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Spacetime, Symmetries, and Observers at the interface between Quantum Theory and Gravity

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Abstract

The incompatibility between Quantum Theory and General Relativity gives rise to several logical conundrums, and different approaches to the Quantum Gravity problem have arisen in trying to solve them. Most of these approaches share the expectation that the smooth spacetime of General Relativity, with local symmetries given by the Poincaré algebra, should be replaced by a *fundamentally discretised* structure, with a minimum localisation limit for spacetime events. The existence of such a scale requires a deformation of the local symmetries to retain the *covariance* of physical laws.

Such deformations naturally arise as the symmetries of non-commutative spacetimes, which are models realising the idea of a *quantum spacetime*, where the transformations between reference frames become non-classical. Interestingly, similar features are found in a seemingly unrelated research area in the foundations of Quantum Theory: the quantum reference frames research programme, in which abstract reference frames are replaced by physical systems that act as reference for the description of physical laws. These systems can be quantum in nature and hence define reference frames with fuzzy properties. Additionally, both the deformation of symmetries and the quantisation of reference frames suggest that measurement apparatuses and observers should acquire non-classical features.

Symmetries, reference frames, observers, and measurement apparatuses: Following the arguments above, it seems that all the pillars of General Relativity, and Physics in general, have to be thoroughly scrutinised in the quest for a theory of Quantum Gravity. This thesis aims to investigate such foundational issues and answer the following questions: How do quantum systems evolve in a spacetime with deformed symmetries? How does the quantisation of reference frames affect the observed physical properties of quantum systems? How do measurement apparatuses behave in a non-classical spacetime and how are communication protocols between observers affected?

The relevance of these questions is rooted in the growing interest that has sparked in recent years in exploring the foundations of Quantum Gravity, independently of what the ultimate full theory will be, to advance the logical understanding of the fundamental principles on which such a theory should be built on. This thesis explores this sector of theoretical physics by focusing on how quantum systems are affected by the deformations of spacetime symmetries and the quantisation of the degrees of freedom describing reference frames. By employing innovative techniques and deriving some significant results, the research work presented in this thesis pushes the boundaries of the theoretical investigations and further stimulates discussions in the foundations of Quantum Gravity.

This thesis is based on the following papers [1–5].

[1] M. Arzano, V. D'Esposito and G. Gubitosi, *Fundamental decoherence from quantum spacetime*, Commun. Phys. **6** (2023) 242 [2208.14119]

[2] G. Amelino-Camelia, V. D'Esposito, G. Fabiano, D. Frattulillo, P.A. Höhn and F. Mercati, *Quantum Euler angles and agency-dependent space-time*, PTEP **2024** (2024) 033A01 [2211.11347]

[3] V. D'Esposito and G. Gubitosi, *Constraints on quantum spacetime-induced decoherence from neutrino oscillations*, Phys. Rev. D (2024) 033A01 [2306.14778]

[4] V. D'Esposito, G. Fabiano, D. Frattulillo and F. Mercati, *Doubly Quantum Mechanics*, [2412.05997]

[5] V. D'Esposito, F. Giacomini and G. Gubitosi, *Quantum clocks resolution affects the detection of mass superpositions for fundamental particles*, In preparation

Prologue

One of the most marvellous skills we have developed in our evolution is the ability to harness in abstract mathematical laws the complexity of the physical world we observe. From ancient times we were sharpening sticks and going after lions, to the present era, when we are expanding the scopes of our investigations to the fundamental constituents and to the very edges of our Universe, we developed more and more complicated and abstract theories to explain the data we collected with our ever-improving probes. This, in turn, led us to reveal (some of) the mechanisms of *Nature*.

All the knowledge acquired during this impressive journey gave physicists the feeling that everything could be described with some mathematical laws at the end of the 19^{th} century. Everything except a handful of phenomena that did not fit into the existing theories. At the beginning of the 20^{th} century, these phenomena[1], which were lacking an accommodation on the chessboard of physical theories, sparked one of the deepest scientific revolutions of human history. The synthesis of this revolution marked the genesis of *General Relativity* and *Quantum Mechanics*. These, together with Special Relativity and Statistical Mechanics, are the pillars of modern physics, the theories on which the majority of the scientific achievements and technological innovations of the last century are built. Furthermore, they also reshaped our philosophical beliefs about the nature of space, time, and the objectiveness of *Reality*.

Special Relativity, developed by Einstein in 1905, was the first step in this revolution [6]. The idea behind this theory is that physical laws should be the same in all inertial reference frames. When referred to electromagnetism, this relativity principle requires the velocity of electromagnetic waves to be the same in all inertial reference frames. The existence of such an invariant velocity scale was in strict contrast with Galilean relativity, and the price to pay for introducing it into the physical laws was to abandon the notion of *absolute simultaneity* and hence absolute time. Moreover, the notions of space and time were unified to form a new entity, *spacetime*. With the derivation of General Relativity in 1915, the principle of relativity was further extended to require that physical laws should be the same in all general reference frames [7]. In setting up this idea, Einstein also provided a new theory of gravitation [8]. Indeed, in General Relativity, Gravity is not intended at the fundamental level as an interaction, rather it is a manifestation of a *non-trivial geometry* of spacetime [9]. The latter loses the status of a static arena on top of which physical phenomena unfold to become the dynamical entity of the theory, whose geometry in a given region depends on the amount, distribution, and type of matter and energy in that region. Vice versa, the dynamics of matter

[1] Some (in)famous examples are: the photoelectric effect, the problem of the propagation velocity of electromagnetic waves in Maxwell's theory of electromagnetism, the UV catastrophe, the discreteness of absorption and emission spectra of atomic elements, the precession of the perihelion of Mercury.

is determined by spacetime geometry. This two-way feedback between spacetime and matter was summarised by John Wheeler with the phrase *"matter tells spacetime how to curve, and curved spacetime tells matter how to move"*.

The second building block of the revolution was Quantum Mechanics that, unlike General Relativity, was the result of a collective effort. Indeed, even though the fathers of Quantum Mechanics are typically considered to be Max Planck and Niels Bohr, significant contributions were made by Albert Einstein, Louis de Broglie, Max Born, Paul Dirac, Werner Heisenberg, Wolfgang Pauli, Erwin Schrödinger, John von Neumann, and many others. The first phenomenological observations acting as a driving force for the development of the theory were the so-called "UV catastrophe", the photoelectric effect, and the discrete absorption and emission spectra of atomic elements. The first problem was tackled by Max Planck, who introduced the idea of *quantised energy* [10]. This was later used by Einstein [2] and Bohr to explain the photoelectric effect [12] and the discreetness of the atomic spectra [13], respectively. Starting from these ideas, the formalism of Quantum Mechanics was developed, turning the latter into a full-fledged theory, which reshaped our understanding of the atomic world and opened the path to the subatomic realm. From a foundational point of view, Quantum Mechanics describes a *World of probabilities* [14]. In fact, physical systems are described by *states* that only contain information on the probability of obtaining a given result after a measurement. In general, physical systems can be in superposition of different positions, momenta, or any other physical observable, measurements being the procedures that settle them into states with definite values of these observables[3]. It is then evident that measurements in Quantum Mechanics change the states of physical systems. This means that, in general, making a measurement of an observable and then making a measurement of a second observable does not yield the same results that would be obtained by changing the order of the measurements. This is the *complementarity* aspect of Quantum Mechanics, which follows from the *uncertainty principles*: there exist some complementary observables whose measurements produce different results if performed in different order. Moreover, if an observer performs a measurement of a given observable and then a second measurement of a complementary observable, there is no inference that can be made about the result that a subsequent measure of the first observable would yield. Therefore, another novelty introduced by Quantum Mechanics is the notion of *agency*: the *operationally meaningful* properties of physical systems *depend on the choices* made by the observers.

The scientific success of these two theories was granted by their extraordinary predictive power. With General Relativity, the observed precession of the perihelion of Mercury was explained [15], and a number of new phenomena were predicted, such as gravitational lensing and the existence of black holes and gravitational waves. While the first was already observed in 1979 [16], the latter two were directly observed only in recent years, a century after the development of the theory, with the efforts of the Event Horizon Telescope (2019) [17] and the LIGO and Virgo Collaborations (2016) [18], respectively. On top of this, General Relativity opened the doors of cosmology, making it possible to study the evolution of our Universe

[2]Who is therefore sometimes considered to be the third father of Quantum Mechanics, although he then strongly criticised the direction that physicists took for the development of the theory [11].

[3]Throughout this thesis we adopt the Copenhagen interpretation and do not deal with the issue of the interpretation of Quantum Mechanics.

from the beginning of time to the present era, passing through the formation of galaxies, stars, and planets. Quantum Mechanics, on the other hand, was responsible of some great technological advancements, such as, among others, the discovery of superconductivity, tunnel field effect transistors, LEDs and medical diagnostic apparatuses like the positron emission tomography (PET) and the magnetic nuclear resonance (MNR). Today, great efforts are being made for the creation of quantum computers and the development of quantum information theory and quantum cryptography. On the foundational side, the attempts to create a quantum theory compatible with the relativistic prescriptions of Special Relativity led to the discovery of Quantum Field Theory. In this formalism, the Standard Model of particle physics was developed [19], providing the most accurate description to date of the fundamental building blocks of our Universe: leptons, gauge bosons, and quarks.

Considering the impact that General Relativity and Quantum Theory had and that they were developed during the same historical period, it is astonishing to realise how *fundamentally incompatible* they are. On the one hand, there is the smooth world of General Relativity, which describes Gravity at the macroscopic scale, on the other the fuzzy probabilistic realm of Quantum Theory. The interface between the two theories is the scope of interest of the *Quantum Gravity* research field. The crucial point that arises at this interface can be understood through a gedankenexperiment proposed by Feynman at the Chapel Hill conference in 1957 [20]. In this proposal, a quantum particle sources a gravitational field and is prepared in a superposition of two positions. Feynman argued that the gravitational field has to be in a quantum superposition of the two possible configurations associated with the two particle positions. He stated that "if you believe in Quantum Mechanics up to any level, then you have to believe in gravitational quantisation to describe this experiment". Since in General Relativity the gravitational field is a manifestation of spacetime geometry, this gedankenexperiment is also setting the stage for the notion of *quantum spacetime*, namely the idea that the smooth spacetime structure described by the Riemannian geometry of General Relativity has to be replaced by some more fundamental fabric, capable of harnessing the quantum nature of spacetime in the Quantum Gravity regime. The latter could have striking consequences for some foundational aspects of physics, such as causality and the notion of time. Indeed, since the spacetime structure defines the causal relations between events, as well as the flow of time, a spacetime with a quantum nature could in principle generate indefinite causal structures and allow time to flow in a quantum superposition [21]. Other foundational questions, which will be relevant for this thesis, regard the role of symmetries and observers in a quantum spacetime. Indeed, if we accept the notion of quantum spacetime, we have to be ready to address the problem of defining reference frames, which in general will be related by non-classical symmetries, and observers related by such symmetries could be intended as *quantum observers*. Moreover, by building on the quantum-mechanical idea of agency, this could challenge the existence of an objective spacetime that all observers agree upon: how spacetime is perceived by observers may be influenced by their measurement choices. Conversely, the quantum properties of spacetime will likely affect the spectrum of possible operations available to an agent.

It is not unexpected that so many fundamental issues arise when considering the interface between Quantum Theory and Gravity. On one hand, General Relativity only describes the effects of classical matter on the spacetime structure by means of the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,, \tag{1}$$

where the left-hand side regards the geometry of spacetime, and the right-hand side is the source term, constituted by the stress-energy tensor of the (classical) matter/energy distribution sourcing the gravitational field. In these equations, there is no room for quantum matter. On the other hand, the Standard Model of particle physics, and Quantum Theory in general, does not include Gravity in its description[4]. The reason for this is that the gravitational effects caused by quantum systems are negligibly small compared to those caused by other fundamental interactions, in the regime in which we have tested Quantum Theory so far. This regime is determined by the energy scale reached today at LHC $E_{\text{LHC}} \approx 10$ TeV. The Quantum Gravity scale is not known and can only be deduced heuristically. One possible way to do this is to consider at which scale the localisation limits of General Relativity and Quantum Theory are saturated for a physical system. In Quantum Theory, physical systems cannot be localised better than their Compton wavelength $\Delta r \gtrsim \frac{\hbar}{mc}$, m being the mass; in General Relativity, a physical system cannot be localised better than its Schwarzschild radius $\Delta r \gtrsim \frac{Gm}{c^2}$. Requiring that these two constraints are satisfied, we get (see fig. 1)

$$\Delta r \gtrsim \ell_P = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35}\,\text{m}\,. \tag{2}$$

This is the so-called Planck length, commonly understood as the fundamental localisation limit for physical systems. It can also be interpreted as the breakdown scale of General Relativity, which is haunted by the emergence of divergences in some regimes. These are called *singularities*, and examples are given by black holes and the singularity at the beginning of time, the Big Bang. The existence of such singularities is commonly accepted and, under some hypotheses [23], it has also been rigorously demonstrated to be a necessity in the General Relativity framework for cosmological models. The problem is that we do not actually know what happens when distances and time intervals smaller than Planckian are involved, and the validity of General Relativity in these regimes, where quantum-mechanical and gravitational effects become comparable (2), is highly questionable. It is therefore plausible that, in order to investigate the early universe regime, as well as to get a fundamental characterisation of black holes, a complete theory of Quantum Gravity is needed.

The Planck length (2) can be easily converted into Planck energy $E_P = c^4 \ell_P \approx 10^{19}$ GeV, time $t_P = c^{-1}\ell_P \approx 10^{-44}$ s, or mass $M_P = c^2 \ell_P \approx 10^{19}$ GeV$c^{-2} = 10^{-5}$ g. Performing the same conversion for the LHC energy scale, $\ell_{\text{LHC}} \approx 10^{-20}$ m, we get a hint on why the quest for a theory of Quantum Gravity is often considered to be the hardest problem in theoretical physics: from the beginning of our evolution to the present day we were able to build a "microscope"

[4]With some exceptions. For instance, it is possible to consider the effects of an external Newtonian potential in Quantum Mechanics or to consider a Quantum Field Theory on a curved background [22], but in these cases the gravitational content is either reduced to a simple potential (Quantum Mechanics) or to a background on which physics unfolds (Quantum Field Theory on curved spacetime). The back-reaction of quantum systems on the structure of the gravitational field, hence the dynamics of spacetime in the presence of quantum sources, is never taken into account.

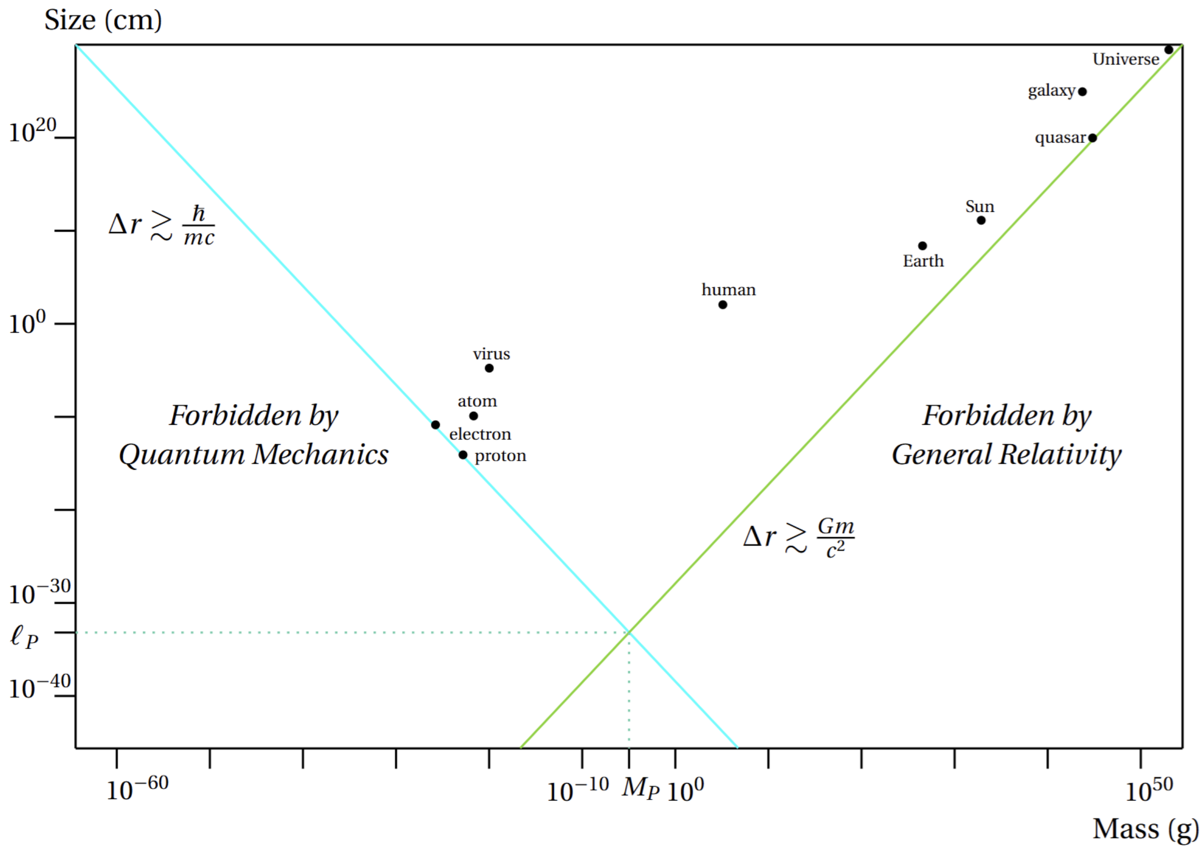

Figure 1: The emergence of the Planck scale from the interface between the physical domains forbidden by Quantum Mechanics and General Relativity. Image adapted from [24]

that, starting from our everyday scale 10^0 m, allowed us to reach the microscopic world at 10^{-20} m; however, this is 15 orders of magnitude away from the putative Quantum Gravity scale ℓ_P. For this reason, so far there are no experimental observations that can serve as hints for theoretical investigations, as happened for the 20$^{\text{th}}$ century revolution, and probably there will never be direct observations at the Planck scale, for a galaxy-spanning particle accelerator would be needed to reach energies of the order of E_P. Since experimental observations offer little to no guidance, Quantum Gravity research was purely speculative in its infancy [25]. This caused a blooming of different theoretical approaches to the Quantum Gravity problem, each of them starting from a subset of the logical conundrums that emerge at the interface between Quantum Theory and Gravity [26]. Some of them, like String Theory [27–30], try to find a unified theory of Gravity and all other fundamental interactions, the so-called "Theory of Everything". Other approaches, such as Loop Quantum Gravity [31, 32] or Causal Dynamical Triangulations [33], "just" focus instead on the problem of the quantisation of Gravity.

All the frameworks listed above are approaches to the Quantum Gravity problem that are aimed at deriving a full-fledged theory. However, there are other avenues that can be followed. For example, several studies have focused on the development of phenomenological models that rely on expected features of spacetime at the Planck scale. This possibility has been majorly investigated within the frameworks of non-commutative spacetime [34, 35] and deformation [36, 37] or breaking [38] of spacetime symmetries. Over the last three decades, this led to the establishment of the Quantum Gravity phenomenology research field [39–42], whose aim is to derive putative Quantum Gravity effects that can be somehow "amplified" to the level of becoming potentially observable with the experimental sensitivities reached, for instance, in multi-messenger astrophysics [43].

Another more recent path that is being followed, which could be referred to as the "First Principles approach", investigates the interface between Quantum Theory and Gravity with methodologies characterising the field of Quantum Foundations and tools from Quantum Information Theory [44]. This is done in such a way as to derive logical constraints on the non-classical nature of the gravitational field [45] or to experimentally reach a physical regime in which the effects of the gravitational field sourced by a mass in a quantum superposition is measured [46–51], realising the gedankenexperiment proposed by Feynman at the Chapel Hill conference in 1957 [20]. This is feasible since the Quantum Gravity scale relevant for such experiments is the Planck mass, which, unlike the Planck length and time, is surprisingly close to our everyday world $M_P \approx 10^{-5}$ g [52]. This new phenomenological window is quite interesting, since it would allow to perform table-top experiments to test the quantum nature of Gravity. Indeed, while astrophysical analyses have enough experimental sensitivity to test some Quantum Gravity effects, the problem lies in the scarcity of data deriving from the total absence of control on the sources. These analyses are promising [43,53], but will require some time to collect a significant amount of data. Conversely, reaching the necessary sensitivity for table-top experiments would give the possibility of performing controlled and replicable tests of Quantum Gravity effects.

The research work discussed in this thesis falls within the approaches described in the last two paragraphs. The focus will be on quantum spacetime, its symmetries, the effects that can be derived from them, and quantum observers. The manuscript is divided into two parts.

In the first part, we review all the mathematical structures and physical notions necessary to discuss the original work presented in the second part. The first chapter focuses on quantum spacetime, specifically non-commutative spacetime, and its symmetries, described in the language of Hopf algebras. The phenomenological opportunities offered by some models are then described. The second chapter reviews some quantum information tools at the interface between Quantum Theory and Gravity, such as quantum clocks and quantum reference frames, discussing what kind of questions can be investigated with these tools.

As anticipated, the second part contains the original work of the thesis and is based on [1–5]. The third chapter is devoted to quantum spacetime-induced decoherence. In the first part of the chapter, we present a review of the quantum decoherence programme. Then we present the results of [1], in which a decoherence process is derived for the evolution of quantum systems evolving in a spacetime with deformed symmetries, *i.e.* a deformed time translation. There exits several Quantum Gravity-induced decoherence models in the literature, and one possible phenomenological window to test them is given by neutrino oscillations. This latter opportunity is investigated in the final part of the chapter, based on [3], in which we show that the decoherence process originating from stochastic metric perturbations is ruled out using KamLAND and Super-Kamiokande data. The fourth chapter is devoted to quantum reference frames and is based on [5]. We consider the problem of mass superselection in Quantum Mechanics and show that a mass superselection rule emerges for fundamental particles from the properties of quantum reference frames transformations and finite-resolution quantum clocks. This superselection rule singles out flavour neutrinos as the only fundamental particles that can be observed in superposition. The last chapter is based on [2,4] and investigates the problem of quantum observers. We focus on a deformation of the rotation sector of spacetime symmetries described by the quantum group $SU_q(2)$, deformation of the standard $SU(2)$ Lie group. This gives rise to a framework in which observers cannot reconstruct an objective notion of space(time), which becomes *agency-dependent*, and measurement devices exhibit non-classical features exemplified by a new notion of quantum probability.

Part I

Prolegomena

First Chapter

Quantum spacetime and quantum symmetries

As anticipated in the Prologue, the ultimate goal of the Quantum Gravity research field is to find a fundamental description of the spacetime structure at the quantum level. This should be achieved by replacing classical spacetime with some quantum structure, which is usually referred to as *quantum spacetime*. However, there is no general consensus on what this term actually means. A concrete realisation of this notion, which will be considered in this thesis, is provided by non-commutative geometry [34, 54–56]. This idea builds on an analogy with the quantisation of the phase space of a point particle. In that context, the tool used for this quantisation is based on the correspondence of position and momenta with non-commuting operators on the Hilbert space of the states describing the point particle. The notion of a quantum spacetime, therefore, could be described by promoting the spacetime coordinates to non-commuting operators acting on a Hilbert space whose elements are the states that describe, in some sense, the quantum structure of spacetime. This structure is called *non-commutative spacetime*. Of course, a non-commutative spacetime cannot possibly grasp all the physical properties of a full theory of Quantum Gravity. Physical theories built on the idea of spacetime non-commutativity are therefore intended as effective models that could arise from a full theory of Quantum Gravity in some physical regimes. For instance, this is the case for Loop Quantum Gravity [57] and String Theory [58]. In this sense, spacetime non-commutativity can be interpreted as a "low-energy remnant" of full Quantum Gravity theories, from which Quantum Gravity effects can be derived in physical regimes of interest both for theoretical reasons [1, 56, 59, 60] and phenomenological purposes [37, 40, 42, 43].

Some physical implications of this approach concern the localisability of events, the role of observers, and spacetime symmetries. Specifically, since spacetime coordinates become non-commuting operators, it is not surprising that some limits on the localisability of spacetime events emerge [59]. It simply comes from the application of the Heisenberg uncertainty principle to the non-vanishing commutators between the spacetime coordinates. This can also be seen as a further motivation to investigate spacetime non-commutativity since, as anticipated in the Prologue, there exists a conflict between the localisation processes of spacetime events in Quantum Theory and General Relativity. Indeed, the former reproduces a sharp localisation in the limit of very massive particles, which is not compatible with General Relativity because it would alter the spacetime structure and eventually lead to the creation of black holes, while the latter yields a sharp localisation in the limit of massless particles, which is incompatible with Quantum Theory since it would produce quantum fluctuations that would prevent a sharp localisation. This impossibility of sharply localising spacetime events can be effectively reproduced with a non-commutative spacetime structure, in which the minimum localisation scale, typically identified with the Planck scale, appears in the commutators between the spacetime coordinates. Moreover, the very notion of reference frames, and thus observers, acquires some non-classical features in this framework. This, in turn, affects the transformation laws between reference frames, giving rise to deformations of spacetime symmetries [36,61]. Such deformations make the non-commutative spacetime structure relativistically invariant, meaning that all observers agree on the type of non-commutativity between spacetime coordinates. Spacetime non-commutativity is therefore implemented as a relativistically invariant law, which reproduces at the quantum level the realisation of the Planck scale as an invariant scale of Nature, giving rise to a framework with two relativistic invariant scales, the velocity of light in vacuum c and the Planck length ℓ_P[1]. This is the so-called *Doubly Special Relativity* framework [36,37,61].

The above arguments allow us to anticipate that the symmetries of non-commutative spacetime acquire some non-classical features and, for this reason, can be understood as *quantum symmetries*. These are the main protagonists of this chapter, which reviews the technical tools necessary to describe them, *i.e.* Hopf algebras, and provides some illustrative examples, following [59, 62]. In this chapter only, we will adopt natural units by setting $c = \hbar = 1$, unless otherwise stated.

1.1 Mathematical preliminaries

For a commutative spacetime, symmetries are described in the language of Lie algebras. When considering a non-commutative spacetime, it is necessary to resort to some richer structures given by Hopf algebras. This section gives a brief review of this topic.

1.1.1 Universal enveloping algebras

A Lie algebra $\mathfrak{L}$ is a non-associative algebra over a field $\mathbb{K}$ with a product given by the Lie bracket

$$[\,,\,] : \mathfrak{L} \times \mathfrak{L} \mapsto \mathfrak{L}\,. \tag{1.1}$$

This product satisfies the following properties

$$\begin{cases} \left[a\mathfrak{g} + b\mathfrak{h}, \mathfrak{f}\right] = a\left[\mathfrak{g}, \mathfrak{f}\right] + b\left[\mathfrak{h}, \mathfrak{f}\right] \qquad \text{(linearity)} \\ \left[\mathfrak{g}, \mathfrak{h}\right] = -\left[\mathfrak{h}, \mathfrak{g}\right] \qquad \text{(anti-symmetry)} \\ \Big[\mathfrak{g}, [\mathfrak{h}, \mathfrak{f}]\Big] + \Big[\mathfrak{h}, [\mathfrak{f}, \mathfrak{g}]\Big] + \Big[\mathfrak{f}, [\mathfrak{g}, \mathfrak{h}]\Big] = 0 \qquad \text{(Jacobi identity)} \end{cases} \qquad \forall\, a, b \in \mathbb{K},\ \mathfrak{g}, \mathfrak{h}, \mathfrak{f} \in \mathfrak{L}\,. \tag{1.2}$$

If a Lie algebra is finite-dimensional with dimension n, it is possible to choose a set of n linearly independent elements of the algebra $\{\mathfrak{g}_i\}_{i=1}^n$ that generate, through linear combinations,

[1]Notice that this realises a theory that has the same degree of symmetry as Special Relativity but that is not Special Relativity. This is similar to the symmetry structure of the de Sitter spacetime, which is a maximally symmetric spacetime with an intrinsic length scale given by its curvature radius.

all the elements of the algebra. This set forms a basis for the algebra and, since the product of two Lie algebra elements lies in the algebra, the Lie bracket between two generators can be expressed as a linear combination of the generators

$$\left[\mathfrak{g}_i, \mathfrak{g}_j\right] = C_{ij}{}^k \mathfrak{g}_k \,, \tag{1.3}$$

where $C_{ij}{}^k$ are called structure constants. These uniquely define the Lie algebra.

It is possible to build a Lie algebra $\mathfrak{L}_{\mathscr{A}}$ from a unital associative algebra $\mathscr{A}$ by defining the Lie bracket as

$$\left[\mathfrak{a}, \mathfrak{b}\right] = \mathfrak{a} \cdot \mathfrak{b} - \mathfrak{b} \cdot \mathfrak{a} \,, \quad \forall\, \mathfrak{a}, \mathfrak{b} \in \mathscr{A} \,, \tag{1.4}$$

where $\cdot$ is the associative product of $\mathscr{A}$. In this way, the bilinearity and the antisymmetry are automatically satisfied, and the Jacobi identity follows from the associativity of the product. The inverse process, namely writing down a unital associative algebra starting from a Lie algebra, does not yield a unique algebra. Moreover, the construction of these associative algebras typically depends on the structure constants of the Lie algebra. However, it is possible to have a universal definition of associative algebra that is independent of the particular Lie algebra from which it is constructed. This is called *universal enveloping algebra* $U[\mathfrak{L}]$. This is the most general unital associative algebra whose Lie algebra, obtained by (1.4), is $\mathfrak{L}$. The universality is obtained as follows. Given a Lie algebra $\mathfrak{L}$ and a unital associative algebra built from it $\mathscr{A}_{\mathfrak{L}}$, we can consider two algebra homomorphisms

$$\begin{aligned} f &: U[\mathfrak{L}] \to \mathfrak{L} \,, \\ g &: \mathfrak{L} \to \mathscr{A}_{\mathfrak{L}} \,, \end{aligned} \tag{1.5}$$

then there exists a unique homomorphism $h : U[\mathfrak{L}] \to \mathscr{A}_{\mathfrak{L}}$ such that $h = g \circ f$.

A universal enveloping algebra can also be constructed directly as follows. Let $\mathscr{T}(\mathfrak{L})$ be the tensor algebra on the vector space of $\mathfrak{L}$

$$\mathscr{T}(\mathfrak{L}) = \mathbb{K} \oplus \mathfrak{L} \oplus \mathfrak{L} \otimes \mathfrak{L} \oplus \ldots = \bigoplus_{n=0}^{\infty} \mathfrak{L}^{\otimes n} \,, \tag{1.6}$$

and let $\mathscr{I}$ be the two-sided ideal of $\mathscr{T}(\mathfrak{L})$ generated by the elements

$$\mathfrak{a} \otimes \mathfrak{b} - \mathfrak{b} \otimes \mathfrak{a} - \left[\mathfrak{a}, \mathfrak{b}\right] \,, \quad \mathfrak{a}, \mathfrak{b} \in \mathfrak{L} \,. \tag{1.7}$$

Then the universal enveloping algebra can be defined as the quotient

$$U[\mathfrak{L}] = {}^{\mathscr{T}(\mathfrak{L})}\!\big/_{\mathscr{I}} \,. \tag{1.8}$$

1.1.2 Hopf algebras

To introduce the notion of Hopf algebras, it necessary to first talk about co-algebras and bi-algebras.

Co-algebra

A co-algebra is a vector space $\mathscr{C}$ over a field $\mathbb{K}$ endowed with two $\mathbb{K}$-linear maps, the co-product Δ and the co-unit ε

$$\Delta : \mathscr{C} \to \mathscr{C} \otimes \mathscr{C} \ , \ \ \varepsilon : \mathscr{C} \to \mathbb{K} \ , \tag{1.9}$$

satisfying the following properties

$$\begin{aligned} &\left(\Delta \otimes \mathbb{1}\right) \circ \Delta = \left(\mathbb{1} \otimes \Delta\right) \circ \Delta \ \text{ (co-associativity)} \ , \\ &\left(\varepsilon \otimes \mathbb{1}\right) \circ \Delta = \mathbb{1} = \left(\mathbb{1} \otimes \varepsilon\right) \circ \Delta \ \text{ (co-unity)} \ , \end{aligned} \tag{1.10}$$

A co-algebra can be seen as a dual structure to a unital associative algebra. This can be seen by writing down the axioms defining a co-algebra and those defining an algebra with commutative diagrams. Similarly to (1.10), a unital associative algebra $\mathscr{A}$ can be defined as a vector space over a field $\mathbb{K}$ endowed with two $\mathbb{K}$-linear maps, the product and the unit

$$\nabla : \mathscr{A} \times \mathscr{A} \to \mathscr{A} \ , \ \ \eta : \mathbb{K} \to \mathscr{A} \ , \tag{1.11}$$

satisfying

$$\begin{aligned} &\nabla \circ \left(\nabla \otimes \mathbb{1}\right) = \nabla \circ \left(\mathbb{1} \otimes \nabla\right) \ \text{ (associativity)} \ , \\ &\nabla \circ \left(\eta \otimes \mathbb{1}\right) = \mathbb{1} = \nabla \circ \left(\mathbb{1} \otimes \eta\right) \ \text{ (unity)} \ , \end{aligned} \tag{1.12}$$

where $\mathbb{1}$ is the unit element with respect to ∇. These two maps are dual to the co-product and the co-unit, meaning that

$$\begin{tikzcd} \mathscr{A} \otimes \mathscr{A} \arrow[d, "\nabla"'] & \mathscr{C} \otimes \mathscr{C} & \mathbb{K} \arrow[d, "\eta"'] & \mathbb{K} \\ \mathscr{A} & \mathscr{C} \arrow[u, "\Delta"'] & \mathscr{A} & \mathscr{C} \arrow[u, "\varepsilon"'] \end{tikzcd}$$

so that the diagrams defining the two sets of maps are unchanged if all the arrows are reversed, and $\eta \leftrightarrow \varepsilon$, $\nabla \leftrightarrow \Delta$, $\mathscr{A} \leftrightarrow \mathscr{C}$. This is true also for the axioms defining the algebra and the co-algebra that can be written as

$$\begin{tikzcd} \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \arrow[r, "\mathbb{1} \otimes \nabla"] \arrow[d, "\nabla \otimes \mathbb{1}"'] & \mathscr{A} \otimes \mathscr{A} \arrow[d, "\nabla"] \\ \mathscr{A} \otimes \mathscr{A} \arrow[r, "\nabla"'] & \mathscr{A} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{C} \otimes \mathscr{C} \otimes \mathscr{C} & \mathscr{C} \otimes \mathscr{C} \arrow[l, "\mathbb{1} \otimes \Delta"'] \\ \mathscr{C} \otimes \mathscr{C} \arrow[u, "\Delta \otimes \mathbb{1}"] & \mathscr{C} \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}$$

$$
\begin{tikzcd}
\mathscr{A}\otimes\mathscr{A} \arrow[r, "\nabla"] \arrow[d, "\nabla"'] & \mathscr{A} \arrow[d, "\mathbb{1}\otimes\eta"] \\
\mathscr{A} \arrow[r, "\eta\otimes\mathbb{1}"'] & \mathscr{A}\otimes\mathscr{A}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathscr{C}\otimes\mathscr{C} & \mathscr{C} \arrow[l, "\Delta"'] \\
\mathscr{C} \arrow[u, "\Delta"] & \mathscr{C}\otimes\mathscr{C} \arrow[l, "\varepsilon\otimes\mathbb{1}"] \arrow[u, "\mathbb{1}\otimes\varepsilon"']
\end{tikzcd}
$$

Finally, there exists also a dual notion to that of an Abelian algebra. A Lie algebra is Abelian if the product is commutative. This requirement can be written as

$$\nabla\circ\tau = \nabla \ , \quad \tau\big(\mathfrak{a}\otimes\mathfrak{b}\big) = \mathfrak{b}\otimes\mathfrak{a}\,, \tag{1.13}$$

where $\tau:\mathscr{A}\otimes\mathscr{A}\to\mathscr{A}\otimes\mathscr{A}$ is the flip map. In the same way, a co-algebra is co-commutative if

$$\tau\circ\Delta = \Delta \tag{1.14}$$

namely

$$\Delta\mathfrak{a} = \mathfrak{a}_{(1)}\otimes\mathfrak{a}_{(2)} = \mathfrak{a}_{(2)}\otimes\mathfrak{a}_{(1)} \ , \quad \forall\,\mathfrak{a}\in\mathscr{C}\,, \tag{1.15}$$

where we used the Sweedler notation for the co-product

$$\Delta\mathfrak{a} = \mathfrak{a}_{(1)}\otimes\mathfrak{a}_{(2)} = \sum_i \mathfrak{a}^{(i)}_{(1)}\otimes\mathfrak{a}^{(i)}_{(2)}\,. \tag{1.16}$$

Bi-algebra

A vector space over a field $\mathbb{K}$ endowed with a product ∇, unit η, co-product Δ, co-unit ε, and an identity with respect to the product $\mathbb{1}$ is both a unital algebra and a co-algebra. Such a structure is a bi-algebra $\mathscr{B}$ over $\mathbb{K}$ if[2]

$$\begin{aligned}
&\Delta\Big[\nabla\big(\mathfrak{a},\mathfrak{b}\big)\Big] = \nabla\big(\Delta\mathfrak{a},\Delta\mathfrak{b}\big) \ , \quad \varepsilon\big(\mathfrak{a}\cdot\mathfrak{b}\big) = \varepsilon(\mathfrak{a})\varepsilon\big(\mathfrak{b}\big) \ , \quad \forall\,\mathfrak{a},\mathfrak{b}\in\mathscr{B}\,,\\
&\Delta\mathbb{1} = \mathbb{1}\otimes\mathbb{1} \ , \quad \varepsilon\big(\mathbb{1}\big) = 1 \ , \quad 1\in\mathbb{K}\,.
\end{aligned} \tag{1.17}$$

Hopf algebra

An Hopf algebra $\mathscr{H}$ is a bi-algebra over a field $\mathbb{K}$ endowed with a $\mathbb{K}$-linear map $S:\mathscr{H}\to\mathscr{H}$, called antipode, that satisfies the following compatibility condition

$$\nabla\circ\big(S\otimes\mathbb{1}\big)\circ\Delta = \nabla\circ\big(\mathbb{1}\otimes S\big)\circ\Delta = \mathbb{1}\varepsilon\,, \tag{1.18}$$

[2]The fist condition simply means that $\big(\mathfrak{a}\cdot\mathfrak{b}\big)_{(1)}\otimes\big(\mathfrak{a}\cdot\mathfrak{b}\big)_{(2)} = \big(\mathfrak{a}_{(1)}\cdot\mathfrak{b}_{(1)}\big)\otimes\big(\mathfrak{a}_{(2)}\cdot\mathfrak{b}_{(2)}\big)$.

that can be written with a commutative diagram as

$$\begin{tikzcd}
 & \mathscr{H}\otimes\mathscr{H} \arrow[r, "S\otimes\mathbb{1}"] & \mathscr{H}\otimes\mathscr{H} \arrow[dr, "\nabla"] & \\
\mathscr{H} \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \mathbb{K} \arrow[r, "\eta"] & \mathscr{H} \\
 & \mathscr{H}\otimes\mathscr{H} \arrow[r, "\mathbb{1}\otimes S"'] & \mathscr{H}\otimes\mathscr{H} \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

This compatibility condition implies

$$S\Big[\nabla\big(\mathfrak{a},\mathfrak{b}\big)\Big] = \nabla\Big[S(\mathfrak{a}),S\big(\mathfrak{b}\big)\Big]\ ,\ \ S\big(\mathbb{1}\big) = \mathbb{1}\ ,\ \ \big(S\otimes S\big)\circ\Delta = \tau\circ\Delta\circ S\,. \tag{1.19}$$

Universal enveloping algebra as an Hopf algebra

Let $U[\mathfrak{L}] = \mathrm{Span}\big(\mathbb{1},\mathfrak{g}_1,\mathfrak{g}_2,\ldots\big)$ be a universal enveloping algebra with generators $\mathfrak{g}_1,\mathfrak{g}_2,\ldots$. It is possible to give a structure of a co-algebra to $U[\mathfrak{L}]$ by defining

$$\begin{aligned}
&\Delta\mathfrak{g}_i = \mathfrak{g}_i\otimes\mathbb{1} + \mathbb{1}\otimes\mathfrak{g}_i\ ,\ \ \varepsilon\big(\mathfrak{g}_i\big) = 0\ ,\ \ \forall\, i\,,\\
&\Delta\mathbb{1} = \mathbb{1}\otimes\mathbb{1}\ ,\ \ \eta(\lambda) = \lambda\mathbb{1}\ ,\ \ \forall\,\lambda\in\mathbb{K}\,.
\end{aligned} \tag{1.20}$$

The co-product and co-unit defined in this way are called *primitive* and automatically give to $U[\mathfrak{L}]$ the structure of a bi-algebra. To have an Hopf algebra, we define a primitive antipode as

$$S\big(\mathfrak{g}_i\big) = -\mathfrak{g}_i\ ,\ \ S\big(\mathbb{1}\big) = \mathbb{1}\,. \tag{1.21}$$

1.1.3 Duality

Given two vector spaces over $\mathbb{C}$, they are said to be dual if there exists a non-degenerate sesquilinear form

$$\langle\cdot,\cdot\rangle : X\times Y\to\mathbb{C}\,, \tag{1.22}$$

that defines a scalar product between the two spaces. Given a vector space V, it is possible to define its dual V^* as the space of functionals over V

$$V^* := \mathrm{Hom}\big(V,\mathbb{C}\big)\ ,\ \ \big\langle f,x\big\rangle = f(x)\ ,\ \ x\in V\,,\ f\in V^*\,. \tag{1.23}$$

With this notion of duality, it is straightforward to understand the reason why co-algebras and algebras are intended as dual concepts. Indeed, given a co-algebra $\mathscr{C}$ over $\mathbb{C}$ it is possible to define two maps $\nabla:\mathscr{C}^*\times\mathscr{C}^*\to\mathscr{C}^*$ and $\eta:\mathbb{C}\to\mathscr{C}^*$ on the dual vector space $\mathscr{C}^*$. These are obtained as

$$\Big\langle\nabla\big(\mathfrak{h},\mathfrak{g}\big),\mathfrak{a}\Big\rangle := \big\langle\mathfrak{h}\otimes\mathfrak{g},\Delta\mathfrak{a}\big\rangle\ ,\ \ \big\langle\eta(1),\mathfrak{g}\big\rangle := \varepsilon\big(\mathfrak{g}\big)\ ,\ \ \forall\,\mathfrak{h},\mathfrak{g}\in\mathscr{C}^*\ \mathfrak{a}\in\mathscr{C}\,, \tag{1.24}$$

from which it descends that these adjoint maps are a multiplication and a unit map, making $\mathscr{C}^*$ a unital associative algebra. Of course, the inverse is also true, namely a co-algebra can be defined by duality from a unital associative algebra. This means that it is possible to define the notion of dual Hopf algebras. Two Hopf algebras $\mathscr{H}$ and $\mathscr{H}^*$ are said to be dual if the co-algebra of $\mathscr{H}$ is dual to the algebra of $\mathscr{H}^*$ and vice versa

$$\begin{aligned} \Big\langle \nabla(\mathfrak{g},\mathfrak{h}),\mathfrak{a}\Big\rangle &= \langle \mathfrak{g}\otimes\mathfrak{h},\Delta^*\mathfrak{a}\rangle \ , \ \langle \eta(1),\mathfrak{a}\rangle = \varepsilon^*(\mathfrak{a}) \ , \ \forall\, \mathfrak{g},\mathfrak{h}\in\mathscr{H},\ \mathfrak{a}\in\mathscr{H}^* \\ \Big\langle \mathfrak{g},\nabla^*(\mathfrak{a},\mathfrak{b})\Big\rangle &= \langle \Delta\mathfrak{g},\mathfrak{a}\otimes\mathfrak{b}\rangle \ , \ \langle \mathfrak{g},\eta^*(1)\rangle = \varepsilon(\mathfrak{a}) \ , \ \forall\, \mathfrak{g},\mathfrak{h}\in\mathscr{H},\ \mathfrak{a}\in\mathscr{H}^* \end{aligned} \tag{1.25}$$

and the antipodes are also dually paired

$$\langle S(\mathfrak{g}),\mathfrak{a}\rangle = \langle \mathfrak{g},S^*(\mathfrak{a})\rangle \ , \ \forall\, \mathfrak{g}\in\mathscr{H},\ \mathfrak{a}\in\mathscr{H}^* \tag{1.26}$$

If two Hopf algebras are dual to each other, the propositions regarding their operations are also dually paired. For example, if $\mathscr{H}^*$ is co-commutative then

$$\Big\langle \nabla(\mathfrak{g},\mathfrak{h}),\Delta\mathfrak{a}\Big\rangle = \langle \mathfrak{g},\mathfrak{a}_{(1)}\rangle\langle \mathfrak{h},\mathfrak{a}_{(2)}\rangle = \langle \mathfrak{g},\mathfrak{a}_{(2)}\rangle\langle \mathfrak{h},\mathfrak{a}_{(1)}\rangle \ \ \forall\, \mathfrak{a}\in\mathscr{H}^* \Rightarrow \nabla(\mathfrak{g},\mathfrak{h}) = \nabla(\mathfrak{h},\mathfrak{g}) \,, \tag{1.27}$$

namely $\mathscr{H}$ is commutative. The inverse is also true, therefore we can state that $\mathscr{H}$ is commutative if and only if $\mathscr{H}^*$ is co-commutative, a statement that is synthesised with "*commutativity* $\Longleftrightarrow$ *co-commutativity*".

Action of a Hopf algebra on an algebra

The left action of a Hopf algebra $\mathscr{H}$ on a unital associative algebra $\mathscr{A}$ is defined as

$$\triangleright : \mathscr{H}\times\mathscr{A}\to\mathscr{A} \ , \ \mathfrak{h}\triangleright\mathfrak{a}\in\mathscr{A} \ , \ \mathfrak{h}\in\mathscr{H},\ \mathfrak{a}\in\mathscr{A}\ . \tag{1.28}$$

This left action is compatible with the product and the co-unity of $\mathscr{H}$, menaing that

$$\nabla(\mathfrak{g},\mathfrak{h})\triangleright\mathfrak{a} = \mathfrak{g}\triangleright(\mathfrak{h}\triangleright\mathfrak{a}) \ , \ \mathfrak{h}\triangleright\mathbb{1} = \varepsilon(\mathfrak{h})\mathbb{1}\ . \tag{1.29}$$

It is possible to define the action of a Hopf algebra on an algebra (or a co-algebra) in such a way that the algebraic structures change covariantly under this action. This is called *covariant action* and is defined as

$$\mathfrak{h}\triangleright\nabla(\mathfrak{a},\mathfrak{b}) = \nabla(\mathfrak{h}_{(1)}\triangleright\mathfrak{a},\mathfrak{h}_{(2)}\triangleright\mathfrak{b}) \ , \ \forall\, \mathfrak{h}\in\mathscr{H},\ \mathfrak{a},\mathfrak{b}\in\mathscr{A}\ , \tag{1.30}$$

for an algebra $\mathscr{A}$ and

$$\Delta(\mathfrak{h}\triangleright\mathfrak{a}) = (\mathfrak{h}_{(1)}\triangleright\mathfrak{a}_{(1)})\otimes(\mathfrak{h}_{(2)}\triangleright\mathfrak{a}_{(2)}) = \Delta\mathfrak{h}\triangleright\Delta\mathfrak{a} \ , \ \forall\, \mathfrak{h}\in\mathscr{H},\ \mathfrak{a},\in\mathscr{C}\ , \tag{1.31}$$

if the covariant action is defined on a co-algebra $\mathscr{C}$. Of course, one can also define a right action and, in the case of two dually paired Hopf algebras $\mathscr{A}$ and $\mathscr{A}^*$, a left action over $\mathscr{A}$ automatically defines a dual right action over $\mathscr{A}^*$

$$\langle \mathfrak{a},\mathfrak{h}\triangleright\mathfrak{b}\rangle = \langle \mathfrak{a}\triangleleft^*\mathfrak{h},\mathfrak{b}\rangle \ , \ \forall\, \mathfrak{a}\in\mathscr{A}^*,\mathfrak{b}\in\mathscr{A},\mathfrak{h}\in\mathscr{H}\ . \tag{1.32}$$

Duality between a Lie group and its Lie algebra

To understand the role of symmetries in physical theories, the relation between a Lie group and its Lie algebra is of paramount importance. This relation can be obtained by means of some duality conditions, as specified, *e.g.*, in [63]. Given an affine algebraic group G over a field $\mathbb{K}$, there exists a canonical Hopf algebra structure on the algebra of regular functions on G, $\mathbb{K}[G]$, given by

$$\Delta f(g_1 \otimes g_2) = f(g_1 g_2)\,,\ \ S(f)(g) = f\left(g^{-1}\right),\ \ \varepsilon(f) = f(\mathbb{1}_G)\ \ ,\ \ \forall\, f \in \mathbb{K}[G]\ g_1, g_2, g \in G\,. \tag{1.33}$$

If $\mathfrak{L}$ is the Lie algebra of G, then the algebra $\mathbb{K}[G]$ is the Hopf algebra dual of the universal enveloping algebra $U[\mathfrak{L}]$ with dual pairing $\langle,\rangle : U[\mathfrak{L}] \otimes \mathbb{K}[G] \to \mathbb{K}$ given by

$$\langle \mathfrak{a}, f\rangle = \left(\mathscr{R}(\mathfrak{a}) f\right)(\mathbb{1}_G) = \left(\mathscr{L}(\mathfrak{a}) f\right)(\mathbb{1}_G)\,, \tag{1.34}$$

where $\mathscr{R}$ ($\mathscr{L}$) is the canonical representation of $U[\mathfrak{L}]$ as the algebra of right (left) invariant differential operators in $\mathbb{K}[G]$.

1.2 Classical spacetime and its symmetries

Hopf algebras not only offer a natural tool to describe quantum symmetries as intended in this chapter, namely as the symmetries of non-commutative spacetimes, but it is also possible to recast the theory of spacetime symmetries in the classical, commutative spacetime case in the language of Hopf algebras. This is reviewed in this section to lay the ground for the analogous formulation of symmetries in the non-commutative setting.

1.2.1 The Poincaré group and algebra

The symmetries of the classical Minkowski spacetime are given by the Poincaré group $\mathscr{P}$, defined by

$$\begin{gathered}(a',\Lambda')\cdot(a,\Lambda) = (\Lambda'\cdot a + a', \Lambda'\cdot\Lambda)\,,\ \ \forall\,(a,\Lambda),(a',\Lambda') \in \mathscr{P}\,,\\[2ex] (a,\Lambda)\cdot(0,I_4) = (0,I_4)\cdot(a,\Lambda) = (a,\Lambda)\,,\ \ I_4 = \mathrm{diag}(1,1,1,1)\,,\\[2ex] (a,\Lambda)\cdot(-\Lambda^T\cdot a, \Lambda^T) = (-\Lambda^T\cdot a, \Lambda^T)\cdot(a,\Lambda) = (0, I_4)\,,\end{gathered} \tag{1.35}$$

where $\Lambda \in SO(1,3)$ describes and element of the Lorentz group, namely a boost and/or rotation, and $a \in \mathbb{R}^4$ describes a spacetime translation. The algebra of functions over this group, $\mathbb{C}[\mathscr{P}]$, is an associative, unital, and commutative algebra defined by

$$(f\cdot g)(X) = f(X)g(X)\,,\ \ \forall\, X \in \mathscr{P}\,,\ \ (\alpha f + \beta g)(X) = \alpha f(X) + \beta g(X)\,,\ \ \mathbb{1}(X) = 1\,, \tag{1.36}$$

that encodes the information on the group structure of $\mathscr{P}$ if it is enriched with the structure of Hopf algebra via

$$\Delta f(X, X') = f(X\cdot X')\,,\ \ S(f)(X) = f\left(X^{-1}\right),\ \ \epsilon(f) = f(\mathbb{1})\,. \tag{1.37}$$

Indeed, it is possible to recover the product and inverse laws of $\mathscr{P}$ from the co-product and the antipode respectively.

The parameters defining a Poincaré transformation, namely the matrix elements $\Lambda^\mu{}_\nu$ and the four-vector a^μ form a basis for the algebra of functions $\mathbb{C}[\mathscr{P}]$. These functions satisfy

$$\begin{gathered}\Delta\Lambda^\mu{}_\nu = \Lambda^\mu{}_\rho \otimes \Lambda^\rho{}_\nu\,,\ \Delta a^\mu = \Lambda^\mu{}_\nu \otimes a^\nu + a^\mu \otimes \mathbb{1}\,,\\ S\big(\Lambda^\mu{}_\nu\big) = \left(\Lambda^{-1}\right)^\mu{}_\nu\,,\ S\big(a^\mu\big) = -\left(\Lambda^{-1}\right)^\mu{}_\nu a^\nu\,,\\ \epsilon\big(\Lambda^\mu{}_\nu\big) = \delta^\mu{}_\nu\,,\ \epsilon\big(a^\mu\big) = 0\,.\end{gathered} \tag{1.38}$$

The quotient of $\mathbb{C}[\mathscr{P}]$ with the Hopf sub-algebra $SO(1,3)$ spanned by the $\Lambda^\mu{}_\nu$ functions gives the algebra of functions over Minkowski spacetime $\mathbb{C}[\mathscr{M}]$, whose basis functions are the co-ordinates x^μ with

$$\Delta x^\mu = x^\mu \otimes \mathbb{1} + \mathbb{1} \otimes x^\mu\,,\ S\big(x^\mu\big) = -x^\mu\,,\ \epsilon\big(x^\mu\big) = 0\,. \tag{1.39}$$

The action of a Poincaré transformation is a map from the spacetime coordinates x^μ to a new set of coordinates x'^μ, equivalently given by

$$x'^\mu = a^\mu + x^\nu \Lambda^\mu{}_\nu = a^\mu + \Lambda^\mu{}_\nu x^\nu\,. \tag{1.40}$$

This is the standard description of Poincaré transformations of coordinates, however if we want to describe the same action in the Hopf-algebraic picture outlined above, we see that spacetime coordinates and Poincaré group transformations are elements of different algebras. Therefore, to properly describe the action of a Poincaré transformation on spacetime coordinates we have to think about this as a map from $\mathbb{C}[\mathscr{M}]$ to $\mathbb{C}[\mathscr{M}]\otimes\mathbb{C}[\mathscr{P}]$ or $\mathbb{C}[\mathscr{P}]\otimes\mathbb{C}[\mathscr{M}]$, depending on the ordering. This is realised by the right (left) *co-action*

$$\begin{aligned}&\Delta_R : \mathbb{C}[\mathscr{M}] \to \mathbb{C}[\mathscr{M}] \otimes \mathbb{C}[\mathscr{P}] \qquad && \Delta_R\big[x^\mu\big] = \mathbb{1} \otimes a^\mu + x^\nu \otimes \Lambda^\mu{}_\nu\,,\\ &\Delta_L : \mathbb{C}[\mathscr{M}] \to \mathbb{C}[\mathscr{P}] \otimes \mathbb{C}[\mathscr{M}] && \Delta_L\big[x^\mu\big] = a^\mu \otimes \mathbb{1} + \Lambda^\mu{}_\nu \otimes x^\nu\,.\end{aligned} \tag{1.41}$$

In general, we can write the right co-action of a function $f(x) \in \mathbb{C}[\mathscr{M}]$ using the Sweedler notation as

$$\Delta_R\big[f(x)\big] = f^{(1)}(x) \otimes A^{(2)}_f\,,\ f^{(1)}(x) \in \mathbb{C}[\mathscr{M}]\,,\ A^{(2)}_f \in \mathbb{C}[\mathscr{P}]\,. \tag{1.42}$$

If $\mathscr{L}$ is the Lie algebra of $\mathscr{P}$, then the right co-action of $\mathbb{C}[\mathscr{P}]$ on $\mathbb{C}[\mathscr{M}]$ defines a left action of the universal enveloping algebra $U[\mathscr{L}]$ on $\mathbb{C}[\mathscr{M}]$ as [64]

$$T \triangleright f(x) := f^{(1)}(x)\Big(T, A^{(2)}_f\Big)\,,\ T \in U[\mathscr{L}]\,,\ f(x) \in \mathbb{C}[\mathscr{M}]\,, \tag{1.43}$$

where $(\cdot,\cdot) : U[\mathscr{L}] \otimes \mathbb{C}[\mathscr{P}] \to \mathbb{C}$ is a sesquilinear form defining a duality between $U[\mathscr{L}]$ and $\mathbb{C}[\mathscr{P}]$.

