly obtained from the Lamb vector and velocity fields (Fig. 4.2.27b). The former is characterised by larger oscillations in the region of the shock wave, in particular above its top end, which are not observed on the indirectly computed vorticity field. The observed oscillations suggest inaccuracies on flow vorticity computed by numerical differentiation of the discrete velocity field, and one should also consider that, in primitive variables flow solvers, the vorticity balance equation is usually not part of the Partial Differential Equations (PDEs) system which is numerically solved. Those inaccuracies are reduced by the indirect vorticity calculation (in 2D), which relies on the Lamb vector, obtained through Crocco's equation (momentum balance), and on fluid velocity, obtained from momentum and mass balance equations.

Concerning the spurious vorticity waves observed in the unsteady compressible regime, in Ch. 3 Sec. 3.2.5 their relation to flow dilatation waves was discussed. Here below a picture similar to that included in Fig. 3.2.3b is obtained using the inverse vorticity calculation method described in Sec. 4.2.4. In Fig. 4.2.28, it is compared to the snapshot of the velocity divergence, at the same time step. It can be observed how the vorticity waves contour plot (Fig. 4.2.28a) now shows a pattern similar to that of dilatation waves (Fig. 4.2.28b), which was less evident using the standard vorticity calculation approach (Fig. 3.2.3b).

4.2.5 LOCAL PRODUCTION OF THE AERODYNAMIC FORCE

For steady high-Reynolds number flows, Eq. (4.1) – or, better, its RANS extension (Appendix E) – can be used to visualise the local production of the aerodynamic force. In fact, in this case the term $\underline{F}_\mu$ is negligible [112, 119] and all other non-zero terms in Eq. (4.1) only contain volume integrals, with exception of $\underline{F}_S$. Nevertheless, Ostieri and Tognaccini [141] also derived an equivalent expression for $\underline{F}_S$ only involving volume integrals (Ch. 2, Sec. 2.1.5). It is recalled hereafter, for the sake of completeness:

$$\underline{F}_{SV} = - \int_V \underline{\nabla} (r \cdot \rho \underline{\ell}) dV + \int_V \underline{\nabla} \cdot (r \rho \underline{\ell}) dV$$

Therefore, the whole aerodynamic force can be obtained integrating, over the control volume domain, a *local force production*, represented by the sum of all quantities under volume integral sign. On a discrete basis, the elementary contribution to this domain integral of each grid cell of the computational domain is visualised in Fig. 4.2.29 and Fig. 4.2.30, with reference to lift and drag respectively. Again the Lamb vector contribution in $\underline{F}_\ell$ and $\underline{F}_{SV}$ can either be computed

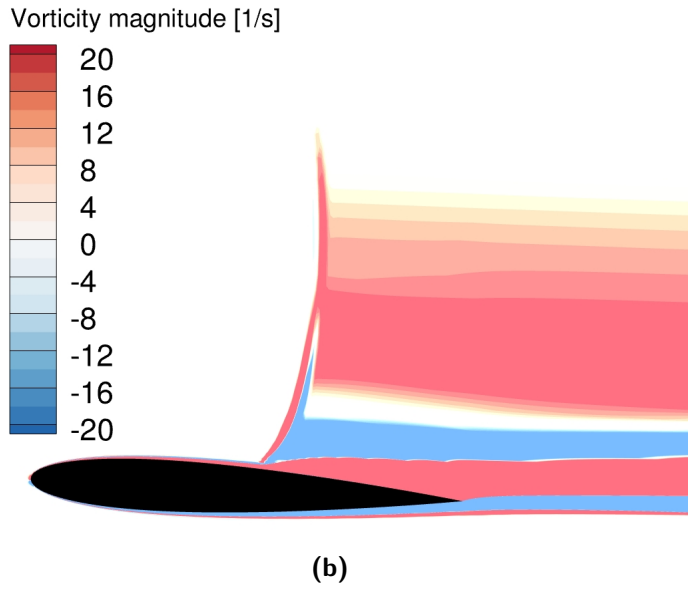
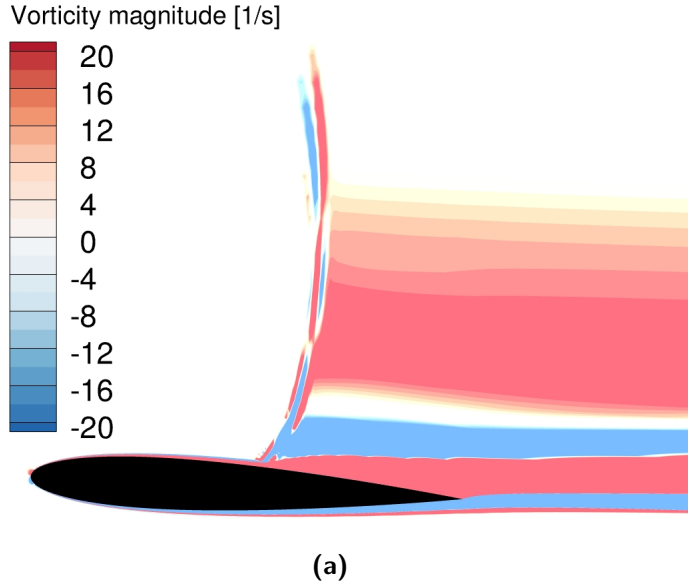


Figure 4.2.27: Contour plot visualisation of the vorticity magnitude directly obtained differentiating the velocity field (4.2.27a) or indirectly computed from the Lamb vector (4.2.27b), for the steady viscous transonic flow-field ($M_\infty = 0.8$; $Re_\infty = 9 \times 10^6$; $\alpha = 3^\circ$).

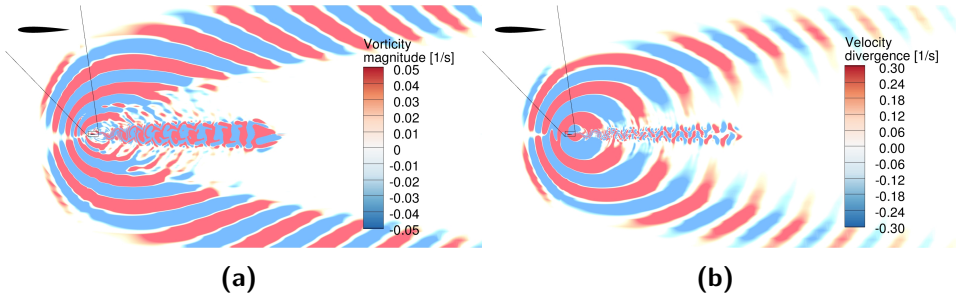


Figure 4.2.28: Flow vorticity (a) and velocity divergence (b) around a plunging NACA0012 ($k = 1$; $\tilde{h} = 0.1$; $M_\infty = 0.5$; $Re_\infty = 9 \times 10^6$; $\tilde{\alpha}_e = 0^\circ$). Snapshot at the end of 10 complete oscillations, using inverse vorticity calculation.

from flow vorticity (Fig. 4.2.29a and Fig. 4.2.30a) or from Crocco's equation (Fig. 4.2.29b and Fig. 4.2.30b), and results illustrate how the local aerodynamic force production is well confined in the second case compared to the first one. Differences are visible on lift (for example at the shock foot, Fig. 4.2.29) and a dramatic improvement is observed on drag (Fig. 4.2.30), where the local drag production at the top of the shock wave region is hard to be physically justified (Fig. 4.2.30a) and the spurious drag production visible around the leading edge is also reduced by the new mixed approach (Fig. 4.2.30b). The neatness of drag production distributions is particularly relevant for Lamb-vector-based drag decomposition methods aiming to separate viscous, wave and spurious drag contributions [83, 110, 125]. In particular, at the shock foot, the distinction between the boundary layer contribution and the shock wave contribution to the local drag is facilitated with the proposed unified method, which is clearly not straightforward in the vorticity-based approach, where the local drag production is spread between the two zones (Fig. 4.2.30a).

In Fig. 4.2.31 the total drag visualisation is also decomposed isolating lift-induced and parasite contributions, using Mele et al. [117] definitions expressed using volume integrals only:

$$D_i = \underline{e}_x \cdot \left(\underline{F}_\ell + \underline{F}_{m_p} \right)$$

$$D_p \approx \underline{e}_x \cdot \underline{F}_{SV} \quad (\text{for high Reynolds number flows})$$

It is interesting to observe that the shock wave wake contributes to the breakdown, but its effects almost compensate on the overall drag (Fig. 4.2.30b).

4.2.6 SENSITIVITY TO THE CONTROL VOLUME SIZE IN UNSTEADY RESULTS

In Ch. 3 dedicated to unsteady force analysis, the Lamb vector was calculated using its definition $\underline{\ell} = \underline{\omega} \times \underline{V}$, i.e. with the standard vorticity-based approach. A significant loss of accuracy was identified in all force formulas when extending the selected control volume far from the body, in the region of computed vortical structures (airfoil unsteady wake and, for compressible flows,

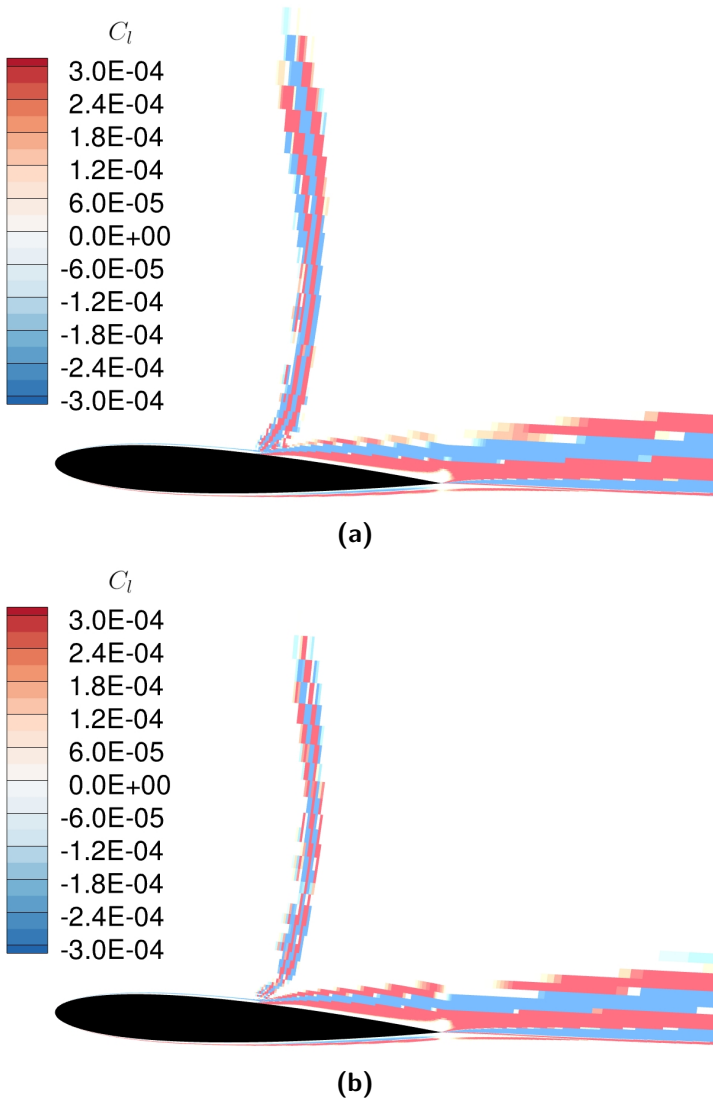


Figure 4.2.29: Lift coefficient of individual grid cells, computing the Lamb vector from flow vorticity (4.2.29a) or from Crocco's equation (4.2.29b), for the steady viscous transonic flow-field ($M_\infty = 0.8$; $Re_\infty = 9 \times 10^6$; $\alpha = 3^\circ$).

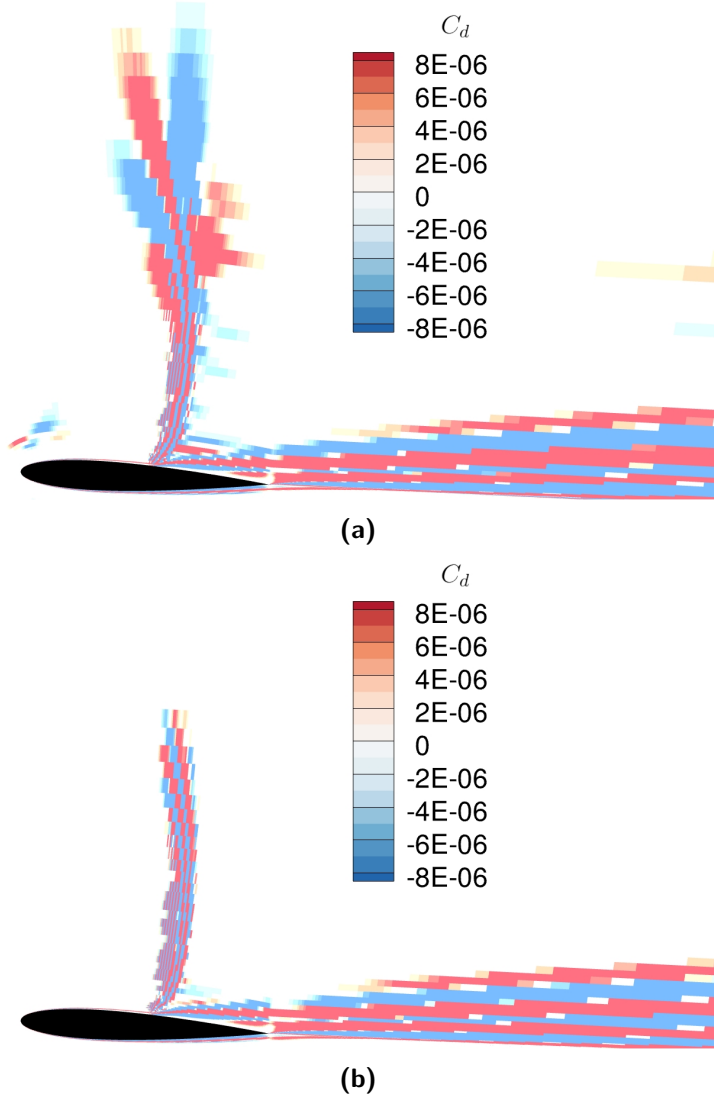


Figure 4.2.30: Drag coefficient of individual grid cells, computing the Lamb vector from flow vorticity (4.2.30a) or from Crocco's equation (4.2.30b), for the steady viscous transonic flow-field ($M_\infty = 0.8$; $Re_\infty = 9 \times 10^6$; $\alpha = 3^\circ$).

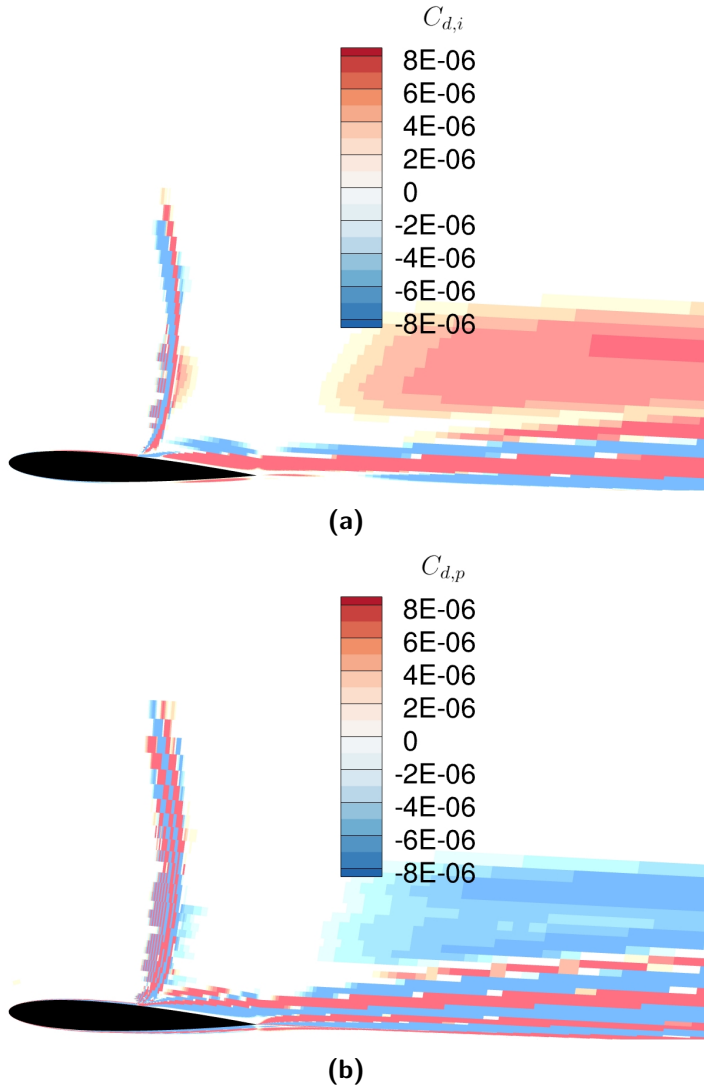


Figure 4.2.31: Lift-induced (4.2.31a) and parasite (4.2.31b) drag coefficient of individual grid cells (obtained computing the Lamb vector from Crocco's equation), for the steady viscous transonic flow-field ($M_\infty = 0.8$; $Re_\infty = 9 \times 10^6$; $\alpha = 3^\circ$).

spurious vorticity waves). On small control volumes, the observed inaccuracies were not so large, although not negligible on drag, especially in viscous cases.

The scope of this section is to analyse the impact of the Lamb vector calculation method on the far-field force variation with the control volume size, for the pitching or plunging viscous cases at $Re_\infty = 9 \times 10^6$ and $k = 1$ in incompressible and compressible subsonic ($M_\infty = 0.5$) flow conditions (Tab. 3.2.1). URANS simulations used the refined grid (Fig. 3.2.2) providing higher resolution especially in the unsteady wake region, to better analyse the far-field force variation with the control volume size. The energy equation was solved also in the incompressible case (nevertheless, negligible temperature variations in the entire flow-field were observed).

Figures from 4.2.32 to 4.2.39 provide comparison of the two approaches, for a control volume selected with $Z_W = 10.0^\dagger$.

In Ch. 3 (Sec. 3.2.4) it was observed that, for pitching oscillations, the presence of apparent forces in the non-inertial force formula leads to a divergent behaviour of computed errors as the control volume is progressively enlarged. This is avoided using the new thermodynamic-based approaches (Fig. 4.2.32 and 4.2.33) as apparent forces are either computed twice and with opposite signs (which of course makes their global contribution vanish in $\underline{F}_b^{thd}$) or not computed at all ($\underline{F}_b^{THD}$) and the divergence of errors disappears in both cases. The new force formula is even more accurate, with no appreciable error on computed lift for any of the investigated control volume size. Nevertheless, errors on drag are still non-negligible and higher accuracy can be obtained, for this pitching aerofoil, working in the inertial reference frame (Fig. 4.2.34 and 4.2.35). In this case, in fact, errors obtained using the proposed thermodynamic approaches are acceptable on drag as well as on lift, with the new force formula providing accurate results almost on the entire investigated range.

Analysing plunging oscillations (Figures 4.2.36 and 4.2.37), it is clear how higher accuracy on drag can be obtained using the thermodynamic-based method ($\underline{F}_b^{thd}$), at least in the regions where the mesh is fine enough to sufficiently capture the unsteady wake (i.e. $X_W < 2$), or where the external CV surface (Σ) does not intercept any vortical structure (i.e. $X_W > 50$). In the same regions, lift is also accurately computed, with improvements with regards to the standard vorticity-based method ($\underline{F}_b^{vrt}$) for $X_W > 50$ (the vortex force volume integral is now accurately computed, Fig. 4.2.36). Using the new exact force formula ($\underline{F}_b^{THD}$) further improves drag accuracy and completely eliminates errors on far-field lift calculation even in the wake region with coarse computational grid.

A very similar behaviour is observed in compressible subsonic conditions (Fig. 4.2.38 and 4.2.39) where the control volume was now selected with $Z_W = 100.0$ to avoid vorticity waves intercepting its external surface (Σ).

[†]To maintain coherence with the notation used for the compressible case illustrated in the following figures, here the symbol $\underline{F}_b$ was used also for incompressible analyses although, of course, the specific expression of the compressibility correction (i.e. $\underline{F}_a$ of $\underline{F}_b$) is not relevant for incompressible cases, as they both vanish. Transonic variants of the force formula ($\underline{F}_c$, $\underline{F}_{bc}$ and $\underline{F}_d$) are also not meaningful for incompressible flows, then the notation could be restored to the general symbol $\underline{F}_{ff}$.

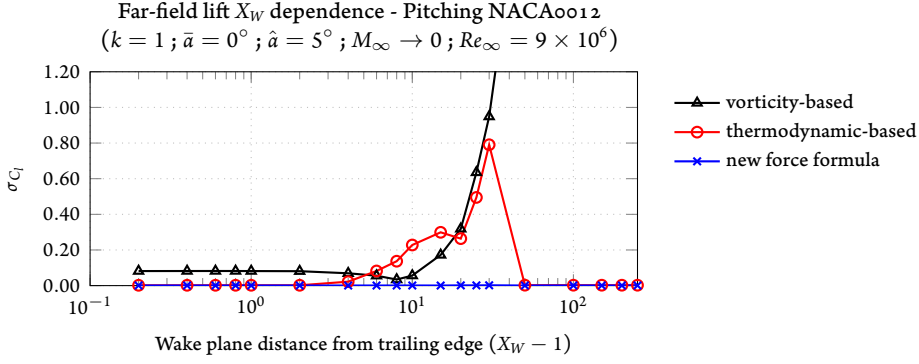


Figure 4.2.32: Comparison of vorticity-based and thermodynamic-based Lamb vector approaches on the total far-field lift variation with wake plane position ($\underline{F}_b^*$ non-inertial, $Z_W = 10.0$, pitching motion).

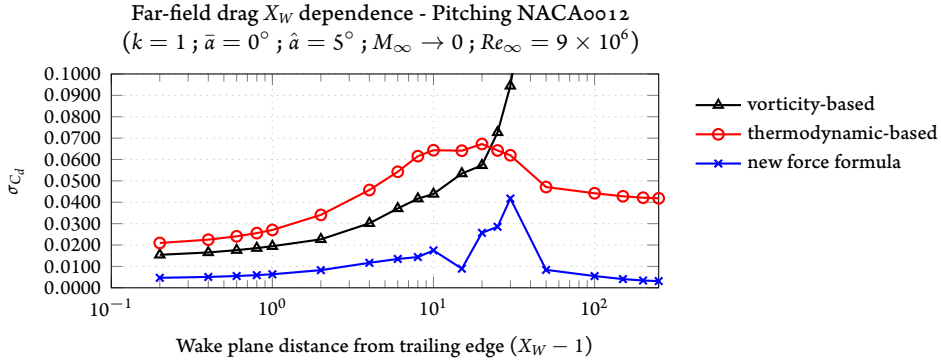


Figure 4.2.33: Comparison of vorticity-based and thermodynamic-based Lamb vector approaches on the total far-field drag variation with wake plane position ($\underline{F}_b^*$ non-inertial, $Z_W = 10.0$, pitching motion).

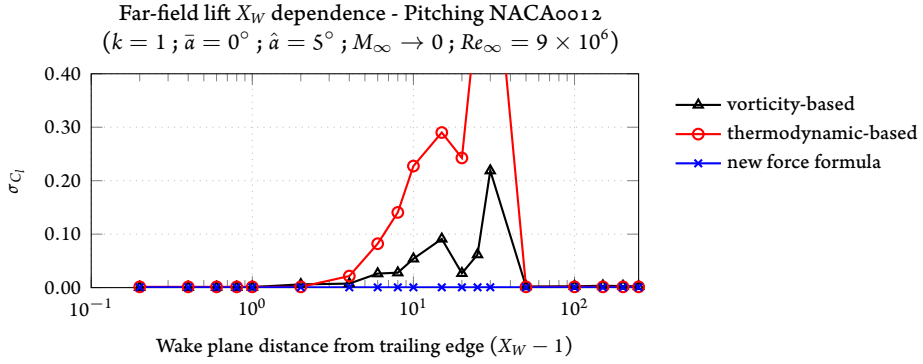


Figure 4.2.34: Comparison of vorticity-based and thermodynamic-based Lamb vector approaches on the total far-field lift variation with wake plane position (E_b^* inertial, $Z_W = 10.0$, pitching motion).

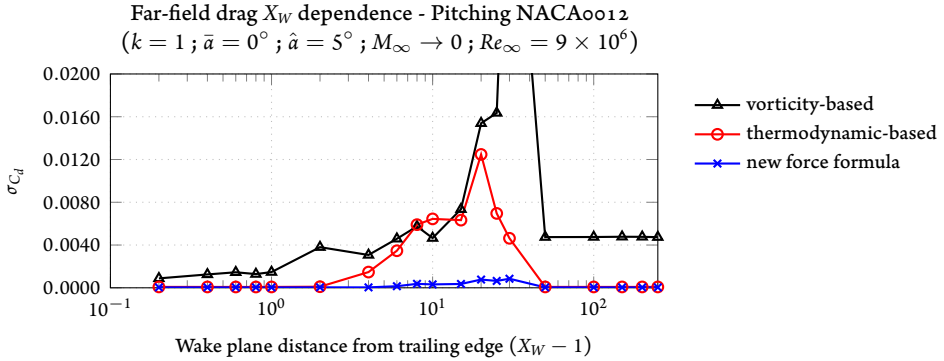


Figure 4.2.35: Comparison of vorticity-based and thermodynamic-based Lamb vector approaches on the total far-field drag variation with wake plane position (E_b^* inertial, $Z_W = 10.0$, pitching motion).

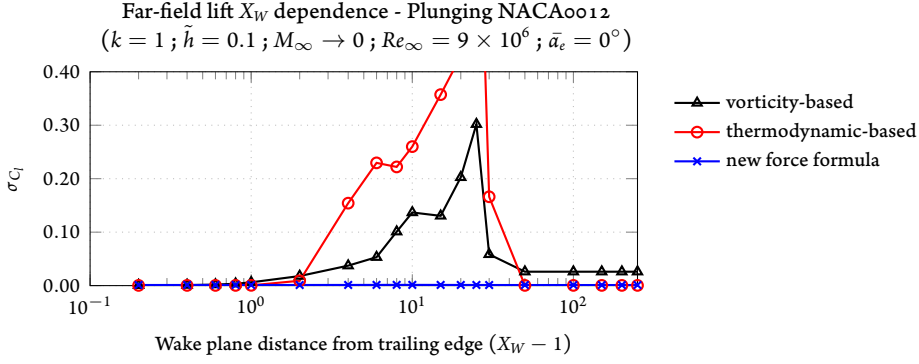


Figure 4.2.36: Comparison of vorticity-based and thermodynamic-based Lamb vector approaches on the total far-field lift variation with wake plane position ($\underline{F}_b^*$ non-inertial, $Z_W = 10.0$, plunging motion).

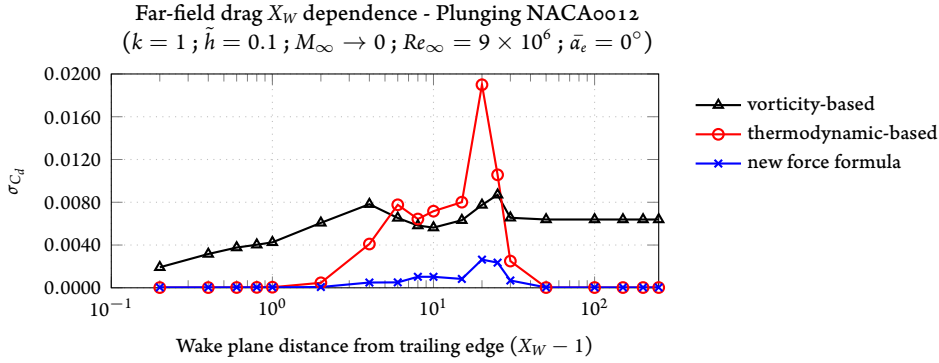


Figure 4.2.37: Comparison of vorticity-based and thermodynamic-based Lamb vector approaches on the total far-field drag variation with wake plane position ($\underline{F}_b^*$ non-inertial, $Z_W = 10.0$, plunging motion).

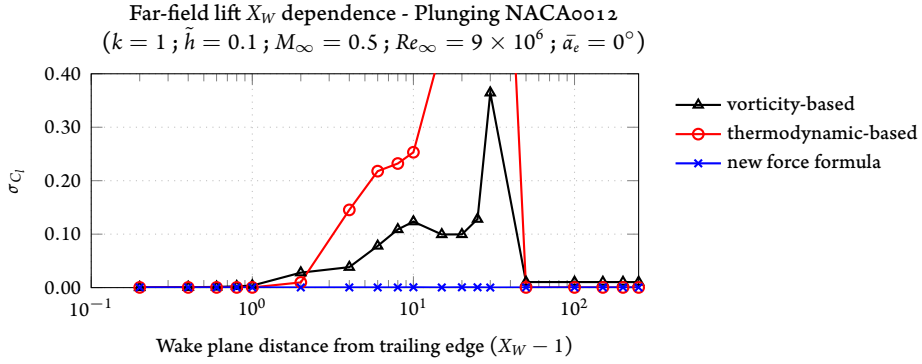


Figure 4.2.38: Comparison of vorticity-based and thermodynamic-based Lamb vector approaches on the total far-field lift variation with wake plane position ($\underline{F}_b^*$ non-inertial, $Z_W = 100.0$, plunging motion).

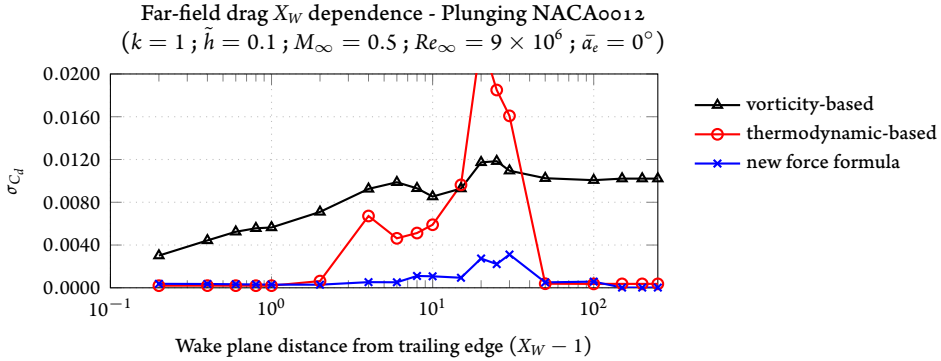


Figure 4.2.39: Comparison of vorticity-based and thermodynamic-based Lamb vector approaches on the total far-field drag variation with wake plane position ($\underline{F}_b^*$ non-inertial, $Z_W = 100.0$, plunging motion).

4.3 SUMMARY

- A new unified thermodynamic/vortical approach was conceived, developed and tested [122]. The author proposed a mixed approach to far-field force/moment prediction, where exact Lamb-vector-based formulas are used in combination with a thermodynamic-based calculation of the Lamb vector, through Crocco's equation.
- The resulting hybrid method (which does not require an explicit vorticity calculation) provides results in far better agreement with regards to the reference near-field force data when compared to standard vorticity-based approaches, especially in presence of shock waves, where inaccuracies of domain integrals involving the Lamb vector were systematically reported by different authors.
- The proposed unified approach also led to the formulation of new *exact* force/moment formulas, representing an innovative thermodynamic method, able to compute the total force in generally unsteady viscous and compressible flows (not just the irreversible drag alone, as typically obtained by standard thermodynamic methods). The new force formula is based on a far-field expression of pressure forces on the body (the only ones deserving a phenomenological decomposition) while friction stresses are retained in their standard *near-field* expression. It does not contain an explicit dependence on time derivatives and on apparent forces due to relative motion in non-inertial reference frames. This can also be obtained in a far-field expression for the *whole* aerodynamic force: it is achieved simply avoiding simplification of viscous terms (which are therefore still computed by far-field integrals). The independence on time derivatives and apparent forces is a peculiarity of the near-field formula, while the possibility to get rid of such terms in far-field expressions was not a trivial result, as the momentum balance naturally brings them into play. This is obtained thanks to the alternative expression of unsteady terms, derived in Ch. 3, which allowed simplifications otherwise not straightforward using the standard expression.
- A similar structure was observed between the new exact formula and the steady inertial version of the baseline Lamb-vector-based formula. The latter represents a far-field expression of the whole aerodynamic force. By comparison, if only the pressure force is expressed by far-field integrals, the exact formula is obtained simply replacing $\underline{E}_t$ with $\underline{F}_t$ and $-\underline{\ell}$ with $\underline{G}$. Above all, the validity of the hybrid far-field/near-field formula obtained in this way is not limited to steady flows in inertial frames of references, but it holds exactly for unsteady flows in generally non-inertial frames.
- Compared to classical thermodynamic methods, the new formulas introduced in this chapter do not suffer of any ill-definition in their integral expressions. This is an issue of all drag formulas based on velocity decompositions (Ch. 1, Sec. 1.4.1.2), which involve square roots of quantities that can drop below zero. No satisfactory solution to this problem has been achieved yet, especially for unsteady flows around rotating bodies [135], while the

new thermodynamic methods derived from the proposed unified approach overcome this limitation.

- For steady flows, the relation with Oswatitsch's entropy drag formula was also highlighted.
- The new far-field aerodynamic force formulas, derived from the Navier-Stokes equations, were then extended to turbulent flows governed by the RANS equations, and applied to post-process two-dimensional numerical flow-fields from both steady and unsteady CFD simulations for a NACA0012 aerofoil (either fixed or oscillating with a pure-plunging motion), in a subsonic or transonic, viscous or inviscid, flow. As a further validation case for the newly derived exact formulas, a three dimensional application to an elliptic wing in transonic and viscous flow at high Reynolds number was also considered.
- The accuracy and domain sensitivity of total force prediction obtained through the baseline far-field formula considered in Ch. 4 were significantly improved by the new thermodynamic-based calculation of the Lamb vector, for all analysed cases. In particular, analysing the unsteady far-field force on variable integration domains (where poor results were obtained with the vorticity-based method except on very small CVs), accuracy on drag was recovered using the thermodynamic-based method in the regions where the adopted mesh is fine enough to sufficiently capture the unsteady wake, or where the external CV surface does not intercept any vortical structure. Using the new exact force formula further improves drag accuracy and completely eliminates errors on far-field lift calculation, even in the wake region solved with a coarse computational grid.
- Moreover, using the new Lamb vector calculation, the local aerodynamic force production is well confined to viscous and shock regions, which was not observed for standard approaches. This is a relevant aspect for Lamb-vector-based drag decomposition methods aiming to separate viscous, wave and spurious drag contributions.
- Finally it was shown that, in the case the Lamb vector is computed from Crocco's equation, an indirect calculation of the flow vorticity can also be obtained, as an alternative to the direct differentiation of the velocity field. This resulted, for example, in computed vorticity waves around plunging aerofoils with pattern similar to flow dilatation waves (as required by the vorticity balance equation, far from the body), which was not obtained using the direct approach. However, this indirect vorticity calculation only allows for computing the transverse vorticity field. In three-dimensional flows, where a physical streamwise vorticity might be present, the Lamb vector knowledge would therefore not suffice to indirectly compute the total vorticity. In this case, besides the Lamb vector, the additional knowledge of flow helicity density would be necessary to indirectly calculate the whole vorticity.

Conclusions

FINDINGS

A comprehensive illustration of the main aerodynamic force/moment theories developed so far for both steady and unsteady flows was first provided (Ch. 1), with emphasis on identifying links and interconnections between the different approaches, homogeneously included in the same theoretical picture. The physical decompositions offered by each method were highlighted, after having accurately analysed the definitions involved in the corresponding breakdowns. In particular, the employed definitions for the different forms of aerodynamic drag, identified on a physical and phenomenological basis, were first discussed, highlighting their objective or non-objective characteristics in either real or numerically computed flow-fields.

This dissertation focused on *Lamb-vector-based* methods for the analysis and decomposition of the aerodynamic force in the steady and unsteady regimes. The need for sharpening and improving such methods moved from previous studies, highlighting that *Lamb-vector-based* force predictions did not yet reach a mature stage (for steady [56, 81] and especially for unsteady flows [140, 142]) compared to the superior accuracy offered, for example, by thermodynamic methods. Moreover, the research steps leading to their development were convoluted, with different formulations and decompositions proposed in the past, involving several degrees of freedom in the application of the methods and some aspects (as, for example, wave drag identification) which were still questioned.

The research activity, subject of this dissertation, started therefore with a preliminary effort focused on the identification of all the critical aspects and spurious effects influencing the robustness, accuracy and reliability of the force computation and breakdown in *Lamb-vector-based* methods [124], considering both the steady (Ch. 2) and the unsteady regime (Ch. 3). Some of the highlighted problems affected the majority of numerical applications published in the past, although very rarely discussed.

As a pre-requisite, the many different *Lamb-vector-based* force formulas available from literature were homogeneously organised in a general form, which is valid for either laminar or turbulent mean flows, viscous or inviscid regime, fixed or moving bodies, inertial or non-inertial reference frames. This also included the derivation of extended versions of some of the formulas, initially restricted to viscous and steady regimes, to deal with inviscid and/or unsteady flows [127]. Similarly, generalised far-field formulations were also derived for the aerodynamic moment.

Conclusions

Solutions to some of the open issues identified in the preliminary phase (such as the separation of viscous and wave drag from the total parasite contribution, or the accuracy of unsteady inertial formulas) were proposed and verified through numerical applications in Ch. 2 and Ch. 3 [124, 127].

It was evidenced that, to decompose parasite drag in viscous and wave contributions, Lamb-vector integrals cannot be limited to the wake plane only, but flow entrainment in the viscous wake needs to be taken into account, and its effect was quantified by integrals extended at the interface between the viscous and the outer flow regions.

A new methodology to perform parasite drag decomposition in viscous and wave contributions, accounting for the flow entrainment in the viscous region, was proposed and tested on transonic flow-fields around the RAE2822 and NACA0012 aerofoils. Results were compared to viscous and wave drag estimates obtained using consolidated thermodynamic-based methods, and a benchmarking was provided including recent results also obtained using different decomposition approaches [69, 117, 172].

Inconsistencies experienced in previous works and related to the drag decomposition in lift-induced and parasite contributions (with ambiguities on the physical or non-physical role of a non-zero induced drag found in two-dimensional flows) have been definitively clarified. For the first time, the relationship between reversible/irreversible drag definitions and lift-induced/parasite drag contributions has been clearly addressed, removing the ambiguity that tended to identify the two in previous works. Using Lamb-vector-based lift-induced drag definitions, a non-zero spurious induced drag was clearly identified post-processing two-dimensional numerical solutions, increasing in magnitude as the Reynolds number is reduced of the Mach number increased, and vanishing as the distance of the outer control volume boundary from the aerofoil surface is significantly increased. This represents a bias which limits the accuracy of the drag breakdown achievable through classical or Lamb-vector-based methods, and highlights that the proposed direct definitions of lift-induced drag in viscous and compressible flows still need to be sharpened.

A new expression was derived for the integrals containing time derivative of the velocity field. This does not involve the computation of spatial gradients and its use eliminates the inaccuracy observed in the simulation of plunging aerofoils for Lamb-vector-based force formulas in inertial reference frames.