The Lie algebra of the Poincaré group

The co-action of a Lie group G can be expressed in terms of the action of the corresponding Lie algebra $\mathscr{L}$ by means of the *exponential map*

$$e : \mathscr{L} \to G\,, \tag{1.44}$$

so that the right co-action of the Poincaré group on $f(x) \in \mathbb{C}[\mathscr{M}]$ can be written as[3]

$$\Delta_R\big[f(x)\big] = e^{\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}}e^{ia^\mu P_\mu}f(x) = f\big(a^\mu + x^\nu \Lambda^\mu{}_\nu\big)\,, \tag{1.45}$$

where the second equality comes from a Taylor expansion of $f(x)$ and from the fact that the Lie algebra generators have to satisfy the Leibniz rule, therefore their exponentiation is a homomorphism for the coordinates product. To derive the differential representations of these generators, we consider infinitesimal transformations. If we take $\omega^{\mu\nu} = 0$ and $a^\mu \neq 0$ we get

$$f(x) + ia^\mu P_\mu f(x) = f(x) + a^\mu \frac{\partial f(x)}{\partial x^\mu}\,, \tag{1.46}$$

therefore the differential representations of the spacetime translations generators read

$$P_\mu = -i\partial_\mu\,. \tag{1.47}$$

Taking $\omega^{i0} \neq 0$, $\omega^{0i} \neq 0$ and $\omega^{ij} = a^\mu = 0$ we get[4]

$$\frac{i}{2}\Big(\omega^{i0}M_{i0} + \omega^{0i}M_{0i}\Big)f(x) = f(x) + \zeta^i\left(x_0\frac{\partial f(x)}{\partial x^i} - x_i\frac{\partial f(x)}{\partial x^0}\right), \tag{1.48}$$

therefore the differential representations of Lorentz boosts generators is given by

$$\omega^{i0} = -\omega^{0i} = \zeta^i\,,\;\; M_{i0} = -M_{0i} = i\big(x_0\partial_i - x_i\partial_0\big)\,. \tag{1.49}$$

Finally, by taking $\omega^{i0} = a^\mu = 0$, $\omega^{ij} \neq 0$ and $\omega^{ji} \neq 0$ we get

$$\frac{i}{2}\Big(\omega^{ij}M_{ij} + \omega^{ji}M_{ji}\Big)f(x) = f(x + \boldsymbol{\omega}\wedge\boldsymbol{x}) \sim f(x) + \boldsymbol{\omega}\wedge\boldsymbol{\nabla}f(x)\,, \tag{1.50}$$

therefore the differential representations of spatial rotations generators read

$$\omega^{ij} = -\omega^{ji}\,,\;\; M_{ij} = i\Big(x_j\partial_i - x_i\partial_j\Big)\,, \tag{1.51}$$

so that we can write the differential representations of the Poincaré algebra generators as

$$P_\mu = -i\partial_\mu\,,\;\; M_{\mu\nu} = i\Big(x_\nu\partial_\mu - x_\mu\partial_\nu\Big)\,. \tag{1.52}$$

From these differential representations we can derive the Lie algebra $\mathscr{L}_{\mathscr{P}}$ of the Poincaré group, given by

$$\begin{gathered}\Big[P_\mu, P_\nu\Big] = 0\,,\;\; \Big[M_{\mu\nu}, P_\rho\Big] = i\Big(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu\Big)\,,\\ \Big[M_{\mu\nu}, M_{\rho\sigma}\Big] = i\Big(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}\Big)\,.\end{gathered} \tag{1.53}$$

[3]Here and in the following we omit the tensor products in the commutative setting where no ambiguity arises. To be precise we should write $e^{\frac{i}{2}M_{\mu\nu}\otimes\omega^{\mu\nu}}e^{iP_\mu\otimes a^\mu}[f(x)] = f(\mathbb{1}\otimes a^\mu + x^\nu \otimes \Lambda^\mu{}_\nu)$. Analogously, equations like (1.46) should be written as $f(x)\otimes\mathbb{1} + iP_\mu \triangleright f(x)\otimes a^\mu = f(x)\otimes\mathbb{1} + i\frac{\partial f(x)}{\partial x^\mu}\otimes a^\mu$.

[4]ζ^i is the infinitesimal boost parameter (rapidity). In 1+1 dimensions this parameterises a pure boost matrix as $\Lambda = \begin{pmatrix}\cosh\zeta & \sinh\zeta\\ \sinh\zeta & \cosh\zeta\end{pmatrix} \sim \begin{pmatrix}1 & \zeta\\ \zeta & 1\end{pmatrix}$ where the last approximation holds for small ζ.

Duality between Poincaré group and algebra

Using the scalar product defined in (1.43) it is possible to define a duality between the universal enveloping algebra $U[\mathscr{L}]$ and the algebra of functions over $\mathscr{P}$, $\mathbb{C}[\mathscr{P}]$. The relations between the basis elements of the two algebras are

$$\begin{aligned} &\left(P_\mu, a_\nu\right) = -i\eta_{\mu\nu}\,, && \left(M_{\mu\nu}, a^\rho\right) = 0\,, && \left(\mathbb{1}, g\right) = \epsilon(g)\,, \\ &\left(P_\mu, \Lambda^\rho{}_\nu\right) = 0\,, && \left(M_{\mu\nu}, \omega^{\rho\sigma}\right) = i\left(\delta^\rho{}_\mu\delta^\sigma{}_\nu - \delta^\rho{}_\nu\delta^\sigma{}_\mu\right), && \left(T, \mathbb{1}\right) = \epsilon(T)\,. \end{aligned} \tag{1.54}$$

These relations, together with the duality conditions for products and co-products

$$\begin{aligned} \left(T_1 \otimes T_2, g\right) &= \left(T_1 \otimes T_2, \Delta g\right) = \left(T_1, g^{(1)}\right)\left(T_2, g^{(2)}\right), \\ \left(T, g_1 \otimes g_2\right) &= \left(\Delta T, g_1 \otimes g_2\right) = \left(T^{(1)}, g_1\right)\left(T^{(2)}, g_2\right), \end{aligned} \tag{1.55}$$

allow to compute the scalar product between any two elements of the two algebras.

Duality between spacetime coordinates and spacetime symmetries

In the beginning of this subsection we have defined the algebra of functions over spacetime as the quotient

$$\mathbb{C}[\mathscr{M}] = {}^{\mathbb{C}[\mathscr{P}]}\!\big/_{SO(1,3)}\,. \tag{1.56}$$

This algebra, with basis functions given by the spacetime coordinates x^μ, has the structure of a commutative Hopf algebra with a co-algebra given by (1.39). By means of the dual pairing introduced in (1.22), this Hopf algebra is dual to the sub-algebra of spacetime translations spanned by the generators P_μ. Specifically, the duality relation between the spacetime translations generators P_μ and the spacetime coordinates x^μ reads

$$\left\langle P_\mu, x^\nu \right\rangle = i\delta_\mu{}^\nu\,. \tag{1.57}$$

From this duality, it is possible to derive the algebra of spacetime coordinates from the algebra of symmetries and vice versa, so that spacetime coordinates and symmetry algebra are intimately related

Spacetime coordinates algebra $\Longleftrightarrow$ *Spacetime symmetry algebra*

For instance, from the duality relations between products and co-products, and more precisely from the "commutativity $\Longleftrightarrow$ co-commutativity" property and the fact that the generators P_μ define a Lie sub-algebra of $\mathscr{P}$, we immediately get that the Lie brackets between spacetime coordinates must be be given by

$$\left[x^\mu, x^\nu\right] = 0\,, \tag{1.58}$$

which simply describe a commutative, classical spacetime structure. Therefore, the structure of classical spacetime is described by the duality between the Hopf algebra of spacetime translations

$$\left[P_\mu, P_\nu\right] = 0\,,\;\; \Delta P_\mu = P_\mu \otimes \mathbb{1} + \mathbb{1} \otimes P_\mu\,,\;\; \epsilon(P_\mu) = 0\,,\;\; S(P_\mu) = -P_\mu\,, \tag{1.59}$$

and the commutative Hopf algebra of spacetime coordinates

$$\Delta x^\mu = x^\mu \otimes \mathbb{1} + \mathbb{1} \otimes x^\mu\,,\;\; \left[x^\mu, x^\nu\right] = 0\,,\;\; \epsilon\left(x^\mu\right) = 0\,,\;\; S\left(x^\mu\right) = -x^\mu\,. \tag{1.60}$$

1.2.2 The (centrally extended) Galilei group and algebra

In the previous subsection, we outlined the Hopf algebraic description of the Poincaré group and algebra, describing the spacetime symmetries for relativistic systems. If we are instead interested in studying the non-relativistic regime, spacetime symmetries are given by the Galilei group and algebra. A Galilean transformation is given by

$$\boldsymbol{x}' = R\cdot\boldsymbol{x}+\boldsymbol{a}+\boldsymbol{v}\,t\,,\;\; t' = t+b\,, \tag{1.61}$$

where $R \in SO(3)$ describes a rotation, $\boldsymbol{v}, \boldsymbol{a} \in \mathbb{R}^3$ are a velocity and a spatial displacement respectively, describing a Galilean boost and spatial translation, and $b \in \mathbb{R}$ describes a time translation. A general Galilean transformation can then be parameterised by these 10 parameters, and the group structure of the Galilei group $\mathscr{G}$ can be obtained by considering the composition of two transformations. Specifically, the group structure is

$$\begin{aligned}
&\left(b',\boldsymbol{a}',\boldsymbol{v}',R'\right)\left(b,\boldsymbol{a},\boldsymbol{v},R\right) = \left(b+b',\boldsymbol{a}'+R'\cdot\boldsymbol{a}+b\,\boldsymbol{v}',\boldsymbol{v}'+R'\cdot\boldsymbol{v},R'R\right),\\
&\left(b,\boldsymbol{a},\boldsymbol{v},R\right)\cdot\left(0,\boldsymbol{0},\boldsymbol{0},I_3\right) = \left(0,\boldsymbol{0},\boldsymbol{0},I_3\right)\cdot\left(b,\boldsymbol{a},\boldsymbol{v},R\right) = \left(b,\boldsymbol{a},\boldsymbol{v},R\right),\\
&\left(b,\boldsymbol{a},\boldsymbol{v},R\right)\cdot\left(-b,-R^{-1}\cdot(\boldsymbol{a}-b\,\boldsymbol{v}),-R^{-1}\cdot\boldsymbol{v},R^{-1}\right) = \left(0,\boldsymbol{0},\boldsymbol{0},I_3\right),\\
&\forall\left(b',\boldsymbol{a}',\boldsymbol{v}',R'\right),\left(b,\boldsymbol{a},\boldsymbol{v},R\right)\in\mathscr{G}\,,\;\; I_3 = \mathrm{diag}\left(1,1,1\right).
\end{aligned} \tag{1.62}$$

The whole Hopf algebraic theory developed for the relativistic case can be carried out also in the Galilean setting. However, every algebraic or co-algebraic structure can be derived from the relativistic case because Galilean symmetries are an İnönü-Wigner contraction of the Poincaré symmetries. This is achieved by explicitly writing the c factors in the generators of the Poincaré algebra, group parameters and spacetime coordinates as

$$P_0 = \frac{H}{c}\,,\;\; N_i = c\,G_i\,,\; x^0 = c\,t\,,\;\; a^0 = c\,b\,,\;\; \zeta^i = \frac{v^i}{c}\,. \tag{1.63}$$

Then, by taking the $c\to\infty$ limit the Galilean Hopf algebraic structures are obtained.

However, there is an important difference between the Poincaré and the Galilean symmetries, besides the different physical regimes in which they are respectively valid. This has to do with the representation of the algebra of symmetries on the phase space of physical systems, both in Classical and Quantum Mechanics. For classical systems, it is not possible to have a true Poisson bracket realisation of the Galilei algebra, indeed it is possible to have such a canonical realisation only up to a neutral element. This is nothing more than a true Poisson bracket realisation of the central extension of the Galilei algebra. For quantum systems, it is not possible to have an irreducible unitary representation of the Galilei algebra. The action of the Galilean generators is represented on the Hilbert space of quantum systems by an irreducible unitary projective representation of the algebra. This is equivalent to an irreducible unitary representation of a central extension of the Galilei algebra. For both classical and quantum systems, therefore, we have that the correct symmetry algebra is the central extension of the Galilei algebra [65–67]. More details about the representation theory of the Galilei group in Quantum Mechanics can be found in chapter A. From a physical point of view, the centre of this algebra is the mass M. This central extension can be derived from the relativistic case as well. It is necessary to extend the Poincaré algebra and then consider the contraction, so that

$$P_0 = \frac{H}{c} + Mc\,, \tag{1.64}$$

the other quantities being the same as (1.63). Then, once again, the limit $c \to \infty$ is taken and the Hopf algebraic structures of the central extension of the Galilei algebra are obtained. For instance, the Lie brackets between the generators of the central extended Galilei algebra read

$$\begin{gathered}[H,P_i] = [H,J_i] = 0 \ , \ \left[G_i,G_j\right] = \left[P_i,P_j\right] = 0 \ , \ \left[G_i,P_j\right] = i\,M\delta_{ij} \ , \ [G_i,H] = i\,P_i \ , \\ \left[J_i,G_j\right] = i\,\epsilon_{ij}{}^k G_k \ , \ \left[J_i,P_j\right] = i\,\epsilon_{ij}{}^k P_k \ , \ [J_n,J_m] = i\,\epsilon_{nm}{}^k J_k \ , \ [M,\cdot] = 0\,. \end{gathered} \tag{1.65}$$

Notice that, for both quantum and classical systems, the only physically meaningful case is with $M \neq 0$ [65,67]. A detailed discussion of the origin of the central extension as the symmetry algebra of Quantum Mechanics can be found in chapter A.

Bargmann argument for a mass superselection rule

The fact that the correct symmetry group (algebra) for quantum-mechanical systems is given by the central extension of the Galilei group (algebra) has an important physical consequence. To understand this, we consider a quantum-mechanical system that is in a superposition of two different masses, m_1 and m_2. We write the state of this system as $|\psi\rangle = \frac{1}{\sqrt{2}}\Big(|\psi_1\rangle + |\psi_2\rangle\Big)$, where the mass operator M in (1.65) acts as $M|\psi_i\rangle = m_i|\psi_i\rangle$. Consider now the following chain of transformations: a spatial translation, a Galilean boost, the inverse translation, and the inverse boost. The overall transformation is the identity on spacetime coordinates, as can be immediately seen from (1.62). Since this transformation is the identity on spacetime, we should expect that the action of its representation on the Hilbert space of a quantum system would give at most a global phase. However, from (1.65) and the Baker-Campbell-Hausdorff formula[5] we have

$$e^{-i\boldsymbol{v}\cdot\boldsymbol{G}}e^{-i\boldsymbol{a}\cdot\boldsymbol{P}}e^{i\boldsymbol{v}\cdot\boldsymbol{G}}e^{i\boldsymbol{a}\cdot\boldsymbol{P}} = e^{-i\boldsymbol{v}\cdot\boldsymbol{G}-i\boldsymbol{a}\cdot\boldsymbol{P}-\frac{1}{2}v^i a^j\left[G_i,P_j\right]}e^{i\boldsymbol{v}\cdot\boldsymbol{G}+i\boldsymbol{a}\cdot\boldsymbol{P}-\frac{1}{2}v^i a^j\left[G_i,P_j\right]} = e^{-iM\boldsymbol{a}\cdot\boldsymbol{v}}\,. \tag{1.66}$$

From this we see that if we act with our transformation, on the superposition $|\psi\rangle$ we end up with

$$|\psi\rangle \mapsto |\psi'\rangle = \frac{1}{\sqrt{2}}\Big(e^{i\,m_1\boldsymbol{a}\cdot\boldsymbol{v}}|\psi_1\rangle + e^{i\,m_2\boldsymbol{a}\cdot\boldsymbol{v}}|\psi_2\rangle\Big)\,, \tag{1.67}$$

meaning that the two states $|\psi_1\rangle$ and $|\psi_2\rangle$ acquire a relative phase, namely $|\psi\rangle$ is changed by the transformation. The fact that this transformation represents the identity on spacetime led Bargmann to conclude that superpositions of different masses are not allowed in a Galilean invariant Quantum Theory. This is known as *Bargmann superselection rule*.

[5] Given a Lie group $\mathscr{G}$ and the exponential map from the Lie algebra to the Lie group $e : \mathscr{L} \to \mathscr{G}$, the Baker-Campbell-Hausdorff formula gives the solution to the equation $e^X e^Y = e^Z$ for $X, Y \in \mathscr{L}$. Specifically, $Z = X + Y + \frac{1}{2}[X,Y] + \ldots$, where the other terms of this infinite series are omitted since they all vanish in our present case.

Even though the argument above seems to be robust, it is not quite true that the transformation considered is the identity. Indeed, it is possible to show that, from a physical point of view, the $e^{-iM\boldsymbol{a}\cdot\boldsymbol{v}}$ factor is a leftover of the special-relativistic time dilation that survives in the non-relativistic $c\to\infty$ limit. Specifically, it is the $c\to\infty$ limit of the difference between the proper times elapsed in the reference frame at rest and the reference frame undergoing the chain of transformations [68]. Therefore, no mass superselection rule is actually needed in non-relativistic Quantum Mechanics. We will also provide a rigorous algebraic proof of this statement in chapter 4.

1.3 Non-commutative spacetime and its (quantum) symmetries

In this section[6], we outline how the symmetry structures of classical spacetime, the Poincaré group and (Hopf) algebra, are deformed in a quantum, non-commutative spacetime. In general, a non-commutative spacetime is described by

$$\left[x^\mu, x^\nu\right] = i\,\Theta^{\mu\nu}(x)\,. \tag{1.68}$$

Of course, the algebra of functions over this spacetime is thus deformed

$$\mathbb{C}[\mathscr{M}] \to \mathbb{C}_q[\mathscr{M}] \tag{1.69}$$

into a non-commutative Hopf algebra, or *quantum group*. The same happens, without changing the structure and properties of $\mathscr{P}$, to the algebra of functions over the Poincaré group, which is itself deformed into a quantum group described by non-commutative basis functions $\Lambda^\mu{}_\nu$ and a^μ, $\mathbb{C}[\mathscr{P}] \to \mathbb{C}_q[\mathscr{P}]$. The relation between the two algebras is not changed, namely $\mathbb{C}_q[\mathscr{M}] = {}^{\mathbb{C}_q[\mathscr{P}]}\!\big/_{SO_q(1,3)}$ provided that the (deformed) Lorentz group is still a Hopf sub-algebra of $\mathbb{C}_q[\mathscr{P}]$. The action of $\mathbb{C}_q[\mathscr{P}]$ on $\mathbb{C}_q[\mathscr{M}]$ is still given by the right co-action, which is an algebra homomorphism so that the commutation rules between the coordinates are not changed by a spacetime coordinate transformation

$$x'^\mu = \Delta_R[x^\mu] = \mathbb{1}\otimes a^\mu + x^\nu\otimes\Lambda^\mu{}_\nu \in \mathbb{C}_q[\mathscr{M}]\otimes\mathbb{C}_q[\mathscr{P}] \Rightarrow \left[x'^\mu, x'^\nu\right] = i\,\Theta(x')\,. \tag{1.70}$$

From this, it immediately follows that

$$\left[x'^\mu, x'^\nu\right] = \mathbb{1}\otimes[a^\mu, a^\nu] + x^\rho\otimes\left(\left[a^\mu, \Lambda^\nu{}_\rho\right] - \left[a^\nu, \Lambda^\mu{}_\rho\right]\right) + x^\sigma x^\rho\otimes\left(\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma - \Lambda^\nu{}_\rho\Lambda^\mu{}_\sigma\right), \tag{1.71}$$

therefore a non-commutative spacetime structure necessarily implies that the group parameters are non-commutative, or equivalently

$$\Theta^{\mu\nu}(x) = 0 \iff [a^\mu, a^\nu] = \left[a^\mu, \Lambda^\rho{}_\sigma\right] = \left[\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma\right] = 0\,. \tag{1.72}$$

On a generic function, the right co-action reads

$$\Delta_R[f(x)] = f^{(1)}(x)\otimes A_f^{(2)}\,,\ \ f^{(1)}(x)\in\mathbb{C}_q[\mathscr{M}]\,,\ \ A_f^{(2)}\in\mathbb{C}_q[\mathscr{P}]\,. \tag{1.73}$$

[6] We will set $\hbar = 1$ and $c = 1$ in this section and in the next one.

In general, choosing a basis $\{g_1,\dots,g_n\}$ for $\mathbb{C}_q[\mathscr{P}]$, (1.73) can be written as a power series in this basis elements. However, since the elements of $\mathbb{C}_q[\mathscr{P}]$ are non-commutative, an ordering prescription is needed so that

$$\Delta_R[f(x)] = \sum_{\alpha_1,\dots,\alpha_n=0}^{\infty} f^w_{\alpha_1,\dots,\alpha_n} \otimes \Omega_w\left[g_1^{\alpha_1},\dots,g_n^{\alpha_n}\right], \tag{1.74}$$

where $\Omega_w\left[g_1^{\alpha_1},\dots,g_n^{\alpha_n}\right]$ is called *Weyl map* and defines the ordering prescription, while

$$f^w_{\alpha_1,\dots,\alpha_n} = T^w_{\alpha_1,\dots,\alpha_n} \triangleright f(x)\,. \tag{1.75}$$

$T^w_{\alpha_1,\dots,\alpha_n} : \mathbb{C}_q[\mathscr{M}] \to \mathbb{C}_q[\mathscr{M}]$ are the operators that constitute the universal enveloping algebra $U_q[\mathscr{L}]$, the Hopf algebra dual to $\mathbb{C}_q[\mathscr{P}]$. As in the commutative case, the left action of these operators on functions is induced by the right co-action (1.73)

$$T \triangleright f(x) := f^{(1)}(x)\left(T, A_f^{(2)}\right), \quad T \in U_q[\mathscr{L}]\,, \quad f(x) \in \mathbb{C}_q[\mathscr{M}]\,. \tag{1.76}$$

As a Hopf algebra, $U_q[\mathscr{L}]$ is independent on the choice of the ordering, namely the Weyl map. It is only the basis of $U_q[\mathscr{L}]$ that depends on the choice of the ordering chosen by Ω_w.

The Weyl map is essential to derive the Hopf algebra $U_q[\mathscr{L}]$ from the first-order expansion in the parameters of the right co-action Δ_R. Indeed, if these parameters are non-commutative, the very notion of "first order" makes no sense without an ordering prescription since, using the commutation relations, it would be possible to change the power counting in the parameters. For instance, if $[a^\mu, a^\nu] = i\alpha^{\mu\nu}$, then it would be possible to write $a^0a^1 = a^1a^0 + i\alpha^{01}$, so that a zero-order term appears in the second-order expression a^0a^1. Choosing an ordering with a Weyl map Ω_w avoids this problem, so that the right co-action can be written as

$$\begin{aligned}\Delta_R[f(x)] &= \sum_{n,m} \frac{i^{n+m}}{2^m n! m!} M^w_{\rho_1\sigma_1}\dots M^w_{\rho_m\sigma_m} P^w_{\mu_1}\dots P^w_{\mu_n} \triangleright f(x) \otimes \Omega_w\left(a^{\mu_1}\dots a^{\mu_n}\omega^{\rho_1\sigma_1}\dots\omega^{\rho_m\sigma_m}\right)\\ &:= \Omega_w\left(e^{\frac{i}{2}M^w_{\rho\sigma}\otimes\omega^{\mu\nu}} e^{iP^w_\mu\otimes a^\mu}\right)[f(x)\otimes \mathbb{1}]\,,\end{aligned} \tag{1.77}$$

for a Weyl map $\Omega_w(a^\mu\omega^{\rho\sigma}) = \omega^{\rho\sigma}a^\mu$. The operators $P^w_\mu, M^w_{\rho\sigma}$ are the ordering-dependent generators of the Hopf algebra $U[\mathscr{L}]$ dual to $\mathbb{C}_q[\mathscr{P}]$. Changing the ordering prescription amounts to a non-linear change of these generators, which is a legitimate operation to perform in a universal enveloping algebra. The co-products and antipodes of these generators, as well as the commutators between them, are found by compatibility with the product of $\mathbb{C}_q[M]$, duality with $\mathbb{C}_q[\mathscr{P}]$, and compatibility with the composition law of $\mathbb{C}_q[\mathscr{P}]$, respectively.

When acting on a product of functions, the right co-action gives

$$\begin{aligned}\Delta_R[f(x)g(x)] &= \Omega_w\left(e^{\frac{i}{2}M^w_{\rho\sigma}\otimes\omega^{\mu\nu}} e^{iP^w_\mu\otimes a^\mu}\right)[f(x)g(x)\otimes\mathbb{1}] = \Delta_R[f(x)]\Delta_R[g(x)] =\\ &= \Omega_w\left(e^{\frac{i}{2}M^w_{\rho\sigma}\otimes\omega^{\mu\nu}} e^{iP^w_\mu\otimes a^\mu}\right)[f(x)\otimes\mathbb{1}]\,\Omega_w\left(e^{\frac{i}{2}M^w_{\rho\sigma}\otimes\omega^{\mu\nu}} e^{iP^w_\mu\otimes a^\mu}\right)[g(x)\otimes\mathbb{1}]\,,\end{aligned} \tag{1.78}$$

since it is a homomorphism for the product in $\mathbb{C}_q[\mathscr{M}]$. Being the co-product of the generators dual to the product of $\mathbb{C}_q[\mathscr{M}]$, the above expression can be rewritten as

$$\Delta_R[f(x)g(x)] = \nabla_{12}\Big\{\Omega_w\Big(e^{\frac{i}{2}\Delta M^w_{\rho\sigma}\otimes\omega^{\mu\nu}}e^{i\,\Delta P^w_\mu\otimes a^\mu}\Big)[f(x)\otimes g(x)\otimes\mathbb{1}]\Big\}, \tag{1.79}$$

where $\nabla_{12}:\mathbb{C}_q[\mathscr{M}]\otimes\mathbb{C}_q[\mathscr{M}]\otimes\mathbb{C}_q[\mathscr{P}]\to\mathbb{C}_q[\mathscr{M}]\otimes\mathbb{C}_q[\mathscr{P}]$ represents the product of $\mathbb{C}_q[\mathscr{M}]$ over the triple tensor product as $\nabla_{12}\{f(x)\otimes g(x)\otimes A\} = f(x)g(x)\otimes A,\ A\in\mathbb{C}_q[\mathscr{P}]$. Enforcing (1.3) we get

$$\Omega_w\Big(e^{\frac{i}{2}\Delta M^w_{\rho\sigma}\otimes\omega^{\mu\nu}}e^{i\,\Delta P^w_\mu\otimes a^\mu}\Big) = \Omega_w\Big(e^{\frac{i}{2}M^w_{\rho\sigma}\otimes\mathbb{1}\otimes\omega^{\mu\nu}}e^{i\,P^w_\mu\otimes\mathbb{1}\otimes a^\mu}\Big)\Omega_w\Big(e^{\frac{i}{2}\mathbb{1}\otimes M^w_{\rho\sigma}\otimes\omega^{\mu\nu}}e^{i\,\mathbb{1}\otimes P^w_\mu\otimes a^\mu}\Big), \tag{1.80}$$

which gives the link between the co-products of the generators of $U_q[\mathscr{L}]$ and the right co-action of $\mathbb{C}[\mathscr{P}]$ on $\mathbb{C}[\mathscr{M}]$. Notice that, if the group parameters a^μ and $\omega^{\rho\sigma}$ are commutative, then the co-products are primitive.

For what concerns the commutator, we can enforce that the right co-action is a homomorphism even with respect to the group composition, the latter being unaffected by the quantum deformation of the algebra of functions $\mathbb{C}[\mathscr{P}]$. By doing so we have

$$\begin{aligned}\Delta_R\Big[(\Lambda',a')\cdot(\Lambda,a)\Big] &= \Omega_w\Big(e^{\frac{i}{2}M^w_{\rho\sigma}\otimes\omega^{\mu\nu}}e^{i\,P^w_\mu\otimes a^\mu}\Big)\Big[\mathbb{1}\otimes(\Lambda'\cdot a+a',\Lambda'\cdot\Lambda)\Big] =\\ &= \Delta_R\Big[(\Lambda',a')\Big]\Delta_R\Big[(\Lambda,a)\Big] =\\ &= \Omega_w\Big(e^{\frac{i}{2}M^w_{\rho\sigma}\otimes\omega^{\mu\nu}}e^{i\,P^w_\mu\otimes a^\mu}\Big)\Big[\mathbb{1}\otimes(\Lambda',a')\Big]\Omega_w\Big(e^{\frac{i}{2}M^w_{\rho\sigma}\otimes\omega^{\mu\nu}}e^{i\,P^w_\mu\otimes a^\mu}\Big)\Big[\mathbb{1}\otimes(\Lambda,a)\Big].\end{aligned} \tag{1.81}$$

The product of two group elements is dual to the co-product of the group parameters, therefore the above expression can be rewritten as

$$\begin{aligned}&\Omega_w\Big(e^{\frac{i}{2}M^w_{\rho\sigma}\otimes\omega^{\mu\nu}}e^{i\,P^w_\mu\otimes a^\mu}\Big)\Big[\mathbb{1}\otimes(\Lambda',a')\cdot(\Lambda,a)\Big] =\\ &= \nabla_{23}\Bigg\{\Omega_w\Big(e^{\frac{i}{2}M^w_{\rho\sigma}\otimes\Delta\omega^{\mu\nu}}e^{i\,P^w_\mu\otimes\Delta a^\mu}\Big)\Big[\mathbb{1}\otimes(\Lambda',a')\otimes(\Lambda,a)\Big]\Bigg\},\end{aligned} \tag{1.82}$$

where $\nabla_{23}:U_q[\mathscr{L}]\otimes\mathscr{P}\otimes\mathbb{P}\to U_q[\mathscr{L}]\otimes\mathscr{P}$ represents the product of $\mathbb{P}$ over the triple tensor product as $\nabla_{23}\{T\otimes(\Lambda',a')\otimes(\Lambda,a)\} = T\otimes(\Lambda',a')\cdot(\Lambda,a)$, $T\in U_q[\mathscr{L}]$. From this we get

$$\Omega_w\Big(e^{\frac{i}{2}M^w_{\rho\sigma}\otimes\Delta\omega^{\mu\nu}}e^{i\,P^w_\mu\otimes\Delta a^\mu}\Big) = \Omega_w\Big(e^{\frac{i}{2}M^w_{\rho\sigma}\otimes\omega^{\mu\nu}\otimes\mathbb{1}}e^{i\,P^w_\mu\otimes a^\mu\otimes\mathbb{1}}\Big)\Omega_w\Big(e^{\frac{i}{2}M^w_{\rho\sigma}\otimes\mathbb{1}\otimes\omega^{\mu\nu}}e^{i\,P^w_\mu\otimes\mathbb{1}\otimes a^\mu}\Big), \tag{1.83}$$

where Δa^μ and $\Delta\omega^{\rho\sigma}$ are the co-products reflecting the composition law of $\mathscr{P}$ in (1.38)[7].

1.3.1 κ-Poincaré group and algebra

In what follows, we focus on a specific non-commutative spacetime structure given by the κ-Minkowski spacetime [59, 69–72]

$$\Big[x^0,x^j\Big] = \frac{i}{\kappa}x^j = i\,\lambda\,x^j\,,\quad\Big[x^i,x^j\Big] = 0\,,\quad\kappa^{-1} := \lambda\,. \tag{1.84}$$

[7] The co-product of $\omega^{\rho\sigma}$ can be derived from the composition law of the Lorentz subgroup.

Even without the Hopf algebraic description discussed above, it is immediately understood that a simple Poincaré transformation cannot leave these commutators invariant. In fact, if we consider a spacetime translation $x^{\mu\prime} = x^\mu + a^\mu$ then the commutators become

$$\left[x^{0\prime}, x^{j\prime}\right] = \left[x^0 + a^0, x^j + a^j\right] = i\lambda x^j \neq i\lambda x^{j\prime} = i\lambda\left(x^j + a^j\right). \tag{1.85}$$

However, if $\left[a^0, a^j\right] = i\lambda a^j$, then the commutators are relativistically invariant and the scale λ, typically identified with the Planck length ℓ_P emerges as an invariant length scale of the theory. In fact, this is one of the commutators that arises when enforcing the homomorphism property of the right co-action of $\mathbb{C}_q[\mathscr{P}]$ on $\mathbb{C}_q[\mathscr{M}]$. This yields the commutation relations defining the κ-Poincaré group [73, 74]

$$\begin{gathered}\left[a^0, a^j\right] = i\lambda a^j\,, \quad \left[a^i, a^j\right] = 0\,, \quad \left[\Lambda^\mu{}_\nu, \Lambda^\rho{}_\sigma\right] = 0\,,\\ \left[a^\mu, \Lambda^\rho{}_\sigma\right] = -i\lambda\left\{\left(\Lambda^\rho{}_0 - \delta^\rho{}_0\right)\Lambda^\mu{}_\sigma + \left(\Lambda^0{}_\sigma - \delta^0{}_\sigma\right)\eta^{\rho\mu}\right\}.\end{gathered} \tag{1.86}$$

An interesting first property that derives from these commutation relations is that it is not possible to perform some "pure" transformations [62, 75]. Indeed, by interpreting the group parameters as operators acting on a Hilbert space $\mathscr{H}_{\mathscr{P}}$, a non-vanishing commutator can be associated to an uncertainty relation between these parameters [62]. For example, to perform a pure spatial translation, we would need a (sharply) vanishing a^0 parameter, but if we have $a^0 = 0$ without uncertainty, $\Delta a^0 = 0$, then from the commutation relation $\left[a^0, a^j\right] = i\lambda a^j$ we would need $\left\langle a^j \right\rangle = 0$ to have a sharp spatial translation, where the uncertainties and the expectation values are computed on states in $\mathscr{H}_{\mathscr{P}}$. The same argument applies for pure boosts.

The generators of the κ-Poincaré group constitute the κ-Poincaré Hopf algebra. The co-products of these generators and the commutators among them can be derived, upon choosing an ordering prescription, by means of (1.3) and (1.83). If the following order is chosen

$$\begin{aligned}\Delta_R[f(x)] &= \sum_{n,m,k=0}^{\infty} f^{\mu_1\nu_1,\dots,\mu_n\nu_n}_{j1,\dots,j_m,k}(x) \otimes \omega_{\mu_1\nu_1}\dots\omega_{\mu_n\nu_n} a_{j_1}\dots a_{j_m} a_0^k\,,\\ f^{\mu_1\nu_1,\dots,\mu_n\nu_n}_{j1,\dots,j_m,k}(x) &= \frac{i^{n+m}(-i)^k}{2^n n!m!k!}\left(M_{\mu_1\nu_1}\dots M_{\mu_n\nu_n} P_{j_1\dots P_{j_m}} P_0^k\right) \triangleright f(x)\,,\end{aligned} \tag{1.87}$$

then the commutators and the co-products read[8]

$$\left[P_\mu, P_\nu\right] = 0\,, \quad \left[J_i, P_j\right] = i\epsilon_{ij}{}^k P_k\,, \quad \left[J_i, N_j\right] = i\epsilon_{ij}{}^k N_k\,, \quad \left[J_i, P_0\right] = 0\,, \quad \left[J_l, J_m\right] = i\epsilon_{lm}{}^n J_n\,, \tag{1.88}$$

$$\left[N_i, N_j\right] = -i\epsilon_{ij}{}^k J_k\,, \quad \left[N_i, P_0\right] = iP_i\,, \quad \left[N_i, P_j\right] = i\delta_{ij}\left(\frac{1 - e^{-2\lambda P_0}}{2\lambda} + \frac{\lambda}{2}\boldsymbol{P}^2\right) - i\lambda P_i P_j\,,$$

$$\begin{gathered}\Delta P_0 = P_0 \otimes \mathbb{1} + \mathbb{1} \otimes P_0\,, \quad \Delta P_j = P_j \otimes \mathbb{1} + e^{-\lambda P_0} \otimes P_j\,, \quad \Delta J_i = J_i \otimes \mathbb{1} + \mathbb{1} \otimes J_i\,,\\ \Delta N_i = N_i \otimes \mathbb{1} + e^{-\lambda P_0} \otimes N_i + \lambda \epsilon_i{}^{jk} P_j \otimes J_k\,.\end{gathered} \tag{1.89}$$

[8] $J_i := \epsilon_i{}^{jk} M_{jk}$, $N_i = M_{i0}$.

The antipodes can be derived by the co-products by enforcing the consistency relations that define a Hopf algebra. These antipodes are

$$S(P_0) = -P_0\,,\;\; S(P_i) = -e^{\lambda P_0}P_i\,,\;\; S(J_i) = -J_i\,,\;\; S(N_i) = -e^{\lambda P_0}N_j + \lambda\epsilon_i^{\;jk}P_jJ_k\,. \tag{1.90}$$

The last thing that is left to notice is that, since the commutators of the algebra are deformed, the Casimirs of the algebra will be deformed too, in general. This is the case for the mass Casimir, that is given by

$$C = \left[\frac{2}{\lambda}\sinh\left(\frac{\lambda}{2}P_0\right)\right]^2 - e^{\lambda P_0}\boldsymbol{P}^2\,. \tag{1.91}$$

The deformations appearing in the symmetry algebra that we have discussed in this subsection are more than just algebraic manipulations. As we will see in the last section of this chapter, as well as in chapter 3, they can be physically interpreted as giving rise to testable effects. Before considering this phenomenological aspect of symmetry deformations, we will discuss some other theoretical features of the framework we have outlined so far.

The problem of the change of basis

The basis of the κ-Poincaré Hopf algebra described by the equations above is called "right-ordered bicrossproduct basis". The reason behind this name comes from the choice that has been made for the ordering of the group parameters, with the parameters of Lorentz boosts on the left and the parameters of translations on the right (specifically, the time translation parameter is the last on the right, hence the name of the basis). Different choices for the ordering correspond to non-linear changes of basis for the generators of the algebra. Of course, such changes of basis yield, in general, different commutators and co-products. An example is given by the so-called *classical basis* [76]. This basis is characterised by an algebra sector, i.e. commutators and Casimirs, that is the same as the Poinacré algebra (this is the reason why it is called "classical"), while all deformations are in the co-algebra sector. Specifically, the co-products and the antipodes are given by

$$\begin{gathered}\Delta P'_0 = P'_0\otimes\Pi_0 + \Pi_0^{-1}\otimes P'_0 + \lambda P'_i\Pi_0^{-1}\otimes P'_i\,,\;\; \Delta P'_i = P'_i\otimes\Pi_0 + \mathbb{1}\otimes P'_i\,,\\ \Delta J_i = J_i\otimes\mathbb{1} + \mathbb{1}\otimes J_i\,,\;\; \Delta N_i = N_i\otimes\mathbb{1} + \Pi_0^{-1}\otimes N_i + \lambda\epsilon_i^{\;jk}P'_j\Pi_0^{-1}\otimes J_k\,,\\ S(P'_0) = -P'_0 + \lambda\boldsymbol{P}'^2\Pi_0^{-1}\,,\;\; S\left(P'_i\right) = -P'_i\Pi_0^{-1}\,,\\ S(J_i) = -J_i\,,\;\; S(N_i) = -\Pi_0N_i + \lambda\epsilon_i^{\;jk}P'_jJ_k\end{gathered} \tag{1.92}$$

where the new translation generators are related to those of the bicrossproduct basis by

$$P_0 = \frac{1}{\lambda}\log\Pi_0\,,\;\; P_i = P'_i\Pi_0^{-1}\,, \tag{1.93}$$

and

$$\begin{gathered}\Pi_0 = \lambda P'_0 + \sqrt{1-\lambda^2P'_\mu P'^\mu}\,,\;\; \Pi_0^{-1} = \frac{\sqrt{1-\lambda^2P'_\mu P'^\mu} - \lambda P'_0}{1-\lambda^2\boldsymbol{P}'^2}\,,\\ \Delta\Pi_0 = \Pi_0\otimes\Pi_0\,,\;\; \Delta\Pi_0^{-1} = \Pi_0^{-1}\otimes\Pi_0^{-1}\,,\;\; S(\Pi_0) = \Pi_0^{-1}\end{gathered} \tag{1.94}$$

Obviously, the algebraic and co-algebraic sectors of the basis presented here are completely different from those of the bicrossproduct basis. Since the physical effects predicted by a deformation of symmetries depend on the form of the commutators, co-products etc., of the generators of the algebra, this means that, in general, different bases give rise to different predictions, i.e. different physical theories. Should this be regarded as a problem? To correctly address this issue, it is instructive to look at it from a different angle. The algebraic deformations of the spacetime symmetries sector discussed in this chapter admit a dual description in terms of curved momentum spaces [77, 78]. In the context of curved momentum spaces, a change of basis in the algebra amounts to a diffeomorphism in momentum space. Contrary to General Relativity, in which diffeomorpshism invariance is built-in as a physical principle of the theory, for curved momentum spaces no such invariance is needed in principle. Different coordinates on momentum space yield, in general, different predictions [79] and could therefore describe different physical models. Only by performing experiments that can verify or falsify the physical predictions of the various models could it be possible to determine which, if any, curved momentum space describes the physical laws of our world.

The Relative Locality regime of Quantum Gravity

The idea that momentum space might be curved is not new in physics, see [80] and references therein. In the context of gravitational physics, a curved momentum space arises in $1+2$ dimensions with Gravity coupled to particles. In $1+2$ dimensions the gravitational field only has topological degrees of freedom and it is argued that there could exist a topological phase of Quantum Gravity in $1+3$ dimensions as well [80]. From a physical point of view, the framework in which such a curved structure on momentum space emerges can be understood as a "non-quantum, non-gravitational" regime of Quantum Gravity, in which $\hbar \to 0$, $G \to 0$, but their ratio, giving the Planck mass or Planck energy, is fixed. This is the *Relative Locality* framework [81], in which the momentum space is the fundamental structure, described by a curved pseudo-Riemannian manifold, and to each point of the manifold a (flat and commutative) momentum-dependent spacetime structure is reconstructed as the tangent space to that point. This can also be seen as a dual picture with respect to General Relativity, in which spacetime is the fundamental structure, described by a pseudo-Riemannian curved manifold, and to each point of this manifold a spacetime-dependent (flat) momentum space is assigned as the tangent space to that point. This effectively realises the idea of *Born reciprocity* [82], a principle by which the laws of physics should admit a duality symmetry between spacetime and momentum space. In the framework of Relative Locality, every algebraic notion such as co-products, the mass Casimir, etc. can be related to a geometrical quantity (connection, geodesic distance, curvature, torsion, ...) of a curved momentum space [83], and this realises the duality between the Hopf algebraic and the geometrical descriptions of deformed symmetries. From a physical point of view, the most striking result of Relative Locality is already written in its name: Locality is not absolute as in Special Relativity, it becomes an observer-dependent notion. Specifically, all observers describe interaction events distant from them as delocalised in spacetime, whereas near events are perceived as localised. This is in analogy with Special Relativity, that is obtained from Galilean relativity by introducing an invariant velocity scale, and this results in the loss of the absoluteness of simultaneity (but locality is still absolute). In Relative Locality, introducing an invariant energy scale results in the loss of the absoluteness of locality.

1.3.2 Localisability and observers

In the framework of Relative Locality, starting from a de Sitter momentum space [77] it is possible to show that the Poisson brackets between the coordinates of events have the same form of the κ-Minkowski commutation relations [83]. Therefore, the Relative Locality regime could be interpreted as a "$\hbar \to 0$ relic" of the κ-Minkowski non-commutative spacetime, which should be recovered upon quantisation (replacing the Poisson brackets with the commutator times $i\hbar$). From this point of view, can the relativity of locality be described directly in the context of κ-Minkowski spacetime? This is what we will discuss here, following [59].

In the previous subsection, we have anticipated that, in a non-commutative spacetime, coordinates become operators acting on a Hilbert space. We denote this Hilbert space as $\mathscr{H}_\kappa$. The three operators x^i form a complete set of observables on $\mathscr{H}_\kappa$ that, as a functional space, is therefore given by $L^2\left(\mathbb{R}^3\right)$. The fact that the position and the time operator satisfy $\left[x^0, x^i\right] = i\lambda x^i$ means that they are incompatible observables that formally satisfy the uncertainty relation

$$\Delta x^0 \Delta x^i \geq \frac{|\lambda|}{2}\left|\left\langle x^i \right\rangle\right| , \tag{1.95}$$

on the states in $\mathscr{H}_\kappa$. A representation of these four operators in $L^2\left(\mathbb{R}^3\right)$ is given by

$$\begin{aligned} \hat{x}^i \psi(x) &= x^i \psi(x) , \\ \hat{x}^0 \psi(x) &= i\lambda\left(r\partial_r + \frac{3}{2}\right)\psi(x) , \end{aligned} \tag{1.96}$$

where we have introduced the hat to denote the operators and distinguish them from their representations. As we can see, the representation of x^0 only contains the radial coordinate r. This suggests that all the non-commutativity of κ-Minkowski can be captured by the radial and time coordinates. In fact, by performing the change of coordinates $x^3 = r\cos\theta$, $re^{i\phi}\sin\theta = x^1 + i\,x^2$ we end up with

$$\left[\cos\theta, x^0\right] = \left[e^{i\phi}, x^0\right] = 0 , \quad \left[x^0, r\right] = i\lambda r \implies \Delta x^0 \Delta r \geq \frac{|\lambda|}{2}\left|\langle r\rangle\right| . \tag{1.97}$$

For this reason, we can write any function in $L^2\left(\mathbb{R}^3\right)$ as a spherical harmonics expansion $\psi(x) = \sum_{l,m}\psi_{l,m}(r)\,Y_{l,m}\left(\theta,\phi\right)$ and focus only on the radial components.

The states described by these wavefuntions can be interpreted as "states of the geometry", describing spacetime events in κ-Minkowksi. The uncertainty relation in (1.97) tells us that, in general, it is not possible to find states that sharply localise both the spatial and the time coordinates. However, there are wave functions in $L^2\left(\mathbb{R}^3\right)$ that minimise this uncertainty. A particular limit of these states yields an improper state $|o\rangle$ that localises both the time and spatial coordinates at the origin $r = 0$. This can happen, contrary to standard Quantum Mechanics, because the commutator in (1.97) is proportional to the expectation value of an operator and not a constant as for the phase space of Quantum Mechanics $\left[q, p\right] = i$. This means that every observer can localise the origin of their reference frame with the state $|o\rangle$ at any time. However, there is no preferred frame. Everything is relativistic and it is possible to change reference frame by using the right (or left) co-action of the κ-Poincaré group on the wave functions describing spacetime events. Of course, it is not possible to sharply change reference frame since events distant from the origin cannot be localised sharply. The new observer can localise sharply their origin and cannot localise sharply the origin of the other observer. This picture is similar to the Relative Locality framework, but the relation between the two approaches is still not completely clear.

The discussion of this subsection introduces an important notion: *Quantum observers*. For the purposes of this thesis, this term denotes observers that are related by quantum symmetries. This notion is concretely realised in κ-Minkowski, in which the states of the geometry describing spacetime events in general do not allow to sharply localise events because of the non-vanishing commutator between the spacetime coordinates. There is only a point that a given (quantum) observer Alice can localise, her spatial origin. Since the other points in space cannot be sharply localised, any other observer Bob cannot be sharply localised by Alice, and the picture is perfectly relativistic in that Bob can localise his origin sharply, but not Alice's position.

1.4 Phenomenological tests

In the previous sections of this chapter, we reviewed some theoretical aspects of the framework of non-commutative spacetime and deformation of spacetime symmetries, exploiting the language of Hopf algebras. It is now time to see what kind of phenomenological analyses can be devised to test the predictions of these models. The idea behind such tests is to "amplify" the small effects predicted by the theory. As we anticipated, the experiments that are performed nowadays are several orders of magnitude away from the Planck scale, and if reaching this scale was necessary to observe Quantum Gravity effects we would never observe them. The possibility of amplifying a physical effect depends on its nature. In particular, it is possible to distinguish between "steep-onset" and "smooth-onset" effects, namely effects that require a "microscope" to be observed and effects that can be "amplified" and be observed even with instruments that do not reach the sensitivity to appreciate the same effect without amplification.

Steep-onset effects are those that appear only at a given scale. For example, a hydrogen atom is ionised only with photons with energies $E \geq 13.6$ eV. If only photons with energies smaller than this value were available, we could not ionise the hydrogen atom, no matter how many photons we used, because the energy of the photons does not cumulate in the process and the effect cannot be amplified in this way. If this was the case for Quantum Gravity, then we would need a Galaxy-spanning particle accelerator (the microscope) to reach the Planck scale and witness Quantum Gravity effects. In this scenario, we would be of course doomed to perform only theoretical analyses.

Smooth-onset effects, on the other hand, do not have a threshold scale above which they appear and can be amplified. The most paradigmatic example of this type of effect is the Brownian motion. In 1827 the botanic Robert Brown observed a strange motion performed by pollen particles immersed in a sample of water. He did not have a microscope sensitive enough to obscure water molecules, and he did not know about the existence of molecules in the first place. However, he was (unknowingly) observing the effects of the collisions between pollen particles and the water molecules, as explained by Einstein almost 80 years later. This was possible, even though the ratio between the mass of a single molecule of water and a pollen particle is roughly 10^{-14}, because the motion of the pollen particle was the result of an Avogadro number ($\approx 10^{23}$) of collisions. The amplifier is the number of molecules.

If the right amplifier can be found, there could be the possibility to test the predictions of those approaches to QG that are characterised by smooth-onset effects. One such example is given by interpreting the mass Casimir operator of the spacetime symmetry algebra as defining the mass parameter for particles. Therefore, if the Casimir is deformed, the dispersion relation for particles will be deformed as well and, in general, takes the form $f(E,p,m)=0$. Typically, the dispersion relation for photons is such that the velocity of light in vacuum becomes energy-dependent and takes the form $c(E)=1-\eta\left(\frac{E}{E_P}\right)^n$ (typically $n=1$ or $n=2$). Such Quantum Gravity corrections to the velocity of particles are of course negligible for the energies that can be reasonably achieved but, if a high energetic particle, e.g. a photon, is emitted from a distant source and travels for enough time, the difference between the arrival time and the expected arrival time without Quantum Gravity effects can become macroscopic and observable with current technologies. This difference is called *time delay*. These investigations can be performed with astrophysical sources [39–42], so that the smallness of the Planck scale effects can be overcome with the largeness of the Universe, realising the idea of using the Universe itself as a laboratory.

The phenomenological test described above was first proposed in [39]. The idea behind this is to consider astrophysical sources such as gamma ray bursts (GRBs) and take the low energy photons as the trigger signal of these sources, then consider the time of arrival of the most energetic photons. In the assumption that the two photons are emitted (nearly) simultaneously[9], if a difference in the arrival time of the high-energy photon and the trigger time of the GRB is observed, this can be a measurement of time delay caused by a deformed dispersion relation. An example is given by the dispersion relation coming from the deformed κ-Poincaré symmetries in the bicrossproduct basis. There are basically two possible dispersion relations in this case that are given by

$$\begin{aligned} m^2 = C &= \left[\frac{2}{\lambda}\sinh\left(\frac{\lambda}{2}E\right)\right]^2 - e^{\lambda E}\boldsymbol{p}^2\,, \\ \left[\frac{2}{\lambda}\sinh\left(\frac{\lambda}{2}m^2\right)\right]^2 = C &= \left[\frac{2}{\lambda}\sinh\left(\frac{\lambda}{2}E\right)\right]^2 - e^{\lambda E}\boldsymbol{p}^2\,. \end{aligned} \tag{1.98}$$

The first comes from interpreting the Casimir as the mass of particles, while the second comes from requiring that the mass is the rest energy of particles, and also has a geometrical interpretation in the curved momentum space framework [83]. In any case, the first order in λ, which is the only thing that matters for phenomenological purposes, is the same for the two dispersion laws. Moreover, the two cases are identical for photons. At first order in λ we get $E=\sqrt{m^2+p^2}+\frac{\lambda p^2}{2}$ so that the energy-dependent Quantum Gravity correction to the parti-

[9]This assumption is, in principle, challengeable. In fact, there is the possibility that GRBs are affected by an intrinsic lag in the emission of low- and high-energy photons [84] and this lag can be of the same order as the time delay effect (order of magnitude $\sim 0.01\,\mathrm{s} - 1\,\mathrm{s}$). For this reason, in some analyses high-energy neutrinos are considered, for which time delays are of the order of days or even months, so that any possible intrinsic lag is completely negligible.

cles velocity reads $v(E) = v_0(E) + \lambda E$, where $v_0(E) \approx 1 - \frac{m^2}{E^2}$ is the special-relativistic velocity for relativistic particles. Notice that there is no a priori indication on the sign of λ and, in principle, not even on its magnitude. Therefore, it is possible to write λ as a free parameter $-\eta$ (to be fitted with experiments) times the expected value of the Quantum Gravity scale $\ell_P = E_P^{-1}$. The velocity of particles then becomes $v(E) = v_0(E) - \eta \frac{E}{E_P}$ with $\eta \in \mathbb{R}$. Therefore, if L is the propagation distance, the time delay Δt is given by

$$t = \frac{L}{v(E)} \approx L\left(1 + \frac{m^2}{E^2} + \eta \frac{E}{E_P}\right) = t_{SR} + \Delta t \Longrightarrow \Delta t = \eta L \frac{E}{E_P}\,, \tag{1.99}$$

where t_{SR} is the special-relativistic time of travel. Notice that, to carry out this analysis, particles are supposed to be classical particles propagating on a commutative spacetime with deformed relativistic symmetries, in the spirit of the curved momentum space framework described in the previous section.

The parameter η in (1.99) above essentially quantifies how far the Quantum Gravity scale is from the Planck scale. This parameter is expected to be not much larger (or smaller) than 1. For this reason, even with energies of the order $E \sim 10$ GeV we would need a travel distance of $L \sim 10^{25}$ m $\sim 10^9$ ly to get a time delay of $\Delta t \sim 0.1$ s. This is why it is necessary to turn to astrophysical sources. However, the discussion above with the κ-Poincaré symmetries is performed in a flat spacetime context, while a framework contemplating an expanding spacetime would be required to correctly describe the propagation of photons (and other particles) for astrophysical distances. A proposal for a time delay in a cosmological setting was first derived in [85] in the case of Lorentz invariance violations (LIV). This proposal is the one that has been phenomenologically investigated ever since, and the expression for the time delay reads

$$\Delta t\big(E, z\big) = \eta \frac{E}{E_P} \int_0^z \mathrm{d}\zeta\, \frac{1+\zeta}{H(\zeta)}\,, \tag{1.100}$$

where a dependence from the redshift z of the source appears and the Hubble parameter for the ΛCDM model is[10] $H(\zeta) = H_0\sqrt{\Omega_\Lambda + (1+\zeta)^3\Omega_m}$. In the context of DSR theories, the most general formula for time delays in a (deformed) FLRW background[11], at first order in the Planck scale, was derived in [87]. Only three possible forms of redshift dependence are allowed by relativistic consistency, on of which is precisely (1.100). Of course, the most general

[10] H_0, Ω_Λ and Ω_m are the Hubble constant, the cosmological constant and the matter fraction respectively.

[11] As anticipated above, particles are considered to be classical and to be propagating on a commutative spacetime background, therefore we are effectively in the limit $\hbar, G \to 0$ with their ratio kept fixed. How is it possible to describe a curved setting in such a regime? The answer is in the procedure used to derive the deformed FLRW setting. To describe this, it is important to notice that it is meaningful to talk about deformation of symmetries only when there is an algebra of symmetries that can be deformed. For the classical FLRW spacetime, this is not the case since it is not maximally symmetric. However, the FLRW spacetime can be reconstructed as a succession of de Sitter slicings at fixed time [86], and the de Sitter spacetime is maximally symmetric. For the deformed case it is possible to first deform the symmetries of de Sitter spacetime and then consider the slicing procedure to reconstruct a deformed FLRW. Notice that de Sitter spacetime can be obtained as a solution to Einstein equations even in the limit $G \to 0$, as the only source needed by this spacetime structure is a positive cosmological constant. Therefore, the idea is to deform symmetries locally, which is in any case the only reasonable thing to do since an algebra of (classical) symmetries for general curved spacetimes does not exist, in the regime $\hbar, G \to 0$, and then reconstruct a general curved spacetime through the slicing method. In the case of FLRW, the most efficient local slicing is with de Sitter spacetimes, where "local" means "at fixed times".

formula for the time delay is a linear combination of these three terms, hence depends on three free parameters.

The possibility of testing putative Quantum Gravity effects using astrophysical sources such as GRBs sparked the research on Quantum Gravity phenomenology almost 30 years ago [39]. In these three decades, the phenomenological opportunity offered by deformed in-vacuo dispersion relations has been the subject of numerous studies that focused on photons from astrophysical sources [88–91], high-energy astrophysical neutrinos [92–94], and both messengers combined [95].

1.5 Summary: non-classical reference frames

In this chapter, we reviewed the basics of the non-commutative spacetime formalism, focusing on the ensuing deformation of spacetime symmetries that is described in the language of Hopf algebras. We have first considered classical spacetime symmetries, and we have discussed how the theory of spacetime symmetries in the classical spacetime setting can be recast in the language of Hopf algebras. We then analysed the main features of the "quantisation" of symmetries that originates from the non-commutativity between spacetime coordinates. In particular, we emphasised that such deformations of symmetries can give rise to testable effects that can be looked for in astrophysical and cosmological messengers. We also observed that non-commutativity between spacetime coordinates intrinsically introduces a notion of quantum observers because of the limits on localisation of spacetime events that it places.

We want to make here an important final remark on the main formal feature of the deformation of spacetime symmetries, *i.e.* the quantisation of the parameters defining symmetry transformations that are promoted to operators acting on Hilbert spaces. Promoting the parameters of spacetime coordinate transformations to operators effectively amounts to replacing classical reference frames with some form of non-classical reference frames. This is to be expected since, as we discussed above, non-commutative spacetime symmetries require observers to acquire non-classical features, hence the reference frames that they define can be expected to behave non-classically. In fact, the problem of introducing *quantum reference frames* in Quantum Gravity has been investigated for many years [31,96–101]. More recent approaches [51,102–126] to the problem of quantum reference frames have focused on the operational perspective in which abstract reference frames are abandoned in favour of a concrete notion of reference frames defined by physical systems that act as a reference for the description of physical laws of other systems. Given that physical systems are ultimately quantum, such a framework inherently replaces classical reference frames with reference frames whose degrees of freedom are quantised, *i.e.* quantum reference frames. Remarkably, the transformations between different reference frames are described by unitary operators in which the transformation parameter is itself promoted to an operator that acts on the Hilbert spaces of the physical systems playing the role of references. This has sparked some interest in trying to understand whether these two types of non-classical symmetry transformations are two sides of the same coin, at least in some low-energy regime [109].

The next chapter is entirely devoted to quantum reference frames and other topics that are useful for investigating the interface between Quantum Theory and Gravity with the logical reasoning of Quantum Information Theory.

Second Chapter

The Quantum Theory and Gravity interface: Quantum Information tools

Traditionally, quantum effects in Gravity were believed to take place only in the high-energy regime, typically identified with the Planck scale $E_P \sim 10^{19} GeV$ (or $\ell_P \sim 10^{-35} m$). Many proposed theories of Quantum Gravity predict Planck scale effects that are far away from the energy scales we can test nowadays, and the only hope to test some of these effects, as discussed in the previous chapter, is to find some amplifiers. For this reason, it is interesting to also explore the low-energy regime, where Quantum Gravity can be treated perturbatively and it is possible to meaningfully define quantum particles [127–129]. In this regime, we can think of concrete scenarios with direct physical meaning that can be used to develop phenomenological tests. This is interesting because some foundational questions for the Quantum Gravity problem already arise in this regime, in which we can investigate some expected features such as the notion of quantum time (and quantum clocks), the existence of indefinite causal structures, the lack of classical reference frames, and so forth. Moreover, it could even be possible to devise specific experiments to prove whether the gravitational field must be quantum in nature [130, 131], analogously to what has been done for the electromagnetic field [132].

In this emerging research field, sometimes denoted as *gravitational quantum physics*, tools from Quantum Information theory are used to answer fundamental questions in scenarios where Gravity is relevant, and this gives the opportunity to explore theoretical, conceptual, and observational questions that arise when combining Gravity and Quantum Theory in the low-energy regime. An important feature of this approach is its intrinsic operational nature, which allows to straightforwardly translate abstract theoretical scenarios into physically meaningful laboratory procedures. This provides an appropriate framework for a dialogue with quantum optics experimentalists, who are planning phenomenological tests at the interface between Quantum Theory and Gravity in regimes that are hitherto undreamt of but could be accessible in the near future [48, 51, 52, 119, 130, 133–138]. On the foundational side, this research field is characterised by a major focus on conceptual questions rather than on the construction of a full theory of Quantum Gravity. These are investigated by means of thought experiments that are apt to question the mutual logical consistency between distinctive elements of Quantum Theory and General Relativity (*e.g.* entanglement, the superposition principle, the equivalence principle, time dilation, and so forth) in concrete physical scenarios. To make sense of the operational procedures that are devised to deduce properties of spacetime or non-classical spacetime in such a framework, it is necessary to implement devices that are sensitive to how time and distances change under the influence of Gravity. *Quantum clocks* are an example of such devices. They not only allow to test putative quantum properties of spacetime, but also to test some features of classical Gravity. In addition to the possible advantages in communication protocols involving quantum clocks, from a foundational perspective using them to test some fundamental principles of physics, such as the equivalence principle, gives the opportunity to question the applicability of these principles beyond the assumptions that led to their current formulation. This, in turn, could yield some hints in understanding how far the applicability of these principles can be pushed and what it is necessary to modify in more general situations.

In this kind of logical reasoning drawn from Quantum Information theory, observers and reference frames play a central role in describing physical systems and measurement apparatuses and procedures. The notion of *reference frame* is indeed at the heart of Physics for every observation and theoretical prediction is relative to a given reference frame. Typically, reference frames in physical theories are considered as abstract entities that serve only the purpose of describing physical laws. However, in actual experimental setups, reference frames are real physical systems that act as references for other systems. For instance, the position of a particle in a laboratory is relative to some other object, such as a rigid optical table equipped with precise and equally rigid rulers used to measure distances. To implement this operational approach in physical theories, we are forced to consider reference frames as actual physical systems with their own dynamics and even interactions with the surrounding systems. However, if Quantum Theory ultimately describes every physical system, then we are forced to consider the possibility that the systems acting as reference can be quantum in nature, resulting in the notion of *quantum reference frames* (QRFs). Reference frames can acquire quantum features in several inequivalent ways, corresponding to which type of quantum effects are relevant in specific physical contexts. This has led to a bloom of different approaches that have developed in parallel in the two communities of Quantum Gravity [31, 96–101] and Quantum Information [139–145]. In recent years, convergence between the two approaches has begun [51, 102–126, 146–148]. This revealed similarities and differences between the approaches and improved our understanding of which type of formulation is more suitable to describe a certain physical situation.

We will focus on this more recent approach to QRFs, which sits at the interface between Quantum Information and Quantum Gravity, introduced in [102]. The reason why QRFs could arise in Quantum Gravity research comes from the observation that when Gravity acquires non-classical features we cannot have a classical spacetime and, in the absence of the latter, it seems questionable to let reference frames retain a classical nature. In fact, we have already seen in the previous chapter an example of this in the context of non-commutative spacetimes. The symmetries of the latter acquire quantum features, hence describe the transformation laws between non-classical reference frames [59]. Although reference frames are abstract entities in this latter context, some possible connections between the two frameworks are starting to emerge [109, 149]. For these reasons, it seems that investigating the issues that arise in switching from classical to quantum reference frames could be paramount in shedding light on some of the logical conundrums that arise in laying down the foundations of a Quantum Gravity theory.

This chapter will give a review of the two fundamental tools that were presented in this introduction, namely quantum clocks and quantum reference frames.

2.1 Quantum clocks

The characteristic feature of the clocks that we use everyday is the correlation between their dynamical states and some physical events that allow to tell the time. For instance, we can observe the movement of the hands of a wristwatch throughout the day and associate to every position of the hands a different time. A quantum clock is nothing more than a physical system with this feature that obeys the laws of Quantum Theory, namely it is prepared in an initial state that changes upon evolution according to some Hamiltonian. The simplest example of a quantum clock is given by a two-level system, as this is not by far the most realistic model of quantum clock that can be considered but the essential behaviour can be captured already by this simple model. The Hamiltonian for such a clock reads

$$H_C = E_0 \left|0\right\rangle\!\left\langle 0\right| + E_1 \left|1\right\rangle\!\left\langle 1\right| , \tag{2.1}$$

where E_0 and E_1 are the two energy levels (with $E_1 > E_0$ and $\Delta E := E_1 - E_0$) and $|0\rangle$ and $|1\rangle$ their respective eigenvectors. Of course, since the state needs to evolve dynamically for the system to properly function as a clock, the initial state of the clock cannot be one of the two eigenvectors, since in that case the evolution would only add a global phase and we would not have a dynamical change of the state of the system. A good choice for the initial state is given by $\left|\psi_0\right\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$. Evolving this state with $\hat{H}_C$, at time t we find, up to a global phase

$$\left|\psi_t\right\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-\frac{i}{\hbar}t\Delta E}|1\rangle\right). \tag{2.2}$$

Representing the state of the clock on the Bloch sphere, its evolution identifies a circular motion on the equator so that, if we had infinite resolution of the relative phase, the position of the quantum state on the Bloch sphere could be correlated with the abstract time t appearing in the Schrödinger equation. This system therefore defines an ideal clock. The only *caveat* is that this is a periodic clock with period $T = \frac{2\pi\hbar}{\Delta E}$. To extend the functioning of the clock beyond T it could be possible to add a memory that keeps track of the "winding number" of the state or consider a quantum clock built out of a N-level system. However, in realistic scenarios, we do not have infinite resolution of the phase. In these cases, the precision of the clock can be identified with the time that it takes for the clock to make a "tick", given by the so-called *orthogonalisation time*, the time after which the state of the clock evolves to the state orthogonal to the initial state. This is given by

$$t_\perp = \frac{\pi\hbar}{\Delta E} . \tag{2.3}$$

We see that the larger the energy gap, the more precise the clock is. Ideally, if this energy gap were infinitely large, the clock would be arbitrarily precise.

2.1.1 Quantum clocks as probe of spacetime

What does this have to do with Gravity and spacetime? It is well known that gravitational fields produce time dilation effects and this effect is tested nowadays with extreme precision [49]. Therefore, it is conceivable that systems acting as quantum clocks could be used to probe the structure of spacetime, both classical and quantum. On the classical side, this could provide the first genuine general-relativistic effect observed in quantum-mechanical

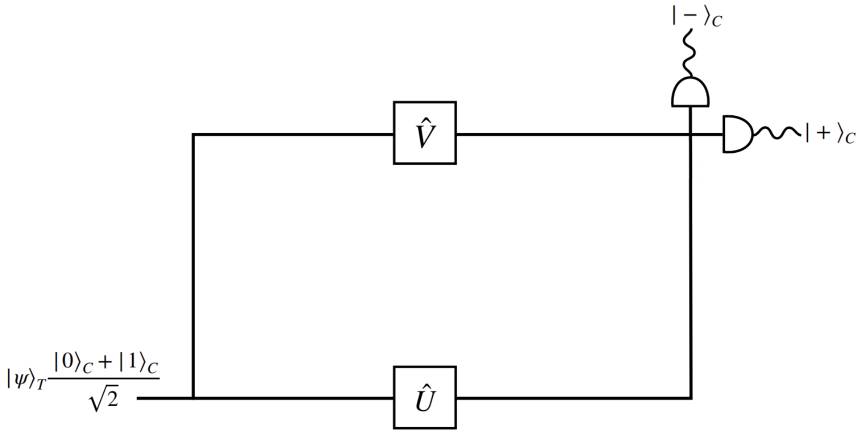

Figure 2: A Mach-Zender interferometer. The quantum state $|\psi\rangle_T$ goes through the two paths and which path information is encoded in the (control) states $|0\rangle_C$ and $|1\rangle_C$. By using a beam splitter, this information is canceled, namely a measurement in the $|\pm\rangle_C$ basis is performed.

systems. We start by considering the formalism of quantum clocks in classical spacetime, as done in [150]. Specifically, we will consider an example in which a quantum clock is prepared in a superposition of different paths into a Mach-Zender interferometer.

Mach-Zender interferometers

Before analysing the scenario pictured above, we introduce the basic notion of Mach-Zender interferometer. First, let us make a distinction between a target state and a control state. The target state is the state to which we apply the operations in the interferometer. The control state typically just describes which path the system takes in the interferometer and is not going to be affected by operations, it is just a flag that monitors where the system has been. Referring to fig. 2, if $|\psi\rangle_T$ is the initial target state and $\frac{1}{\sqrt{2}}(|0\rangle_C+|1\rangle_C)$ is the control state, $|0\rangle_C$ ($|1\rangle_C$) signaling the passage of the system in the lower (upper) branch of the interferometer, then the evolution of the state through the interferometer is

$$|\psi\rangle_T \otimes \frac{|0\rangle_C+|1\rangle_C}{\sqrt{2}} \to \frac{1}{\sqrt{2}}\left(\widehat{U}|\psi\rangle_T \otimes |0\rangle_C + \widehat{V}|\psi\rangle_T \otimes |1\rangle_C\right). \tag{2.4}$$

Therefore, if we perform a measure in the $\{|0\rangle, |1\rangle\}$ basis, we will know which path the system has gone through and we will not have a state in superposition, thus we will not observe interference. In technical terms we say that acquiring *which-path information* destroys interference. To observe interference, the reason an interferometer is built for, we have to destroy the which-path information, for instance by inserting a beam splitter at the second intersection of the two paths. Formally, this is described by performing a measure in the $\{|-\rangle_C, |+\rangle_C\}$

defined by[1]

$$|\pm\rangle_C = \frac{|0\rangle_C \pm |1\rangle_C}{\sqrt{2}} \,. \tag{2.5}$$

In this way we can observe interference, indeed rewriting the final state before the measurement in this basis we get

$$\frac{1}{2}\Big(\hat{U}|\psi\rangle_T + \hat{V}|\psi\rangle_T\Big) \otimes |+\rangle_C + \frac{1}{2}\Big(\hat{U}|\psi\rangle_T - \hat{V}|\psi\rangle_T\Big) \otimes |-\rangle_C \,, \tag{2.6}$$

hence the probabilities in the $|\pm\rangle_C$ basis are

$$P(\pm) = \frac{1}{4}\Big|\hat{U}|\psi\rangle_T \pm \hat{V}|\psi\rangle_T\Big|^2 = \frac{1}{2} \pm \frac{1}{2}\mathrm{Re}\Big\{{}_T\langle\psi|\hat{U}^\dagger\hat{V}|\psi\rangle_T\Big\} \,. \tag{2.7}$$

The form of this probability signals the appearance of an interference pattern on the target state.

Time dilation as which-path monitor

We are now ready to study an example of how quantum clocks can be used to test general-relativistic effects. The system we will consider is given by a quantum clock and a Mach-Zender interferometer placed vertically in the gravitational field of the Earth with the two horizontal paths in fig. 2 at different heights. A quantum clock can be regarded as a quantum system with some internal degrees of freedom which keep track of how time ticks. Of course, such a system will also be characterised by some external degrees of freedom describing the dynamics in a given laboratory (for our purposes, they describe the dynamics in a gravitational field). The dynamics of these external degrees of freedom is described by a Hamiltonian that can be formally derived, see [136, 150, 151]. However, a heuristic analysis is sufficient to capture the main features. A quantum particle in a gravitational potential $\phi(x)$ is described, in the laboratory frame, by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + m\phi(\hat{x}) \,. \tag{2.8}$$

It is important to understand the meaning of the mass parameter that appears in this Hamiltonian. In fact, in Special an General Relativity, the internal energy of a system contributes to its mass. To take this into account for the quantum clock we can heuristically replace m with

[1]This is equivalent to describing the action of the beam splitters as matrices. Specifically, consider the situation in which the system is inserted from the left as in fig. 2. Then we can describe the state as a vector $\psi_T = \begin{pmatrix}1\\0\end{pmatrix}$. The beam splitters basically yields a probability $\frac{1}{2}$ of finding the system in either of the two branches, and can be represented by $B = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$. We can model the phase shift acquired during the propagation in the two arms as $P = \begin{pmatrix}1 & 0\\ 0 & e^{i\Delta}\end{pmatrix}$. The final state after the second beam splitter is given by $\psi = BPB\psi_T = ie^{i\frac{\Delta}{2}}\begin{pmatrix}-\sin\frac{\Delta}{2}\\ \cos\frac{\Delta}{2}\end{pmatrix}$. This yields probabilities $P_+ = \cos^2\frac{\Delta}{2}$ and $P_- = \sin^2\frac{\Delta}{2}$ which are the same as in (2.7) with $\mathrm{Re}\Big\{{}_T\langle\psi|\hat{U}^\dagger\hat{V}|\psi\rangle_T\Big\} = \cos\Delta$.

$m + \frac{\widehat{H}_C}{c^2}$, thus making the mass an operator on the Hilbert space of the clock. If we consider slowly moving particles, we can neglect $\frac{\widehat{H}_C}{c^2}$ in the kinetic term so that the Hamiltonian reads

$$\widehat{H} = \frac{\widehat{p}^2}{2m} + \left(m + \frac{\widehat{H}_C}{c^2}\right)\phi(\widehat{x})\,. \tag{2.9}$$

This Hamiltonian, which can be derived formally by means of an action principle [136, 150, 151], shows a coupling between internal and external degrees of freedom. This tells us that the internal and external degrees of freedom will become entangled. Since the external degrees of freedom, in the Mach-Zender example we are considering, correspond to the position occupied by the system in the two arms, this means that which-path information will be encoded into the internal degrees of freedom. We will now see this in more detail, describing what the effect of such entanglement is.

In the case under study, the internal degrees of freedom of the system describe how time ticks. Therefore, the internal degrees of freedom of the clock will be related to its proper time. We call $|\tau\rangle$ the generic state of the internal degrees of freedom of the clock, whose Hilbert space is the target space, thus the one on which quantum operations will be applied. The generic state of the external degrees of freedom is denoted by $\left|\psi\right\rangle$ and will be the control state, labelling the path that the clock takes. The total state is given by $\left|\Psi_{\text{in}}\right\rangle = \left|\psi_{\text{in}}\right\rangle \otimes \left|\tau_{\text{in}}\right\rangle$ at initial time, where

$$\left|\psi_{\text{in}}\right\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,,\quad \left|\tau_{\text{in}}\right\rangle = \frac{1}{\sqrt{2}}\left(\left|E_0\right\rangle + \left|E_1\right\rangle\right). \tag{2.10}$$

The final state will in general take the form

$$\left|\Psi_f\right\rangle = \frac{1}{\sqrt{2}}\left(e^{i\varphi_0}\,|0\rangle \otimes \left|\tau_0\right\rangle + e^{i\varphi_1}\,|1\rangle \otimes \left|\tau_1\right\rangle\right), \tag{2.11}$$

where $\left|\tau_i\right\rangle$ is the evolution of $\left|\tau_{\text{in}}\right\rangle$ when the system is on path i. It refers to the time the clock is telling just before recombining the path, the two times being in general different because the clock has spent some time on different paths. By tracing out the clock's degrees of freedom, the computation of the probabilities $P_\pm$ in (2.7) yields

$$P_\pm = \frac{1}{2} \pm \frac{1}{2}\left|\left\langle\tau_1\middle|\tau_2\right\rangle\right|\cos\left(\Delta\varphi + \beta\right), \tag{2.12}$$

where $\left|\left\langle\tau_0\middle|\tau_1\right\rangle\right| = \left\langle\tau_1\middle|\tau_2\right\rangle e^{i\beta}$ and $\Delta\varphi$ is the phase shift coming from the free motion on the two different paths. This probability oscillates with an amplitude $\mathcal{V}$, called *visibility*, defined formally as

$$\mathcal{V} := \frac{\max P_\pm - \min P_\pm}{\max P_\pm + \min P_\pm}\,. \tag{2.13}$$

In our case, $\mathcal{V} = \left|\left\langle\tau_1\middle|\tau_2\right\rangle\right|$. Therefore, we see that a drop in the visibility occurs whenever $\left|\left\langle\tau_1\middle|\tau_2\right\rangle\right| \neq 1$. This is a result of complementarity: the entanglement between internal and external degrees of freedom allows to acquire which-path information when the internal degrees of freedom are physically accessible. Specifically, the probability to correctly guess which path the clock has taken, called distinguishability, is defined as $\mathscr{D}^2 = 1 - \mathscr{V}^2$. This is non-zero when $\left|\langle\tau_1|\tau_2\rangle\right| \neq 1$. Of course, if we cannot access the internal degrees of freedom, meaning that we cannot resolve the proper time difference, we have $\left|\langle\tau_1|\tau_2\rangle\right| = 1$ and no which-path information would be accessible. This is telling us that this effect is a genuine general-relativistic one. Indeed, while the additional phase shift β that occurs in (2.12) can be described simply with Newtonian Gravity, the drop in visibility described by $\left|\langle\tau_1|\tau_2\rangle\right| \neq 1$ arises because of the possibility to resolve the difference between the proper times elapsed on the two paths, a genuine general-relativistic predictions that cannot be described simply with a Newtonian picture. In particular, this can be seen deriving the expression of the phase difference in (2.12). By evolving the initial state as

$$\left|\Psi_f\right\rangle = e^{-\frac{i}{\hbar}\int \mathrm{d}t\,\hat{H}}\left|\psi_{\text{in}}\right\rangle , \tag{2.14}$$

we get

$$\mathscr{V} = \left|\langle\tau_0|\tau_1\rangle\right| = \left|\cos\left(\frac{T\Delta\phi\Delta E}{2\hbar c^2}\right)\right| = \left|\cos\left(\frac{\Delta\tau}{t_\perp}\frac{\pi}{2}\right)\right| , \; \beta = \left(mc^2 + \left\langle \widehat{H}_C\right\rangle\right)\frac{T\Delta\phi}{\hbar c^2} , \tag{2.15}$$

where $\Delta\phi$ is the gravitational (Newtonian) potential difference between the two horizontal paths in the Schwarzschild spacetime sourced by the Earth, $\Delta\tau$ is the ensuing time dilation effect, $t_\perp$ is the orthogonalisation time of the clock, and T is the time spent by the clock in the horizontal part of the interferometer.

The last expression for the visibility features only the time dilation $\Delta\tau$ and the orthogonalization time of the clock, which respectively tell us information about the paths and the internal features of the clock. Therefore, it is conceivable to apply this formula, at least as a first-order approximation, to more general cases with arbitrary clock initial states and Hamiltonian, as well as to more general gravitational fields. This also clearly shows that the effect is a genuine consequence of General Relativity and cannot be explained simply by classical Newtonian Gravity: it requires a time dilation $\Delta\tau$, which is a general-relativistic effect, to produce a visible effect. If we did not have the clock degree of freedom, the only effect would be the phase shift β, similarly to the Aharonov-Bohm effect. However, this phase shift cannot be interpreted as a genuinely gravitational effect because it would occur with any kind of potential. Conversely, the reduction in visibility occurs only with clock degrees of freedom and is an effect of the proper time difference along the two paths. We also see that, if the clock is initially in an eigenstate of the clock Hamiltonian, namely $t_\perp = \infty$, we have $\mathscr{V} = 1$. This is not at all surprising since, if the clock is initially in an eigenstate of the clock Hamiltonian, the internal degrees of freedom will not evolve, thus there would be no notion of proper time of the clock since there would be no ticking of the latter. Finally, whenever the difference between the proper times measured by the clock on the two paths is equal to the orthogonalization time $t_\perp$ of the clock, the information about the paths accessible via the time dilation will lead to a decrease in visibility due to complementarity. This occurs even after having recombined the two paths, trying to "erase" which-path information[2]: fully erasing such information is impossible since the internal degrees of freedom get entangled with the external ones, thus the clock carries which-path information in its proper time.

[2] This is done by measuring in the $|\pm\rangle$ basis.

The scenario discussed above is just an example of the compatibility of General Relativity (here represented by one of its most relevant predictions, gravitational time dilation) and Quantum Mechanics (exemplified by the complementarity principle). Even more interestingly, it shows how the principle of complementarity helps in interpreting the experiment: the clock accumulates which-path information in its proper time, and this causes a drop in visibility in the interference pattern.

2.1.2 Quantum clocks in non-classical spacetime: entanglement through Gravity

In this subsection, we will study how quantum clocks behave in non-classical spacetime and how entanglement arises in the interaction of two quantum clocks due to gravitational time dilation.

In General Relativity, an ideal clock is assigned to each worldline. Hence, in General Relativity spacetime is pictured as a latticework of ideal clocks, telling the time at which events occur. Being ideal, the gravitational effects of one clock on the flow of time of other clocks is ignored, namely there is no back-reaction of the clocks on the structure of spacetime. However, this picture is inconsistent in that it treats physical systems and clocks on different footings, while a more accurate analysis should consider clocks as physical systems that affect the spacetime structure. Of course, these effects are expected to be extremely small, but, from a fundamental perspective, this effect is there. Specifically, we might ask what happens when quantum clocks are used in this context. Being quantum clocks in superposition of energies, from the equivalence principle we get that they are in a superposition of masses, thus they give rise to a superposition of spacetimes, effectively realising a non-classical spacetime. Since the proper time measured by a clock depends on the spacetime structure it is in, we can expect that a superposition of spacetimes leads to a fuzzy time dilation and to the emergence of entanglement, through time dilation itself, between the two clocks. This means that coherence for the single clock will be lost, and the precision with which a quantum clock can measure time is thus limited. We will now review this scenario, illustrated in [152].

Consider two quantum clocks whose trajectories can be well approximated by a semi-classical static trajectory in the reference frame of a far-away observer, whose proper time is called t. In this reference frame, let r be the distance between the two clocks. Let A and B be the two clocks that can be considered identical, namely they are described by the same Hamiltonian (hence the same energy levels). For simplicity, we set $E_0 = 0$ and $E_1 = \Delta E$, so that the spacetime corresponding to the energy E_0 is flat spacetime. A and B are initially prepared in the separable state

$$\left|\psi_{\text{in}}\right\rangle = \frac{1}{2}\left(\left|E_0\right\rangle_A + \left|E_1\right\rangle_A\right) \otimes \left(\left|E_0\right\rangle_B + \left|E_1\right\rangle_B\right). \tag{2.16}$$

Given that A and B are considered approximately static, the dynamics will be defined by the internal dynamics and the gravitational interaction between them. If $\widehat{H}_A$ and $\widehat{H}_B$ are the internal Hamiltonians of A and B, from the equivalence principle, analogously to what has been done in the previous section, this interaction can be written as

$$\widehat{H}_{\text{int}} = -\frac{G\left(m_A + \frac{\widehat{H}_A}{c^2}\right)\left(m_B + \frac{\widehat{H}_B}{c^2}\right)}{r} \simeq -\frac{G}{c^4 r}\widehat{H}_A\widehat{H}_B\,, \tag{2.17}$$

where m_i are the "static masses" and the last approximation holds when these masses are negligible when compared to the "dynamical masses" given by the internal energies. Therefore, we can heuristically consider

$$\widehat{H} = \widehat{H}_A + \widehat{H}_B - \frac{G}{c^4 r}\widehat{H}_A\widehat{H}_B \tag{2.18}$$

to be the Hamiltonian describing the evolution of the joint two-clocks system in the reference frame of the far-away observer. This can be also derived more formally, as discussed in [152]. The evolution of the state of the two clocks under this Hamiltonian reads

$$\begin{aligned}\left|\psi_f\right\rangle = e^{-\frac{i}{\hbar}\widehat{H}}\left|\psi_{\text{in}}\right\rangle = \frac{1}{2}\left|E_0\right\rangle_A \otimes \left(\left|E_0\right\rangle_B + e^{-\frac{i}{\hbar}t\Delta E}\left|E_1\right\rangle_B\right) + \\ + \frac{1}{2}e^{-\frac{i}{\hbar}t\Delta E}\left|E_1\right\rangle_A \otimes \left[\left|E_0\right\rangle_B + e^{-\frac{i}{\hbar}t\Delta E\left(1-\frac{G\Delta E}{c^4 r}\right)}\left|E_1\right\rangle_B\right].\end{aligned} \tag{2.19}$$

We immediately see that the clocks get entangled via the gravitational interaction. If we trace out clock A we get the mixed state

$$\begin{aligned}\widehat{\rho}_B = \frac{1}{2}\Bigg\{\left|E_0\middle\rangle\middle\langle E_0\right| + \left|E_1\middle\rangle\middle\langle E_1\right| + \frac{1}{2}e^{-\frac{i}{\hbar}t\Delta E}\left(1 + e^{\frac{i}{\hbar}\frac{G(\Delta E)^2}{c^4 r}t}\right)\left|E_1\middle\rangle\middle\langle E_0\right| + \\ + \frac{1}{2}e^{\frac{i}{\hbar}t\Delta E}\left(1 + e^{-\frac{i}{\hbar}\frac{G(\Delta E)^2}{c^4 r}t}\right)\left|E_0\middle\rangle\middle\langle E_1\right|\Bigg\},\end{aligned} \tag{2.20}$$

whose purity is

$$\text{Tr}\left\{\widehat{\rho}_B^2\right\} = \frac{1}{2}\left[1 + \cos^2\left(\frac{G(\Delta E)^2}{\hbar c^4 r}t\right)\right]. \tag{2.21}$$

The minimal value of this purity is $\frac{1}{2}$, corresponding to a maximally mixed state for clock B, or a maximally entangled state for the joint system. Such a value is reached at time

$$t_{\text{mix}} = \frac{\pi\hbar c^4 r}{G(\Delta E)^2}. \tag{2.22}$$

Note that this value is unaffected if we consider non-negligible masses m_A and m_B in that the additional terms would act as identity operators on each subsystem, hence no additional entanglement is generated by such terms. As we approach this time, the reduced state of clock B becomes more and more mixed and it ceases to function as a proper clock, since it would start to give random answers upon being interrogated on its time. Of course, since the joint evolution is still unitary, after t_{mix} the purity of the state will increase again, as can be easily seen from (2.21). The system is also perfectly symmetrical upon exchanging A and B, hence the same arguments hold for clock A.

We now focus on the nature of the entanglement arising in (2.19). Specifically, from the phase factor in the last term of (2.19) we find that the rate at which time runs for one clock depends on the energy of the other clock: if clock A is in the state $|E_0\rangle$, then clock B runs as if it were in flat spacetime; if clock A is in the state $|E_1\rangle$, then clock B runs according to the dilated time $t\left(1-\frac{G\Delta E}{c^4 r}\right)$. The entanglement is therefore mediated by the gravitational time dilation effect, and it yields an uncertainty in the time dilation experienced by either of the clocks. This uncertainty is given by $\Delta\tau = \frac{G\Delta E}{c^4 r} t$. This can be made arbitrarily small by decreasing the value of the energy gap ΔE of clock A. However, we know that the precision of a clock is given by the orthogonalisation time (2.3), inversely proportional to the energy gap. Therefore, if we want to be precisely measurable at A, we lose precision for clock B and *vice versa*. We can express this in the form an "uncertainty principle"

$$t_\perp \Delta\tau = \frac{\pi\hbar G}{c^4 r} t\,. \tag{2.23}$$

The entanglement between two clocks therefore implies that there are fundamental limitations to the measurability of time as recorded by them: two nearby clocks cannot measure time with arbitrary precision, even in principle.

2.1.3 How do quantum clocks tell the time?

In all of the above considerations, we described everything in terms of a far-way observer picture. The state of the clocks, the distances and so on were written with respect to a far-away observer, which is not influenced by the states of the clocks and can write everything in terms of a Minkowski metric. However, this is a somewhat unsatisfactory description in that it is quite unpractical. What we would prefer to have is a quantum clock "local" picture, namely a picture in which we can describe the properties of the clock in its reference frame and in which we can "stand" on different clocks describing the quantum dynamics from their point of view. We will now try to see if there is a way to make sense of the perspective of the clocks and how they tell the time.

The Page-Wootters formalism

The most common setting is the Page-Wootters formalism [153,154]. It takes inspiration from the Wheeler-DeWitt equation, although it has nothing to do with a constrained gravitational theory. The idea is to describe quantum systems and the clocks used to tell the time in the same formalism. In the original idea there was still a difference between "clocks" and "systems", however nowadays only "systems" are considered and some of them can be considered to act as "clocks", hence describing the two types of systems on the same footing. Consider two systems A and B with Hamiltonians $\widehat{H}_A$ and $\widehat{H}_B$. We can build the operator

$$\widehat{C} = \widehat{H}_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \widehat{H}_B\,, \tag{2.24}$$

which can we naively think of as the total Hamiltonian of the system. In addition to these Hamiltonians, time operators $\widehat{T}_i$ are introduced such that[3]

$$\left[\widehat{T}_i, \widehat{H}_i\right] = i\hbar\,. \tag{2.25}$$

[3] The problem of introducing a time operator canonically conjugated to a Hamiltonian will be discussed in the following.

Then, the physical states of the joint system are those that satisfy the constraint

$$\widehat{C}\left|\Psi_{\text{phys}}\right\rangle = 0\,. \tag{2.26}$$

These are referred to as *timeless states* since, if we regard $\widehat{C}$ as the total Hamiltonian, this constraint enforces the physical states to not evolve with respect to it. Indeed, the general form of a physical state is given by

$$\left|\Psi_{\text{phys}}\right\rangle = \int \mathrm{d}\tau\, e^{\frac{i}{\hbar}\tau\widehat{C}}\left|\phi\right\rangle_{AB}\,, \tag{2.27}$$

where $\left|\phi\right\rangle_{AB}$ is a generic state of the joint system and the integration over τ yields $\delta\left(\widehat{C}\right)$, hence it automatically enforces the constraint to be solved. Then if we consider the standard dynamical evolution we have

$$e^{-\frac{i}{\hbar}\tau'\widehat{C}}\left|\Psi_{\text{phys}}\right\rangle = \int \mathrm{d}\tau\, e^{\frac{i}{\hbar}(\tau-\tau')\widehat{C}}\left|\phi\right\rangle_{AB} = \left|\Psi_{\text{phys}}\right\rangle\,, \tag{2.28}$$

hence physical states do not evolve. This is analogous to the timeless wave functionals for the Wheeler-DeWitt equation. Nonetheless, a dynamics for the single systems can still be recovered. Let $|t\rangle_i$ be the eigenstates of $\widehat{T}_i$ defined as

$$\widehat{T}_i\,|t\rangle_i = t\,|t\rangle_i\;,\quad {}_i\!\left\langle t'\middle|t\right\rangle_i = \delta\left(t-t'\right), \tag{2.29}$$

so that $\widehat{T}_i$ can be decomposed as $\widehat{T}_i = \int \mathrm{d}t\; {}_i|t\rangle\langle t|_i$. From (2.25) we have that

$$e^{-\frac{i}{\hbar}\widehat{H}_i s}\,\widehat{T}_i\, e^{\frac{i}{\hbar}\widehat{H}_i s} = \widehat{T}_i - s\mathbb{1}_i\,, \tag{2.30}$$

thus the clock states satisfy the "covariance condition" $e^{-\frac{i}{\hbar}\widehat{H}_i t'}\,|t\rangle_i = \left|t+t'\right\rangle_i$, so that every clock state state can be obtained from a "seed" state $|0\rangle_i$ by means of $|t\rangle_i = e^{-\frac{i}{\hbar}\widehat{H}_i t}\,|0\rangle_i$.
The time dependent state of one of the two systems $\left|\psi(t)\right\rangle_i$ is recovered by conditioning the physical state $|\Psi\rangle_{\text{phys}}$ of the joint system on being at time t via a projection on the time eigenvectors of the other system $|t\rangle_j$. For instance, if we want to describe the dynamics of B with respect to the time told by the clock A, the state of the system B is given by[4]

$$\left|\psi(t)\right\rangle_B = {}_A\!\left\langle t\middle|\Psi_{\text{phys}}\right\rangle\,. \tag{2.31}$$

Then, considering that $\widehat{C}\left|\Psi_{\text{phys}}\right\rangle = 0$ and that $\mathbb{1}_A\otimes\widehat{H}_B$ acts as an identity on $|t\rangle_A$ we have

$$\begin{aligned}
i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle_B &= i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,{}_A\!\left\langle t\middle|\Psi_{\text{phys}}\right\rangle = i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,{}_A\!\left\langle 0\middle|e^{\frac{i}{\hbar}t\widehat{H}_A\otimes\mathbb{1}_B}\middle|\Psi_{\text{phys}}\right\rangle = \\
&= -\,{}_A\!\left\langle 0\middle|e^{\frac{i}{\hbar}t\widehat{H}_A\otimes\mathbb{1}_B}\,\widehat{H}_A\otimes\mathbb{1}_B\middle|\Psi_{\text{phys}}\right\rangle = -\,{}_A\!\left\langle t\middle|\widehat{H}_A\otimes\mathbb{1}_B\middle|\Psi_{\text{phys}}\right\rangle = \\
&= -\,{}_A\!\left\langle t\middle|\widehat{C}-\mathbb{1}_A\otimes\widehat{H}_B\middle|\Psi_{\text{phys}}\right\rangle = \widehat{H}_B\left|\psi(t)\right\rangle_B\,,
\end{aligned} \tag{2.32}$$

[4]In the formalism of the density operator, the same equation reads $\widehat{\rho}_B(t) = \mathrm{Tr}_A\left\{\widehat{\rho}_{AB}\,|t\rangle\langle t|_A\right\}$.

where we used the definition of $\left|\psi(t)\right\rangle_B$, the constraint condition (2.26) and $|t\rangle_A = e^{\frac{i}{\hbar}t\widehat{H}_A\otimes\mathbb{1}_B}|0\rangle_A$. The first and the last term give the Schrödinger equation for system B, therefore we have

$$ {}_A\left\langle t\middle|\widehat{H}_A+\widehat{H}_B\middle|\Psi_{\rm phys}\right\rangle = 0 \Longrightarrow i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi(t)\right\rangle_B = \widehat{H}_B\left|\psi(t)\right\rangle_B\,, \tag{2.33}$$

This is the dynamics of system B from the perspective of system A (which is acting as a clock). In this sense, the dynamics emerges from the correlations between the two clocks: there is no global dynamics for the joint system but there exists a dynamical evolution when one of the two systems is looked from the perspective of the other. This emergence of dynamics from the correlations can be made explicit by noting that the states $|t\rangle_A$ provide a resolution of the identity for system A, hence we can write the physical states as

$$ |\Psi\rangle_{\rm phys} = \int \mathrm{d}t\,|t\rangle_A\otimes\left|\psi(t)\right\rangle_B\,. \tag{2.34}$$

This decomposition is also referred to as *history state*, since it is written in terms of the whole history of clock A (the integration over all times). To each time of clock A a state $\left|\psi(t)\right\rangle_B$ of system B is associated. We therefore see that the physical states describe completely the temporal evolution of the system B by means of correlations between the latter and the degrees of freedom of the clock system. Specifically, if $\widehat{U}_B(t) = e^{-\frac{i}{\hbar}t\widehat{H}_B}$ is the propagator obtained from the Schrödinger equation for system B, then we have

$$ \left|\Psi_{\rm phys}\right\rangle = \int \mathrm{d}t\,|t\rangle_A\otimes\widehat{U}_B(t)\left|\psi(0)\right\rangle = \widehat{\mathbb{U}}\left|TimeLine\right\rangle_A\otimes\left|\psi(0)\right\rangle_B\,, \tag{2.35}$$

where $\left|\psi(0)\right\rangle_B$ is the initial state of B at time $t=0$, $\left|TimeLine\right\rangle_A$ is the improper state representing the entire history of clock A, given by

$$ \left|TimeLine\right\rangle_A = \int \mathrm{d}t\,|t\rangle_A\,, \tag{2.36}$$

and $\widehat{\mathbb{U}}$ is the unitary operator

$$ \widehat{\mathbb{U}} = \int \mathrm{d}t\,|t\rangle\langle t|_A\otimes\widehat{U}_B(t) = \widehat{U}_B\left(\widehat{T}\right) = e^{-\frac{i}{\hbar}\widehat{T}\otimes\widehat{H}_B}\,. \tag{2.37}$$

Problems with a time operator

In reviewing the Page-Wooters formalism, we introduced a time operator that closes a canonical commutation relation with the Hamiltonian of the quantum clock. Research on finding a time operator has been going on since the dawn of Quantum Mechanics, but already in 1933 Pauli raised some criticisms on the possibility of finding such an operator. In fact, if an operator $\widehat{T}$ that closes a canonical commutation relation with a Hamiltonian can be found, the spectrum of the latter should necessarily be unbounded from below. This argument can be recast in the form of a theorem as done in [155]. Specifically:

If a Hamiltonian $\widehat{H}$ is bounded from below, then there is no time operator $\widehat{T}$ such that $\left[\widehat{T},\widehat{H}\right] = i\hbar$ and the following three conditions all hold

1. $|t_n\rangle$ is an eigenstate of $\hat{T}$ such that $t_0 < t_1 < \cdots < t_n < \cdots$
2. For all n, there exists $m > n$ such that the amplitude of going from t_n to t_m in time t, *i.e.* $\langle t_m| e^{-\frac{i}{\hbar}\hat{H}t} |t_n\rangle$, is non-vanishing.
3. For all $m > n$ and for all $t > 0$, the amplitude of going from t_m to t_n vanishes.

This proves the impossibility of having a "Heraclitian" time (*i.e.* a *monotonically perfect clock* always evolving forward) in Quantum Mechanics.

The solution to the conundrum raised by Pauli and the theorem above can be found between the lines of the theorem itself. In fact, the clock (*i.e.* the time operator) that is impossible to build is an ideal one. However, ideal clocks are not physical systems, as only finite-size clocks with some intrinsic uncertainty exist in Nature. For this reason, the canonically conjugated Hamiltonian and time operators are idealisations that simplify technical computations, and they should not be regarded as something that is fundamentally realisable. Real quantum clocks are, in general, N-level systems that oscillate between their energy levels. Only in the (non-physical) $N \to \infty$ limit, with the spacing between the energy levels vanishing, can a Hamiltonian with a continuous spectrum be approximately recovered. In realistic scenarios, there will always be finite-size effects, which depend on the particular model of the clock that we consider. We will not delve into the details of any specific model, for which we refer to [156, 157].

Real clocks as N-level systems

In general, a good clock can be defined as a quantum system with a lower-bounded Hamiltonian with a discrete spectrum [158]. In the following, we assume an equally-spaced energy spectrum, but the generalisation to non-equally spaced energies is straightforward. The Hamiltonian of the clock in this scenario reads

$$H = \sum_{n=0}^{s} E_n |E_n\rangle\langle E_n| \;,\; \langle E_n|E_m\rangle = \delta_{n,m} \,, \tag{2.38}$$

where $s+1$ is the dimension of the clock space and E_n are the clock energy levels that, without loss of generality, can be written as $E_n = n\Delta$, Δ being the energy gap between two successive energy levels. In the clock space, the following time states can be defined

$$|\tau_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{-\frac{i}{\hbar}E_n\tau_m} |E_n\rangle \;,\; \langle \tau_n|\tau_m\rangle = \delta_{n,m} \,, \tag{2.39}$$

which provide a resolution of the identity $\mathbb{1} = \sum_{n=0}^{s} |\tau_n\rangle\langle\tau_n|$. This time states allow to define a Hermitian time operator $\widehat{T}$ as [158, 159]

$$\widehat{T} = \sum_{n=0}^{s} \tau_n |\tau_n\rangle\langle\tau_n| \,, \tag{2.40}$$

so that the time states are eigenstates of this operator and the Hamiltonian is the generator of shifts in time while this operator is the generator of shifts in energy

$$\begin{aligned} |\tau_n\rangle &= e^{-\frac{i}{\hbar}\widehat{H}\tau_n} |\tau_0 = 0\rangle \,, \\ |E_m\rangle &= e^{-\frac{i}{\hbar}\widehat{T}E_m} |E_0 = 0\rangle \,. \end{aligned} \tag{2.41}$$

Being the spectrum finite, the set of times is finite too. Without loss of generality, this set can be taken to range from $\tau_0 = 0$ to $\tau_{s+1} := \mathfrak{T}$ with times defined as $\tau_n = \frac{n}{s+1}\mathfrak{T}$. In particular, a cyclic condition has to be enforced on the time states, such that $\left|\tau_{s+1}\right\rangle \equiv \left|\tau_0\right\rangle$. This condition relates the period of the clock $\mathfrak{T}$, which is the maximum value of time that the clock can read and the time it takes to "reset" and come back to the initial state, with the energy gap Δ as

$$\mathfrak{T} = \frac{2\pi}{\Delta} . \tag{2.42}$$

Therefore, the spacing between two successive time values reads

$$\delta\tau = \tau_{m+1} - \tau_m = \frac{\mathfrak{T}}{s+1} = \frac{2\pi}{(s+1)\Delta} . \tag{2.43}$$

From this we see that the range $\mathfrak{T}$ of clock depends on the energy difference between the energy levels: the smaller such spacing is the greatest the range and the difference between two successive times are. The latter also depends on the dimension of the clock space $s+1$. Specifically, in the limit $s \to \infty$ we obtain a continuous clock, for which $\delta\tau = 0$. In this case, the clock values will continuously range in the interval $\left[0, \mathfrak{T}\right]$ and the range can be made infinite by considering the limit of vanishing spacing between energy levels $\Delta \to 0$, in which case the spectrum of the Hamiltonian becomes continuous too. In the case of a continuous flow of time, the Schrödinger equation for the system BS, with respect to time τ, can also be recovered [158].

With the construction outlined above, a physically meaningful time operator can be introduced in Quantum Mechanics in the Page-Wooters formalism. The problem of having an unbounded from below Hamiltonian is resolved by replacing ideal clocks with physical ones, characterised by discrete energy levels and time readings. The approximation of ideal clocks is then justified since the continuous limit can be easily recovered [158].

2.2 Quantum reference frames

In this section, we review the basic notions of the Quantum Reference Frames (QRFs) formalism. Although there are different approaches to the problem, we consider the so-called "foundational" approach introduced in [102], whose core idea is to abandon the notion of abstract reference frames in favour of a more operational definition of reference frames that become associated with physical systems that are, in general, quantum in nature.

From kindergarten physics, we know that a reference frame is a set of axes and a state of motion that we use to represent physics, *e.g.* to fix the point of view from which a measurement is carried out. In standard Quantum Mechanics, the change of reference frame is given by the action of a unitary operator on the state of physical systems, $\left|\psi'\right\rangle = \widehat{U}\left|\psi\right\rangle$. Then the Schrödinger equation reads

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi'\right\rangle = i\hbar\widehat{U}\frac{\mathrm{d}}{\mathrm{d}t}\left|\psi\right\rangle + i\hbar\frac{\mathrm{d}\widehat{U}}{\mathrm{d}t}\left|\psi\right\rangle = \left(\widehat{U}\widehat{H} + i\hbar\frac{\mathrm{d}\widehat{U}}{\mathrm{d}t}\right)\widehat{U}^\dagger\left|\psi'\right\rangle = \widehat{H}'\left|\psi'\right\rangle , \tag{2.44}$$

therefore the covariance of the Schrödinger equation is expressed by

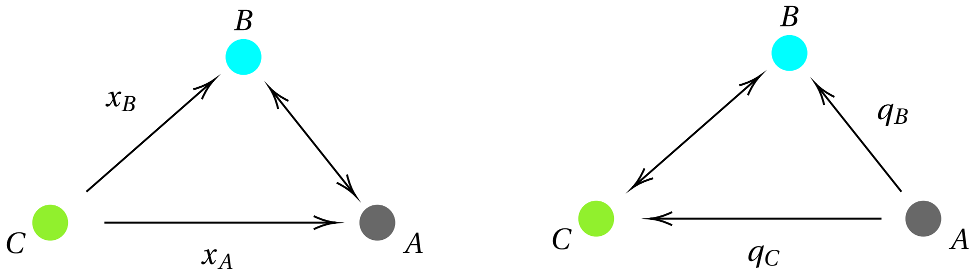

Figure 3: Canonical transformation between relative coordinates. Image adapted from [102].

$$\widehat{H}' = \widehat{U}\widehat{H}\widehat{U}^{\dagger} + i\hbar\frac{\mathrm{d}\widehat{U}}{\mathrm{d}t}\widehat{U}^{\dagger} \,. \tag{2.45}$$

The transformation is a symmetry if $\widehat{H}' = \widehat{H}$, namely the Hamiltonian is invariant in form. The defining characteristic of these transformations in standard Quantum Mechanics is that the reference frame transformations only depend on fixed parameters. For instance, spatial translations and boosts are given by

$$\widehat{U}_T = e^{\frac{i}{\hbar}x_0\widehat{p}} \,, \ \widehat{U}_B = e^{\frac{i}{\hbar}v\widehat{G}} \,, \ \widehat{G} = \widehat{p}t - m\widehat{x} \,, \tag{2.46}$$

where x_0 and v are the relative position and relative velocity of the initial and final reference frame. These are not dynamical variables, rather they are just parameters, meaning that such a transformation corresponds to considering reference frames as abstract entities, without any dynamical law or physical content.

Relationalism in QRFs

In the framework introduced in [102], a fully relational construction is developed. Indeed, to abandon the notion of abstract reference frames, all the relevant physical quantities have to be relative to some physical system acting as a reference. This means that only the relational degrees of freedom, *e.g.* relative positions and velocities, are physically meaningful in such a framework. The typical setup consists of three systems, A, B, and C, of which two act as "reference systems" and one act as "observed system". To fix the ideas, we consider A and C to define the QRFs and B to be the observed system. Of course, everything is symmetric under the exchange of any pair of systems. This scenario is depicted in fig. 3. We will consider the problem of defining QRFs for position and momentum in the following, therefore the physical observables we will be interested in are the positions and the momenta of physical systems. For example, from the perspective of C, there are two quantities that are important, the relative distance between A and C and the relative distance between B and C. In this perspective, the states that characterise two of the three physical systems with respect to the third contain this relational information. In the reference frame of A, the relevant quantities are instead the relative positions of B and C with respect to A, as depicted in fig. 3, and the transformation that links the two sets of relational variables is given by

$$x_A \mapsto -q_C \,, \ x_B \mapsto q_B - q_C \,. \tag{2.47}$$

Then a question arises: is it possible to "attach" a reference frame to a physical system which is in superposition of some classical states? The answer is going to be positive [102].

2.2.1 Quantum reference frame transformations

In standard Quantum Mechanics, if we want to change from a reference frame with origin in C to a reference frame with origin in a different position translated by a quantity α, then we have to apply the translation

$$\widehat{U}_T(\alpha) = e^{\frac{i}{\hbar}\alpha\widehat{p}_B} \tag{2.48}$$

to the state of system B. This acts on position eigenstates as $e^{\frac{i}{\hbar}\alpha\widehat{p}_B}|x\rangle_B = |x-\alpha\rangle_B$ and on general states by linearity

$$e^{\frac{i}{\hbar}\alpha\widehat{p}_B}\left|\psi\right\rangle_B = \int \mathrm{d}x_B\, \psi(x_B)\left|x_B-\alpha\right\rangle_B\,, \tag{2.49}$$

If we want now to consider the reference frame with origin in A and the latter is described, with respect to C, by a generic quantum state, it is in general not possible to translate the reference frame by a fixed amount, since system A can in general be in superposition of different positions. To overcome this difficulty, we have to take seriously the linearity of Quantum Theory and consider a generalisation of the above translation transformation. Specifically, we enlarge the Hilbert space by adding to the Hilbert space of system B the Hilbert space of system A, and generalise the translation transformation to

$$\widehat{U}_T\left(\widehat{x}_A\right) = e^{\frac{i}{\hbar}\widehat{x}_A\otimes\widehat{p}_B}\,, \tag{2.50}$$

which can be regarded as a superposition of translations, "quantum-controlled" by the state of system A. In fact, the transformed state in the new reference frame will be given by[5]

$$e^{\frac{i}{\hbar}\widehat{x}_A\otimes\widehat{p}_B}\left|\phi\right\rangle_A\left|\psi\right\rangle_B\,. \tag{2.51}$$

There is one last thing we have to implement. In the reference frame of A we describe the systems B and C, hence we need a "swap" operator which maps A's Hilbert space and observables into C ones. For translations, the latter is given by

$$\widehat{P}_{AC}\widehat{x}_A\widehat{P}^\dagger_{AC} = -\widehat{q}_C\,,\ \widehat{P}_{AC}\widehat{x}_B\widehat{P}^\dagger_{AC} = \widehat{q}_B\,,\ \widehat{P}_{AC}\widehat{p}_A\widehat{P}^\dagger_{AC} = -\widehat{\pi}_C\,,\ \widehat{P}_{AC}\widehat{p}_B\widehat{P}^\dagger_{AC} = \widehat{\pi}_B\,, \tag{2.52}$$

which acts as in (2.47) on the position eigenstates

$$\widehat{P}_{AC}|x\rangle_A = |-x\rangle_C\,,\ \widehat{P}_{AC}\left|p\right\rangle_A = \left|-p\right\rangle_C\,. \tag{2.53}$$

The overall transformation is therefore given by

$$\widehat{S}_x = \widehat{P}_{AC}e^{\frac{i}{\hbar}\widehat{x}_A\otimes\widehat{p}_B}\,,\ \widehat{S}_x : \mathcal{H}^{(C)}_A\otimes\mathcal{H}^{(C)}_B \to \mathcal{H}^{(A)}_B\otimes\mathcal{H}^{(A)}_C\,, \tag{2.54}$$

where $\mathcal{H}^{(J)}_I$ is the Hilbert space of system I as described with respect to system J. This unitary operator is such that the transformation is canonical, namely

$$\widehat{S}_x\widehat{x}_A\widehat{S}^\dagger_x = -\widehat{q}_C\,,\ \widehat{S}_x\widehat{p}_A\widehat{S}^\dagger_x = -\left(\widehat{\pi}_C+\widehat{\pi}_B\right),\ \widehat{S}_x\widehat{x}_B\widehat{S}^\dagger_x = \widehat{q}_B\,,\ \widehat{S}_x\widehat{p}_B\widehat{S}^\dagger_x = \widehat{\pi}_B\,, \tag{2.55}$$

[5] Note that no ordering issue arises since the two operators $\widehat{x}_A$ and $\widehat{p}_B$ act on different Hilbert spaces.

$\widehat{q}_I$ and $\widehat{\pi}_I$ being canonically conjugate variables. Moreover, the action of $\widehat{S}_x$ on generic states is

$$\begin{aligned} |\psi\rangle_{AB}^{(C)} &= \int \mathrm{d}x_A \,\mathrm{d}x_B \,\psi_{AB}(x_A, x_B) \,|x_A\rangle_A |x_B\rangle_B \;, \\ |\psi\rangle_{BC}^{(A)} &= \widehat{S}_x |\psi\rangle_{AB}^{(C)} = \int \mathrm{d}q_B \,\mathrm{d}q_C \,\psi_{BC}(-q_C, q_B - q_C) \,|q_B\rangle_B |q_C\rangle_C \;. \end{aligned} \tag{2.56}$$

The above equation resembles the form of a quantum-controlled transformation depending on the state of system A. The implications of this are quite striking: in general, superposition and entanglement become (quantum) reference frame dependent notions [102, 103, 115]. To see this, we can consider a couple of simple examples. Consider first the case in which A is in a superposition of two positions while B is localised, namely $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\Big(|x_1\rangle_A + |x_2\rangle_A\Big) \otimes |x_0\rangle_B$, which is a product state corresponding to

$$\psi_{AB}(x_A, x_B) = \frac{1}{\sqrt{2}}\Big[\delta(x_A - x_1) + \delta(x_A - x_2)\Big]\delta(x_B - x_0) \;. \tag{2.57}$$

Therefore, from (2.56) we have

$$|\psi\rangle_{AB}^{(C)} = \frac{1}{\sqrt{2}}\Big(|x_1\rangle_A + |x_2\rangle_A\Big) \otimes |x_0\rangle_B \mapsto |\psi\rangle_{BC}^{(A)} = \frac{1}{\sqrt{2}}\Big(|x_0 - x_1\rangle_B |-x_1\rangle_C + |x_0 - x_2\rangle_B |-x_2\rangle_C\Big) \;. \tag{2.58}$$

The transformed state above is clearly an entangled state, hence entanglement is a QRF dependent property. Consider now the case in which $|\psi\rangle_{AB} = \int \mathrm{d}x \,|x\rangle_A |x + L\rangle$, which corresponds to $\psi_{AB}(x, x_B) = \delta(x_B - x - L)$. From (2.56) we get

$$|\psi\rangle_{AB}^{(C)} = \int \mathrm{d}x \,|x\rangle_A |x + L\rangle \mapsto |\psi\rangle_{BC}^{(A)} = \left(\int \mathrm{d}q \,|q\rangle_C\right) |L\rangle_B \;, \tag{2.59}$$

hence while systems A and B are both in a superposition and entangled with respect to C, when changing to the reference frame of A we have a product state in which B is localised. This clearly shows that superposition is a QRF dependent notion. Of course, when changing reference frame from C to A, it is always possible to go back to the description in terms of C by means of the inverse transformations, the two descriptions being therefore completely equivalent under QRF transformations. Finally, although superposition and entanglement are QRF dependent, it is possible to construct a quantity, which is the sum of the entanglement entropy and the coherence entropy, which is invariant under QRF transformations [160].