Additionally, a method was proposed for oscillating aerofoils identifying, in the far-field Lamb-vector-based axial force expression, the individual force terms providing thrust and those providing drag. Indeed, the proposed breakdown in unsteady induced force and parasite drag allows for TDB in two-dimensional flows, and a pole-independent axial force decomposition was also derived. It is interestingly highlighted that the decomposition is achieved using a control volume that tends to separate the bound vorticity on the body surface from the free vorticity in the unsteady wake. The proposed vortical approach to the TDB problem also gives a new insight on the physical phenomenon of thrust generation on flapping wings, complementary to that provided by Knoller and Betz (related to non-zero effective angles of attack): thrust is obtained as the axial

force induced by the unsteady lift, i.e. by the time-variation of the aerofoil circulation and associated unsteady trailing vorticity. In subsonic flow, the decomposition provides satisfactory results when thrust and drag contributions are compared respectively to the inviscid force (thrust only) and to the steady force (drag only). Nonetheless, it is also highlighted that the computed unsteady parasite drag exhibits very slight variations with the effective angle of attack experienced by the aerofoil during its plunging oscillation, while strong incidence effects are instead observed on the steady drag. Decomposition results in transonic flow were also analysed, although in this case the inviscid force does not purely represent a thrust contribution, but also includes wave drag.

Then, a new unified thermodynamic/vortical method was conceived, developed and tested [122]. The author proposed a mixed approach to far-field force/moment prediction, where exact Lamb-vector-based formulas are used in combination with a thermodynamic-based calculation of the Lamb vector, through Crocco's equation. This is fundamental in post-processing CFD data from flow solvers working on primitive variables. In this case, in fact, the standard approach, which consists in computing the Lamb vector from its definition, involves the numerical differentiation of the discrete velocity field to compute the flow vorticity. This gives no guarantee that it satisfies the vorticity balance equation, usually not part of the Partial Differential Equations (PDEs) system which is numerically solved. Momentum equation is instead satisfied, and Crocco's theorem is an expression of the momentum balance.

From the operative point of view, the resulting hybrid method (which does not require an explicit vorticity calculation) allowed for solving most of the problems frequently encountered in the numerical application of Lamb-vector-based methods on CFD solutions obtained from momentum-based flow solvers (and highlighted in the early steps of this doctoral research), namely: the accuracy of force formulas in presence of shock waves, the non-physical dependence of total force coefficients on the control volume selection, a spurious local force production in non-rotational flow-field regions, etc..

From the theoretical point of view, the proposed approach represents a step forward in the direction of unifying far-field theories and also led to the formulation of a new *exact* force formula, providing an innovative thermodynamic method, able to compute the total force (not just drag alone) in generally unsteady viscous and compressible flows. The new formula is based on a far-field expression of pressure forces on the body (the only ones deserving a phenomenological decomposition) while friction stresses are retained in their standard *near-field* expression. It involves the integration of the vector field $\underline{G} = \underline{\nabla}H - T\underline{\nabla}s$ which represents, for isothermal flows, the variation of total Gibbs free energy. It is related to the gradient of flow exergy ε (referred to the free-stream state) by: $\underline{G} = \underline{\nabla}\varepsilon - \Delta T \underline{\nabla}s$.

This hybrid near-field/far-field formula does not contain an explicit dependence on time derivatives and on apparent forces due to relative motion in non-inertial reference frames, which impact is twofold: operationally, it represents an advantage for any post-processing routine; physically, it tells us that a far-field decomposition of the pressure force is possible without involving unsteady or apparent terms at all, even for unsteady flows or in non-inertial frames.

Conclusions

The unified approach also allows deriving a far-field expression independent of time derivatives and apparent forces for the *whole* aerodynamic force: it is obtained simply avoiding simplifying viscous terms, which are therefore still computed by far-field integrals. The independence on time derivatives and apparent forces is a peculiarity of the near-field formula, while the possibility to get rid of such terms in far-field expressions is not a trivial result, as the momentum balance naturally brings them into play. This is obtained thanks to the alternative expression of unsteady terms, derived in Ch. 3, which allowed simplifications otherwise not straightforward using the standard expression.

Compared to classical thermodynamic methods, the new formulas derived from the proposed unified approach do not suffer of any ill-definition in their integral expressions (an issue of all drag formulas based on velocity decompositions: Ch. 1, Sec. 1.4.1.2). No satisfactory solution to this problem has been achieved yet [135], while the new thermodynamic methods introduced in this doctoral research overcome this limitation.

Moreover, the proposed unified method eliminates the need to use some of the Lamb-vector-based formulations analysed in this dissertation (namely, “*bc*” and “*d*”). In fact, they were based on artefacts to recover accuracy in presence of shock waves (at the cost of introducing other issues) which are not necessary anymore as the unified formulas are accurate in subsonic as well as in transonic flows. A further reduction to two (i.e. formulation “*a*” and “*b*”) is possible recalling that formulation “*c*” is indeed not truly Lamb-vector-based but hybrid classical, the vortex force having disappeared in this far-field expression (Ch. 2). Then, for steady flows, considering that the mid-field property is only guaranteed by formulation “*b*” (Ch. 2), this finally reduces to one the set of relevant Lamb-vector-based force formulas.

The unified approach was also extended to the computation of the aerodynamic moment and a new exact moment formula based on the $\underline{G}$ field integration was similarly derived.

Finally it was shown that, in the case the Lamb vector is computed from Crocco’s equation, an indirect calculation of the flow vorticity can also be obtained, as an alternative to the direct differentiation of the velocity field. This resulted, for example, in computed vorticity waves around plunging aerofoils with pattern similar to flow dilatation waves (as required by the vorticity balance equation, far from the body), which was not obtained using the direct approach. However, this indirect vorticity calculation only allows for computing the transverse vorticity field. In three-dimensional flows, where a physical streamwise vorticity might be present, the Lamb vector knowledge would therefore not suffice to indirectly compute the total vorticity. In this case, besides the Lamb vector, the additional knowledge of flow helicity density would be necessary to indirectly calculate the whole vorticity.

FUTURE PERSPECTIVES

Further developments of the method are envisaged in identifying a physical and phenomenological correlation for the different terms contributing to the new exact force formula. Indeed it has

already been highlighted that a new breakdown of the pressure force is allowed by this formula, suggesting that interesting links could be identified with recent studies being conducted on Partial Pressure Fields [177, 178], which as well provide a decomposition of pressure forces.

It was also mentioned that the newly identified vector $\underline{G} = \underline{\nabla}H - T\underline{\nabla}s$ represents, for isothermal flows, the variation of total Gibbs free energy. It is related to the gradient of flow exergy ε (referred to the free-stream state) by: $\underline{G} = \underline{\nabla}\varepsilon - \Delta T \underline{\nabla}s$. This suggests another direction for further investigations, as different far-field methods based on the energy/exergy balance [8, 9, 43] were recently developed and studied. These however are scalar equations and those methods therefore cannot provide information on the total force vector. The new exact force formula of the proposed unified method, instead, is a vector relation, linking the total force to flow exergy and providing information on both lift and drag.

Since not only the resultant force but also the aerodynamic moment was addressed, deriving exact far-field formulations for both quantities, this opens the door to the analysis and decomposition of the moment as well.

Additionally, the axial force decomposition illustrated in Ch. 3, allowing to define a TDB for oscillating aerofoils, could also be investigated to potentially provide TDB on isolated propellers in axial and subsonic flow, comparing the isolated thrust with the total axial force obtained in inviscid conditions.

Finally, as the new formulas were derived in an exact form from the Navier-Stokes equations, future applications of the proposed method may also address, for example, supersonic flows or turbulent flows resolved by Direct Numerical Simulations.

Appendices



Notable Transformations and Derivations

A.1 DERIVATIVE MOMENT TRANSFORMATIONS (DMTs)

Let $\mathcal{V}$ be a subset of $\mathbb{R}^{\mathcal{N}}$ ($\mathcal{N} = 2, 3$ is the space dimension) where a differentiable vector field $\underline{f}$ is defined and a regular boundary $\partial\mathcal{V}$ constitutes its frontier. The following integral identities are applicable:

$$\int_{\mathcal{V}} \underline{f} d\mathcal{V} = \int_{\mathcal{V}} \underline{r} \times (\underline{\nabla} \times \underline{f}) d\mathcal{V} - \int_{\partial\mathcal{V}} \underline{r} \times (\underline{n} \times \underline{f}) dS \quad (\text{A1})$$

with $\underline{r}$ defined by Eq. (2.2). If $\underline{f}$ is the gradient of a scalar function, $\underline{f} = \underline{\nabla}\varphi$, Eq. (A1) reduces to:

$$\int_{\partial\mathcal{V}} \varphi \underline{n} dS = - \int_{\partial\mathcal{V}} \underline{r} \times (\underline{n} \times \underline{\nabla}\varphi) dS \quad (\text{A2})$$

Additional DMT identities are included in [222, pp. 698-705] (this section only reports those effectively used in this dissertation) and a rigorous derivation of Eq. (A1) and Eq. (A2) can be found in [138], including the general proof of Eq. (A2) which does not lie on Eq. (A1). It is also worth mentioning that Eq. (A1) can be derived as a particular case of the more general identity published by the Italian mathematician Pietro Burgatti [24] (the proof is provided in [136]).

All these identities are invariant with respect to a change in the origin (pole) of the scaled position vector $\underline{r}$. Therefore, in the application of Eq. (A1) and Eq. (A2), the pole in $\underline{r}$ can be arbitrarily chosen, usually depending on convenience.

Appendix A. Notable Transformations and Derivations

Eq. (A1) can be regarded as a multi-dimensional version of the integration by parts formula [222, pp. 593, 700], as it relates the integral of $\underline{f}$ to the same integral of the first moment of its derivatives (through the curl of $\underline{f}$, in this case) plus a boundary term*.

Similar formulas can also be obtained for the integral of the first moment of $\underline{f}$, to be related to the same integral of the second moment of $\underline{f}$ derivatives, plus boundary terms. Denoting with $\hat{\underline{r}}$ the (non-scaled) position vector referred to the DMT pole ($\hat{\underline{r}} = (\mathcal{N} - 1) \underline{r} = \underline{x} - \underline{x}_p$), the following integral identities can be proved to hold for both 2D and 3D fields:

$$\int_{\mathcal{V}} \hat{\underline{r}} \times \underline{f} d\mathcal{V} = -\frac{1}{2} \int_{\mathcal{V}} \hat{r}^2 (\underline{\nabla} \times \underline{f}) d\mathcal{V} + \frac{1}{2} \int_{\partial\mathcal{V}} \hat{r}^2 (\underline{n} \times \underline{f}) dS \quad (\text{A3})$$

$$\int_{\partial\mathcal{V}} \hat{\underline{r}} \times \underline{n} \varphi dS = \frac{1}{2} \int_{\partial\mathcal{V}} \hat{r}^2 (\underline{n} \times \underline{\nabla} \varphi) dS \quad (\text{A4})$$

In these identities† it is implicitly assumed that the pole is the same for all moments involved on both sides of the identity. Therefore, when Eq. (A1) and (A2) are used to compute the aerodynamic force (Sec. A.4), the pole can be arbitrarily chosen as mentioned above, while when the aerodynamic moment is computed using Eq. (A3) and (A4), the DMT pole is inherently coincident with the reference pole where the moment calculation is required (see Sec. A.4).

It is explicitly observed that identities (A3) and (A4) are similar to Eq. (A1) and (A2), in the sense that moments can be substituted on the LHS and identities are recovered by replacing the operator $\underline{r} \times$, on the RHS, with $-\hat{r}^2/2$.

A.2 REFERENCE FRAME TRANSFORMATIONS

Let us consider a non-inertial reference frame $O'(x'y'z')$, with the same clock of an inertial frame $O(xyz)$: $t' = t$. The relative motion of $O'(x'y'z')$ with regards to the inertial frame is described by the translational velocity of its origin $\underline{U}(t)$ and by its angular velocity $\underline{\Omega}(t)$ around the origin O' . Specifying with a prime superscript (') the variables in the $O'(x'y'z')$ reference, the relationship between derivatives computed in the two frames are given by the following relations (detailed

*a similar formula, which instead involved the divergence of $\underline{f}$, can also be derived [222, p. 700].

†derived and discussed into details in [222, pp. 700-705].

derivation in [199, pp. 3-6, in Italian]):

$$\underline{\nabla} f = \underline{\nabla}' f \quad (\text{superscript ' will be therefore omitted for spatial gradients}) \quad (\text{A5})$$

$$\frac{Df}{Dt} = \frac{Df}{Dt'} \quad (\text{A6})$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t'} - \underline{V}_T \cdot \underline{\nabla} f \quad (\text{A7})$$

$$\frac{D\underline{b}}{Dt} = \frac{D\underline{b}}{Dt'} + \underline{\Omega} \times \underline{b} \quad (\text{A8})$$

$$\frac{\partial \underline{b}}{\partial t} = \frac{\partial \underline{b}}{\partial t'} - \underline{V}_T \cdot \underline{\nabla} \underline{b} + \underline{\Omega} \times \underline{b} \quad (\text{A9})$$

$$\frac{D\underline{V}}{Dt} = \frac{D\underline{V}'}{Dt'} + \frac{d\underline{U}}{dt} + \frac{d\underline{\Omega}}{dt} \times \underline{x}' + \underline{\Omega} \times (\underline{\Omega} \times \underline{x}') + 2\underline{\Omega} \times \underline{V}' \quad (\text{A10})$$

where f and $\underline{b}$ denote a scalar and a vector field respectively, while $\underline{V}$, $\underline{V}'$ and

$$\underline{V}_T = \underline{V} - \underline{V}' = \underline{U} + \underline{\Omega} \times \underline{x}' \quad (\text{A11})$$

represent the absolute fluid velocity ($\underline{V}$), the relative fluid velocity ($\underline{V}'$) and the transport speed ($\underline{V}_T$) at relative frame position ($\underline{x}'$) with regards to the inertial reference.

These statements allow us to rewrite the continuity and momentum balance equations in non-inertial reference frames as:

$$\frac{1}{\rho} \frac{D\rho}{Dt'} = -\underline{\nabla} \cdot \underline{V}' \quad (\text{A12})$$

$$\rho \frac{D\underline{V}'}{Dt'} = \rho \underline{a}' = \underline{\nabla} \cdot \underline{\tau} + \rho \left[\underline{g} - \frac{d\underline{U}}{dt} - \frac{d\underline{\Omega}}{dt} \times \underline{x}' - \underline{\Omega} \times (\underline{\Omega} \times \underline{x}') - 2\underline{\Omega} \times \underline{V}' \right] \quad (\text{A13})$$

In [199, p. 6, in Italian] it is also shown how the stress tensor, for Newtonian fluids, is also invariant for inertial or non-inertial reference frames, therefore prime superscript is omitted on $\underline{\tau}$.

It is explicitly observed that the notation with prime superscripts used above is strictly necessary in Eqs. from (A5) to (A10), where the need to distinguish the two reference frames in the same equation is required. When there is no need to distinguish quantities, as they are all evaluated in the same frame of reference (as for Eq. (A12) or (A13)), prime superscripts can be omitted, and this will be done in other document sections. In such cases, it is intended that, for example, $\underline{V}$ is considered as, respectively, the absolute or the relative fluid velocity and partial time derivatives should be interpreted as appropriate for the selected reference frame (i.e. respectively at constant absolute or relative position). Apparent body forces due to the relative frame motion will of course be zero in the case of inertial reference frames.

Appendix A. Notable Transformations and Derivations

A.3 SHIFTING TIME DERIVATIVE AND INTEGRAL SIGNS

A.3.1 REYNOLDS TRANSPORT THEOREM

Let $\mathcal{F}$ be a tensor of any rank and $\mathcal{V}(t)$ a time-dependent control volume. Then the theorem states that:

$$\frac{d}{dt} \int_{\mathcal{V}(t)} \mathcal{F} d\mathcal{V} = \int_{\mathcal{V}(t)} \frac{\partial \mathcal{F}}{\partial t} d\mathcal{V} + \int_{\partial \mathcal{V}(t)} \mathcal{F} (\underline{V}_{\partial \mathcal{V}} \cdot \underline{n}) d\mathcal{V} \quad (\text{A14})$$

with $\underline{V}_{\partial \mathcal{V}}$ being the velocity of the control volume boundary [222, p. 24].

A.3.2 ARIS IDENTITY

Let $\underline{\Phi}$ be a tensor of rank higher than zero and $S(t)$ a time-dependent closed surface. Then Aris proved the following identity [7, p. 63]:

$$\frac{d}{dt} \int_{S(t)} \underline{n} \cdot \underline{\Phi} dS = \int_{S(t)} \underline{n} \cdot \left[\frac{\partial \underline{\Phi}}{\partial t} + \underline{V}_S (\underline{\nabla} \cdot \underline{\Phi}) \right] dS \quad (\text{A15})$$

with $\underline{V}_S$ being the velocity of the moving surface ([138]).

A.4 DERIVATION OF THE ORIGINAL LAMB-VECTOR-BASED FORMULA

A.4.1 AERODYNAMIC FORCE

Here below a derivation of the original far-field force formula based on the Lamb-vector is included, alternative to that provided by the pioneering authors [222, pp. 617-621]. A general non-inertial frame of reference will be considered and the formula will be derived for unsteady compressible flows around bodies with non-zero wall velocities (e.g. moving objects, slip flows or non-impermeable walls). The flow will be assumed laminar, although generalisations of the final force formulas to the turbulent regime are also included in Appendix E.

The starting point is always represented by the integral balance equation for linear momentum, although now considered in the form of Newton's second law (combination of continuity and momentum balance), which allows translating Eq. (1.2) into:

$$\underline{F}_{ff} = - \int_{\mathcal{V}} \rho \underline{a} d\mathcal{V} - \int_{\Sigma} \left(p \underline{n} - \underline{\tau}_v \cdot \underline{n} \right) dS + \int_{\mathcal{V}} \rho \underline{f}_m d\mathcal{V} \quad (\text{A16})$$

where $\underline{a}$ is the fluid particle acceleration and $\underline{f}_m$ is defined by Eq. (3.5). Considering the differential vector identity $\underline{V} \cdot \underline{\nabla} \underline{V} = \underline{\nabla} (V^2/2) + (\underline{\nabla} \times \underline{V}) \times \underline{V}$, the (opposite) inertial force of the fluid

particle ($\rho \underline{a}$) can be written as:

$$\begin{aligned}\rho \underline{a} &= \rho \frac{D\underline{V}}{Dt} = \rho \frac{\partial \underline{V}}{\partial t} + \rho \underline{V} \cdot \underline{\nabla} \underline{V} = \rho \frac{\partial \underline{V}}{\partial t} + \rho \underline{\nabla} (V^2/2) + \rho (\underline{\nabla} \times \underline{V}) \times \underline{V} \\ &= \rho \frac{\partial \underline{V}}{\partial t} + \underline{\nabla} \left(\rho \frac{V^2}{2} \right) - \frac{V^2}{2} \underline{\nabla} \rho + \rho \underline{\ell}\end{aligned}$$

Using this expression in the far-field force formula, and getting rid of the gradient term through Eq. (A1), leads to:

$$\begin{aligned}\underline{F}_{ff} &= - \int_{\mathcal{V}} \underline{r} \times \left[\underline{\nabla} \times \left(\rho \underline{\ell} - \frac{V^2}{2} \underline{\nabla} \rho \right) \right] d\mathcal{V} + \int_{\partial \mathcal{V}} \underline{r} \times (\underline{n} \times \rho \underline{a}) dS - \int_{\Sigma} p \underline{n} dS + \\ &+ \int_{\Sigma} \underline{\tau}_v \cdot \underline{n} dS + \int_{\mathcal{V}} \rho \underline{f}_m d\mathcal{V} - \int_{\mathcal{V}} \underline{r} \times \left[\underline{\nabla} \times \left(\rho \frac{\partial \underline{V}}{\partial t} \right) \right] d\mathcal{V}\end{aligned}$$

Recalling that $\partial \mathcal{V} = \Sigma \cup \partial B$ and applying again Eq. (A1) to the vector field $\underline{f} = \rho \underline{\ell} - \frac{V^2}{2} \underline{\nabla} \rho$, it becomes:

$$\begin{aligned}\underline{F}_{ff} &= - \int_{\mathcal{V}} \left(\rho \underline{\ell} - \frac{V^2}{2} \underline{\nabla} \rho \right) d\mathcal{V} - \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\rho \underline{\ell} - \frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS + \\ &- \int_{\partial B} \underline{r} \times \left[\underline{n} \times \left(\rho \underline{\ell} - \frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS + \int_{\Sigma} \underline{r} \times (\underline{n} \times \rho \underline{a}) dS + \int_{\partial B} \underline{r} \times (\underline{n} \times \rho \underline{a}) dS + \\ &- \int_{\Sigma} p \underline{n} dS + \int_{\Sigma} \underline{\tau}_v \cdot \underline{n} dS + \int_{\mathcal{V}} \rho \underline{f}_m d\mathcal{V} - \int_{\mathcal{V}} \underline{r} \times \left[\underline{\nabla} \times \left(\rho \frac{\partial \underline{V}}{\partial t} \right) \right] d\mathcal{V}\end{aligned}$$

Then, introducing momentum equation (Eq. (A13)) in the fourth integral of the above equation, we obtain:

$$\begin{aligned}\underline{F}_{ff} &= - \int_{\mathcal{V}} \left(\rho \underline{\ell} - \frac{V^2}{2} \underline{\nabla} \rho \right) d\mathcal{V} - \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\rho \underline{\ell} - \frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS + \\ &- \int_{\partial B} \underline{r} \times \left[\underline{n} \times \left(\rho \underline{\ell} - \frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS - \int_{\Sigma} \underline{r} \times (\underline{n} \times \underline{\nabla} p) dS + \\ &+ \int_{\Sigma} \underline{r} \times (\underline{n} \times \underline{\nabla} \cdot \underline{\tau}_v) dS + \int_{\Sigma} \underline{r} \times (\underline{n} \times \rho \underline{f}_m) dS + \int_{\partial B} \underline{r} \times (\underline{n} \times \rho \underline{a}) dS + \\ &- \int_{\Sigma} p \underline{n} dS + \int_{\Sigma} \underline{\tau}_v \cdot \underline{n} dS + \int_{\mathcal{V}} \rho \underline{f}_m d\mathcal{V} - \int_{\mathcal{V}} \underline{r} \times \left[\underline{\nabla} \times \left(\rho \frac{\partial \underline{V}}{\partial t} \right) \right] d\mathcal{V}\end{aligned}$$

where the static pressure contribution is eliminated with the help of identity (A2). The two integrals over the body skin can be combined and re-elaborated using the inertial force expression

Appendix A. Notable Transformations and Derivations

and, again, Eq. (A2), obtaining:

$$\begin{aligned} \int_{\partial B} \underline{r} \times \left[\underline{n} \times \left(\rho \underline{a} - \rho \underline{\ell} + \frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS &= \int_{\partial B} \underline{r} \times \left(\underline{n} \times \rho \frac{\partial \underline{V}}{\partial t} \right) dS + \\ &+ \int_{\partial B} \underline{r} \times \left[\underline{n} \times \underline{\nabla} \left(\rho \frac{V^2}{2} \right) \right] dS = \int_{\partial B} \underline{r} \times \left(\underline{n} \times \rho \frac{\partial \underline{V}}{\partial t} \right) dS - \int_{\partial B} \rho \frac{V^2}{2} \underline{n} dS \end{aligned} \quad (A17)$$

The final far-field force expression therefore becomes:

$$\begin{aligned} \underline{F}_{ff} &= - \int_V \rho \underline{\ell} dV + \int_V \frac{V^2}{2} \underline{\nabla} \rho dV - \int_\Sigma \underline{r} \times (\underline{n} \times \rho \underline{\ell}) dS + \int_\Sigma \underline{r} \times \left[\underline{n} \times \frac{V^2}{2} \underline{\nabla} \rho \right] dS + \\ &+ \int_\Sigma \underline{r} \times \left(\underline{n} \times \underline{\nabla} \cdot \underline{\tau}_v \right) dS + \int_\Sigma \underline{r} \times (\underline{n} \times \rho \underline{f}_m) dS + \int_\Sigma \underline{\tau}_v \cdot \underline{n} dS + \int_V \rho \underline{f}_m dV + \\ &- \int_V \underline{r} \times \left[\underline{\nabla} \times \left(\rho \frac{\partial \underline{V}}{\partial t} \right) \right] dV + \int_{\partial B} \underline{r} \times \left(\underline{n} \times \rho \frac{\partial \underline{V}}{\partial t} \right) dS - \int_{\partial B} \rho \frac{V^2}{2} \underline{n} dS \end{aligned}$$

where, on the right-hand-side, the contributions of $\underline{F}_\ell$, $\underline{F}_\rho$, $\underline{F}_S$, $\underline{F}_\mu$, $\underline{F}_t$ and $\underline{F}_m$ as defined in Ch. 2, Sec. 2.1.1 and Ch. 3, Sec. 3.1.1 can be clearly identified.

Each of these individual contributors can vanish in particular cases, e.g.: inertial frames of reference and no gravitational effects ($\underline{F}_m = \underline{0}$); steady flows ($\underline{F}_t = \underline{0}$); incompressible flows ($\underline{F}_\rho = \underline{0}$); inviscid flows ($\underline{F}_\mu = \underline{0}$).

A.4.2 AERODYNAMIC MOMENT

Here below a derivation of the original far-field moment formula based on the Lamb-vector is included, alternative to that provided by the pioneering authors [222, pp. 617-621]. A general non-inertial frame of reference will be considered and the formula will be derived for unsteady compressible flows around bodies with non-zero wall velocities (e.g. moving objects, slip flows or non-impermeable walls). The flow will be assumed laminar, although a generalisation of the final moment formula to the turbulent regime is straightforward, based on what derived for the force formula in Appendix E.

The starting point is represented by the integral balance equation of extrinsic fluid particle moment (which, for non-micropolar fluids, provides the whole angular momentum balance). It is obtained by taking the cross-product of the linear momentum balance (in the form of Newton's second law) by $\hat{\underline{r}}$. The corresponding definition for the far-field aerodynamic moment, with regards to the pole P , is therefore given by (Eq. A16 left-multiplied by $\hat{\underline{r}}$ in the cross product way):

$$\underline{M}_{ff} = - \int_V \hat{\underline{r}} \times \rho \underline{a} dV - \int_\Sigma \hat{\underline{r}} \times (\underline{p} \underline{n} - \underline{\tau}_v \cdot \underline{n}) dS + \int_V \hat{\underline{r}} \times \rho \underline{f}_m dV \quad (A18)$$

where the dependence of $\underline{M}_{ff}$ on the chosen pole is included in $\hat{\underline{r}}$.

From now on, the same derivation used for the force formula can be followed, at condition of using identities (A3) and (A4) instead of (A1) and (A2), respectively. The final far-field moment

A.4. Derivation of the original Lamb-vector-based formula

formula, expressed in terms of the Lamb vector, is given by:

$$\begin{aligned} \underline{M}_{ff} = & - \int_{\mathcal{V}} \hat{\underline{r}} \times \rho \underline{\ell} d\mathcal{V} + \int_{\mathcal{V}} \hat{\underline{r}} \times \frac{V^2}{2} \underline{\nabla} \rho d\mathcal{V} + \frac{1}{2} \int_{\Sigma} \hat{r}^2 (\underline{n} \times \rho \underline{\ell}) dS + \\ & - \frac{1}{2} \int_{\Sigma} \hat{r}^2 \left(\underline{n} \times \frac{V^2}{2} \underline{\nabla} \rho \right) dS - \frac{1}{2} \int_{\Sigma} \hat{r}^2 (\underline{n} \times \underline{\nabla} \cdot \underline{\tau}_v) dS - \frac{1}{2} \int_{\Sigma} \hat{r}^2 (\underline{n} \times \rho \underline{f}_m) dS + \\ & + \int_{\Sigma} \hat{\underline{r}} \times (\underline{\tau}_v \cdot \underline{n}) dS + \int_{\mathcal{V}} \hat{\underline{r}} \times \rho \underline{f}_m d\mathcal{V} + \frac{1}{2} \int_{\mathcal{V}} \hat{r}^2 \left[\underline{\nabla} \times \left(\rho \frac{\partial \underline{V}}{\partial t} \right) \right] d\mathcal{V} + \\ & - \frac{1}{2} \int_{\partial B} \hat{r}^2 \left(\underline{n} \times \rho \frac{\partial \underline{V}}{\partial t} \right) dS - \int_{\partial B} \hat{\underline{r}} \times \rho \frac{V^2}{2} \underline{n} dS \end{aligned}$$

It can be written in compact form as:

$$\underline{M}_{ff} = \underline{M}_{\ell} + \underline{M}_{\rho} + \underline{M}_{\mathcal{S}} + \underline{M}_{\mu} + \underline{M}_t + \underline{M}_m \quad (\text{A19})$$

once the following definitions have been set (according to the colour highlights used above):

$$\underline{M}_{\ell} = - \int_{\mathcal{V}} \hat{\underline{r}} \times \rho \underline{\ell} d\mathcal{V} - \int_{\partial B} \hat{\underline{r}} \times \rho \frac{V^2}{2} \underline{n} dS \quad (\text{A20})$$

$$\underline{M}_{\rho} = \int_{\mathcal{V}} \hat{\underline{r}} \times \frac{V^2}{2} \underline{\nabla} \rho d\mathcal{V} - \frac{1}{2} \int_{\Sigma} \hat{r}^2 \left(\underline{n} \times \frac{V^2}{2} \underline{\nabla} \rho \right) dS \quad (\text{A21})$$

$$\underline{M}_{\mathcal{S}} = \frac{1}{2} \int_{\Sigma} \hat{r}^2 (\underline{n} \times \rho \underline{\ell}) dS \quad (\text{A22})$$

$$\underline{M}_{\mu} = - \frac{1}{2} \int_{\Sigma} \hat{r}^2 (\underline{n} \times \underline{\nabla} \cdot \underline{\tau}_v) dS + \int_{\Sigma} \hat{\underline{r}} \times (\underline{\tau}_v \cdot \underline{n}) dS \quad (\text{A23})$$

$$\underline{M}_t = \frac{1}{2} \int_{\mathcal{V}} \hat{r}^2 \left[\underline{\nabla} \times \left(\rho \frac{\partial \underline{V}}{\partial t} \right) \right] d\mathcal{V} - \frac{1}{2} \int_{\partial B} \hat{r}^2 \left(\underline{n} \times \rho \frac{\partial \underline{V}}{\partial t} \right) dS \quad (\text{A24})$$

$$\underline{M}_m = - \frac{1}{2} \int_{\Sigma} \hat{r}^2 (\underline{n} \times \rho \underline{f}_m) dS + \int_{\mathcal{V}} \hat{\underline{r}} \times \rho \underline{f}_m d\mathcal{V} \quad (\text{A25})$$

It is worth observing that these individual constituents of the moment formula can be obtained from the homologous terms appearing in the force formula ($\underline{F}_{\ell}$, $\underline{F}_{\rho}$, $\underline{F}_{\mathcal{S}}$, $\underline{F}_{\mu}$, $\underline{F}_t$ and $\underline{F}_m$) by the *similarity rule* highlighted in Sec. A.1, i.e. by taking (non-scaled) moments of the vector fields under integral signs and, where integrals already involve moments, by replacing the operator $\underline{r} \times$ with $-\hat{r}^2/2$.

Scaling factors $(1/(\mathcal{N} - 1))$ do not appear in position vectors for the moment formula, which is therefore independent of the space dimension (2D or 3D).

The dependence of $\underline{M}_{ff}$ on the chosen pole is included in $\hat{\underline{r}}$ and the same pole was used to apply DMTs.

According to what done for the force formula by Mele and Tognaccini [119], the compressibility correction ($\underline{M}_{\rho}$) can also be expressed in an equivalent form which guarantees, for steady

flows, the mid-field property of the moment formula:

$$\begin{aligned}\underline{M}_{m_p} &= \frac{1}{2} \int_{\mathcal{V}} \hat{r}^2 \left\{ \underline{\nabla} \times \left[\rho \underline{\nabla} \left(\frac{V^2}{2} \right) \right] \right\} d\mathcal{V} = -\frac{1}{2} \int_{\mathcal{V}} \hat{r}^2 \left[\underline{\nabla} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] d\mathcal{V} = \\ &= \frac{1}{2} \int_{\mathcal{V}} \hat{r}^2 \left[\underline{\nabla} \rho \times \underline{\nabla} \left(\frac{V^2}{2} \right) \right] d\mathcal{V}\end{aligned}\quad (\text{A26})$$

Moreover, similarly to what illustrated in Ch. 3 (Sec. 3.1.6), an alternative and mathematically equivalent form of unsteady moment terms ($\underline{M}_t$) can be obtained by adding and subtracting the integral $\frac{1}{2} \int_{\Sigma} \hat{r}^2 \left(\underline{n} \times \rho \frac{\partial \underline{V}}{\partial t} \right) dS$ and applying identity (A3). This leads to the expression:

$$\underline{M}_t^* = - \int_{\mathcal{V}} \hat{r} \times \left(\rho \frac{\partial \underline{V}}{\partial t} \right) d\mathcal{V} + \frac{1}{2} \int_{\Sigma} \hat{r}^2 \left(\underline{n} \times \rho \frac{\partial \underline{V}}{\partial t} \right) dS \quad (\text{A27})$$

which does not involve spatial gradients of flow variables and allows the aerodynamic moment formula to be written as:

$$\underline{M}_{ff} = \underline{M}_{\ell} + \underline{M}_{m_p} + \underline{M}_S + \underline{M}_{\mu} + \underline{M}_t^* + \underline{M}_m \quad (\text{A28})$$

This expression represents the baseline formula used to derive the new exact moment formula in Ch. 4, Sec. 4.1.4.

A.5 LINKS BETWEEN VORTEX FORCE THEORY AND UNSTEADY LINEAR THEORIES

The aim of this section is to prove that, under the same hypotheses used in Ch. 1, Sec. 1.2.2 (non-dissipative incompressible 2D slightly perturbed flow, initially irrotational), the vortex force theory of Wu et al. [221] encompasses unsteady linear theories as a particular case, therefore establishing links between the two formulations. Of course, due to the impulse nature of linear theories, the connection between 2D Lamb vector methods and impulse formulations (highlighted in Ch 3, Sec. 3.1.7) is a key element to bridge the two. Once this connection is identified, the generality of Wu et al. [221] vortex force theory can be exploited to extend unsteady force analysis to non-linear, viscous and compressible flows.

Linear theories were studied in inertial reference frames, therefore the incompressible Lamb-vector force formula to start with is Eq. 3.16, which is here recalled for the sake of completeness:

$$\underline{F}_{ff} = \underline{F}_{\ell} + \underline{F}_t + \underline{F}_S + \cancel{\underline{F}_{\mu}}^0$$

where it was considered that $\underline{F}_{\mu}$ vanishes for inviscid flows. The control volume is selected with its external surface Σ intercepting the aerofoil trailing edge. This allows excluding the wake discontinuity ∂W from the CV boundary $\partial \mathcal{V}$, which would otherwise be composed by $\partial B \cup \partial W \cup \Sigma$, as

represented in Fig. A.5.1a. With this CV choice, the wake plane can run without being interrupted by the wake discontinuity (Fig. A.5.1b). In fact, Kutta's conditions ensures that the velocity field is continuous at the trailing edge (and equal to $\underline{V}_\infty$, for slightly perturbed flows).

The other terms of the far-field formula are therefore given by[‡]:

$$\underline{F}_\ell = -\rho_\infty \int_V \underline{\ell} dV - \rho_\infty \int_{\partial B} \frac{V^2}{2} \underline{n} dS = -\rho_\infty \int_{-b}^b \frac{(V^\pm)^2}{2} \underline{n}^\pm dx \quad (\text{A29})$$

$$\begin{aligned} \underline{F}_t &= -\rho_\infty \int_V \underline{r} \times \frac{\partial \underline{\omega}}{\partial t} dV + \rho_\infty \int_{\partial B} \underline{r} \times \left(\underline{n} \times \frac{\partial V}{\partial t} \right) dS = \\ &= \rho_\infty \int_{-b}^b \underline{r} \times \left(\underline{n}^\pm \times \frac{\partial V^\pm}{\partial t} \right) dx \end{aligned} \quad (\text{A30})$$

$$\underline{F}_S = -\rho_\infty \int_\Sigma \underline{r} \times (\underline{n} \times \underline{\ell}) dS = -\rho_\infty [\underline{r} \times (\underline{n} \times \underline{\ell})] |_{(b,0,t)} \quad (\text{A31})$$

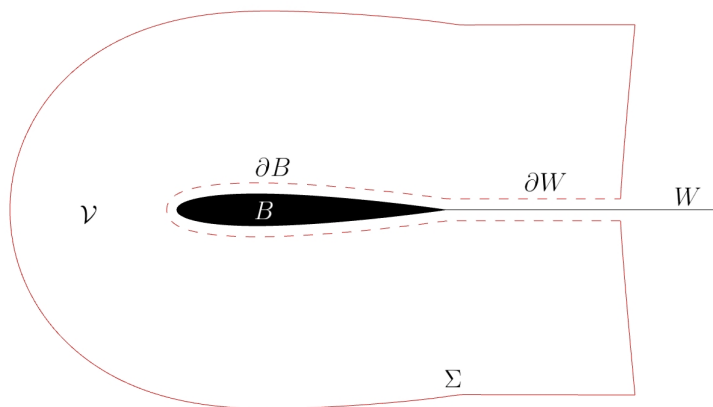
Under the abovementioned assumptions, the entire flow-field is irrotational outside of vortical sheets concentrated at the body wall and along its infinitely-thin unsteady wake, therefore domain integrals in Eq. (A29) and (A30) were cancelled to zero.

Linear theories studied the unsteady motion of a flat plate which, under small perturbation hypothesis, can be considered at $z = 0$. Therefore, integrals on ∂B were specialised to the upper and lower surfaces of the x-axis segment between $-b$ and b , where quantities are indicated with ⁺ (upper) or ⁻ (lower) superscripts. The reference frame is selected as in Sec. 1.2.2, i.e. x-axis chordwise, aligned with $\underline{V}_\infty$, with origin at the mid-chord point. Over the Σ surface, the Lamb vector is zero except at the intersection with the wake line (plate trailing edge), where the vorticity is concentrated (the free vorticity is therein the only non-zero singularity). The Lamb vector can therefore be described on Σ as a distribution with Dirac impulse at the trailing edge (where $\underline{\ell} = \underline{\gamma}_w \times \underline{V}_\infty$) and Eq. (A31) can be computed using Lebesgue integration.