The examples above are, of course, simple cases in which we focused only on translations. In general, there can be arbitrarily many types of QRFs transformations that can be generically described as superpositions of transformations of the form

$$\widehat{S} = \sum_i \widehat{U}_{\alpha_i} \otimes |\alpha_i\rangle\langle\alpha_i| \;, \tag{2.60}$$

where $\widehat{U}_{\alpha_i}$ is a standard reference frame transformation [6] which depends on the state of a quantum system $|\alpha_i\rangle$ which encodes the relevant information for the transformation[7]. Notice that, in any case, this transformation must be unitary. A unitary transformation is, by

[6] It can be a spatial or time translation, a Galilean or Lorentz boost, a transformation to an accelerated reference frame, a rotation, or any other physically meaningful reference frame transformation.

[7] This can be the position, velocity, acceleration and so on, depending on the transformation.

definition, a canonical transformation, because it preserves the symplectic structure of the phase space. This ensures that a Hamiltonian system is mapped to a Hamiltonian system. However, as we see from (2.55), the price to pay to keep the transformation canonical is that not every physical quantity can be mapped to its relative counterpart. For example, a transformation that maps positions and momenta into relative positions and relative momenta is not canonical, because it changes the structure of the Poisson brackets. This means that we have to choose some variables that are physcally meaningful for some reasons (*e.g.*, because there exists an operational notion of measurement that allows us to identify them). These will be the quantities that will be mapped to relative variables, and this basically determines the form of the swap operator, then the transformation will be completed canonically.

Dynamics in QRFs

Until now, we did not talk about dynamics. The time evolution of systems A and B in the reference frame C is governed by the Liouville-von Neumann equation

$$i\hbar\frac{\mathrm{d}\hat{\rho}^{(C)}_{AB}}{\mathrm{d}t} = \left[\hat{H}^{(C)}_{AB}, \hat{\rho}^{(C)}_{AB}\right] . \tag{2.61}$$

When changing to the reference frame of A with a generic transformation S we have

$$i\hbar\frac{\mathrm{d}\hat{\rho}^{(A)}_{BC}}{\mathrm{d}t} = \left[\hat{H}^{(A)}_{BC}, \hat{\rho}^{(A)}_{BC}\right] , \tag{2.62}$$

with

$$\begin{aligned}\hat{\rho}^{(A)}_{BC} &= \hat{S}\hat{\rho}^{(C)}_{AB}\hat{S}^\dagger ,\\ \hat{H}^{(A)}_{BC} &= \hat{S}\hat{H}^{(C)}_{AB}\hat{S}^\dagger + i\hbar\frac{\mathrm{d}\hat{S}}{\mathrm{d}t}\hat{S}^\dagger .\end{aligned} \tag{2.63}$$

This transformation is called an *extended symmetry* if the Hamiltonian is invariant in form

$$\hat{S}\hat{H}^{(C)}_{AB}\big(\{m_i, \hat{x}_i, \hat{p}_i\}_{i=A,B}\big)\hat{S}^\dagger + i\hbar\frac{\mathrm{d}\hat{S}}{\mathrm{d}t}\hat{S}^\dagger = \hat{H}^{(A)}_{BC}\big(\{m_i, \hat{q}_i, \hat{\pi}_i\}_{i=B,C}\big) . \tag{2.64}$$

In general, to satisfy this requirement, some additional structures have to be added to the transformations $\hat{S}$. Indeed, imagine that we want to change reference frame from C to A at time t. To do so, we have to take the new reference frame in the position of A at time $t = 0$, therefore we have to first evolve back in time system A in reference frame C, then perform the change of reference through a QRF transformation, and finally evolve forward in time system C in the reference frame A. A general QRF transformation at time t is therefore given by [102]

$$\hat{S} = e^{-\frac{i}{\hbar}\hat{H}^{(A)}_C t}\hat{P}^{(S)}_{AC}\prod_k e^{\frac{i}{\hbar}\hat{f}^k_A(t)\otimes\hat{O}^k_B}e^{\frac{i}{\hbar}\hat{H}^{(C)}_A t} , \tag{2.65}$$

where $\hat{f}^k_A(t)$ are parametric operator functions of time which are determined by the transformation and depend on the position of A in the reference frame C and its derivatives. $\hat{O}^k_B$ are the operators that define the transformation, and $\hat{P}^{(S)}_{AC}$ is also transformation-dependent.

As an example, we can take translations. For these transformations, we have $\hat{O}_B = \hat{p}_B$, $\hat{f}_A(t) = \hat{x}_A$ and $\hat{P}^{(S)}_{AC} = \hat{P}^{(x)}_{AC} = \hat{P}_{AC}$ defined in (2.52), hence

$$\hat{S}^{(x)} = e^{-\frac{i}{\hbar}\hat{H}^{(A)}_C t}\hat{P}_{AC}e^{\frac{i}{\hbar}\hat{x}_A\otimes\hat{p}_B}e^{\frac{i}{\hbar}\hat{H}^{(C)}_A t} . \tag{2.66}$$

From this we see

$$\hat{S}\hat{H}_{AB}^{(C)}\hat{S}^\dagger + i\hbar\frac{\mathrm{d}\hat{S}}{\mathrm{d}t}\hat{S}^\dagger = \hat{S}\hat{H}_{B}^{(C)}\hat{S}^\dagger + \hat{S}\hat{H}_{A}^{(C)}\hat{S}^\dagger + i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-\frac{i}{\hbar}\hat{H}_C^{(A)}t}\hat{P}_{AC}e^{\frac{i}{\hbar}\hat{x}_A\otimes\hat{p}_B}e^{\frac{i}{\hbar}\hat{H}_A^{(C)}t}\right]\hat{S}^\dagger =$$
$$= \hat{H}_B^{(A)} + \hat{S}\hat{H}_A^{(C)}\hat{S}^\dagger + i\hbar\frac{i}{\hbar}\left(-\hat{H}_C^{(A)} + \hat{S}\hat{H}_A^{(C)}\hat{S}^\dagger\right) = \hat{H}_B^{(A)} + \hat{H}_C^{(A)} = \hat{H}_{BC}^{(A)}\,. \quad (2.67)$$

For Galilean boosts we have $\hat{f}_A = \frac{\hat{p}_A}{m_A}$, $\hat{O}_B = \hat{G}_B = \hat{p}_B t - m_B\hat{x}_B$ and, with computations analogous to the ones performed above, it is shown that the parity swap operator is

$$\hat{P}_{AC}^{(v)} = \hat{P}_{AC}\exp\left\{\frac{i}{\hbar}\log\left(\sqrt{\frac{m_C}{m_A}}\right)(\hat{x}_A\hat{p}_A + \hat{p}_A\hat{x}_A)\right\}. \quad (2.68)$$

Therefore, Galilean boosts are given by

$$\hat{S}^{(v)} = e^{-\frac{i}{\hbar}\hat{H}_C^{(A)}t}\hat{P}_{AC}e^{\frac{i}{\hbar}\log\left(\sqrt{\frac{m_C}{m_A}}\right)(\hat{x}_A\hat{p}_A+\hat{p}_A\hat{x}_A)}e^{\frac{i}{\hbar}\frac{\hat{p}_A}{m_A}\otimes\hat{G}_B}e^{\frac{i}{\hbar}\hat{H}_A^{(C)}t} \quad (2.69)$$

in order to be symmetries of the free Hamiltonian. This strange form of the parity swap operator is such that the velocity of A in the reference frame of C is opposite to the velocity of C in the reference frame of A. Indeed, the operator $\hat{P}_{AC}^{(v)}$ act as a rescaling of the phase space coordinates. To see this, notice that

$$\begin{aligned}\hat{P}_{AC}^{(v)} &= \hat{P}_{AC}e^{\frac{i}{\hbar}\log\left(\frac{m_C}{m_A}\right)\hat{x}_A\hat{p}_A}e^{\frac{1}{2}\log\left(\frac{m_C}{m_A}\right)}\,,\\ \hat{P}_{AC}^{(v)\dagger} &= e^{-\frac{i}{\hbar}\log\left(\frac{m_C}{m_A}\right)\hat{x}_A\hat{p}_A}e^{-\frac{1}{2}\log\left(\frac{m_C}{m_A}\right)}\hat{P}_{AC}^\dagger\,,\end{aligned} \quad (2.70)$$

therefore

$$\hat{P}_{AC}^{(v)}\hat{p}_A\hat{P}_{AC}^{(v)\dagger} = \hat{P}_{AC}e^{\frac{i}{\hbar}\log\left(\frac{m_C}{m_A}\right)\hat{x}_A\hat{p}_A}\hat{p}_A e^{-\frac{i}{\hbar}\log\left(\frac{m_C}{m_A}\right)\hat{x}_A\hat{p}_A}\hat{P}_{AC}^\dagger\,. \quad (2.71)$$

By expanding the exponential, calling $\alpha = \log\left(\frac{m_C}{m_A}\right)$ and $\hat{\beta} = \frac{i}{\hbar}\alpha\hat{x}_A\hat{p}_A$, we have[8]

$$\begin{aligned}e^{\hat{\beta}}\hat{p}_A &= \sum_{n=0}^{\infty}\frac{1}{n!}\hat{\beta}^n\hat{p}_A = \hat{p}_A\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{1}{n!}(-\alpha)^m\hat{\beta}^{n-m} = \hat{p}_A\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{m!k!}(-\alpha)^m\hat{\beta}^k =\\ &= \hat{p}_A e^{-\alpha}e^{\hat{\beta}}\,.\end{aligned} \quad (2.72)$$

With this we have

$$\hat{P}_{AC}^{(v)}\hat{p}_A\hat{P}_{AC}^{(v)\dagger} = \hat{P}_{AC}\hat{p}_A e^{-\alpha}e^{-\alpha}e^{\hat{\beta}}e^{-\hat{\beta}}\hat{P}_{AC}^\dagger = -e^{-\alpha}\hat{\pi}_C = -\frac{m_A}{m_C}\hat{\pi}_C\,, \quad (2.73)$$

which means that the velocity of A in the reference frame of C is opposite to the velocity of C in the reference frame of A. With analogous calculations it is possible to show that

$$e^{\hat{\beta}}\hat{x}_A = \hat{x}_A e^{\alpha}e^{\hat{\beta}} \Rightarrow \hat{P}_{AC}^{(v)}\hat{x}_A\hat{P}_{AC}^{(v)\dagger} = -\frac{m_C}{m_A}\hat{q}_C\,. \quad (2.74)$$

[8] Note that $\hat{\beta}\hat{p}_A = \hat{p}_A\left(\hat{A}-\alpha\right)$, $\hat{\beta}^2\hat{p}_A = \hat{p}_A\left(\hat{\beta}^2 - 2\alpha\hat{\beta} + \alpha^2\right)$, $\hat{\beta}^3\hat{p}_A = \hat{p}_A\left(\hat{\beta}^3 - 3\alpha\hat{\beta}^2 + 3\alpha^2\hat{\beta} - \alpha^3\right)$, and so on, therefore $\hat{\beta}^n\hat{p}_A = \hat{p}_A\sum_{m=0}^{n}\binom{n}{m}(-\alpha)^m\hat{\beta}^{n-m}$.

2.2.2 The algebraic structure of QRF transformations

In standard Quantum Mechanics, the generators of transformations between different reference frames form a Lie algebra of relativistic symmetries. This is given by the (centrally extended) Galilei algebra and by the Poincaré algebra in non-relativistic and relativistic Quantum Mechanics, respectively. Of course, we cannot expect that the QRFs transformations introduced above close the corresponding classical algebra. Indeed, for instance, it is clear that (2.66) and (2.69) describe in general superpositions of translations and Galilean boosts. Can the QRFs transformations be interpreted in terms of generators closing a symmetry algebra? The answer is positive, at least for the non-relativistic case [109]. Consider, as in the previous examples, two quantum systems A and B described in the reference frame of a third system C and restrict to one spatial dimension. We want now to describe the algebraic structure of the transformations that describe the change of reference frame from C to A. For the two systems A and B, consider two copies of the Heisenberg algebra

$$\left[\widehat{x}_A, \widehat{p}_A\right] = i\kappa\,,\ \left[\widehat{x}_B, \widehat{p}_B\right] = i\hbar\,, \tag{2.75}$$

with different constants κ and $\hbar$. Of course, to describe symmetries and preserve the relational feature of the framework, we need to have $\kappa = \hbar$. We keep both constants, however, in order to keep track of the quantum features of the reference frame, and to better understand how a "classical reference frame limit" can be achieved.

Relational Lie algebra

We start by considering a simple example, specifically we set $t = 0$ and consider a time-independent scenario. In this case, the operator $\widehat{G}_B$ simply reads $\widehat{G}_B = -m_B\widehat{x}_B$. Since QRFs transformations act on the tensor product space, the generators of translations and boosts will be given by

$$\widehat{P}_{AB} = \widehat{x}_A \otimes \widehat{p}_B \tag{2.76}$$

and

$$\widehat{K}_{AB} = \frac{\widehat{p}_A}{m_A} \otimes \widehat{G}_B \tag{2.77}$$

respectively. However, if we naively generalised the standard translation and boost transformations as

$$\widehat{U}_T = e^{\frac{i}{\hbar}\widehat{x}_A \otimes \widehat{p}_B}\,,\ \widehat{U}_G = e^{\frac{i}{\hbar}\frac{\widehat{p}_A}{m_A}\otimes\widehat{G}_B}\,, \tag{2.78}$$

we would get

$$\widehat{U}_G\widehat{p}_A\widehat{U}_G^{\dagger} = \widehat{p}_A\,,\ \widehat{U}_G\widehat{p}_B\widehat{U}_G^{\dagger} = \widehat{p}_B + \frac{m_A}{m_C}\widehat{p}_A\,. \tag{2.79}$$

This, of course, would spoil the relational meaning of the phase space coordinates. In addition, note that if we compute the commutator between the two generators $\widehat{P}_{AB}$ and $\widehat{K}_{AB}$ we get

$$\left[\widehat{K}_A B, \widehat{P}_{AB}\right] = i\kappa\frac{m_B}{m_A}\widehat{D}_B - i\hbar\frac{m_B}{m_A}\widehat{D}_A\,, \tag{2.80}$$

where

$$\widehat{D}_A := \frac{1}{2}\left(\widehat{x}_A\widehat{p}_A + \widehat{p}_A\widehat{x}_A\right)\otimes \mathbb{1}_B\,,\ \widehat{D}_B := \mathbb{1}_A \otimes \frac{1}{2}\left(\widehat{x}_B\widehat{p}_B + \widehat{p}_B\widehat{x}_B\right)\,. \tag{2.81}$$

Observe that the operator $\widehat{D}_A$ is exactly what is needed to restore the relational character of the transformations. Indeed, as already observed previously, it holds that

$$e^{\frac{i}{\kappa}\alpha\widehat{D}_A}\widehat{p}_A e^{-\frac{i}{\kappa}\alpha\widehat{D}_A} = e^{-\alpha}\widehat{p}_A\,, \tag{2.82}$$

therefore, by setting $\alpha = \log\left(\frac{m_C}{m_A}\right)$ and by applying the parity swap (2.52) we have

$$\widehat{p}_A \mapsto -\frac{m_A}{m_C}\widehat{\pi}_C\,, \tag{2.83}$$

hence restoring the relational meaning of the phase space coordinates. The operators $\widehat{D}_A$, $\widehat{D}_B$, $\widehat{K}_{AB}$ and $\widehat{P}_{AB}$ are then found to satisfy the algebra

$$\begin{gathered}\left[\widehat{K}_A B, \widehat{P}_{AB}\right] = i\kappa\frac{m_B}{m_A}\widehat{D}_B - i\hbar\frac{m_B}{m_A}\widehat{D}_A\,,\ \left[\widehat{D}_I, \widehat{P}_{AB}\right] = i\hbar\delta_{BI}\widehat{P}_{AB} - i\kappa\delta_{AI}\widehat{P}_{AB}\,,\\ \left[\widehat{D}_A, \widehat{D}_B\right] = 0\,,\ \left[\widehat{D}_I, \widehat{K}_{AB}\right] = i\kappa\delta_{AI}\widehat{K}_{AB} - i\hbar\delta_{BI}\widehat{K}_{AB}\,,\end{gathered} \tag{2.84}$$

where δ_{IJ} is the Kronecker delta. This 4-dimensional algebra is called *relational Lie algebra* $\mathscr{R}(4)$. This is built from the naively deduced generators $\widehat{P}_{AB}$ and $\widehat{K}_{AB}$ and $\widehat{D}_I$, $I = A, B$. These two generators are deduced from the commutator between $\widehat{P}_{AB}$ and $\widehat{K}_{AB}$ and are exactly the operators introduced in (2.69) to preserve the relational property of the boost transformation. This algebra can be simplified by introducing the following operators

$$\widehat{D} := \frac{\kappa}{m_A}\mathbb{1}_A \otimes m_B\widehat{D}_B - \frac{\hbar}{m_A}\widehat{D}_A \otimes m_B\mathbb{1}_B\,,\ \widehat{D}^* := \frac{\kappa}{m_A}\mathbb{1}_A \otimes m_B\widehat{D}_B + \frac{\hbar}{m_A}\widehat{D}_A \otimes m_B\mathbb{1}_B\,. \tag{2.85}$$

In this way, we get

$$\begin{gathered}\left[\widehat{P}_{AB}, \widehat{K}_{AB}\right] = -i\widehat{D}\,,\ \left[\widehat{P}_{AB}, \widehat{D}\right] = -2i\kappa\hbar\frac{m_B}{m_A}\widehat{P}_{AB}\,,\\ \left[\widehat{K}_{AB}, \widehat{D}\right] = 2i\kappa\hbar\frac{m_B}{m_A}\widehat{K}_{AB}\,,\ \left[\widehat{D}^*, \cdot\right] = 0\,,\end{gathered} \tag{2.86}$$

therefore the algebra is the direct sum $so(2,1) \oplus \mathbb{R}$. The classical reference frame limit can be achieved by taking the $\kappa \to 0$ limit, in which case the operators $\widehat{x}_A$ and $\widehat{p}_A$ become numbers. In this limit, the subalgebra generated by $\{\widehat{P}_{AB}, \widehat{K}_{AB}, \widehat{D}\}$ yields a three-dimensional Heisenberg algebra generated by $\{\widehat{p}_B, \widehat{G}_B, m_B\mathbb{1}_B\}$ which satisfies the commutation relations of the centrally extended Galilei algebra.

Time doesn't hear if you ask it to wait: Dynamical Lie algebra

We now analyse the time-dependent counterpart of the algebra derived above. In this case, the operator $\widehat{G}_B$ is given by $\widehat{G}_B = \widehat{p}_B t - m_B\widehat{x}_B$. By repeating the same arguments of the previous paragraph, we end up with the following 7-dimensional algebra

$$\begin{gathered}\left[\widehat{P}_{AB}, \widehat{K}_{AB}\right] = i\hbar\frac{m_B}{m_A}\widehat{D}_A - i\kappa\frac{m_B}{m_A}\widehat{D}_B + 2i\kappa\frac{m_B}{m_A}\widehat{Q}_B t\,,\ \left[\widehat{P}_{AB}, \widehat{D}_I\right] = i\left(\kappa\delta_{AI} - \hbar\delta_{BI}\right)\widehat{P}_{AB}\\ \left[\widehat{P}_{AB}, \widehat{Q}_I\right] = i\frac{\kappa}{m_A}\delta_{AI}\widehat{T}\,,\ \left[\widehat{P}_{AB}, \widehat{T}\right] = 2i\kappa m_B\widehat{Q}_B\,,\ \left[\widehat{K}_{AB}, \widehat{Q}_I\right] = -i\frac{\hbar}{m_A}\delta_{BI}\widehat{T}\\ \left[\widehat{K}_{AB}, \widehat{D}_I\right] = -i\kappa\delta_{AI}\widehat{K}_{AB} + i\delta_{BI}\left(\hbar\widehat{K}_{AB} - 2\frac{\hbar}{m_A}t\widehat{T}\right),\ \left[\widehat{K}_{AB}, \widehat{T}\right] = -2i\hbar m_B\widehat{Q}_A\\ \left[\widehat{D}_I, \widehat{Q}_J\right] = 2i\delta_{IJ}\left(\kappa\delta_{AI}\widehat{Q}_A + \hbar\delta_{BI}\widehat{Q}_B\right),\ \left[\widehat{D}_A, \widehat{D}_B\right] = 0\,,\\ \left[\widehat{D}_I, \widehat{T}\right] = i\left(\kappa\delta_{AI} + \hbar\delta_{BI}\right)\widehat{T}\,,\ \left[\widehat{Q}_I, \widehat{T}\right] = i\left(\hbar\delta_{BI} + \kappa\delta_{AI}\right),\end{gathered} \tag{2.87}$$

where $\widehat{Q}_A = \frac{\widehat{p}_A^2}{2m_A} \otimes \mathbb{1}_B$, $\widehat{Q}_B = \mathbb{1}_A \otimes \frac{\widehat{p}_B^2}{2m_B}$, $\widehat{T} = \widehat{p}_A \otimes \widehat{p}_B$, while all the other generators but $\widehat{G}_B$ have the same form as before. The above algebra is called *dynamical Lie algebra* $\mathscr{D}(7)$. As in the time-independent case, the introduction of the additional generators $\widehat{Q}_A$, $\widehat{D}_I$ and $\widehat{T}$ restores the relational character of the phase space variables, as well as ensuring that the QRFs transformations are extended symmetries of the full Hamiltonian

$$\widehat{H} = \frac{\widehat{p}_A^2}{2m_A} \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \frac{\widehat{p}_B^2}{2m_B} \,. \tag{2.88}$$

With this definition of the generators, the translation in (2.66) and the Galilean boost in (2.69) can be written as products of group elements obtained via the exponential map and the parity swap operator[9]

$$\begin{aligned} \widehat{S}^{(x)} &= \widehat{P}_{AC} e^{\frac{i}{\hbar}\frac{m_A}{m_C}\widehat{Q}_A t} e^{\frac{i}{\hbar}\widehat{P}_{AB}} e^{\frac{i}{\hbar}\widehat{Q}_A t} \,, \\ \widehat{S}^{(v)} &= \widehat{P}_{AC} e^{\frac{i}{\hbar}\frac{m_A}{m_C}\widehat{Q}_A t} e^{\frac{i}{\hbar}\log\left(\frac{m_C}{m_A}\right)\widehat{D}_A} e^{\frac{i}{\hbar}\widehat{K}_{AB}} e^{\frac{i}{\hbar}\widehat{Q}_A t} \,. \end{aligned} \tag{2.89}$$

For what concerns the classical reference frame limit, this is once again given by $\kappa \to 0$, which is the same as saying that the position and momentum of system A become commutative functions. In this limit, the generators $\{\widehat{P}_{AB}, \widehat{K}_{AB}, \widehat{Q}_B\}$ have classical counterparts

$$\widehat{P}^c_{AB} = x_A \otimes \widehat{p}_B \,, \; \widehat{K}^c_{AB} = \frac{p_A}{m_A} \otimes \left(\widehat{p}_B t - m_B \widehat{x}_B\right) , \; \widehat{Q}^c_B = \mathbb{1}_A \otimes \frac{\widehat{p}_B^2}{2m_B} \,. \tag{2.90}$$

By straightforward calculations, it can be shown that the commutators between these operators are the same as the $\kappa \to 0$ limit of (2.87). Therefore, the operators

$$\widehat{P}_0 = \widehat{Q}^c_B \,, \; \widehat{G} = \frac{m_A}{p_A} \widehat{K}^c_{AB} \,, \; \widehat{P} = -\frac{1}{x_A} \widehat{P}^c_{AB} \,, \; \widehat{M} = m_B \mathbb{1}_B \,, \tag{2.91}$$

define a representation of the $(1+1)$ centrally extended Galilei Lie algebra.

2.2.3 Perspective-neutral approach to QRFs

Until now, we have developed the framework of QRFs by describing everything from the perspective of a given reference frame from the start and then discussing how to change between different perspectives. Another possibility is to adopt a perspective-neutral approach from the beginning and then study the reduction to a specific perspective of a given quantum system [103]. Consider three quantum systems, A, B, and C in a simple $1+1$-dimensional model. Imagine the system being invariant under spatial translations. This property can be described by requiring that the total momentum of the system vanishes. As already discussed within the Page-Wooters formalism, such a condition is a constraint that has to be enforced on the physical states of the system

$$\widehat{P} |\Psi\rangle^{\text{phys}}_{ABC} = 0 \Rightarrow |\Psi\rangle^{\text{phys}}_{ABC} = \frac{1}{2\pi} \int \mathrm{d}\alpha \, e^{i\alpha \widehat{P}} |\phi\rangle_{ABC} \,, \tag{2.92}$$

[9] Note that $e^{-\frac{i}{\hbar}\widehat{Q}_C t} \widehat{P}_{AC} = \widehat{P}_{AC} e^{\frac{i}{\hbar}\frac{m_A}{m_C}\widehat{Q}_A t}$, where $\widehat{Q}_C$ is defined in the same way as $\widehat{Q}_A$ and $\widehat{Q}_B$, namely $\widehat{Q}_C = \mathbb{1}_B \otimes \frac{\widehat{\pi}_C^2}{2m_C}$. We also set $\kappa = \hbar$.

where $|\phi\rangle_{ABC}$ is a generic state of the kinematical Hilbert space and $\hat{P} = \sum_i \hat{p}_i$ is the total momentum. If

$$|\phi\rangle_{ABC} = \int \mathrm{d}p_A \,\mathrm{d}p_B \,\mathrm{d}p_C \,\phi_{ABC}(p_A, p_B, p_C)\,|p_A\rangle\,|p_B\rangle\,|p_C\rangle \tag{2.93}$$

is a generic state in the kinematical Hilbert space, then

$$\begin{aligned}|\Psi\rangle^{\text{phys}}_{ABC} &= \int \mathrm{d}p_B \,\mathrm{d}p_C \,\psi_{BC|A}(p_B, p_C)\,|-p_B - p_C\rangle\,|p_B\rangle\,|p_C\rangle = \\ &= \int \mathrm{d}p_A \,\mathrm{d}p_C \,\psi_{AC|B}(p_A, p_C)\,|p_A\rangle\,|-p_A - p_C\rangle\,|p_C\rangle = \\ &= \int \mathrm{d}p_A \,\mathrm{d}p_B \,\psi_{AB|C}(p_A, p_B)\,|p_A\rangle\,|p_B\rangle\,|-p_A - p_B\rangle\ ,\end{aligned} \tag{2.94}$$

where $\psi_{AB|C}(p_A, p_B) := \phi_{ABC}(p_A, p_B, -p_B - p_C)$ and analogously for the other momentum distributions. Then of course the physical states contain some redundancy, given by the projection onto the null-momentum states. This redundancy can be removed by reducing to the perspective of one of the three quantum systems and then disregarding the system's degrees of freedom, which will be taken to be the redundant ones. Specifically, imagine that we want to take A's perspective. We define the trivialisation operator

$$\widehat{\mathcal{T}}_{BC|A} := e^{\frac{i}{\hbar}\hat{x}_A \otimes (\hat{p}_B \otimes \mathbb{1}_C + \mathbb{1}_B \otimes \hat{p}_C)}\ , \tag{2.95}$$

which maps the physical states (2.92) to

$$|\psi\rangle_{BC|A} = |p_A = 0\rangle \otimes |\psi\rangle^{(A)}_{BC} := |p_A = 0\rangle \otimes \int \mathrm{d}p_B \,\mathrm{d}p_C \,\psi_{BC|A}(p_B, p_C)\,|p_B\rangle\,|p_C\rangle\ . \tag{2.96}$$

Technically, this operator does not commute with the constraint, hence it actually maps out of the physical Hilbert space. Nevertheless, it is shown that it is actually a rewriting of the same information content in a different basis. Indeed, with simple computations, we have

$$\widehat{\mathcal{T}}_{BC|A}\hat{P}\widehat{\mathcal{T}}^{\dagger}_{BC|A} = \hat{p}_A\ , \tag{2.97}$$

hence the constraint is mapped to $\hat{p}_A$, making (2.96) still a physical state. After this trivialisation of the constraint, there is no relevant information, in the physical system A, about the original state. Therefore, we can consider A to be redundant and interpret the state $|\psi\rangle^{(A)}_{BC}$ as the state of the physical systems B and C with respect to A which corresponds to the perspective-neutral physical state $|\Psi\rangle^{\text{phys}}_{ABC}$. To achieve this reduction, we simply project the state (2.96) onto $|x_A = 0\rangle_A$. This has the physical meaning of fixing the position of the physical system A, thus taking it to be the origin of a reference frame from which all distances are taken. The reduced state is therefore

$$|\psi\rangle^{(A)}_{BC} = 2\pi\ {}_A\langle x_A = 0|\psi\rangle_{BC|A} = \int \mathrm{d}p\ {}_A\langle p|\psi\rangle_{BC|A} = \int \mathrm{d}p_B \,\mathrm{d}p_C \,\psi_{BC|A}(p_B, p_C)\,|p_B\rangle\,|p_C\rangle\ . \tag{2.98}$$

Moreover, it can be shown that the combination of the trivialisation operator and this projection is an isometry. Now, of course the same derivation can be performed by taking another system's perspective, say system C. Then it is possible to change from one perspective to the other and the transformation that allows to do that is found to be exactly (2.66). Hence we have recovered, from a perspective-neutral framework, the QRFs transformations introduced previously by considering superpositions of translations.

2.2.4 From space to spacetime: Quantum clocks and spacetime QRFs

We reviewed the QRFs framework in simple scenarios. Specifically, we started from a simple Galilean setting, without any gravitational content, and devised a recipe to generalise reference frame transformations to cases in which the reference frames are attached to quantum systems. As discussed previously, there are some arguments in the Quantum Gravity literature that hint towards a quantisation of the degrees of freedom defining reference frames, and one nice property of the framework adopted in the QRFs literature is that it allows to operationally discuss physical quantities well-defined in the laboratory, which is something really hard to do in a full theory of Quantum Gravity. For this reason, it is interesting to study the connections between the two fields and, to do so, it is paramount to first analyse the generalisations of the QRFs framework to gravitational settings.

Timeless and relational formulation

To fix the ideas, we consider a system of N quantum particles with internal degrees of freedom and a mass M which is the source of a gravitational field in a $1+1$ dimensional model [117]. Each particle has a mass m_I, $I=1,\dots,N$ and is described by a state living in the Hilbert space $\mathcal{H}_I = \mathcal{H}^I_{\text{ext}} \otimes \mathcal{H}^I_{\text{int}}$, where $\mathcal{H}^I_{\text{ext}}$ encodes the external degrees of freedom (*e.g.* the position) while $\mathcal{H}^I_{\text{int}}$ is the Hilbert space of the internal degrees of freedom. These degrees of freedom allow us to formulate a fully relational formalism in which one of the particles is chosen as QRF, and the dynamics of the other particles is described with respect to the quantum clock defined by the internal degrees of freedom of the particle chosen as QRF. In such a framework, both the state of the external degrees of freedom and the clock state play a role in describing the dynamics of the other particles. Specifically, the external degrees of freedom fix the QRF in space, while the internal degrees of freedom describe the clock, ticking according to the proper time in the particle's frame.

In the framework discussed above, space and time are treated on equal footing by means of a timeless formulation in which the state of the whole system is fully constrained, hence does not evolve, and dynamics emerges from internal correlations of physical clocks by conditioning on the state of the particle chosen as QRF. Then, the physical state of the N particles and the mass M, denoted as $|\Psi\rangle_{\text{ph}}$, satisfies

$$\widehat{C}\,|\Psi\rangle_{\text{ph}} = 0\,, \tag{2.99}$$

where $\hat{C}$ is (a set of) first-class constraints, defined on the Hilbert space of the whole system, encoding both the dynamics of the particles in spacetime and the global symmetries of the model. With this condition, the evolution of the state $|\Psi\rangle_{\text{ph}}$ is "frozen".

The constraint appearing in (2.99) is obtained by summing all the contributions to the dynamics of the N particles, which come from the particles' dispersion relations, and by enforcing the conservation of the energy-momentum. The latter requirement is analogous to what is done in a perspective-neutral framework in (2.92). In a general-relativistic picture, the dispersion relation of a particle is given by $\tilde{C}_I = -g^{\mu\nu}p^I_\mu p^I_\nu - m_I^2c^2$. The gravitational field sourced by the mass M can be regarded as a Newtonian limit of a more general metric field, therefore in our model we have

$$g^{00} = -1 + \frac{2\phi\big(\mathbf{x}-\mathbf{x}_M\big)}{c^2}\,,\;\; g^{01} = g^{10} = 0\,,\;\; g^{11} = 1\,, \tag{2.100}$$

where

$$\phi(\mathbf{x}-\mathbf{x}_M) = \frac{GM}{|\mathbf{x}-\mathbf{x}_M|} . \tag{2.101}$$

With this form of the metric, the dispersion relation reads

$$\tilde{C}_I = -g^{00}(\mathbf{x}_I - \mathbf{x}_M) p_0^{I2} - \mathbf{p}_I^2 - m_I^2 c^2 . \tag{2.102}$$

We can rewrite this constraint as

$$\tilde{C}_I = \left[\sqrt{-g^{00}(\mathbf{x}_I - \mathbf{x}_M)}\, p_0^{I2} - \omega_p^I\right]\left[\sqrt{-g^{00}(\mathbf{x}_I - \mathbf{x}_M)}\, p_0^{I2} + \omega_p^I\right], \tag{2.103}$$

with $\omega_p^I = \sqrt{\mathbf{p}_I^2 + m_I^2 c^2}$. By enforcing the energies to be positive, $p_0 \geq 0$, we therefore have a new constraint that, upon quantisation, reads

$$\widehat{C}_I = \sqrt{-g^{00}(\hat{\mathbf{x}}_I - \hat{\mathbf{x}}_M)}\, \hat{p}_0^I - \widehat{\omega}_p^I , \;\; \widehat{\omega}_p^I = \sqrt{\hat{\mathbf{p}}_I^2 + m_I^2 c^2} . \tag{2.104}$$

The weak-field approximation is crucial for this quantisation to be carried without ambiguities, since g^{11} is a constant and g^{00} is time independent, hence no ordering ambiguity arises. If $|\psi_I\rangle_{\text{ph}}$ is the physical state of particle I that solves the constraint, by conditioning $\widehat{C}|\psi_I\rangle_{\text{ph}} = 0$ with respect to the time x_I^0 we get the Schrödinger equation for particle I in the weak field limit and with special-relativistic corrections. Specifically, by considering a regime in which[10]

$$\left[\sqrt{-g^{00}(\hat{\mathbf{x}}_I - \hat{\mathbf{x}}_M)}, \widehat{\omega}_p^I\right] = 0 , \tag{2.105}$$

we have

$${}_I\left\langle x_I^0\middle|\widehat{C}\middle|\psi_I\right\rangle_{\text{ph}} = 0 \Rightarrow i\hbar \frac{\mathrm{d}}{\mathrm{d}x_I^0}\left|\psi_I\left(x_I^0\right)\right\rangle = \frac{1}{\sqrt{-g^{00}(\hat{\mathbf{x}}_I - \hat{\mathbf{x}}_M)}}\widehat{\omega}_p^I \left|\psi_I\left(x_I^0\right)\right\rangle , \tag{2.106}$$

where $\left|\psi_I\left(x_I^0\right)\right\rangle = {}_I\left\langle x_I^0\middle|\psi_I\right\rangle_{\text{ph}}$. Defining $x_I^0 = c t_I$ and expanding perturbatively we get

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t_I}\left|\psi_I(t_I)\right\rangle = \left[m_I c^2 + \frac{\hat{p}_I^2}{2m_I} - \frac{\hat{p}_I^4}{8m_I^3 c^2} + m_I \phi(\hat{\mathbf{x}}_I - \hat{\mathbf{x}}_M)\right]\left|\psi_I(t_I)\right\rangle . \tag{2.107}$$

Therefore, at the level of the single particle, the dynamics emerges from the entanglement between the time and spatial degrees of freedom.

The formalism outlined above realises the first task, namely to have a timeless formulation in which dynamics is an emergent feature. However, we do not yet have a relational formalism. To realise this, we need to eliminate the external structure. This is done by considering two more constraints on top of the N constraints $\widehat{C}_I$. These additional constraints encode the conservation of the energy-momentum for the whole system and are given by[11]

$$\hat{f}_0 = \sum_{I=1}^{N}\left[\hat{p}_0^I + \Delta(\hat{\mathbf{x}}_I - \hat{\mathbf{x}}_M, \hat{\mathbf{p}}_I)\frac{\widehat{H}_I}{c}\right] + \hat{p}_0^M , \;\; \hat{f}_1 = \sum_{I=1}^{N}\hat{p}_I + \hat{p}_M , \tag{2.108}$$

[10]This is equivalent to the requirement that all the terms $\frac{\phi(\hat{x}_I - \hat{x}_M)\hat{p}_I^2}{m_I^2 c^4}$ are negligible.

[11]We consider that the state of the mass M in the initial coordinates x^μ is chosen so that the physical configuration of the mass M does not dynamically evolve. Nevertheless, the mass M contributes with its energy and momentum to the definition of constraints, but does not define a dynamical constraint.

where

$$\Delta\big(\hat{\mathbf{x}}_I - \hat{\mathbf{x}}_M, \hat{\mathbf{p}}_I\big) = \sqrt{-g_{00}\big(\hat{\mathbf{x}}_I - \hat{\mathbf{x}}_M\big)}\left(1 + \frac{\hat{\mathbf{p}}_I^2}{m_I^2 c^2}\right)^{-\frac{1}{2}} \tag{2.109}$$

is denoted as *worldline operator* and is derived in [117]. This operator simply encodes the relativistic time dilation from the rest frame to an arbitrary frame. Notice that both the energies coming from the motion of the particles and the internal energies are considered in the constraint $\hat{f}_0$, which reduces to a Page-Wooters constraint when the external degrees of freedom are disregarded and only the internal state of the particles is considered. As a result of this construction, the full constraint is written as

$$\widehat{C} = \sum_{I=1}^{N} \mathcal{N}_I \widehat{C}_I + z^\mu \hat{f}_\mu \,, \tag{2.110}$$

where $\hat{z}^\mu$ and $\mathcal{N}_I$, $I = 1, \ldots, N$, are Lagrange multipliers. Notice that, if we consider the clocks to be non-interacting, all the constraints are first-class, meaning that they commute with each other. Specifically, $\left[\hat{C}_I, \hat{f}_1\right] = \left[\hat{f}_0, \hat{f}_1\right] = 0$ hold exactly, while $\left[\hat{C}_I, \hat{f}_0\right] = 0$ holds in our regime because of $\left[\sqrt{-g^{00}\big(\hat{\mathbf{x}}_I - \hat{\mathbf{x}}_M\big)}, \hat{\omega}_p^I\right] = 0$ for all I. The physical states that satisfy (2.99) can therefore be written as

$$|\Psi\rangle_{\rm ph} \propto \int \mathrm{d}^N \mathcal{N}\, \mathrm{d}^2 z\, e^{\frac{i}{\hbar}\sum_{I=1}^{N} \mathcal{N}_I \widehat{C}_I} e^{\frac{i}{\hbar} z^\mu \hat{f}_\mu} |\Phi\rangle \,, \tag{2.111}$$

where

$$|\Phi\rangle = \int \prod_{I=1}^{N} \Big[\mathrm{d}\mu(x_I)\, \mathrm{d}E_I\Big]\, \mathrm{d}^2 x_M\, \Phi\big(x_1, \ldots, x_N, x_M; E_1, \ldots, E_N\big) \big|x_1, \ldots, x_N, x_M\big\rangle \big|E_1, \ldots, E_N\big\rangle \,, \tag{2.112}$$

with $x_I = (x_I^0, \mathbf{x}_I)$. In the above, $\mathrm{d}\mu(x_I) = \sqrt{-g_{00}\big(\hat{\mathbf{x}}_I - \hat{\mathbf{x}}_M\big)}\, \mathrm{d}^2 x_I$ is the covariant integration measure and $\big|E_I\big\rangle$ are the eigenstates of the internal Hamiltonians $\widehat{H}_I$.

We can now take the perspective of a specific particle, for instance particle 1, by mapping the phase space observables to the relational observables in the perspective of particle 1 and then requiring that the origin of the reference frame coincides with the position of the particle. This is done by performing a projection operation onto the state $\big|q_1 = 0\big\rangle_1$. This procedure yields the relational state from the perspective of particle 1, which reads

$$\big|\psi\big\rangle^{(1)} = {}_1\big\langle q_1 = 0\big|\widehat{\mathcal{T}_1}\big|\Psi\big\rangle_{\rm ph} \,, \tag{2.113}$$

where the operator $\widehat{\mathcal{T}_1}$ implements the mapping of the initial positions to the relative positions to particle 1 and is given by

$$\widehat{\mathcal{T}_1} = e^{\frac{i}{4\hbar}\log\left[-g_{00}(\hat{\mathbf{x}}_M)\right]\sum_{I=1}^{N}\left(\hat{x}_I^0 \hat{p}_0^I + \hat{p}_0^I \hat{x}_I^0\right)} e^{\frac{i}{\hbar}\hat{\mathbf{x}}_1\left(\hat{f}_1 - \hat{\mathbf{p}}^1\right)} e^{\frac{i}{\hbar}\hat{x}_1^0\left(\hat{f}_0 - \hat{p}_0^1\right)} \,. \tag{2.114}$$

The last two terms of this operator are responsible for the mapping onto relative coordinates, while the first term transforms the metric to a locally inertial metric field at the location of particle 1. The details can be found in [117]. The dynamics is now encoded in the new constraints

$$\widehat{C}_I' = \widehat{\mathcal{T}_1}\widehat{C}_I \widehat{\mathcal{T}_1}^\dagger = \sqrt{\frac{g_{00}\big(\hat{\mathbf{q}}_I - \hat{\mathbf{q}}_M\big)}{g_{00}\big(\hat{\mathbf{q}}_M\big)}}\, \hat{k}_0^I - \hat{\omega}_k^I \,, \tag{2.115}$$

where $\hat{q}_I$ and $\hat{q}_M$ are the new spacetime coordinates of the particles relative to particle 1 and $\hat{k}_I$ and $\hat{k}_M$ the momenta operators canonically conjugate to them. Therefore the new metric field, in the perspective of particle 1, is given by

$$g'_{00}(\hat{\mathbf{q}}_I, \hat{\mathbf{q}}_M) = \widehat{\mathcal{T}}_1 \frac{g_{00}(\hat{\mathbf{x}}_I - \hat{\mathbf{x}}_M)}{g_{00}(\hat{\mathbf{x}}_I - \hat{\mathbf{x}}_M)} \widehat{\mathcal{T}}_1^\dagger , \tag{2.116}$$

which sets the metric in the position of particle 1 to be the Minkowski metric. Therefore, $\widehat{\mathcal{T}}_1$ realises a transformation to the Quantum Locally Inertial Frame (QLIF) of particle 1.

With some calculations, it is possible to show that the state in (2.113) can be written as

$$|\psi\rangle^{(1)} = \int \mathrm{d}\tau_1 \, e^{-\frac{i}{\hbar}\widehat{H}^{(1)}\tau_1} \left|\psi_0^{(1)}\right\rangle |\tau_1\rangle , \tag{2.117}$$

where $\widehat{H}^{(1)}$ is the Hamiltonian of the system in the perspective of particle 1 and $|\tau_1\rangle$ is the state of the clock 1, defined by $\widehat{T}_1 |\tau_1\rangle = \tau_1 |\tau_1\rangle$, with $\left[\widehat{T}_I, \widehat{H}_J\right] = i\hbar\delta_{IJ}$. The state $\left|\psi_0^{(1)}\right\rangle$ formally plays the role of an initial state and is related to the state $|\phi\rangle$ in (2.113), see [117] for the details. Indeed, the state in (2.117) can be interpreted as an history state which associates to any time eigenstate $|\tau_1\rangle$ a state of the remaining particles at time τ_1.

Of course, it is possible to describe the system in the perspective of any other particle. If we want to take the perspective of particle 2, another operator $\widehat{\mathcal{T}}_2$ is defined analogously to $\widehat{\mathcal{T}}_1$ and the state of the system in the perspective of particle 2 is given by

$$|\psi\rangle^{(2)} = {}_2\langle r_2 = 0|\widehat{\mathcal{T}}_2|\Psi\rangle_{\text{ph}} = \int \mathrm{d}\tau_2 \, e^{-\frac{i}{\hbar}\widehat{H}^{(2)}\tau_2} \left|\psi_0^{(2)}\right\rangle |\tau_2\rangle . \tag{2.118}$$

It is possible to map one perspective into the other with the operator $\hat{\mathcal{T}}_{12} = \hat{\mathcal{T}}_2 \hat{\mathcal{T}}_1^\dagger$ by means of

$$|\psi\rangle^{(2)} = {}_2\langle r_2 = 0|\widehat{\mathcal{T}}_{12}|\psi\rangle^{(1)} \otimes |p_1 = 0\rangle_1 . \tag{2.119}$$

Superposition of gravitational redshift and relativistic time dilation

We now focus on a model with only two particles (and the mass M) and consider measurements made in the QRF of particle 2. Analogously to the measurement procedure in [112], we change the constraints to

$$\begin{gathered}
\widehat{C}_I = \sqrt{-g^{00}(\hat{\mathbf{x}}_I - \hat{\mathbf{x}}_M)}\, \hat{p}_0^I - \hat{\omega}_p^I , \quad \hat{f}_1 = \hat{\mathbf{p}}_1 + \hat{\mathbf{p}}_2 + \hat{\mathbf{p}}_M , \\
\hat{f}_0^Q = \hat{p}_0^1 + \hat{p}_0^2 + \hat{p}_0^M + \Delta(\hat{\mathbf{x}_1} - \hat{\mathbf{x}_M}, \hat{\mathbf{p}}_1) \frac{\widehat{H}_1}{c} + \Delta(\hat{\mathbf{x}_2} - \hat{\mathbf{x}_M}, \hat{\mathbf{p}}_2) \left[\frac{\widehat{H}_2}{c} + \delta\left(\widehat{T}_2 - \tau_2^*\right) \frac{\widehat{Q}_2}{c} \right] ,
\end{gathered} \tag{2.120}$$

where the measurement $\widehat{Q}_2$ is assumed to commute with every constraint and to occur at time τ_2^* in the QRF of particle 2. We can define the state of the system in the perspective of particle 1 in the same way as done in (2.113), with the replacement of $\hat{f}_0$ with $\hat{f}_0^Q$. In this case, the $\widehat{\mathcal{T}}_1$ operator is denoted as $\widehat{\mathcal{T}}_{Q,1}$. The resulting state is given by

$$|\psi\rangle^{(1)} = \int \mathrm{d}\tau_1\, T\left\{ e^{-\frac{i}{\hbar}\int_0^{\tau_1}\mathrm{d}s\left[\widehat{H}^{(1)}+\Delta_{12}\delta\left(\widehat{T}_2+\Delta_{12}s-\tau_2^*\right)\widehat{Q}_2\right]}\right\} \left|\psi_0^{(1)}\right\rangle |\tau_1\rangle\,, \tag{2.121}$$

where

$$T\left[\widehat{O}(t_1)\widehat{O}(t_2)\right] = \theta(t_1-t_2)\widehat{O}(t_1)\widehat{O}(t_2) + \theta(t_2-t_1)\widehat{O}(t_2)\widehat{O}(t_1) \tag{2.122}$$

is the time ordering operator and

$$\Delta_{12} = \frac{\Delta\left(\hat{\mathbf{q}}_2-\hat{\mathbf{q}}_M, \hat{\mathbf{k}}_2\right)}{\sqrt{-g_{00}\left(\hat{\mathbf{q}}_M\right)}\,\hat{\gamma}_{\Sigma k,1}^{-1}} = \sqrt{\frac{g_{00}\left(\hat{\mathbf{q}}_2-\hat{\mathbf{q}}_M\right)}{g_{00}\left(\hat{\mathbf{q}}_M\right)}}\,\frac{\hat{\gamma}_2^{-1}}{\hat{\gamma}_{\Sigma k,1}^{-1}} \tag{2.123}$$

is the worldline operator of particle 2 in the perspective of clock 1, with

$$\hat{\gamma}_{\Sigma k,1} = \sqrt{1+\frac{\left(\sum_I \hat{\mathbf{k}}_I + \hat{\mathbf{k}}_M\right)^2}{m_1^2c^2}}\,,\quad \hat{\gamma}_I = \sqrt{1+\frac{\hat{\mathbf{k}}_I^2}{m_I^2c^2}} \tag{2.124}$$

encoding the effects of the special-relativistic time dilation. Notice that the metric field appearing in this expression is exactly the metric field $g'_{00}\left(\hat{\mathbf{q}}_2, \hat{\mathbf{q}}_M\right)$ in the QLIF of particle 1.

To understand what the effects of the quantum nature of the reference frame 1 are, we specify the state in (2.121) in three different physical regimes: Galilean, in which particles are considered to be non-relativistic and no gravitational field is present; Newtonian, in which particles are considered to be non-relativistic but there is a mass M sourcing a gravitational field; special-relativistic in which particles are considered to be relativistic but no gravitational field is present.

- In the Galilean case, there is no source of gravitational field and the particles are non-relativistic. Therefore we have $\Delta_{12} = 1$ and the history state in (2.121) can be written as

$$\begin{aligned}|\psi\rangle^{(1)} = &\int_{-\infty}^{\tau_2^*}\mathrm{d}\tau_1\, e^{-\frac{i}{\hbar}\widehat{H}^{(1)}\tau_1}\left|\psi_0^{(1)}\right\rangle|\tau_1\rangle + \\ &+\int_{\tau_2^*}^{+\infty}\mathrm{d}\tau_1\, e^{-\frac{i}{\hbar}\widehat{H}^{(1)}\left(\tau_1-\tau_2^*\right)}e^{-\frac{i}{\hbar}\widehat{Q}_2}e^{-\frac{i}{\hbar}\widehat{H}^{(1)}\tau_2^*}\left|\psi_0^{(1)}\right\rangle|\tau_1\rangle\,.\end{aligned} \tag{2.125}$$

- In the special-relativistic regime, there is no source of gravitational field but particles are special-relativistic, therefore the worldline operator reduces to

$$\Delta_{12} = \frac{\hat{\gamma}_{\Sigma k,1}}{\hat{\gamma}_2}\,. \tag{2.126}$$

  The history state (2.121) becomes

$$\begin{aligned}|\psi\rangle^{(1)} = \int \mathrm{d}\mathbf{k}_2\,\mathrm{d}t_2 \Bigg\{&\int_{-\infty}^{\tau_k}\mathrm{d}\tau_1\, e^{-\frac{i}{\hbar}\widehat{H}^{(1)}\tau_1}\psi_0^{(1)}(\mathbf{k}_2,t_2)\,|\mathbf{k}_2,t_2\rangle + \\ &+\int_{\tau_k}^{+\infty}\mathrm{d}\tau_1\, e^{-\frac{i}{\hbar}\widehat{H}^{(1)}(\tau_1-\tau_k)}e^{-\frac{i}{\hbar}\widehat{Q}_2}\psi_{\tau_k}^{(1)}(\mathbf{k}_2,t_2)\,|\mathbf{k}_2,t_2\rangle\Bigg\}|\tau_1\rangle\,,\end{aligned} \tag{2.127}$$

where

$$\left|\psi^{(1)}_{\tau_k}\right\rangle = \int \mathrm{d}t'\, e^{-\frac{i}{\hbar}\widehat{H}^{(1)}\tau_k}\psi^{(1)}_0(\mathbf{k}_2, t')\left|\mathbf{k}_2, t'\right\rangle , \tag{2.128}$$

and

$$\tau_k = \Delta^{-1}_{12}(\mathbf{k}_2)\tau^*_2 \tag{2.129}$$

is the (momentum-dependent) special-relativistic time dilation of time τ^*_2.

- In the Newtonian case, particles are considered to be non-relativistic but there is a source of gravitational field. Such a source has mass M and the worldline operator becomes

$$\Delta_{12} = \sqrt{-g'_{00}(\hat{\mathbf{q}}_2, \hat{\mathbf{q}}_M)} = \sqrt{\frac{g_{00}(\hat{\mathbf{q}}_2 - \hat{\mathbf{q}}_M)}{g_{00}(\hat{\mathbf{q}}_M)}} , \tag{2.130}$$

therefore the history states (2.121) reads

$$\begin{aligned}\left|\psi\right\rangle^{(1)} = \int \mathrm{d}\mathbf{q}_2\, \mathrm{d}^2 q_M\, \mathrm{d}t_2 \Bigg\{ \int_{-\infty}^{\tau_q} \mathrm{d}\tau_1\, e^{-\frac{i}{\hbar}\widehat{H}^{(1)}\tau_1}\psi^{(1)}_0(\mathbf{q}_2, q_M, t_2)\left|\mathbf{q}_2, q_M, t_2\right\rangle + \\ + \int_{\tau_q}^{+\infty} \mathrm{d}\tau_1\, e^{-\frac{i}{\hbar}\widehat{H}^{(1)}\left(\tau_1-\tau_q\right)} e^{-\frac{i}{\hbar}\widehat{Q}_2}\psi^{(1)}_{\tau_q}(\mathbf{q}_2, q_M, t_2)\left|\mathbf{q}_2, q_M, t_2\right\rangle \Bigg\} \left|\tau_1\right\rangle ,\end{aligned} \tag{2.131}$$

where

$$\left|\psi^{(1)}_{\tau_q}\right\rangle = \int \mathrm{d}t'\, e^{-\frac{i}{\hbar}\widehat{H}^{(1)}\tau_q}\psi^{(1)}_0(\mathbf{q}_2, q_M, t')\left|\mathbf{q}_2, q_M, t'\right\rangle , \tag{2.132}$$

and

$$\tau_q = \Delta^{-1}_{12}(\mathbf{q}_2, \mathbf{q}_M)\tau^*_2 \tag{2.133}$$

is the (position-dependent) gravitational redshift of time τ^*_2.

From these physical regimes, we understand that the framework we adopted can describe two interesting phenomena: the *superposition of gravitational redshifts* and the *superposition of special-relativistic time dilations*. Starting from (2.125), we see that in the Galilean case no new effect arises, indeed the measurement happening at time τ^*_2 in the perspective of clock 2 occurs at the same time in the perspective of clock 1. In (2.127), we see that special-relativistic effects cause a time dilation from the perspective of clock 1, which therefore does not see the measurement occurring at the same time of clock 2, as can be seen from (2.129). This is nothing new [12] but, on top of this, we also see that clock 1 sees the measurement as being a delocalised event in time. Indeed, in the second integral in (2.127), we see that the time τ_k at which clock 1 sees the measurement taking place depends on the momentum of particle 2 as seen in the reference frame of clock 1. Since the relational degrees of freedom between the two particles are, in general, in a quantum superposition of different momenta, clock 1 observes a *superposition of special-relativistic time dilations*, hence perceives the event as being *delocalised in time*. Finally, in (2.131) some analogous considerations can be made for (2.131). The time τ_q at which clock 1 sees the measurement taking place is gravitationally

[12]The only difference is that in the time dilation factor in (2.129) there is a term that depends on particle 1 dependent and does not appear classically.

redshifted, as can be seen in (2.132) and in the second integral in (2.131). Since time depends on the position of particle 2 as seen in the reference frame of clock 1, the latter perceives the measurement as a delocalised event in time in this case as well, the difference being that in this case this delocalisation results from a *superposition of gravitational redshifts*. Ultimately, the latter emerges because, in general, the relational degrees of freedom between the two particles are in a quantum superposition of different positions.