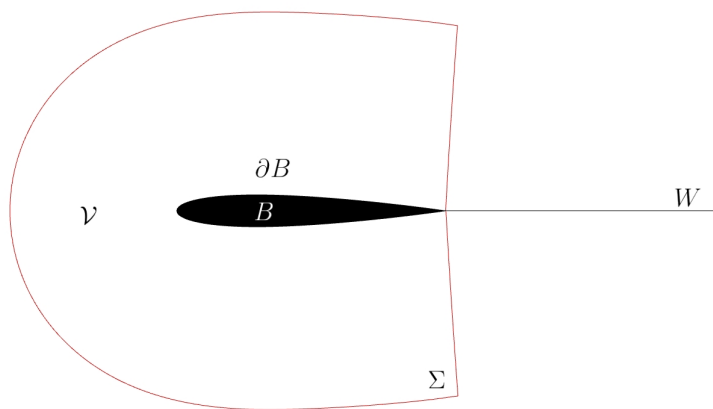
Assuming the DMT pole at the mid-chord point (reference frame origin) and recalling that $\underline{n}^\pm = \mp \underline{e}_z$ over the upper and lower flat plate surfaces (invariant in time, under small perturbations hypothesis), where the velocity vector $\underline{V}^\pm$ is given by $(V_\infty + u^\pm) \underline{e}_x + w \underline{e}_z$, lift can be expressed

[‡]In the case where the CV is instead chosen as in Fig. A.5.1a, $\underline{F}_\ell$ and $\underline{F}_t$ include another term, identical to the body skin integral but computed over the ∂W boundary (instead of ∂B), while $\underline{F}_S = \underline{0}$ (in this case the wake plane is interrupted and therefore Σ does not intercept any singularity).

[§]No distinction has been done between upper and lower w velocities as, for small disturbances, the relative transverse fluid velocity is zero at the impermeable wall and the plate motion velocities are of course the same on both sides of the wall.



(a) Generic wake plane position
(dashed lines collapse onto the body and wake boundaries).



(b) Wake plane intercepting the airfoil trailing edge.

Figure A.5.1: Integration domain for 2D slightly perturbed unsteady inviscid flows.

by:

$$\begin{aligned}
 l &= \underline{F}_{ff} \cdot \underline{e}_z = \\
 &= -\rho_\infty \int_{-b}^b \frac{(V^\pm)^2}{2} (\underline{n}^\pm \cdot \underline{e}_z) dx + \rho_\infty \int_{-b}^b \underline{r} \times \left(\underline{n}^\pm \times \frac{\partial V^\pm}{\partial t} \right) \cdot \underline{e}_z dx - \rho_\infty V_\infty b \gamma_w(b, t) = \\
 &= \frac{\rho_\infty}{2} \int_{-b}^b [V_\infty^2 + \cancel{(u^+)^2} + 2V_\infty u^+ - V_\infty^2 - \cancel{(u^-)^2} - 2V_\infty u^-] dx + \\
 &\quad - \rho_\infty \int_{-b}^b \frac{\partial}{\partial t} [x(V_\infty + u^+) - x(V_\infty + u^-)] dx - \rho_\infty V_\infty b \gamma_w(b, t) = \\
 &= \rho_\infty V_\infty \int_{-b}^b (u^+ - u^-) dx - \rho_\infty \frac{d}{dt} \int_{-b}^b x(u^+ - u^-) dx - \rho_\infty V_\infty b \gamma_w(b, t)
 \end{aligned} \tag{A32}$$

where higher order velocity perturbation terms were neglected, based on the assumption that $u/V_\infty \ll 1$ and $w/V_\infty \ll 1$. In the last row, time derivative was moved outside of the integral sign despite the moving integration domain, because the boundary integral in the Reynolds transport theorem [222, p. 24] would be identically zero, since the frontier velocities cancel-out on opposite sides of the flat plate.

Taking into account that $u^+ - u^- = \gamma_b$, Eq. (A32) reduces to Eq. (1.19) of Ch. 1, which proves the objective of this section. The time variation of the fluid impulse in $\mathcal{V}$ appears in the second term of Eq. (A32) and the derivation from vortex force theory provided herein also justifies the symbols used in Eq. (1.20).

Choosing the DMT pole at the trailing edge, Eq. (1.18) is instead recovered. It is observed that, in this case, the $\underline{F}_S$ term gives no lift contribution as expected, being the pole located on the wake plane with $\underline{n} = \underline{e}_x$ [81] (see also Sec. 2.1.1).

With an analogous procedure, links can be established for the aerodynamic moment as well. Indeed, starting from Eq. (A19), for inviscid and incompressible flows in inertial frames of reference the formula reduces to:

$$\underline{M}_{ff} = \underline{M}_\ell + \underline{M}_t + \underline{M}_S$$

Under the same hypothesis on the control volume selection and assuming a 2D slightly perturbed

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flow over the flat plate with straight wake aligned with the free-stream, it is:

$$\begin{aligned}\underline{M}_\ell &= -\rho_\infty \int_V \hat{\underline{r}} \times \underline{\ell} dV - \rho_\infty \int_{\partial B} \hat{\underline{r}} \times \frac{V^2}{2} \underline{n} dS = \\ &= -\rho_\infty \int_{-b}^b (\underline{x} - \underline{x}_p) \times \frac{(V^\pm)^2}{2} \underline{n}^\pm dx\end{aligned}\quad (\text{A33})$$

$$\begin{aligned}\underline{M}_t &= \frac{\rho_\infty}{2} \int_V \hat{\underline{r}}^2 \left(\underline{\nabla} \times \frac{\partial \underline{V}}{\partial t} \right) dV - \frac{\rho_\infty}{2} \int_{\partial B} \hat{\underline{r}}^2 \left(\underline{n} \times \frac{\partial \underline{V}}{\partial t} \right) dS = \\ &= -\frac{\rho_\infty}{2} \int_{-b}^b (x^2 + x_p^2 - 2xx_p) \left(\underline{n}^\pm \times \frac{\partial \underline{V}^\pm}{\partial t} \right) dx\end{aligned}\quad (\text{A34})$$

$$\underline{M}_s = \frac{\rho_\infty}{2} \int_\Sigma \hat{\underline{r}}^2 (\underline{n} \times \underline{\ell}) dS = \frac{\rho_\infty}{2} [(x^2 + x_p^2 - 2xx_p) (\underline{n} \times \underline{\ell})] |_{(b,0,t)} \quad (\text{A35})$$

Therefore, based on the same considerations used in the lift formula, the plate pitching moment is given by:

$$\begin{aligned}m &= \underline{M}_f \cdot \underline{e}_y = \\ &= -\rho_\infty \int_{-b}^b (\underline{x} - ab\underline{e}_x) \times \frac{(V^\pm)^2}{2} \underline{n}^\pm \cdot \underline{e}_y dx + \\ &\quad - \frac{\rho_\infty}{2} \int_{-b}^b [x^2 + (ab)^2 - 2xab] \left(\underline{n}^\pm \times \frac{\partial \underline{V}^\pm}{\partial t} \right) \cdot \underline{e}_y dx + \\ &\quad + \frac{\rho_\infty V_\infty}{2} [b^2 + (ab)^2 - 2ab^2] \gamma_w(b, t) = \\ &= -\rho_\infty V_\infty \int_{-b}^b (x - ab)(u^+ - u^-) dx + \\ &\quad + \frac{\rho_\infty}{2} \frac{d}{dt} \int_{-b}^b [x^2 + (ab)^2 - 2xab] (u^+ - u^-) dx + \\ &\quad + \frac{\rho_\infty V_\infty}{2} [b^2 + (ab)^2 - 2ab^2] \gamma_w(b, t) = \\ &= -\rho_\infty V_\infty \int_{-b}^b x \gamma_b dx + \frac{\rho_\infty}{2} \frac{d}{dt} \int_{-b}^b x^2 \gamma_b dx + \rho_\infty V_\infty \frac{b^2}{2} \gamma_w(b, t) + abl + \\ &\quad + \frac{(ab)^2}{2} \rho_\infty \frac{d\Gamma}{dt} + \frac{(ab)^2}{2} \rho_\infty V_\infty \gamma_w(b, t)\end{aligned}\quad (\text{A36})$$

where the lift expression provided by Eq. (A32) was taken into account. The last two terms in the moment formula were cancelled-out since, for the Kelvin theorem, it is:

$$-\frac{d\Gamma}{dt} = V_\infty \gamma_w(b, t)$$

then the formula reduces to Eq. (1.21) of Ch. 1, which proves the result.

A.6 ASYMPTOTIC BEHAVIOUR OF PRESSURE AND DENSITY TERMS IN $\underline{E}_{\text{BETZ}}$

The asymptotic behaviour of flow quantities in unbounded external aerodynamics has been deeply analysed by Liu et al. [102]. Starting from the linearized Navier-Stokes equations, they showed for example that an exponential decay of the flow velocity is obtained once the fluid vorticity and dilatation are assumed to have a compact structure in the analysed domain[¶]. This domain however is very large and includes the starting vorticity, with the flow being inherently unsteady.

Considering a steady flow model, which means limiting the analysis to a true subset of the complete domain [217], and assuming a high-Reynolds number regime (which allows neglecting viscous effects far from the body and wake) the asymptotic behaviour of the velocity field was studied by Batchelor [12, pp. 114-124]. He proved, on a kinematic basis, that an algebraic decay of the velocity perturbation ($\delta \underline{V} = (u, v, w)^T$) is obtained in the case of compact vorticity and dilatation (which however in this case needs to be assumed by hypothesis). Far from the body and wake, therefore, u , v and w behave as $\propto 1/r^2$, where r is the distance from perturbation sources. Surface integrals involving velocity perturbations raised to powers higher than one can then be neglected as the integration domain is extended, since the area grows as $\propto r^2$.

For steady compressible flows, density variations in the isentropic flow region far from the body and wake can be modelled at second-order on the velocity by the following relation, derived independently by Drela [44, p. 168] and Smith [181]:

$$\frac{\rho - \rho_\infty}{\rho_\infty} = -\frac{1}{2} \frac{M_\infty^2}{V_\infty^2} \{2uV_\infty + [1 - (2 - \gamma)M_\infty^2]u^2 + v^2 + w^2\} \quad (\text{A37})$$

Smith [181] also derived a binomial expansion of the compressible Bernoulli equation to express static pressure variations in terms of the velocity perturbations, at second-order [177]:

$$p - p_\infty = \frac{1}{2} \rho_\infty \{-2uV_\infty - [(1 - M_\infty^2)u^2 + v^2 + w^2]\} \quad (\text{A38})$$

In Ch. 1, Sec. 1.4.2, the following definition was made:

$$\underline{E}_{\text{BETZ}} \stackrel{\text{def}}{=} \int_{\Sigma} (p_{t,\infty} - p_t) \underline{n} \, dS + \int_{\Sigma} \frac{1}{2} V_\infty^2 (\rho - \rho_\infty) \underline{n} \, dS + \int_{\Sigma} \underline{\tau}_{\underline{\Xi}vt} \cdot \underline{n} \, dS \quad (\text{A39})$$

The scope of this section is to prove that, for high-Reynolds number flows and far from the body and wake, the sum of the first two integrals on the LHS of the above expression can be restricted on the wake plane W . Indeed it will be shown that the contributions on $\Sigma \setminus W$ for the first and second integrals are approximately equal and opposite in sign, as numerically verified by Fournis et al. [57].

Eq. (A37) and (A38) can be used at the scope, when $\Sigma \setminus W$ is assumed far from the body and

[¶]This was first proved to hold, on a dynamic basis, for incompressible flows by Wu [219] and then generalised to unsteady viscous and compressible flows by Liu et al. [100].

Appendix A. Notable Transformations and Derivations

wake. In this case, in fact, it is possible to write:

$$\begin{aligned}
 \int_{\Sigma \setminus W} (p_{t,\infty} - p_t) \underline{n} \, dS &= \int_{\Sigma \setminus W} \left(p_\infty + \frac{1}{2} \rho_\infty V_\infty^2 - p - \frac{1}{2} \rho V^2 \right) \underline{n} \, dS \approx \\
 &\approx \int_{\Sigma \setminus W} \left[\rho_\infty u V_\infty + \frac{1}{2} \rho_\infty V_\infty^2 - \frac{1}{2} \left(\rho_\infty - \frac{M_\infty^2 u}{V_\infty} \right) \cdot \right. \\
 &\quad \cdot (V_\infty^2 + u^2 + 2V_\infty u + v^2 + w^2) \left. \right] \underline{n} \, dS \approx \\
 &\approx \int_{\Sigma \setminus W} \left(\cancel{\rho_\infty u V_\infty} + \cancel{\frac{1}{2} \rho_\infty V_\infty^2} - \cancel{\frac{1}{2} \rho_\infty V_\infty^2} - \cancel{\rho_\infty u V_\infty} + \frac{1}{2} M_\infty^2 u V_\infty \right) \underline{n} \, dS
 \end{aligned}$$

and

$$\int_{\Sigma \setminus W} \frac{1}{2} V_\infty^2 (\rho - \rho_\infty) \underline{n} \, dS \approx - \int_{\Sigma \setminus W} \frac{1}{2} M_\infty^2 u V_\infty \underline{n} \, dS$$

where terms of order $\mathcal{O}(\delta V^2)$ or higher were automatically neglected as their integral contribution vanishes as $\Sigma \setminus W$ is extended far from the body and wake.

Being the two integrals approximations equal and opposite in sign, this proves the possibility to write:

$$\int_{\Sigma} \left[p_{t,\infty} - p_t + \frac{1}{2} V_\infty^2 (\rho - \rho_\infty) \right] \underline{n} \, dS \approx \int_W \left[p_{t,\infty} - p_t + \frac{1}{2} V_\infty^2 (\rho - \rho_\infty) \right] \underline{n} \, dS$$

under the specified assumptions. This gives no lift component once the wake surface is assumed as a plane orthogonal to $\underline{V}_\infty$.

B

Sensitivity study of the steady force formulas

This study was performed on two-dimensional test-cases, considering the majority of the analysed flow conditions for the NACA0012 aerofoil, listed in Section 2.2.1 (Table 2.2.1).

B.1 VARIATION OF THE CONTROL VOLUME SIZE

Results of the sensitivity analysis are hereafter reported focusing on the lifting cases (asymmetric flow-fields), although data at zero incidence are also included for comparison when relevant in the discussion below (e.g. in transonic flow conditions).

B.1.1 NEARLY-INCOMPRESSIBLE INVISCID FLOW

In this case, there's no dissipative effect (except that related to the artificial diffusion of the numerical scheme) and the total force is provided by the vortex force only ($\underline{F}_\ell$) and, in particular, by the additional body surface term of the inviscid formulation (2^{nd} term on the RHS of Eq. (2.21)). In fact, the effectively inviscid flow, in absence of shock waves and with uniform free-stream conditions, is both omoenthalpic and omoentropic, then the Lamb vector is zero according to *Crocco's* theorem everywhere within the flow-field except for the infinitely thin vortex sheet bounding the aerofoil. Therefore, being the flow also incompressible, all volume and far-field surface integrals appearing in Eq. (2.6), (2.12), (2.18), (2.19) give no contribution to the aerodynamic force and

Appendix B. Sensitivity study of the steady force formulas

the classical inviscid near-field formula is recovered from the body surface integral in Eq. (2.21), considering the validity of *Bernoulli's* theorem in incompressible flow regime.

For a fixed distance of the integration domain from the body boundary (Z_W), a generally good agreement of all far-field vortical force formulations with near-field results is observed on total lift (Figure B.1.1a) as soon as the control volume chordwise dimension (X_W) is extended downstream the aerofoil trailing edge ($X_W \geq 1$). In Figure B.1.1b all individual force constituents are zero except for the vortex force ($\underline{F}_\ell$) which is entirely given by the body boundary integral ($\underline{F}_\ell^{\partial B}$, 2^{nd} term on the RHS of Eq. (2.21)). The agreement on total drag is very good for all formulations (differences with regards to the result of near-field drag results are lower than one drag count) except for the one using volume integrals only (formulation “d”), which estimates however differ from near-field data by less than 2 drag counts (Figure B.1.2a). Looking at the individual drag constituents, a very slight compressibility correction is visible (Figure B.1.2b) - being the flow regime actually nearly, yet not exactly, incompressible - which is balanced by an opposite vortex force contribution resulting in a zero lift-induced drag coefficient. The slightly non-zero total drag coefficient is due to the vortex forces circulation moment term ($\underline{F}_S$ or $\underline{F}_{SV}$) revealing the presence of a spurious wake with non-zero vorticity downstream of the aerofoil. An analysis of the different contributions in $\underline{F}_{SV}$ revealed that the body surface integral is negligible and volume integrals are providing almost the whole drag in formulation “d”, being therefore more affected by spurious effects than the surface integral $\underline{F}_S$. Drag variations stabilise within 1 count limit approximately one chord downstream of the trailing edge.

For a fixed wake plane location (≈ 15 chords downstream of aerofoil trailing edge), small oscillations on total lift and drag are observed when varying the integration domain size between 0 and 2 chords in the normal-to-chord direction (Figure B.1.3). This occurs because, for small values of Z_W , the thin spurious wake downstream of the aerofoil is intercepting the control volume boundary mostly at its upper and lower boundaries, where the grid resolution is coarser. As Z_W is increased, the wake is fully captured at the wake plane, where the mesh size is much smaller. This phenomenon will be further highlighted describing results for one of the viscous cases reported in the following, where the (physical) aerofoil wake is thicker and oscillations are larger.

The sensibility to the integration domain size is further analysed with regards to the decomposition of drag in lift-induced and parasite contributions as described in Section 2.1.7. Both contributions can be considered zero (values smaller than 1 drag count) with the exception for the parasite drag coefficient computed using volume integrals (formulation “d”), which is slightly overestimated (B.1.4).

B.1. Variation of the control volume size

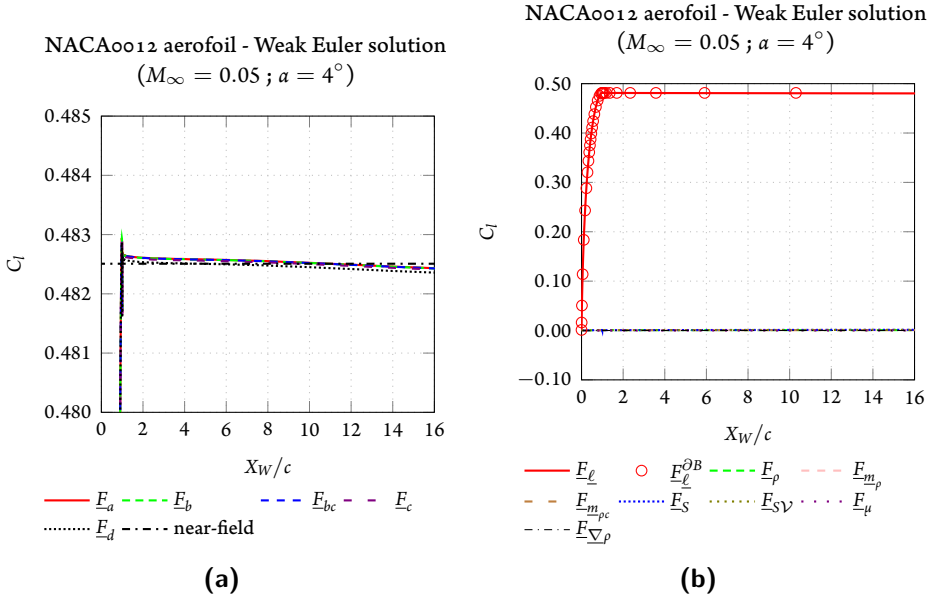


Figure B.1.1: Sensitivity to wake plane location of total force (Fig. B.1.1a) and individual constituents (Fig. B.1.1b) of different Lamb vector-based lift formulations, in inviscid nearly-incompressible flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

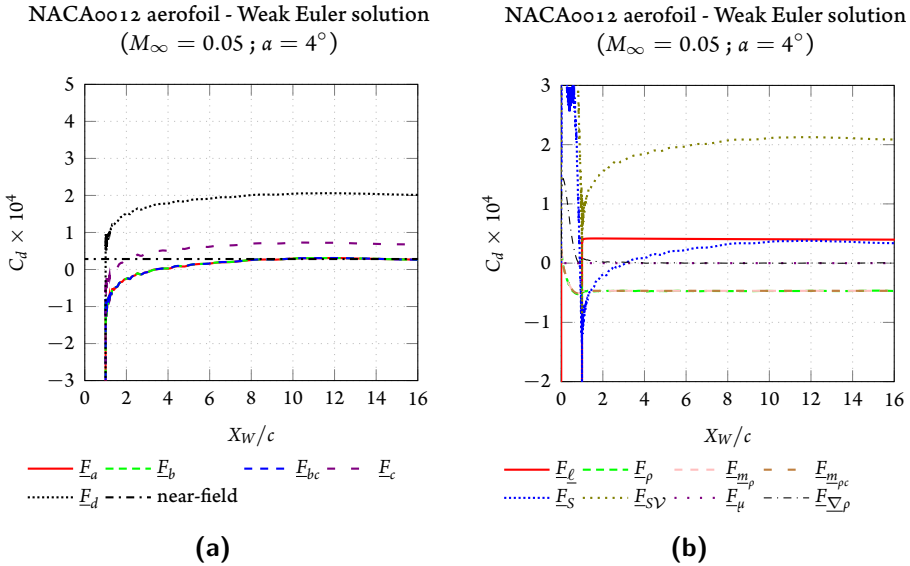


Figure B.1.2: Sensitivity to wake plane location of total force (Fig. B.1.2a) and individual constituents (Fig. B.1.2b) of different Lamb vector-based drag formulations, in inviscid nearly-incompressible flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

Appendix B. Sensitivity study of the steady force formulas

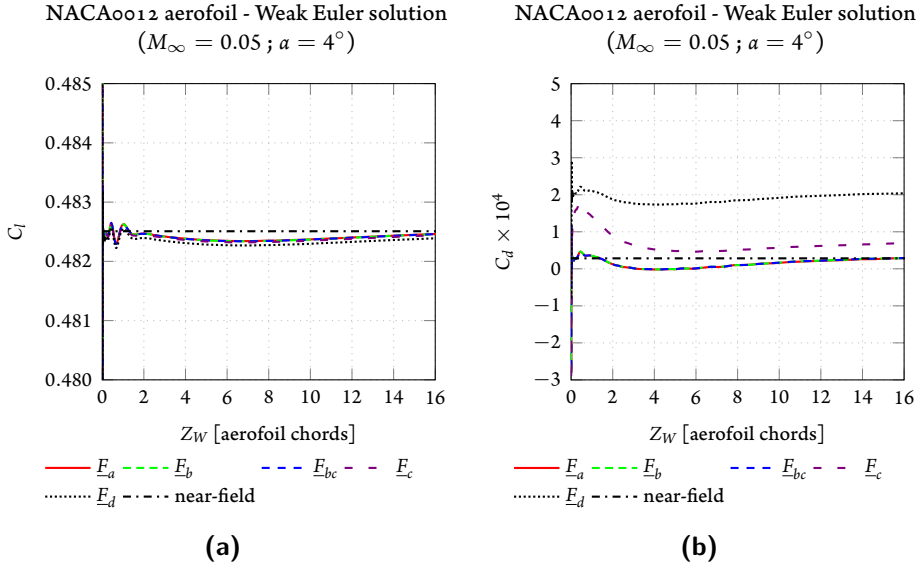


Figure B.1.3: Sensitivity to integration domain size, at fixed wake plane location, of total lift (Fig. B.1.3a) and drag (Fig. B.1.3b), in inviscid nearly-incompressible flow ($X_W \approx 16c$; fine mesh; pole at aerofoil l.e.).

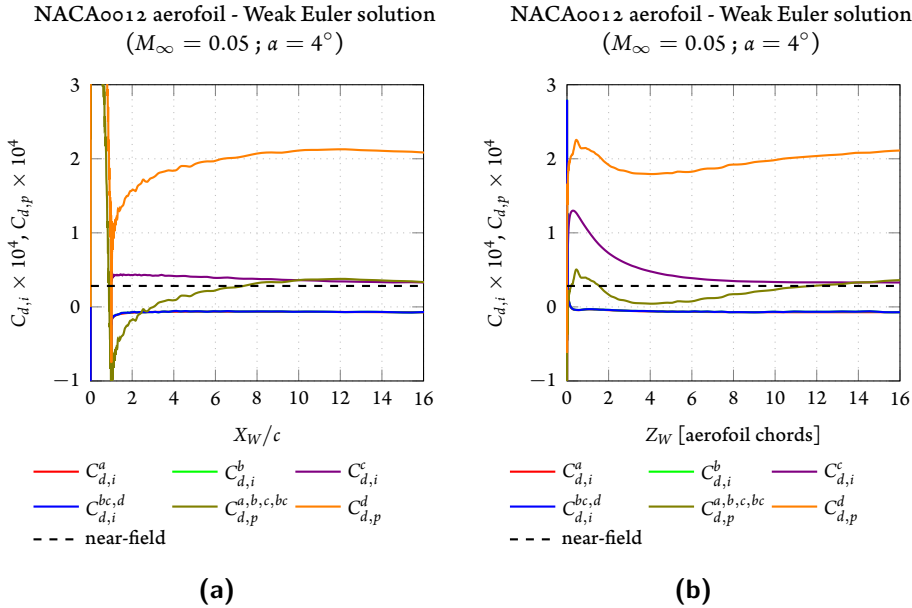


Figure B.1.4: Sensitivity of different Lamb vector-based formulations of lift-induced and parasite drag coefficients to the integration domain size, in inviscid nearly-incompressible flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

B.1.2 NEARLY-INCOMPRESSIBLE VISCOUS FLOW

In this case viscous dissipation is active, the Lamb vector is non-zero in the boundary layer and in the viscous wake, therefore its volume and far-field surface integrals ($\underline{F}_\ell$ and $\underline{F}_S$ or $\underline{F}_{SV}$) give non-zero contributions to the total force (Figure B.1.5b, B.1.6b). The effects of compressibility ($\underline{F}_\rho$, $\underline{F}_{m_p}$ and $\underline{F}_{m_{pc}}$) is negligible being the flow regime nearly-incompressible at the simulated Mach number, as well as the effect of terms explicitly depending on viscosity and turbulent fluctuations ($\underline{F}_\mu$, Eq. (2.4)) as proved by Marongiu and Tognaccini for high-Reynolds number flows [112]. Basically, the only contribution to lift is provided by the vortex force (Figure B.1.5b) and almost the entire contribution to drag is provided by vortex forces circulation moment at far-field (Figure B.1.6b). This is justified because, as illustrated by Fournis et al. [55], being in this case compressibility corrections and viscous terms ($\underline{F}_\mu$) negligible, vortex forces circulation moment alone represents a compressible Betz's formula (giving no contribution to lift, as numerically verified in [55]) and the vortex force alone represents the generalised Kutta-Joukowski theorem and extended of Maskell's formula contributions, which drag component is the induced drag [55].

A very good agreement with near-field results is provided by all formulations on total lift coefficients (Figure B.1.5a) while total drag coefficient estimated using “c” formulation (based on far-field surface integrals only) is closer than others to near-field computed value (Figure B.1.6a). For a fixed distance of the integration domain from the body boundary (Z_W), total drag variations stabilise approximately 5 chord downstream of aerofoil trailing edge.

As observed also in the inviscid case, for a fixed wake plane location (≈ 15 chords downstream of aerofoil trailing edge), oscillations on total lift and drag are observed when varying the integration domain size between 0 and 4 chords in the normal-to-chord direction (Figure B.1.7a, B.1.8a), with significant variations in the same range of the vortex force ($\underline{F}_\ell$) and its circulation moment at far-field surface ($\underline{F}_S$ or $\underline{F}_{SV}$) as illustrated in Figure B.1.7b and Figure B.1.8b. Again, this phenomenon will be further analysed in the following, when describing results of the viscous case at subsonic Mach number ($M_\infty = 0.5$).

analysing drag decomposition in lift-induced and parasite contributions (Figure B.1.9), slight negative values are obtained for the induced drag when using “a”, “b”, “bc” and “d” formulations, while zero induced drag is predicted by “c” formulation based on far-field surface integrals only. It is observed that, for the viscous case, induced drag coefficients are slightly lower than corresponding values computed for the inviscid case at the same Mach number (Figure B.1.9b, B.1.4).

Appendix B. Sensitivity study of the steady force formulas

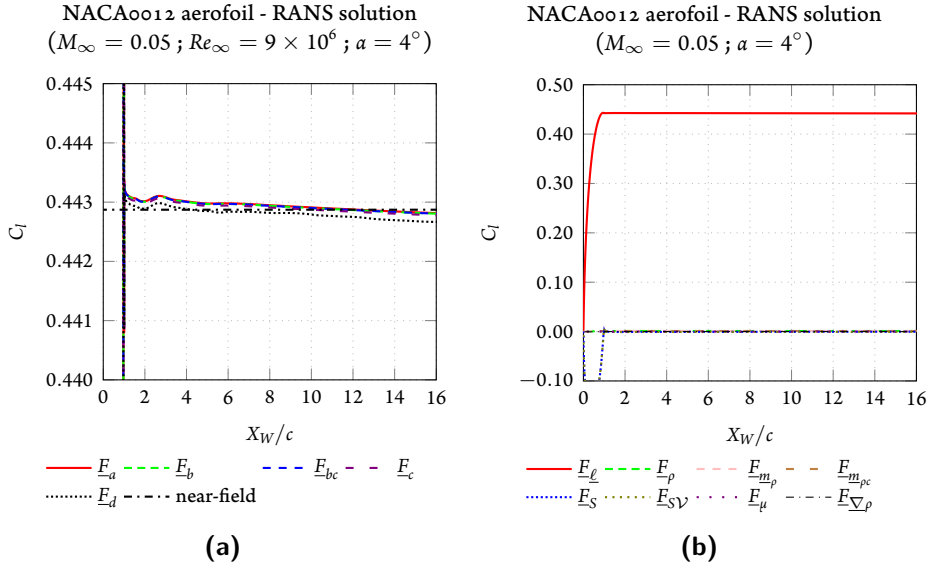


Figure B.1.5: Sensitivity to wake plane location of total force (Fig. B.1.5a) and individual constituents (Fig. B.1.5b) of different Lamb vector-based lift formulations, in viscous nearly-incompressible flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

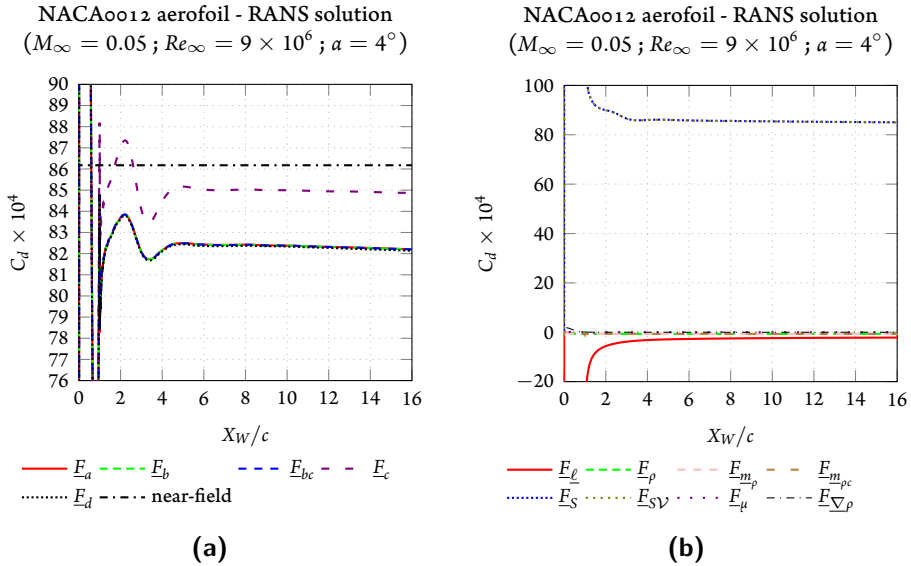


Figure B.1.6: Sensitivity to wake plane location of total force (Fig. B.1.6a) and individual constituents (Fig. B.1.6b) of different Lamb vector-based drag formulations, in viscous nearly-incompressible flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

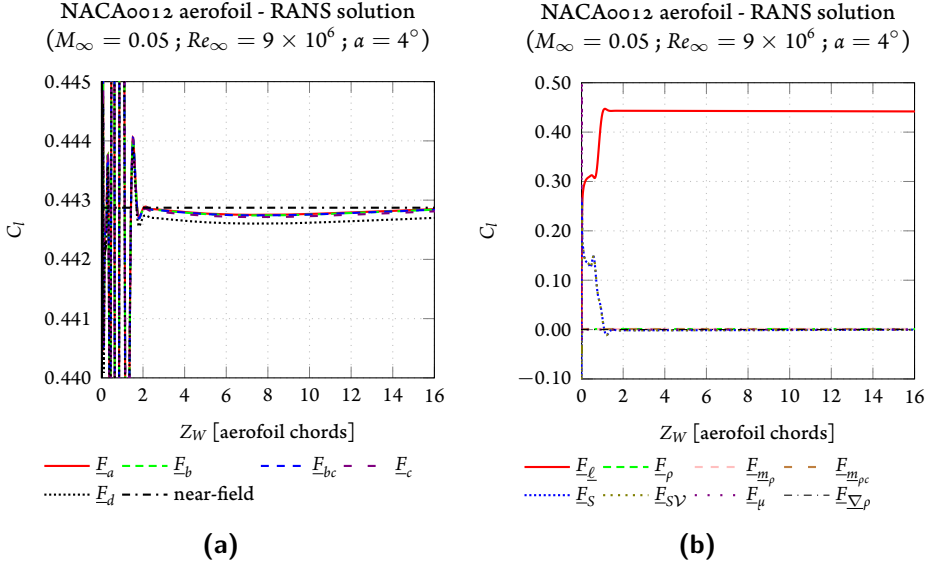


Figure B.1.7: Sensitivity to integration domain size, at fixed wake plane location, of total force (Fig. B.1.7a) and individual constituents (Fig. B.1.7b) of different Lamb vector-based lift formulations, in viscous nearly-incompressible flow ($X_W \approx 16c$; fine mesh; pole at aerofoil l.e.).

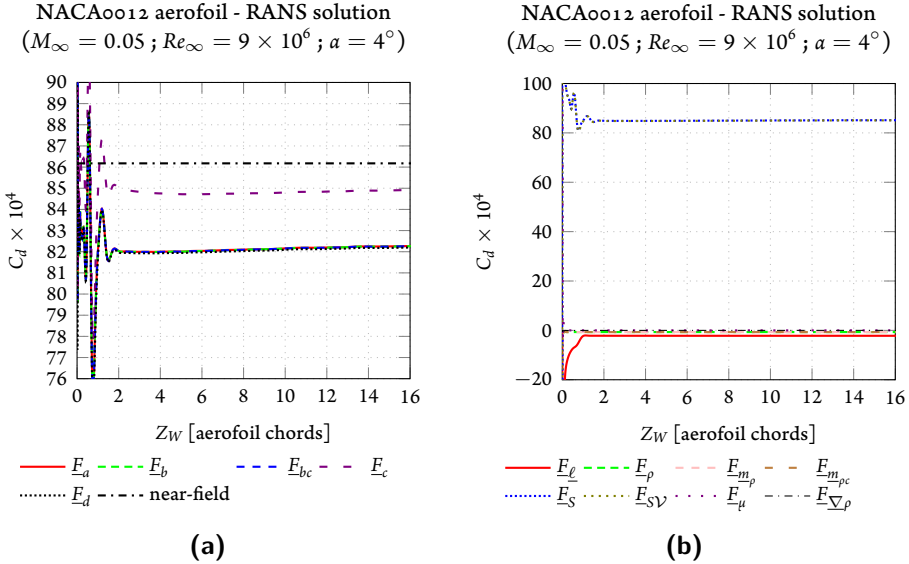


Figure B.1.8: Sensitivity to integration domain size, at fixed wake plane location, of total force (Fig. B.1.8a) and individual constituents (Fig. B.1.8b) of different Lamb vector-based drag formulations, in viscous nearly-incompressible flow ($X_W \approx 16c$; fine mesh; pole at aerofoil l.e.).

Appendix B. Sensitivity study of the steady force formulas

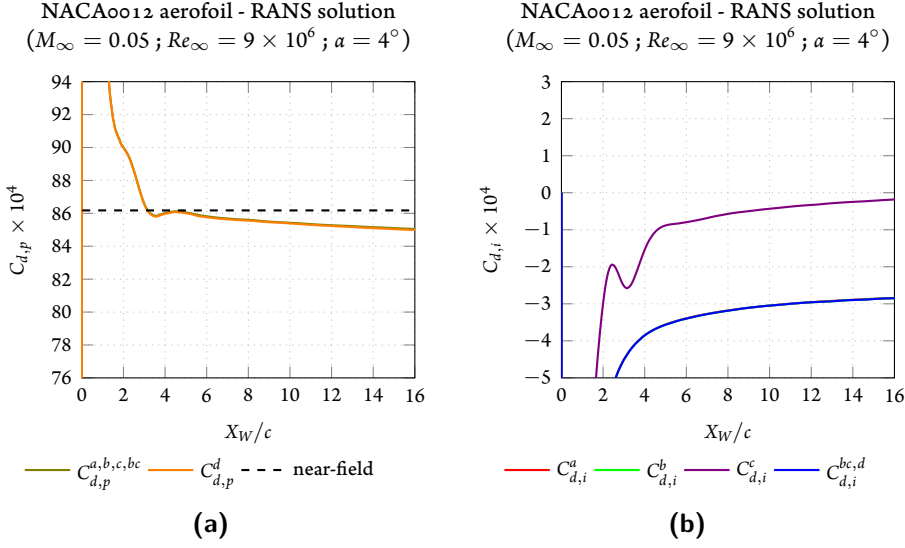


Figure B.1.9: Sensitivity of different Lamb vector-based formulations of lift-induced and parasite drag coefficients to the integration domain size, in viscous nearly-incompressible flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.). Fig. B.1.9a: parasite drag; Fig. B.1.9b: induced drag.

B.1.3 SUBSONIC INVISCID FLOW

In this case, there's no dissipative effect (except that related to the artificial diffusion of the numerical scheme) and the Lamb vector is zero everywhere within the flow-field (except in the infinitely thin vortex sheet bounding the aerofoil). However, differently from the already illustrated case at $M_\infty = 0.05$, compressibility effects are now not negligible then the aerodynamic lift is not yet provided by the vortex force alone, as obtained for the nearly-incompressible case (Figure B.1.10b). All formulations provided a very good matching with near field data (Figures B.1.10a and B.1.11a) with relatively larger discrepancies observed for the formulation "d" on total drag coefficient, although limited within 1 drag count. As for the inviscid case in nearly-incompressible flow conditions, a perfect balance between vortex force and compressibility corrections was observed on drag (Figure B.1.11b), resulting in zero induced drag coefficients (Figure B.1.13). Variations with Z_W are negligible after 2 chords distance in the normal-to wall direction (Figure B.1.12).

B.1. Variation of the control volume size

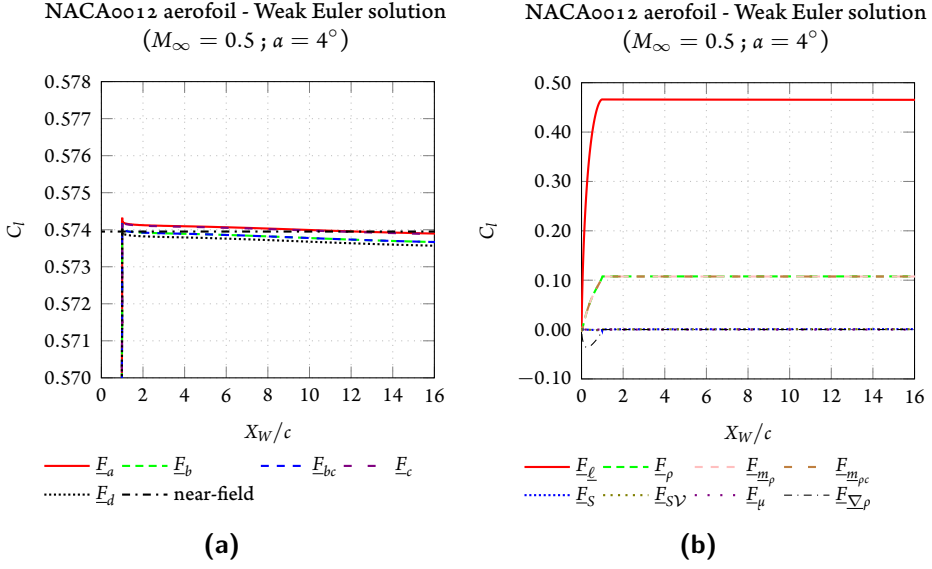


Figure B.1.10: Sensitivity to wake plane location of total force (Fig. B.1.10a) and individual constituents (Fig. B.1.10b) of different Lamb vector-based lift formulations, in inviscid subsonic flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil i.e.).

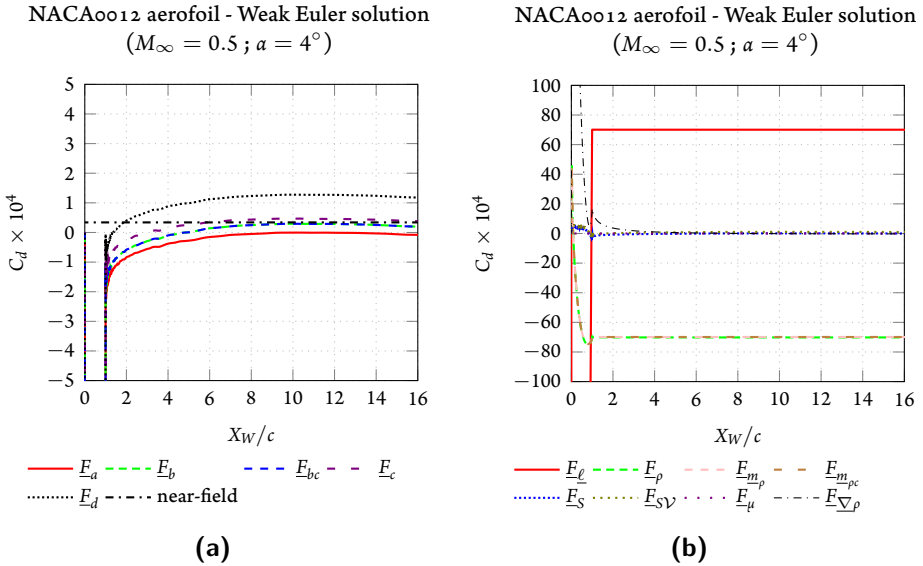


Figure B.1.11: Sensitivity to wake plane location of total force (Fig. B.1.11a) and individual constituents (Fig. B.1.11b) of different Lamb vector-based drag formulations, in inviscid subsonic flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil i.e.).

Appendix B. Sensitivity study of the steady force formulas

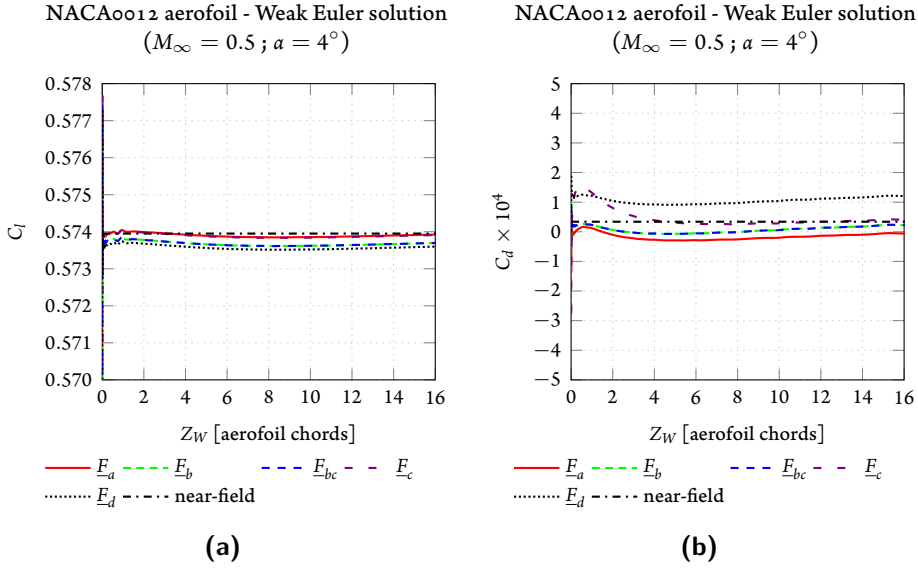


Figure B.1.12: Sensitivity to integration domain size, at fixed wake plane location, of total lift (Fig. B.1.12a) and drag (Fig. B.1.12b), in inviscid subsonic flow ($X_W \approx 16c$; fine mesh; pole at aerofoil l.e.).

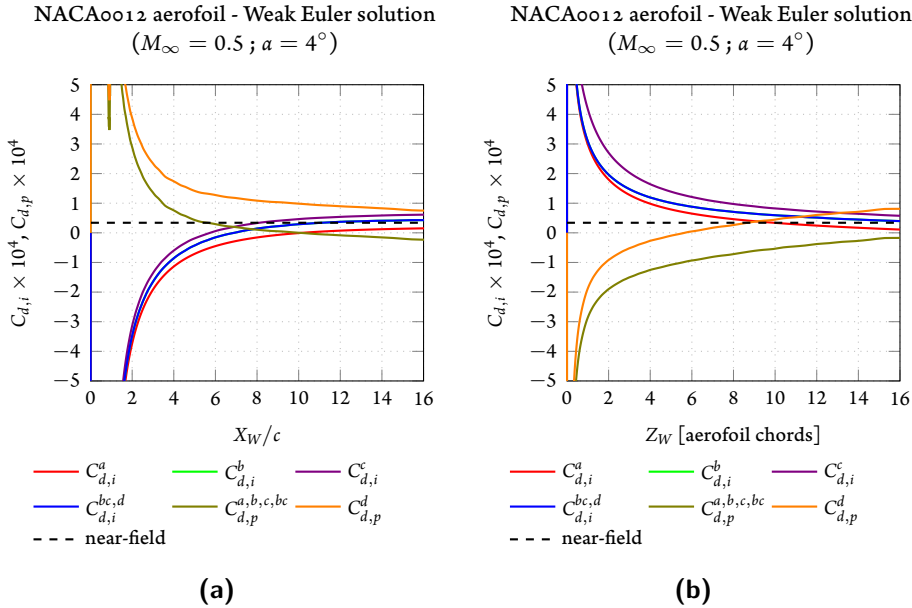


Figure B.1.13: Sensitivity of different Lamb vector-based formulations of lift-induced and parasite drag coefficients to the integration domain size, in inviscid subsonic flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

B.1.4 SUBSONIC VISCOUS FLOW (HIGH REYNOLDS NUMBER)

In this case viscous dissipation is active and compressibility effects are not negligible. As observed also for the viscous nearly-incompressible case, the agreement of far-field results versus near field data is very good on total lift and better for formulation “c” on total drag (Figure B.1.14a, B.1.15a), with vortex force and compressibility correction both contributing to aerofoil lift (Figure B.1.14b) but not exactly compensating each other on drag (Figure B.1.15b), resulting in slightly negative induced drag coefficients (Figure B.1.19), as already observed in the case of viscous nearly-incompressible flow, while zero lift-induced drag is instead predicted by formulation “c”. Total drag values stabilise approximately 4 chords downstream of the aerofoil.

The origin of total lift and drag oscillations observed when varying the integration domain size in normal-to-chord direction (Z_W) up to 2 aerofoil chords is now better highlighted looking at the relative position of the integration domain external boundaries with regards to the aerofoil viscous wake, visualised in Figure B.1.17 using a contour plot of flow entropy variation. Is it clear that, for small values of Z_W , the viscous wake is captured on the upper portion of the integration domain boundary, where the mesh resolution is coarser. As Z_W is increased and the integration domain intercepts the viscous wake on its downstream boundary (with finer mesh resolution), rapid variations of lift constituents are observed (Figure B.1.16b), then computed values stabilise at $Z_W \approx 1.5c$.

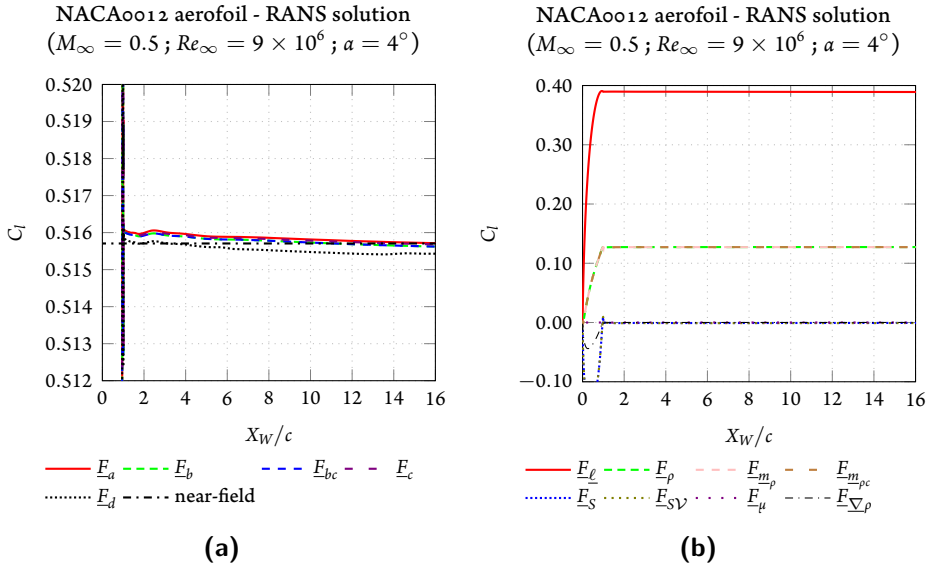


Figure B.1.14: Sensitivity to wake plane location of total force (Fig. B.1.14a) and individual constituents (Fig. B.1.14b) of different Lamb vector-based lift formulations, in viscous subsonic flow (high Reynolds number, $Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

Appendix B. Sensitivity study of the steady force formulas

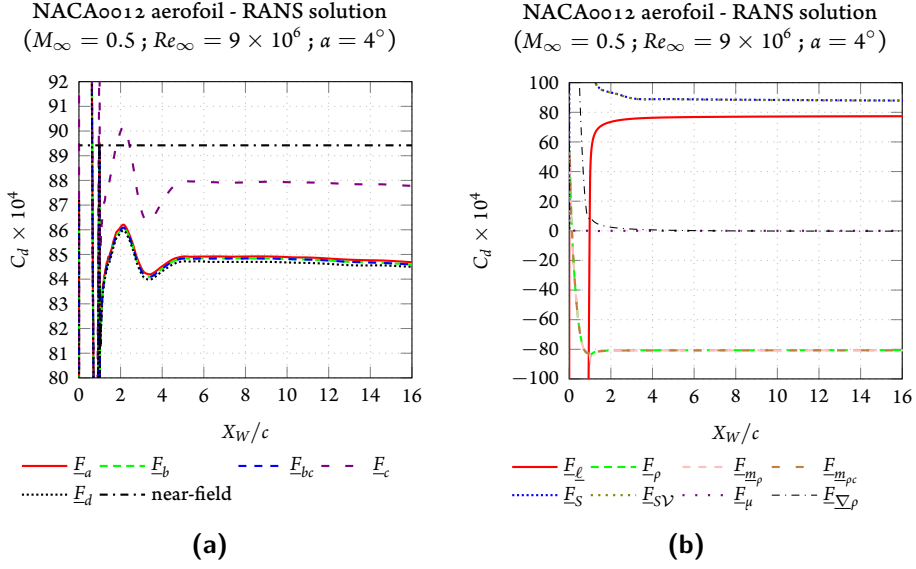


Figure B.1.15: Sensitivity to wake plane location of total force (Fig. B.1.15a) and individual constituents (Fig. B.1.15b) of different Lamb vector-based drag formulations, in viscous subsonic flow (high Reynolds number, $Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

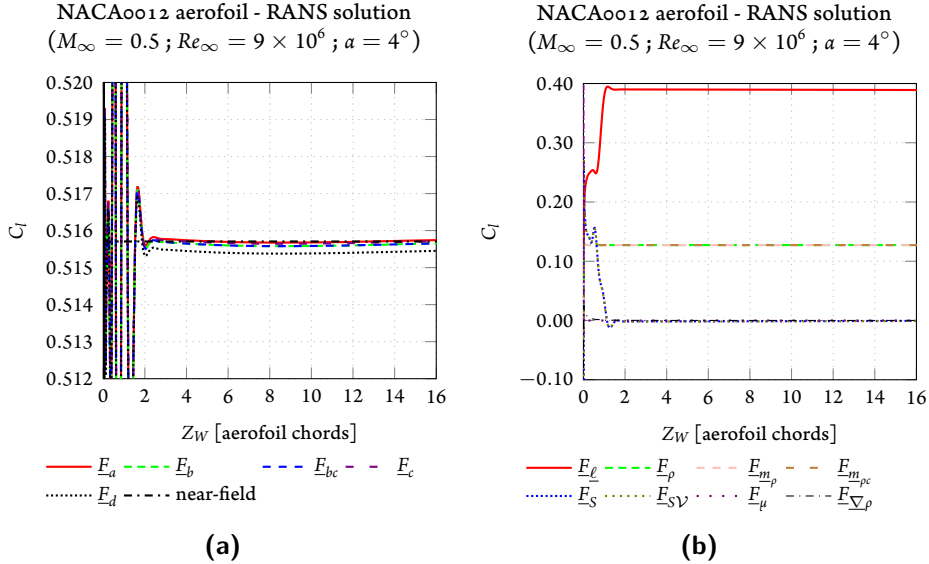


Figure B.1.16: Sensitivity to integration domain size, at fixed wake plane location, of total force (Fig. B.1.16a) and individual constituents (Fig. B.1.16b) of different Lamb vector-based lift formulations, in viscous subsonic flow (high Reynolds number, $X_W \approx 16c$; fine mesh; pole at aerofoil l.e.).

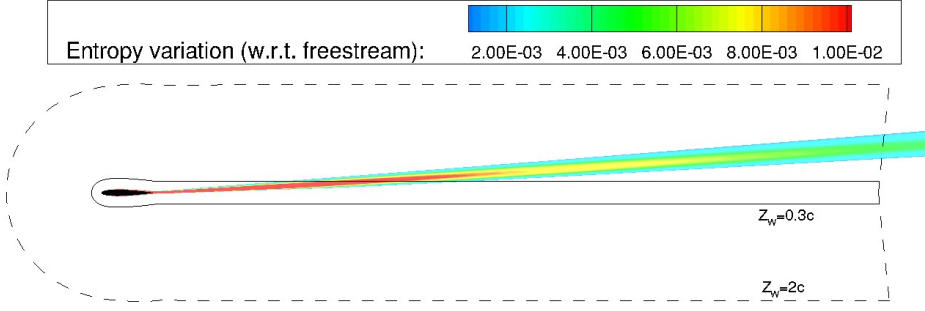


Figure B.1.17: Relative position of integration domain external boundaries with regards to the aerofoil wake ($M_\infty = 0.5$; $Re_\infty = 9 \times 10^6$; $\alpha = 4^\circ$).

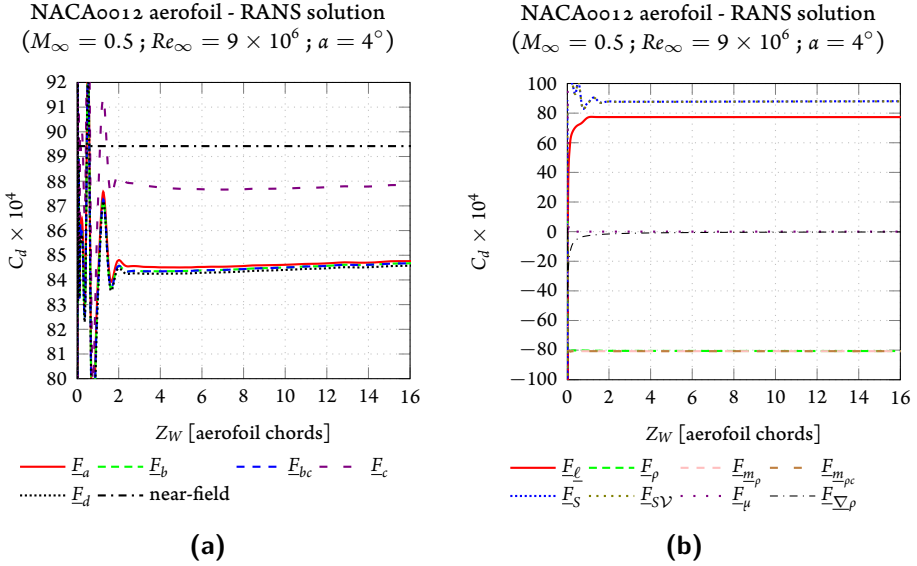


Figure B.1.18: Sensitivity to integration domain size, at fixed wake plane location, of total force (Fig. B.1.18a) and individual constituents (Fig. B.1.18b) of different Lamb vector-based drag formulations, in viscous subsonic flow (high Reynolds number, $X_W \approx 16c$; fine mesh; pole at aerofoil i.e.).

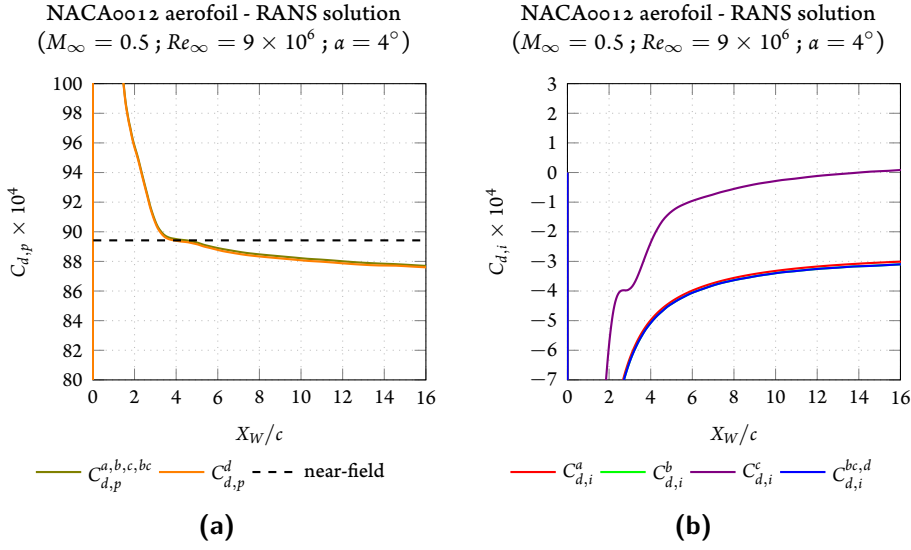


Figure B.1.19: Sensitivity of different Lamb vector-based formulations of lift-induced and parasite drag coefficients to the integration domain size, in viscous subsonic flow (high Reynolds number, $Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.). Fig. B.1.19a: parasite drag; Fig. B.1.19b: induced drag.

B.1.5 SUBSONIC VISCOUS FLOW (LOW REYNOLDS NUMBER)

The laminar flow separation on the aerofoil upper surface (occurring at approximately 45% of the chord length) is responsible for the slightly negative lift coefficient, at positive incidence, observed in Figure B.1.20a. A slight contribution of E_S to total lift is also visible from Figure B.1.20b, being the wake plane not exactly orthogonal to the freestream direction (Section 2.1.7).

From Figure B.1.21a, all formulations seem to provide estimates of total drag with non-negligible differences with regards to the near field value.

It is worth mentioning that, in laminar flow cases, all terms in Lamb-vector based formulations were computed without neglecting any contribution, being in this case null the last integral of Eq. (2.4) (only present in turbulent flows). Nevertheless, due to numerical reasons, the aerodynamic force is not constant as the control volume is extended in the X_W direction (Figures B.1.20a and B.1.21a) downstream of the trailing edge (as theoretically it should be) and convergence of total force coefficients is slower than in the turbulent flow case. No significant differences are instead observed with domain variations in the Z_W direction, with regards to the analogous subsonic case at high Reynolds number (Figures B.1.22 and B.1.23).

Analysing drag decomposition in lift-induced and parasite contributions (Figure B.1.24), all formulations provided negative induced drag and parasite drag higher than total near field drag in control volumes smaller than 15 chord lengths around the aerofoil. This is in line with what was already observed by Kang et al. [81]. They used formulation “b” of lift-induced drag (without

the additional compressibility correction $\underline{F}_{\nabla p}$, which is however negligible as visible from Figure B.1.23b) and a wake plane location between 2 and 3 chord lengths downstream of the aerofoil trailing edge (corresponding to X_W between 3 and 4) and observed negative induced drag coefficients in the order of $\approx -100d.c.$ at $Re_\infty = 1 \times 10^4$ ([81], Fig. 21, pag. 4636). This value however provides a rough estimate only, as they simulated subsonic flow conditions at $M_\infty = 0.3$ (instead of $M_\infty = 0.5$) and $Re_\infty = 1 \times 10^3$, $Re_\infty = 1 \times 10^6$ and $Re_\infty = 1 \times 10^7$ (i.e. value at $Re_\infty = 1 \times 10^4$ is interpolated).

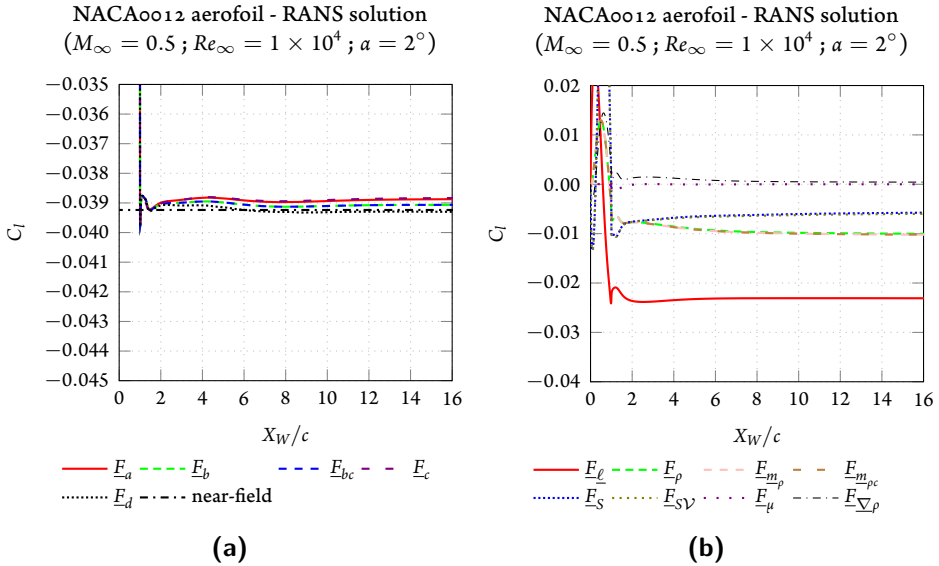


Figure B.1.20: Sensitivity to wake plane location of total force (Fig. B.1.20a) and individual constituents (Fig. B.1.20b) of different Lamb vector-based lift formulations, in viscous subsonic flow (low Reynolds number, $Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

Appendix B. Sensitivity study of the steady force formulas

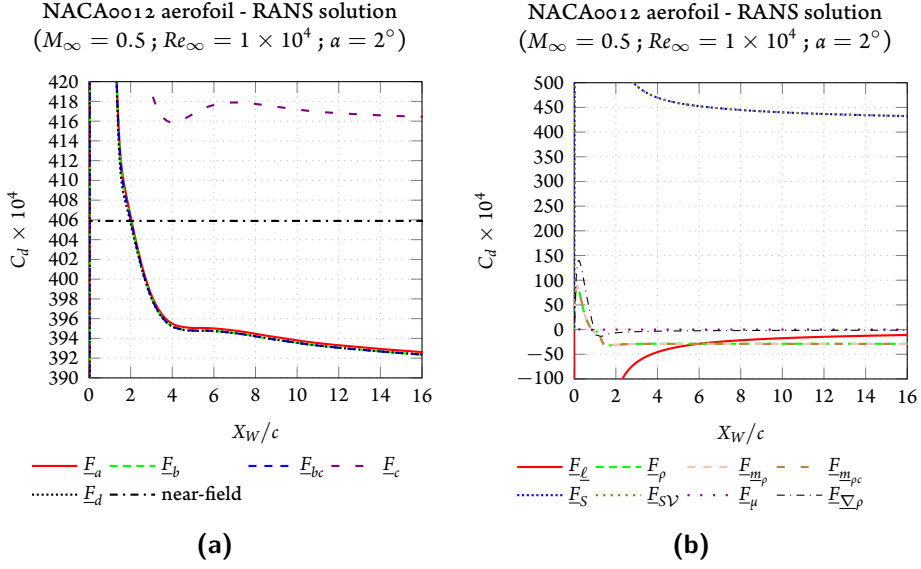


Figure B.1.21: Sensitivity to wake plane location of total force (Fig. B.1.21a) and individual constituents (Fig. B.1.21b) of different Lamb vector-based drag formulations, in viscous subsonic flow (low Reynolds number, $Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

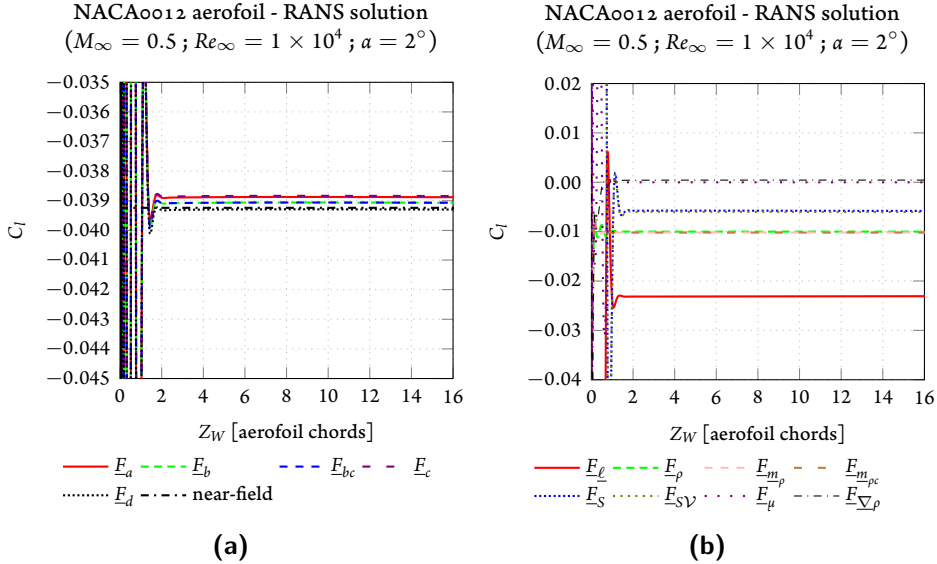


Figure B.1.22: Sensitivity to integration domain size, at fixed wake plane location, of total force (Fig. B.1.22a) and individual constituents (Fig. B.1.22b) of different Lamb vector-based lift formulations, in viscous subsonic flow (low Reynolds number, $X_W \approx 16c$; fine mesh; pole at aerofoil l.e.).

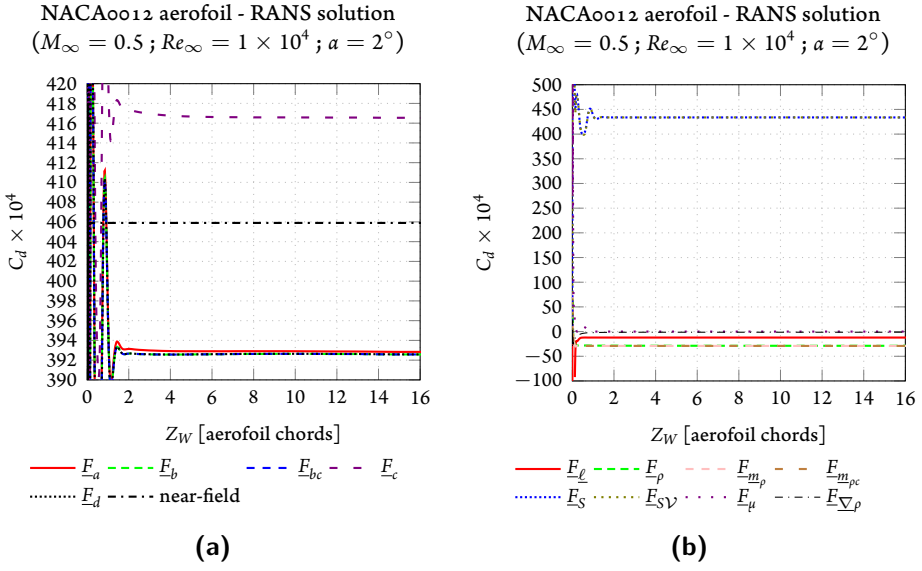


Figure B.1.23: Sensitivity to integration domain size, at fixed wake plane location, of total force (Fig. B.1.23a) and individual constituents (Fig. B.1.23b) of different Lamb vector-based drag formulations, in viscous subsonic flow (low Reynolds number, $X_W \approx 16c$; fine mesh; pole at aerofoil l.e.).

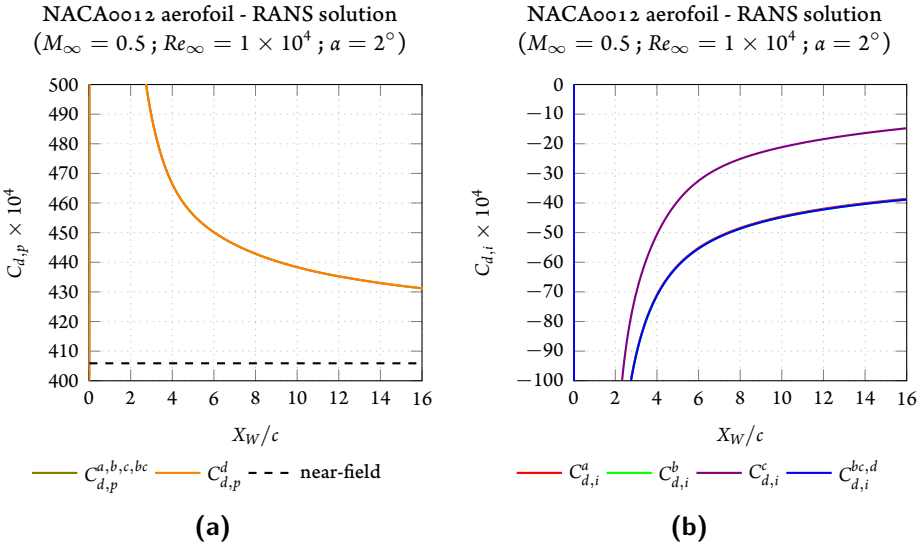


Figure B.1.24: Sensitivity of different Lamb vector-based formulations of lift-induced and parasite drag coefficients to the integration domain size, in viscous subsonic flow (low Reynolds number, $Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.). Fig. B.1.24a: parasite drag; Fig. B.1.24b: induced drag.

B.1.6 TRANSONIC INVISCID FLOW

In this case, apart from numerical dissipation, the only drag source is due to the strong shock wave on the aerofoil upper side. Compressibility correction contribution to total lift is larger than vortex force lift component (Figure B.1.25b) and all far-field formulations provide acceptable total lift estimates when compared to near-field data (Figure B.1.25a). The same is not true on total drag, where formulations “a” and “b” provide much higher values than others. This was already reported by Mele and Tognaccini [119] with regards to formulation “b”. Mele et al. introduced the hybrid “bc” formulation [117] to avoid volume integration in the shock wave region, where high-resolution numerical schemes may blend towards lower order accuracy. Of course “c” formulation is even better in this case, as no volume integration is performed, while formulation “a” suffers of similar inaccuracies as formulation “b” because both the vortex force and one term of the corresponding compressibility correction are computed using volume integrals. It is observed that correct values of drag compressibility correction ($E_{m_{pc}}$) don’t exactly counterbalance the vortex force term (Figure B.1.28b), resulting in slightly negative induced drag coefficients (Figure B.1.29) as already observed in previously illustrated viscous cases. However, while in mentioned cases formulation “c” did not provide negative induced drag coefficient, all formulations now exhibit this behaviour in the analysed inviscid transonic case at 2° angle of attack. This is not observed at zero incidence angle (symmetric flow-field) for the same transonic flow conditions using formulation “c” (Figure B.1.30). Parasite drag represents therefore a far-field prediction of total (wave) drag at zero incidence, while at $\alpha = 2^\circ$ the non-zero lift-induced contribution to total drag cannot be ignored for any of the analysed Lamb vector-based formulations.

At fixed wake plane location ($X_W \approx 16c$), variations of far-field results with the integration domain size in normal-to-chord direction (Z_W) are visible up to $\approx 1c$ (where the control volume boundary is intersecting the shock wave) and up to $\approx 3c$ as far as formulation “bc” is concerned, the control volume boundary intersecting in this case the selected region around the shock wave excluded from volume integration (Figures B.1.27 and B.1.28).

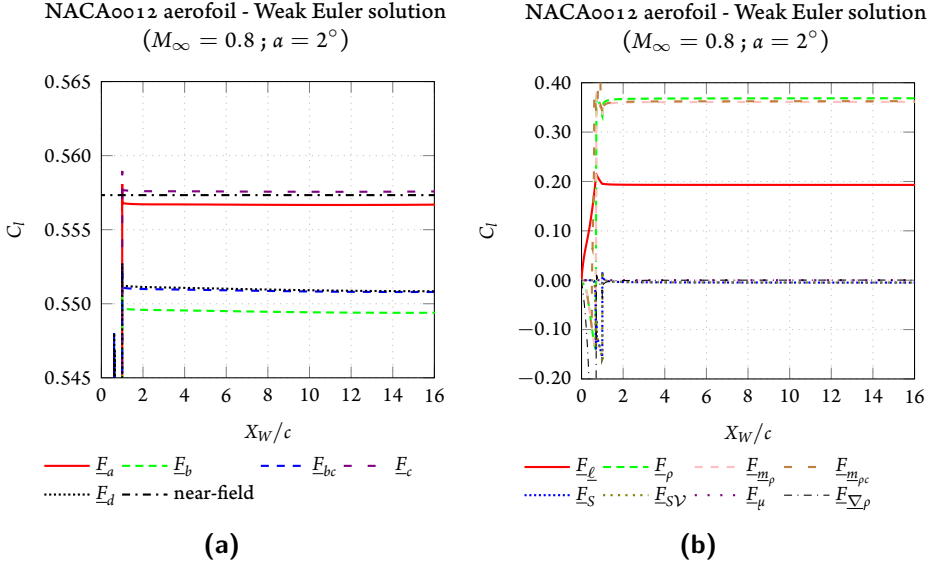


Figure B.1.25: Sensitivity to wake plane location of total force (Fig. B.1.25a) and individual constituents (Fig. B.1.25b) of different Lamb vector-based lift formulations, in inviscid transonic flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

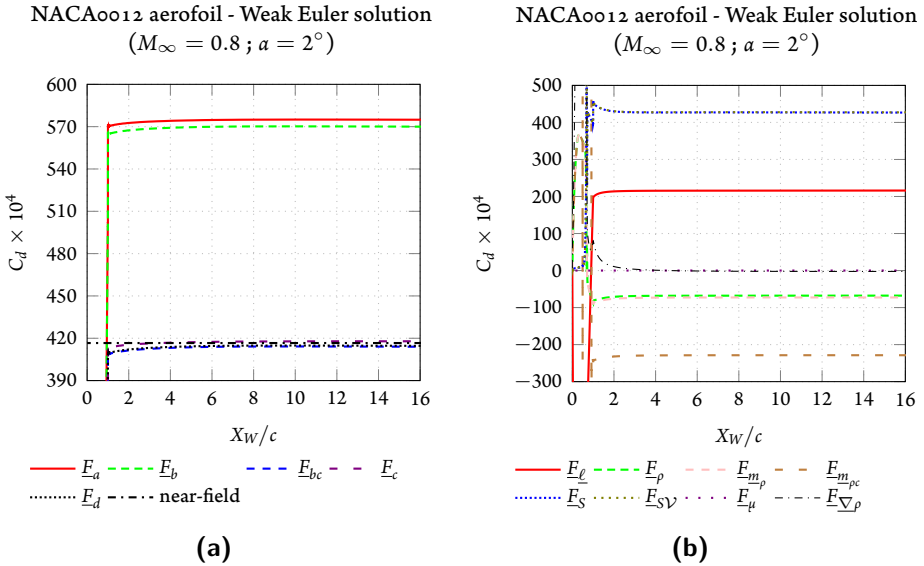


Figure B.1.26: Sensitivity to wake plane location of total force (Fig. B.1.26a) and individual constituents (Fig. B.1.26b) of different Lamb vector-based drag formulations, in inviscid transonic flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

Appendix B. Sensitivity study of the steady force formulas

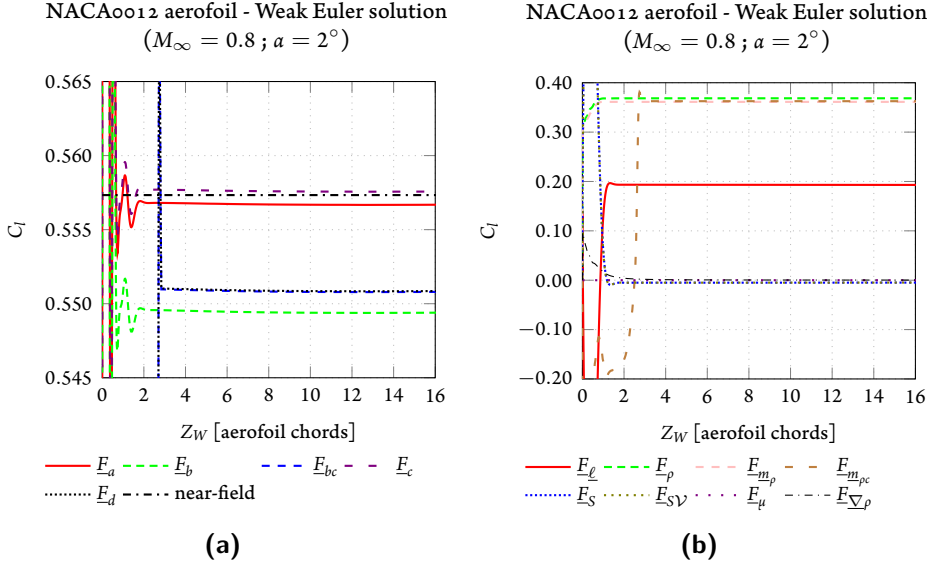


Figure B.1.27: Sensitivity to integration domain size, at fixed wake plane location, of total force (Fig. B.1.27a) and individual constituents (Fig. B.1.27b) of different Lamb vector-based lift formulations, in inviscid transonic flow ($X_W \approx 16c$; fine mesh; pole at aerofoil i.e.).