2.3 Summary: Quantum observers

In this chapter, we have reviewed some of the most useful tools to investigate the Quantum Theory and Gravity interface in the language of Quantum Information Theory. These are, specifically, quantum clocks and quantum reference frames. We have discussed some examples of operational procedures to observe both classical general-relativistic effects, *e.g.* the gravitational time dilation, and non-classical spacetime effects with quantum clocks. We then showed how dynamics can emerge from quantum correlations between quantum clocks in the relational and timeless framework of the Page-Wooters formalism. We finally introduced some basic notions of the quantum reference frames framework, discussing both its foundational approach and the group properties of these transformations in the $1+1$ dimension case. We also discussed the notion of spacetime quantum reference frame examining some features that emerge when space and time are put on the same footing in this framework, *e.g.* the relativity of time localisation of events in spacetime originating from superpositions of gravitationally-redshited or special-relativistic dilated times.

This last effect is only one example out of several of this kind that hint towards the notion of quantum observers. In fact, observers who describe physical events with a quantum reference frame are not expected to behave classically, since such a description is affected by quantum effects that can even give rise to quantum superpositions of causal orders as in the gravitational quantum switch [112,161,162]. While these observations are relevant for the development of a theory of quantum observers, it is important to notice that there is no need to talk about observers in the QRFs framework, as the purpose of this research field is to derive a general method to map the description of physical systems between different coordinate systems that are in a quantum relation with each other. Talking about observers would introduce an additional structure that would complicate the discussion even further. However, when a theory of quantum observers will be fully developed, it could be expected that these will be observers whose reference frames have quantum degrees of freedom, subject to the effects that we outlined in this chapter. However, QRFs are only one side of the coin. To fully develop a theory of quantum observers, it will also be paramount to define a notion of *quantum measurement apparatuses*, intended as measurement devices that do not produce classical readings [13], and to delve into the problem of the *agency-dependence of spacetime*[14]. This is a largely unexplored area in the foundations of Quantum Gravity that will be investigated in the last chapter of this thesis.

[13]These could manifest both as non-classical outputs and with outputs with non-classical probabilities.

[14]This term is used to denote the possible influence that the observers' measurement choices have in reconstructing spacetime through physical events.

Interlude

In the first part of the thesis, we have introduced all the topics that are relevant for what follows. The discussions around these topics have already highlighted how the interface between Quantum Theory and Gravity really challenges the most fundamental pillars of our understanding of the physical realm. *Symmetries, reference frames, observers, and measurement apparatuses*. The logical conundrums at the heart of the Quantum Gravity problem, from the “first principles” point of view, originate from the incompatibility between the different ways these foundational elements enter, and are characterised in, the two theories. This thesis opened with the aim of exploring exactly this issue. How do quantum systems evolve in a spacetime with deformed symmetries? How does the quantisation of reference frames affect the observed physical properties of quantum systems? How do measurement apparatuses behave in a non-classical spacetime and how are communication protocols between observers affected? It is now time to delve into these questions.

Part II

Original work

Third Chapter

Quantum spacetime-induced decoherence

In this chapter, we investigate quantum decoherence mechanisms that originate from quantum spacetime properties, both from a theoretical and a phenomenological point of view, presenting the results in [1] and [3]. First, we consider the effects of spacetime symmetries deformations on the dynamics of quantum-mechanical systems, following the results presented in [1]. The main result derived in [1] is that, in general, the evolution of quantum systems in a spacetime equipped with deformed symmetries, specifically with deformed time translations, is not given by the standard Liouville-von Neumann equation. From a specific deformation, this equation is modified into a Lindblad equation that describes a decoherence mechanism, which ultimately arises from the sheer properties of spacetime, rather than from interactions with an environment, and, for this reason, is an example of *fundamental decoherence*. After this, we present the phenomenological analysis of some quantum spacetime decoherence models presented in [3], in which we studied the effects of these models on neutrino oscillation phenomena. Using neutrino oscillation data from long-baseline reactors and atmospheric neutrino observations, we establish stringent constraints on the energy scale governing the strength of the decoherence induced by stochastic metric fluctuations, which amount to, respectively, $E_{QG} \geq 2.6 \cdot 10^{34}$ GeV and $E_{QG} \geq 2.5 \cdot 10^{55}$ GeV. These constraints are strong enough to rule out this decoherence model.

Since decoherence is the main character of this chapter, the first section is devoted to the introduction of the main ideas of the *quantum decoherence programme*. The focus is then shifted to a formal rewriting of some properties of Quantum Mechanics in terms of Hopf algebraic structures. This paves the way for the subsequent part of the chapter, which is devoted to the discussion of the results presented in [1]. The final part of the chapter is dedicated to the results in [3].

3.1 The Quantum Decoherence programme

Right after the development of Quantum Mechanics, it was already crystal clear that some effects observed at microscopic scales seemed completely absent in our everyday world. For instance, in the microscopic world, physical systems can be, and are in general observed to be, in superpositions of states describing distinct values of physical observables, *e.g.* a particle being in a superposition of two different positions. On the contrary, not only are such superpositions never observed in our macroscopic world, but even the possibility of just transfer quantum features to macroscopic systems by coupling them with microscopic ones, which is allowed by the laws of Quantum Theory, leads to paradoxes. The most paradigmatic example in this regard is given by Schrödinger's cat paradox [163], a gedankenexperiment in which a cat is closed in a box with a vial of poison, a hammer, and an unstable atom. The decay of the atom triggers the hammer that breaks the vial, hence the poison is released and the cat dies. In the quantum-mechanical description of this setup, the atom is at all times in a superposition of "decayed" and "not decayed" states, thus the cat should be in a superposition of "being dead" and "being alive". Then, if an observer opens the box, Quantum Mechanics predicts that the state of the system collapses into one of the two states. It would therefore seem that the observer decides the fate of the cat, hence giving rise to the paradox: What was the state of the cat before opening the box? Can the cat be dead before opening the box and alive afterward? Are these questions meaningful at all? Paradoxes such as the one just illustrated are at the heart of the so-called *quantum-to-classical* transition problem that, in the early days of Quantum Mechanics, urged physicists to refuse the application of the theory to the macroscopic realm, setting a clear separation between the quantum and the classical world known as the *Heisenberg cut* [14].

3.1.1 No system is truly isolated

Since Quantum Theory describes our Universe to a more fundamental level, the emergence of the macroscopic world from the microscopic quantum realm remained a long-standing problem for the foundations of Quantum Mechanics. Only in the 1970s and 1980s was this problem finally addressed [164–167], when it was understood that the paradoxes could be solved by not trying to force preconceived classical concepts onto Quantum Mechanics. One of these was the idealisation of *isolated systems*, namely that it should always be possible to isolate any physical system from interactions with the surrounding environment, so that the objective nature of the system can be observed.

While for classical phenomena it is possible to neglect the effects of some interactions with an environment, for quantum features this is not the case. For example, consider the interactions stemming from the collisions of a football at rest with air molecules. These collisions produce no observable effect on the classical motion of the football (it will remain at rest), since the momentum transfer from every collision is negligible and, moreover, these are isotropic, so the net transferred momentum is zero on average. However, if we consider the football to be, in principle, a quantum object, from the laws of Quantum Mechanics, the scattering interactions yield the emergence of *quantum correlations* between the football and the air molecules, and these are the real cause for the loss of "quantumness"[1]: the coherence "leaks" into the environment. Air molecules act as a decohering source for the football, although the interactions are small enough not to perturb the classical motion. This is nothing more than the manifestation of *quantum entanglement*, which requires a rethinking of the notion of *locality*. Indeed, while Quantum Mechanics is still a local theory, namely interactions are all local, these can produce non-local states. This is a feature completely absent in the macroscopic world, where the properties of a physical system are always preserved within the system itself. This fact justifies the idealisation of isolated systems in classical physics: shielding a physical system from the environment just reduces noise effects without changing its

[1] On the contrary, microscopic systems do not interact much with the environment, hence they retain their quantum features.

nature, thus the physical system retains its individuality. Vice versa, in Quantum Theory a state describing two systems A and B

$$\left|\psi_{AB}\right\rangle = \sum_{ij} c_{ij} \left|A_i\right\rangle \otimes \left|B_j\right\rangle \tag{3.1}$$

in general cannot be written in terms of states describing the single systems

$$\left|\Psi_{AB}\right\rangle = \sum_{ij} a_i b_j \left|A_i\right\rangle \otimes \left|B_j\right\rangle = \left|\psi_A\right\rangle \otimes \left|\psi_B\right\rangle , \tag{3.2}$$

no matter how spatially separated they are. This means that the two systems have lost their individuality and new physical properties exist within the non-local quantum correlations between them. Contrary to the classical case, interactions with the environment can therefore create entanglement between the latter and a physical system, changing its nature. These interactions cannot be neglected if they are too "weak" and are not to be thought of as simple perturbations that should be minimised to describe the true nature of physical systems. They are instead the cause of the (typically irreversible) "delocalisation" of quantum coherence, which is the origin of quantum features, into the system-environment state. This is the core idea of *environmental decoherence*.

Ultimately, the purely quantum-mechanical phenomenon of quantum entanglement[2] is the root of the emergence of the classical world from the underlying quantum substrate. Of course, it is not yet clear whether the decoherence induced by environmental interactions can explain the effective emergence of all classical phenomena. This is still highly debated in the physics and philosophy community, although there are no doubts about the robustness of the formalism of Quantum Decoherence since, in the last three decades, several experiments have witnessed the action of decoherence and the gradual transition between quantum and classical regimes in different physical setups [169–177], which has also become a problem to overcome to realise *mesoscopic* quantum superpositions [46, 47] and fault-tolerant quantum computers [178, 179].

In the following subsections, we will review the concepts introduced in this introduction in a more formal manner, following [180].

3.1.2 The emergence of the classical world from quantum correlations

To understand how decoherence actually works, it is very illustrative to analyse the famous double-slit experiment, which, in the words of Feynman [181], is a phenomenon "which has in it the heart of Quantum Mechanics; in reality, it contains the only mystery". In the double-slit experiment, particles pass through a screen with two slits. After the passage, they hit a detector screen, where they leave a spot. Then two different behaviours are observed:

[2] This is the standard interpretation of environmental decoherence. However, there are some examples of decoherence models that do not require the generation of quantum entanglement [168]. Since the subsequent presentation of environmental decoherence serves only the purpose of introducing the general ideas of the framework, we will not deal with this issue and will simply follow the standard approach.

- The "wave" scenario, in which an interference pattern is observed on the screen, corresponding to the probability density

$$\varrho(x) = \frac{1}{2}\left|\psi_1(x) + \psi_2(x)\right|^2 = \frac{1}{2}\Big(\left|\psi_1(x)\right|^2 + \left|\psi_2(x)\right|^2\Big) + \mathrm{Re}\big\{\psi_1(x)\psi_2^*(x)\big\}\,, \tag{3.3}$$

where $\psi_i(x)$ is the wave function that describes the passage through the slit $i = 1,2$.

- The "particle" scenario, where a detector is placed at one of the slits to observe whether the particles pass through it and the pattern created by the particles on the screen corresponds to the probability density

$$\varrho(x) = \frac{1}{2}\Big(\left|\psi_1(x)\right|^2 + \left|\psi_2(x)\right|^2\Big)\,. \tag{3.4}$$

In the second case, the probability density is just the classical sum of the two probability densities, whereas in the first case, there is interference between the two probability amplitudes. This experiment is a clear manifestation of complementarity between *which-path information* and *interference*: either the "particle-like" or the "wave-like" behaviour is observed.

Between the two opposite cases described above, there is also the possibility of gaining only partial which-path information, avoiding the emergence of a full classical behaviour. To do so, we must properly describe the measurement processes that occur in the detectors at the slits. Specifically, we will assign a quantum state to the "composite" detector, consisting of the two detectors, before measurement, called the "ready" state and denoted by $|r\rangle$. We write the state of the particles passing through the slit i as $\left|\psi_i\right\rangle$, and the states of the detectors after the passage of the particle through the slit i are denoted by $\left|i\right\rangle$, so that if a particle is prepared in such a way as to pass with certainty through the slit i, the state of the detector-particle system is $\left|\psi_i\right\rangle \otimes \left|i\right\rangle$. Therefore, if a particle is prepared in a superposed state $\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\Big(\left|\psi_1\right\rangle + \left|\psi_2\right\rangle\Big)$, the evolution of the particle-detectors system is given by

$$\left|\psi\right\rangle \otimes |r\rangle \longrightarrow \frac{1}{\sqrt{2}}\Big(\left|\psi_1\right\rangle \otimes |1\rangle + \left|\psi_2\right\rangle \otimes |2\rangle\Big)\,. \tag{3.5}$$

We see that an entanglement has been generated between the composite detector and the particle. The state of the particle alone is given by the reduced density operator

$$\begin{aligned}\widehat{\rho}_P &= \frac{1}{2}\,\mathrm{Tr}_{\text{detector}}\Big\{\Big(\left|\psi_1\right\rangle \otimes |1\rangle + \left|\psi_2\right\rangle \otimes |2\rangle\Big)\Big(\left\langle\psi_1\right| \otimes \langle 1| + \left\langle\psi_2\right| \otimes \langle 2|\Big)\Big\} \\ &= \frac{1}{2}\Big(\left|\psi_1\right\rangle\!\left\langle\psi_1\right| + \left|\psi_2\right\rangle\!\left\langle\psi_2\right| + \langle 2|1\rangle \left|\psi_1\right\rangle\!\left\langle\psi_2\right| + \langle 1|2\rangle \left|\psi_2\right\rangle\!\left\langle\psi_1\right|\Big)\,,\end{aligned} \tag{3.6}$$

therefore the probability density on the screen at position x is

$$\varrho(x) = \langle x|\widehat{\rho}_P|x\rangle = \frac{1}{2}\Big(\left|\psi_1(x)\right|^2 + \left|\psi_2(x)\right|^2\Big) + \mathrm{Re}\big\{\psi_1(x)\psi_2^*(x)\,\langle 2|1\rangle\big\}\,. \tag{3.7}$$

The last term in this probability density describes the interference pattern, and the visibility of such interference is modulated by the overlap of the detector states $|1\rangle$ and $|2\rangle$. The more are these states distinguishable, namely the more wich-path information we can acquire, the more this overlap decreases. It is easy to see that in the case of perfect measurement, in which

$\langle 2|1\rangle = 0$, we recover the particle behaviour of (3.4). On the contrary, in the case of complete absence of which-path information, $\langle 2|1\rangle = 1$, we recover the wave behaviour (3.3). Between these two extreme cases, (3.7) describes the possibility of obtaining only partial which-path information so that it is still possible to observe some interference. With this description, we understand that the reason behind the loss of "quantumness" is to be searched for in the creation of entanglement between the particle and the measuring apparatus.

The example above is crucial as it can easily be generalised to explain the idea of the *environment as a which-path monitor.* Consider a macroscopic system that can be in two different positions. The air molecules (and even the photons!) around it scatter on its surface, and their subsequent motion, of course, depends on the position of this system. Therefore, it is easy to understand that the air molecules contain "which-path information" [3], hence they play the same role of the composite detector in the double-slit experiment, the states $|1\rangle$ and $|2\rangle$ being now the post-scattering states. If the number of air molecules is N, which for macroscopic systems is, of course, a large number, then these states are

$$\left|i\right\rangle = \bigotimes_{k=1}^{N} \left|e_k^{(i)}\right\rangle , \tag{3.8}$$

where $\left|e_k^{(i)}\right\rangle$ is the state of the k-th air molecule after being scattered from the system in position i, and $\left\langle e_k^{(2)}\middle|e_k^{(1)}\right\rangle = \epsilon \lesssim 1$. Therefore, $\langle 2|1\rangle = \epsilon^N \sim 0$ for large N, and any interference between the two different positions is completely damped, meaning that the macroscopic system cannot be observed in superposition of the two positions. This occurs because the evolution (3.5) spreads the superposition, initially confined to the system, to the overall system-environment state. Hence, the coherence of the initial superposition describing the state of the system is now "delocalised" into the environment and becomes a shared property of the system-environment state. In this process, the environment behaves as a monitoring process that, however, does not require an actual observer: To decohere the system, it is sufficient that which-path information is encoded in the environment and thus could be in principle retrieved.

3.1.3 von Neuamn measurements

As we have seen in the previous subsection, many interactions can be understood as measurement processes, and this is crucial to enlightening the quantum-to-classical transition process through decoherence. The measurement processes described above are *ideal quantum measurements* that are described within the *von Neumann scheme.* In this framework, two physical systems are considered: a system S, with Hilbert space $\mathscr{H}_S$ and basis vectors $\left\{\left|s_n\right\rangle\right\}$, and a measurement apparatus A that performs measurements, formally represented by a Hilbert space $\mathscr{H}_A$. The former (latter) is typically considered microscopic (macroscopic). We can consider states $\left|s_n\right\rangle$ to be the eigenstates of an observable, measured by the apparatus, corresponding to the possible outcomes of the measurements, namely the eigenvalues of the observable. We can also imagine that the apparatus has some kind of pointer that starts from

[3]The term "which-path information" is used in a broad sense to denote generally "which-state information". In this case, this is a "which-position information".

a fixed position before the measurements and reaches a position n after measuring the result corresponding to $\left|s_n\right\rangle$. We can describe this situation by denoting the pre-measurement state, the "ready" state, as $\left|a_r\right\rangle$, while the post-measurements states are $\left|a_n\right\rangle$. Therefore, the generic measurement process in the von Neumann scheme is given by

$$\mathscr{H}_S \otimes \mathscr{H}_A \ni \left|s_n\right\rangle \otimes \left|a_r\right\rangle \longrightarrow \left|s_n\right\rangle \otimes \left|a_n\right\rangle \in \mathscr{H}_S \otimes \mathscr{H}_A \,, \tag{3.9}$$

which establishes a one-to-one correspondence between the states of the system and the post-measurement states of the apparatus. An important assumption is that the measurement does not change the state of the system, namely it is an *ideal* measurement.

If we consider system S to be in a general state before measurement, $\left|\psi\right\rangle_S = \sum_n c_n \left|s_n\right\rangle$, then the evolution of the system-environment state is given by linearity as

$$|\Psi\rangle = \left(\sum_n c_n \left|s_n\right\rangle\right) \otimes \left|a_r\right\rangle \longrightarrow \sum_n c_n \left|s_n\right\rangle \otimes \left|a_n\right\rangle \,, \tag{3.10}$$

which shows the dynamical emergence of entanglement. If we now consider the reduced state of the system, this will be given by the trace over the apparatus of the density operator describing the system-environment state

$$\begin{aligned} \widehat{\rho}_S = \mathrm{Tr}_A\left\{|\Psi\rangle\langle\Psi|\right\} &= \sum_{n,m} c_n c_m^* \left|s_n\right\rangle\!\left\langle s_m\right| \mathrm{Tr}\left\{\left|a_n\right\rangle\!\left\langle a_m\right|\right\} = \\ &= \sum_{n,m} c_{nm} \sum_k \left\langle \phi_k \middle| a_n\right\rangle \left\langle a_m \middle| \phi_k\right\rangle \left|s_n\right\rangle\!\left\langle s_m\right| \,, \end{aligned} \tag{3.11}$$

where $\left\{\left|\phi_k\right\rangle\right\}$ is an orthonormal basis of $\mathscr{H}_A$ and $c_{nm} := c_n c_m^*$. Therefore, we see that $\widehat{\rho}_S$ is given by

$$\widehat{\rho}_S = \sum_{n,m} c_{nm} \left\langle a_m \middle| a_n\right\rangle \left|s_n\right\rangle\!\left\langle s_m\right| \,. \tag{3.12}$$

Local damping of interference: the environment as a measuring device

The state (3.12), which is a generalisation of (3.6), can help us better understand how the quantum-to-classical transition works. Indeed, if we consider the expectation value of a local observable $\widehat{O} = \widehat{O}_S \otimes \mathbb{1}_A$ of the system S, this is given by

$$\left\langle \widehat{O} \right\rangle = \mathrm{Tr}\left\{\widehat{O}\,|\Psi\rangle\langle\Psi|\right\} = \mathrm{Tr}_S\left\{\widehat{O}_S \widehat{\rho}_S\right\} = \sum_{n,m} c_{nm} \left\langle a_m \middle| a_n\right\rangle \left\langle s_m \middle| \widehat{O}_S \middle| s_n\right\rangle \,. \tag{3.13}$$

From this we see that the possibility of witnessing any interference between states $\left|s_m\right\rangle$ and $\left|s_n\right\rangle$ with a local observable depends on the overlap of states $\left|a_n\right\rangle$ and $\left|a_m\right\rangle$. The more distinguishable these states are, the more the off-diagonal terms of the density operator decohere. In the limit of perfect distinguishability, $\left\langle a_m \middle| a_n\right\rangle \approx 0$, a complete loss of "quantumness" characterises the system S. Interference terms are still present at the global level but become *locally unobservable* in (3.12), which becomes diagonal for $\left\langle a_m \middle| a_n\right\rangle \approx 0$. This occurs for any system that interacts with a macroscopic environment. Indeed, while it would still be possible to witness interference with global measurements in the overall system-environment superposition (3.10), in practice it is impossible to include all environmental degrees of freedom in the measurement process for macroscopic environments. Moreover, due to the large number of degrees of freedom, the creation of system-environment entanglement is *irreversible* for all practical purposes. As a consequence, the *delocalisation of quantum coherence* into the joint system-environment state induced by the inevitable interactions with the environment, which thus behaves as a "which-state monitor", is irreversible. This is exactly the process of environmental decoherence. The only way to preserve quantum features would be to shield physical systems from their environments, but this is practically impossible for macroscopic objects. On the contrary, microscopic systems do not interact much with the environment and quantum features are preserved[4].

Superselection rules

Since decoherence arises as a consequence of continuous system-environment interactions, it is not surprising that the study of decoherence is intrinsically linked to the infamous measurement problem. The latter can be decomposed into three parts.

1. The problem of the non-observability of interference: Why is it difficult to observe interference properties, in particular on macroscopic scales?
2. The problem of the preferred basis: Why do there appear to be some preferred physical quantities?
3. The problem of outcomes: Why do measurements have outcomes and what selects the specific outcome out of all the possibilities?

From the discussions in the previous subsections and the examples presented in the next subsection, it should be clear why decoherence essentially solved the first problem. On the contrary, the possibility of solving the last problem within this framework is a matter of debate. This is easily understood by looking at (3.12): There is nothing in this state that singles out a specific outcome of the measurement, rather it describes a probability distribution of possible outcomes. For this reason, states such as (3.12) are sometimes referred to as *pre-measurement states*. This issue is to be ascribed to the fact that the problem of outcomes is intrinsically related to the choice of a specific interpretation of Quantum Mechanics. For this reason, we will not deal with this problem.

What deserves clarification here is the second problem, which can be understood by looking at (3.9). In principle, there is no reason to prefer the basis states $|s_n\rangle$ to expand the states of the system S, and in fact we could choose another basis $|s'_n\rangle$ for such expansion. In this case, there would be another set of states of the apparatus $|a'_n\rangle$ associated with this other basis, and all subsequent discussions would repeat in the same way. Although the choice of the basis is purely mathematical, the emergence of classical properties is a very tangible physical phenomenon. What sets the preferred basis? Why are macroscopic systems always observed in definite positions but not in superpositions thereof? The answer to this problem is to be searched in the form of system-environment interactions [182–184]. Due to these, some states

[4]For instance, the number of air molecules scattering off an atom is ridiculously small compared to the number of air molecules scattering off a macroscopic object like a car.

are more prone to decoherence. Such preferred states are not predetermined, they emerge dynamically as the states that are least sensitive to the interactions with the environment, *i.e.* the states that become least entangled with it. Such states are called *pointer states* [182], because they can be interpreted as those that correspond to the physical quantities that can be most easily "read off". This process is called *environment-induced superselection*, in analogy with other superselection rules in physics that axiomatically prohibit the existence of some superpositions. For the case of environmental superselection, this is not set axiomatically, rather it is the environment that dynamically defines the pointer states and the observable properties of physical systems.

To understand how this superselection emerges, consider the total Hamiltonian of the system-environment

$$\widehat{H} = \widehat{H}_S + \widehat{H}_{\mathscr{E}} + \widehat{H}_{\text{int}} \,. \tag{3.14}$$

We can now consider some important cases. First, we consider the so-called *quantum measurement limit*, in which the dynamics is dominated by the interaction Hamiltonian, namely the energy scales involved in the interaction part are much larger than the energy scales of the self-Hamiltonians $\widehat{H}_S$ and $\widehat{H}_{\mathscr{E}}$. This is a remarkable limiting case since it applies to a variety of physical situations and allows us to understand quite easily the superselection process. As remarked above, pointer states are those that get least entangled with the environment upon evolution. Therefore, we are interested in finding the states $|s_n\rangle$ such that a product state $|s_n\rangle \otimes |a_0\rangle$ at time $t=0$ remains in the product form $|s_n\rangle \otimes |a_n(t)\rangle$ at later times $t>0$. Thus, we need

$$e^{-\frac{i}{\hbar}\widehat{H}_{\text{int}}t}\,|s_n\rangle \otimes |a_0\rangle = \lambda_i\,|s_n\rangle \otimes e^{-\frac{i}{\hbar}\widehat{H}_{\text{int}}t}\,|a_0\rangle := |s_n\rangle \otimes |a_n(t)\rangle \,. \tag{3.15}$$

Therefore we see that the pointer states in the quantum-measurement limit are simply the eigenstates of the part of the interaction Hamiltonian pertaining to the Hilbert space of the system. This can also be expressed in terms of *pointer observables*, which are nothing more than the superselected observables of the system, linear combinations of projectors onto pointer states

$$\widehat{O}_S = \sum_n o_n\,|s_n\rangle\langle s_n| \,. \tag{3.16}$$

Since $|s_n\rangle$ are eigenstates of the Hamiltonian, every projector $|s_n\rangle\langle s_n|$ commutes with it, therefore pointer observables satisfy the *commutativity criterion*

$$\left[\widehat{O}_S, \widehat{H}_{\text{int}}\right] = 0 \,. \tag{3.17}$$

Typically, the interaction Hamiltonian can be written as $\widehat{H}_{\text{int}} = \widehat{S} \otimes \widehat{E}$, and in these cases the pointer observables are those that commute with $\widehat{S}$, or the latter itself if it is Hermitian, in which cases the pointer states are the eigenstates of $\widehat{S}$. A frequent scenario is $\widehat{S} = f(\widehat{x})$, which corresponds to a continuous monitoring of the position of the system by the environment, leading to decoherence of spatial superpositions. In general, the interaction Hamiltonian can always be decomposed as

$$\widehat{H}_{\text{int}} = \sum_\alpha \widehat{S}_\alpha \otimes \widehat{E}_\alpha \,, \tag{3.18}$$

and if the $\widehat{S}_\alpha$ are Hermitian, this Hamiltonian describes the simultaneous monitoring of these observables. Pointer states with such decomposition are given by simultaneous eigenstates of all the $\widehat{S}_\alpha$, $\widehat{S}_\alpha \left|s_n\right\rangle = \lambda_n^{(\alpha)} \left|s_n\right\rangle$. If the eigenvalues $\lambda_n^{(\alpha)}$ do not depend on n, namely $\left|s_n\right\rangle$ are degenerate eigenstates of all $\widehat{S}_\alpha$, then every state which is a superposition of the states $\left|s_n\right\rangle$ will not get entangled with the environment as it is easily shown

$$\begin{aligned} e^{-\frac{i}{\hbar}\widehat{H}_{\mathrm{int}}t}\left|\psi\right\rangle \otimes \left|a_0\right\rangle = e^{-\frac{i}{\hbar}\widehat{H}_{\mathrm{int}}t}\left(\sum_n c_n \left|s_n\right\rangle\right) \otimes \left|a_0\right\rangle = \left(\sum_n c_n \left|s_n\right\rangle\right) \otimes e^{-\frac{i}{\hbar}t\sum_\alpha \lambda^{(\alpha)}\widehat{E}_\alpha}\left|a_0\right\rangle \\ = \left|\psi\right\rangle \otimes \left|a_\psi(t)\right\rangle \,. \end{aligned} \tag{3.19}$$

If the states $\left|s_n\right\rangle$ are also orthonormal, they define a subspace of the Hilbert space of system S in which all states do not get entangled with the environment. Such subspaces are called *decoherence-free subspaces*. The existence of such subspaces is a manifestation of a dynamical symmetry of the interaction. Of course, if the self-Hamiltonian $\widehat{H}_S$ cannot be neglected, to define a decoherence-free subspace it is necessary that the states $\left|s_n\right\rangle$ are not projected out the subspace via the internal evolution.

Another interesting scenario is the so-called *quantum limit of decoherence* [185], in which the highest energies available in the environment are smaller than the energy differences between the eigenstates of the system. In this case, the evolution of the environment is slow compared to that of the system, therefore the former can only monitor those observables that are constants of the motion. Typically, such a quantity is the Hamiltonian of the system, which leads to the superselection of the energy.

Of course, it is also possible to have intermediate regimes in which $\widehat{H}_{\mathrm{int}}$ and $\widehat{H}_S$ are comparable. In these cases, the two Hamiltonians can lead to superseleciton of non-commuting observables, in which cases the preferred states will be a “compromise” between the two sets. For instance, if the interaction Hamiltonian provides a monitoring of the position and the self-Hamiltonian yields a superselection of momenta, the pointer states will be localised in phase space.

As for the applicability of these limiting cases, the quantum-measurement limit efficiently describes the interactions of macroscopic systems with scattering particles. In these cases, the interaction Hamiltonian will be given by force laws that typically depend on some power of the distance between the object and the particle, hence it will commute with the position operator. From this we understand why macroscopic objects are observed to be in definite positions. The quantum limit can be applied for microscopic systems like atoms that are typically observed to be in eigenstates of the self-Hamiltonian, namely those that are observed to be superselected in the energy. Note that such a scenario is very different from a situation in which the presence of the environment can be simply neglected. Indeed, without the environment, there would not be continuous monitoring of the energy leading to a suppression of the coherence between different energy states. Another remarkable example is the superselection of the charge for charged particles that interact with their own Coulomb field that constitutes the environment [186]. In all these examples, the preferred basis is not chosen so as to match our experience of which physical quantities are observed as superselected. Selection occurs because of the structure of the system-environment interaction. Accordingly, the space of possible quantum superpositions is strongly reduced because the interaction laws depend only on a handful of physical quantities (position, momentum, charge, ...).

3.1.4 Simple examples of environmental decoherence

For the discussion outlined above, it is no surprise that, in quantum decoherence research, several efforts are made to derive specific models for the system-environment interactions in different physical scenarios. Such models focus on describing the time evolution of environmental states and the ensuing influence on the density operators of physical systems. Typically, it is found that the overlap between different environmental states decays exponentially as

$$\langle a_m(t) | a_n(t) \rangle \propto e^{-t/\tau_D} \,, \tag{3.20}$$

where τ_d is the characteristic (model-dependent) decoherence timescale. This exponential decay of the overlap then translates, by means of (3.12), into a suppression of coherences between off diagonal elements of the density operator of the system interacting with the environment. This shows that the dynamics of the local system alone is effectively *non-unitary*. As we shall see shortly, even the interactions with a thermal bath of photons as weak as the cosmic microwave background leads to a fast decoherence of macroscopic objects. This shows that such decoherence processes are not only utterly effective, but that it is also nearly impossible to avoid them.

A two-level model

Consider a two-level system S that interacts with an environment $\mathcal{E}$ of N other two-level systems. For simplicity, we refer to these systems as "spins" in the following. The basis states of S are indicated by $|0\rangle$ and $|1\rangle$, while the basis states of the environment are $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$, with $i = 1, \ldots, N$. We assume that the intrinsic dynamics of the system and the environment can be neglected, hence the dynamics is fully specified by the interaction Hamiltonian, which is chosen to be

$$\widehat{H} = \widehat{H}_{\text{int}} = \frac{1}{2}\big(|0\rangle\langle 0| - |1\rangle\langle 1|\big) \otimes \left[\sum_{i=1}^{N} g_i \Big(|\uparrow\rangle\langle\uparrow|_i - |\downarrow\rangle\langle\downarrow|_i\Big) \bigotimes_{j\neq i} \widehat{\mathbb{1}}_j\right] = \frac{1}{2}\hat{\sigma}_z \otimes \left(\sum_{i=1}^{N} g_i \hat{\sigma}_z^{(i)} \bigotimes_{j\neq i} \widehat{\mathbb{1}}_j\right), \tag{3.21}$$

where $\widehat{\mathbb{1}}_i = |\uparrow\rangle\langle\uparrow|_i + |\downarrow\rangle\langle\downarrow|_i$ is the identity operator on system i and $\hat{\sigma}_z$ is the Pauli z-spin operator. This Hamiltonian describes a linear interaction between the system S and each of the environmental spins in such a way that the environment monitors the z-spin component of the system, which is therefore the superselected observable. We also notice that the Hamiltonian commutes with $\hat{\sigma}_z$, so there is no exchange of energy between the system and the environment. This means that this Hamiltonian describes a model of *decoherence without dissipation*, showing that decoherence is a truly quantum effect with no classical counterpart. For convenience, we can write the Hamiltonian as

$$\widehat{H}_{\text{int}} = \frac{1}{2}\hat{\sigma}_z \otimes \widehat{E} \,, \tag{3.22}$$

where the 2^N eigenstates of $\widehat{E}$ are labeled by

$$|n\rangle = |\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes |\downarrow\rangle_3 \otimes \ldots \otimes |\uparrow\rangle_N \,, \tag{3.23}$$

with $0 < n < 2^N - 1$ and eigenvalues

$$\epsilon_n = \sum_{i=1}^{N} (-1)^{n_i} g_i \,, \tag{3.24}$$

where $n_i = 1$ ($n_i = 0$) if the i-th environmental spin is the $|\downarrow\rangle_i$ $\left(|\uparrow\rangle_i\right)$ state. Therefore, a generic pure state of the overall system cab be written as

$$|\Psi\rangle = \sum_{n=0}^{2^N-1} \left(a_n|0\rangle + b_n|1\rangle\right) \otimes |n\rangle \;. \tag{3.25}$$

Of course, since $\hat{\sigma}_z|0\rangle = |0\rangle$ and $\hat{\sigma}_z|1\rangle = -|1\rangle$, the action on the Hamiltonian on the states of the system is given by

$$\widehat{H}|0\rangle \otimes |n\rangle = \frac{\epsilon_n}{2}|0\rangle \otimes |n\rangle \;, \;\; \widehat{H}|1\rangle \otimes |n\rangle = -\frac{\epsilon_n}{2}|1\rangle \otimes |n\rangle \;, \tag{3.26}$$

therefore, if the state of $S\mathscr{E}$ at $t = 0$ is uncorrelated

$$\left|\Psi(0)\right\rangle = \left(a|0\rangle + b|1\rangle\right) \otimes \sum_{n=0}^{2^N-1} c_n|n\rangle \;, \tag{3.27}$$

upon evolution this state becomes

$$\left|\Psi(t)\right\rangle = a|0\rangle \otimes \left|\mathscr{E}_0(t)\right\rangle + b|1\rangle \otimes \left|\mathscr{E}_1(t)\right\rangle \;, \tag{3.28}$$

with

$$\left|\mathscr{E}_0(t)\right\rangle = \left|\mathscr{E}_1(-t)\right\rangle = \sum_{n=0}^{2^N-1} c_n e^{-it\frac{\epsilon_n}{2\hbar}}|n\rangle \;. \tag{3.29}$$

Since some correlations have emerged between the states $|0\rangle$, $|1\rangle$ and the states $\left|\mathscr{E}_0(t)\right\rangle$, $\left|\mathscr{E}_1(t)\right\rangle$, interference between the states $|0\rangle$ and $|1\rangle$ becomes increasingly quenched as the overlap $\left\langle\mathscr{E}_1(t)\middle|\mathscr{E}_0(t)\right\rangle$ increases (which means that the amount of which-state information that could be recovered from the environment increases). The latter is given by

$$r(t) := \left\langle\mathscr{E}_1(t)\middle|\mathscr{E}_0(t)\right\rangle = \sum_{n=0}^{2^N-1} \left|c_n\right|^2 e^{-\frac{i}{\hbar}\epsilon_n t} \,, \tag{3.30}$$

called *decoherence factor*. Of course, $\left|c_n\right|^2 \leq 1$ and $\sum_{n=0}^{2^N-1}\left|c_n\right|^2 = 1$. This factor appears in the reduced density operator of the system as

$$\widehat{\rho}_S(t) = \mathrm{Tr}_{\mathscr{E}}\left\{\left|\Psi(t)\right\rangle\!\left\langle\Psi(t)\right|\right\} = |a|^2|0\rangle\langle 0| + \left|b\right|^2|1\rangle\langle 1| + ab^* r(t)|0\rangle\langle 1| + a^* b r^*(t)|1\rangle\langle 0| \,, \tag{3.31}$$

therefore the off-diagonal terms vanish as $r(t) \to 0$, as it should be. From this we also see that there is no dissipation, since the diagonal terms of the density operator are not changed by the interaction, namely $\mathrm{Tr}\left\{\widehat{\rho}_S(t)\right\} = 1 \;\forall\, t$. Of course, the specific form of interference damping depends on the state of the environment and the couplings of the interaction, namely c_n and g_i, respectively. However, the coherence factor (3.30) can be interpreted as the sum of 2^N vectors in the complex plane with different phases. At any given time, this can be seen as a two-dimensional random walk with 2^N steps with lengths given by $\left|c_n\right|^2$ following the direction specified by the phase $\epsilon_n t$. It then follows that the average squared length of the coherence factor will be proportional to the average step length, which scales as

$$\left\langle \left|r(t)\right|^2 \right\rangle \propto \left\langle |c|^2 \right\rangle = 2^{-N} \,. \tag{3.32}$$

This shows an exponential decay of the coherence with the size of the environment. Moreover, it can also be shown [183, 187] that for sufficiently large N and several distributions of environmental couplings g_i, the coherence factor $r(t)$ is approximately described by a Gaussian decay with time

$$r(t) \approx e^{-\Gamma^2 t^2} \,, \tag{3.33}$$

where the decay constant Γ depends on the initial state of the environment and the distribution of g_i. However, as long as the number of degrees of freedom of the environment is finite, there will always be a *recurrence time* such that the coherence factor returns to the initial value $r(\tau_{\rm rec}) = r(0) = 1$. Of course, this time strongly depends on the initial state of the environment and the distribution of g_i. For a highly symmetric case in which the state of all environmental systems is given by $\frac{1}{\sqrt{2}}\left(\left|\uparrow\right\rangle_i + \left|\downarrow\right\rangle_i\right)$ and the couplings are all equal, $g_i = g \,, \ \forall\, i$, then

$$r(t) = \cos^N\left(\frac{g t}{\hbar}\right) , \tag{3.34}$$

which is periodic with period $\frac{2\pi\hbar}{g}$ and shows that full "re-coherence" will occur at times $\tau_{\rm rec} = n\frac{\hbar}{g} \,, \ \ n \in \mathbb{N}$. Another highly symmetric scenario is when the environment happens to be in an eigenstate of the Hamiltonian $|n\rangle$, in which case no decoherence occurs at all. Of course, such highly "ordered" scenarios are not realistic. In physically relevant cases, the recurrence time is of Poincaré type, *i.e.* $\tau_{\rm rec} \propto N!$, and it exceeds the lifetime of the Universe for realistic (but finite) environments [183], hence the loss of coherence is irreversible for all practical purposes.

Localisation due to scattering

The spatial localisation of macroscopic systems induced by the scattering of environmental particles is really one of the most compelling reasons that sparked the interest in the quantum-to-classical transition, and it ultimately is the heart of the decoherence research programme. In fact, environmental scattering and the emission of thermal radiation are the dominant effects for the decoherence of macroscopic objects. Through scattering, environmental particles provide continuous monitoring of the position of physical systems, which is therefore the superselected observable. Thus, these interactions delocalise the coherence between spatially separated components of the states of physical systems, leading to decoherence in position space. Of course, the scattering cross section is bigger for larger systems, and hence spatial decoherence is faster for macroscopic systems.

We now consider an example of the above-mentioned scattering process, without providing all the details that can be found in [180]. As usual, consider a system S and an environment (which is now made up of scattering particles) $\mathscr{E}$ that are initially uncorrelated

$$\widehat{\rho}(0) = \widehat{\rho}_S(0) \otimes \widehat{\rho}_{\mathscr{E}}(0) \,. \tag{3.35}$$

The centre of mass of system S is denoted by $\boldsymbol{x}$, so the different states of system S are labelled by the eigenstates of the position operator, $|\boldsymbol{x}\rangle$. A state in superposition of different $|\boldsymbol{x}\rangle$ describes a system S with a centre of mass in superposition of different positions. The scattering model is derived [180] by requiring the following assumptions:

- At initial time, S and $\mathscr{E}$ are uncorrelated (this assumption is considered in practically all decoherence models).
- The centre of mass of S is not perturbed by the scattering interactions (this is a requirement of "macroscopicity" for S).
- The particle distribution is isotropic.
- The scattering interaction is invariant under translations of $S\mathscr{E}$.
- The scattering rate is faster than the characteristic evolution time of S under the self-Hamiltonian $\widehat{H}_S$.

All of these assumptions can be justified in the physical context of interest. The only assumption that can be limiting is the second, which means that the scattering model cannot be applied when the masses of the scattering particles are similar to the mass of S, hence excluding the cases in which the latter is a mesoscopic object.

With these assumptions, it can be shown that, over timescales that are short compared to the internal dynamics of S, the evolution of the off-diagonal matrix elements $\rho_S\left(\boldsymbol{x},\boldsymbol{x}',t\right)$ of the density operator of S is given by

$$\rho_S\left(\boldsymbol{x},\boldsymbol{x}',t\right)=\begin{cases}\rho_S(\boldsymbol{x},\boldsymbol{x}',0)e^{-\Lambda|\boldsymbol{x}-\boldsymbol{x}'|^2 t}\,, & \lambda_0 \gg \left|\boldsymbol{x}-\boldsymbol{x}'\right| \\ \rho_S(\boldsymbol{x},\boldsymbol{x}',0)e^{-\Gamma t}\,, & \lambda_0 \ll \left|\boldsymbol{x}-\boldsymbol{x}'\right|\end{cases}\,, \tag{3.36}$$

where λ_0 is the typical de Broglie wavelength of the scattering particles. This different behaviour depending on λ_0 is easily explained. If the particle wavelengths are small compared to the spatial separation $\Delta x := \left|\boldsymbol{x}-\boldsymbol{x}'\right|$, each scattering process can resolve this separation, therefore the localisation rate will be independent of Δx. Vice versa, if the wavelengths are larger than Δx, a large number of particles is required to acquire significant which-path information, and this number increases as the separation Δx increases. Of course, the short-wavelength limit provides a lower bound for the decoherence rate, since it corresponds to a maximum of which-path information that can be acquired by the environment. Therefore, we focus on the long-wavelength limit, which is incidentally also the case of interest for the scattering of photons. The decoherence time in this case, for a separation Δx, is clearly given by

$$\tau_D\left(\Delta x\right)=\frac{1}{\Lambda\left(\Delta x\right)^2}\,. \tag{3.37}$$

For the scattering of thermal photons at temperature T and assuming system S to be a dielectric sphere of radius R [167], the constant Λ is found to be

$$\Lambda \approx 10^{20}\left(\frac{R}{cm}\right)^6\left(\frac{T}{K}\right)^9 \text{ cm}^{-2}\text{s}^{-1}\,. \tag{3.38}$$

If we consider room temperature ($T \sim 300$ K) photons scattering off a dust grain with $R \sim 10^{-3}$ cm, the decoherence time for a separation $\Delta x \sim 10^{-6}$ cm (three orders of magnitude smaller than the radius) is $\tau_D \sim 10^{-12}$ s. Even if we consider the same object simply immersed in the cosmic microwave background $T = 3K$, the decoherence time for a separation of the same size as S, $\Delta x \sim 10^{-3}$ cm, is $\tau_D \sim 1$ s, which still corresponds to a pretty efficient decoherence process. Of course, if the size of the object is decreased, these decoherence times become increasingly longer. For example, for a large molecule with $R \sim 10^{-6}$ cm and a separation comparable to the size of the object $\Delta x \sim 10^{-6}$ cm, we get $\tau_D = 10^6$ s for photons at room temperature.

These examples show that decoherence is an ubiquitous effect and also that decoherence mechanisms can be very efficient. We also see that decoherence is just a consequence of the application of standard Quantum Mechanics to the unavoidable interactions of physical systems with their environments. Therefore, decoherence is not something distinct from Quantum Theory, it is built-in the theory itself, and cannot be neglected. Only by properly taking into account the effects of decoherence is it possible to realistically describe physical systems.

3.1.5 Master equations

In the formalism described above, to compute the evolution of the density matrix of a physical system S interacting with an environment $\mathscr{E}$, it is necessary to first compute the overall dynamics and then trace out the degrees of freedom of $\mathscr{E}$

$$\widehat{\rho_S}(t) = \mathrm{Tr}_{\mathscr{E}}\left\{U(t)\hat{\rho}_{S\mathscr{E}}(0)U^\dagger(t)\right\}. \tag{3.39}$$

However, for any realistic scenario, this is seldom feasible since determining the exact dynamics of the system-environment combination is typically impossible. Moreover, we are not interested in the evolution of the environment, the only thing we care about is the influence of the environment on the system. For this reason, to study the evolution, it is possible to derive, under some assumptions, some approximate equations called *master equations*. Since the trace operation is non-unitary, these equations will also describe, in general, a non-unitary dynamics. Such master equations allow us to replace (3.39) with

$$\hat{\rho}_S(t) = \hat{V}(t)\left[\hat{\rho}_S(0)\right], \tag{3.40}$$

where $\hat{V}(t)$ is called *dynamical map*. Basically, it is a *superoperator*[5] representing a *channel* from the initial to the final system state. In the following, we restrict our attention to master equations that can be written as first-order differential equations that are local in time, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_S(t) = \hat{\mathfrak{L}}\left[\hat{\rho}_S(t)\right] = -\frac{i}{\hbar}\left[\hat{H}'_S, \hat{\rho}_S(t)\right] + \mathfrak{D}\left[\hat{\rho}_S(t)\right], \tag{3.41}$$

where the first term is the standard (unitary) Liouville-von Neumann part, and the second term represents decoherence, and possibly also dissipation, caused by the environment. The

[5] If $\mathscr{D}(\mathscr{H}_S) \ni \hat{\rho}_S : \mathscr{H}_S \to \mathscr{H}_S$ is a trace-class operator, then $\mathscr{D}\left[\mathscr{D}(\mathscr{H}_S)\right] \ni \hat{V} : \mathscr{D}(\mathscr{H}_S) \to \mathscr{D}(\mathscr{H}_S)$.

only difference in the unitary term is the presence of a shifted Hamiltonian $\widehat{H}'_S$ instead of $\widehat{H}_S$ because, in general, the interaction with an environment will change the energy levels of the unperturbed Hamiltonian. If the environment induces no decoherence, $\mathfrak{D}\left[\widehat{\rho}_S(t)\right] = 0$, then this change in energy levels, called the Lamb shift, will be the only effect induced by the environment.

An important class of master equations, which can be applied to several physical contexts, is the so-called *Born-Markov* master equation. The assumptions on which the derivation of this class of equations is based are the following.

1. *The Born approximation*, which can be applied when the system-environment coupling is sufficiently weak and the environment is large enough so that the change in its density operator can be neglected.

2. *The Markov approximation*, which can be applied when "memory effects" of the environment can be neglected, namely self-correlations within the environment decay faster than the characteristic timescale of the evolution of the system.

Physically, the Born approximation corresponds to a weak-coupling limit between a physical system and a much larger environment, for which the back-action of the system can be neglected. This approximation is therefore reasonable for many cases of interest, in which the changes of the system-environment as a whole are negligibly small compared to the evolution of the system alone. The Markov approximation is similar to the notion of Markovian processes in classical probability theory. A stochastic process is Markovian if the probability of an event is independent of previous events, namely the system has no "memory of its past". A Markovian environment is therefore an environment in which the dynamically formed self-correlations (namely correlations among parts of the environment) decay on a timescale τ_{corr} which is much shorter than the characteristic evolution time of system S, τ_S. This assumption is reasonable if the system-environment coupling is weak and the environment is at a sufficiently high temperature.

A particular class of Markovian master equations, which will be of interest in the following, are the so-called *Lindblad* master equations, or *Lindbladians*. They are derived by requiring that positivity of the density operator be preserved throughout the evolution

$$\left\langle\psi\middle|\widehat{\rho}_S(t)\middle|\psi\right\rangle \geq 0\,, \quad \forall \left\{\left|\psi\right\rangle, t\right\} \in \mathscr{H}_S \times \mathbb{R}^+\,. \tag{3.42}$$

This is a physically motivated requirement, since the diagonal matrix elements of the density operator are typically interpreted as occupation probabilities. It has been shown [188, 189] that the most general master equation satisfying this requirement takes the form[6]

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}_S(t) = -\frac{i}{\hbar}\left[\widehat{H}'_S, \widehat{\rho}_S(t)\right] + \frac{1}{2}\sum_{\alpha,\beta}\gamma_{\alpha\beta}\left\{\left[\widehat{S}_\alpha, \widehat{\rho}_S(t)\widehat{S}^\dagger_\beta\right] + \left[\widehat{S}_\alpha\widehat{\rho}_S(t), \widehat{S}^\dagger_\beta\right]\right\}, \tag{3.43}$$

where the operators $\widehat{S}_\alpha$ are those appearing in the decomposition of the interaction Hamiltonian $\widehat{H}_{\text{int}} = \sum_\alpha \widehat{S}_\alpha \otimes \widehat{E}_\alpha$, and $\gamma_{\alpha\beta}$ are time-independent coefficients that contain the physical

[6] A simple derivation of this equation can be found in chapter IX of [190].

information about the decoherence parameters. It is important to note that this equation is of Markovian type, but not all Markovian equations can be recast in this form. In general, an additional assumption, the rotating-wave approximation, is needed. If a master equation is not a Lindbladian, in general, it is not guaranteed that the positivity condition is satisfied. However, the inability to write a master equation in this form does not necessarily mean that (3.42) is violated. An example is given by the quantum Brownian motion, for which the exact, non-Markovian, master equation can be derived. Being non-Markovian, this equation is, of course, not a Lindbladian, but it can be shown that (3.42) is satisfied [191].

The Lindblad equation can be simplified by diagonalising the matrix $\Gamma = (\gamma_{\alpha\beta})$, which is always possible since this matrix is positive. If $\lambda_\mu \geq 0$ are its eigenvalues, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}_S(t) = -\frac{i}{\hbar}\left[\widehat{H}'_S, \widehat{\rho}_S(t)\right] - \frac{1}{2}\sum_\mu \lambda_\mu \left\{\widehat{L}^\dagger_\mu \widehat{L}_\mu \widehat{\rho}_S(t) + \widehat{\rho}_S(t)\widehat{L}_\mu \widehat{L}^\dagger_\mu - 2\widehat{L}_\mu \widehat{\rho}_S(t)\widehat{L}^\dagger_\mu\right\}, \tag{3.44}$$

where $\widehat{L}_\mu$ are called Lindblad generators. These are not Hermitian, in general, namely they are not necessarily physical observables. When they are Hermitian, the equation is further simplified

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}_S(t) = -\frac{i}{\hbar}\left[\widehat{H}'_S, \widehat{\rho}_S(t)\right] - \frac{1}{2}\sum_\mu \lambda_\mu \left[\widehat{L}_\mu, \left[\widehat{L}_\mu, \widehat{\rho}_S(t)\right]\right]. \tag{3.45}$$

If the Lindblad generators are chosen to be dimensionless, λ_μ have the dimension of inverse time and therefore are to be regarded as the decoherence rates. In fact, all the physics of the decoherence process is hidden in these coefficients.

3.2 Fundamental decoherence from quantum symmetries

The introduction to decoherence of the previous section has set the stage for the presentation of the results of [1, 3]. In [1], we develop a decoherence model for quantum systems by considering the action of deformed symmetries (specifically, deformed time translations) on the evolution of quantum systems, while in [3] we derive some strong constraints for another quantum spacetime-induced decoherence model using neutrino oscillations. These results fall under the umbrella of the studies on decoherence carried out in the context of Quantum Gravity and Gravity in general. In the next subsection, we review this topic [192].

3.2.1 Decoherence in (Quantum) Gravity

As we extensively discussed in the first part of the chapter, interactions with an environment generally lead to decoherence processes for quantum systems. An idea that dates back to Penrose and Diósi is that the gravitational field is essentially an ubiquitous environment, with which every system interacts [193, 194]. Contrary to the environments discussed above, interactions with the gravitational field cannot be shielded, even in principle, simply because every system *necessarily is* in spacetime. Since Gravity is universal, and also because the magnitude of gravitational effects increases with the mass, Diósi and Penrose proposed the idea that Gravity could be the ultimate explanation for the emergence of classical behaviours in the macroscopic regime.

Moreover, the possibility that the interplay between quantum and gravitational effects can generate decoherence of quantum systems has also emerged in the earliest investigations on

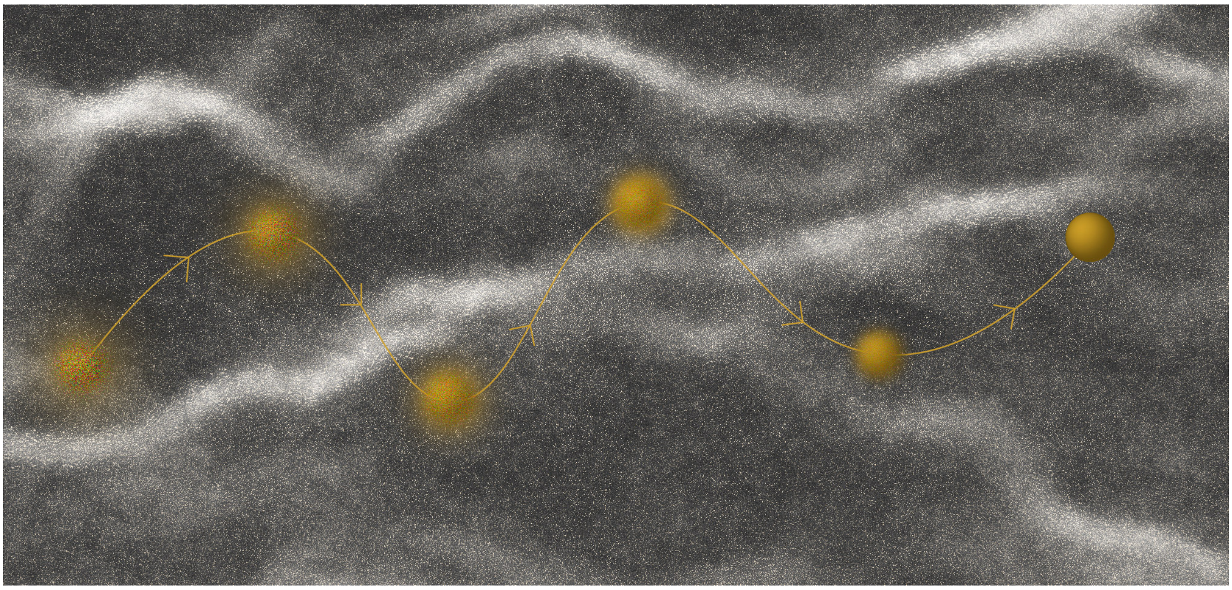

Figure 4: Artistic representation of fundamental decoherence originating from quantum spacetime. The latter is represented as a fuzzy medium in which quantum particles propagate. A particle is initially represented as a fuzzy cloud that, upon evolution (represented by the trajectory), becomes more and more sharp. Here, localisation in space is represented but quantum gravitational effects can cause localisation in other relevant variables, *e.g.* energy and momentum.