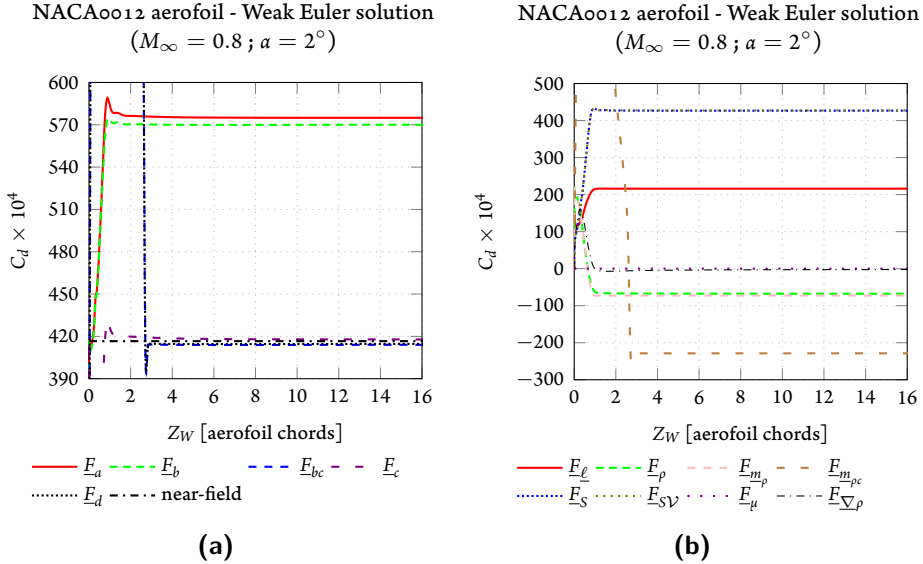


Figure B.1.28: Sensitivity to integration domain size, at fixed wake plane location, of total force (Fig. B.1.28a) and individual constituents (Fig. B.1.28b) of different Lamb vector-based drag formulations, in inviscid transonic flow ($X_W \approx 16c$; fine mesh; pole at aerofoil i.e.).

B.1. Variation of the control volume size

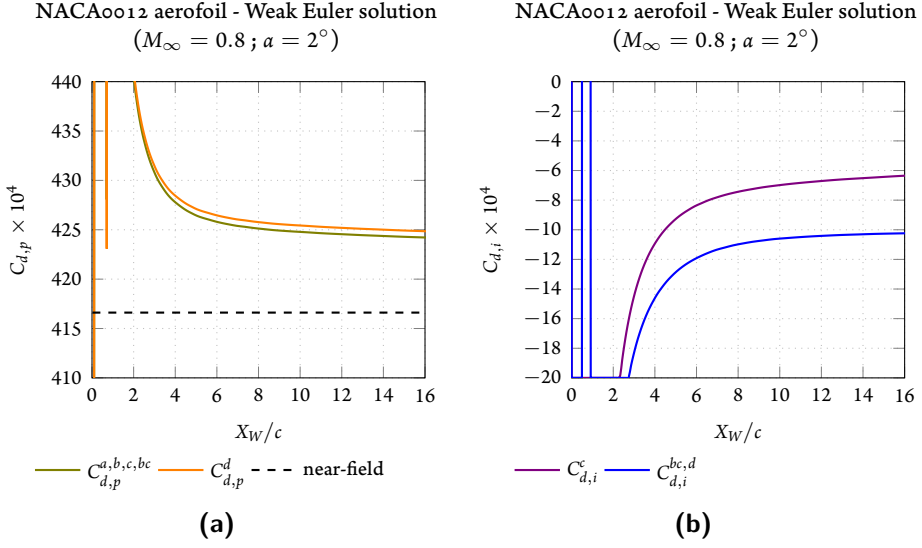


Figure B.1.29: Sensitivity of different Lamb vector-based formulations of lift-induced and parasite drag coefficients to the integration domain size, in inviscid transonic flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.). Fig. B.1.29a: parasite drag; Fig. B.1.29b: induced drag.

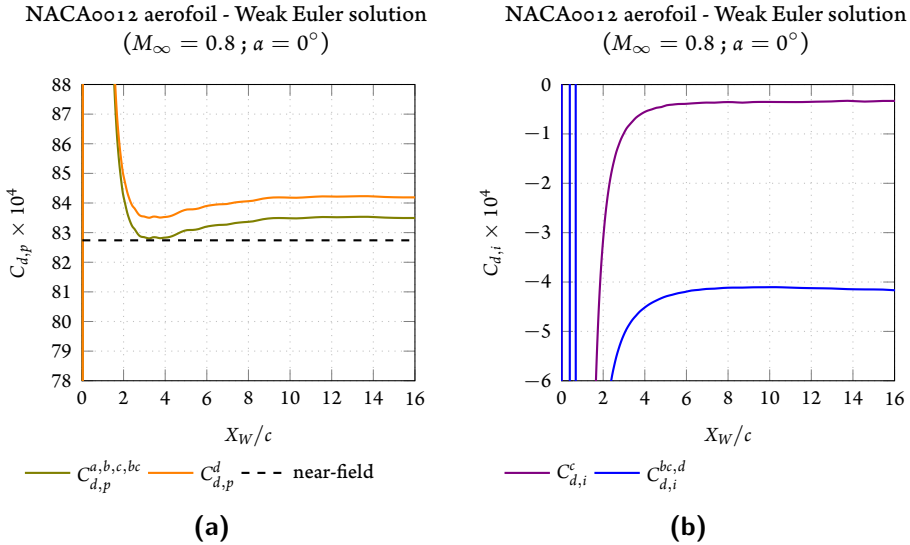


Figure B.1.30: Sensitivity of different Lamb vector-based formulations of lift-induced and parasite drag coefficients to the integration domain size, in inviscid transonic flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.). Fig. B.1.30a: parasite drag; Fig. B.1.30b: induced drag.

B.1.7 TRANSONIC VISCOUS FLOW

In this case both viscous and wave drag are present. Comments and observations discussed above for other cases with one effect active at the time (viscosity or shocks), similarly apply together to this viscous transonic case, with formulation “c” providing the best agreement with regards to near-field total lift and drag values (Figure B.1.31, B.1.32, B.1.33, B.1.34) and negative values observed for lift-induced drag coefficients of all analysed Lamb vector-based formulations (Figure B.1.35). A similar behaviour is observed, in this viscous transonic case, also at zero incidence angle (Figure B.1.36).

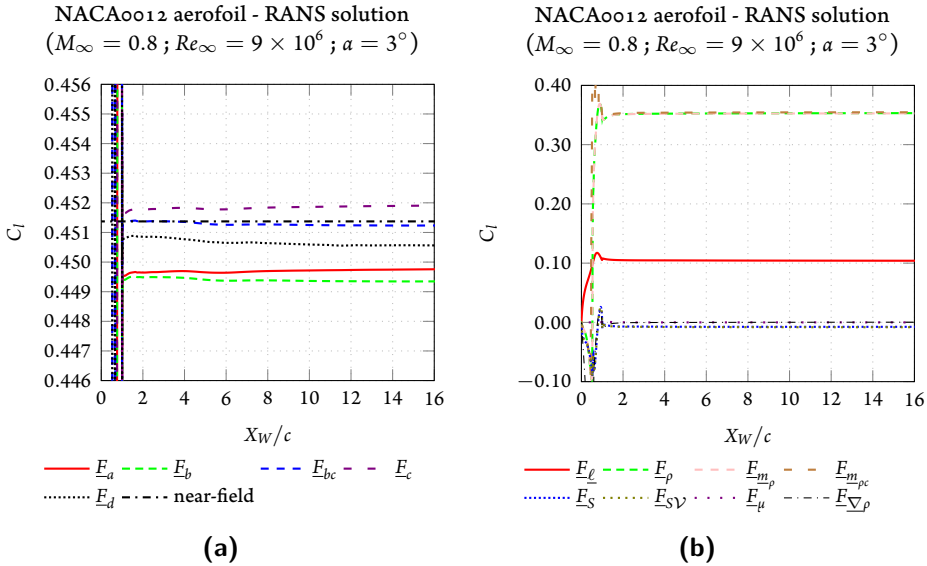


Figure B.1.31: Sensitivity to wake plane location of total force (Fig. B.1.31a) and individual constituents (Fig. B.1.31b) of different Lamb vector-based lift formulations, in viscous transonic flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

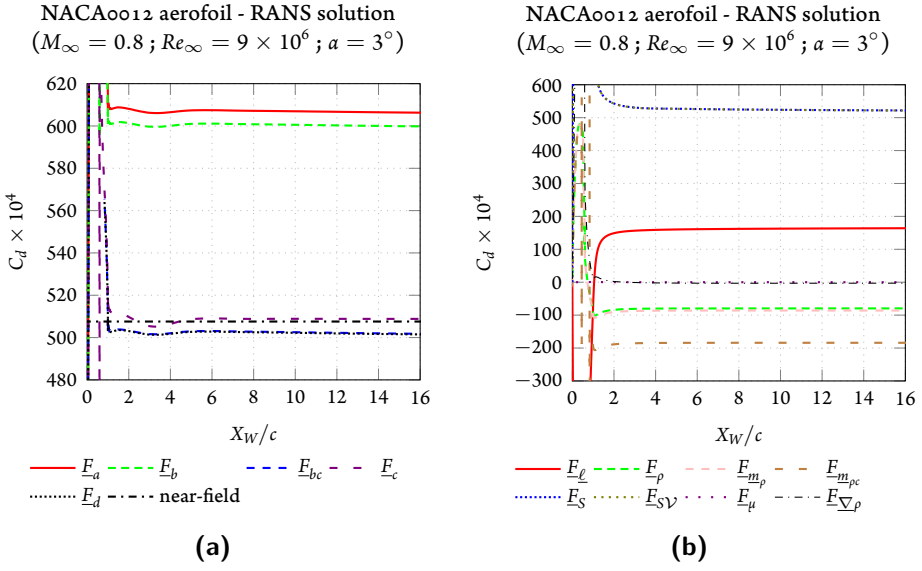


Figure B.1.32: Sensitivity to wake plane location of total force (Fig. B.1.32a) and individual constituents (Fig. B.1.32b) of different Lamb vector-based drag formulations, in viscous transonic flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.).

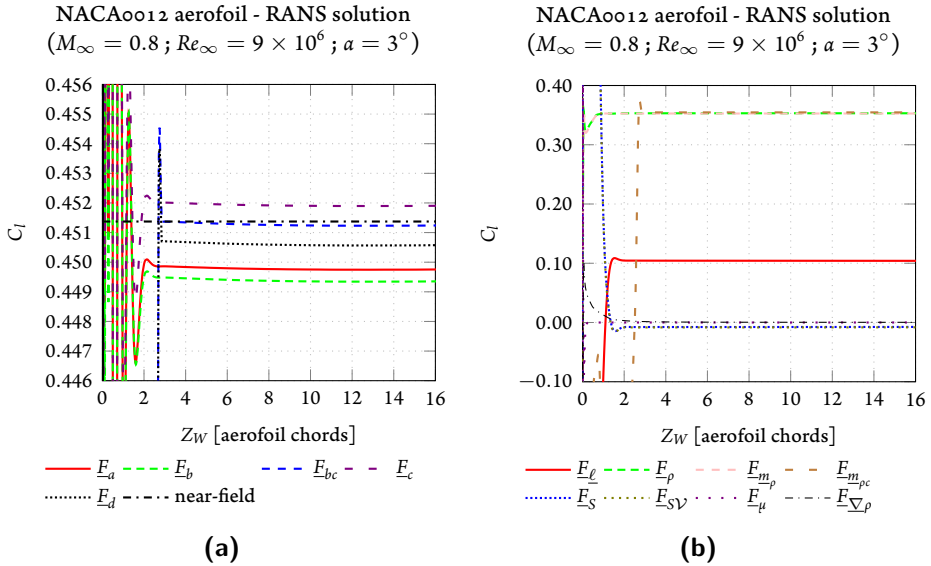


Figure B.1.33: Sensitivity to integration domain size, at fixed wake plane location, of total force (Fig. B.1.33a) and individual constituents (Fig. B.1.33b) of different Lamb vector-based lift formulations, in viscous transonic flow ($X_W \approx 16c$; fine mesh; pole at aerofoil l.e.).

Appendix B. Sensitivity study of the steady force formulas

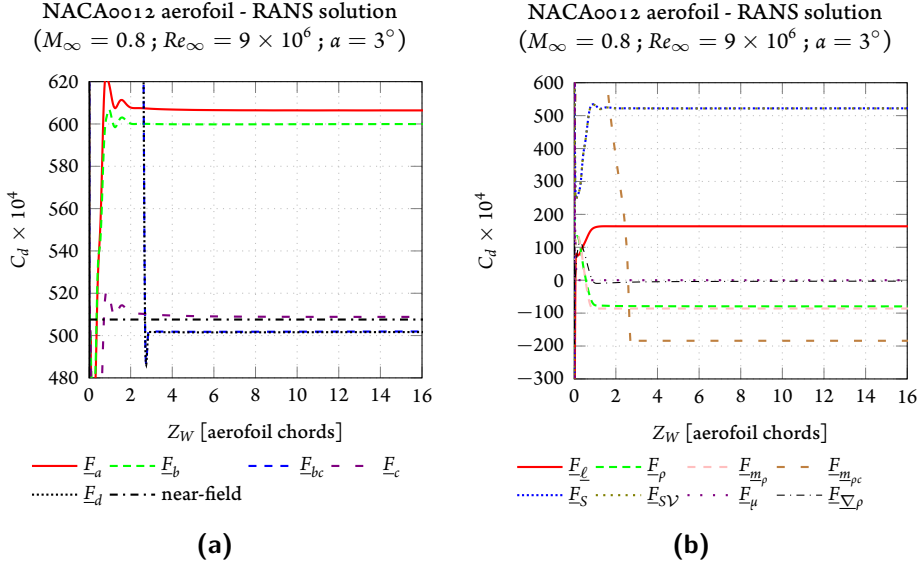


Figure B.1.34: Sensitivity to integration domain size, at fixed wake plane location, of total force (Fig. B.1.34a) and individual constituents (Fig. B.1.34b) of different Lamb vector-based drag formulations, in viscous transonic flow ($X_W \approx 16c$; fine mesh; pole at aerofoil l.e.).

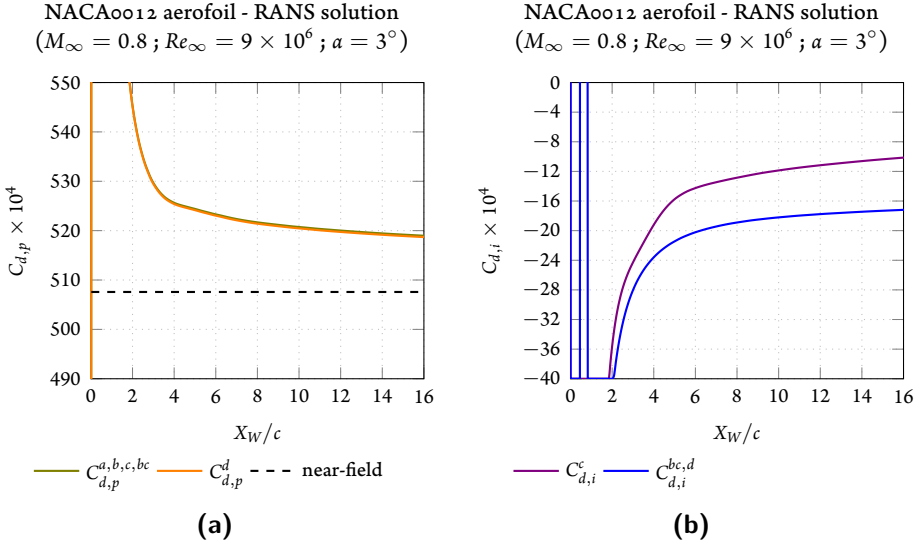


Figure B.1.35: Sensitivity of different Lamb vector-based formulations of lift-induced and parasite drag coefficients to the integration domain size, in viscous transonic flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.). Fig. B.1.35a: parasite drag; Fig. B.1.35b: induced drag.

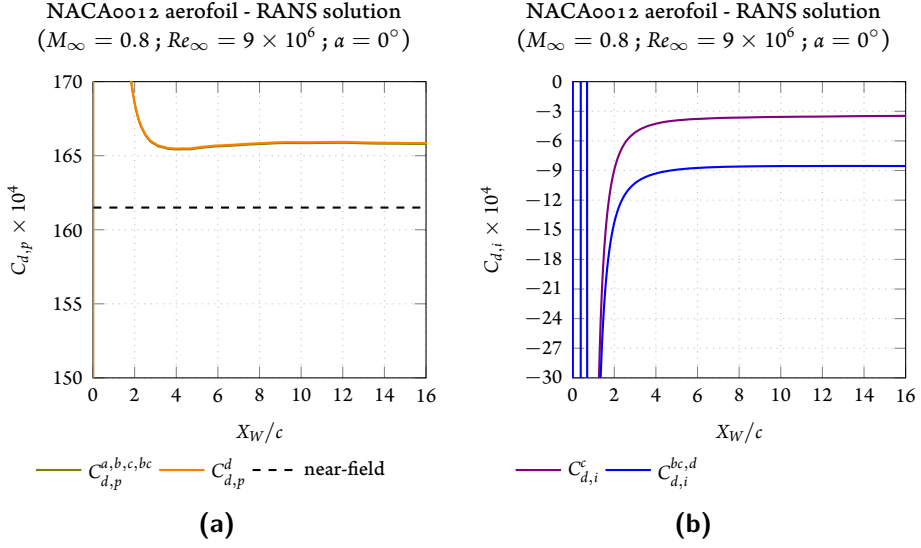


Figure B.1.36: Sensitivity of different Lamb vector-based formulations of lift-induced and parasite drag coefficients to the integration domain size, in viscous transonic flow ($Z_W \approx 15c$; fine mesh; pole at aerofoil l.e.). Fig. B.1.35a: parasite drag; Fig. B.1.36b: induced drag.

B.2 VARIATION OF THE DMT POLE LOCATION

The variation with the DMT pole coordinates of total lift and drag, as well as of the drag decomposition in lift-induced and parasite contributions, is herein reported for the analysed NACA0012 cases where non-negligible sensitivities were observed (asymmetric flow-fields).

Appendix B. Sensitivity study of the steady force formulas

B.2.1 NEARLY-INCOMPRESSIBLE INVISCID FLOW

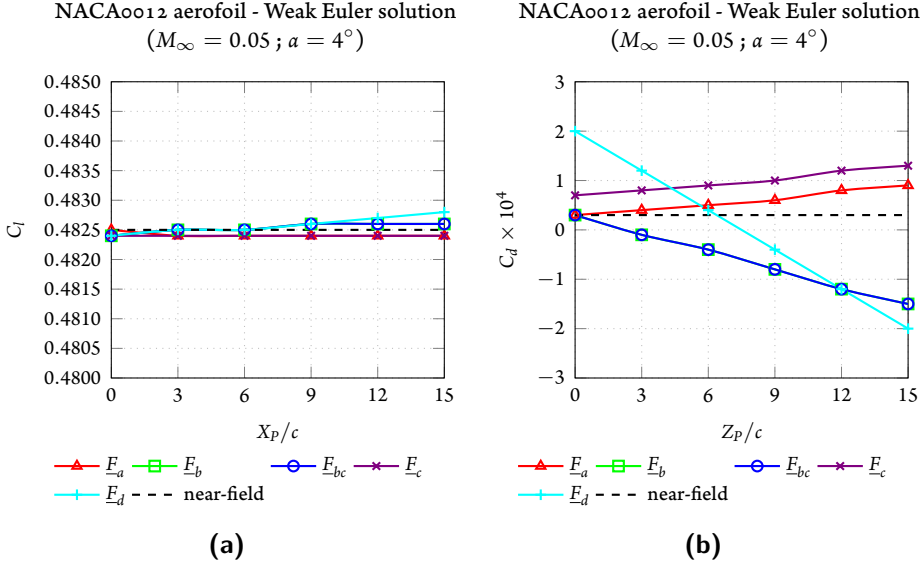


Figure B.2.1: Sensitivity to pole location of different Lamb vector-based formulations of total lift (Fig. B.2.1a) and total drag (Fig. B.2.1b), in inviscid nearly-incompressible flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh).

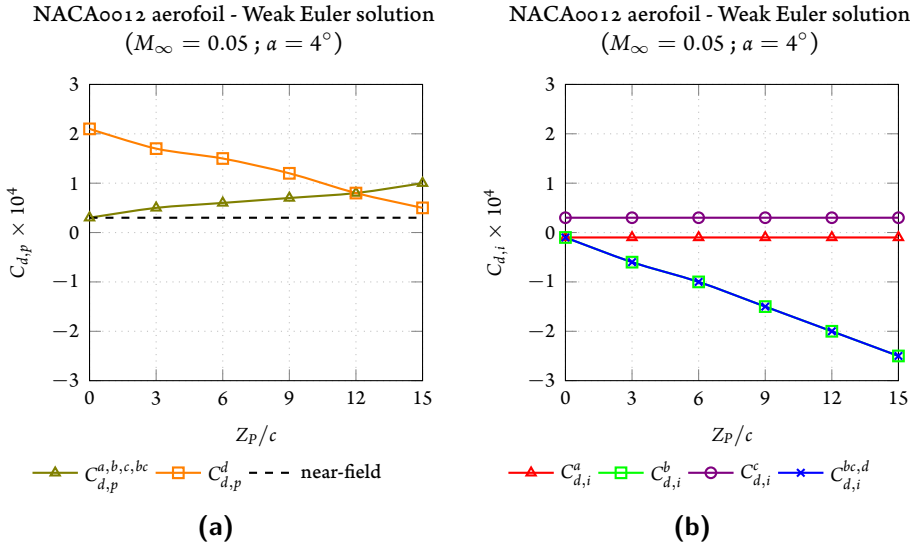


Figure B.2.2: Sensitivity to pole location of different Lamb vector-based formulations of parasite drag (Fig. B.2.2a) and lift-induced drag (Fig. B.2.2b), in inviscid nearly-incompressible flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh).

B.2.2 NEARLY-INCOMPRESSIBLE VISCOUS FLOW

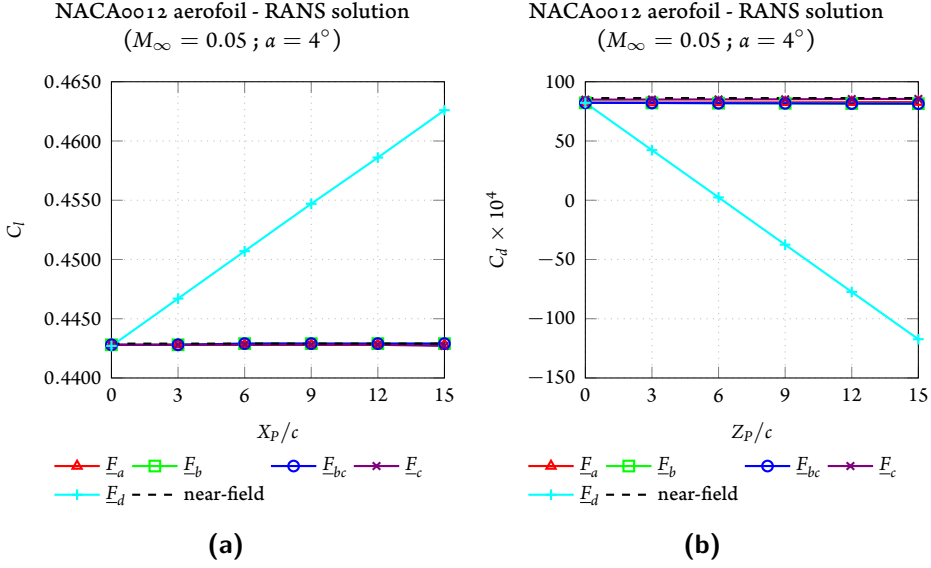


Figure B.2.3: Sensitivity to pole location of different Lamb vector-based formulations of total lift (Fig. B.2.3a) and total drag (Fig. B.2.3b), in viscous nearly-incompressible flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh).

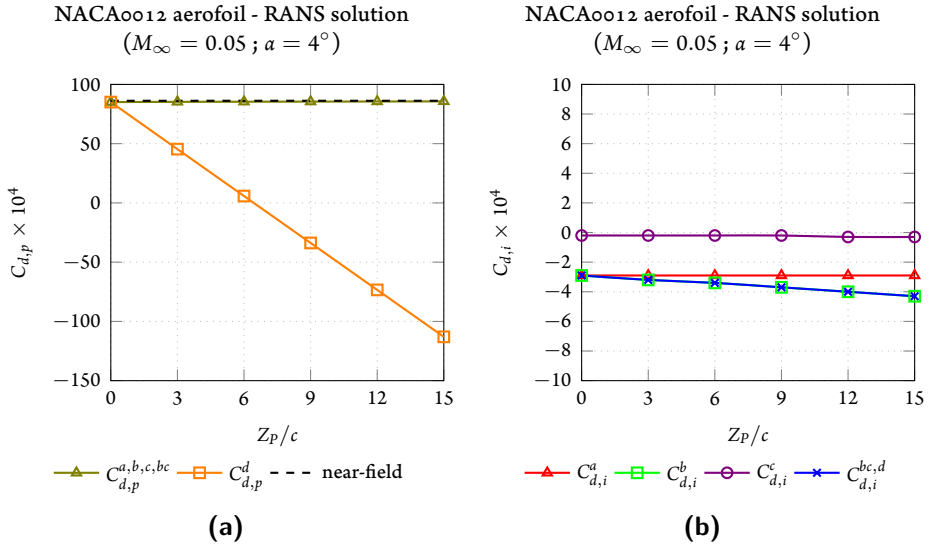


Figure B.2.4: Sensitivity to pole location of different Lamb vector-based formulations of parasite drag (Fig. B.2.4a) and lift-induced drag (Fig. B.2.4b), in viscous nearly-incompressible flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh).

Appendix B. Sensitivity study of the steady force formulas

B.2.3 SUBSONIC INVISCID FLOW

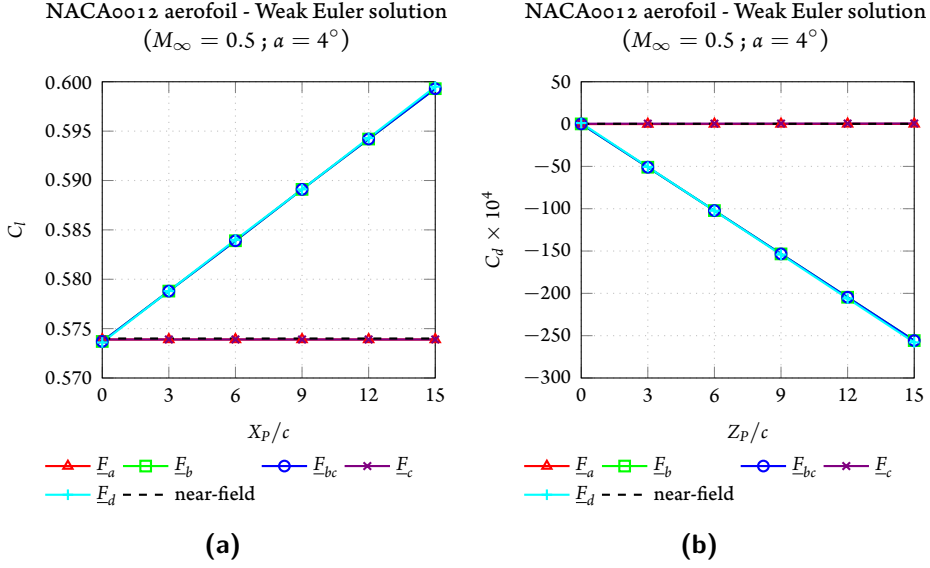


Figure B.2.5: Sensitivity to pole location of different Lamb vector-based formulations of total lift (Fig. B.2.5a) and total drag (Fig. B.2.5b), in inviscid subsonic flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh).

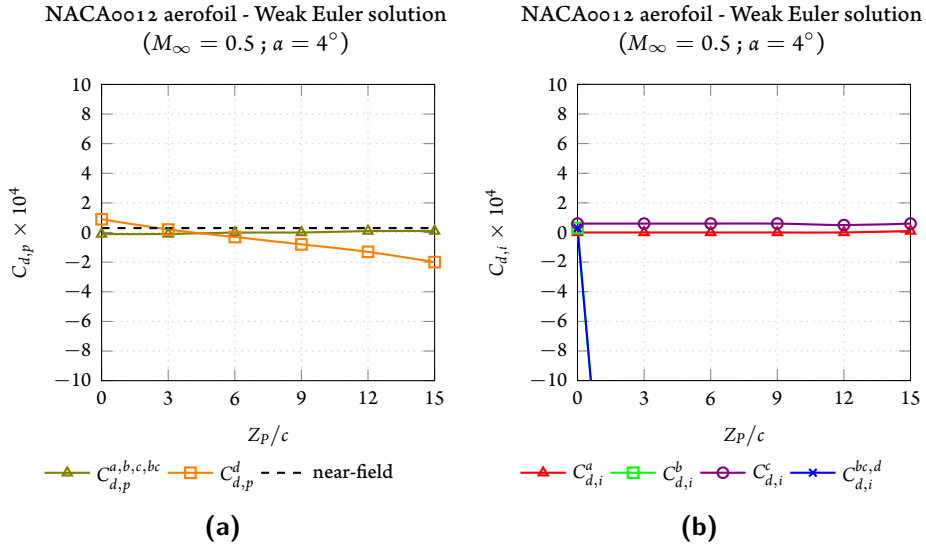


Figure B.2.6: Sensitivity to pole location of different Lamb vector-based formulations of parasite drag (Fig. B.2.6a) and lift-induced drag (Fig. B.2.6b), in inviscid subsonic flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh).

B.2.4 SUBSONIC VISCOUS FLOW (HIGH REYNOLDS NUMBER)

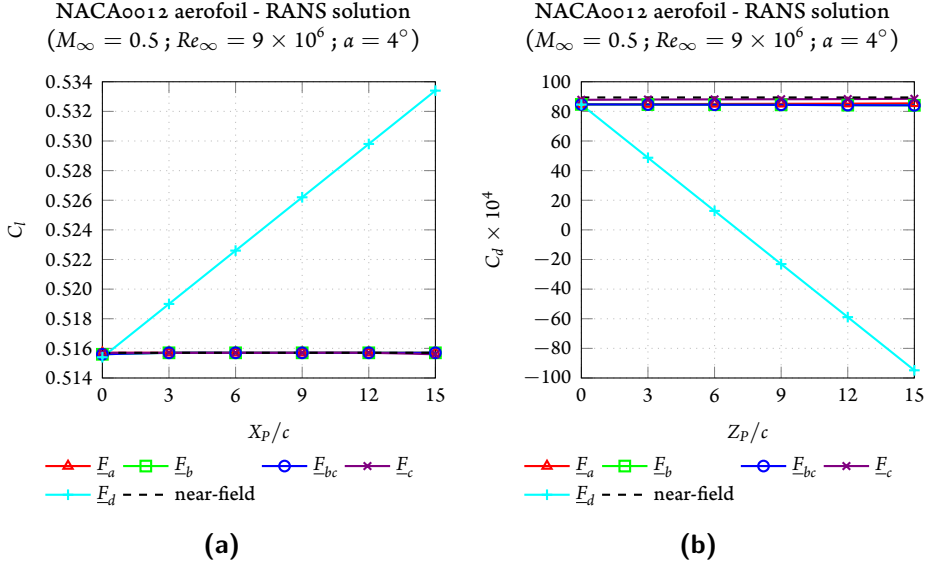


Figure B.2.7: Sensitivity to pole location of different Lamb vector-based formulations of total lift (Fig. B.2.7a) and total drag (Fig. B.2.7b), in viscous subsonic flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh).

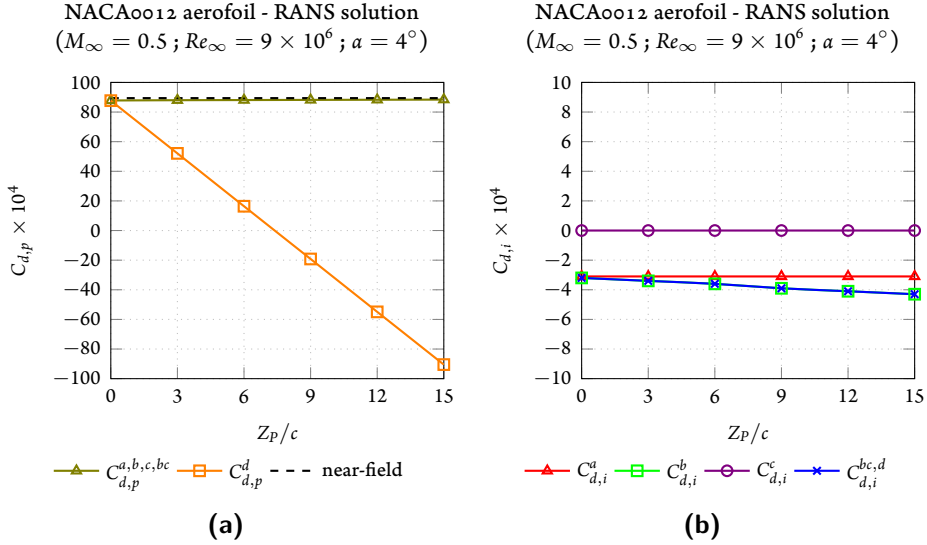


Figure B.2.8: Sensitivity to pole location of different Lamb vector-based formulations of parasite drag (Fig. B.2.8a) and lift-induced drag (Fig. B.2.8b), in viscous subsonic flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh).

Appendix B. Sensitivity study of the steady force formulas

B.2.5 SUBSONIC VISCOUS FLOW (LOW REYNOLDS NUMBER)

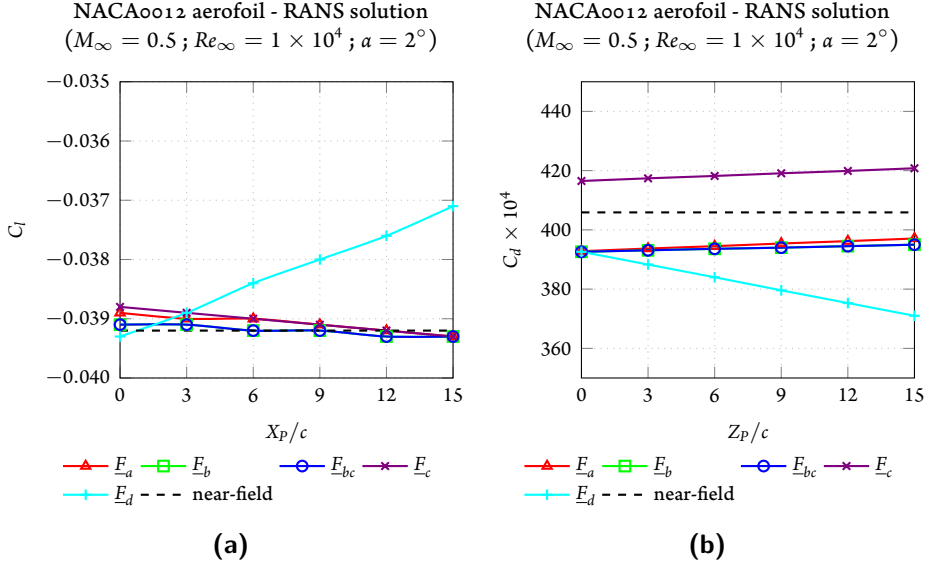


Figure B.2.9: Sensitivity to pole location of different Lamb vector-based formulations of total lift (Fig. B.2.9a) and total drag (Fig. B.2.9b), in viscous subsonic flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh).

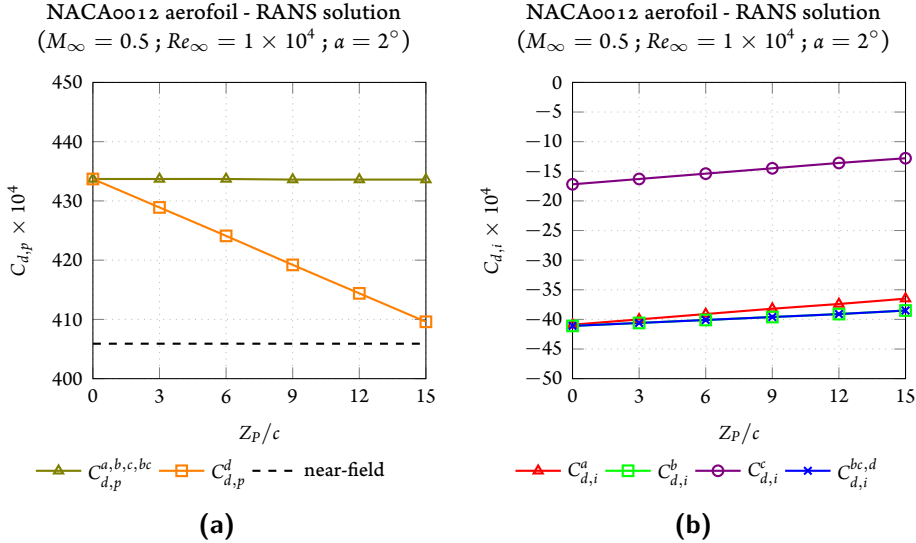


Figure B.2.10: Sensitivity to pole location of different Lamb vector-based formulations of parasite drag (Fig. B.2.10a) and lift-induced drag (Fig. B.2.10b), in viscous subsonic flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh).

B.2.6 TRANSONIC INVISCID FLOW

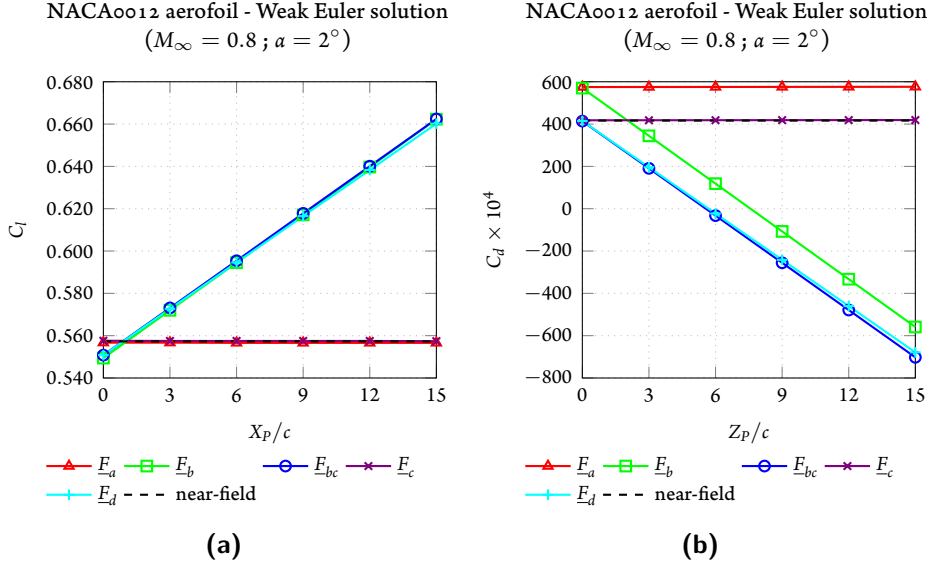


Figure B.2.11: Sensitivity to pole location of different Lamb vector-based formulations of total lift (Fig. B.2.11a) and total drag (Fig. B.2.11b), in inviscid transonic flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh).

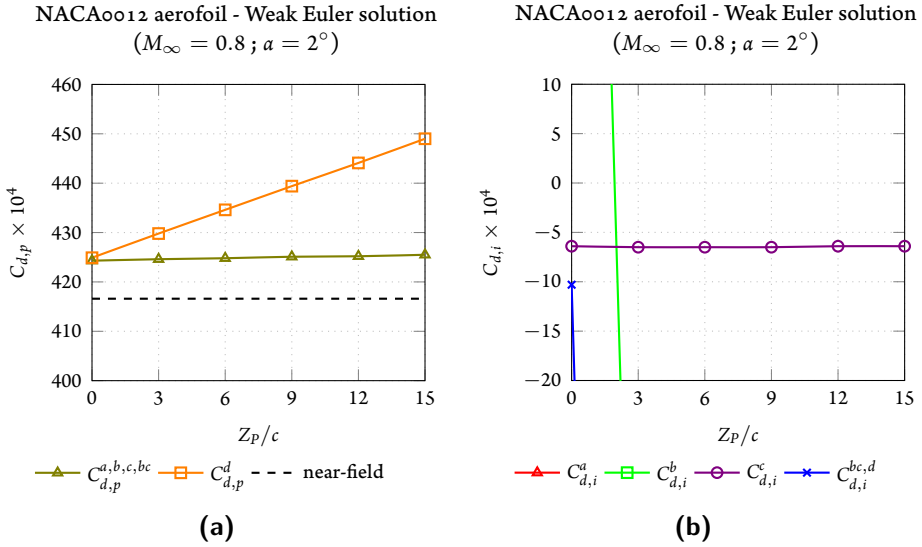


Figure B.2.12: Sensitivity to pole location of different Lamb vector-based formulations of parasite drag (Fig. B.2.12a) and lift-induced drag (Fig. B.2.12b), in inviscid transonic flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh).

B.2.7 TRANSONIC VISCOUS FLOW

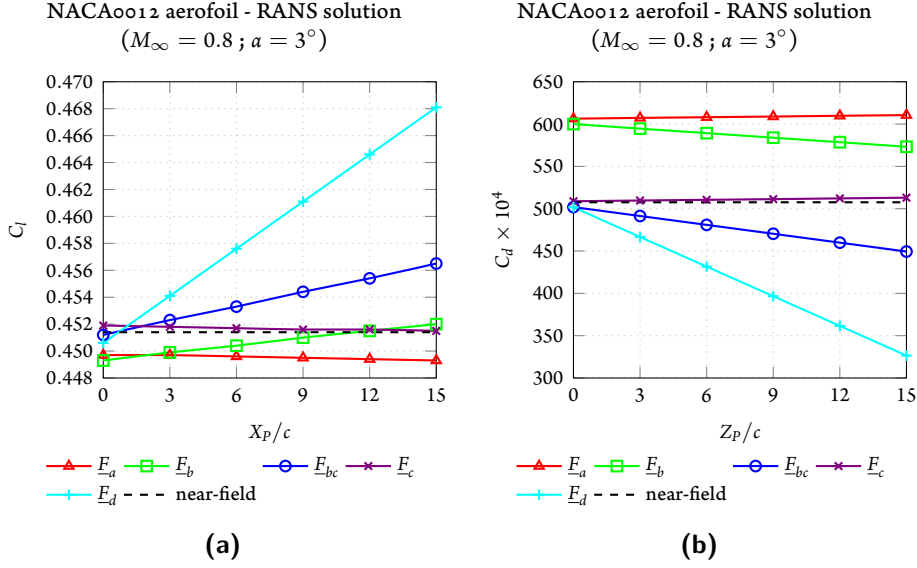


Figure B.2.13: Sensitivity to pole location of different Lamb vector-based formulations of total lift (Fig. B.2.13a) and total drag (Fig. B.2.13b), in viscous transonic flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh).

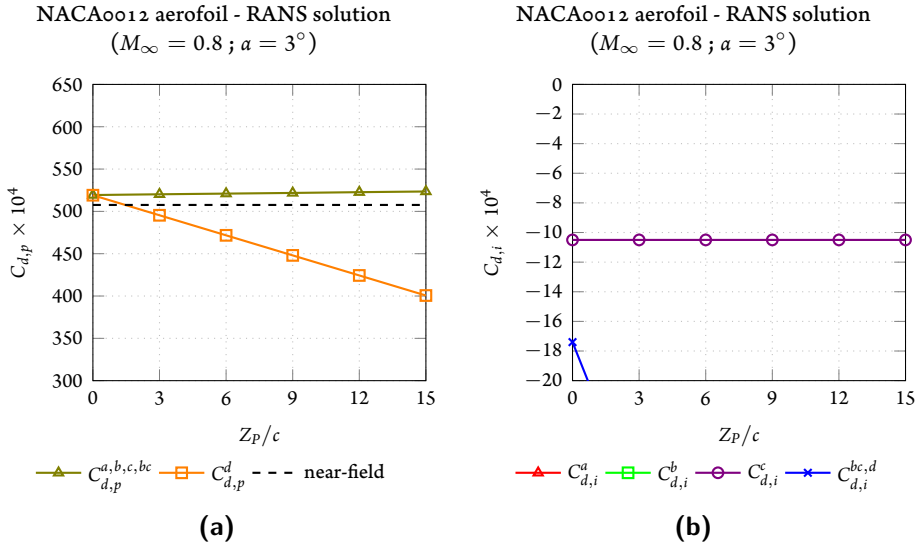


Figure B.2.14: Sensitivity to pole location of different Lamb vector-based formulations of parasite drag (Fig. B.2.8a) and lift-induced drag (Fig. B.2.8b), in viscous transonic flow ($X_W \approx 16c$; $Z_W \approx 15c$; fine mesh).



URANS solver validation in known flow conditions

For plunging oscillations in incompressible flow, benchmarking with reference data is possible, to validate the computed results.

Euler solutions can be compared to analytical results obtained by the linear theories of Theodorsen [194] and Garrick [61] (valid for small perturbations), as done by Sticchi et al. [188] using the same flow solver (*SU2*) as in the present study and detailed in Ch. 3, Sec. 3.2.3.

To validate unsteady *SU2* RANS solutions, instead, the selected test-case is a NACA0012 aerofoil undergoing sinusoidal plunging oscillations, with amplitude $\hat{h} = 0.175c$ and immersed in an incompressible and viscous flow at $Re_\infty = 2 \times 10^4$. Experiments were performed for this test-case by Heathcote et al. [70] using a free-surface closed-loop water tunnel and considering different values of the oscillation frequency. Numerical simulations are also available for the same test conditions from the work of Young and Lai [229] and, more recently, of Wu et al. [227] and Liu et al. [99]. The latter source is particularly relevant for the actual validation purposes as the authors used the *SU2* flow solver and provided benchmarking of their computed results against the experimental data from Heathcote et al. [70] and against numerical calculations performed with different flow solvers.

Nevertheless, as the results validated by Liu et al. [99] were obtained using a different version of the code (v7.0.7) and a different turbulence model (Sparlar-Allmaras) with regards to those

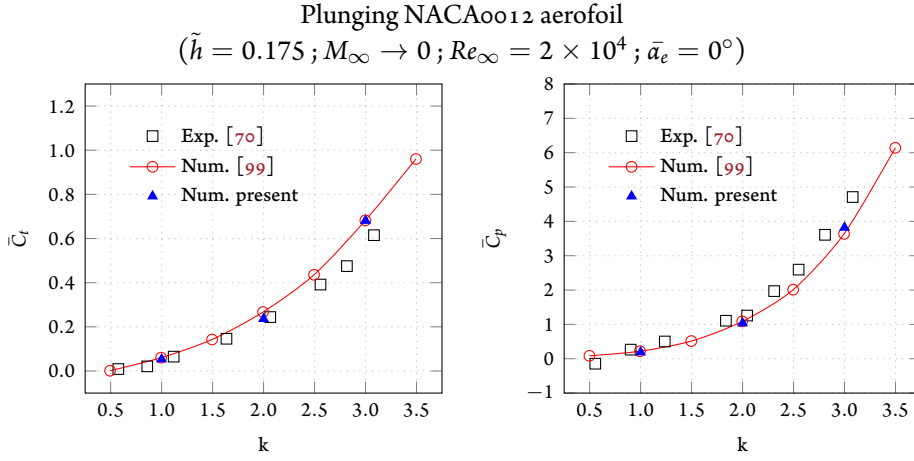


Figure C.0.1: Comparison of the present results with literature experimental and numerical data: non-dimensional time-averaged thrust acting on the NACA0012 aerofoil (left) and power input (right) sustaining the prescribed plunging oscillation.

adopted here (v7.3.1 and $k - \omega$ SST, see Sec. 3.2.1), a validation study for the actually employed RANS solver was also performed and included hereafter, considering three different values of the oscillation reduced frequency ($k = 1$, $k = 2$ and $k = 3$). The analysed time period is taken after ten computed oscillation cycles.

Results obtained by the current simulation methodology and flow solver are compared in Fig. C.o.1 and C.o.2 against the experimental data of Heathcote et al. [70] and the previously published numerical computations by Liu et al. [99]. A satisfactory agreement of predicted time-averaged thrust and power coefficients with literature data can be observed at all computed values of the oscillation frequency.

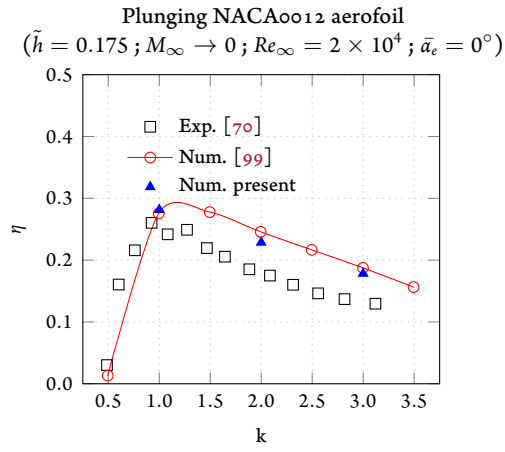


Figure C.0.2: Comparison of the present results with literature experimental and numerical data: propulsive efficiency of the oscillating NACA0012 aerofoil in pure plunging motion.

D

Unsteady grid convergence analyses

D.1 NEAR-FIELD FORCE IN INCOMPRESSIBLE LAMINAR FLOW

D.1.1 SPATIAL RESOLUTION STUDY

A grid sensitivity study on a plunging NACA0012 aerofoil was recently published [188] using the same flow solver, free-stream conditions and oscillation parameters. Hereafter an additional spatial grid convergence analysis is presented for the unsteady viscous simulation at unitary reduced frequency. The computational mesh illustrated in Fig. 3.2.2 (*fine* mesh) was coarsened down by taking one-over-two nodes (*medium* mesh) or one-over-four nodes (*coarse* mesh) along each of the two grid directions.

The mesh refinement study reports time histories of the computed lift and drag coefficients provided by the flow solver* (Fig. D.1.1,D.1.2) as well as the time-averaged drag coefficient ($\bar{C}_d$) obtained for the three grids over one complete oscillation (Fig. D.1.3). Here, the generic cell size was referred to the corresponding grid spacing on the fine mesh. The ratio between them identifies the three grid levels on the abscissa in Fig. D.1.3, where $h = 1$ represents the fine mesh. The

*In this and in the next appendix section, the near-field force will always be directly obtained as an output of the flow solver (i.e. it is not computed by the post-processor). According to the notation used in Ch. 2, a superscript ^{NF} should be used in force coefficients. Nevertheless, since no confusion would be generated in this case where no other force coefficients are considered, superscripts will be omitted to simplify the notation.

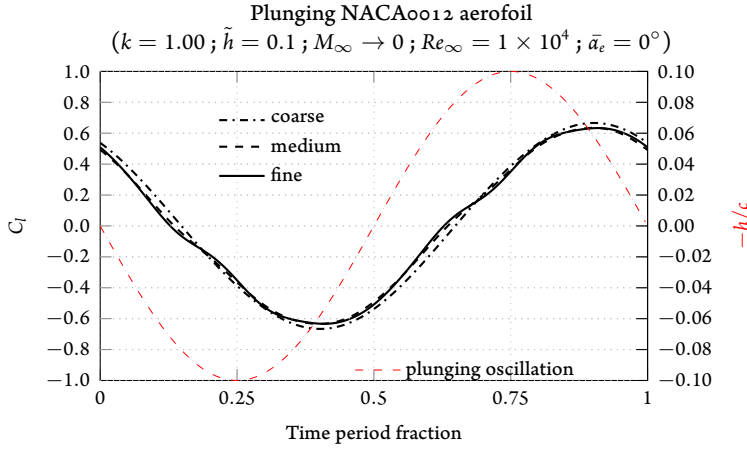


Figure D.1.1: Time history of computed lift coefficients, on different grid levels.

selected time period is the same used in Section 4.2.3 (i.e. the fifth oscillation cycle).

The Grid Convergence Index (GCI) methodology, first proposed by Roache in [163] and further described in [164, pp. 107-142][47], is used to analyse the mesh sensitivity of the averaged drag coefficient. As described in [180], the GCI is a measure of the percentage by which the computed value is away from the asymptotic numerical value, theoretically obtained on an infinitely-refined grid. It also indicates how much the solution would change with a further refinement of the mesh. The outcome of the grid convergence analysis is reported in Table D.1.1, where the asymptotic range of convergence of the monitored quantity ($\bar{C}_d$) is also provided, allowing to verify that its value is approximately one, as required by the adopted methodology [163, 180].

The fine mesh level was therefore selected in this work as a good compromise between flow solution accuracy and computational cost.

Table D.1.1: Unsteady grid convergence data.

Grid Level	Mesh Size [cells]	$\bar{C}_d$	Order of Convergence	Richardson's extrapolation	GCI	Asymptotic Range of Convergence Indicator
coarse	22459	0.0166	2.2	0.0127	7.95%	0.950
medium	91050	0.0135				
fine	364200	0.0129			1.84%	

D.2 NEAR-FIELD FORCE IN SUBSONIC TURBULENT FLOW

D.2.1 SPATIAL RESOLUTION STUDY

Hereafter a grid convergence analysis is presented for the unsteady viscous simulation at $M_\infty = 0.5$ (Tab. 3.2.1, Sec. 3.2.1). The computational mesh illustrated in Fig. 3.2.2 (*fine* mesh) was

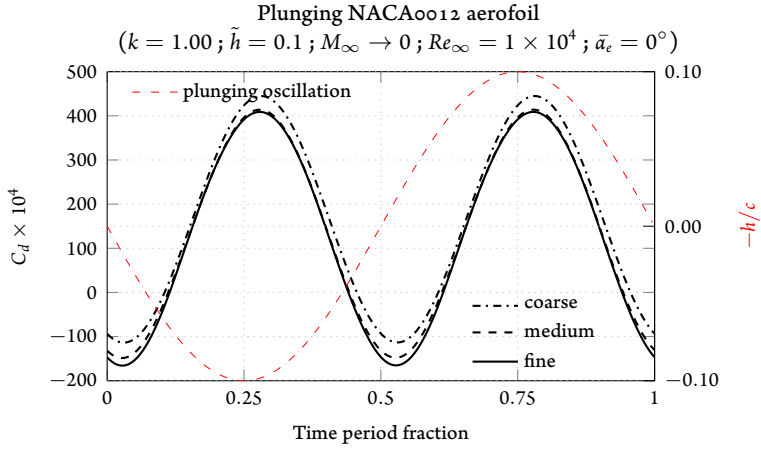


Figure D.1.2: Time history of computed drag coefficients, on different grid levels.

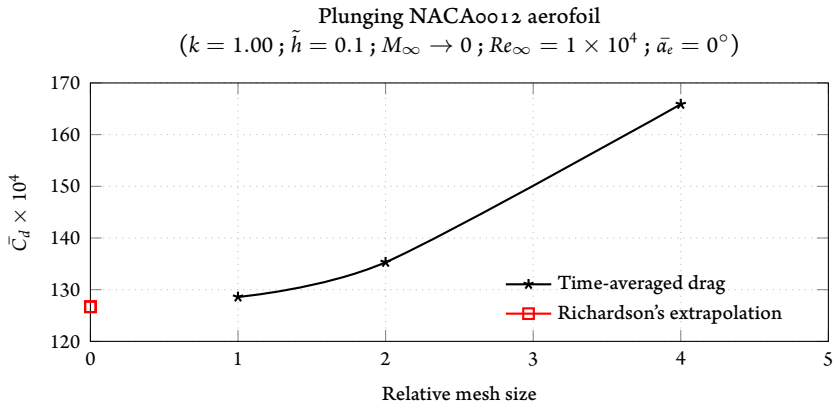


Figure D.1.3: Grid convergence of the time-averaged drag coefficient.

Appendix D. Unsteady grid convergence analyses

coarsened down by keeping one-every-two nodes (*medium* mesh) or one-every-four nodes (*coarse* mesh) along each of the two grid directions (Fig. D.2.1).

Considering the plunging aerofoil as a propulsion system, its performance is characterised by the time-averaged thrust coefficient generated on the aerofoil:

$$\bar{C}_t = \frac{1}{T} \int_t^{t+T} C_t dt = -\frac{1}{T} \int_t^{t+T} C_d dt$$

as a consequence of the provided power input to sustain the oscillatory motion, which for pure plunging is expressed, in non-dimensional form, by:

$$\bar{C}_p = \frac{1}{T} \int_t^{t+T} C_p dt = \frac{1}{T} \int_t^{t+T} C_l \frac{\dot{h}}{V_\infty} dt$$

resulting in a propulsive efficiency of the system quantified by $\eta = \bar{C}_t / \bar{C}_p$.

The mesh refinement study reports time histories of the instantaneous power and thrust coefficients, computed from lift and drag forces provided by flow solver (Fig. D.2.2, D.2.3), as well as the time-averaged thrust coefficient ($\bar{C}_t$) and propulsive efficiency (η) obtained for the three grids over one complete oscillation (Fig. D.2.4). Here, the generic cell size was referred to the corresponding grid spacing on the fine mesh. The ratio between them identifies the three grid levels on the abscissa in Fig. D.2.4, where $h = 1$ represents the fine mesh. The selected time period is the fifth computed oscillation cycle.

A converging trend of computed results as the mesh is refined can be observed in all figures. Then, the Grid Convergence Index (GCI) methodology, first proposed by Roache in [163] and further described in [164, pp. 107-142][47], is used to quantify the mesh sensitivity of the averaged thrust coefficient. As detailed in [180], the GCI is a measure of the percentage by which the computed value is away from the asymptotic numerical value, theoretically obtained on an infinitely-refined grid. It also indicates how much the solution would change with a further refinement of the mesh. The outcome of the grid convergence analysis is reported in Table D.2.1, where the asymptotic range of convergence of the monitored quantity ($\bar{C}_t$) is also provided, allowing to verify that its value is approximately one, as required by the adopted methodology [163, 180].

The fine mesh level was therefore selected in this work as a good compromise between flow solution accuracy and computational cost.

Table D.2.1: Unsteady spatial grid convergence data.

Spatial Grid Level	Mesh Size [cells]	$\bar{C}_t$	Order of Convergence	Richardson's extrapolation	GCI	Asymptotic Range of Convergence Indicator
coarse	30350	0.0213	1.7	0.0234	3.47%	1.019
medium	121400	0.0228				
fine	485600	0.0232			1.04%	

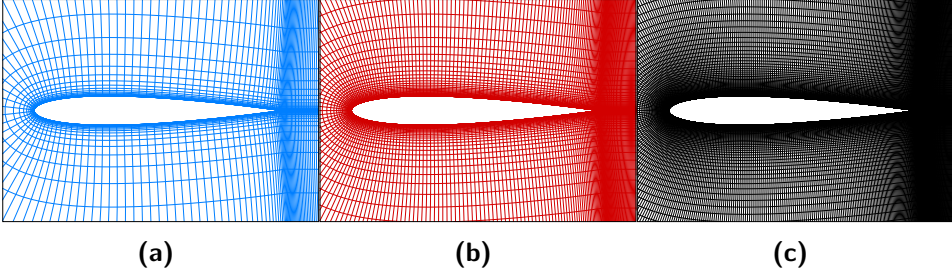


Figure D.2.1: Zoomed view, in the aerofoil region, of the coarse (a), medium (b) and fine (c) meshes used for the spatial grid convergence analysis.

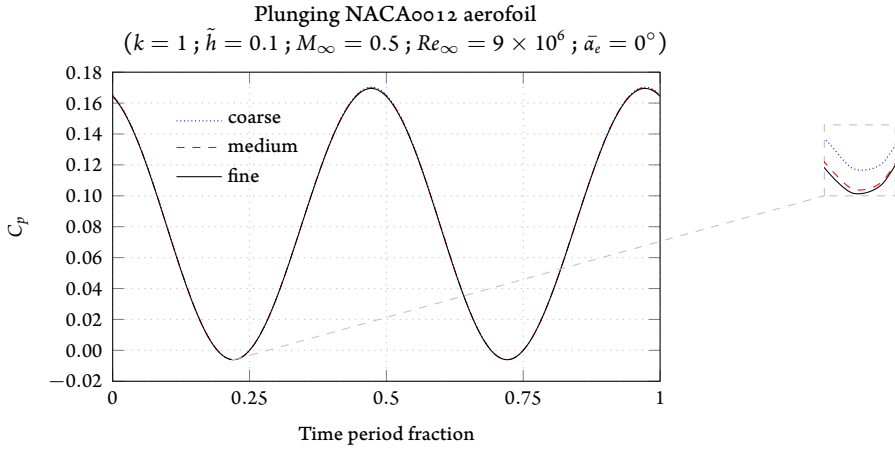


Figure D.2.2: Time histories of computed power coefficients, on different grid levels.

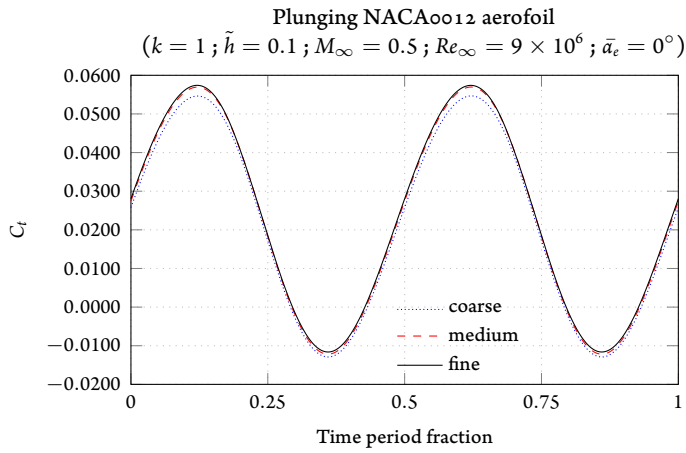


Figure D.2.3: Time histories of computed thrust coefficients, on different grid levels.

Appendix D. Unsteady grid convergence analyses

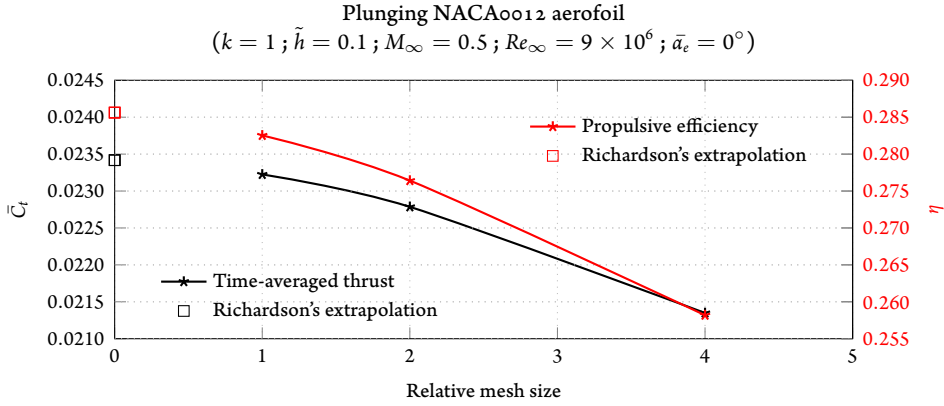


Figure D.2.4: Spatial grid convergence of the time-averaged thrust coefficient and propulsive efficiency.

D.2.2 TIME RESOLUTION STUDY

For the same flow conditions used in the spatial resolution study (viscous case at $M_\infty = 0.5$, Tab. 3.2.1, Sec. 3.2.1) and using the fine mesh (485600 cells), the sensitivity of computed results to the physical time step size (Δt) was analysed considering three different values: $\Delta t = \mathcal{T}/500$, $\Delta t = \mathcal{T}/250$ and $\Delta t = \mathcal{T}/125$.

Time histories of thrust and power coefficients computed after four complete oscillation cycles (Fig. D.2.6 and Fig. D.2.5) indicate a very slight sensitivity of computed results to the time step size, as evidenced also by the trends of time-averaged quantities (Fig. D.2.7), with grid convergence indices well below 1% already for the medium time step grid ($\mathcal{T}/250$, Table D.2.2). Nevertheless, for unsteady simulations included in Sec. 3.2.1, the smaller time step was selected ($\mathcal{T}/500$).

Table D.2.2: Unsteady temporal grid convergence data.

Temporal Grid Level	Time Step Size	$\bar{C}_t$	Order of Convergence	Richardson's extrapolation	GCI	Asymptotic Range of Convergence Indicator
coarse	$\mathcal{T}/125$	0.0231	0.7	0.0233	0.51%	1.002
medium	$\mathcal{T}/250$	0.0232				
fine	$\mathcal{T}/500$	0.0232			0.32%	

D.3 FAR-FIELD FORCE IN SUBSONIC TURBULENT FLOW

The aim of the study reported herein is to analyse and to evaluate the impact of spatial grid refinements on the integral momentum balance which (analytically) guarantees the equivalence between near-field and far-field force expressions. Considering the unsteady viscous case at $M_\infty = 0.5$ (Tab. 3.2.1, Sec. 3.2.1), URANS simulations were run for the three grid levels already described in Sec. D.2.1. Then, near-field forces obtained from surface stresses integration were com-

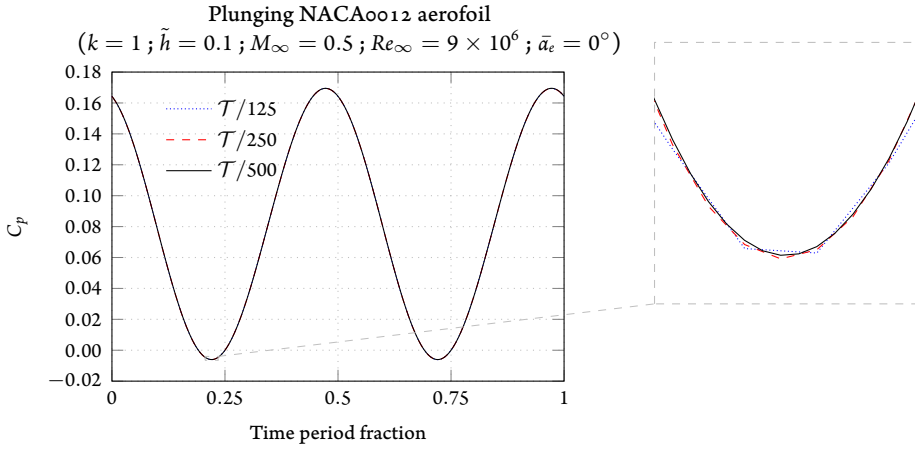


Figure D.2.5: Time histories of computed power coefficients, for different time steps.

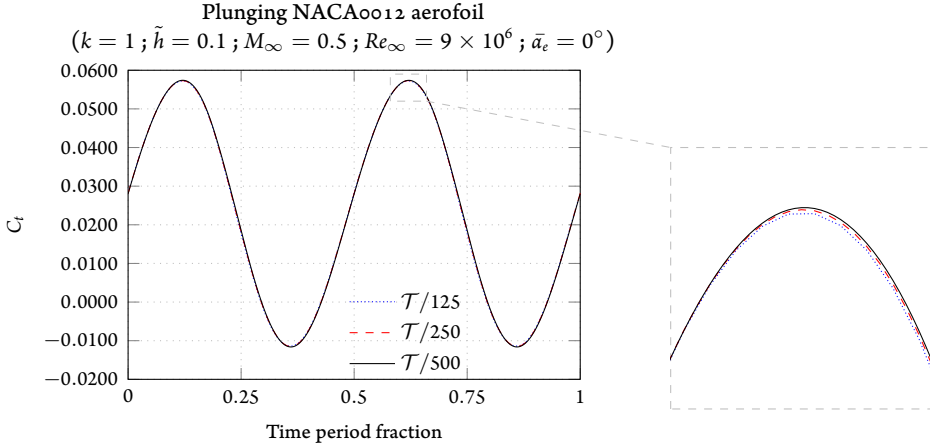


Figure D.2.6: Time histories of computed thrust coefficients, for different time steps.

Appendix D. Unsteady grid convergence analyses

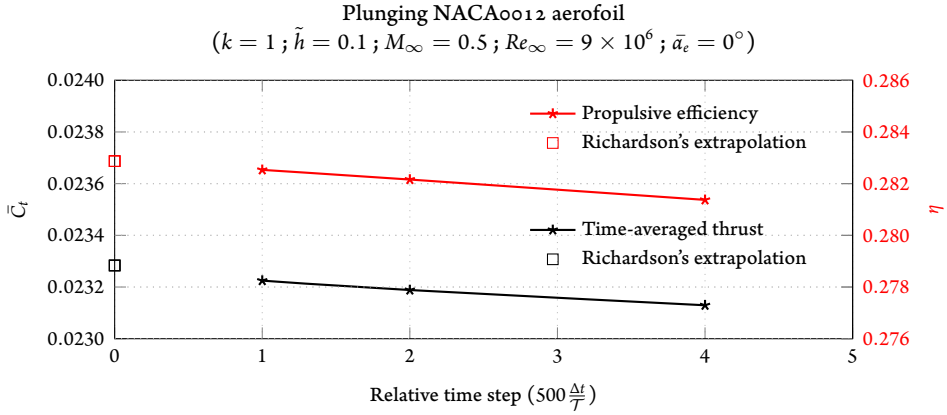


Figure D.2.7: Temporal grid convergence of the time-averaged thrust coefficient and propulsive efficiency.

pared to the corresponding far-field data, computed using the three different formulas illustrated in Sec. 3.1 (steady variants) and selecting the non-inertial reference frame along with the alternative unsteady terms expression ($\underline{F}_t^*$). Similarly to what done in Sec. 3.2.5, the post-processing considered the time period starting after ten complete aerofoil oscillations and a control volume sized by $X_W = 1.0$ and $Z_W = 0.2$. Lift and drag time series standard deviation defined by Eq. (3.57) were used to measure the distance of unsteady force results (far-field vs. near-field). The convergence of one force method onto the other can be appreciated from data included in Table D.3.1, where standard deviations computed on the three grid levels (coarse, medium and fine) are reported. Nevertheless, different sensitivities are observed for the three compared formulations, with $\underline{F}_c^*$ matching near-field data by a few drag counts difference already on the coarse mesh. Other formulations integrating spatial gradients of flow variables in the CV interior ($\underline{F}_a^*$ and $\underline{F}_b^*$) are much more sensitive to the spatial grid resolution.

Table D.3.1: Standard deviation of far-field force coefficient time series, with regards to near-field data, in viscous subsonic flow and on different grid levels.

Formulation ¹	fine		medium		coarse	
	σ_{C_l}	σ_{C_d}	σ_{C_l}	σ_{C_d}	σ_{C_l}	σ_{C_d}
$\underline{F}_a^*$	4.75e-04	3.99e-04	1.68e-03	9.31e-04	7.12e-03	1.62e-03
$\underline{F}_b^*$	5.11e-04	4.28e-04	1.83e-03	1.05e-03	7.06e-03	2.01e-03
$\underline{F}_c^*$	4.74e-04	2.61e-04	1.68e-03	3.81e-04	7.16e-03	4.43e-04

¹ Non-inertial reference frame.

E

Turbulent flow regime

E.1 VORTICITY-BASED LAMB VECTOR METHOD

Mele and Tognaccini [119] extended the validity of Eq. (4.10) to turbulent flows governed by the Reynolds-Averaged Navier-Stokes (RANS) equations. They focused on statistically stationary flows, although a generalisation to unsteady mean flows and to non-inertial frames of reference is straight-forward and introduced hereafter. Expressing a random property f as $f = \bar{f} + f'$ or $f = \tilde{f} + f''$, where $\bar{f}$ and $\tilde{f}$ are, respectively, the *ensemble* and *Favre* average of f (the latter defined as: $\tilde{f} = \overline{\rho f / \bar{\rho}}$), while f' and f'' are the corresponding fluctuations, the formula expressing the mean far-field force was proved to hold in the same form of Eq. (4.10), i.e.:

$$\bar{\mathbf{F}}_b^{vrt} = \bar{\mathbf{F}}_{\underline{\ell}}^{vrt} + \bar{\mathbf{F}}_{\underline{m}_p} + \bar{\mathbf{F}}_t^* + \bar{\mathbf{F}}_S^{vrt} + \bar{\mathbf{F}}_{\underline{\mu}} + \bar{\mathbf{F}}_m \quad (\text{E1})$$

where the averaged individual terms on the right-hand side are defined from their instantaneous expressions (Eqs. (4.8), (4.3), (4.4), (4.9), (4.7)) replacing ρ , $\underline{\mathbf{V}}$ and $\underline{\omega}$ with their corresponding ensemble (for mass density) or Favre (for fluid velocity and vorticity) averages, while $\bar{\mathbf{F}}_{\underline{\mu}}$ is defined including turbulent fluctuations as:

$$\begin{aligned} \bar{\mathbf{F}}_{\underline{\mu}} = & \int_{\Sigma} \underline{\mathbf{r}} \times \left[\underline{\mathbf{n}} \times \nabla \cdot \left(\underline{\underline{\tau}}_{\underline{\nu}} + \underline{\underline{\tau}}_{\underline{\mathbf{R}}} \right) \right] dS + \int_{\Sigma} \left(\underline{\underline{\tau}}_{\underline{\nu}} + \underline{\underline{\tau}}_{\underline{\mathbf{R}}} \right) \cdot \underline{\mathbf{n}} dS + \\ & + \int_{\mathcal{V}} \underline{\mathbf{r}} \times \left(\underline{\nabla} \times \tilde{\underline{\delta}}_{\rho V} \right) d\mathcal{V} \end{aligned} \quad (\text{E2})$$

Appendix E. Turbulent flow regime

with $\underline{\tau}_R = -\overline{\rho \underline{V}'' \underline{V}''}$ indicating the Reynolds stress tensor and $\underline{\tilde{\delta}}_{\rho V} = \overline{\underline{V}'' \underline{\nabla} \cdot (\rho \underline{V}'')}$.

Therefore, when referring to the turbulent formula in the document, average symbols will be omitted (except for turbulent fluctuation terms) to simplify the notation.

E.2 THERMODYNAMIC-BASED LAMB VECTOR METHOD

Crocco's equation for turbulent flows can be written as:

$$\bar{\rho} \frac{\partial \tilde{V}}{\partial t} + \bar{\rho} \underline{\ell} = \bar{\rho} \left(\tilde{T} \underline{\nabla} \tilde{s} + \widetilde{T'' \underline{\nabla} s''} - \underline{\nabla} \tilde{H} \right) + \underline{\nabla} \cdot \underline{\tilde{\tau}}_v + \tilde{\rho} \underline{\tilde{f}}_{-m} \quad (\text{E}_3)$$

where the non-linear product of static temperature and entropy gradient generates the second term on the right-hand side containing turbulent fluctuations.

To extend the thermodynamic-based Lamb vector method to turbulent flows, we take the Reynolds average of Eq. (4.13):

$$\bar{\underline{F}}_b^{thd} = \bar{\underline{F}}_{\underline{\ell}}^{thd} + \bar{\underline{F}}_{m_p} + \bar{\underline{F}}_t^* + \bar{\underline{F}}_S^{thd} + \bar{\underline{F}}_{\mu} + \bar{\underline{F}}_m \quad (\text{E}_4)$$

and proceed in a similar way as in [119] to calculate the individual terms on the right-hand side averaging their instantaneous expressions (Eqs. (4.8),(4.3),(4.4),(4.9),(4.6),(4.7)) and substituting $\bar{\rho} \underline{\ell}$ with its equivalent expression obtained through Eq. (E3). For example, for the first term, we have:

$$\begin{aligned} \bar{\underline{F}}_{\underline{\ell}}^{thd} &= -\int_{\mathcal{V}} \rho \underline{\ell} d\mathcal{V} - \int_{\partial B} \frac{\rho V^2}{2} \underline{n} dS = -\int_{\mathcal{V}} \bar{\rho} \underline{\ell} d\mathcal{V} - \int_{\partial B} \frac{\bar{\rho} V^2}{2} \underline{n} dS = \\ &= -\int_{\mathcal{V}} \left[\bar{\rho} \left(\tilde{T} \underline{\nabla} \tilde{s} + \widetilde{T'' \underline{\nabla} s''} - \underline{\nabla} \tilde{H} \right) + \underline{\nabla} \cdot \underline{\tilde{\tau}}_v + \tilde{\rho} \underline{\tilde{f}}_{-m} - \bar{\rho} \frac{\partial \tilde{V}}{\partial t} \right] d\mathcal{V} + \\ &\quad - \int_{\partial B} \frac{\bar{\rho} \tilde{V}^2}{2} \underline{n} dS \end{aligned} \quad (\text{E}_5)$$

where, in the body skin integral, turbulent velocity fluctuations from $\tilde{V}^2 = \tilde{V}^2 + \widetilde{\underline{V}'' \cdot \underline{V}''}$ were not included as null at the wall. For the compressibility correction term, taking into account that

$\underline{r} \times \left[\underline{\nabla} \rho \times \underline{\nabla} \left(\frac{V^2}{2} \right) \right] = \underline{r} \times \left\{ \underline{\nabla} \times \left[\rho \underline{\nabla} \left(\frac{V^2}{2} \right) \right] \right\}$, we can write:

$$\begin{aligned}
 \bar{\underline{F}}_{\underline{m}_\rho} &= - \overline{\int_{\mathcal{V}} \underline{r} \times \left\{ \underline{\nabla} \times \left[\rho \underline{\nabla} \left(\frac{V^2}{2} \right) \right] \right\} d\mathcal{V}} - \overline{\int_{\partial B} \underline{r} \times \left[\underline{n} \times \left(\frac{V^2}{2} \underline{\nabla} \rho \right) \right] dS} = \\
 &= - \int_{\mathcal{V}} \underline{r} \times \left\{ \underline{\nabla} \times \left[\bar{\rho} \underline{\nabla} \left(\frac{\tilde{V}^2}{2} \right) \right] \right\} d\mathcal{V} - \int_{\mathcal{V}} \underline{r} \times [\underline{\nabla} \times (\bar{\rho} \underline{\nabla} k_t)] d\mathcal{V} + \\
 &\quad - \int_{\partial B} \underline{r} \times \left[\underline{n} \times \left(\frac{\tilde{V}^2}{2} \underline{\nabla} \bar{\rho} \right) \right] dS = \\
 &= - \int_{\mathcal{V}} \underline{r} \times \left[\underline{\nabla} \bar{\rho} \times \underline{\nabla} \left(\frac{\tilde{V}^2}{2} \right) \right] d\mathcal{V} - \int_{\mathcal{V}} \underline{r} \times [\underline{\nabla} \times (\bar{\rho} \underline{\nabla} k_t)] d\mathcal{V} + \\
 &\quad - \int_{\partial B} \underline{r} \times \left[\underline{n} \times \left(\frac{\tilde{V}^2}{2} \underline{\nabla} \bar{\rho} \right) \right] dS \tag{E6}
 \end{aligned}$$

where $k_t = \left(\widetilde{\underline{V}'' \cdot \underline{V}''} \right) / 2$ is the turbulent kinetic energy, with velocity fluctuations now contributing off the wall through the volume integral and still not on the body skin.