Quantum Gravity. The original motivation was provided by the Hawking's proposal [195] that the evaporation of a black hole evolves pure states into mixed states. This is at the heart of the so-called black hole information paradox, which is still today at the centre of a heated debate aimed at establishing whether the evolution of the quantum states of a black hole and the emitted radiation is unitary [196–202].

From these pioneering ideas, the field of gravitationally-induced decoherence research was born, and several Quantum Gravity-inspired framework were also proposed over the years [192]. The key notion is that Gravity could give rise to environmental decoherence processes, if the interactions between the gravitational degrees of freedom (either classical or quantum) and those of the quantum system are taken into account [203–208]. However, there can also be different types of gravitationally-induced decoherence. In fact, an intriguing possibility is that classical or Quantum Gravity effects might cause departures from standard Quantum Theory caused by the sheer properties of spacetime itself, giving rise to decoherence processes even in the absence of interactions with an environment, see fig. 4. This type of decoherence is called fundamental, since it would affect quantum systems independently of their environments. Unlike environmental decoherence processes, for which information on quantum coherence is delocalised in the environment, in the case of fundamental decoherence such information is lost, since there is no environment to (from) which it could have been transferred (be retrieved).

Since a full theory of Quantum Gravity is still missing, any theoretical description of quantum gravitational decoherence is necessarily incomplete, and physical predictions in general depend on assumptions about the behaviour of Gravity at the quantum scale. For this reason, decoherence effects are typically discussed in very different ways, and either the proposals are based on specific models, resulting in model-dependent effects, or quantum-gravitational features are introduced heuristically, based on some general ideas about Gravity at the Planck scale. The model proposed in [1] certainly falls into the first category. In fact, we propose a rigorous derivation of a decoherence process in a full-fledged model of quantum spacetime, namely spacetime non-commutativity and the ensuing deformation of spacetime symmetries. Examples of other models in the literature can be grouped into categories:

- Some authors build their intuition on gravitational decoherence starting from the idea that, independently of the resolution of the Quantum Gravity riddle, it is expected that spacetime as a dynamical variable fluctuates in a stochastic way. These stochastic fluctuations, which are the cause of decoherence in these models, can be implemented in different ways [209–211].

- Since decoherence is a dynamical feature, it is not a surprise that it is possible to derive some decoherence mechanisms by adding some non-trivial features to the flow of time, such as replacing ideal clocks with realistic clocks [212] or taking into account the effects of gravitational time dilation [151].

- Another idea that is shared among some authors is that spacetime, at the quantum level, should manifest a foamy structure. This foaminess, of course, modifies the evolution of quantum systems, leading to decoherence [207, 208].

- Finally, one of the first ideas in Quantum Gravity research is that a quantum structure of spacetime should manifest itself by enforcing a fundamental limit on localisation. Depending on the nature of this minimal length, some decoherence mechanisms can be derived [213, 214].

Regardless of the specific different realisations in these models, the core question consistently seems to be the same: Does gravitational decoherence enforce a fundamental limit on the scales at which quantum features of nature can be observed?

3.2.2 Hopf-algebraic rewriting of Quantum Mechanics

The decoherence model proposed in [1] is derived in the framework of non-commutative spacetime and deformation of spacetime symmetries, introduced in the first chapter chapter 1, which are typically described with the language of Hopf algebras. The mathematical structures arising in this Hopf-algebraic framework have a surprisingly simple interpretation in the context of Quantum Mechanics, as we shall outline briefly in the following.

Given a quantum particle, the physical observables are given by Hermitian operators acting on its Hilbert space $\mathscr{H}$ and, considering an observable $\widehat{O}$ acting on this Hilbert space, its action on multiparticle states, namely states in the tensor product $\mathscr{H}^{\otimes n}$, is given by the so-called *second quantised* operator [215]

$$\mathrm{d}\Gamma\left(\widehat{O}\right) = \mathbb{1}\circ\Pi_0+\widehat{O}\circ\Pi_1+\left(\widehat{O}\otimes\mathbb{1}+\mathbb{1}\otimes\widehat{O}\right)\circ\Pi_2+\left(\widehat{O}\otimes\mathbb{1}\otimes\mathbb{1}+\mathbb{1}\otimes\widehat{O}\otimes\mathbb{1}+\mathbb{1}\otimes\mathbb{1}\otimes\widehat{O}\right)\circ\Pi_3+\dots\,, \tag{3.46}$$

where Π_n is the projector onto $\mathscr{H}^{\otimes n}$. This expression can be easily rewritten by defining[7]

$$\Delta_n := \left(\Delta \otimes \mathbb{1}\right) \circ \Delta_{n-1}\,,\;\; n \geq 2\,,\;\; \Delta_1 \widehat{O} := \Delta \widehat{O} = \widehat{O} \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{O}\,,\;\; \Delta_0 \widehat{O} := \mathbb{1}\,, \tag{3.47}$$

so that

$$\mathrm{d}\Gamma = \sum_{n=0}^{\infty} \left(\Delta_n \widehat{O}\right) \circ \Pi_n\,. \tag{3.48}$$

From this we see that we can interpret the co-product as the action of a single-particle observable on a two-particle state. To make this more clear, we can consider an example. The simplest type of observable is given by the generators of (spacetime) translations, namely the Hamiltonian and spatial momentum. Considering the latter, if particle states are labelled by the eigenvalues of this operator[8], then the action of the i-th component of the momentum operator on a two-particle state is given by

$$\Delta \widehat{P}_i \left|\boldsymbol{k}\right\rangle \otimes \left|\boldsymbol{q}\right\rangle = \left(k_i + q_i\right) \left|\boldsymbol{k}\right\rangle \otimes \left|\boldsymbol{q}\right\rangle\,, \tag{3.49}$$

namely, it simply yields the i-th component of the total momentum of the two-particle state. If the co-product of the momentum is not trivial, then the composition law will be deformed, and in general will be written as

$$\Delta \widehat{P}_i \left|\boldsymbol{k}\right\rangle \otimes \left|\boldsymbol{q}\right\rangle = \left(\boldsymbol{k} \oplus \boldsymbol{q}\right)_i \left|\boldsymbol{k}\right\rangle \otimes \left|\boldsymbol{q}\right\rangle\,. \tag{3.50}$$

Another important action of the observables of a physical system is given by the right action on the dual Hilbert space $\mathscr{H}^*$, which is defined as

$$\left(\widehat{P}_i \left\langle \boldsymbol{q}\right|\right)\left|\boldsymbol{k}\right\rangle = -\left\langle \boldsymbol{q}\right|\widehat{P}_i\left|\boldsymbol{k}\right\rangle \;\Rightarrow\; \widehat{P}_i \left\langle \boldsymbol{k}\right| = -k_i \left\langle \boldsymbol{k}\right|\,. \tag{3.51}$$

This is not to be confused with the conjugate of the left action

$$\left\langle \boldsymbol{k}\right| \widehat{P}_i = \left(\widehat{P}_i \left|\boldsymbol{k}\right\rangle\right)^{\dagger} = k_i \left|\boldsymbol{k}\right\rangle\,. \tag{3.52}$$

The difference is easily understood by considering these actions in the position representation, in which momentum eigestates are represented by plane waves, and the translation generator is the derivative. Therefore

$$\begin{aligned} \widehat{P}_i \left|\boldsymbol{k}\right\rangle &\to -i\hbar \overrightarrow{\frac{\partial}{\partial x^i}} e^{\frac{i}{\hbar}\boldsymbol{x}\cdot\boldsymbol{k}} = k_i\, e^{\frac{i}{\hbar}\boldsymbol{x}\cdot\boldsymbol{k}} \\ \widehat{P}_i \left\langle \boldsymbol{k}\right| &\to -i\hbar \overrightarrow{\frac{\partial}{\partial x^i}} e^{-\frac{i}{\hbar}\boldsymbol{x}\cdot\boldsymbol{k}} = -k_i\, e^{-\frac{i}{\hbar}\boldsymbol{x}\cdot\boldsymbol{k}} \end{aligned} \tag{3.53}$$

while

$$\left\langle \boldsymbol{k}\right| \widehat{P}_i = \left(\widehat{P}_i \left|\boldsymbol{k}\right\rangle\right)^{\dagger} \to e^{-\frac{i}{\hbar}\boldsymbol{x}\cdot\boldsymbol{k}} \left(i\hbar \overleftarrow{\frac{\partial}{\partial x^i}}\right) = k_i\, e^{-\frac{i}{\hbar}\boldsymbol{x}\cdot\boldsymbol{k}}\,, \tag{3.54}$$

[7] For example, $\Delta_3 \widehat{O} = \widehat{O} \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{O} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \widehat{O}$

[8] Particle states can be regarded as the representations of the symmetry group of the underlying spacetime structure. In both the relativistic and non-relativistic regime, these states can then be labelled by the momentum (and the mass, which defines the mass shell hyperboloid or paraboloid, respectively).

were the arrows denote the direction in which the derivates act. Although the right- and left-actions are different, it is possible to use a map to write the first as a left-action. This map is nothing more than the antipode. In fact, by defining $S(\widehat{P}_i) = -\widehat{P}_i$, we get

$$\widehat{P}_i \langle \boldsymbol{k}| = -k_i \langle \boldsymbol{k}| = \langle \boldsymbol{k}| (-k_i) = \langle \boldsymbol{k}| S(\widehat{P}_i) . \tag{3.55}$$

From a physical point of view, the antipode can be seen as defining the opposite momentum. For example, in the position representation $\langle \boldsymbol{x}|\boldsymbol{k}\rangle = e^{\frac{i}{\hbar}\boldsymbol{x}\cdot\boldsymbol{k}}$ while $\langle \boldsymbol{k}|\boldsymbol{x}\rangle = e^{-\frac{i}{\hbar}\boldsymbol{x}\cdot\boldsymbol{k}}$, which is simply a plane wave with opposite momentum. Since the latter, given a particle with a momentum $\boldsymbol{k}$, can also be defined as the momentum that a second particle must have to produce a two-particle state with zero total momentum

$$\boldsymbol{q} \mid \Delta\widehat{P}_i |\boldsymbol{k}\rangle \otimes |\boldsymbol{q}\rangle = 0 , \tag{3.56}$$

in general having a deformed composition law for the momenta means that how the opposite momentum is defined is also deformed.

The last operation that we need to take into account is the action of symmetry generators on operators. In Quantum Mechanics this is given by the *adjoint action*

$$\mathrm{ad}_{\widehat{H}}(\widehat{A}) = -i\left[\widehat{H}, \widehat{A}\right] , \tag{3.57}$$

where, in this example, we have considered the Hamiltonian $\widehat{H}$, so that the adjoint action gives the time evolution of $\widehat{A}$ in the Heisenberg picture. Is there a way to write such an action in terms of the co-algebraic structures defined above? The answer turns out to be positive

$$\mathrm{ad}_{\widehat{H}}(\widehat{A}) = (\mathrm{id}\otimes S)\circ\Delta\widehat{H}\diamond\widehat{A} = \left(\widehat{H}\otimes\mathbb{1} - \mathbb{1}\otimes\widehat{H}\right)\diamond\widehat{A} = -i\left[\widehat{H}, \widehat{A}\right] , \tag{3.58}$$

where id is the identity map $\mathrm{id}(\widehat{O}) = \widehat{O}$ and the diamond map is defined as

$$\left(\widehat{H}\otimes\mathbb{1}\right)\diamond\widehat{A} := -i\widehat{H}\widehat{A} , \quad \left(\mathbb{1}\otimes\widehat{H}\right)\diamond\widehat{A} := -i\widehat{A}\widehat{H} . \tag{3.59}$$

This is the definition of the *quantum adjoint action* [216]. If we have non-trivial coalgebra structures, given a generator of the symmetry algebra $\widehat{G}$, $(\mathrm{id}\otimes S)\Delta\widehat{G}$ in general is a sum of terms of the form $\widehat{G}_1^n\otimes\widehat{G}_2^m$, where $\widehat{G}_i$ are some generators of the symmetry algebra, and n and m are some powers. Therefore, for non-trivial coalgebra structures, we generalise the adjoint action of a generic symmetry generator $\widehat{G}$ on an operator $\widehat{A}$ to[9]

$$\mathrm{ad}_{\widehat{G}}(\widehat{A}) := (\mathrm{id}\otimes S)\circ\Delta\widehat{G}\diamond\widehat{A} , \quad \left(\widehat{G}_1^n\otimes\widehat{G}_2^m\right)\diamond\widehat{A} := (-i)^{n+m}\widehat{G}_1^n\,\widehat{A}\,\widehat{G}_2^m . \tag{3.60}$$

With this adjoint action we can define the dynamical law of quantum systems, namely the Liouville-von Neumann equation, directly from the algebraic and co-algebraic structures of the spacetime symmetry algebra

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}(t) = \frac{1}{\hbar}\mathrm{ad}_{\widehat{H}}\left(\widehat{\rho}(t)\right) = -\frac{i}{\hbar}\left[\widehat{H}, \widehat{\rho}(t)\right] . \tag{3.61}$$

[9] In [216] the quantum adjoint action is defined for the abstract algebra. Here, we consider the representation of this algebra on the Hilbert space of a quantum system. For this reason, the quantum adjoint action has to satisfy some additional properties, such as preserving the hermiticity and positivity of the density operator upon evolution. Therefore, we introduce the powers of $-i$ in the definition of the diamond map. This will become clearer in the next subsection.

This completes the Hopf-algebraic rewriting of Quantum Mechanics, specifically of the properties we are interested in. From this we understand that the algebraic properties of the spacetime symmetry algebra define the structure of Quantum Theory:

- The representations of the algebra define the particle states with definite momentum and mass. In both the relativistic[10] and non-relativistic cases, the spacetime symmetry group can be written as a semidirect product of a non-Abelian part and the group of spacetime translations. Therefore, the representations are obtained with the little group method (cf. chapter A) and are labelled by the eigenvalues of the spatial translations (the momentum) and the mass, which defines the mass shell, $m^2 = E^2 - \boldsymbol{p}\cdot\boldsymbol{p}$ for the Poincaré group and $2mE = \boldsymbol{p}\cdot\boldsymbol{p} + 2mu$ for the Galilei group, where u is the internal energy [67]. These are just the orbits in the $(E,\boldsymbol{p})$ space under the action of the non-Abelian part of the group, namely the points that can be reached from a reference point $(E_0,\boldsymbol{p}_0)$ under the action of rotations and boosts.

- The coalgebra structures define the action on the multiparticle states (co-product), dual Hiblert space states (antipode), and operators (adjoint action).

- The dynamical law is nothing more than the adjoint action of the time translation generator on density operators.

If the symmetry algebra is the standard Galilei (or Poincaré) algebra, then the structure of Quantum Mechanics is recovered. However, with the formal Hopf-algebraic definitions we have given, we can introduce quantum spacetime-inspired deformations in the symmetry algebra and see the effects that such deformations have on quantum systems.

3.2.3 Here comes the decoherence

We are finally ready to discuss the results of [1]. We consider the deformation of spacetime symmetries given by the κ-Poincaré algebra that arises as the symmetry algebra of κ-Minkowski non-commutative spacetime, described in chapter 1. The latter is characterised by a non-commutativity between the spatial coordinates and the time coordinate

$$\left[x^0, x^i\right] = i\, t_P x^i = i\frac{\hbar}{c\kappa} x^i\,, \quad \left[x^i, x^j\right] = 0\,, \tag{3.62}$$

where $c\kappa$ is typically identified with the Planck energy, $c\kappa = E_P$ and $t_P = \ell_P c^{-1}$ is the Planck time. As discussed in chapter 1, the symmetry algebra of this non-commutative spacetime can be written in different bases, which in general predict different physical effects. The basis we choose to analyse here is the so-called *classical basis* [76], which is characterised by an undeformed algebra sector, with all deformations confined in the coalgebra. This is a particularly interesting basis to study in a quantum-mechanical framework, since single-particle

[10] The reader could be worried that considering relativistic systems with the formalism of first quantisation is inconsistent. This framework should be intended to be applied to a physical regime in which particle creation and annihilation are negligible. In such a context, it is justified to consider relativistic systems as described by a first quantised theory. As a matter of fact, this is done in a variety of contexts in the literature [117, 150, 217, 218].

states, defined as representations of the algebra, are the standard ones, and only the action of symmetries on operators and on multi-particle states is deformed.

The only coalgebra structures we will need in the following concern the (spacetime) translation sector, and are given by

$$\begin{aligned} \Delta\widehat{P}_0 &= \widehat{P}_0 \otimes \widehat{\Pi}_0 + \widehat{\Pi}_0^{-1} \otimes \widehat{P}_0 + \frac{1}{\kappa}\widehat{P}_i \widehat{\Pi}_0^{-1} \otimes \widehat{P}^i\,, \\ \Delta\widehat{P}_i &= \widehat{P}_i \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{P}_i\,, \\ S\big(\widehat{P}_0\big) &= -\widehat{P}_0 + \frac{1}{\kappa}\widehat{\boldsymbol{P}}^2\,\widehat{\Pi}_0^{-1}\,,\ \ S\big(\widehat{P}_i\big) = -\widehat{P}_i\,\widehat{\Pi}_0^{-1}\,, \end{aligned} \tag{3.63}$$

where

$$\begin{aligned} \widehat{\Pi}_0 &= \frac{1}{\kappa}\widehat{P}_0 + \sqrt{1 - \frac{1}{\kappa^2}\widehat{P}^\mu \widehat{P}_\mu}\,, \\ \widehat{\Pi}_0^{-1} &= \frac{\sqrt{1-\frac{1}{\kappa^2}\widehat{P}^\mu\widehat{P}_\mu} - \frac{1}{\kappa}\widehat{P}_0}{1-\frac{1}{\kappa^2}\widehat{\boldsymbol{P}}^2}\,, \\ \widehat{P}^\mu\widehat{P}_\mu &= -\widehat{P}_0^2 + \widehat{\boldsymbol{P}}^2 = -\widehat{P}_0^2 + \widehat{P}_i\widehat{P}^i\,, \end{aligned} \tag{3.64}$$

and P_μ have the physical dimensions of momentum. In the following we consider the effects of these deformations at first order in the Quantum Gravity scale κ^{-1}. Therefore, the deformed structures will be given by

$$\begin{aligned} \Delta\widehat{H} \quad &= \widehat{H} \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{H} + \tfrac{1}{M_P}\widehat{P}_i \otimes \widehat{P}^i\,, \\ S\big(\widehat{P}_0\big) \quad &= -\widehat{P}_0 + \tfrac{1}{M_P}\widehat{\boldsymbol{P}}^2\,, \\ \Delta\widehat{P}_i \quad &= \widehat{P}_i \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{P}_i + \tfrac{1}{M_P c^2}\widehat{P}_i \otimes \widehat{P}_0\,, \\ S\big(\widehat{P}_i\big) \quad &= -\widehat{P}_i + \tfrac{1}{M_P c^2}\widehat{P}_i\widehat{P}_0\,, \end{aligned} \tag{3.65}$$

where $M_P = c^{-1}\kappa$ is the Planck mass and $\widehat{H} := c\widehat{P}_0$ is the generator of time translations with respect to $t = c^{-1}x^0$.

With this deformed coalgebra, we can immediately read off the deformed composition laws for momenta

$$\begin{aligned} \Delta\widehat{P}_i\,\big|\boldsymbol{k}\big\rangle \otimes \big|\boldsymbol{q}\big\rangle = \big(\boldsymbol{k}\oplus\boldsymbol{q}\big)_i\,\big|\boldsymbol{k}\big\rangle \otimes \big|\boldsymbol{q}\big\rangle \;\Rightarrow\; \big(\boldsymbol{k}\oplus\boldsymbol{q}\big)_i = k_i + q_i + \frac{1}{M_P c^2}k_i E(\boldsymbol{q})\,, \\ \Delta\widehat{H}\,\big|\boldsymbol{k}\big\rangle \otimes \big|\boldsymbol{q}\big\rangle = \big(E(\boldsymbol{k}) \oplus E(\boldsymbol{q})\big)\,\big|\boldsymbol{k}\big\rangle \otimes \big|\boldsymbol{q}\big\rangle \;\Rightarrow\; E(\boldsymbol{k}) \oplus E(\boldsymbol{q}) = E(\boldsymbol{k}) + E(\boldsymbol{q}) + \frac{1}{M_P}k_i q^i\,, \end{aligned} \tag{3.66}$$

where, for on-shell free particles with mass m, $E(\boldsymbol{q}) = \sqrt{m^2+\boldsymbol{q}^2}$. Moreover, by making use of (3.60) we can derive the deformed time evolution for the density operators that describe the states of quantum particles. Since we want to preserve a probabilistic interpretation for the matrix elements of density operators, we define this evolution in such a way as to preserve the hermiticity and the positivity:

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}(t) := \frac{1}{2\hbar}\left\{\mathrm{ad}_{\widehat{H}}\big(\widehat{\rho}(t)\big) + \Big[\mathrm{ad}_{\widehat{H}}\big(\widehat{\rho}(t)\big)\Big]^\dagger\right\}. \tag{3.67}$$

In fact, with the definition given in (3.60), we show that this equation is a Lindbladian. The basic ingredient of this equation is $\mathrm{ad}_{\widehat{H}}\big(\widehat{\rho}(t)\big)$ which, at first order in M_P^{-1}, reads

$$\begin{aligned}\mathrm{ad}_{\widehat{H}}\big(\widehat{\rho}(t)\big) &= (\mathrm{id}\otimes S)\Delta\widehat{H}\diamond\widehat{\rho}(t) = (\mathrm{id}\otimes S)\left(\widehat{H}\otimes\mathbb{1}+\mathbb{1}\otimes\widehat{H}+\frac{1}{M_P}\widehat{P}_i\otimes\widehat{P}^i\right)\diamond\widehat{\rho}(t)\simeq\\ &\simeq -i\left[\widehat{H},\widehat{\rho}(t)\right]-\widehat{\rho}(t)\widehat{\boldsymbol{P}}^2-\frac{1}{M_P}\widehat{P}_i\widehat{\rho}(t)\widehat{P}^i\,,\end{aligned}\tag{3.68}$$

thus the evolution equation (3.67) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}(t) = -\frac{i}{\hbar}\left[\widehat{H},\widehat{\rho}(t)\right]-\frac{1}{2\hbar M_P}\left[\widehat{P}_i,\left[\widehat{P}^i,\widehat{\rho}(t)\right]\right]\,,\tag{3.69}$$

which is a Lindblad equation (3.45) with Lindblad generators given by the spatial components of momentum, $\widehat{L}_i=\widehat{P}_i$, and a matrix of decoherence coefficients proportional to the identity, $\lambda_i = (2\hbar M_P)^{-1}\ \forall\, i$. Hence, this equation, interpreted as the evolution equation of the density operator of a quantum system living in a non-commutative spacetime whose symmetries are those of a quantum deformation of the Poincaré algebra, describes a decoherence mechanism in the momentum basis, which is therefore the superselected basis, see section 3.1.3. While for environmental decoherence the superselection rules depend on the form of the interaction with the environment, for decoherence induced by deformed symmetries the superselected observables depend on the deformations. Since the deformation of the time translation generator in (3.65) is given by the spatial components of the momentum, these appear as the superselected observables.

We can easily solve (3.69) for a free particle. If $\left|m,\boldsymbol{p}\right\rangle$ are the basis states of the Hilbert space of the particle, where m is the mass, (3.69) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{pq}(t) = \left[-\frac{i}{\hbar}\Big(E(\boldsymbol{p})-E(\boldsymbol{q})\Big)-\frac{1}{2\hbar M_P}(\boldsymbol{p}-\boldsymbol{q})^2\right]\rho_{pq}(t)\,,\tag{3.70}$$

where $\rho_{pq}(t):=\left\langle m,\boldsymbol{p}\middle|\widehat{\rho}(t)\middle|m,\boldsymbol{q}\right\rangle$. The solution therefore is simply given by

$$\rho_{pq}(t) = \rho_{pq}(0)\exp\left[-\frac{i}{\hbar}t\Big(E(\boldsymbol{p})-E(\boldsymbol{q})\Big)-\frac{t}{2\hbar M_P}(\boldsymbol{p}-\boldsymbol{q})^2\right]\,.\tag{3.71}$$

From this solution we see that the diagonal elements are conserved, while the off-diagonal terms are exponentially damped, with a (momentum-dependent) decoherence time given by

$$\tau_D\big(\boldsymbol{p},\boldsymbol{q}\big) = \frac{2\hbar M_P}{\big(\boldsymbol{p}-\boldsymbol{q}\big)^2}\,.\tag{3.72}$$

The ensuing decoherence rate is suppressed by the quantum deformation scale M_P and is amplified by increasing values of $\boldsymbol{p}-\boldsymbol{q}$, in such a way that the decoherence process becomes increasingly more efficient as the energies involved approach the Planck scale.

This exponential damping effectively realises a localisation in momentum, hence confirming that the spatial components of momentum are the superselected observables, and it also results in a loss of purity over time, as can be seen by studying the time evolution of the linear entropy

$$S(t) = 1-\mathrm{Tr}\left\{\widehat{\rho}^2\right\}\,.\tag{3.73}$$

The time derivative of this quantity is simply

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= -2\,\mathrm{Tr}\left\{\widehat{\rho}\,\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}(t)\right\} = \frac{1}{\hbar}\,\mathrm{Tr}\left\{2i\left[\widehat{H},\widehat{\rho}(t)\right] + \frac{1}{M_P}\widehat{\rho}(t)\left[\widehat{P}_i,\left[\widehat{P}^i,\widehat{\rho}(t)\right]\right]\right\} = \\ &= \frac{1}{\hbar M_P}\,\mathrm{Tr}\left\{\widehat{\rho}(t)\left[\widehat{P}_i,\left[\widehat{P}^i,\widehat{\rho}(t)\right]\right]\right\} = \frac{1}{\hbar M_P}\,\mathrm{Tr}\left\{\sum_i \widehat{O}_i(t)\widehat{O}_i^\dagger(t)\right\} \geq 0\,,\end{aligned} \tag{3.74}$$

where we used $\mathrm{Tr}\left\{\widehat{\rho}(t)\left[\widehat{H},\widehat{\rho}(t)\right]\right\} = 0$ and $\widehat{O}_i(t) := \left[\widehat{\rho}(t),\widehat{P}_i\right]$. Therefore, we see that (3.69) indeed leads to an increase in the entropy (or a decrease in the purity) of the system. Using (3.71) we can also derive the asymptotic behaviour

$$S(t) = 1 - \int \mathrm{d}^3p\,\mathrm{d}^3q\left|\rho_{pq}(0)\right|^2 e^{-\frac{t}{\kappa}(\boldsymbol{p}-\boldsymbol{q})^2} \xrightarrow{t\to\infty} 1 - \int \mathrm{d}^3p\left|\rho_{pp}(0)\right|^2 . \tag{3.75}$$

Of course, the increase in the entropy stops when the off-diagonal terms are washed out, making the density operator diagonal in the momentum basis, so that $\widehat{O}_i = \left[\widehat{\rho}(t),\widehat{P}_i\right] = 0$ and $\frac{\mathrm{d}S}{\mathrm{d}t} = 0$.

3.2.4 Galilean limit

In addition to the result discussed above, we can also derive the effects on the evolution for non-relativistic particles by considering the Galilean limit of (3.63). The non-relativistic version of this algebra is the κ-Galilei algebra [219]. This non-relativistic limit is achieved by explicitly writing off the units of c in the algebra, and taking the limit $c \to \infty$. We have[11]

$$\widehat{P}_0 = \frac{\widehat{H}}{c}\,,\ \kappa = c\,M_P\,. \tag{3.76}$$

With these definitions, (3.63) reads

$$\begin{aligned}\Delta\widehat{H}_0 &= \widehat{H}_0\otimes\widehat{\Pi}_0 + \widehat{\Pi}_0^{-1}\otimes\widehat{H}_0 + \frac{1}{M_P}\widehat{P}_i\widehat{\Pi}_0^{-1}\otimes\widehat{P}^i\,,\\ \Delta\widehat{P}_i &= \widehat{P}_i\otimes\mathbb{1} + \mathbb{1}\otimes\widehat{P}_i\,,\\ S\big(\widehat{H}_0\big) &= -\widehat{H}_0 + \frac{1}{M_P}\widehat{\boldsymbol{P}}^2\,\widehat{\Pi}_0^{-1}\,,\ S\big(\widehat{P}_i\big) = -\widehat{P}_i\,\widehat{\Pi}_0^{-1}\,,\end{aligned} \tag{3.77}$$

with

$$\begin{aligned}\widehat{\Pi}_0 &= \frac{1}{M_P c^2}\widehat{H}_0 + \sqrt{1 - \frac{1}{c^4 M_P^2}\left(-\widehat{H}^2 + c^2\widehat{P}_i\widehat{P}^i\right)}\,,\\ \widehat{\Pi}_0^{-1} &= \frac{\sqrt{1-\frac{1}{c^4M_P^2}\left(-\widehat{H}^2 + c^2\widehat{P}_i\widehat{P}^i\right)} - \frac{1}{M_Pc^2}\widehat{H}_0}{1 - \frac{1}{c^2M_P^2}\widehat{P}_i\widehat{P}^i}\,.\end{aligned} \tag{3.78}$$

[11] For the complete algebra, we would also have $\widehat{N}_i = c\,\widehat{B}_i$ for the boosts generators.

Therefore, $c \to \infty$ yields $\widehat{\Pi}_0 \to \mathbb{1}$, $\widehat{\Pi}_0^{-1} \to \mathbb{1}$, hence

$$\begin{aligned} \Delta \widehat{H}_0 &= \widehat{H}_0 \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{H}_0 + \frac{1}{M_P} \widehat{P}_i \otimes \widehat{P}^i \,, \\ \Delta \widehat{P}_i &= \widehat{P}_i \otimes \mathbb{1} + \mathbb{1} \otimes \widehat{P}_i \,, \\ S\big(\widehat{H}_0\big) &= -\widehat{H}_0 + \frac{\widehat{\boldsymbol{P}}^2}{M_P} \,, \;\; S\big(\widehat{P}_i\big) = -\widehat{P}_i \,, \end{aligned} \tag{3.79}$$

The composition laws for momenta thus become

$$\begin{aligned} &\Delta \widehat{P}_i \left|\boldsymbol{k}\right\rangle \otimes \left|\boldsymbol{q}\right\rangle = \left(\boldsymbol{k} \oplus \boldsymbol{q}\right)_i \left|\boldsymbol{k}\right\rangle \otimes \left|\boldsymbol{q}\right\rangle \;\Rightarrow\; \left(\boldsymbol{k} \oplus \boldsymbol{q}\right)_i = k_i + q_i \,, \\ &\Delta \widehat{H} \left|\boldsymbol{k}\right\rangle \otimes \left|\boldsymbol{q}\right\rangle = \left(E(\boldsymbol{k}) \oplus E(\boldsymbol{q})\right) \left|\boldsymbol{k}\right\rangle \otimes \left|\boldsymbol{q}\right\rangle \;\Rightarrow\; E(\boldsymbol{k}) \oplus E(\boldsymbol{q}) = E(\boldsymbol{k}) + E(\boldsymbol{q}) + \frac{1}{M_P} k_i q^i \,, \end{aligned} \tag{3.80}$$

where now $E(\boldsymbol{p}) := \frac{\boldsymbol{p}^2}{2m}$ for free particles. It is also clear that the deformed equation (3.69) remains the same in this non-relativistic limit. Therefore this is the evolution equation for quantum particles both in the relativistic (at first order in M_P^{-1}) and in the non-relativistic cases.

Even though the result is the same in the two cases, in the non-relativistic limit we can derive another interesting result. In fact, the decoherence time (3.72) can be used to derive a fundamental constraint on the mass of quantum particles. To do so, we observe that if a system is in a state with a given energy spread δE, the state is characterised by a lifetime [220]

$$\tau_c \gtrsim \frac{\hbar}{\delta E} \,. \tag{3.81}$$

Decoherence cannot be observed if the decoherence time is longer than the lifetime of the state of the system. Thus, for decoherence to be observable, the decoherence time and the lifetime must satisfy the condition $\tau_D \leq \tau_c$. Conversely, if we do not observe decoherence in a given quantum system prepared in a state with energy spread δE, we have the opposite bound, $\tau_D \geq \tau_c \gtrsim \hbar\big(\delta E\big)^{-1}$, namely,

$$\frac{\left(\delta \boldsymbol{p}\right)^2}{\delta E} \lesssim 2 M_P \,. \tag{3.82}$$

For quantum states whose wave function is localised in spatial momentum, $\frac{|\langle \boldsymbol{p} \rangle|}{|\delta \boldsymbol{p}|} \gg 1$ (from which $\frac{\delta \boldsymbol{p}^2}{(\delta \boldsymbol{p})^2} \sim 1$), we have from (3.82) that

$$m \lesssim M_P \,. \tag{3.83}$$

This bound sets a limit on the mass of quantum systems. Specifically, it is not possible to create highly delocalised states in position for quantum systems with a mass greater than the Planck mass M_P. This can be regarded as a more rigorous derivation of an expected feature of quantum-gravitational physics, namely that the Planck mass constitutes a breakdown scale of Quantum Mechanics, as is shown from heuristic arguments, for instance, in [52, 214]. Notice that had the momentum dependence in (3.72) been different, e.g. linear, we would not have been able to derive such an absolute bound, which is a genuine implication of quantum spacetime properties.

Before closing this section, let us remark that the limit (3.83) should be taken seriously only for elementary quantum systems. For composite systems, it has been suggested that the effects of symmetry deformation might be suppressed by some power of the number of constituents [221–223].

3.2.5 The emergence of the arrow of Time?

It is well known that decoherence is related to the problem of the emergence of a time reversal asymmetry [224–226]. The problem of the emergence of an "arrow of Time" is deeply connected to the interpretation of Quantum Mechanics, the problem of the wave function collapse, the measurement problem, and other somewhat philosophical questions [224]. However, we will not discuss these issues here and will only make a quick comment on (3.69). The ideas presented below are not part of [1] because they were sparked after the publication of the article as a consequence of a question raised by Claus Kiefer in the review process. We thank Claus Kiefer for his question that led to the discussion presented in the following.

Non-unitary evolution typically breaks the time reversal symmetry of the Liouville-von Neumann equation. If $\widehat{\mathbb{T}}$ is the anti-unitary time reversal operator and $\widehat{\rho}(t)$ is the solution to the Liouville-von Neumann equation, it is easy to see that the time reversal transformation changes this density operator as

$$\widehat{\mathbb{T}}\widehat{\rho}(t)\widehat{\mathbb{T}}^{-1} = \widehat{\mathbb{T}}e^{-\frac{i}{\hbar}\widehat{H}t}\widehat{\rho}(0)e^{\frac{i}{\hbar}\widehat{H}t}\widehat{\mathbb{T}}^{-1} = e^{\frac{i}{\hbar}\widehat{H}t}\widehat{\rho}(0)e^{-\frac{i}{\hbar}\widehat{H}t} = \widehat{\rho}(-t)\,, \tag{3.84}$$

therefore applying $\widehat{\mathbb{T}}$ to the Liouville-von Neumann equation and then changing $t \to -t$ leaves the equation invariant (assuming that $\widehat{H}$ is invariant under time reversal)

$$\widehat{\mathbb{T}}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}(t)\widehat{\mathbb{T}}^{-1} = -\frac{i}{\hbar}\widehat{\mathbb{T}}\left[\widehat{H},\widehat{\rho}(t)\right]\widehat{\mathbb{T}}^{-1} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}(-t) = \frac{i}{\hbar}\left[\widehat{H},\widehat{\rho}(-t)\right] \xrightarrow{t\to -t} \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}(t) = -\frac{i}{\hbar}\left[\widehat{H},\widehat{\rho}(t)\right]. \tag{3.85}$$

Of course, this property does not hold, in general, for a non-unitary evolution equation like the Lindbladians (3.45). In particular, this is the case for (3.69), as is immediately shown

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}(-t) = \widehat{\mathbb{T}}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}(t)\widehat{\mathbb{T}}^{-1} = \frac{i}{\hbar}\left[\widehat{H},\widehat{\rho}(-t)\right] - \frac{1}{2\hbar M_P}\left[\widehat{P}_i,\left[\widehat{P}^i,\widehat{\rho}(-t)\right]\right], \tag{3.86}$$

therefore, by mapping $t \to -t$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}(t) = -\frac{i}{\hbar}\left[\widehat{H},\widehat{\rho}(t)\right] + \frac{1}{2\hbar M_P}\left[\widehat{P}_i,\left[\widehat{P}^i,\widehat{\rho}(t)\right]\right], \tag{3.87}$$

which is not the same equation as (3.69). Of course, the solution to this equation is simply given by (3.71) with $t \to -t$ in the non-unitary part of the exponent. This leads to a non-physical scenario in which, for example, the purity of the density operator exceeds 1 as time grows. This is to be expected since equation (3.69) is not symmetric under time-reversal and its solution contains an exponential damping factor that yields an exponential growth when time is reversed.

What is the physical origin of this time asymmetry? In environmental decoherence this can be traced back to the neglecting of the environmental degrees of freedom: The evolution of the system-environment combination is perfectly unitary, and if we were able to have access to all environmental degrees of freedom, we would see the evolution as symmetric under time reversal. Since this is impossible for all practical purposes, an effective time asymmetry emerges in the dynamics of the system alone. In our case, however, there is no environment, and the evolution is therefore fundamentally non-unitary: The emergence of the arrow of Time in (3.69) seems to be fundamental and not a consequence of the impossibility of controlling the degrees of freedom of some environment. This time asymmetry can be explained once again with deformed symmetries. In fact, in the standard Liouville-von Neumann case, the time-reversed evolution equation is generated by $-\widehat{H}$, which we can rewrite as $S(\widehat{H})$. By making the ansatz that this operator generates the time-reversed evolution also in the deformed case, we define

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}(-t) := \frac{1}{2\hbar}\left\{\mathrm{ad}_{S(\widehat{H})}\left(\widehat{\rho}(-t)\right) + \left[\mathrm{ad}_{S(\widehat{H})}\left(\widehat{\rho}(-t)\right)\right]^{\dagger}\right\}. \tag{3.88}$$

This definition yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\rho}(-t) = \frac{i}{\hbar}\left[\widehat{H}, \widehat{\rho}(-t)\right] - \frac{1}{2\hbar M_P}\left[\widehat{P}_i, \left[\widehat{P}^i, \widehat{\rho}(-t)\right]\right], \tag{3.89}$$

which is exactly (3.87). From this we understand that the reason for the emergence of the time-reversal asymmetry is simply $S(\widehat{H}) \neq -\widehat{H}$. Therefore, in this model, Time asymmetry originates, at the fundamental level, from a deformation of spacetime symmetries.

Of course, the model presented in this chapter is derived within a very specific framework, non-commutative spacetime, which should not be intended as a full theory of Quantum Gravity, rather it can be regarded as some effective limit of such a theory. Therefore, here we are not arguing that we derived a definite explanation for the emergence of the tyme-reversal asymmetry. The results presented above are just to show that a full theory of Quantum Gravity, among other things, could also yield a definitive answer for the foundational long-standing problem of the emergence of the arrow of Time in the physical world. As a matter of fact, a simple model, such as the one presented in this chapter, already hints towards this direction.

3.3 Constraints on quantum spacetime-induced decoherence from neutrino oscillations

In the previous section, we presented a new model of quantum spacetime-induced decoherence originating from a deformation of spacetime symmetries. In general, propagation in a quantum spacetime can affect the pattern of neutrino oscillations. Specifically, this may be ascribed to deformations of the neutrino dispersion relation [227], minimal length and generalised uncertainty principle scenarios [228, 229], CPT violations [208, 230, 231], Lorentz invariance violations [232, 233] or generalised quantum-mechanical evolution equations leading to decoherence mechanisms and collapse models [234–236]. Some of these models cause deviations from the standard neutrino oscillation pattern, while others [230, 232, 233] imply a damping of the total neutrino fluxes (the two possibilities are discussed and compared in [237]).

In this section, we perform a phenomenological analysis of Quantum Gravity-induced decoherence models by investigating their effects on neutrino oscillations, following [3]. We investigate how neutrino oscillations are affected by fundamental decoherence, a generic feature emerging in Quantum Gravity research [1, 192, 209, 211–213, 238]. To account for decoherence, the usual Schrödinger equation describing propagation of neutrino wavepackets is replaced by a general Lindblad equation [189]. We fully develop the relevant formalism and show that a Lindblad-type evolution introduces an exponential damping factor into the oscillation probability that quenches the oscillations, without altering the neutrino fluxes. The functional dependence of the damping factor on the neutrino energy characterising different neutrino experiments and on neutrino masses depends on the specific decoherence model. We compute it explicitly for a number of quantum spacetime-induced decoherence mechanisms originating from different sources: non-commutative spacetime [1], stochastic fluctuations of a minimal length [213], stochastic fluctuations of metric perturbations around Minkowski spacetime [211], and the interaction with a thermal background of gravitons [206]. We discuss the relevance of the resulting oscillation damping for measurements exploiting reactor, solar, atmospheric, and astrophysical neutrino.

Because the experiment baseline generally enters linearly in the oscillation damping factor, one could naively assume that long-baseline experiments might provide stronger constraints on the effect. However, for very long baselines neutrino wavepackets decohere already in the standard oscillation scenario because of the different propagation velocities of the mass eigenstates. Therefore, oscillations are washed out and flavour eigenstates decohere into mass eigenstates [43, 218, 239, 240], thus rendering the fundamental decoherence effect irrelevant.

We find that the most sensitive probes for fundamental decoherence are reactor neutrinos and atmospheric neutrinos. Their sensitivity to decoherence depends on the dependence of the effect on neutrino energies and masses. Among the models we considered, we find that the only one that can be significantly constrained using these observations is the one where decoherence is induced by metric fluctuations [211]. For this model, we establish stringent constraints on the Quantum Gravity scale governing the strength of the effect. Even though this is the only model that can be meaningfully constrained using neutrino oscillations, for completeness, we discuss the dependence on the relevant experimental parameters of all the models we considered, to show in detail why neutrino oscillations do not offer a good phenomenological window for them.

3.3.1 Neutrino oscillation in the standard scenario

Neutrino oscillations are commonly described within the plane wave approximation [218], which requires specific assumptions concerning the conditions at production (equal energy, equal momenta, equal velocity, or energy-momentum conservation). This is reasonable in the standard neutrino oscillation picture, and all possible assumptions produce the correct oscillation probability. However, it is not suited for decoherence analyses since, typically, the decoherence terms in the evolution break the equivalence of the assumptions, resulting in ambiguous results. For this reason, in the following, we describe neutrino evolution within the wave packet formalism. We start by briefly reviewing the derivation in the standard scenario following [218], and then generalise the analysis by introducing decoherence effects. We remark that a fully consistent treatment of neutrino oscillations should require a relativistic quantum field theory framework. However, the correct (standard) oscillation probability can also be derived with a first quantisation approach, see, *e.g.*, [218]. It is then conceivable to discuss possible decoherence effects affecting the standard oscillation pattern within the same framework, as extensively done in the literature [208, 227, 229–231, 234, 240–243].

The wave packet corresponding to a generic neutrino flavour state $\left|\nu_\gamma\right\rangle$ can be written as a superposition of mass states wave packets $|\psi_i\rangle \otimes |\nu_i\rangle$ as

$$\left|\nu_\gamma\right\rangle = \sum_i U^*_{\gamma i} |\psi_i\rangle \otimes |\nu_i\rangle = \sum_i U^*_{\gamma i} \int \mathrm{d}^3 p\, \psi_i(\boldsymbol{p}) |\boldsymbol{p}\rangle \otimes |\nu_i\rangle \,, \tag{3.90}$$

where U is the neutrino mixing matrix and $|\boldsymbol{p}\rangle$ are momentum eigenstates. If the neutrino state is described by a density operator $\rho(t)$ at time t, such that $\rho(0) = \left|\nu_\beta\right\rangle\left\langle\nu_\beta\right|$, the transition probability from the initial flavour state β to a different flavour state α within time t is given by

$$P(\beta \to \alpha; t) = \mathrm{Tr}\left\{\rho(t) \left|\nu_\alpha\right\rangle\left\langle\nu_\alpha\right|\right\} . \tag{3.91}$$

Taking into account the time evolution of the density matrix

$$\begin{aligned}
\rho(t) &= e^{-\frac{i}{\hbar}Ht}\left[\left|\nu_\beta\right\rangle\left\langle\nu_\beta\right|\right] = \\
&= \sum_{i,j} U^*_{\beta i} U_{\beta j} \int \mathrm{d}^3 p\, \mathrm{d}^3 q\, \psi_i(\boldsymbol{p}) \psi^*_j(\boldsymbol{q}) e^{-\frac{i}{\hbar}\left[E_i(\boldsymbol{p}) - E_j(\boldsymbol{q})\right]t} |\boldsymbol{p}\rangle\langle\boldsymbol{q}| \otimes \left|\nu_i\right\rangle\left\langle\nu_j\right|
\end{aligned} \tag{3.92}$$

and decomposing the flavour states as in (3.90),

$$\left|\nu_\alpha\right\rangle\left\langle\nu_\alpha\right| = \sum_{k,l} U^*_{\alpha k} U_{\alpha l} |\phi_k\rangle\langle\phi_l| \otimes |\nu_k\rangle\langle\nu_l| \,, \tag{3.93}$$

the transition probability at time t reads

$$P(\beta \to \alpha; t) = \sum_{i,j} U^*_{\beta i} U_{\beta j} U^*_{\alpha j} U_{\alpha i} \int \mathrm{d}^3 p\, \mathrm{d}^3 q\, \psi_i(\boldsymbol{p}) \psi^*_j(\boldsymbol{q}) \phi_j(\boldsymbol{q}) \phi^*_i(\boldsymbol{p}) e^{-\frac{i}{\hbar}\left[E_i(\boldsymbol{p}) - E_j(\boldsymbol{q})\right]t} , \tag{3.94}$$

where we used the relation

$$\mathrm{Tr}\left\{\left(|\boldsymbol{p}\rangle\langle\boldsymbol{q}| \otimes \left|\nu_i\right\rangle\left\langle\nu_j\right|\right)\left(|\phi_k\rangle\langle\phi_l| \otimes |\nu_k\rangle\langle\nu_l|\right)\right\} = \phi_k(\boldsymbol{q}) \phi^*_l(\boldsymbol{p}) \delta_{kj} \delta_{li} \,. \tag{3.95}$$

Note that the plane wave description can be recovered from (3.94) by setting $\phi_i(\boldsymbol{p}) = e^{\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{x}}$, $\psi_i(\boldsymbol{p}) = \delta^{(3)}(\boldsymbol{p} - \boldsymbol{p}_i)$ and performing one of the standard space-to-time conversion techniques [218].

Since in any neutrino oscillation experiment the propagation distance is much larger than the wave packet size, the problem is effectively one-dimensional, and only the component of momenta in the direction between the source and the detector is relevant. If the wave packets of the produced and detected neutrinos are peaked, respectively, in $x = 0$ and $x = L$, the corresponding momentum distribution functions are $\psi_i(p) = f_i^{(S)}(p - p_i)$ and $\phi_i(p) =$

$f_i^{(D)}(p-p_i')e^{-\frac{i}{\hbar}pL}$. Assuming that the momentum distribution functions are sharply peaked around the mean momenta p_i (at emission) and p_i' (at detection), one can expand the energies $E_{i,j}(p)$ about the mean momenta at first order:

$$E_{i,j}(p) \simeq E_{i,j}(p_{i,j}) + (p - p_{i,j})v_{g_{i,j}}\,, \tag{3.96}$$

where $v_{g_{i,j}} = p_{i,j}c^2/E_{i,j}(p_{i,j})$ is the group velocity of the wave packet. Then the transition probability (3.94) reads

$$\begin{aligned} P(\beta\to\alpha;t) = \sum_{i,j} U^*_{\beta i}U_{\beta j}U^*_{\alpha j}U_{\alpha i}\, e^{-\frac{i}{\hbar}\Delta E_{ij}t+\frac{i}{\hbar}\Delta p_{ij}L}\int \mathrm{d}p\,\mathrm{d}q\, f_i^{(S)}(p)f_j^{(S)*}(q) \\ \cdot f_j^{(D)}(q+\delta_j)e^{-\frac{i}{\hbar}qL}f_i^{(D)*}(p+\delta_i)e^{\frac{i}{\hbar}pL}e^{-\frac{i}{\hbar}\left[pv_{g_i}-qv_{g_j}\right]t}\,, \end{aligned} \tag{3.97}$$

where we further shifted the integration variables $p - p_i \to p$ and $q - p_j \to q$ and introduced the notation $\delta_i = p_i - p_i'$, $\Delta E_{ij} = E_i(p_i) - E_j(p_j)$ and $\Delta p_{ij} = p_i - p_j$.

Because the emission and arrival times of neutrinos are not measured in typical oscillation experiments, the transition probability (3.97) is averaged over time [218] and a normalisation factor is introduced so that the probability is normalised, $\sum_{\alpha,\beta}P(\beta\to\alpha)=1$:

$$\begin{aligned} P(\beta\to\alpha) = \mathcal{N}\sum_{i,j} U^*_{\beta i}U_{\beta j}U^*_{\alpha j}U_{\alpha i}\, e^{\frac{i}{\hbar}\Delta p_{ij}L}\int \mathrm{d}t\int \mathrm{d}p\,\mathrm{d}q\, f_i^{(S)}(p)f_j^{(S)*}(q) \\ \cdot f_j^{(D)}(q+\delta_j)f_i^{(D)*}(p+\delta_i)e^{\frac{i}{\hbar}(p-q)L}e^{-\frac{i}{\hbar}\left[pv_{g_i}-qv_{g_j}+\Delta E_{ij}\right]t}\,, \end{aligned} \tag{3.98}$$

where $\mathcal{N}$ is the normalisation factor. Integration over the time variable results in $\delta(pv_{g_i} - qv_{g_j} + \Delta E_{ij})$. This is simply enforcing an equal-energy condition on the wave packet components, $\delta\Big(E_i(p) - E_j(q)\Big)$. This condition arises because only the equal-energy components are not washed out by the rapidly oscillating factor in the integration. Therefore, integration over time can be replaced by enforcing the equal energy condition on the wave packets components. This produces equivalent results as to the time integration when no decoherence is in place (the complete proof can be found in section 6 of [218]), and it is the only one that gives well-defined results once decoherence processes are taken into account, as we show in the following section.

Calling $r_{ij} = v_{g_i}/v_{g_j}$, one finally obtains

$$\begin{aligned} P(\beta\to\alpha) = \sum_{i,j} U^*_{\beta i}U_{\beta j}U^*_{\alpha j}U_{\alpha i}\, e^{i\phi_{ij}}\frac{\mathcal{N}}{v_{g_j}}\int \mathrm{d}p\, f_i^{(S)}(p)f_j^{(S)*}(r_{ij}p+\Delta E_{ij}/v_{g_j}) \\ \cdot f_j^{(D)}(r_{ij}p+\Delta E_{ij}/v_{g_j}+\delta_j)f_i^{(D)*}(p+\delta_i)e^{\frac{i}{\hbar}p(1-r_{ij})L}\,, \end{aligned} \tag{3.99}$$

where

$$\phi_{ij} := \left(\Delta p_{ij} - \frac{\Delta E_{ij}}{v_{g_j}}\right)\frac{L}{\hbar} \simeq \left(\Delta p_{ij} - \frac{\Delta E_{ij}}{v_{g_{ij}}}\right)\frac{L}{\hbar}\,, \tag{3.100}$$

is the phase factor and the last approximation, in which $v_{g_{ij}} := (v_{g_i} + v_{g_j})/2$, can be performed because the difference between v_{g_i} and $v_{g_{ij}}$ is of order Δm^2_{ij} and ΔE_{ik} is already of order

Δm_{ij}^2. For relativistic or quasi-degenerate neutrinos, the momentum difference can be expanded in terms of the energy difference and the mass difference Δm_{ij}^2

$$\Delta p_{ij} \simeq \frac{\Delta E_{ij}}{v_{g_{ij}}} - \frac{c^2 \Delta m_{ij}^2}{2\, p_{ij}} , \tag{3.101}$$

where p_{ij} is the average momentum. This yields the standard phase

$$\phi_{ij} = -L \frac{c^2 \Delta m_{ij}^2}{2\hbar p_{ij}} . \tag{3.102}$$

In general, the last phase factor in eq. (3.99) can be rapidly oscillating, and this would produce a loss of coherence for the neutrino wave packets. However, in several experimental setups of interest, this effect is negligible. This is the case where two conditions are met.

The *propagation coherence condition* ensures that the distance L travelled by the neutrinos is smaller than the distance over which the wave packets corresponding to different mass eigenstates separate due to the difference in their group velocities. This condition is satisfied if

$$L \ll l_{\text{coh}} = \sigma_X \frac{v_{g_{ij}}}{\Delta v_{g_{ij}}} , \tag{3.103}$$

where $\sigma_X = \max\left\{\sigma_x^S, \sigma_x^D\right\}$ is the maximum wave packet spatial width between the source (S) and the detector (D), and $\Delta v_{g_{ij}} = \left| v_{g_i} - v_{g_j} \right|$.

The *interaction coherence condition*, related to the coherence properties in the neutrino production and detection processes [218], ensures that the two momentum distributions overlap in the domain of the integral. This condition reads

$$\Delta E_{ij} \frac{\sigma_X}{\hbar v_{g_{ij}}} \ll 1 . \tag{3.104}$$

When conditions (3.103) and (3.104) are satisfied, we can set $r_{ij} = 1$ in the integral (3.99) and neglect the term ΔE_{ij} inside the arguments of the distribution function. This leads to the standard neutrino oscillation formula typically derived within the plane wave assumption:

$$P(\beta \to \alpha) = \sum_{i,j} U_{\beta i}^* U_{\beta j} U_{\alpha j}^* U_{\alpha i}\, e^{i\phi_{ij}} , \tag{3.105}$$

with ϕ_{ij} given by (3.102).

3.3.2 Fundamental decoherence in neutrino oscillation

Fundamental decoherence causes a non-unitary evolution of the neutrino wave packet. This can be described by generalising the Hamiltonian evolution via a Lindblad equation [189],

$$\hbar \partial_t \rho = \mathfrak{L}[\rho] , \tag{3.106}$$

where the evolution operator is $\mathfrak{L}[\rho] = -i[H,\rho] + \sum_k \left(\mathscr{L}_k \rho \mathscr{L}_k^\dagger - \frac{1}{2}\left\{\mathscr{L}_k^\dagger \mathscr{L}_k, \rho\right\}\right)$ and $\mathscr{L}_k$ are the Lindblad operators, that depend on the specific decoherence mechanism, as we discuss later in this section. Then $e^{-\frac{i}{\hbar}Ht}$ in (3.92) is replaced by $e^{\frac{t}{\hbar}\mathfrak{L}}$, yielding the density operator evolution

$$\begin{aligned}\rho(t) = e^{\frac{t}{\hbar}\mathfrak{L}}\left[\left|\nu_\beta\right\rangle\left\langle\nu_\beta\right|\right] &= \sum_{i,j} U^*_{\beta i}U_{\beta j}\int \mathrm{d}^3p\,\mathrm{d}^3q\,\psi_i(\boldsymbol{p})\psi_j^*(\boldsymbol{q})e^{\frac{t}{\hbar}\mathfrak{L}}\left[\,|\boldsymbol{p}\rangle\langle\boldsymbol{q}|\otimes\left|\nu_i\right\rangle\left\langle\nu_j\right|\right] \\ &= \sum_{i,j} U^*_{\beta i}U_{\beta j}\int \mathrm{d}^3p\,\mathrm{d}^3q\,\psi_i(\boldsymbol{p})\psi_j^*(\boldsymbol{q})e^{-\frac{i}{\hbar}\left[E_i(\boldsymbol{p})-E_j(\boldsymbol{q})\right]t}e^{-\frac{t}{\hbar}\mathscr{L}_{ij}(\boldsymbol{p},\boldsymbol{q})}|\boldsymbol{p}\rangle\langle\boldsymbol{q}|\otimes\left|\nu_i\right\rangle\left\langle\nu_j\right|\,,\end{aligned} \tag{3.107}$$

where in the last line the Lindblad operators are taken to commute with the momentum operator, an assumption that is consistent with the specific decoherence models we consider in this section, and $\mathscr{L}_{ij}(\boldsymbol{p},\boldsymbol{q})$ is a model-dependent positive function of $\boldsymbol{p}$ and $\boldsymbol{q}$ resulting from the action of the Lindblad operators $\mathscr{L}_k$ on the $|\boldsymbol{p}\rangle\langle\boldsymbol{q}|$ eigenstates. Restricting to the one dimensional approximation, and following similar steps as in the previous section, we obtain an expression similar to (3.98), with an additional damping exponential in the integral

$$\begin{aligned}P_{QG}(\beta\to\alpha;t) = \mathscr{N}\sum_{i,j}U^*_{\beta i}U_{\beta j}U^*_{\alpha j}U_{\alpha i}\,e^{\frac{i}{\hbar}\Delta p_{ij}L}\int \mathrm{d}p\,\mathrm{d}q\,f_i^{(S)}(p)f_j^{(S)*}(q) \\ \cdot f_j^{(D)}(q+\delta_j)e^{\frac{i}{\hbar}(p-q)L}f_i^{(D)*}(p+\delta_i)e^{-\frac{i}{\hbar}\left[pv_{g_i}-qv_{g_j}+\Delta E_{ij}\right]t}e^{-\frac{t}{\hbar}\mathscr{L}_{ij}(p+p_i,q+p_j)}\,,\end{aligned} \tag{3.108}$$

where $P_{QG}(\beta\to\alpha;t)$ denotes the deformed probability. Averaging over time would produce a divergent probability, since $\mathscr{L}_{ij}(p,q)\geq 0\;\forall\,p,q$. This can be traced back to the fact that averaging over time in the presence of decoherence is inherently an ill-defined procedure, since decoherence processes break the time symmetry of quantum-mechanical laws. The time interval $]-\infty,0[$ corresponds to a non-physical "recoherence", which is the cause of the divergent probability. We therefore directly enforce the equal-energy condition on wave packet components, as discussed in the previous section. Further substituting the remaining time variable in the damping factor with $\frac{L}{v_{g_{ij}}}$, the transition probability over a propagation distance L reads

$$\begin{aligned}P_{QG}(\beta\to\alpha;L) = \mathscr{N}\sum_{i,j}U^*_{\beta i}U_{\beta j}U^*_{\alpha j}U_{\alpha i}\,e^{i\phi_{ij}}\frac{2\pi}{v_{g_j}}\int \mathrm{d}p\,f_i^{(S)}(p)f_j^{(S)*}(r_{ij}\,p+\Delta E_{ij}/v_{g_j})\cdot \\ \cdot f_j^{(D)}(r_{ij}\,p+\Delta E_{ij}/v_{g_j}+\delta_j)f_i^{(D)*}(p+\delta_i)e^{\frac{i}{\hbar}p(1-r_{ij})L}e^{-D_{ij}\left(p+p_i,\,r_{ij}p+p_j-v_{g_j}^{-1}\Delta E_{ij}\right)}\,,\end{aligned} \tag{3.109}$$

where $D_{ij}(p,q) := v_{g_{ij}}^{-1}L\mathscr{L}_{ij}(p,q)$ is the decoherence-induced damping factor. If the coherence conditions discussed at the end of the previous section are satisfied, the normalised transition probability reads

$$P_{QG}(\beta\to\alpha;L) = \sum_{i,j}U^*_{\beta i}U_{\beta j}U^*_{\alpha j}U_{\alpha i}\,e^{i\phi_{ij}}e^{-D_{ij}(p_i,p_j)}\,. \tag{3.110}$$

This general formula applies to any decoherence model described by Lindblad-type evolution, provided that the Lindblad operators commute with the momentum operator. The effect of decoherence is to introduce a damping factor in the transition probability, such that when the decoherence effect is dominant no oscillation pattern in the neutrino propagation is observed. In subsection 3.3.2 we specialise this formula to different models of Quantum Gravity-induced fundamental decoherence.

Two flavours analysis

Most neutrino oscillation experiments are sensitive to the oscillation between two neutrino flavours, depending on their sensitivity to different mass-squared differences. In the two-flavour case, the matrix U in (3.110) is parameterised in terms of the mixing angle θ as

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{3.111}$$

and the oscillation phase (3.102) reduces to $\phi = -L\frac{c^2\Delta m^2}{2\hbar p}$, with Δm^2 the mass-squared difference and p the average momentum. Then the standard transition probability (3.105) takes the well-known form

$$P_{\text{std}}(\alpha \to \beta) = \sin^2 2\theta \sin^2 \frac{\phi}{2} \tag{3.112}$$

and the transition probability accounting for fundamental decoherence, (3.110), reduces to

$$P_{QG}(\alpha \to \beta) = \sin^2 2\theta \left[\frac{1}{2}\left(1-e^{-D}\right) + e^{-D}\sin^2\frac{\phi}{2}\right] = e^{-D}P_{\text{std}}(\alpha\to\beta) + \frac{1}{2}\left(1-e^{-D}\right)\sin^2 2\theta\,, \tag{3.113}$$

with $D = v_g^{-1}\frac{L}{\hbar}\mathscr{L}(p_i, p_j)$, where $v_g = \frac{pc^2}{E} = \frac{pc^2}{\sqrt{p^2c^2+m^2c^4}}$ is the average group velocity of the two mass eigenstates, m is the average mass, p_i, p_j are the mean momenta of such states and p is the average momentum.

For the analysis reported in the following sections, it is useful to consider the survival probability

$$P_{QG}(\alpha\to\alpha) \equiv 1 - P_{QG}(\alpha\to\beta) = e^{-D}P_{\text{std}}(\alpha\to\alpha) + \left(1-e^{-D}\right)\left(1-\frac{1}{2}\sin^2 2\theta\right), \tag{3.114}$$

where

$$P_{\text{std}}(\alpha\to\alpha) = 1 - P_{\text{std}}(\alpha\to\beta) = 1 - \sin^2 2\theta\sin^2\frac{\phi}{2}\,. \tag{3.115}$$

This is consistent with the result reported in [242], where, however, different possible forms for the damping factor were considered with respect to [3].

Damping factor for different decoherence models

In this section, we specialise the general formulas (3.109) and (3.114), that account for the effects of a Lindlbad-type evolution on neutrino oscillations, to a number of decoherence mechanisms that find support in Quantum Gravity research. Specific decoherence models can be distinguished according to the functional dependence of the damping factor on the neutrino energies and masses. In the following, we focus on four models, whose properties are summarised in table 3.1, in which fundamental decoherence is induced by different possible features of quantum spacetime: non-commutative spacetime [1], stochastic fluctuations of a minimal length [213], stochastic fluctuations of the metric around Minkowski spacetime [211], and the interaction with a thermal background of gravitons [206].

Starting from the Lindblad operators characterising the evolution of quantum systems in each of these models, the damping factor $D_{ij}(p,q)$ appearing in the oscillation probability

(3.109) and the resulting oscillation probability (3.110) can be computed as described in the previous section. Notice that while in the original papers defining the various models decoherence is assumed to be suppressed by the Planck scale E_P, in the following we replace this with a generic Quantum Gravity scale E_{QG}, to be constrained by observations. Moreover, the mass appearing in the Lindblad operators needs to be understood as being an operator that acts on flavour neutrinos states, the latter being superpositions of the mass operator eigenstates.

Decoherence induced by non-commutative spacetime [1] leads to the transition probability

$$P(\beta \to \alpha) = \sum_{i,j} U^*_{\beta i} U_{\beta j} U^*_{\alpha j} U_{\alpha i}\, e^{i\phi_{ij}}\, e^{-\frac{Lc^6}{8\hbar v_{g\,ij} E_{QG}\, p^2_{ij}}\left(\Delta m^2_{ij}\right)^2} . \tag{3.116}$$

In the two-flavour approximation, the damping factor reads

$$D = \frac{Lc^6}{8\hbar v_g E_{QG}\, p^2} (\Delta m^2)^2 . \tag{3.117}$$

Decoherence induced by stochastic fluctuations of the metric [211] results in the transition probability

$$P(\beta \to \alpha) = \sum_{i,j} U^*_{\beta i} U_{\beta j} U^*_{\alpha j} U_{\alpha i}\, e^{i\phi_{ij}}\, e^{-\frac{LE^6_{ij}\left(\Delta m^2_{ij}\right)^2}{4\hbar v_{g\,ij} E_{QG}\, c^4 m_i^4 m_j^4}} . \tag{3.118}$$

Then the two-flavour damping factor is

$$D = \frac{LE^6\left(\Delta m^2\right)^2}{4\hbar v_g E_{QG}\, c^4 m_i^4 m_j^4} . \tag{3.119}$$

Depending on whether we consider the quasi-degenerate mass eigenstates m_1 and m_2 or the mass eigenstates m_1 (or m_2) and m_3, we can write the product of the masses in terms of the average mass or the mass-squared difference. If the two mass eigenstates of interest are m_1 and m_2, as is the case for reactor neutrinos, we have $m_i m_j \sim m^2$, m being the average mass. If the mass eigenstates of interest are m_1 (or m_2) and m_3, as it happens to be the case for atmospheric neutrinos, then $m_i m_j = m_{\min}\sqrt{m^2_{\min} + \Delta m^2}$, $m_{\min}$ being the minimum mass between the two mass eigenstates.

The decoherence process induced by a fluctuating minimal length [213] gives the transition probability

$$P(\beta \to \alpha) = \sum_{i,j} U^*_{\beta i} U_{\beta j} U^*_{\alpha j} U_{\alpha i}\, e^{i\phi_{ij}}\, e^{-\frac{16Lc^4}{\hbar v_{g\,ij} E^5_{QG}} E^4_{ij}\left(\Delta m_{ij}\right)^2} , \tag{3.120}$$

and the damping factor in the two-flavour approximation

$$D = \frac{16Lc^4}{\hbar v_g E^5_{QG}} E^4 \left(\Delta m\right)^2 . \tag{3.121}$$

Model	Physical source of decoherence	Lindblad operators	Damping factors
A)	Deformation of symmetries [1]	$\mathscr{L}_j = \frac{c}{\sqrt{E_{QG}}} P_j$	$D = \frac{Lc^6\left(\Delta m^2\right)^2}{8\hbar v_g E_{QG}\, p^2}$
B)	Metric perturbations [211]	$\mathscr{L} = \frac{\boldsymbol{p}^2}{2m^2c^2\sqrt{E_{QG}}} H$	$D = \frac{LE^6\left(\Delta m^2\right)^2}{4\hbar v_g E_{QG}\, c^4 m_i^4 m_j^4}$
C)	Fluctuating minimal length [213]	$\mathscr{L} = \frac{4\sqrt{2}mc^2}{\sqrt{E_{QG}^5}} H^2$	$D = \frac{16LE^4c^4(\Delta m)^2}{\hbar v_g E_{QG}^5}$
D)	Gravitons at temperature T [206]	$\mathscr{L} = \frac{\sqrt{k_B T}}{E_{QG}} H$	$D = 0$

Table 3.1: Lindblad operators and damping factors for different Quantum Gravity-induced decoherence models. P_j is the $j-$th component of the momentum operator, E_{QG} is the Quantum Gravity scale, k_B is the Boltzmann constant, m is the mass and H is the free Hamiltonian. The Lindlbad operator for the model [211] is computed within a relativistic version of the model, see (3.130).

Depending on whether we consider the quasi-degenerate mass eigenstates or not, we can approximate the mass difference squared with the squared mass difference or write the former in terms of the latter. For the quasi-degenerate mass eigenstates, we have $\left(\Delta m\right)^2 \sim \Delta m^2$. For the non-degenerate mass eigenstates we have instead

$$\left(\Delta m\right)^2 = \left(-m_{\min} + \sqrt{m_{\min}^2 + \Delta m^2}\right)^2 \tag{3.122}$$

A thermal background of gravitons [206] yields the exponential factor

$$\exp\left\{-t\frac{k_B T}{2\hbar E_{QG}^2}\left(E_i(p) - E_j(q)\right)^2\right\} \tag{3.123}$$

in the evolved density operator (3.107). Therefore, this model predicts no contribution to the probability after the equal-energy condition is imposed and has no observable effect on neutrino oscillations. Therefore, neutrino oscillations cannot provide a phenomenological test ground for this model. We summarise the results described above in table 3.1. We note that, among the models we consider, the only one that can be recast in terms of the phenomenological models analysed in [241] is the first in table 3.1. Specifically, the third model in table 3.1, which turns out to be the most interesting one for our purposes, is not included in the analysis of [241].

Relativistic model with stochastic metric fluctuations

To study the effects of decoherence induced by stochastic metric perturbations on neutrino oscillations, we derive here a stochastic Schrödinger equation resulting from the metric fluctuations defined in [211], adapted to a relativistic regime.

Consider a particle evolving freely in a spacetime with stochastic perturbations. Denoting by τ the proper time of the particle and by t the laboratory time, the Schrödinger equation in the laboratory time reads [150]

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi\rangle = \dot{\tau} H |\psi\rangle \,, \tag{3.124}$$

where H is the Hamiltonian with respect to the proper time τ and

$$\dot{\tau} = \frac{1}{c}\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}x^\nu}{\mathrm{d}t}} \,. \tag{3.125}$$

Writing the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $\left|h_{\mu\nu}\right| \ll \left|\eta_{\mu\nu}\right|$ being a (time-dependent) stochastic perturbation, we get

$$\dot{\tau} = \gamma^{-1}\sqrt{1 - \frac{h_{\mu\nu}p^\mu p^\nu}{m^2c^2}} \sim \gamma^{-1}\left(1 - \frac{h_{\mu\nu}p^\mu p^\nu}{2m^2c^2}\right), \tag{3.126}$$

where γ is the Lorentz factor and the last expansion comes from $\left|h_{\mu\nu}\right| \ll \left|\eta_{\mu\nu}\right|$. By calling $H_0 = \gamma^{-1} H$ the free Hamiltonian in the laboratory frame and $H_s = -\frac{h_{\mu\nu}p^\mu p^\nu}{2m^2c^2} H_0$ the stochastic correction, we get a stochastic Schrödinger equation

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi\rangle = \left(H_0 + H_s\right) |\psi\rangle \,. \tag{3.127}$$

By following the same techniques in [213] (namely, by means of a cumulant expansion [244]) and assuming the following form for the correlator of the stochastic fluctuations [211]

$$\left\langle h_{ij}(t) h_{mn}(t')\right\rangle = \tau_c \delta_{ij}\delta_{mn}\delta(t - t') \,, \tag{3.128}$$

with all the other correlators being 0 and with τ_c being the characteristic time scale of the perturbations, we get the Lindblad equation

$$\partial_t \rho = -\frac{i}{\hbar}\left[H_0, \rho\right] - \frac{\tau_c}{\hbar^2}\left[\frac{\boldsymbol{p}^2}{2m^2c^2} H_0, \left[\frac{\boldsymbol{p}^2}{2m^2c^2} H_0, \rho\right]\right] . \tag{3.129}$$

By setting the scale of the fluctuations to be the Quantum Gravity scale $\tau_c = \hbar E_{QG}^{-1}$, we see that the Lindblad operator defining this equation is given by

$$\mathscr{L} = \frac{\boldsymbol{p}^2}{2m^2c^2}\frac{H_0}{\sqrt{E_{QG}}} \,, \tag{3.130}$$

which leads to a damping factor

$$D_{ij} = \frac{L E_{ij}^6 \left(\Delta m_{ij}^2\right)^2}{4\hbar v_{g_{ij}} E_{QG}\, c^4 m_i^4 m_j^4} \,. \tag{3.131}$$

3.3.3 Experimental constraints on fundamental decoherence

Different neutrino oscillation experiments, involving astrophysical, atmospheric, solar, or reactor neutrino, span a wide range of propagation lengths and energy. In this section, we start by discussing the potential sensitivity to decoherence of these classes of neutrino oscillation experiments. This depends on the specific model of fundamental decoherence that is considered, since the damping factors that are derived from each model have different dependencies on the neutrino energies and masses. We show that only the decoherence induced by metric perturbations [211] may affect neutrino oscillations to an observable level, assuming that the decoherence effect is suppressed by the Planck energy scale, and we use atmospheric and reactor experiments to establish constraints on the model.

Sensitivity of different neutrino oscillation experiments to fundamental decoherence

Before considering the sensitivity of neutrino oscillation experiments to specific fundamental decoherence models, two general considerations are in order.

The standard neutrino oscillation pattern, governed by the phase factor ϕ of (3.102), depends on the ratio $\frac{E}{L}$: when $\frac{E}{L} \sim \Delta m^2$ oscillations are visible, while $\frac{E}{L} \ll \Delta m^2$ is the fast oscillation regime, where it is only possible to measure the average probability. In this fast oscillation regime, the standard survival probability for a two-flavour oscillation (3.112) reads

$$\left\langle P_{\mathrm{std}}(\alpha \to \alpha)\right\rangle = 1 - \frac{1}{2}\sin^2 2\theta\,. \tag{3.132}$$

When accounting for decoherence, the average over the oscillations in (3.114) gives the same result as in the standard case

$$\left\langle P_{QG}(\alpha \to \alpha)\right\rangle = \left\langle P_{\mathrm{std}}(\alpha \to \alpha)\right\rangle = 1 - \frac{1}{2}\sin^2 2\theta\,, \tag{3.133}$$

thus Lindblad-type decoherence has no observable effects in this regime. This is the regime characterising solar neutrino oscillations, which are therefore not relevant for this kind of decoherence studies.

Because the damping factor in the oscillation probability, D_{ij}, depends linearly on the propagation length L (see table 3.1), one would naively expect that the best setup to test fundamental decoherence is provided by astrophysical neutrino observations, for which L is the largest. However, in this regime, the fundamental decoherence mechanism competes with the standard decoherence induced by the difference in propagation velocities of different neutrino mass eigenstates, discussed in section 3.3.1. For illustrative purposes, we discuss this by focusing on the model in [1], for which the damping factor is $D_{ij}(p,q) = \frac{Lc^2}{2\hbar v_{g_{ij}} E_{QG}}\left(p-q\right)^2$, but similar arguments hold for any Lindblad-type decoherence that leads to the oscillation probability (3.109). We also consider a simple case in which the shape factors are Gaussian distributions with the same variance at the source and the detector for all the mass eigenstates, with $\delta_i = \delta_j = 0$ and $\Delta E_{ij} = 0$. Specifically, we set

$$f_i^{(S)}(p) \propto e^{-\frac{p^2}{2\sigma_p^2}}\,, \tag{3.134}$$

where the normalisation is unimportant since it is absorbed in the factor $\mathcal{N}$ in (3.109). With these choices, and approximating v_{g_j} with $v_{g_{ij}}$, eq. (3.109) yields (we here set $\hbar = c = 1$ to simplify the notation)

$$P(\beta \to \alpha) = \mathcal{N} 2\pi\sqrt{2\pi} \sum_{i,j} \left\{ U^*_{\beta i} U_{\beta j} U^*_{\alpha j} U_{\alpha i}\, e^{i\phi_{ij}\left[1-\frac{L(1-r_{ij})^2\sigma_p^2}{2E_{QG}(1+r_{ij}^2)v_{g_{ij}}+L(1-r_{ij})^2\sigma_p^2}\right]} \right.$$
$$\left. \cdot \frac{e^{-\frac{L^2(1-r_{ij})^2 v_{g_{ij}}\sigma_p^2+\frac{2(1+r_{ij}^2)\phi_{ij}^2}{LE_{QG}}}{4(1+r_{ij}^2)v_{g_{ij}}+\frac{2L(1-r_{ij})^2\sigma_p^2}{E_{QG}}}}}{\sqrt{v_{g_{ij}}\left(2(1+r_{ij}^2)\,v_{g_{ij}}+\frac{L(1-r_{ij})^2\sigma_p^2}{E_{QG}}\right)}} \right\}. \tag{3.135}$$

The exponential damping factor in (3.135) takes two contributions: one, which vanishes when $E_{QG} \to \infty$, is due to the Lindblad deformation of the evolution equation; the other one, proportional to $(1-r_{ij})^2$, is due to the velocity difference between mass eigenstates. Although the velocity difference is rather small and its effect can be partially controlled by decreasing σ_p through suitable detection techniques [240], the decoherence it induces is strong enough to wash out oscillations over propagation length scales typical of astrophysical neutrinos. Therefore, the effects of Quantum Gravity-induced decoherence are not observable.

For atmospheric and reactor neutrinos the standard decoherence effect we just discussed is negligible. Moreover, the full oscillation pattern is visible and is not averaged out, contrary to what happens for solar neutrinos. Therefore, in the following, we focus on this kind of neutrino observations to study fundamental decoherence.

We start by computing, for each of the models in table 3.1, the theoretical ranges for neutrino energies that would produce significant decoherence effects in oscillations when using the mass and length parameters typical of reactor and atmospheric neutrino experiments, and assuming $E_{QG} = E_P \sim 10^{19}$ GeV. For simplicity, we work in a two-flavour setup and require that the corresponding damping factor satisfies the condition $D \gtrsim 1$, which ensures that the decoherence effect is large enough to affect the oscillation pattern [234]. We then compare these energy ranges with the realistic energies characterising reactor and atmospheric neutrinos, in order to identify the models that can be effectively constrained with this kind of observations.

For long-baseline reactor experiments, we take $L \sim 10^5$ m as a reference propagation distance and set $\Delta m^2 \sim (\Delta m)^2 \sim 10^{-5}$ eV$^2/c^4$ [245], since such experiments are sensitive to the mass-squared difference between the quasi-degenerate mass eigenstates m_1 and m_2. The damping factors of the various decoherence models also depend on the neutrino mass value, for which we take two possible reference values, $m \sim 10^{-2}$ eV$/c^2$ and $m \sim 1$ eV$/c^2$, the maximum value allowed by cosmological data [246]. Once these parameters are fixed, the requirement $D \gtrsim 1$ identifies a range in neutrino momentum p for each decoherence model. These are listed in table 3.2. Considering that the typical energy of reactor neutrinos, such as those detected by the KamLAND experiment [245], is in the range [1, 10] MeV [245], only decoherence induced by stochastic metric fluctuations [211] could be significantly constrained using this kind of data.

Model	**Observability window for $m = 10^{-2}$ eV/c^2**	**Observability window for $m = 1$ eV/c^2**
A)	$p \lesssim 10^{-8}\,\mathrm{eV}/c$	$p \lesssim 10^{-8}\,\mathrm{eV}/c$
B)	$p \lesssim 10^{-22}\,\mathrm{eV}/c$ or $p \gtrsim 100\,\mathrm{eV}/c$	$p \lesssim 10^{-23}\,\mathrm{eV}/c$ or $p \gtrsim 10^{4}\,\mathrm{eV}/c$
C)	$p \lesssim 10^{-147}\,\mathrm{GeV}/c$ or $p \gtrsim 10^{24}\,\mathrm{GeV}/c$	$p \lesssim 10^{-143}\,\mathrm{GeV}/c$ or $p \gtrsim 10^{24}\,\mathrm{GeV}/c$
D)	//	//

Table 3.2: Observability windows for the decoherence models listed in table 3.1 using reactor neutrino experiments, for different values of the neutrino mass. Details of the computations are provided in the main text. Note that when two ranges are shown for p, this comes from solving the quadratic equation that comes from the condition $D \gtrsim 1$.

A similar analysis can be applied to atmospheric neutrino oscillation experiments, that are sensitive to the mass-squared difference between eigenstates $m_{1,2}$ and m_3, so that $\Delta m^2 \sim 10^{-3}\,\mathrm{eV}^2/c^4$ [247]. We assess the sensitivity of these experiments considering again two possible values of the lowest-mass eigenstate, $m_{\min} \sim 10^{-2}\,\mathrm{eV}/c^2$ and $m_{\min} \sim 1\,\mathrm{eV}/c^2$. Atmospheric neutrinos can propagate over a wide range of distances, spanning several orders of magnitude, from 10 Km to 10^4 Km [247]. For the scope of estimating the sensitivity of this kind of observations we take the minimum and the maximum possible values. The observability windows are reported in table 3.3 and table 3.4, respectively. Again, we see that only decoherence induced by stochastic metric fluctuations [211] might be significantly constrained using atmospheric neutrino oscillation data.

In summarising the results of this subsection, solar and astrophysical neutrinos cannot be used to test Lindblad-type decoherence. Reactor and atmospheric neutrino experiments can in principle be used to this aim. However, not all models predict a level of decoherence that might be detected by these experiments if the energy scale governing the strength of decoherence is set to the Planck scale. Among the four models we considered, only the one in which decoherence is induced by metric fluctuations can be meaningfully constrained using these experiments. In the following subsections, we derive the constraints on the energy scale associated to this model.

Before we conclude this section, we would like to comment on the concerns expressed in [243], where the possibility of testing Quantum Gravity-induced decoherence with neutrino oscillation experiments was questioned. The Quantum Gravity model considered in [243] (see also [242]) had a similar form as the one of [213] (since the Lindblad operator depends on the energy). In fact, we find that the damping factor associated with this model is heavily suppressed, in accordance with the estimates provided in [243]. For other models, such as the one in [211], similar arguments to those used in [243] do not lead to such a large suppression of the damping factor and, in fact, lead to estimates of the damping factor consistent with those we report in table 3.1.

Constraints from long-baseline reactor neutrinos

We base our analysis on the data used by the KamLAND collaboration for precision measurements of neutrino oscillation parameters concerning the mass eigenstates m_1 and m_2 [245]. Within this two-flavour approximation, the survival probability accounting for Lindblad-type decoherence is (3.114), and the damping factor D for decoherence induced by metric fluc-

Model	Observability window for $m_{\min} = 10^{-2}$ eV/c^2	Observability window for $m_{\min} = 1$ eV/c^2
A)	$p \lesssim 10^{-8}$ eV/c	$p \lesssim 10^{-7}$ eV/c
B)	$p \lesssim 10^{-19}$ eV/c or $p \gtrsim 10^{2}$ eV/c	$p \lesssim 10^{-20}$ eV/c or $p \gtrsim 10^{3}$ eV/c
C)	$p \lesssim 10^{-145}$ GeV/c or $p \gtrsim 10^{23}$ GeV/c	$p \lesssim 10^{-141}$ GeV/c or $p \gtrsim 10^{24}$ GeV/c
D)	//	//

Table 3.3: Observability windows for the decoherence models listed in table 3.1 using atmospheric neutrino experiments, for different values of the neutrino mass $L = 10\,\mathrm{Km}$. Details of the computations are provided in the main text.

Model	Observability window for $m_{\min} = 10^{-2}$ eV/c^2	Observability window for $m_{\min} = 1$ eV/c^2
A)	$p \lesssim 10^{-7}$ eV/c	$p \lesssim 10^{-6}$ eV/c
B)	$p \lesssim 10^{-16}$ eV/c or $p \gtrsim 10$ eV/c	$p \lesssim 10^{-18}$ eV/c or $p \gtrsim 10^{3}$ eV/c
C)	$p \lesssim 10^{-142}$ GeV/c or $p \gtrsim 10^{22}$ GeV/c	$p \lesssim 10^{-138}$ GeV/c or $p \gtrsim 10^{23}$ GeV/c
D)	//	//

Table 3.4: Observability windows for the decoherence models listed in table 3.1 using atmospheric neutrino experiments, for different values of the neutrino mass and $L = 10^4\,\mathrm{Km}$. Details of the computations are provided in the main text.

tuations [211] is given by (3.119)[12]. As can be seen in fig. 5, the neutrino oscillations in the KamLAND experiment are damped with respect to the pattern of a neutrino beam travelling in vacuo with a fixed baseline. This is expected since several reactors are involved in the experiment, placed at different distances from the detectors, and the neutrino beam travels through matter. Including these effects would require a more refined analysis and a complete knowledge of the KamLAND experiment. However, such a detailed modelling goes beyond the scope of [3], since it is not needed when the goal is to provide a conservative bound on Quantum Gravity-induced decoherence. To this aim, in fact, it is sufficient to ask that decoherence is not as strong as to dampen oscillations on its own more than what the known damping effects do. Indeed, if this were not the case, the damping stemming from quantum-spacetime effects, combined with this standard damping, would completely quench the oscillations. In fig. 5 we also plot the survival probability (3.114), computed taking into account the appropriate values for the experimental parameters as detailed in the caption of the figure. The only free parameter is the Quantum Gravity scale E_{QG}, and to produce the plot, we fixed it to a reference value $E_{QG} = 10^{24}$ GeV for illustrative purposes. We can see that the decoherence effects are stronger at higher energies (lower values of $\frac{L}{E}$), whereas at lower energies the deformed survival probability approaches the standard one in vacuo. Therefore, the regime most sensitive to test decoherence is the one corresponding to the first oscillation period, so we use data points in the range $\frac{L}{E} \in [10, 45]$ Km MeV^{-1} to provide a constraint on the model, see fig. 6. This restriction is justified by the fact that if the quantum spacetime-induced damping had observable effects in the remaining part of the $\frac{L}{E}$ interval, it would also

[12] Given the energies at play in this regime, the velocity factor is $v_g \simeq 1$.

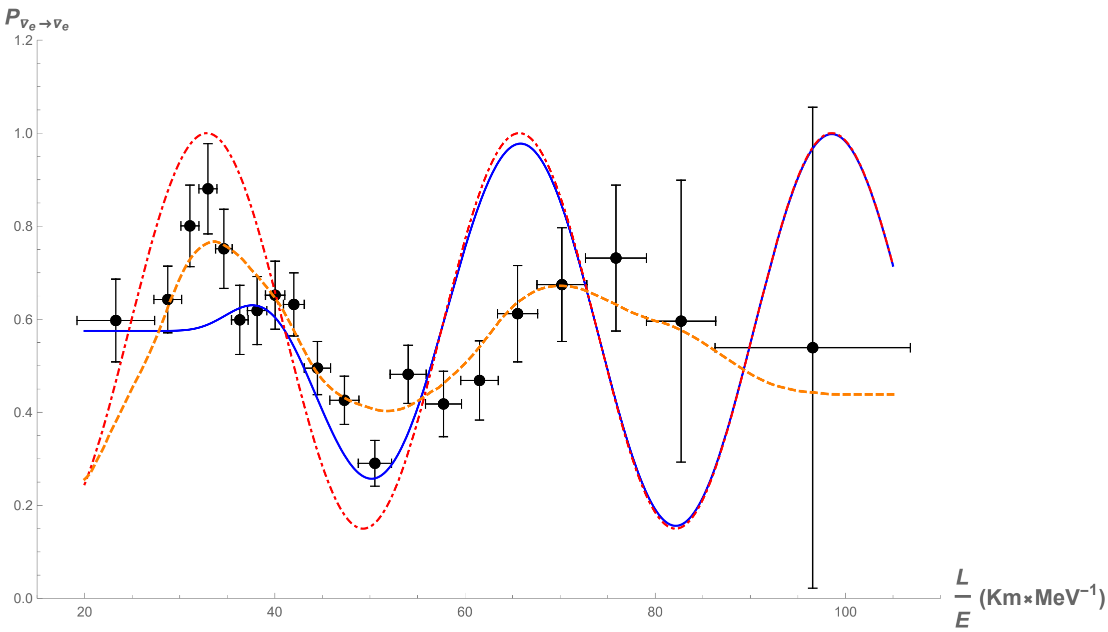

Figure 5: We plot the standard survival probability (red dot-dashed curve) in vacuo, given by (3.115), the Quantum Gravity-deformed probability with damping factor (3.119) (blue solid line), the data points from KamLAND experiment [245] (black dots with error bars) and the KamLAND best fit for such data (orange dashed curve). The oscillation parameters are set to the PDG values in [248] $\sin^2 2\theta = 0.85$ and $\Delta m^2 = 7.53\cdot 10^{-5}\text{eV}^2/c^4$. The damping factor (3.119) is computed by setting $L = 180$ Km, $E_{QG} = 10^{34}$ GeV and $m = 1$ eV$/c^2$.

have a much stronger effect on the oscillation pattern in the considered $\frac{L}{E}$ interval, thus being incompatible with the data in that regime.

Our constraint is derived by asking that the difference between the χ^2 obtained by fitting the deformed probability (3.114), with D given by (3.119), to KamLAND data and the χ^2 corresponding to KamLAND best fit line is not larger than 2.7, corresponding to a 90% confidence level for a difference of 1 in the counting of degrees of freedom for the two fits. In the expression for the damping factor (3.119), we set the oscillation parameters to be the values reported by the Particle Data Group (PDG) in [248]. We also set $L = 180$ Km as done in the original oscillation analysis by the KamLAND collaboration [245]. Moreover, since the constraint on the Quantum Gravity scale is stronger for smaller neutrino masses (see (3.119)), a conservative bound is found by setting $m = 1$ eV$/c^2$, namely:

$$E_{QG} \geq 2.6\cdot 10^{34}\,\text{GeV}\,. \tag{3.136}$$

Constraints from atmospheric neutrinos

We use data collected by the Super-Kamiokande collaboration and reported in [247]. The relevant $\frac{L}{E}$ range is $[1, 10^4]$ Km GeV^{-1}, so that the observed oscillation pattern is sensitive to the parameters relative to the mass eigenstates m_1 and m_3. As can be seen in fig. 7, the fast oscillations regime starts for $\frac{L}{E} \gtrsim 10^3$ Km GeV^{-1}. In this regime, the data only probe the average oscillations, and the averaged Quantum Gravity-deformed survival probability converges to the standard one, see (3.133). For this reason, we limit our analysis to data in the range

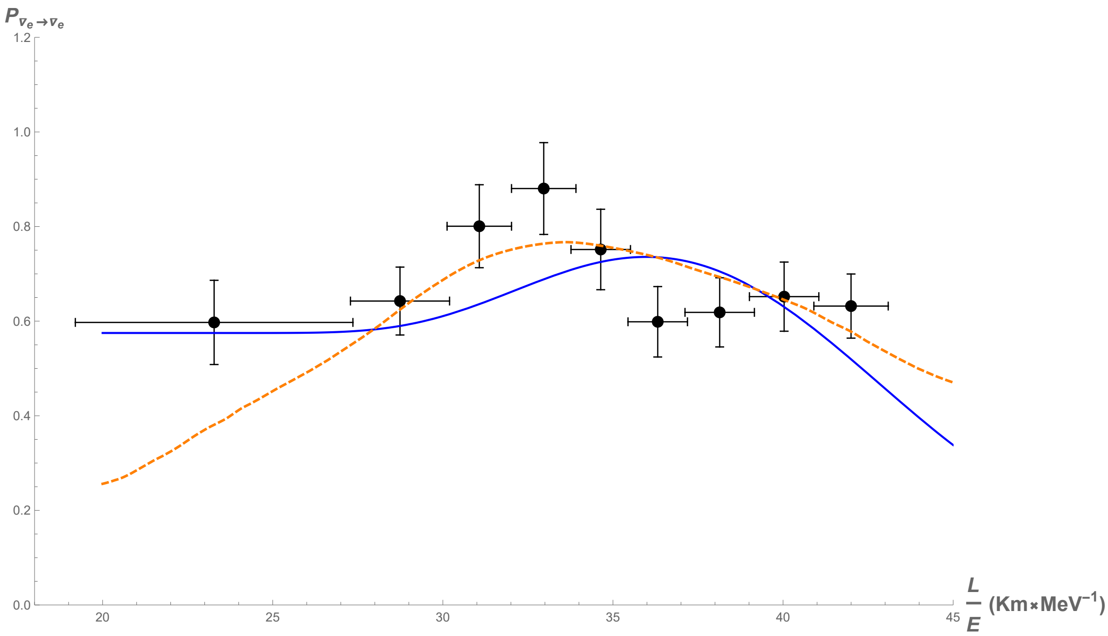

Figure 6: We plot the Quantum Gravity-deformed probability with damping factor (3.119) (blue solid line) with the value of $E_{QG} = 2.6 \cdot 10^{34}$ GeV corresponding to the upper limit we set as described in the main text, the data points from KamLAND experiment [245] in the range $\frac{L}{E} \in [10, 45]$ Km MeV^{-1} (black dots with error bars) and the KamLAND best fit for the full data set reported in fig. 5 (orange dashed curve). The oscillation and the decoherence parameters are the same as for fig. 5.

$\frac{L}{E} \in [1, 10^3]$ Km GeV^{-1}, where the oscillation pattern is visible and decoherence produces observable effects. We compare the χ^2 obtained by fitting the data with the deformed probability (3.114), with D given by (3.119)[13], and the χ^2 corresponding to the standard oscillation pattern with oscillation parameters taken from the PDG [248]. We require that the difference between the two χ^2 is less than 2.7, corresponding to a 90% confidence level for the 1 degree of freedom difference between the two models. A conservative constraint is found by setting $m_{\min} = 1$ eV$/c^2$ in (3.119), since the constraint on the Quantum Gravity scale would be stronger with smaller values of the mass. Moreover, each data point in fig. 7 results from oscillations over a distance that spans several orders of magnitude, roughly from $L = 10$ Km to $L = 10^4$ Km. Since the constraint on the Quantum Gravity scale is stronger for higher travel distances (see (3.119)), we take a conservative approach again and set $L = 10$ Km in our analysis. With this, we find

$$E_{QG} \geq 2.5 \cdot 10^{55}\ \text{GeV}\,. \tag{3.137}$$

[13] Given the energies at play in this regime, the velocity factor is $v_g \simeq 1$.

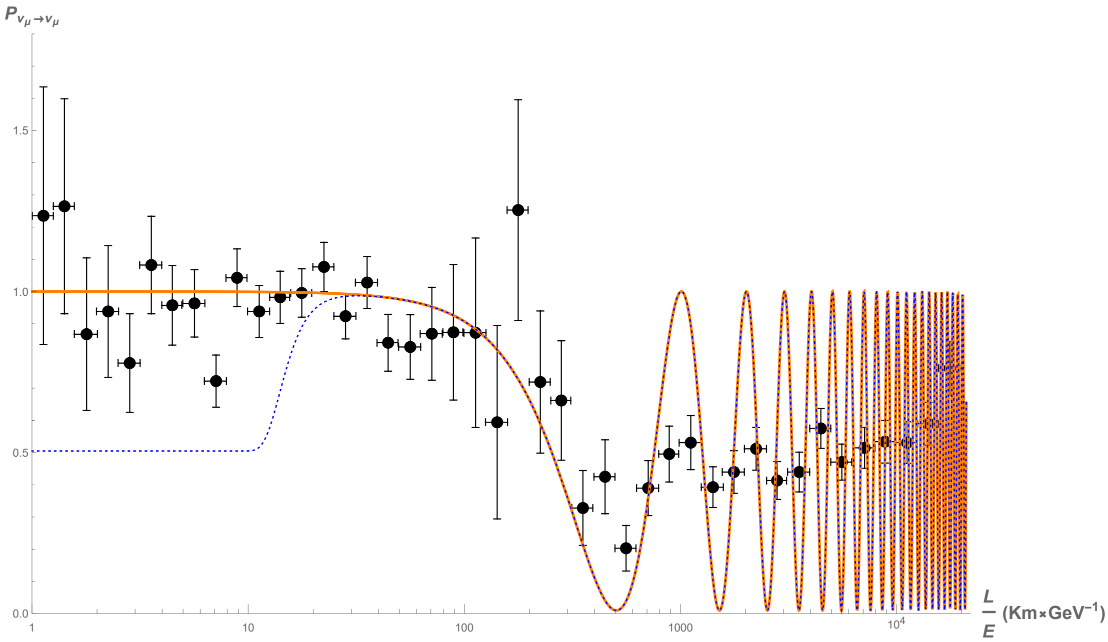

Figure 7: We plot the standard oscillation probability (solid orange curve), the Quantum Gravity-deformed probability (dotted blue curve) with damping factor (3.119), with $m_{\min} = 1\,\mathrm{eV}/c^2$, $L = 10$ Km and $E_{QG} = 10^{49}\,$GeV, chosen for illustrative purposes, and the data collected in [247]. The values of the oscillation parameters used for the plot are those reported in the PDG [248], $\Delta m^2_{23} = 2.45 \cdot 10^{-3}\ \mathrm{eV}^2/c^4$ and $\sin^2 2\theta_{23} = 0.99$. The $\frac{L}{E}$ axis is in logarithmic scale.

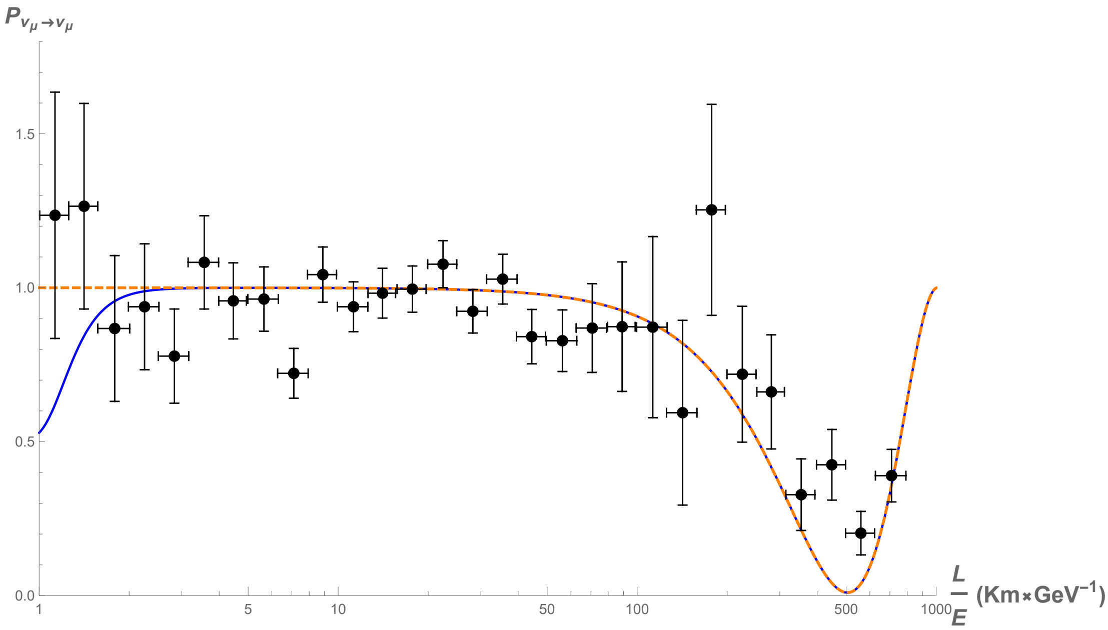

Figure 8: We plot the standard oscillation probability (solid orange curve), the Quantum Gravity-deformed probability (blue solid curve) with damping factor (3.119), with $E_{QG} = 2.5 \cdot 10^{55}\,$GeV, corresponding to our upper limit (3.137), and the data collected in [247] in the range $\frac{L}{E} \in [1, 10^3]$ Km GeV^{-1}. The values of the oscillation and the decoherence parameters are the same as in fig. 5.

3.4 Discussion

This chapter was devoted to the study of decoherence processes in Quantum Gravity. In the first part, we reviewed the environmental decoherence programme to provide the relevant notions for the results of [1] and [3]. Then, we provided an overview of decoherence studies in Quantum Gravity to embed our work in the relevant literature. The second section was dedicated to [1], in which we studied the emergence of a decoherence mechanism from a deformation of spacetime symmetries. An important novelty of (3.69) is that it yields a localisation process in momentum, while, to the best of our knowledge, all other models in the literature describe localisation in energy or in position [192]. The final section was dedicated to [3], in which we provided a phenomenological analysis of some Quantum Gravity-induced decoherence models using neutrino oscillation data. The remarkable result of [3] is that it rules out one of such models, namely the decoherence process arising from stochastic metric perturbations [211], generalised to its relativistic version in [3].

In our work, we showed that a Lindblad equation arises naturally as the evolution equation of quantum systems living in a non-commutative spacetime, where the symmetry generators are deformed and described by non-trivial Hopf algebras. A first hint towards this direction was given in [249] where, however, the time evolution is not of the Lindblad type and the positivity of the density operator is not preserved. Our results yield a fundamental decoherence mechanism by which the density operator of a quantum system becomes localised in the momentum basis. Although our decoherence process is fundamental and unavoidable, since there is no interaction with an environment, it would still be interesting to understand what type of environment could lead to the Lindblad equation we derived.

Our analysis shows that there exists a consistent framework modelling putative quantum spacetime effects in which pure quantum states can evolve into mixed states. This fact is particularly suggestive when seen from the perspective of the black hole information paradox. Indeed, the possibility that the evolution of quantum fields on a black hole background might not be given by ordinary quantum evolution has been dismissed over the years as problematic and lacking links with fundamental theories of Quantum Gravity [250]. A natural development on this side would be to check whether the non-unitary evolution we derived can be recast in the form of a qubit toy model of black hole evaporation [251] thus providing the first fully worked out model of non-unitary black hole evolution derived from first principles.

Another interesting development of the line of thought that led us to our result would be to try to derive a model that could explain the quantum-to-classical transition of primordial cosmological perturbations [252–257]. In fact, the Fourier modes of such perturbations are expected to have Planckian energy at sufficiently early times. On these scales, the decoherence time (3.72) is of the order of the Planck time, making the decoherence process very effective. In this scenario, one does not need to invoke any of the additional decoherence mechanisms that have been considered in the past, such as the interaction of the perturbations with external fields, and the emergence of classicality in the large-scale structures that we observe today would be derived from the sheer properties of (quantum) spacetime, namely its (deformed) symmetries.

Finally, we outlined an intriguing discussion concerning the emergence of the arrow of Time. Indeed, the equation we derived is of course not symmetric under time reversal, as is obvious for a non-unitary evolution. However, differently from the "effective" asymmetry that characterises environmental decoherence models, in our case, this asymmetry is fundamental because it does not arise as a consequence of the absence of control over environmental degrees of freedom. Instead, such a time-reversal asymmetry emerges because the generator of the time-reversed evolution is deformed, hence yielding a fundamental arrow of Time.

Besides the possible implications for the foundational issues just discussed, fundamental decoherence models might open up several windows to experimentally access the Quantum Gravity regime. Notice that such decoherence processes are not expected to replace the experimentally well-established environmental decoherence [177, 258], and, in fact, they typically provide negligible contributions to the typical experimental setups devised to study standard environmental decoherence effects. However, they can become relevant and observable in specific regimes, as discussed below.

An example of such experimental opportunities concerns tabletop experiments with optomechanical cavities [213, 259]. In these setups, two cavities are realised by means of four mirrors, one of which is movable. Inside the cavities, two atoms or molecules in a harmonic potential are prepared in an entangled state and are hit by an external laser source (realising a Raman single-photon source). The excitations of these sources lead to single-photon excitations and the radiation so obtained in the cavities allows to study the vibrational modes of the oscillators. In the presence of an environment, the entanglement between the two oscillators is degraded with a rate $\Lambda_{\text{heat}} = \frac{k_B T}{\hbar Q}$ from mechanical heating at temperature T; the presence of additional gravitational decoherence mechanisms would increase such a rate to $\Lambda_{\text{heat}} + \Lambda_{\text{grav}}$, where in our case $\Lambda_{\text{grav}} = \frac{(\delta \boldsymbol{p})^2}{2\hbar M_P}$. Adapting the analysis presented in [213] to the peculiarities of the fundamental decoherence mechanism we uncovered, we find that the experimental parameters that will be achievable in the near future, like temperatures $T \sim 10\,\text{mK}$ and quality factors $Q \sim 10^6$ [260, 261], will allow to reach Planck-scale sensitivity on the quantum deformation scale, $M_P \sim 10^{-8}\,\text{g}$, for quantum systems with mass $M \sim 10^{-25}\,\text{g}$.

Another physical framework that allows to test such decoherence models involves modifications of the oscillation rates for neutrinos. Generic decoherence mechanisms induce modifications in these rates [262–264] that are driven by the exponential damping factor $e^{-\Gamma t}$, with $\Gamma = \tau_D^{-1}$. This factor produces a significant contribution to the oscillation probability roughly when $e^{-\Gamma t} \lesssim e^{-1}$ [262]. In our case $\Gamma \sim \left(\hbar M_P\right)^{-1} (\delta \boldsymbol{p})^2 \sim \left(\hbar M_P\right)^{-1} (\delta m)^2$. Taking $M_P \sim 10^{-8}\,\text{g}$ and $(\delta m)^2 \sim 10^{-3}\,\text{eV}^2$ [264], we find $\Gamma \sim 10^{-15}\,\text{s}^{-1}$. Therefore, the factor $e^{-\Gamma t}$ produces relevant effects on neutrino oscillations when they propagate over cosmological times $t \gtrsim 10^{15}\,s \sim 10^8\,\text{yr}$. However, when neutrinos propagate over such large distances, standard decoherence effects, which are much stronger than Quantum Gravity-induced damping, occur because of the different propagation velocities between different mass eigenstates [43, 218, 239, 240]. Therefore, neutrino oscillations cannot be used as a phenomenological test for our model, but they can be of use for other decoherence models.

We investigated this latter phenomenological opportunity for different models of decoherence induced by quantum spacetime effects. We showed that fundamental decoherence, originating from quantum properties of spacetime, can alter the pattern of oscillation of neutrinos, causing the emergence of a damping factor in the neutrino transition probability. This might be significant in principle for long baseline-reactor neutrino experiments and for atmospheric neutrino observations. These are general results concerning all quantum spacetime-induced decoherence models that can be described in terms of a Lindblad-type equation for the evolution of neutrinos. However, whether a specific model affects observations to a detectable level depends on the specific decoherence-inducing mechanism. These predict different forms of the damping factor, characterised by the dependence on neutrino energies and masses. Among the models we considered, we found that only the one in which decoherence is induced by stochastic metric fluctuations [211] can be significantly constrained using reactor neutrinos and atmospheric neutrinos. With these, we set conservative constraints on the scale governing the strength of the decoherence process that are several orders of magnitude stronger than the Planck scale, $E_{QG} \geq 2.6 \cdot 10^{34}$ GeV and $E_{QG} \geq 2.5 \cdot 10^{55}$ GeV from the KamLAND reactor neutrino experiment and the Super-Kamiokande atmospheric neutrino observatory, respectively. These constraints would require the scale governing the model to be unrealistically large, so that the hypothesis that the decoherence induced by stochastic metric perturbations affects neutrino oscillations can be considered to be ruled out.

Fourth Chapter

Quantum reference frames and superposition of mass states

Fundamental particles (*i.e.* not composite) are generally considered to have fixed masses. Formally, they are identified with irreducible representations of the relevant symmetry group, namely the Poincaré or the Galilei group for relativistic and non-relativistic Quantum Mechanics, respectively. See chapter A for a review of the representation theory of the Galilei group. In both cases, the mass is one of the parameters that specify the representation, and hence it is one of the relevant quantum numbers that physically define fundamental particles [67, 265, 266]. For this reason, it is in principle possible to have quantum states for physical systems and fundamental particles that are in a quantum superposition of different mass eigenstates.

As discussed in chapter 1, in non-relativistic Quantum Mechanics, mass superpositions lead to an apparently troublesome effect, namely the fact that a particular chain of reference frame transformations that yields the identity on spacetime coordinates, results in a relative phase between different mass components. This leads Bargmann to enforce a superselection rule on the mass in non-relativistic Quantum Mechanics [66], known as the *Bargmann superselection rule*. It was later shown by Greenberger [68] that, from the physical point of view, this effect is just a remnant of special-relativistic time dilation, therefore it should be expected. Hence, the enforcement of a mass superselection rule is not justified.

In relativistic Quantum Mechanics and quantum field theory, fundamental particles are typically described as mass eigenstates. However, this is not the case for all fundamental particles. For instance, neutrinos can be observed in states that are not eingenstates of the mass: flavour neutrinos oscillations not only signal that such particles are massive but are also an imprint of the fact that they propagate as coherent mass superpositions [218, 267]. Flavour neutrinos are the only fundamental particles that exhibit this feature, and no other fundamental particles, such as charged leptons, are observed to undergo flavour oscillations. One possible explanation for this can be looked for in the experimental sensitivity of current oscillation experiments. In fact, while in some processes, such as pion decays, flavour neutrinos can be produced as coherent mass superpositions and charged leptons cannot, in other processes, such as W boson decays, both types of particles can be produced in mass superpositions, but the coherence length of mass superposed charged leptons is too small to allow for the detection of these superpositions with current experimental setups [268]. Of course, the disparity between the two types of particles originates from the huge difference between the neutrinos and the charged leptons masses. Although this argument is compelling, it is possible to ask if there could be other possibilities to explain this discrepancy, namely that flavour neutrinos are the only fundamental particles that are observed in mass superpositions while other particles are superselected in mass. Is there a fundamental explanation for this *emergent* superselection rule for the mass? Does the different behaviour among particles depend on how we observe them, or is it an *intrinsic* property of the particles themselves?

The reason for investigating this issue lies in the observation that when relativistic particles, such as flavour neutrinos, are in mass superposition, the standard Lorentz boost transformations are not suitable to define their rest frames, since these transformations depend on the mass of the particles. For this reason, the rest frames of flavour neutrinos are intrinsically quantum reference frames [269]. In the quantum reference frames formalism (cf. chapter 2) [51, 102–126], reference frames are not abstract entities, rather they are defined by physical systems that act as relational references. Given that physical systems are ultimately quantum in nature, the formalism opens up the possibility of defining physical quantities with respect to physical systems that are in quantum superpositions, but the reference physical systems can also be in localised states, and hence behave as classical reference frames. This latter possibility is prevented for the rest frame of flavour neutrinos as these particles are intrinsically superpositions of different mass eigenstates and cannot be localised in mass. From this we understand that the possibility of having fundamental particles in mass superpositions naturally divides such particles into two classes, depending on whether their rest frames are inherently quantum reference frames or not. These observations lead us to ask whether there is a fundamental origin for the mass superselection that does not affect flavour neutrinos.

Superselection rules, in general, can originate from the lack of knowledge that one observer has on the reference frame from which the physical systems they observe come from [270]. In these cases, the observer describes the density operator of a physical system that they receive from an unknown reference frame as an incoherent average over the group of transformations that relate the unknown and the observer's reference frames. This incoherent average procedure leads to *decoherence*, hence superselection rules, for the physical observables conjugated to the transformation generators. Of course, it is possible to consider partially coherent averages by smearing the integral average with a function of the group elements that describes the partial knowledge of the relation between the two reference frames. In the context of quantum reference frames, considering two systems A and B that act as quantum reference frames for the description of a third system S, the incoherent average can be realised by performing a transformation from A to B, then taking a partial trace on the degrees of freedom of A and taking its density operator to be maximally mixed, so that it contains no information on its relation with B. The partially coherent average is obtained in this context by letting the density operator of A be in a generic state, different from the identity, so that it takes the role of the smearing function in the classical reference frame setting.

In this chapter, which is based on [5], we consider two observers, A and B, and a system S. We take A to be a physical system describing the production of the fundamental particle S (*e.g.* the particle whose decay generates S) and B to be the detector/observer that detects S. We investigate the possibility that the observer's lack of knowledge of the relation between their reference frame and the reference frame of A, in which the production of the particle S is described, may be the source of a superselection rule on the mass for the latter. We take the two reference frames A and B to be equipped with a quantum clock each, so that they are spacetime quantum reference frames [117] (cf. chapter 2) and we can consider a general spacetime coordinate transformation between them. We assume that B has no access to the degrees of freedom of A, so that, after the reference frame transformation, the latter are traced out. Moreover, the quantum clocks are assumed not to be perfect, namely their time readings are affected by some finite resolution, which we implement by replacing perfect measurements with coarse-grained ones described by POVMs. We show that a superselection rule on the mass for fundamental particles can emerge, depending on the differences between the masses that are superposed, the average energy of the particles, and the resolution of the clocks of the two spacetime quantum reference frames A and B. We also derive the interval in which these resolutions must lie, such that flavour neutrinos are observed as mass superpositions, while charged leptons are superselected in mass.

This chapter is organised as follows. In section 4.1 we develop an algebraic proof that the phase factor used in the Bargmann argument is not an artefact of the non-relativistic Quantum Mechanics formalism, rather it is a special-relativistic effect that survives in the non-relativistic limit. This establishes, on a formal ground, the inconsistency of the Bargmann argument for a mass superselection rule in non-relativistic Quantum Mechanics. In section 4.2, for the sake of readability, we give a brief review of the tools, some of them already discussed in more detail in chapter 2, that we need to derive our result: quantum reference frames, incoherent averages, and quantum clocks. Finally, in section 4.3 we discuss our main result. We show that a mass superselection rule can emerge for a fundamental particle in the relativistic regime when the observer does not have access to the reference frame in which the particle production process is described. Whether the particle is superselected in mass or not depends on its energy, the mass-squared difference between the mass eigenstates to be superposed, and the resolution of the (quantum) clocks of the two reference frames in which the production and the detection of the particle are described.