For the averaged outer vorticity contribution ($\bar{\underline{F}}_S^{thd}$), we have:

$$\begin{aligned}
 \bar{\underline{F}}_S^{thd} &= - \overline{\int_{\Sigma} \underline{r} \times (\underline{n} \times \rho \underline{\ell}) dS} = - \int_{\Sigma} \underline{r} \times (\underline{n} \times \bar{\rho} \underline{\ell}) dS = \\
 &= - \int_{\Sigma} \underline{r} \times \left\{ \underline{n} \times \left[\bar{\rho} \left(\widetilde{\tilde{T} \underline{\nabla} \tilde{s}} + \widetilde{T'' \underline{\nabla} s''} - \underline{\nabla} \tilde{H} \right) + \underline{\nabla} \cdot \underline{\tilde{\tau}}_{\underline{v}} + \tilde{\rho} \underline{\tilde{f}}_{\underline{m}} - \bar{\rho} \frac{\partial \tilde{V}}{\partial t} \right] \right\} dS \tag{E7}
 \end{aligned}$$

And it is easy to verify that:

$$\bar{\underline{F}}_t^* = - \int_{\mathcal{V}} \bar{\rho} \frac{\partial \tilde{V}}{\partial t} d\mathcal{V} - \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \left(\bar{\rho} \frac{\partial \tilde{V}}{\partial t} \right) \right] dS \tag{E8}$$

$$\bar{\underline{F}}_{\underline{\mu}} = \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \underline{\nabla} \cdot \underline{\tilde{\tau}}_{\underline{v}} \right] dS + \int_{\Sigma} \underline{\tilde{\tau}}_{\underline{v}} \cdot \underline{n} dS \tag{E9}$$

$$\bar{\underline{F}}_{\underline{m}} = \int_{\mathcal{V}} \tilde{\rho} \underline{\tilde{f}}_{\underline{m}} d\mathcal{V} + \int_{\Sigma} \underline{r} \times \left[\underline{n} \times (\tilde{\rho} \underline{\tilde{f}}_{\underline{m}}) \right] dS \tag{E10}$$

$$\underline{\tilde{f}}_{\underline{m}} = - \frac{d\underline{U}}{dt} - \frac{d\underline{\Omega}}{dt} \times \underline{x} - \underline{\Omega} \times (\underline{\Omega} \times \underline{x}) - 2\underline{\Omega} \times \underline{\tilde{V}} \tag{E11}$$

A re-definition of $\bar{\underline{F}}_{\underline{\ell}}^{thd}$, $\bar{\underline{F}}_{\underline{m}_\rho}$ and $\bar{\underline{F}}_S^{thd}$ can be done moving the contribution of turbulent fluctuations from Eqs. (E5)(E6)(E7) into $\bar{\underline{F}}_{\underline{\mu}}$. Applying identity (A1) to the vector field $(\widetilde{T'' \underline{\nabla} s''})$ and

Appendix E. Turbulent flow regime

considering that turbulent fluctuations are zero on the body wall, the new $\bar{\underline{F}}_\mu$ expression is:

$$\begin{aligned} \bar{\underline{F}}_\mu^* = & \int_{\Sigma} \underline{r} \times \left[\underline{n} \times \underline{\nabla} \cdot \underline{\underline{\tau}} \right] dS + \int_{\Sigma} \underline{\underline{\tau}} \cdot \underline{n} dS - \int_{\mathcal{V}} \underline{r} \times [\underline{\nabla} \times (\bar{\rho} \underline{\nabla} k_t)] d\mathcal{V} + \\ & - \int_{\mathcal{V}} \underline{r} \times \left[\underline{\nabla} \times \left(\widetilde{T'' \underline{\nabla} s''} \right) \right] d\mathcal{V} \end{aligned} \quad (\text{E12})$$

This allows to formally write all other terms in the far-field force formula (Eq. (E4)) in a similar way to their laminar expressions (as done also for Eq. (E1)), replacing each quantity with the corresponding ensemble or Favre average as appropriate.

Therefore, when referring to the turbulent formula in the document, average symbols will be omitted (except for turbulent fluctuation terms) to simplify the notation.

It is worth highlighting that viscous/turbulent terms ($\bar{\underline{F}}_\mu$ or $\bar{\underline{F}}_\mu^*$) are different for the vorticity-based or the thermodynamic-based turbulent force formulas (Eq. (E1) or Eq. (E4)).

Finally, for turbulent flows, in the Reynolds-averaged version of the simplified exact formula (Eq. (4.15)) $\bar{\underline{F}}_\tau$ will also include the last two integrals (fluctuation terms) of Eq. (E12).

F

Numerical implementation

The post-processing of numerical solutions illustrated in the previous chapters was approached in different ways.

First, a pilot research code, *UBF2D*, was designed and developed in-house from scratch. It is able to analyse unsteady two-dimensional structured solutions and implements a variety of far-field formulations and numerical schemes, to serve as a virtual laboratory for testing theoretical developments and numerical implementations.

The following extension to three-dimensional and to unstructured solutions posed the investigators in front of different options: the design and development of a new in-house 3D unstructured code, or the implementation of post-processing algorithms in available commercial/public-domain codes, widely used in engineering.

The key factor influencing initial decisions was represented by parallel computing in distributed memory mode, a feature strictly necessary for a powerful code, able to deal with complex configurations and/or unsteady 3D solutions. As a matter of fact, computing time could be an issue also in post-processing (not only in the CFD solution phase) and, in addition, the investigators already experienced memory management difficulties using *UBF2D* for unsteady post-processing, which required a significant effort in reducing the memory footprint of the code for an efficient execution. Memory usage could therefore represent a limitation in generally 3D and unstructured cases (even for steady solutions), unless the code is able to exploit resources distributed on different computational nodes.

Appendix F. Numerical implementation

Such features are nowadays offered by the majority of commercial and public-domain post-processors, while an in-house implementation allowing for distributed memory usage would require adopting a completely different programming paradigm, with regards to the shared memory approach initially used for the pilot code UBF2D.

For the purposes of verification and validation of the theories illustrated in this dissertation, the 3D post-processing was therefore implemented using dedicated routines developed and executed in the *Tecplot*®360 commercial environment [193]. At the same time, investigations also addressed a similar implementation in the public-domain open-source software *ParaView* [84], although without reaching a mature stage yet (details in Sec. F.2.2).

F.1 UBF2D: A PILOT RESEARCH CODE

The following description is not intended as a code reference guide or detailed documentation manual. Instead, it offers an overview on the main features of the software and briefly describes the numerical receipts used in the implementation. Results obtained by the code described herein are illustrated and discussed throughout Chapters 2-4 of this dissertation.

Unsteady Break-Force 2D (UBF2D) is a pilot two-dimensional code developed by the *Italian Aerospace Research Centre (CIRA)* and the *University of Naples Federico II*, for the analysis and decomposition of the aerodynamic force on clean aerofoils. It is written in C++ (roughly 8300 code lines) and uses the public-domain *TecIO* library [192] to manage input and output of structured datasets*. UBF2D relies on local and remote code repositories in a distributed version control systems based on *Git*.

The code is able to post-process a steady or unsteady 2D flow-field, where data are provided in dimensional form (International System of Units) at grid nodes of a structured C-mesh (organised in one or multiple blocks, clock-wise-oriented along the C-direction). UBF2D reads in solution input files, together with a case configuration file (containing asymptotic conditions, reference values, mesh movement specification, numerical settings, etc.).

The accepted solution input format is given by Tecplot sub-zone (load-on-demand) binary files (one file for each solution time) containing at least primitive variables data and grid velocities. Indeed, either fixed or moving meshes are considered, where harmonic pitching and/or plunging oscillations may be prescribed. Similarly, the code is able to deal with either viscous or inviscid solutions, obtained from compressible or incompressible flow solvers.

The aerodynamic force is internally computed using both near-field and far-field formulations, ensuring coherence of the adopted gradient schemes and working variables allocation necessary to fairly compare the two force results. Different far-field formulations are implemented and available to the user: steady thermodynamic methods from Destarac and van der Vooren [42] and from Pa-

*Incidentally, during the initial developing phase, a bug was found in the *TecIO* library (version 2021, release 1), related to writing out cell-centred data in structured blocks. It was reported to *Tecplot Inc.* and subsequently fixed in the following releases of the library.

parone and Tognaccini [148]; several Lamb-vector-based methods developed in the last decades (a summary is provided in [124]) and applicable to inertial or non-inertial reference frames. For incompressible flows, the mixed inertial/non-inertial described in Ch. 3 (Sec. 3.1.7) is also available. Unsteady terms can be computed in the standard or in the alternative way (Ch. 3, Sec. 3.1.6) and the compressibility correction by Mele et al. [119] can be computed using any of the three mathematically equivalent formulations provided by Eq. (2.11). The Lamb vector can be computed in two different ways, i.e. by its vortical or thermodynamic definitions (Ch. 4), then the new exact force formula derived in Ch. 4 (Sec. 4.1.4) is also implemented in the code.

The integration domain used by far-field force methods is always selected as a sub-set of the input grid (identified by three orthogonal grid lines of the C-mesh around the aerofoil) and can be parametrically modified to allow for far-field force analysis on variable control volumes.

UBF2D provides three different options for the computation of spatial gradients from discrete structured data-sets:

- a nodes-based Green-Gauss (G-G) formula, with results allocated at cells-centres.
- 2nd-order finite differences in the computational space associated to the physical curvilinear grid, using nodal values of the working variables (asymmetric stencils are used at grid boundaries to preserve accuracy) and allocating gradient data at grid nodes.
- a Weighted Least Squares (WLS) method, based on cell-centres data and allocating results at cells centres as well. Ghost cells are used at grid boundaries with appropriate values of the working variables.

The last two methods are discrete gradient approximations with step h , where h is the generic mesh spacing. The nodes-based G-G method, for curvilinear grids, degenerates to the finite differences scheme with step $h/2$ and was used in almost all 2D computations included in this dissertation. A detailed discussion on the numerical accuracy of the implemented gradient reconstruction methods can be found in [182, 190].

Time derivatives were always computed using a 2nd-order central finite differences scheme, with asymmetric stencils adopted at time domain boundaries to preserve accuracy.

Volume integrals are numerically computed by simply summing-up cell centre values, multiplied by the cell volume, over the selected integration domain. Similarly, surface integrals are obtained computing the state in the cell surface centre, which can be either obtained by centred averaging based on face nodes, or by linear interpolation of cell-centred values from adjacent cells†.

Different regions can be identified in the integration domain and used for parasite drag decomposition. This is performed by either using deterministic sensors or by supplying external information (provided as cells flagging, in the solution input files), which may come, for example, from other codes using machine learning approaches to separate different regions in the flow-field

†In the text, a general three-dimensional nomenclature is used, despite UBF2D being a 2D code. In this case, obviously, “volumes” should be intended as “areas”, “faces” as “edges”, and so on.

Appendix F. Numerical implementation

(this was done in [171], where UFB2D was used in combination with an auto-encoder, performing the domain decomposition). In the first approach, the shock sensor defined by Lovely and Haimes [109] is used, together with two viscous sensors, designed for fully turbulent flows: i) an eddy viscosity sensor ($F_{bl} = (\mu + \mu_t) / \mu$), which assumes values significantly higher than 1 in the boundary layer and in the viscous wake [148, 197]; ii) turbulence specific dissipation rate (ω), assuming very high values near the body surface and, in particular, in the viscous sub-layer, where the eddy viscosity vanishes. The user can supply appropriate sensor cut-off values in the configuration file.

The code provides screen output (including case settings, near-field pressure and viscous forces, far-field forces obtained by different formulations and their complete decomposition) as well as Tecplot binary output (sub-zone load-on-demand file, containing input and computed variables, including near-field and far-field aerodynamic forces, calculated on all specified integration domains). For unsteady post-processing, a time history file including the main force results is also generated for the user's convenience.

F.1.1 AUXILIARY CODES

Auxiliary codes were also developed to provide interface between the open-source grid generator *Construct2D* (COMputational fluid dyNAMics STRUctured grid CreaTor for 2D aerofoils [155]), the flow solver *SU2* (Stanford University Unstructured [46, 145]) and the UBF2D post-processor.

The pre-processor *Construct2D* generates a mesh file in *NASA Plot3D* structured format [174]. An interface code, *CONVERT2SU2*, was then developed to generate coarser grids from the original mesh file, taking one over n nodes in the input grid ($n > 1$ can be specified by the user), and to convert the resulting structured mesh into the *SU2* native unstructured grid format.

Unstructured flow solutions obtained by *SU2* cannot be directly provided as an input to the structured post-processor UBF2D, due to the different organisation of the data. Nevertheless, due to the original ordered arrangement of the mesh generated during pre-processing, nodes indexing is possible and such information can be recovered from the original structured grid and applied to the *SU2* unstructured flow-fields, resulting in mapped solution files. The interface code *UNS2STR* provides this data conversion, taking care of the nodes duplication at the “wake line” necessary to *cut* the unstructured domain, restoring the C-grid topology.

F.2 IMPLEMENTATION IN COMMERCIAL AND OPEN-SOURCE POST-PROCESSORS

The most popular post-processors, routinely used in the world of computer-aided engineering, since time include all the numerical ingredients to perform a complete analysis of the aerodynamic force, according to the methodologies illustrated in this dissertation.

Nevertheless, the abovementioned tools, as such, do not offer a functionality to directly compute the aerodynamic force from a numerical flow-field around a body of interest. Instead, a dedicated routine need to be implemented and executed in the post-processor environment (either

F.2. Implementation in commercial and open-source post-processors

interactively, or in batch mode) to instruct the software to perform all necessary operations to compute both near-field and far-field force values.

Moreover, their reliability has to be verified and results need to be validated. In fact, such tools are usually (and almost exclusively) employed to *visualise* numerical flow-field information, eventually performing limited manipulation to compute additional quantities from the initial data-set. The accuracy of operations such as spatial gradients calculation, for example, is questioned, and has been investigated in both Tecplot 360 (a proprietary commercial tool) and ParaView (an open-source license-free software).

Investigations involved steady and viscous cases only, considering both structured and unstructured data-sets. The main outcomes of this study (still in progress) are briefly summarised in the following subsections, highlighting consolidated results as well as the identified problems which require further work to be solved. The anticipated conclusion is that, at the moment, reliable results were only obtained using Tecplot 360 on structured data-sets.

F.2.1 TECPLOT 360

Applications involved version 2022.R1 of the software and the Tecplot macro language was used to develop a scripting procedure (roughly 1200 code lines) to post-process 3D numerical flow-fields obtained for the elliptic-planform wing and described in Ch. 2 (Sec. 2.2.1). Both thermodynamic and Lamb-vector methods were implemented, the latter based on a vortical or thermodynamic formulation of the Lamb vector (Ch. 4).

Multi-block structured solutions generated by CIRA ZEN flow solver were successfully analysed and results are illustrated in Ch. 2 and 4.

Subsequently, in the perspective of a further extension of the implemented methodology to complex configurations, unstructured tests were conducted. In particular, the same elliptic wing configuration was simulated, for the same flow conditions, using the unstructured solver SU2[‡].

The “CFD analyzer” tool, in Tecplot 360, was used to compute spatial gradients of flow variables. It implements a finite differences algorithm in the case of ordered data-sets and a least-squares method for unstructured data (details are available in the online software documentation [193]).

While for structured data the accuracy of computed results (both near-field and far-field) was satisfactory since early applications, severe accuracy issues were observed in post-processing unstructured flow-fields. As a matter of fact, even the near-field force was not correctly computed (Tab. F.2.1).

A detailed analysis evidenced how the inaccuracy came from the viscous contribution (C_D^f , Tab. F.2.1) and, ultimately, from the calculated velocity gradient tensor (Fig. F.2.1). Indeed,

[‡]Indeed, to allow for a comparison only based on the different data arrangement (i.e. without involving other effects due to the different flow solvers), similar tests (with similar conclusions) were also performed converting to unstructured data the flow solutions directly obtained using CIRA ZEN code (i.e., the same solutions previously analysed as ordered data-sets).

Table F.2.1: Near-field force computed by SU2 or using Tecplot 360, from the unstructured elliptic wing flow-field ($M_\infty = 0.75$, $Re_\infty = 3 \times 10^6$, $\alpha = 4^\circ$).

	C_L	C_D	$10^4 \times C_D^p$	C_D^f
SU2 output (F_{NF})	0.421	262.5	204.1	58.4
Tecplot 360 ($F_{\eta f}$)	0.421	246.6	204.4	42.2

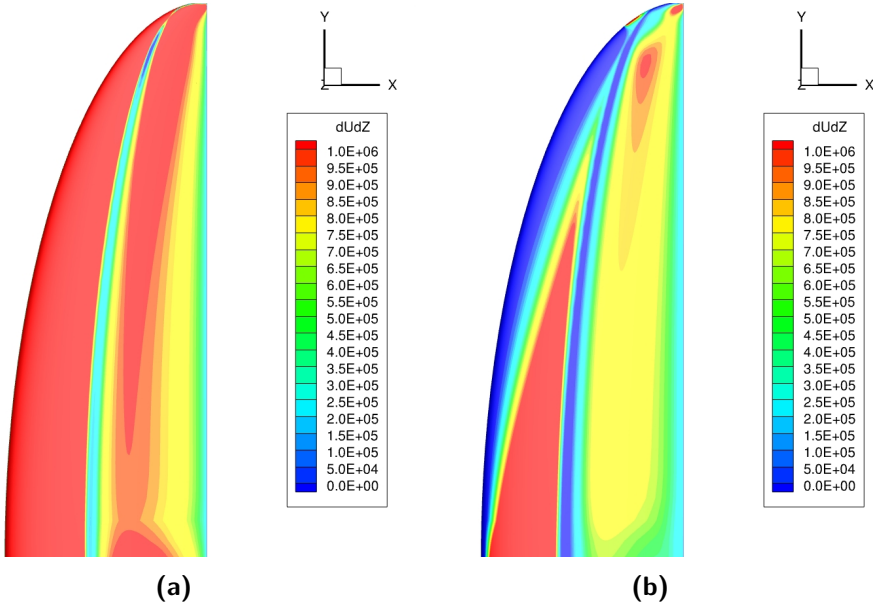


Figure F.2.1: Derivative of the streamwise velocity component (U) with regards to the Z coordinate, on the wall of the elliptic wing ($M_\infty = 0.75$, $Re_\infty = 3 \times 10^6$, $\alpha = 4^\circ$), computed by Tecplot 360 from the structured (a) or the unstructured (b) flow solution.

Table F.2.2: Near-field force computed by SU2 or using Tecplot 360 beta, from the unstructured elliptic wing flow-field ($M_\infty = 0.75$, $Re_\infty = 3 \times 10^6$, $\alpha = 4^\circ$).

	C_L	C_D	$10^4 \times C_D^p$	C_D^f
SU2 output (F_{NF})	0.421	262.5	204.1	58.4
Tecplot 360 beta (F_{nf})	0.421	262.9	204.4	58.5

the least-squares algorithm used by Tecplot 360 on unstructured data is known to be inaccurate for highly stretched cells in near-wall grid layers [190], even using an appropriate weighting matrix. The observed differences were however too large and a ticket with the customer support was created to report the problem, suggesting the implementation of a G-G algorithm. *Tecplot Inc.* confirmed the reported issue and implemented the G-G scheme as a beta feature in the 2024 R1 version of the code, which revealed successful in eliminating the inaccuracy in the near-field force calculation (Tab. F.2.2).

However, far-field results on unstructured data were still not satisfactory. This was due to a partial implementation of the G-G method, initially introduced to compute gradients at grid nodes only, while cell-centred gradients are used in the post-processing routine for the far-field force analysis. As confirmed by *Tecplot Inc.*, further releases of the software should include a full implementation of the G-G algorithm, also including cell types different from hexahedra, which will allow for a general applicability of the developed methodology to complex configurations.

F.2.2 PARAVIEW

ParaView is a popular open-source post-processor relying, among others, on the *VTK* library, collecting software tools for manipulating and displaying scientific data (maintained by the same Company). Applications involved version 5.11.2 of the software, which was compiled on a Linux HPC cluster with MPI (Message Passing Interface) and Python support. Python scripting was then used to develop a code procedure to post-process numerical flow-fields obtained from CFD solvers.

Differently from Tecplot 360, ParaView offers some interesting features, such as the possibility to extract a control volume by exact plane clipping (instead of using grid subsets delimited by existing mesh surfaces), the availability of different algorithms (instead of just one) for the calculations of spatial gradients (in either structured or unstructured solutions), or the definition of embedded user-defined “programmable” operations through python implementation. Of course, the most interesting aspect is represented by the availability of source files, which allows investigating code functionalities into the details and eventually modifying existing features or implementing new ones.

Both 2D and 3D transonic flow-fields were analysed, considering structured solutions generated by CIRA ZEN flow solver, their unstructured-converted versions (Sec. F.2.1), as well as unstructured flow-fields directly obtained using the SU2 code (instead of ZEN).

Table F.2.3: Near-field force computed by SU2 or using ParaView, from the unstructured elliptic wing flow-field ($M_\infty = 0.75$, $Re_\infty = 3 \times 10^6$, $\alpha = 4^\circ$).

	C_L	C_D	$10^4 \times C_D^p$	C_D^f
SU2 output (F_{NF})	0.421	262.5	204.1	58.4
ParaView (F_{nf})	0.421	261.5	204.5	57.0

The implementation in ParaView provided accurate results for the near-field force computed on either structured or unstructured solutions (2D or 3D). Tab. F.2.3 provides an example for the unstructured elliptic wing solution.

Far-field formulas, instead, did not return accurate force values except on 2D structured flow-fields, indicating that further investigations are necessary to identify the reasons of the observed discrepancies with regards to the reference near-field data. This could require manipulating source files to introduce the implementation of a new gradient scheme (G-G method, for example).

Since, at the moment, these preliminary investigations did not lead, apart from very simple cases, to reliable and accurate results, this dissertation does not include any post-processing data obtained from ParaView.

It is nevertheless relevant to mention the work conducted at *ISAE SUPAERO* in the ParaView environment. The involved research team developed a set of open-source plug-in modules to be loaded in ParaView to perform both near-field and far-field post-processing of unstructured CFD and EFD solutions [3]. The complete tool is named *Epsilon* [5] and, in its actual version (v3), it implements steady methods only, mainly devoted to the exergy analysis (this explains the software name) but also including pioneering momentum methods and a few Lamb-vector formulas. Their work is currently under investigation by the author although, at the moment, no sufficient data have been collected to allow for a validation of the code results on some of the test cases analysed in this dissertation.

References

- [1] Abramowitz, M. & Stegun, I. A. (1965). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover Books on Advanced Mathematics. New York, NY, USA: Dover Publications, tenth printing edition.
- [2] Ackeret, J. (1925). Über Luftkräfte auf Flügel, die mit größerer als Schallgeschwindigkeit bewegt werden. *Zeitschrift Flugtech.*, 16, 72–74.
- [3] Aguirre, M. A. (2022). *Exergetic analysis of innovative aircrafts with aero-propulsive coupling*. PhD thesis, École doctorale Mécanique, énergétique, génie civil et procédés (Toulouse).
- [4] Aguirre, M. A. & Duplaa, S. (2019). Exergetic Drag Characteristic Curves. *AIAA Journal*, 57(7), 2746–2757.
- [5] Aguirre, M. A. & Duplaa, S. (2022). Epsilon v3: exergy-based aerodynamic analysis tool for CFD and WTT data. In *Proceedings of the 56th 3AF International Conference on Applied Aerodynamics* Toulouse.
- [6] Aguirre, M. A., Duplaa, S., Carbonneau, X., & Turnbull, A. (2020). Velocity Decomposition Method for Exergy-Based Drag Prediction. *AIAA Journal*, 58(11), 4686–4701.
- [7] Aris, R. (1962). *Vectors, Tensors and the Basic Equations of Fluid Mechanics*. Dover Books on Mathematics. Dover Publications.
- [8] Arntz, A., Atinault, O., & Merlen, A. (2014). Exergy-Based Formulation for Aircraft Aero-propulsive Performance Assessment: Theoretical Development. *AIAA Journal*, 53(6), 1627–1639.
- [9] Arntz, A. & Hue, D. (2015). Exergy-Based Performance Assessment of the NASA Common Research Model. *AIAA Journal*, 54(1), 88–100.
- [10] Bailly, D. (2024). Development of a far-field drag extraction method for supersonic flows. In *AERO 2024* Orléans, France: 3AF - Association Aéronautique et Astronautique de France.
- [11] Banerjee, M. R., Ramadugu, R., & Ansumali, S. (2020). Discrete differential operators on a class of lattices. *Journal of Computational Science*, 44, 101172.

References

- [12] Batchelor, G. K. (1967). *An Introduction to Fluid Dynamics*. Cambridge: Cambridge University Press.
- [13] Berhouini, I., Bailly, D., & Petropoulos, I. (2023a). Exergy Balance Extension to Rotating Reference Frames: Application to a Propeller Configuration. *AIAA Journal*, 61(4), 1790–1806.
- [14] Berhouini, I., Bailly, D., & Petropoulos, I. (2023b). On the Definition of Exergy in the Field of Aerodynamics. *AIAA Journal*, 61(10), 4356–4366.
- [15] Berhouini, I., Bailly, D., & Petropoulos, I. (2024). Exergy Balance Decomposition Between Mechanically- and Thermally-Recoverable Exergy Outflows. *AIAA Journal*, 62(3), 1232–1239.
- [16] Betz, A. (1912). Ein Beitrag zur Erklarung Segelfluges. *Z Flugtech Motorluftschiffahrt*, 3, 269–272.
- [17] Betz, A. (1925). *A method for the direct determination of wing-section drag*. Technical Report NACA TM37, National Advisory Committee for Aeronautics.
- [18] Bicknell, J. (1939). *Determination of the Profile Drag Of an Airplane Wing in Flight At high Reynolds Numbers*. Technical Report 667, National Advisory Committee for Aeronautics, Langley Memorial Aeronautical Laboratory, Headquarters, Navy Building, Washington, D. C.
- [19] Birnbaum, W. (1924). Das ebene Problem des schlagenden Flügels. *Z. angew. Math. Mech.*, 4(4), 277–292.
- [20] Bisplinghoff, R. L., Ashley, H., & Halfman, R. L. (1996). *Aeroelasticity*. Dover Books on Aeronautical Engineering. Dover Publications.
- [21] Bisson, F. (2015). *Aerodynamic Design Optimization Using the Drag Decomposition Method*. Master's thesis, Department of Mechanical Engineering, McGill University, Montreal, Quebec.
- [22] Brodersen, O., Rakowitz, M., Amant, S., Larrieu, P., Destarac, D., & Suttcliffe, M. (2005). Airbus, ONERA, and DLR Results from the Second AIAA Drag Prediction Workshop. *Journal of Aircraft*, 42(4), 932–940.
- [23] Bryant, L. W. & Williams, D. H. (1926). An investigation of the flow of air an aerofoil of infinite span. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, 225(626-635), 199–245.
- [24] Burgatti, P. (1931). Intorno a una formula generale di trasformazione di un integrale di spazio in uno di superficie e alle sue varie deduzioni. *Bolletino della Unione Matematica Italiana*, 10, 1–5.

- [25] Catalano, P., De Rosa, D., Mele, B., Tognaccini, R., & Moens, F. (2020). Performance improvements of a regional aircraft by riblets and natural laminar flow. *Journal of Aircraft*, 57(1), 29–40.
- [26] Cavallaro, R. & Demasi, L. (2016). Challenges, Ideas, and Innovations of Joined-Wing Configurations: A Concept from the Past, an Opportunity for the Future. *Progress in Aerospace Sciences*, 87, 1–93.
- [27] Cella, U., Quagliarella, D., Donelli, R., & Imperatore, B. (2010). Design and test of the UW-5006 transonic natural-laminar-flow wing. *Journal of Aircraft*, 47(3), 783–795.
- [28] Chan, W. M., Chiu, I.-T., & Buning, P. G. (1993). *User's manual for the HYPGEN hyperbolic grid generator and the HGUI graphical user interface*. Technical Memorandum (TM) 19940011311, NASA National Aeronautics and Space Administration, AMES Research Center, MoffettField, California 94035-1000.
- [29] Chao, D. D. & Van Dam, C. P. (2006). Wing Drag Prediction and Decomposition. *Journal of Aircraft*, 43(1), 82–90.
- [30] Cicala, P. (1934). *Le Azioni aerodinamiche sui profili di ala oscillanti in presenza di corrente uniforme*, chapter I. Number 68 in 2. Mem. della R. Accad. delle Sci. di Torino.
- [31] Coder, J. G. & Schmitz, S. (2019). Thermodynamic Decomposition of Compressible Wave Drag in the Euler Equations. In *Proceedings of the AIAA Aviation 2019 Forum*, number 0 in AIAA AVIATION Forum. American Institute of Aeronautics and Astronautics.
- [32] Cook, P. H., McDonald, M. A., & Firmin, M. C. P. (1979). *Aerofoil RAE2822 – Pressure Distributions, and Boundary Layer and Wake Measurements*. Technical Report AGARD TR AR-138, North Atlantic Treaty Organization, Advisory Group For Aerospace Research And Development, Royal Aircraft Establishment, Farnborough, Hants, United Kingdom.
- [33] Crocco, L. (1937). Eine neue Stromfunktion für die Erforschung der Bewegung der Gase mit Rotation. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 17(1), 1–7.
- [34] Cummings, R., Giles, M., & Shrinivas, G. (1996). Analysis of the elements of drag in three-dimensional viscous and inviscid flows. In *Proceedings of the 14th Applied Aerodynamics Conference*, number 0 in Fluid Dynamics and Co-located Conferences. American Institute of Aeronautics and Astronautics.
- [35] Davies-Jones, R. (2017). Roles of Streamwise and Transverse Partial-Vorticity Components in Steady Inviscid Isentropic Supercell-Like Flows. *Journal of the Atmospheric Sciences*, 74(9), 3021–3041.
- [36] Demasi, L. (2024). *Introduction to Unsteady Aerodynamics and Dynamic Aeroelasticity*. Springer Nature Switzerland.

References

- [37] Demasi, L., Monegato, G., Cavallaro, R., & Rybarczyk, R. (2019). Optimum induced drag of wingtip devices: The concept of best winglet design. *AIAA Scitech 2019 Forum*.
- [38] Destarac, D. (2003). *Far-Field/Near-Field Drag Balance and Applications of Drag Extraction in CFD*, chapter 02, (pp. 1–62). VKI Lecture Series, von Karman Inst. for Fluid Dynamics: Waterloosesteenweg 72, 1640 Sint-Genesius-Rode, Belgium.
- [39] Destarac, D. (2011a). Investigating Negative Drag in Grid Convergence for Two-Dimensional Euler Solutions. *Journal of Aircraft*, 48(4), 1468–1470.
- [40] Destarac, D. (2011b). Spurious Far-Field-Boundary Induced Drag in Two-Dimensional Flow Simulations. *Journal of Aircraft*, 48(4), 1444–1455.
- [41] Destarac, D. (2015). Three-Component Breakdown of Spurious Drag in Computational Fluid Dynamics. *Journal of Aircraft*, 52(4), 1336–1344.
- [42] Destarac, D. & van der Vooren, J. (2004). Drag/thrust analysis of jet-propelled transonic transport aircraft; Definition of physical drag components. *Aerospace Science and Technology*, 8(6), 545–556.
- [43] Drela, M. (2009). Power Balance in Aerodynamic Flows. *AIAA Journal*, 47(7), 1761–1771.
- [44] Drela, M. (2014). *Flight Vehicle Aerodynamics*. The MIT Press.
- [45] Ducros, F., Ferrand, V., Nicoud, F., Weber, C., Darracq, D., Gacherieu, C., & Poinso, T. (1999). Large-Eddy Simulation of the Shock/Turbulence Interaction. *Journal of Computational Physics*, 152(2), 517–549.
- [46] Economou, T. D., Palacios, F., Copeland, S. R., Lukaczyk, T. W., & Alonso, J. J. (2015). SU2: An Open-Source Suite for Multiphysics Simulation and Design. *AIAA Journal*, 54(3), 828–846.
- [47] Eça, L., Hoekstra, M., & Roache, P. (2005). Verification of Calculations: an Overview of the Lisbon Workshop. In *Proceedings of the 23rd AIAA Applied Aerodynamics Conference*, number 0 in Fluid Dynamics and Co-located Conferences. American Institute of Aeronautics and Astronautics.
- [48] Fan, Y. & Li, W. (2018). Review of Far-Field Drag Decomposition Methods for Aircraft Design. *Journal of Aircraft*, 56(1), 11–21.
- [49] Fernandez-Feria, R. (2016). Linearized propulsion theory of flapping airfoils revisited. *PRFLUIDS*, 1(8), 084502.
- [50] Fernandez-Feria, R. & Alaminos-Quesada, J. (2018). Unsteady thrust, lift and moment of a two-dimensional flapping thin airfoil in the presence of leading-edge vortices: a first approximation from linear potential theory. *Journal of Fluid Mechanics*, 851, 344–373.

- [51] Filon, L. N. G. (1926). The forces on a cylinder in a stream of viscous fluid. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 113(763), 7–27.
- [52] Fonzi, N., Cavalieri, V., De Gaspari, A., & Ricci, S. (2022). Extended computational capabilities for high-fidelity fluid-structure simulations. *Journal of Computational Science*, 62, 101698.
- [53] Fournis, C. (2021). *Aerodynamic Force Prediction and Breakdown Using the Lamb Vector in Steady Compressible Flows*. PhD thesis, Sorbonne Université.
- [54] Fournis, C. (2022). Wave drag prediction using the Lamb vector in two-dimensional steady transonic flows. In *Proceedings of the AIAA AVIATION 2022 FORUM*, number 0 in AIAA AVIATION Forum. American Institute of Aeronautics and Astronautics.
- [55] Fournis, C., Bailly, D., & Tognaccini, R. (2021a). An invariant vortex-force theory related to classical far-field analyses in transonic flows. In *Proceedings of the AIAA AVIATION 2021 FORUM*, number 0 in AIAA AVIATION Forum. American Institute of Aeronautics and Astronautics.
- [56] Fournis, C., Bailly, D., & Tognaccini, R. (2021b). Compressibility Correction to Kutta–Joukowski and Maskell Formulas Using Vortex-Force Theory. *AIAA Journal*, 59(2), 773–778.
- [57] Fournis, C., Bailly, D., & Tognaccini, R. (2022). Invariant Vortex-Force Theory Extending Classical Aerodynamic Theories to Transonic Flows. *AIAA Journal*, 60(9), 5070–5082.
- [58] Fujisawa, N., Tanahashi, S., & Srinivas, K. (2005). Evaluation of pressure field and fluid forces on a circular cylinder with and without rotational oscillation using velocity data from PIV measurement. *Measurement Science and Technology*, 16(4), 989–996.
- [59] Gariépy, M., Trepanier, J.-Y., & Malouin, B. (2013). Generalization of the Far-Field Drag Decomposition Method to Unsteady Flows. *AIAA Journal*, 51(6), 1309–1319.
- [60] Gariépy, M. & Trépanier, J. Y. (2012). A new axial velocity defect formulation for a far-field drag decomposition method. *Can. Aeronaut. Space J.*, 58(02), 69–82.
- [61] Garrick, I. E. (1936). *Propulsion of a flapping and oscillating airfoil*. Technical report, National Advisory Committee for Aeronautics.
- [62] Garrick, I. E. (1948). *A survey of flutter*. Technical report, Langley Aeronautical Laboratories.
- [63] Gehlert, P., Andreu-Angulo, I., & Babinsky, H. (2023). Vortex force decomposition—forces associated with individual elements of a vorticity field. *Experiments in Fluids*, 64(6), 112.
- [64] Giles, M. B. & Cummings, R. M. (1999). Wake Integration for Three-Dimensional Flow-field Computations: Theoretical Development. *Journal of Aircraft*, 36(2), 357–365.

References

- [65] Glauert, H. (1926). *The Elements of Aerofoil and Airscrew Theory*. Cambridge: Cambridge University Press.
- [66] Glauert, H. (1929). *The Force and Moment on an Oscillating Aerofoil*. Reports and Memoranda 1242, British Aeronautical Research Council.
- [67] Gonzalez, C. & Taha, H. E. (2022). A variational theory of lift. *Journal of Fluid Mechanics*, 941(A58), A58.
- [68] Graham, W. R., Pitt Ford, C. W., & Babinsky, H. (2017). An impulse-based approach to estimating forces in unsteady flow. *Journal of Fluid Mechanics*.
- [69] Hart, P. L. & Schmitz, S. (2022). Partial Pressure Field for Airfoil Wave Drag. *AIAA Journal*, 60(10), 5791–5804.
- [70] Heathcote, S., Wang, Z., & Gursul, I. (2008). Effect of spanwise flexibility on flapping wing propulsion. *Journal of Fluids and Structures*, 24(2), 183–199.
- [71] Huang, G. C. (1994). *Unsteady vortical aerodynamics: theory and applications*. Shanghai Jiaotong Univ. Press.
- [72] Hunt, D. L. (2001). Development and application of farfield drag extraction techniques for complex viscous flows. *The Aeronautical Journal*, 105(1045), 161–169.
- [73] Hunt, D. L., Cummings, R. M., & Giles, M. B. (1999). Wake Integration for Three-Dimensional Flowfield Computations: Applications. *Journal of Aircraft*, 36(2), 366–373.
- [74] Imai, I. (1951). On the Asymptotic Behaviour of Viscous Fluid Flow at a Great Distance from a Cylindrical Body, with Special Reference to Filon's Paradox. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 208(1095), 487–516.
- [75] Jameson, A., Schmidt, W., & Turkel, E. (1981). Numerical solution of the Euler equations by finite volume methods using Runge Kutta time stepping schemes. In *Proceedings of the 14th Fluid and Plasma Dynamics Conference*, number 0 in Fluid Dynamics and Co-located Conferences. American Institute of Aeronautics and Astronautics.
- [76] Jones, B. M. (1936). *The Measurement of Profile Drag by the Pitot-traverse Method*. Technical Report 1688, Her Majesty's Stationery Office, London.
- [77] Jones, K., Dohring, C., & Platzer, M. (1998). Experimental and Computational Investigation of the Knoller-Betz Effect. *Aiaa Journal - AIAA J*, 36, 1240–1246.
- [78] Joukowski, N. E. (1906). On Annexed Vortices. In *Proceedings of the Physical Section of the Natural Science Society*, volume 13 (2) (pp. 12–25). London.
- [79] Kang, L., Liu, L., Su, W., & Wu, J. (2017). A minimum-domain impulse theory for unsteady aerodynamic force with discrete wake. *Theoretical and Applied Mechanics Letters*, 7(5), 306–310.

- [80] Kang, L. L., Liu, L. Q., Su, W. D., & Wu, J. Z. (2018). Minimum-domain impulse theory for unsteady aerodynamic force. *Physics of Fluids*, 30(016107).
- [81] Kang, L. L., Russo, L., Tognaccini, R., Wu, J. Z., & Su, W. D. (2019). Aerodynamic Force Breakdown in Reversible and Irreversible Components by Vortex Force Theory. *AIAA Journal*, 57(11), 4623–4638.
- [82] Katzmayr, R. (1922). Effect of Periodic Changes of Angle of Attack on Behavior of Airfoils. In *NACA Technical Memorandum*.
- [83] Kim, H. & Liou, M.-F. (2019). Flow simulation and drag decomposition study of N3-X hybrid wing-body configuration. *Aerospace Science and Technology*, 85, 24–39.
- [84] Kitware, I. (2023). ParaView. <https://www.paraview.org/>, last accessed: Apr. 2024.
- [85] Knoller, R. & Verein, O. F. (1909). Die Gesetze des Luftwiderstandes. *Flug-und Motortech-nik*, 3(21), 1–7.
- [86] Kok, J. C. (2000). Resolving the Dependence on Freestream Values for the k- Turbulence Model. *AIAA Journal*, 38(7), 1292–1295.
- [87] Kroo, I. (2001). DRAG DUE TO LIFT: Concepts for Prediction and Reduction. *Annu. Rev. Fluid Mech.*, 33(1), 587–617.
- [88] Küssner, H. G. (1936). Zusammenfassender Bericht über den instationären Auftrieb von Flügeln. *Luftfahrtforschung*, 13(12), 410–424.
- [89] Kusunose, K. (1998). Drag prediction based on a wake-integral method. In *Proceedings of the 16th AIAA Applied Aerodynamics Conference*, number 0 in Fluid Dynamics and Co-located Conferences. American Institute of Aeronautics and Astronautics.
- [90] Kusunose, K. (2005). *A wake integration method for airplane drag prediction*. The 21st century COE program international COE of flow dynamics lecture series ; v. 3. Tohoku University Press.
- [91] Kusunose, K. & Crowder, J. (2001). Physical properties of Maskell’s induced drag integral. In *Proceedings of the 39th Aerospace Sciences Meeting and Exhibit*, number 0 in Aerospace Sciences Meetings. American Institute of Aeronautics and Astronautics.
- [92] Kusunose, K., Crowder, J., & Watzlavick, R. (1999). Wave drag extraction from profile drag based on a wake-integral method. In *Proceedings of the 37th Aerospace Sciences Meeting and Exhibit*, number 0 in Aerospace Sciences Meetings. American Institute of Aeronautics and Astronautics.
- [93] Kutta, W. M. (1902). Auftriebskräfte in Strömenden Flüssigkeiten. In *Illustrierte Aero-nautische Mitteilungen*, volume 6 of 133 (pp. 133–135).

References

- [94] Lamb, H. (1877). On the Conditions for Steady Motion of a Fluid. *Proceedings of the London Mathematical Society*, 51-9(1), 91–93.
- [95] Lamb, H. (1945). *Hydrodynamics*. Dover Books on Physics. Dover Publications.
- [96] Lanzetta, M., Mele, B., & Tognaccini, R. (2015). Advances in Aerodynamic Drag Extraction by Far-Field Methods. *Journal of Aircraft*, 52(6), 1873–1886.
- [97] Li, G.-J. & Lu, X.-Y. (2012). Force and power of flapping plates in a fluid. *Journal of Fluid Mechanics*, 712, 598–613.
- [98] Lighthill, M. J. (1963). Introduction: Boundary Layer Theory. In L. Rosenhead (Ed.), *Laminar Boundary Theory*. Oxford University Press.
- [99] Liu, K., Song, B., Wang, Z., Chen, A., & Xue, D. (2023). Effects of dynamical chordwise retracting and stretching on the propulsive performance of a pure plunging flat plate. *AIP Advances*, 13(2), 025347.
- [100] Liu, L., Shi, Y., Zhu, J., Su, W., Zou, S., & Wu, J. (2014a). Longitudinal–transverse aerodynamic force in viscous compressible complex flow. *Journal of Fluid Mechanics*, 756, 226–251.
- [101] Liu, L., Wu, J.-Z., Su, W., & Kang, L. (2017a). Lift and drag in three-dimensional steady viscous and compressible flow. *Physics of Fluids*, 29.
- [102] Liu, L. Q., Kang, L. L., & Wu, J. Z. (2017b). Zonal structure of unbounded external-flow and aerodynamics. *Fluid Dynamics Research*, 49(4), 045508.
- [103] Liu, L. Q., Wu, J. Z., Shi, Y. P., & Zhu, J. Y. (2014b). A dynamic counterpart of Lamb vector in viscous compressible aerodynamics. *Fluid Dynamics Research*, 46(6), 061417.
- [104] Liu, L. Q., Zhu, J. Y., & Wu, J. Z. (2015a). Lift and drag in two-dimensional steady viscous and compressible flow. *Journal of Fluid Mechanics*, 784, 304–341.
- [105] Liu, T., Wang, S., Zhang, X., & He, G. (2015b). Unsteady Thin-Airfoil Theory Revisited: Application of a Simple Lift Formula. *AIAA Journal*, 53(6), 1492–1502.
- [106] Lock, C. N. H., Hilton, William, F., & Goldstein, S. (1940). *Determination of Profile Drag at High Speeds by a Pitot Traverse Method*. Technical Report 4709, Great Brit. Aeronautical Research Council. R & M 1971. London H.M.S.O., n.d. 39 p. incl. illus.
- [107] Lock, R. C. (1985). *Prediction of the Drag of Wings at Subsonic Speeds by Viscous/Inviscid Interaction Techniques*. AGARD R-723, Aircraft Drag Prediction and Reduction.
- [108] Lock, R. C. (1986). The prediction of the drag of aerofoils and wings at high subsonic speeds. *The Aeronautical Journal*, 90(896), 207–226.

- [109] Lovely, D. & Haimes, R. (1999). Shock detection from computational fluid dynamics results. In *Proceedings of the 14th Computational Fluid Dynamics Conference*, number 0 in Fluid Dynamics and Co-located Conferences. American Institute of Aeronautics and Astronautics.
- [110] Malouin, B., Trépanier, J.-Y., & Laurendeau, E. (2016). Installation and interference drag decomposition via RANS far-field methods. *Aerospace Science and Technology*, 54, 132–142.
- [111] Marongiu, C., Catalano, P., Amato, M., & Iaccarino, G. (2004). U-ZEN: A Computational Tool Solving U-Rans Equations for Industrial Unsteady Applications. In *Proceedings of the 34th AIAA Fluid Dynamics Conference and Exhibit*, number 0 in Fluid Dynamics and Co-located Conferences. American Institute of Aeronautics and Astronautics.
- [112] Marongiu, C. & Tognaccini, R. (2010). Far-Field Analysis of the Aerodynamic Force by Lamb Vector Integrals. *AIAA Journal*, 48(11), 2543–2555.
- [113] Marongiu, C., Tognaccini, R., & Ueno, M. (2013). Lift and Lift-Induced Drag Computation by Lamb Vector Integration. *AIAA Journal*, 51(6), 1420–1430.
- [114] Maskell, E. C. (1972). *Progress towards a method for the measurement of the components of the drag of a wing of finite span*. Technical Report Technical Report No. 72232, RAE, United Kingdom.
- [115] Meheut, M. (2013). Thrust and Torque Far-field Analysis of Propeller and Counter Rotating Open Rotor Configurations. In *Proceedings of the 31st AIAA Applied Aerodynamics Conference*, number 0 in Fluid Dynamics and Co-located Conferences. American Institute of Aeronautics and Astronautics.
- [116] Mele, B., Ostieri, M., & Tognaccini, R. (2016). Vorticity Based Breakdown of the Aerodynamic Force in Three-Dimensional Compressible Flows. *AIAA Journal*, 54(4), 1198–1208.
- [117] Mele, B., Ostieri, M., & Tognaccini, R. (2017). Aircraft Lift and Drag Decomposition in Transonic Flows. *Journal of Aircraft*, 54(5), 1933–1944.
- [118] Mele, B., Russo, L., & Tognaccini, R. (2020). Drag bookkeeping on an aircraft with riblets and NLF control. *Aerospace Science and Technology*, 98, 105714.
- [119] Mele, B. & Tognaccini, R. (2014). Aerodynamic force by Lamb vector integrals in compressible flow. *Physics of Fluids*, 26(5), 056104.
- [120] Menter, F. R. (1994). Two-equation eddy-viscosity turbulence models for engineering applications. *AIAA Journal*, 32(8), 1598–1605.
- [121] Minervino, M. & Tognaccini, R. (2022). Wave Drag and Vorticity in Viscous and Inviscid flows. In *Book of Abstracts of the 14th European Fluid Mechanics Conference EFMC14* (pp. 167). Athens, GR.

References

- [122] Minervino, M. & Tognaccini, R. (2023a). A unified thermodynamic/Lamb-vector-based analysis of the aerodynamic force. *Physics of Fluids*, 35(9), 097135.
- [123] Minervino, M. & Tognaccini, R. (2023b). Aerodynamic Force by Lamb Vector Integrals in Unsteady Compressible Flows. In *Proceedings of the 57th 3AF International Conference on Applied Aerodynamics* Bordeaux, FR.
- [124] Minervino, M. & Tognaccini, R. (2023c). On the spurious effects in Lamb-vector-based force decomposition methods. *Aerospace Science and Technology*, 142, 108674.
- [125] Minervino, M. & Tognaccini, R. (2023d). Wave Drag and Vorticity in Two-Dimensional Viscous and Inviscid Flows. In *Proceedings of the AIAA SCITECH 2023 Forum*, number 2023-1032 in AIAA SciTech Forum. American Institute of Aeronautics and Astronautics.
- [126] Minervino, M. & Tognaccini, R. (2024a). A Unified Thermodynamic/Vortical Far-Field Force Method. In *Book of Abstracts of the 1st European Fluid Dynamics Conference EFDC1* Aachen, DE.
- [127] Minervino, M. & Tognaccini, R. (2024b). Aerodynamic force by Lamb vector integrals in unsteady compressible flows. *International Journal of Numerical Methods for Heat & Fluid Flow*, 34(7), 2654–2687.
- [128] Minotti, F. O. (2011). Determination of the instantaneous forces on flapping wings from a localized fluid velocity field. *Physics of Fluids*, 23(11), 111902.
- [129] Moreau, J.-J. (1952). Bilan dynamique d'un écoulement rotationnel. *Journal de Mathématiques Pures et Appliquées*, 32, 1–78.
- [130] Morelli, M., Trapier, S., Lopez de Vega, L., Thollet, W., Petropoulos, I., & Berhouni, I. (2024). Assessment of the Far-Field Exergy Balance Method for Industrial Aerodynamic and Aerothermal Applications. In *AERO 2024 Orléans, France: 3AF - Association Aéronautique et Astronautique de France*.
- [131] Morkovin, M. V. (1962). Effects of Compressibility on Turbulent Flow. In *The Mechanics of Turbulence* (pp. 367). Gordon and Breach.
- [132] Mutangara, N. E., Smith, L., Craig, K. J., & Sanders, D. S. (2024). Potential for Energy Recovery from Boundary-Layer Ingesting Actuator Disk Propulsion. *Journal of Aircraft*, (pp. 1–14).
- [133] Méheut, M. (2006). *Évaluation des Composantes Phénoménologiques de la Traînée d'un Avion à Partir de Résultats Expérimentaux*. PhD thesis, Université de Lille, Lille, France.
- [134] Méheut, M. & Bailly, D. (2008). Drag-Breakdown Methods from Wake Measurements. *AIAA Journal*, 46(4), 847–862.

- [135] Nadal, M., Fournis, C., & Lewis, D. (2024). Unsteady Far-Field Drag Analyses of the ON-ERA HAD-1 Propeller. In *Proceedings of the AIAA AVIATION 2024 Forum*, number 0 in AIAA AVIATION Forum. American Institute of Aeronautics and Astronautics.
- [136] Noca, F. (1996). *On the evaluation of instantaneous fluid-dynamic forces on a bluff body*. Technical Report GALCIT FM96-5, California Institute of Technology. Funding by Office of Naval Research (ONR).
- [137] Noca, F., Shiels, D., & Jeon, D. (1997). Measuring Instantaneous Fluid Dynamic Forces on Bodies, Using Only Velocity Fields and Their Derivatives. *Journal of Fluids and Structures*, 11(3), 345–350.
- [138] Noca, F., Shiels, D., & Jeon, D. (1999). A Comparison Of Methods For Evaluating Time-dependent Fluid Dynamic Forces On Bodies, Using Only Velocity Fields And Their Derivatives. *Journal of Fluids and Structures*, 13(5), 551–578.
- [139] Ostieri, M., Mele, B., & Tognaccini, R. (2017). Exact aerodynamic force decomposition and dynamic force derivatives in non linear flows. In *Proceedings of the 55th AIAA Aerospace Sciences Meeting*, number 0 in AIAA SciTech Forum. American Institute of Aeronautics and Astronautics.
- [140] Ostieri, M., Mele, B., & Tognaccini, R. (2018a). Linear and Nonlinear Decomposition of Aerodynamic Force Acting on an Oscillating Plate. *AIAA Journal*, 56(2), 594–608.
- [141] Ostieri, M. & Tognaccini, R. (2018). On a Recently Proposed Vorticity-Based Definition of Wave Drag. *Journal of Aircraft*, 55(6), 2521–2523.
- [142] Ostieri, M., Tognaccini, R., Bailly, D., & Destarac, D. (2018b). Aerodynamic force and Lamb vector field in compressible unsteady flows. In *Proceedings of the 2018 AIAA Aerospace Sciences Meeting*, number 0 in AIAA SciTech Forum. American Institute of Aeronautics and Astronautics.
- [143] Oswatitsch, K. (1945). Der luftwiderstand als integral des entropiestromes. In *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* (pp. 88–90). Vandenhoeck & Ruprecht.
- [144] Oswatitsch, K. (1980). The Drag as Integral of the Entropy Flow. In W. Schneider & M. Platzer (Eds.), *Contributions to the Development of Gasdynamics: Selected Papers, Translated on the Occasion of K. Oswatitsch's 70th Birthday* (pp. 2–5). Wiesbaden: Vieweg+Teubner Verlag.
- [145] Palacios, F., Alonso, J., Duraisamy, K., Colonno, M., Hicken, J., Aranake, A., Campos, A., Copeland, S., Economon, T., Lonkar, A., Lukaczyk, T., & Taylor, T. (2013). Stanford University Unstructured (SU2): An open-source integrated computational environment for multi-physics simulation and design. In *Proceedings of the 51st AIAA Aerospace Sciences*

References

- Meeting including the New Horizons Forum and Aerospace Exposition*, number 0 in Aerospace Sciences Meetings. American Institute of Aeronautics and Astronautics.
- [146] Paladini, E., Couilleaux, A., Gelain, M., Sirvin, N., & Petropoulos, I. (2024). Exergy analysis of surface heat exchangers for aircraft engine applications. In *AERO 2024* Orléans, France.
- [147] Paparone, L. & Tognaccini, R. (2002). A Method for Drag Decomposition from Cfd Calculations. In *Proceedings of the ICAS2002 Congress*.
- [148] Paparone, L. & Tognaccini, R. (2003). Computational Fluid Dynamics-Based Drag Prediction and Decomposition. *AIAA Journal*, 41(9), 1647–1657.
- [149] Patel, A. & Bhardwaj, R. (2022). Propulsive performance of a two-dimensional elliptic foil undergoing interlinked pitching and heaving. *Physics of Fluids*, 34(11), 111904.
- [150] Petropoulos, I., Bailly, D., Atinault, O., Hantrais-Gervois, J.-L., Wervaecke, C., Boniface, J.-C., Dumont, A., Berhouni, I., Trapier, S., Morelli, M., de Vega, L. L., Thollet, W., Paladini, E., & Couilleaux, A. (2024). Overview of the IDEFFIX project: towards the development of the far-field exergy balance method at an industrial level of complexity. In *AERO 2024* (pp. AERO2024–15). Orléans, France: 3AF - Association Aéronautique et Astronautique de France.
- [151] Platzer, M. F., Jones, K. D., Young, J., & Lai, J. C. S. (2008). Flapping Wing Aerodynamics: Progress and Challenges. *AIAA Journal*, 46(9), 2136–2149.
- [152] Prandtl, L. (1904). Über Flüssigkeitsbewegung bei sehr kleiner Reibung. In *Proceedings of the 3rd International Mathematical Congress* (pp. 485–491). Leipzig.
- [153] Prandtl, L. (1919). *Theory of lifting surfaces*. Technical Report NACA TN-10, National Advisory Committee for Aeronautics. Part 1.
- [154] Prandtl, L. & Tietjens, O. G. (1934). *Applied Hydro- and Aeromechanics*. Dover Publications.
- [155] Prosser, D. (2022). Computational fluid dyNamics STRUctured grid CreaTor for 2D airfoils. Version 2.1. <https://sourceforge.net/projects/construct2d/>, Last accessed on: 05/11/2022.
- [156] Protas, B. (2011). On calculation of hydrodynamic forces for steady flows in unbounded domains. *Journal of Fluids and Structures*, 27(8), 1455–1460.
- [157] Protas, B., Styczek, A., & Nowakowski, A. (2000). An Effective Approach to Computation of Forces in Viscous Incompressible Flows. *Journal of Computational Physics*, 159(2), 231–245.
- [158] Quadrio, M., Chiarini, A., Banchetti, J., Gatti, D., Memmolo, A., & Pirozzoli, S. (2022). Drag reduction on a transonic airfoil. *Journal of Fluid Mechanics*, 942(R2).

- [159] Quagliarella, D. & Cioppa, A. D. (1995). Genetic algorithms applied to the aerodynamic design of transonic airfoils. *Journal of Aircraft*, 32(4), 889–891.
- [160] Quartapelle, L. & Napolitano, M. (1983). Force and moment in incompressible flows. *AIAA Journal*, 21(6), 911–913.
- [161] Rant, Z. (1956). Exergie, ein neues Wort für "technische Arbeitsfähigkeit". *Forschung im Ingenieurwesen*, 22, 36–37.
- [162] Richardson, E. G. (1936). The Physical Principles of Fish Locomotion. *Journal of Experimental Biology*, III(1), 63–74.
- [163] Roache, P. J. (1994). Perspective: A Method for Uniform Reporting of Grid Refinement Studies. *Journal of Fluids Engineering*, 116(3), 405–413.
- [164] Roache, P. J. (1998). *Verification and Validation in Computational Science and Engineering*. Hermosa Publishers.
- [165] Rumsey, C. (2021). Turbulence Modeling Resource - 2D NACA 0012 Airfoil Validation Case. National Aeronautics and Space Administration. https://turbmodels.larc.nasa.gov/naca0012_val.html, Last accessed on: 14/12/2023.
- [166] Ruscio, J., Duplaa, S., Aguirre, M. A., & Binder, N. (2022). Exergy Analysis of Unsteady Flow around an Adiabatic Cylinder in Vortex Shedding Condition. In *Proceedings of the 56th 3AF International Conference on Applied Aerodynamics*.
- [167] Ruscio, J., Duplaa, S., & Binder, N. (2023a). Unsteady Exergy Analysis of an Airfoil (OAT15A) under Transonic Buffet Condition. In *Proceedings of the 57th 3AF International Conference on Applied Aerodynamics*.
- [168] Ruscio, J. P., Duplaa, S., & Binder, N. (2023b). Wake energy recovery potential determination for unsteady flows by exergy analysis. In *Proceedings of the AIAA AVIATION 2023 Forum*, number 0 in AIAA AVIATION Forum. American Institute of Aeronautics and Astronautics.
- [169] Russo, L., Ostieri, M., & Tognaccini, R. (2018). Thrust Extraction from Vorticity Fields in Steady and Unsteady Flows. In *Proceedings of the 2018 Applied Aerodynamics Conference*, number 0 in AIAA AVIATION Forum. American Institute of Aeronautics and Astronautics.
- [170] Russo, L., Tognaccini, R., & Demasi, L. (2020). Box wing and induced drag: Compressibility effects in subsonic and transonic regimes. *AIAA Journal*, 58(6), 2398–2413.
- [171] Sietta, E., Minervino, M., & Tognaccini, R. (2023). Towards a "headache-free" flow region selection. In *Proceedings of the 57th 3AF International Conference on Applied Aerodynamics* Bordeaux, FR.

References

- [172] Saetta, E. & Tognaccini, R. (2023). Identification of Flowfield Regions by Machine Learning. *AIAA Journal*, 61(4), 1503–1518.
- [173] Saffman, P. G. (1993). *Vortex Dynamics*. Cambridge Monographs on Mechanics. Cambridge University Press.
- [174] Salter, J. W. (2021). Plot3d File Format for Grid and Solution Files. National Aeronautics and Space Administration. <https://www.grc.nasa.gov/www/wind/valid/plot3d.html>, Last accessed: Apr. 2024.
- [175] Sanmiguel-Rojas, E. & Fernandez-Feria, R. (2022). Numerical study of the propulsive performance of two-dimensional pitching foils at very high frequencies: scaling laws and turbulence effects. *International Journal of Numerical Methods for Heat and Fluid Flow*, 32(5), 1602–1617.
- [176] Schmitz, S. (2014). Finite Domain Viscous Correction to the Kutta-Joukowski Theorem in Incompressible Flow. *AIAA Journal*, 52(9), 2079–2083.
- [177] Schmitz, S. (2019). Drag Decomposition Using Partial-Pressure Fields in the Compressible Navier-Stokes Equations. *AIAA Journal*, 57(5), 2030–2038.
- [178] Schmitz, S. & Coder, J. G. (2015). Inviscid Circulatory-Pressure Field Derived from the Incompressible Navier-Stokes Equations. *AIAA Journal*, 53(1), 33–41.
- [179] Shen, W.-Z. & Loc, T.-P. (1997). A coupling finite difference/particle method for the resolution of 2D Navier-Stokes equations in velocity-vorticity form. *Aerospace Science and Technology*, 1(2), 97–109.
- [180] Slater, J. W. (2021). Examining Spatial (Grid) Convergence. <https://www.grc.nasa.gov/www/wind/valid/tutorial/spatconv.html>, Last accessed on: 30-05-2022.
- [181] Smith, S. C. (1996). *A Computational and Experimental Study of Nonlinear Aspects of Induced Drag*. Technical Paper 3598, National Aeronautics and Space Administration.
- [182] Sozer, E., Brehm, C., & Kiris, C. C. (2014). Gradient Calculation Methods on Arbitrary Polyhedral Unstructured Meshes for Cell-Centered CFD Solvers. In *Proceedings of the 52nd Aerospace Sciences Meeting*, number 0 in AIAA SciTech Forum. American Institute of Aeronautics and Astronautics.
- [183] Spalart, P. R. (2008). On the far wake and induced drag of aircraft. *Journal of Fluid Mechanics*, 603, 413–430.
- [184] Spalart, P. R. (2015). Extensions of d'Alembert's paradox for elongated bodies. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 471(2181), 20150106.

- [185] Spekreijse, S. P., Boerstoeel, J. W., Vitagliano, P. L., & Kuyvenhoven, J. L. (1992). *Domain modeling and grid generation for multi-block structured grids with application to aerodynamic and hydrodynamic configurations*. Technical Report 92N24411, NASA. Langley Research Center, Software Surface Modeling and Grid Generation Steering Committee.
- [186] Squire, H. B. & Young, A. D. (1937). *The calculation of the profile drag of aerofoils*. Technical Report 1838, Aeronautical Research Committee Reports & Memoranda.
- [187] Steger, J. L. & Baldwin, B. (1973). Shock Waves and Drag in the Numerical Calculation of Compressible, Irrotational Transonic Flow. *AIAA Journal*, 11(7), 903–904.
- [188] Sticchi, E., Minervino, M., & Tognaccini, R. (2023). Aerodynamic Force and Vortex Force on Flapping Wings. In *Proceedings of the AIAA AVIATION 2023 Forum*, number 0 in AIAA AVIATION Forum. American Institute of Aeronautics and Astronautics.
- [189] Stoliker, F. N. (1979). *Guide to In-Flight Thrust Measurement of Turbojets and Fan Engines*. Technical Report AGARD-AG-237, Advisory Group for Aerospace Research and Development. MIDAP Study Group.
- [190] Syrakos, A., Varchanis, S., Dimakopoulos, Y., Goulas, A., & Tsamopoulos, J. (2017). A critical analysis of some popular methods for the discretisation of the gradient operator in finite volume methods. *Physics of Fluids*, 29(12), 127103.
- [191] Taylor, G. I. (1926). Note on the connection between the lift on an aerofoil in a wind and the circulation round it. *Phil. Trans. R. Soc. London A*, 225, 238–246.
- [192] Tecplot, I. (2023a). TecIO library. <https://tecplot.com/products/tecio-library/>, Last accessed: Apr. 2024.
- [193] Tecplot, I. (2023b). Tecplot 360. <https://tecplot.com/>, Last accessed: Apr. 2024.
- [194] Theodorsen, T. (1934). *General Theory of Aerodynamic Instability and the Mechanism of Flutter*. Naca technical report, National Advisory Committee for Aeronautics. NACA-TR-496.
- [195] Thompson, B. A. (1944). *A Simple Method of Computing Cd from Wake Traverses at Highsubsonic Speeds*. Technical report, Ministry of Supply, Aeronautical Research Council, Reports and Memoranda.
- [196] Thomson, W. (1869). On Vortex Motion. *Transactions of the Royal Society of Edinburgh*, 25, 217–260.
- [197] Tognaccini, R. (2003). *Methods for Drag Decomposition, Thrust-Drag Bookkeeping from CFD Calculations*, chapter 02. CFD-Based Aircraft Drag Prediction and Reduction, VKI Lecture Series, von Karman Inst. for Fluid Dynamics: Waterloostraat 72, 1640 Sint-Genesius-Rode, Belgium.

References

- [198] Tognaccini, R. (2005). Drag Computation and Breakdown in Power-on Conditions. *Journal of Aircraft*, 42(1), 245–252.
- [199] Tognaccini, R. (2019). *Lezioni Di Aerodinamica Dell'ala Rotante*. University of Naples Federico II.
- [200] Tognaccini, R., Saetta, E., & Mele, B. (2021). Aerodynamic force breakdown: Thermodynamic vs Vortex-force methods. In *Proceedings of the AIAA AVIATION 2021 FORUM*, number 0 in AIAA AVIATION Forum. American Institute of Aeronautics and Astronautics.
- [201] Toubin, H. & Bailly, D. (2015). Development and Application of a New Unsteady Far-Field Drag Decomposition Method. *AIAA Journal*, 53(11), 3414–3429.
- [202] Toubin, H., Bailly, D., & Costes, M. (2016). Improved Unsteady Far-Field Drag Breakdown Method and Application to Complex Cases. *AIAA Journal*, 54(6), 1907–1921.
- [203] Ueno, M., Yamamoto, K., Tanaka, K., Murayama, M., & Tognaccini, R. (2013). Far-Field Drag Analysis of NASA Common Research Model Simulation. *Journal of Aircraft*, 50(2), 388–397.
- [204] van Dam, C. P. (1999). Recent experience with different methods of drag prediction. *Progress in Aerospace Sciences*, 35(8), 751–798.
- [205] van der Vooren, J. (2006). Streamwise vorticity in viscous, compressible, steady flow about aircraft. *Aerospace Science and Technology*, 10(4), 288–294.
- [206] van der Vooren, J. (2008). On drag and lift analysis of transport aircraft from wind tunnel measurements. *Aerospace Science and Technology*, 12(4), 337–345.
- [207] van der Vooren, J. & Slooff, J. W. (1990). *CFD-Based Drag Prediction; State of the Art, Theory, Prospects*. Lectures Notes NLR TP 90247, Lectures Notes, AIAA Professional Studies Series, Course on Drag-Prediction and Measurement, Dutch Aerospace Lab.
- [208] van Oudheusden, B. W. (2013). PIV-based pressure measurement. *Measurement Science and Technology*, 24(3), 032001.
- [209] Verley, S. (2012). *Evaluation du couple "champ lointain" d'un rotor d'hélicoptère en vol stationnaire: analyse de résultats issus de simulations numériques de mécanique des fluides*. PhD thesis, Université d'Orléans, France. NNT: 2012ORLE2087.
- [210] Von Kármán, T. & Burgers, J. M. (1935). General aerodynamic theory : Perfect fluids. In W. F. Durand (Ed.), *Aerodynamic Theory* (pp. 1–24). Berlin, Heidelberg: Springer Berlin Heidelberg.
- [211] Von Kármán, T. H. & Sears, W. R. (1938). Airfoil Theory for Non-Uniform Motion. *Journal of the Aeronautical Sciences*, 5(10), 379–390.

- [212] Von Kármán, T. V. (1921). Über laminare und turbulente Reibung. *Z. angew. Math. Mech.*, 1(4), 233–252.
- [213] Wagner, H. (1925). Über die Entstehung des dynamischen Auftriebes von Tragflügeln. *Z. angew. Math. Mech.*, 5(1), 17–35.
- [214] Wang, D., Lin, Q., Zhou, C., & Wu, J. (2022). Aerodynamic performance of a self-propelled airfoil with a non-zero angle of attack. *Physics of Fluids*, 34(3), 031901.
- [215] Wang, S., Zhang, X., He, G., & Liu, T. (2015). Evaluation of Lift Formulas Applied to Low-Reynolds-Number Unsteady Flows. *AIAA Journal*, 53(1), 161–175.
- [216] Wu, C. & Kinnas, S. A. (2019). A 3-D VIScous Vorticity Equation (VISVE) Method Applied to Flow Past a Hydrofoil of Elliptical Planform and a Propeller. In *Proceedings of the sixth International Symposium on Marine Propulsors Rome, Italy*.
- [217] Wu, J., Liu, L., & Liu, T. (2018). Fundamental theories of aerodynamic force in viscous and compressible complex flows. *Progress in Aerospace Sciences*, 99, 27–63.
- [218] Wu, J. C. (1981). Theory for Aerodynamic Force and Moment in Viscous Flows. *AIAA Journal*, 19(4), 432–441.
- [219] Wu, J. C. (1982). Problems of general viscous flow. *NASA STI/Recon Technical Report A*, 82, 69–109.
- [220] Wu, J. Z. (1986). Incompressible theory of the interaction between moving bodies and vorticity field- The generation of vorticity by body surfaces and its dissipation. *Acta Aerodynamica Sinica*, 4, 168–176.
- [221] Wu, J.-Z., Lu, X.-Y., & Zhuang, L.-X. (2007). Integral force acting on a body due to local flow structures. *Journal of Fluid Mechanics*, 576, 265–286.
- [222] Wu, J.-Z., Ma, H.-Y., & Zhou, M.-D. (2006). *Vorticity and Vortex Dynamics*. Springer, Berlin, Heidelberg.
- [223] Wu, J. Z. & Wu, J. M. (1989). Vortical Sources of Aerodynamic Force and Moment. In *Aerospace Technology Conference and Exposition: SAE International*.
- [224] Wu, J.-Z. & Wu, J.-M. (1993). Interactions between a solid surface and a viscous compressible flow field. *Journal of Fluid Mechanics*, 254, 183–211.
- [225] Wu, J. Z., Wu, J. M., & Wu, C. J. (1988). A viscous compressible flow theory on the interaction between moving bodies and flow field in the (ω, ϑ) framework. *Fluid Dynamics Research*, 3(1), 203–208.
- [226] Wu, J.-Z., Zhou, Y., Lu, X.-Y., & Fan, M. (1999). Turbulent force as a diffusive field with vortical sources. *Physics of Fluids*, 11(3), 627–635.

References

- [227] Wu, T., Song, B., Song, W., Yang, W., Xue, D., & Han, Z. (2022). Lift performance enhancement for flapping airfoils by considering surging motion. *Chinese Journal of Aeronautics*, 35(9), 194–207.
- [228] Yamazaki, W., Matsushima, K., & Nakahashi, K. (2008). Aerodynamic Design Optimization Using the Drag-Decomposition Method. *AIAA Journal*, 46(5), 1096–1106.
- [229] Young, J. & Lai, J. C. S. (2004). Oscillation Frequency and Amplitude Effects on the Wake of a Plunging Airfoil. *AIAA Journal*, 42(10), 2042–2052.
- [230] Yu, N., Chen, H., & Samant, S. (1983). Inviscid drag calculations for transonic flows. In *Proceedings of the 6th Computational Fluid Dynamics Conference Danvers*, number 6 in Fluid Dynamics and Co-located Conferences. American Institute of Aeronautics and Astronautics.
- [231] Zou, S., Liu, L., & Wu, J. (2021). Numerical validation and physical explanation of the universal force theory of three-dimensional steady viscous and compressible flow. *Physics of Fluids*, 33(036107).
- [232] Zou, S.-F., Wu, J. Z., Gao, A.-K., Liu, L., Kang, L., & Shi, Y. (2019). On the concept and theory of induced drag for viscous and incompressible steady flow. *Physics of Fluids*, 31(6), 065106.

List of Author's Publications

JOURNAL ARTICLES

- Minervino M. and Tognaccini R.. “*Aerodynamic force by Lamb vector integrals in unsteady compressible flows*”, International Journal of Numerical Methods for Heat & Fluid Flow, 34 (7): 2654–2687 (2024). DOI: <https://doi.org/10.1108/HFF-06-2023-0350>.
- Minervino M. and Tognaccini R.. “*On the spurious effects in Lamb-vector-based force decomposition methods*”, Aerospace Science and Technology, 142 (B): 108674 (2023). DOI: <https://doi.org/10.1016/j.ast.2023.108674>.
- Minervino M. and Tognaccini R.. “*A unified thermodynamic/Lamb-vector-based analysis of the aerodynamic force*”, Physics of Fluids, 35 (9): 097135 (2023). DOI: <https://doi.org/10.1063/5.0164384>.
- Minervino M., Andreutti G., Russo L. and Tognaccini R.. “*Drag Reduction by Wingtip-Mounted Propellers in Distributed Propulsion Configurations*”, MDPI Fluids, 7 (7): 212 (2022). DOI: <https://doi.org/10.3390/fluids7070212>.

CONFERENCE PAPERS*

- D’Aniello F. A., Catalano P., Quagliarella D. and Minervino M.. “*Aerodynamic design of a compliant morphing flap for the next generation hybrid-electric regional aircrafts*”, 14th European Aeronautics Science Network (EASN) International Conference on “Innovation in Aviation and Space towards sustainability today & tomorrow”, October 8-11, 2024, Thessaloniki, GR. DOI: Proceedings in press.
- Bown N. W., Simpson R., Baert L., Lepot I., Minervino M., Barbarino M., Mingione G., Giacché D., Drack L., Lammers K., Habing R., Gardarein P., Delrieux Y., Balbo M. and Di Giulio M.. “*Experimental and Analytical Evaluation of Low-Noise Propeller Designs for Large Regional Aircraft*”, AIAA 2024-2637, AIAA SciTech 2024 Forum, January 8-12, 2024, Orlando, FL, USA. DOI: <https://doi.org/10.2514/6.2024-2637>.

*Speaker.

List of Author's Publications

- Sticchi E., Minervino M. and Tognaccini R.. “*Aerodynamic Force and Vortex Force on Flapping Wings*”, AIAA 2023-4090, AIAA AVIATION 2023 Forum, June 12-16, 2023, San Diego, CA & Online, USA. DOI: <https://doi.org/10.2514/6.2023-4090>.
- Saetta E., Minervino M. and Tognaccini R.. “*Towards a “headache-free” flow region selection*”, AERO2023-07, 57th 3AF International Conference on Applied Aerodynamics, March 29-31, 2023, Bordeaux, FR. URL: <https://hdl.handle.net/11588/916288>.
- Minervino M. and Tognaccini R.. “*Aerodynamic Force by Lamb Vector Integrals in Unsteady Compressible Flows*”, AERO2023-37, 57th 3AF International Conference on Applied Aerodynamics, March 29-31, 2023, Bordeaux, FR. URL: <https://hdl.handle.net/11588/916503>.
- Minervino M. and Tognaccini R.. “*Wave Drag and Vorticity in Two-Dimensional Viscous and Inviscid flows*”, AIAA 2023-1032, AIAA SciTech 2023 Forum, January 23-27, 2023, National Harbor, MD & Online, USA. DOI: <https://doi.org/10.2514/6.2023-1032>.
- French A., Mingione G., Schettino A., Roncioni P., Vitagliano P. L. and Minervino M.. “*Parametric Studies and Simulations of a Hydrogen Micromix Combustor*”, GT2022-81784, V10CT32A021, ASME Turbo Expo 2022, June 13-17, 2022, Rotterdam, NL. DOI: <https://doi.org/10.1115/GT2022-81784>.

CONFERENCE ABSTRACTS†

- Minervino M. and Tognaccini R.. “*A Unified Thermodynamic/Vortical Far-Field Force Method*”, 1st European Fluid Dynamics Conference (EFDC1), September 16-20, 2024, Aachen, DE. URL: <https://hdl.handle.net/11588/975143>.
- di Mare F., Donndorf J., Lo Presti F., Quagliarella D., Donelli R., Saccone G., Andreutti G., French A. D., Minervino M., Russo S., Natale N., Mele B., Merola U., Viola N., Ferretto D., Fusaro R., Gori O., Graziani S., Piccirillo G., Fureby C., Bai X., Richter M. and Birken P.. “*MYTHOS: Medium-Range Hybrid Low-Pollution FlexiFuel/Hydrogen Sustainable Engine*”, 13th EASN International Conference, September 5-8, 2023, Salerno, IT. URL: <https://hdl.handle.net/11588/942658>.
- Minervino M. and Tognaccini R.. “*Wave Drag and Vorticity in Viscous and Inviscid flows*”, 14th European Fluid Mechanics Conference (EFMC14), September 13-16, 2022, Athens, GR. URL: <https://hdl.handle.net/11588/898715>.

CONFERENCE POSTERS‡

†Speaker.

‡Speaker.

- Minervino M.. “*A unified thermodynamic/vortical theory for the aerodynamic force analysis*”, PhD International Intensive School, 1st Edition: Towards Zero Emissions Mobility, October 07-11, 2024, Naples, IT. DOI: <http://dx.doi.org/10.13140/RG.2.2.12061.73448>.

List of Author's Publications

Colophon

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