4.1 No mass superselection in non-relativistic Quantum Mechanics

Superselection rules have been debated for a long time in Quantum Mechanics [139]. The reason for this is that there are some physical parameters, such as electric charge, mass, and spin, whose coherent superpositions do not seem to be observed in Nature, even though nothing prevents this possibility in principle. However, this is not entirely true for the mass. In fact, while the majority of particles appear to be superselected in mass, the phenomenon of particle oscillations is caused by the propagation of coherent superpositions of different mass eigenstates. Moreover, from the properties of the algebra of symmetries in non-relativistic Quantum Mechanics, it is possible to develop the *Bargmann argument* to enforce a superselection rule for the mass [66] (cf. chapter 1). However, it has been physically motivated [68] that this argument does not hold and, in this section, we provide a formal derivation of this fact. Specifically, we will algebraically prove that the phase factor appearing in (1.66) is not a mathematical artefact of the algebra of symmetries, rather it is a special-relativistic effect that survives in the non-relativistic regime.

A Galilean transformation of spacetime coordinates is given by (see chapter A for more details on the Galilei group and algebra)

$$\boldsymbol{x}' = R \cdot \boldsymbol{x} + \boldsymbol{a} + \boldsymbol{v}\, t\,, \;\; t' = t + b\,, \tag{4.1}$$

where R is an $SO(3)$ matrix describing a rotation, $\boldsymbol{a} \in \mathbb{R}^3$ and b describe a spatial and a time translation, $\boldsymbol{v} \in \mathbb{R}^3$ is a velocity that parameterises a Galilean boost. These transformations are the symmetries of non-relativistic Quantum Mechanics. Specifically, the symmetry group of Quantum Mechanics is given by a projective unitary representation of the Galilei group, whose algebra is the central extension of the Galilei algebra and is given by

$$\begin{gathered}[H, P_i] = [H, J_i] = 0 \;\;,\;\; \left[G_i, G_j\right] = \left[P_i, P_j\right] = 0 \;\;,\;\; \left[G_i, P_j\right] = i\, M \delta_{ij} \;\;,\;\; [G_i, H] = i\, P_i \;, \\ \left[J_i, G_j\right] = i\, \epsilon_{ij}{}^{k} G_k \;\;,\;\; \left[J_i, P_j\right] = i\, \epsilon_{ij}{}^{k} P_k \;\;,\;\; [J_n, J_m] = i\, \epsilon_{nm}{}^{k} J_k \;\;,\;\; [M, \cdot] = 0 \;.\end{gathered} \tag{4.2}$$

where P_j and H are the spatial and time translations generators, G_j are the generators of Galilean boosts, J_i are the generators of rotations, and M is the center given by the mass. All these generators are Hermitian operators acting on the Hilbert spaces of quantum systems.

Given the algebra above, the Bargmann argument is as follows. Consider a chain of reference frame transformations given by a spatial translation of $\boldsymbol{a}$, a Galilean boost of $\boldsymbol{v}$, a translation of $-\boldsymbol{a}$, and a boost of $-\boldsymbol{v}$. Of course, looking at (4.1), it is immediate to see that the overall transformation is the identity on spacetime coordinates. However, if we consider the representation of this chain of transformations on the Hilbert space of a quantum system, from the algebra (4.2) we get[1]

$$e^{-i\,\boldsymbol{v}\cdot\boldsymbol{G}} e^{-\frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} e^{i\,\boldsymbol{v}\cdot\boldsymbol{G}} e^{\frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} = e^{-i\,\boldsymbol{v}\cdot\boldsymbol{G} - \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P} - \frac{1}{2\hbar} v^i a^j \left[G_i, P_j\right]} e^{i\,\boldsymbol{v}\cdot\boldsymbol{G} + \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P} - \frac{1}{2\hbar} v^i a^j \left[G_i, P_j\right]} = e^{-\frac{i}{\hbar} M \boldsymbol{a}\cdot\boldsymbol{v}}\,, \tag{4.3}$$

where of course the finite transformations are given by the exponentiation of the algebra generators. Therefore, the chain of transformations, which gives the identity on spacetime coordinates, is a phase factor when represented on the Hilbert space of a quantum system. For quantum systems with a definite mass, this factor becomes an irrelevant global phase, so that the chain of transformations is effectively the identity on the quantum system too. However, if we consider a quantum system that is in a superposition of two different masses, m_1 and m_2, then this is no longer the case, as discussed in chapter 1. The fact that this transformation represents the identity on spacetime led Bargmann to conclude that superpositions of different masses are not allowed in a Galilean invariant Quantum Theory [66]. This is known as *Bargmann superselection rule*.

Even if the Bargmann argument seems reasonable, in [68] it has been shown that the phase factor appearing in (4.3) is actually a remnant of the special-relativistic time dilation effect that survives in the non-relativistic limit. To show this, a more general class of reference frame transformations, the extended Galilean transformations that include transformations between reference frames with a relative acceleration, are considered in [68]. Here, we show

[1] To get this result, the Baker-Campbell-Hausdorff is used. Given a Lie group $\mathscr{G}$ and the exponential map from the Lie algebra to the Lie group $e : \mathscr{L} \to \mathscr{G}$, the Baker-Campbell-Hausdorff formula gives the solution to the equation $e^X e^Y = e^Z$ for $X, Y \in \mathscr{L}$. Specifically, $Z = X + Y + \frac{1}{2}[X, Y] + \ldots$, where the other terms of this infinite series are omitted since they all vanish in our present case.

that this phase factor emerges naturally when considering the chain of transformations that yields (4.3) in the relativistic regime, *i.e.* when the generators satisfy the Poincaré algebra, and then taking the non-relativistic limit, so that (4.3) is indeed an effect of the Poincaré symmetries that survives in the non-relativistic regime.

The relevant commutators for the derivation of (4.3) in the relativistic case are

$$\left[N_i, P_j\right] = i\delta_{ij}P_0\,, \ \left[N_i, P_0\right] = iP_i\,, \ \left[P_\mu, P_\nu\right] = 0\,. \tag{4.4}$$

Using the general expression for the Baker-Campbell-Hausdorff formula [271]

$$e^X e^Y = e^Z\,, \ Z = X + \int_0^1 \mathrm{d}t\, g\left(e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y}\right)[Y]\,, \ g(z) := \frac{\log z}{1-\frac{1}{z}} = \sum_{n=0}^{\infty} g_n (z-1)^n\,, \tag{4.5}$$

where $\mathrm{ad}_A[B] := [A,B]$ is the adjoint action and $g(z)$ is an analytic function in the disk given by $\{|z-1| < 1\}$. Considering the chain of transformations in (4.3), in the relativistic case we get

$$T_P(\boldsymbol{\zeta}, \boldsymbol{a}) := e^{-i\boldsymbol{\zeta}\cdot\boldsymbol{N}} e^{-\frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} e^{i\boldsymbol{\zeta}\cdot\boldsymbol{N}} e^{\frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} = \exp\left\{-2i\frac{\boldsymbol{\zeta}\cdot\boldsymbol{a}}{\hbar\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}} h_s(\boldsymbol{\zeta})\left(P_0 + \boldsymbol{\zeta}\cdot\boldsymbol{P}\right)\right\}, \tag{4.6}$$

where $h_s(\boldsymbol{\zeta}) = \sum_{n=0}^{\infty}\sum_{r=0}^{n} g_n (-1)^{n-r}\binom{n}{r}\sinh\left(r\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\right)$ and $\boldsymbol{\zeta}$ is the boost parameter (rapidity). More details on the derivation can be found in section 4.B. Acting with this operator on a momentum eigenstate of an on-shell particle $|\boldsymbol{p}, m\rangle$ we get

$$T_P(\boldsymbol{\zeta}, \boldsymbol{a})\,|\boldsymbol{p}, m\rangle = \exp\left\{-2i\frac{\boldsymbol{\zeta}\cdot\boldsymbol{a}}{\hbar\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}} h_s(\boldsymbol{\zeta})\left(\sqrt{\boldsymbol{p}^2 + m^2c^2} + \boldsymbol{\zeta}\cdot\boldsymbol{p}\right)\right\}|\boldsymbol{p}, m\rangle\,. \tag{4.7}$$

By taking the non-relativistic limit we have $\boldsymbol{\zeta} \approx c^{-1}\boldsymbol{v} \Rightarrow h_s(\boldsymbol{\zeta}) \approx \frac{1}{2}c^{-1}\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}$ and $\boldsymbol{N} = c\boldsymbol{G}$, so that

$$\begin{array}{ccl} \lim\limits_{c\to\infty} T_P(\boldsymbol{\zeta}, \boldsymbol{a})\,|\boldsymbol{p}, m\rangle &=& \lim\limits_{c\to\infty} \exp\left\{-2i\frac{\boldsymbol{\zeta}\cdot\boldsymbol{a}}{\hbar\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}} h_s(\boldsymbol{\zeta})\left(\sqrt{\boldsymbol{p}^2 + m^2c^2} + \boldsymbol{\zeta}\cdot\boldsymbol{p}\right)\right\}|\boldsymbol{p}, m\rangle \\ \parallel & & \parallel \\ T_G(\boldsymbol{v}, \boldsymbol{a})\,|\boldsymbol{p}, m\rangle &=& e^{-\frac{i}{\hbar}m\boldsymbol{a}\cdot\boldsymbol{v}}\,|\boldsymbol{p}, m\rangle = e^{-\frac{i}{\hbar}M\boldsymbol{a}\cdot\boldsymbol{v}}\,|\boldsymbol{p}, m\rangle\,, \end{array} \tag{4.8}$$

where $T_G(\boldsymbol{v}, \boldsymbol{a}) := e^{-i\boldsymbol{v}\cdot\boldsymbol{G}} e^{-\frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} e^{i\boldsymbol{v}\cdot\boldsymbol{G}} e^{\frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}}$. This result formally shows that the phase factor in (4.3) is inherently a consequence of the special-relativistic symmetry algebra that survives in the non-relativistic limit.

From this derivation, as well as from [68], it is clear that the Bargmann argument for a mass superselection rule in non-relativistic Quantum Mechanics does not hold, meaning that mass superpositions are allowed, in principle, both in relativistic and non-relativistic Quantum Mechanics. However, such superpositions are quite uncommon among fundamental (*i.e.* not composite) particles. In particular, the only fundamental particles that are observed to be in mass superpositions are flavour neutrinos. Is there a fundamental origin for the *emergence* of such a superselection for the mass? In the following, we investigate this issue.

4.2 A brief review of some tools

In this section, we will give a brief review of some tools that will be used in section 4.3 to derive the main result of this work. We start with quantum reference frames, namely reference frames whose degrees of freedom, which are used as a reference for other systems, are quantised. Then we review the incoherent group average procedure, which describes the reference frame transformation between the reference frames of two observers that have no knowledge of each other's reference frames. Finally, we discuss the notion of quantum clocks, which are physical systems acting as temporal reference frames. We will discuss how quantum clocks can be used to perform perfect time measurements, in which the time is sharply determined, or coarse-grained time measurement with finite resolution, described by POVMs.

4.2.1 Quantum reference frames

In the quantum reference frames formalism [51, 102–126], reference frames are not assumed to be abstract entities, rather they become associated to physical systems. These can also be quantum in nature, so that they can be in superposition of some classical states, hence defining quantum reference frames. In this subsection, we will denote operators with a hat, contrary to the previous section, to make the difference with the classical reference frame setting more evident.

The simplest setting, which will be enough for our work, involves three physical systems, A, B and S, one of them defining a reference frame in which the other two are described. In standard Quantum Mechanics, if we describe system S in a reference frame centred in A and we want to change from this to a reference frame centred in a position away from A by a quantity α, we have to apply the translation $e^{\frac{i}{\hbar}\alpha\hat{p}_S}$ to the state of system S. This acts on position eigenstates as $e^{\frac{i}{\hbar}\alpha\hat{p}_S}|x\rangle_S = |x-\alpha\rangle_S$ and on general states by linearity. If we want now to centre the reference frame on system B and the latter is described, with respect to A, by a generic quantum state, it is in general not possible to translate the reference frame by a fixed amount, since system B can be, in general, in superposition of different positions. To overcome this difficulty, we consider a generalisation of the above translation transformation. Specifically, we enlarge the Hilbert space by adding to the Hilbert space of system S the Hilbert space of system B, and generalise the translation transformation to $e^{\frac{i}{\hbar}\hat{x}_B\otimes\hat{p}_S}$ [102]. This translation is now "*quantum-controlled*" by the state of the system B. The transformed state in the new reference frame will be given by

$$e^{\frac{i}{\hbar}\hat{x}_B\otimes\hat{p}_S}\left|\phi\right\rangle_B\otimes\left|\psi\right\rangle_S . \tag{4.9}$$

Another thing to take into account is that in the reference frame of A we describe systems B and S, while in the reference frame defined by B the states of A and S are described. For this reason we need a "swap" operator which maps B's Hilbert space and observables into A's ones [102]. This is denoted with $\widehat{\mathscr{P}}_{A\to B}$, therefore the overall transformation is given by

$$\widehat{S}_x = \widehat{\mathscr{P}}_{A\to B}\, e^{\frac{i}{\hbar}\hat{x}_B\otimes\hat{p}_S}\,,\quad \widehat{S}_x : \mathscr{H}_B^{(A)}\otimes\mathscr{H}_S^{(A)} \to \mathscr{H}_A^{(B)}\otimes\mathscr{H}_S^{(B)}\,, \tag{4.10}$$

where $\mathscr{H}_I^{(J)}$ is the Hilbert space of system I as described with respect to system J.

An important remark is that the above considerations are for physical systems at fixed time. If we want to take into account the dynamics and transform the state at time t in the reference frame of A to the state at time t' in the reference frame of B, we need to do some additional steps. Basically, we first have to translate back in time system B, then we need to apply the quantum reference frame transformation to system S (in this case, it will also contain a time translation) and B (by means of the swap operator) and finally we translate at time t' system A in the new reference frame B. In general, the quantum reference frame transformation will be given by [102]

$$\widehat{U} = e^{-\frac{i}{\hbar}\widehat{H}_A t'} \widehat{\mathscr{P}}_{A\to B}\, \widehat{U}_S^{(B)}\, e^{\frac{i}{\hbar}\widehat{H}_B t}\,, \tag{4.11}$$

where $\widehat{U}_S^{(B)}$ is a generic transformation on system S quantum-controlled by the state of system B. In the following, we suppress the hat on operators again to simplify the notation.

4.2.2 Incoherent group average

In the quantum reference frame formalism, the three physical systems are on the same footing and any of them can be considered as the one defining the quantum reference frame in which the other two are described. However, in our work, we will consider A or B as the physical system defining the reference frame ("*observer*") in which the system S is described ("*system*") together with B or A.

An important ingredient for our work is the notion of *knowledge* that one observer has about the other observer's reference frame, and how B describes system S in the absence of such knowledge about A's reference frame [217, 270, 272, 273]. We will first analyse this problem with classical reference frames and then see how to incorporate it in the quantum reference frame setting.

Consider two observers A and B with their classical reference frames. Let us consider that A prepares a quantum system S in a given state. This state is prepared with respect to the reference frame of A: for instance, A can prepare a spin in a superposition of up and down with respect to her z axis, hence the state depends on which axis she chooses to call z. If she sends this system to B, he can recover information about this system only if he knows the relation between his reference frame and A's. In the spin example, B has to know the relative orientation between the two reference frames. If B does not know this relation, the state that he receives is effectively a random state given by an average over all the possible transformations relating the two reference frames. This is called *incoherent group average* [270]. In the spin example, if B does not know the relative orientation, how can he know where A's z axis lies? There is no information, for B, in the statement "this is a spin state in a superposition of up and down with respect to my z axis" if he does not know which such axis is. This incoherent average is given by [270]

$$\rho_S^{(B)} = \int_G \mathrm{d}\mu(g)\, U(g) \rho_S^{(A)} U^\dagger(g)\,, \tag{4.12}$$

where G is the group of transformation and $\mathrm{d}\mu(g)$ is the Haar measure. In the spin example, the state prepared by A is[2]

$$|\psi\rangle_S^{(A)} = \frac{1}{\sqrt{2}}\Big(|\uparrow\rangle + |\downarrow\rangle\Big) \Rightarrow \rho_S^{(A)} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \tag{4.13}$$

and the group is $SU(2)$ parameterised as

$$U = \begin{pmatrix} e^{i\eta}\cos\frac{\theta}{2} & -e^{-i\delta}\sin\frac{\theta}{2} \\ e^{i\delta}\sin\frac{\theta}{2} & e^{-i\eta}\cos\frac{\theta}{2}\end{pmatrix}, \; \mathrm{d}\mu(\theta,\eta,\delta) = \frac{\sin^2\theta}{2\pi^3}\,\mathrm{d}\theta\,\mathrm{d}\eta\,\mathrm{d}\delta\ .$$

If B knows the relation between the two reference frames (that is, he knows the angles η, δ and η) the transformed state is

$$\rho_S^{(B)} = \frac{1}{2}\begin{pmatrix} 1-\sin\theta\cos\big(\delta+\eta\big) & \frac{1}{2}e^{-2i\delta}\left(e^{2i(\delta+\eta)}(\cos\theta+1)+\cos\theta-1\right) \\ \frac{1}{2}e^{-2i\eta}\left(e^{2i(\delta+\eta)}(\cos\theta-1)+\cos\theta+1\right) & \sin\theta\cos\big(\delta+\eta\big)+1\end{pmatrix}, \tag{4.14}$$

while, if he does not know such relation, he effectively describes the state he receives as the average over the rotation group of (4.14), namely $\rho_S^{(B)} = \frac{1}{2}\mathbb{1}_2$. This means that the averaging procedure is a completely decohering channel in this case. In the general case, the Hilbert space of the system will decompose, under the action of a generic symmetry group G, into a direct sum of *charge sectors* $\mathscr{H}_S = \bigoplus_q \mathscr{H}_S^q$, where each charge sector carries an inequivalent representation of G. Each of these charge sectors can be further decomposed into a tensor product $\mathscr{H}_S^q = \mathscr{M}_S^q \otimes \mathscr{N}_S^q$, where $\mathscr{M}_S^q$ are called *decoherence-full subsystems* (or gauge spaces) [274] and $\mathscr{N}_S^q$ are called *decoherence-free subsystems* (or multiplicity spaces) [275, 276]. The result of the average over G on the former is to map every operator to the identity and act on the latter as the identity. Therefore, in the general case, the incoherent average amounts to a fully decohering channel for subsystems $\mathscr{M}_S^q$, while the information is preserved in the subsystems $\mathscr{N}_S^q$ [270]. This means that, depending on the system and the group of transformations, the incoherent average induces some decoherence with respect to some observables and gives rise to superselection rules [139, 277]: Observables associated with the generators of the transformations are superselected, which means that B cannot observe interference for such observables. In the example above, the spin is superselected, since there are no off-diagonal terms in the density matrix.

It is possible to formalise the incoherent averaging process in the context of quantum reference frames [270]. Consider again three systems, A, B, and S, which are in general all quantum systems. Consider A and B as the systems defining two reference frames. The simple idea is that the transformation from A to B is quantum-controlled by the state of the reference frame B and, in the full Hilbert space comprising the reference frame and the system, there are no decoherence or superselection rules for the global observables of the system. However, it is possible to consider the situation in which B does not have access to the degrees of freedom of A. In this case, if $\mathscr{U} : \mathscr{H}_B^{(A)} \otimes \mathscr{H}_S^{(A)} \to \mathscr{H}_A^{(B)} \otimes \mathscr{H}_S^{(B)}$ is the transformation from the

[2] It is possible to consider a generic superposition $|\psi\rangle_S^{(A)} = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$, $\alpha\alpha^* + \beta\beta^* = 1$. The result of the average is the same for any choice of α and β.

reference frame of A to the reference frame of B, the final state after the transformation is

$$\rho_S^{(B)} = \mathrm{Tr}_A\left\{\mathscr{U}\rho_{BS}^{(A)}\mathscr{U}^\dagger\right\}, \tag{4.15}$$

where $\rho_{BS}^{(A)}$ is the initial state of B and S in the reference frame of A and $\rho_S^{(B)}$ is the state of S in the reference frame of B. This procedure amounts to an incoherent average in which the two reference frames have partial knowledge of the relation among them, in which case the incoherent average is to be taken with a distribution function over the group modelling this partial knowledge [274]

$$\rho_S^{(B)} = \int_G \mathrm{d}\mu(g)\, f(g) U(g)\rho_S^{(A)} U^\dagger(g), \tag{4.16}$$

where if $f(g) = \delta(g)$ we have full knowledge and hence no average on the group, while $f(g) = 1\ \forall\, g \in G$ describes a full incoherent average. If A and B have no knowledge of each other's degrees of freedom, which means that $\rho_{BS}^{(A)} \propto \mathbb{1}_B \otimes \rho_S^{(A)}$, then (4.15) yields the equivalent of an incoherent average.

We can consider a simple example to make these statements more clear. Consider A and B as two (non-relativistic) quantum reference frames that do not share the spatial origin. The operator $\mathscr{U}$ is simply the spatial translation operator $\mathscr{U} = \widehat{\mathscr{P}}_{A\to B} e^{\frac{i}{\hbar}\hat{x}_B \otimes \hat{p}_S}$. Therefore, given a generic state of B and S

$$\rho_{BS}^{(A)} = \int_{\mathbb{R}^2} \mathrm{d}x_B\, \mathrm{d}x'_B \int_{\mathbb{R}^2} \mathrm{d}p_S\, \mathrm{d}p'_S\, \rho\left(x_B, p_S, x'_B, p'_S\right) |x_B\rangle\langle x'_B| \otimes \left|p_S\right\rangle\left\langle p'_S\right|, \tag{4.17}$$

we get

$$\rho_S^{(B)} = \int_{\mathbb{R}} \mathrm{d}x_B \int_{\mathbb{R}^2} \mathrm{d}p_S\, \mathrm{d}p'_S\, \rho\left(x_B, p_S, x_B, p'_S\right) e^{\frac{i}{\hbar}x_B\left(p_S - p'_S\right)} \left|p_S\right\rangle\left\langle p'_S\right|. \tag{4.18}$$

If $\rho_{BS}^{(A)} = \mathbb{1}_B \otimes \rho_S^{(A)}$, then $\rho\left(x_B, p_S, x_B, p'_S\right) = \rho_S\left(p_S, p'_S\right)$, hence

$$\rho_S^{(B)} = \int_{\mathbb{R}} \mathrm{d}x_B \int_{\mathbb{R}^2} \mathrm{d}p_S\, \mathrm{d}p'_S\, \rho\left(p_S, p'_S\right) e^{\frac{i}{\hbar}x_B\left(p_S - p'_S\right)} \left|p_S\right\rangle\left\langle p'_S\right| = \int_{\mathbb{R}} \mathrm{d}p_S\, \rho(p_S, p_S)\, |p_S\rangle\langle p_S|, \tag{4.19}$$

which is exactly the result of an incoherent average over spatial translations.

From this we see that, in the context of quantum reference frames, the process of tracing out the degrees of freedom of one reference frame describes the fact that the other reference frames do not have access to its degrees of freedom. This is equivalent to an incoherent average procedure in which the reference frames have partial knowledge of their relation. If there is no knowledge, then the state of the reference frame to which we want to transform has to contain no information, meaning that it should be a maximally mixed state, proportional to the identity. This yields, therefore, the quantum reference frames equivalent of the incoherent average.

4.2.3 Quantum clocks and coarse-grained time measurements

It is possible to get rid of time as an external parameter in Quantum Mechanics and give it an operational meaning with the formalism of quantum clocks [153]. In this way, time becomes a quantity defined by means of some quantum system acting as a clock [112, 158, 278–281]. Given three physical systems A, B and S, each of them can act as a quantum clock defining the time for the evolution of the other two. Choosing system A as the time reference frame, the state of the overall system can be written as a *history state* [154, 279]

$$|\Psi\rangle = \int_{\mathcal{G}_A} \mathrm{d}\tau_A \left|\tau_A\right\rangle_A \otimes \left|\phi(\tau_A)\right\rangle_{BS} = \int_{\mathcal{G}_A} \mathrm{d}\tau_A \, e^{-\frac{i}{\hbar}(H_B+H_S)\tau_A} \left|\tau_A\right\rangle_A \otimes \left|\phi_0\right\rangle_{BS} , \tag{4.20}$$

where $\left|\phi_0\right\rangle_{BS}$ is a generic state of systems B and S at initial time $\tau_A = 0$, $\left|\tau_A\right\rangle_A$ is the state of the clock when it reads time τ_A and $\mathcal{G}_A$ is the clock space, namely the set of values that the clock can read. This can also be a discrete set [158, 280, 281] and the integral symbol has to be regarded as a generic notation to denote the sum over the clock values. If the clock is discrete, this integral reads

$$|\Psi\rangle = \sum_{n=0}^{s} \left|\tau_A^n\right\rangle_A \otimes \left|\phi\left(\tau_A^n\right)\right\rangle_{BS} , \tag{4.21}$$

$s+1$ being the dimension of the clock space. If the clock Hamiltonian is

$$H_A = \sum_{n=0}^{s} E_A^n \left|E_A^n\right\rangle\!\left\langle E_A^n\right| , \tag{4.22}$$

E_A^n being the clock energy levels, then the clock states are orthonormal and provide a resolution of the identity. These can be written as

$$\left|\tau_A^m\right\rangle_A = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{-\frac{i}{\hbar}E_A^n\tau_A^m} \left|E_A^n\right\rangle_A , \ \left\langle\tau_A^n\middle|\tau_A^m\right\rangle = \delta\left(\tau_A^n - \tau_A^m\right) , \ \mathbb{1}_A = \sum_{n=0}^{s} \left|\tau_A^n\right\rangle\!\left\langle\tau_A^n\right| . \tag{4.23}$$

We will focus on physical systems with lower-bounded Hamiltonians with equally spaced energy levels. In this case, the energies can be written as $E_A^n = E_A^0 + n\delta E_A$, δE_A being the spacing, and the times τ_A^n become $\tau_A^n = \frac{n}{s+1}\mathfrak{T}_A$, so that the set of times ranges from $\tau_A^0 = 0$ to $\tau_A^{s+1} = \mathfrak{T}_A$. In this case, an Hermitian time operator can be defined [158, 159]

$$T_A = \sum_{n=0}^{s} \tau_A^n \left|\tau_A^n\right\rangle\!\left\langle\tau_A^n\right| . \tag{4.24}$$

Asking for the resetting of the clock requires the cyclic condition $\left|\tau_A^0 = 0\right\rangle = \left|\tau_A^{s+1}\right\rangle$, which yields $\mathfrak{T}_A = \frac{2\pi\hbar}{\delta E_A}$. Therefore, the smallest time interval is given by

$$\delta\tau_A = \tau_A^{m+1} - \tau_A^m = \frac{2\pi\hbar}{(s+1)\delta E_A} = \frac{\mathfrak{T}_A}{s+1} . \tag{4.25}$$

From this we see that the range $\mathfrak{T}_A$ of clock A depends on the energy difference between the energy levels: the smaller such spacing is the greatest the range and the difference between two successive times are. The latter also depends on the dimension of the clock space $s+1$. Specifically, in the limit $s \to \infty$ we obtain a continuous clock, for which $\delta\tau_A = 0$. In this case, the clock values will continuously range in the interval $\left[0, \mathfrak{T}_A\right]$. In the case of a continuous flow of time, the Schrödinger equation for the system BS, with respect to time τ_A, can also be recovered [158] (c.f. chapter 2).

In the following, we will consider systems with both external degrees of freedom (*i.e.* position, momentum, ...), defining the state of motion, hence a spatial quantum reference frame, and internal degrees of freedom (*i.e.* spin, atomic energy levels, ...), which can be used to define the clock space. Therefore, the Hamiltonian of a system A will be given by a sum of an external degrees of freedom Hamiltonian H_A and a clock Hamiltonian H_{C_A}. Such a system defines a spacetime quantum reference frame [117]. In this way, we have a formalism in which every physical quantity, including time, is defined by means of a quantum state of a physical system. In this case, the quantum reference frame transformation (4.11) is generalised to

$$U = e^{-\frac{i}{\hbar}\left(H_A+H_{C_A}\right)\otimes T_B}\mathscr{P}_{A\to B}\,U_S^{(B)}\,e^{\frac{i}{\hbar}\left(H_B+H_{C_B}\right)\otimes T_A}\,, \tag{4.26}$$

H_{C_I} (T_I) being the clock Hamiltonian (time operator) of clock I.

With quantum clocks it is possible to describe time measurements. A perfect time measurement is given by a projective measurement on the measured time. Specifically, if S is a physical system described by the quantum clock C then

$$\rho_{SC} = \int_{\mathscr{G}_C^2} \mathrm{d}\tau\,\mathrm{d}\tau'\,\rho_S(\tau,\tau')\otimes|\tau\rangle\langle\tau'|_C \tag{4.27}$$

is the history state of the overall system. The state of system S at a given time τ^* reads

$$\rho_S(\tau^*) = \mathrm{Tr}_C\left\{\rho_{SC}\Pi(\tau^*)\right\} = \mathrm{Tr}_C\left\{\rho_{SC}\,|\tau^*\rangle\langle\tau^*|\right\} = \langle\tau^*|\rho_{SC}|\tau^*\rangle\ . \tag{4.28}$$

This describes a perfect measurement of time. A finite resolution measurement can be described with a POVM instead of a projection. This is given by

$$M(\tau^*;\sigma) = \int_{\mathscr{G}_C} \mathrm{d}\tau\, c_\sigma(\tau-\tau^*)\,|\tau\rangle\langle\tau|\ ,\ \int_{\mathscr{G}_C}\mathrm{d}\tau^*\,M(\tau^*;\sigma) = \mathbb{1}\,, \tag{4.29}$$

where $\int_{\mathscr{G}_C}\mathrm{d}\tau^*\,c_\sigma(\tau-\tau^*) = 1$ and the parameter σ is used to reproduce the perfect measurement limit

$$\lim_{\sigma\to0} c_\sigma(\tau-\tau^*) = \delta(\tau-\tau^*)\ \Rightarrow\ \lim_{\sigma\to0} M(\tau^*;\sigma) = |\tau^*\rangle\langle\tau^*|\ . \tag{4.30}$$

With a POVM, the state of system S at time τ^*, with an coarse-grained time measurement, becomes

$$\rho_S(\tau^*) = \mathrm{Tr}_C\left\{\rho_{SC}M(\tau^*;\sigma)\right\} = \int_{\mathscr{G}_C}\mathrm{d}\tau\,c_\sigma(\tau-\tau^*)\,\langle\tau|\rho_{SC}|\tau\rangle\ . \tag{4.31}$$

In the following, we will take the function $M(\tau^*,\sigma)$ to be a Gaussian centred in τ^* with variance σ. We will refer to the variance as the "resolution" of the clock, so that the smaller the variance, the higher the resolution of the clock is.

4.3 Mass superselection from QRF transformation

We now consider a $1+1$-dimensional model in which we have three systems, A, B, and S. The first two will be regarded as the systems defining two quantum reference frames, each of them associated with a finite resolution quantum clock C_A and C_B, while S will be the "observed" (relativistic) system. Specifically, we will consider B as a detector that detects S at a given time. A will be a generic system that describes the production of S (it could even be the target of a collision or a decaying particle that produces S). Moreover, in the setup described above, we suppose that B has no access to the degrees of freedom of A (and vice versa). Therefore, if the state of the BS system as described by A is $\rho_{BS}^{(A)}$, the state of S in the reference frame of B will be given by (4.15). We want to derive the transformation $\mathscr{U}$ when the physical systems involved are relativistic. We assume, moreover, that A and B have fixed masses m_A and m_B, while S is allowed to be in (in principle even continuous) superposition of different mass eigenstates.

The pure *history state* [154] that describes B and S in the reference frame of A is given by

$$\left|\Psi_{BS}\right\rangle^{(A)} = \int_{\mathscr{G}_A^{(A)}} \mathrm{d}\tau_A\, e^{-\frac{i}{\hbar}\left(H_B + H_{C_B} + H_S\right)\tau_A} \phi(\tau_A) \left|\Psi_0\right\rangle_{B,S,C_B} \otimes \left|\tau_A\right\rangle_{C_A}, \tag{4.32}$$

where $\left|\Psi_0\right\rangle_{B,S,C_B}$ is the initial state of systems B with its clock and S, and $\phi(\tau_A)$ is the state of clock A. More explicitly, the state above reads

$$\begin{aligned}\left|\Psi_{BS}\right\rangle^{(A)} = &\int_{\mathscr{G}_A^{(A)}} \mathrm{d}\tau_A \int_{\mathscr{G}_B^{(A)}} \mathrm{d}t_B \int_{\mathbb{R}} \mathrm{d}m_S \int_{\mathbb{R}} \mathrm{d}\mu_{m_B}(p_B) \int_{\mathbb{R}} \mathrm{d}\mu_{m_S}(p_S)\, \psi_B(p_B; t_B)\phi_S(p_S, m_S)\cdot \\ &\cdot e^{-\frac{i}{\hbar}\left(H_B + H_{C_B} + H_S\right)\tau_A} \phi(\tau_A) \left|p_S, m_S\right\rangle_S \otimes \left|p_B\right\rangle_B \otimes \left|t_B\right\rangle_{C_B} \otimes \left|\tau_A\right\rangle_{C_A},\end{aligned} \tag{4.33}$$

where

$$\mathrm{d}\mu_m(p) := \frac{\mathrm{d}p}{2\sqrt{m^2c^2 + p^2}} \tag{4.34}$$

is the Poincaré covariant measure and $\mathscr{G}_J^{(I)}$ and $\mathscr{G}_J^{*(I)}$ are the clock space of clock J in the reference frame I and its dual, *i.e.* the energy space. If the initial state is not pure, we will have more generally

$$\rho_{BS}^{(A)} = \int_{\mathscr{G}_A^{(A)2}} \mathrm{d}\tau_A \mathrm{d}\tau'_A\, e^{-\frac{i}{\hbar}\left(H_B + H_{C_B} + H_S\right)\tau_A} \rho_{B,S,C_B} e^{\frac{i}{\hbar}\left(H_B + H_{C_B} + H_S\right)\tau'_A} \otimes \rho_{C_A}(\tau_A, \tau'_A) \left|\tau_A\right\rangle\!\left\langle\tau'_A\right|_{C_A}, \tag{4.35}$$

and analogously for (4.33).

To describe the change of reference frame from A to B, we analyse all the factors that arise in $\mathscr{U}$ in (4.15) in the setup described above. We do this by considering the action of transformations on a state of the form (4.32), since for the general case (4.35) the action is easily derived from this simpler case. The details of these derivations can be found in section 4.A. The overall transformation is given by

$$\mathscr{U} = e^{-\frac{i}{\hbar}\left(H_A + H_{C_A}\right)\otimes T_B} \mathscr{P}_{A\to B}\, e^{\frac{i}{\hbar} P_S \otimes x_B}\, e^{-\frac{i}{\hbar} H_S \otimes T_B}\, U_S^{\Lambda}\left(\frac{P_B}{m_B}\right) e^{\frac{i}{\hbar}\left(H_B + H_{C_B}\right)\otimes T_A}, \tag{4.36}$$

where $\mathscr{P}_{A\to B}$ is the operator swapping the degrees of freedom of B with respect to A with the degrees of freedom of A with respect to B, $U_S^{\Lambda}\left(\frac{P_B}{m_B}\right)$ is the boost operator, and the other operators realise the spacetime translations. By applying this transformation and tracing over the degrees of freedom of A, we obtain that the final state of S at time τ_B^* after being emitted at time t_A^* in the reference frame of B is given by

$$\begin{aligned}\rho_S(\tau_B^*; t_A^*) = &\int_{\mathbb{R}} \mathrm{d}\mu_{m_A}(\pi_A) \int_{\mathscr{G}_A^{(B)}} \mathrm{d}t_A \int_{\mathscr{G}_B^{(B)}} \mathrm{d}\tau_B \int_{\mathbb{R}^2} \mathrm{d}m_S \mathrm{d}m'_S \int_{\mathbb{R}^2} \mathrm{d}\mu_{m_S}(\pi_S)\, \mathrm{d}\mu_{m'_S}\left(\pi'_S\right)\cdot \\ &\cdot c_{A,\sigma_A}(t_A - t_A^*) c_{B,\sigma_B}(\tau_B - \tau_B^*) e^{-\frac{i}{\hbar}\left\{\left(E_S^{\pi_S} - E_{S'}^{\pi'_S}\right)\left[\gamma\left(\frac{\pi_A}{m_A}\right) - 1\right] - \left(\pi_S - \pi'_S\right)c\beta\left(\frac{\pi_A}{m_A}\right)\right\}\tau_B}\cdot \\ &\cdot e^{-\frac{i}{\hbar}\left[E_S^{\pi_S} - E_{S'}^{\pi'_S} + \left(\pi_S - \pi'_S\right)c\beta\left(\frac{\pi_A}{m_A}\right)\right] t_A} \rho_S\left(\pi_S, \pi'_S, m_S, m'_S; \pi_A\right) \left|\pi_S, m_S\right\rangle\!\left\langle\pi'_S, m'_S\right|_S,\end{aligned} \tag{4.37}$$

where we considered general coarse-grained time measurements for both clocks C_A and C_B. The details of the computations can be found in section 4.A. There are essentially two cases of interest in this general scenario:

1. **Time τ_B^* is measured while B has no information about the time measurement of A.** This is described by a trace over the clock C_A, and this can be effectively realised by integrating over t_A^*, which amounts to performing a trace over the Hilbert space of clock C_A.

2. **Time τ_B^* is not measured while B has information on the time measurement of A.** This case is analogous to the previous one and can be realised by integrating over τ_B^*.

These two cases can be obtained from the general scenario in which B measures τ_B^* and knows t_A^*, in which case the state of the system is exactly (4.37) with both $c_{A,\sigma_A}\left(t_A - t_A^*\right)$ and $c_{B,\sigma_B}\left(\tau_B - \tau_B^*\right)$ being non-trivial functions. In the following, we will choose Gaussian profiles for the functions $c_{A,\sigma_A}\left(t_A - t_A^*\right)$ and $c_{B,\sigma_B}\left(\tau_B - \tau_B^*\right)$. In this way, it is possible to perform the computations in this general setting and then consider some limiting procedures to recover the two cases above. Specifically, by taking

$$c_{B,\sigma_B}(\tau_B - \tau_B^*) = \frac{1}{\sqrt{2\pi}\sigma_B} e^{-\frac{\left(\tau_B - \tau_B^*\right)^2}{2\sigma_B^2}}, \quad c_{A,\sigma_A}(t_A - t_A^*) = \frac{1}{\sqrt{2\pi}\sigma_A} e^{-\frac{\left(t_A - t_A^*\right)^2}{2\sigma_A^2}}, \tag{4.38}$$

the integrals on t_A and τ_B can be easily performed and yield

$$\begin{aligned}
\rho_S(\tau_B^*; t_A^*) = &\int_{\mathbb{R}} \mathrm{d}\mu_{m_A}(\pi_A) \int_{\mathbb{R}^2} \mathrm{d}m_S \,\mathrm{d}m'_S \int_{\mathbb{R}^2} \mathrm{d}\mu_{m_S}(\pi_S)\, \mathrm{d}\mu_{m'_S}\left(\pi'_S\right) \rho_S\left(\pi_S, \pi'_S, m_S, m'_S; \pi_A\right) \cdot \\
&\cdot e^{-\frac{i}{\hbar}\left[E_S^{\pi_S} - E_{S'}^{\pi'_S} + \left(\pi_S - \pi'_S\right) c\beta\left(\frac{\pi_A}{m_A}\right)\right] t_A^*} e^{-\frac{i}{\hbar}\left\{\left(E_S^{\pi_S} - E_{S'}^{\pi'_S}\right)\left[\gamma\left(\frac{\pi_A}{m_A}\right) - 1\right] - \left(\pi_S - \pi'_S\right) c\beta\left(\frac{\pi_A}{m_A}\right)\right\} \tau_B^*} \cdot \\
&\cdot e^{-\left[E_S^{\pi_S} - E_{S'}^{\pi'_S} + \left(\pi_S - \pi'_S\right) c\beta\left(\frac{\pi_A}{m_A}\right)\right]^2 \frac{\sigma_A^2}{2\hbar^2}} e^{-\left\{\left(E_S^{\pi_S} - E_{S'}^{\pi'_S}\right)\left[\gamma\left(\frac{\pi_A}{m_A}\right) - 1\right] - \left(\pi_S - \pi'_S\right) c\beta\left(\frac{\pi_A}{m_A}\right)\right\}^2 \frac{\sigma_B^2}{2\hbar^2}} \cdot \\
&\cdot \left|\pi_S, m_S\right\rangle\left\langle \pi'_S, m'_S\right|_S .
\end{aligned} \tag{4.39}$$

From this result, we see that superpositions between different momenta and different mass states are damped unless the following two conditions are satisfied

$$\begin{aligned}
&-\sqrt{2} \lesssim \left(\Delta E_S^{\pi_S} + c\beta \Delta\pi_S\right)\frac{\sigma_A}{\hbar} \lesssim \sqrt{2}\,, \\
&-\sqrt{2} \lesssim \left[\Delta E_S^{\pi_S}(\gamma - 1) - c\beta\Delta\pi_S\right]\frac{\sigma_B}{\hbar} \lesssim \sqrt{2}\,.
\end{aligned} \tag{4.40}$$

If S is taken to be a relativistic particle $\pi_S, \pi'_S \gg m_S c, m'_S c$ with a narrow momentum distribution $\frac{\left|\pi_S - \pi'_S\right|}{\pi_S} \ll 1$, $\frac{\left|\pi_S - \pi'_S\right|}{\pi'_S} \ll 1$ in the support of $\rho_S\left(\pi_S, \pi'_S, m_S, m'_S; \pi_A\right)$, then for the superposition of different mass eigenstates we get

$$\left|\Delta m_S^2\right| \lesssim \frac{2\sqrt{2}\, E_S \hbar}{c^4 \gamma}\left[\frac{1}{\sigma_A}\left(1 + \frac{\gamma - 1}{|\beta|}\right) + \frac{1}{\sigma_B}\left(1 + \frac{1}{|\beta|}\right)\right]. \tag{4.41}$$

Such a constraint for the mass superposition could be of interest in the context of fundamental particle oscillations. Specifically, we have a constraint on the mass separation that the observer B can observe in superposition depending on the mean energy of the particle E_S, the resolution of the clocks at the source and at the detector, and the boost parameter of A with respect to B. To make the above constraint more readable, we consider the two limiting cases $\sigma_A \to \infty$ and $\sigma_B \to \infty$.

The case in which $\sigma_A \to \infty$ gives the same result as performing the integral over t_A^* in (4.37), namely a trace on the Hilbert space of the clock C_A. In this limit, we get

$$\left|\Delta m_S^2\right| \lesssim \frac{2\sqrt{2}E_S\hbar}{\gamma c^4\sigma_B}\left(1+\frac{1}{|\beta|}\right) \lesssim \frac{2\sqrt{2}E_S\hbar}{\sigma_B c^4}\left(1+\frac{1}{|\beta|}\right). \tag{4.42}$$

In this case we do not get a constraint for the mass superposition in general, since $\left|\Delta m_S^2\right|$ is unbounded for small values of π_A in the integral (4.37). However, if we restrict to the case in which system A is moving with (unknown) relativistic velocity with respect to B, namely if $\pi_A \gtrsim m_A c$, then

$$\left|\Delta m_S^2\right| \lesssim \frac{2\sqrt{2}E_S\hbar}{c^4\sigma_B}\left(1+\frac{1}{|\beta|}\right) \lesssim \frac{2\sqrt{2}E_S\hbar}{c^4\sigma_B}\left(1+\frac{1}{\sqrt{2}}\right) \lesssim \frac{4\sqrt{2}E_S\hbar}{c^4\sigma_B}. \tag{4.43}$$

The other interesting scenario is achieved with $\sigma_B \to \infty$ that yields

$$\left|\Delta m_S^2\right| \lesssim \frac{2\sqrt{2}E_S\hbar}{c^4\sigma_A\gamma}\left(1+\frac{\gamma-1}{|\beta|}\right) \lesssim \frac{4\sqrt{2}E_S\hbar}{c^4\sigma_A}. \tag{4.44}$$

In this case, the mass superposition is constrained independently of the value of π_A. Notice that, in particles oscillations experiments, the measurement time is not observed, and this is typically realised formally with an average over the observation time [3, 218]. This would correspond to the present case in which $\sigma_B \to \infty$. This result can be visualised in fig. 9

The results (4.43) and (4.44) have the same form as the result in [268] in which[3] $\left|\Delta m_S^2\right| c^4 \lesssim 2\sqrt{2}E_S\sigma_E$, where σ_E is the energy uncertainty at production. This is the crucial difference between our results and [268]. In our case, the two constraints (4.43) and (4.44) depend on the resolutions of the cloks C_B and C_A at detection and production. In [268], the quantity σ_E is related to the decay width of the decaying particle that produces S. It is found that charged leptons are not produced as coherent mass superpositions in pion decays and are produced coherently in W decays, then the possibility of observing mass superpositions for charged leptons, in the latter case, is disregarded by means of a standard analysis on the coherence length of a mass superposition. In particular, the coherence length of mass superpositions for charged leptons produced in a W decay is too small to make such a superposition detectable with current technology. In our framework, it is the resolution of the clock that determines the possibility of observing mass superpositions. We can derive the constraints on such resolution so that neutrinos are allowed to be observed in mass superpositions while charged leptons are not. To derive these constraints, we consider the observed mass-squared differences for neutrinos, the typical energies at which they are detected in different oscillation

[3]Notice that σ_E has the dimension of energy, while σ_B and σ_A have dimension of time.

Physical regime	Average energy E_S	Mass-squared differences
Solar neutrinos	$E_S \sim 0.1$ Mev – 10 MeV	$\left\|\Delta m^2_{12}\right\| \sim 10^{-5}\ \mathrm{eV}^2/c^4$, $\left\|\Delta m^2_{13}\right\| \sim \left\|\Delta m^2_{23}\right\| \sim 10^{-3}\ \mathrm{eV}^2/c^4$
Atmospheric neutrinos	$E_S \sim 100$ Mev – 10 TeV	$\left\|\Delta m^2_{12}\right\| \sim 10^{-5}\ \mathrm{eV}^2/c^4$, $\left\|\Delta m^2_{13}\right\| \sim \left\|\Delta m^2_{23}\right\| \sim 10^{-3}\ \mathrm{eV}^2/c^4$
Reactor neutrinos	$E_S \sim 1$ Mev – 1 GeV	$\left\|\Delta m^2_{12}\right\| \sim 10^{-5}\ \mathrm{eV}^2/c^4$, $\left\|\Delta m^2_{13}\right\| \sim \left\|\Delta m^2_{23}\right\| \sim 10^{-3}\ \mathrm{eV}^2/c^4$
Charged leptons in pion decays	$E_S \sim 90$ MeV	$\left\|\Delta m^2_{e\mu}\right\| \sim 100\ \mathrm{MeV}^2/c^4$, $\left\|\Delta m^2_{e\tau}\right\| \sim \left\|\Delta m^2_{\mu\tau}\right\| \sim 1\ \mathrm{GeV}^2/c^4$
Charged leptons in W decays	$E_S \sim 40$ GeV	$\left\|\Delta m^2_{e\mu}\right\| \sim 100\ \mathrm{MeV}^2/c^4$, $\left\|\Delta m^2_{e\tau}\right\| \sim \left\|\Delta m^2_{\mu\tau}\right\| \sim 1\ \mathrm{GeV}^2/c^4$

Table 4.1: Typical energies of neutrinos and charged leptons in different physical regimes. The values of the mass-squared differences for charged leptons are approximated as $\left|\Delta m^2_{e\mu}\right| \sim m^2_\mu$, $\left|\Delta m^2_{e\tau}\right| \sim \left|\Delta m^2_{\mu\tau}\right|$ and the values of the muon and tau masses, as well as the values of the neutrinos mass-squared differences, are taken from [248]. We used the average energies for charged lepton reported in [268] and for neutrino energy ranges we used the values in [282, 283].

regimes [282, 283], the mass-squared differences for charged leptons, and the dypical energies at which they are produced in pion and W decays [268]. These parameters are reported in table 4.1.

By requiring that neutrinos (charged leptons) should (not) be observed in mass superpostions, independently of the energy regime and the mass-squared difference, we get the following constraint for the resolution of the clocks

$$10^{-19}\ \mathrm{s} \lesssim \sigma_{A,B} \lesssim 10^{3}\ \mathrm{s} \tag{4.45}$$

where the upper bound comes from the requirement of being able to witness mass superpositions for flavour neutrinos, while the lower bound enforces the mass superselection for charged leptons. This interval falls well within our current sensitivity. For comparison, the order of magnitude necessary to minimally violate the lower bound is roughly four orders of magnitude away from the shortest pulses of laser light created yet, with a duration of $\sim 10^{-16}$ s [284].

4.4 Discussion

The work presented in this chapter provides new insights into the problem of mass superselection in Quantum Mechanics. We first developed an algebraic proof of the relativistic origin of the phase factor used in the Bargmann argument to enforce a mass superselection rule in non-relativistic Quantum Mechanics, and this formally shows the inconsistency of this argument. Then we presented a novel approach to establish the emergence of a mass superselection rule for fundamental relativistic particles that is based on the properties of quantum reference frame transformations and non-ideal quantum clocks (*i.e.* quantum clocks with finite resolution). Specifically, we considered the spacetime quantum reference frame transformation from the quantum reference frame describing the production of a fundamental particle, taken to be in mass superposition at the source, to the reference frame of the ob-

$$\frac{\left|\Delta m_S^2\right| c^4 \sigma_A}{E_S \hbar} \gg 1\,, \quad \rho_S^{(B)}\left(m_S, m_S'\right) = 0 \qquad\qquad \frac{\left|\Delta m_S^2\right| c^4 \sigma_A}{E_S \hbar} \lesssim 1\,, \quad \rho_S^{(B)}\left(m_S, m_S'\right) \neq 0$$

Figure 9: Cartoon of the main result of this chapter summarised in (4.44), *i.e.* the emergence of a mass superselection rule that depends on the mass-squared difference, the energy of the particle, and the resolution of the clock σ_A. The higher the resolution of the clock, the higher the mass-squared difference that can be observed in superpostion is. The resolution of the clock here is determined in this picture by the thickness of the clock marks (thicker clock marks correspond to higher values of the variance). An analogous graphic representation can be realised for (4.43).

server/detector that detects it[4]. We assumed that the observer does not have access to the other quantum reference frame, so that the state of the particle is actually given by a partial trace over the degrees of freedom of the latter, effectively realising an average of the state of the particle over the Poincaré group of transformations between the two reference frames. The state of the particle detected at a given time in the reference frame of the detector is obtained by means of two POVMs that represent the finite-resolution time readings of the clocks at the source and at the detector. By means of this procedure, we derived a constraint for the mass difference that can be observed to be in superposition from the detector that depends on the resolution of the two clocks, the average energy of the particle, the mass-squared difference itself, and, in the general case, the velocity of the source in the reference frame of the detector. If a particle is observed to be in mass superposition, the resolutions of the clocks do not satisfy the constraint.

In the physically relevant case, when an average over the observation time is taken [218], the constraint no longer depends on the velocity of the source, thus we are able to derive the interval in which the resolution of the clock at the source has to lie in such a way that flavour neutrinos can be detected to be in superposition while charged leptons are superselected in mass. The same constraint applies for the resolution of the clock at the source if the time average is taken on the production time and the source is assumed to have a momentum greater than its mass.

To conclude, we considered a simplified scenario in $1+1$ dimensions with free particles. These assumptions are enough to derive a selective superselection rule for the mass that sin-

[4]There is no quantum clock associated with the particle since the latter is assumed to be fundamental, *i.e.* with no internal structure apt to define a quantum clock. The same results that we obtained can also be derived by considering a quantum clock associated with the particle and taking a partial trace over this clock's degrees of freedom, which would represent the inability to access the internal structure of the particle.

gles out flavour neutrinos as the only fundamental particles that can be observed in mass superpositions. However, it would be interesting to generalise our work to the $1+3$ dimensional setting and include interactions and also a more robust definition of measurement events, along the lines of the perspective-neutral framework presented in [117]. In this way, our formalism could be extended to investigate the emergence of other superselection rules in Quantum Theory, *e.g.* superselection of spin and electric charge [139, 285, 286]. This would establish a relational origin of superselection rules in Quantum Theory, rooted in the properties of quantum reference frame transformations.

4.A Derivation of the transformed state

In this appendix, we show the derivation of the state (4.37).

We start by performing the change of reference frame in the description of the density operator corresponding to (4.33) that is given by[5]

$$
\begin{aligned}
\rho_{BS}^{(A)} = &\int_{\mathscr{G}_A^{(A)2}} \mathrm{d}\tau_A \, \mathrm{d}\tau'_A \int_{\mathscr{G}_B^{(A)2}} \mathrm{d}t_B \, \mathrm{d}t'_B \int_{\mathbb{R}^2} \mathrm{d}m_S \, \mathrm{d}m'_S \int_{\mathbb{R}^2} \mathrm{d}\mu_{m_B}(p_B) \, \mathrm{d}\mu_{m_B}(p'_B) \int_{\mathbb{R}^2} \mathrm{d}\mu_{m_S}(p_S) \, \mathrm{d}\mu_{m'_S}\left(p'_S\right) \cdot \\
&\cdot \rho_{C_A}(\tau_A, \tau'_A) \rho_{BS}\left(p_B, p_S, p'_B, p'_S, m_S, m'_S\right) \rho_{C_B}(t_B, t'_B) e^{-\frac{i}{\hbar}\left(H_B + H_{C_B}\right)\tau_A} e^{-\frac{i}{\hbar}\left(E_S^{p_S}\tau_A - E_{S'}^{p'_S}\tau'_A\right)} \cdot \\
&\cdot \left|p_S, m_S\right\rangle\left\langle p'_S, m'_S\right|_S \otimes |p_B\rangle\langle p'_B|_B \otimes |t_B\rangle\langle t'_B|_{C_B} \otimes |\tau_A\rangle\langle \tau'_A|_{C_A} \, e^{\frac{i}{\hbar}\left(H_B + H_{C_B}\right)\tau'_A} .
\end{aligned}
\tag{4.A.1}
$$

To perform the change of reference frame, we have to perform the following steps:

- First, we need to evolve back in time system B, which is achieved with $e^{i\left(H_B + H_{C_B}\right)\otimes T_A}$. This simply cancels the $e^{-i\left(H_B + H_{C_B}\right)\tau_A}$ factor.
- A Poincaré transformation can be written as a Lorentz boost followed by a spacetime translation. The next step is therefore a Lorentz boost on system S. This is given by an operator $U_S^{\Lambda}\left(\frac{p_B}{m_B}\right)$ that transforms the momentum of the system S as

$$
U_S^{\Lambda}\left(\frac{P_B}{m_B}\right) |p_S, m_S\rangle_S \otimes |p_B\rangle_B = |\pi_S, m_S\rangle_S \otimes |p_B\rangle_B \ , \ \ \pi_S = \gamma\left(\frac{p_B}{m_B}\right)\left[p_S - \beta\left(\frac{p_B}{m_B}\right)\sqrt{m_S^2 c^2 + p_S^2}\right] , \tag{4.A.2}
$$

  where

$$
\gamma\left(\frac{p_B}{m_B}\right) := \sqrt{1 + \frac{p_B^2}{m_B^2 c^2}} = \frac{1}{\sqrt{1 - \beta^2\left(\frac{p_B}{m_B}\right)}} \ , \ \ \beta\left(\frac{p_B}{m_B}\right) := \frac{p_B}{m_B c \gamma\left(\frac{p_B}{m_B}\right)} . \tag{4.A.3}
$$

- After the Lorentz boost, we have a spacetime translation of system S, performed with $e^{\frac{i}{\hbar}P_S \otimes x_B} e^{-\frac{i}{\hbar}\left(H_S + H_{C_S}\right)\otimes T_B}$. This results in

$$
\begin{aligned}
&e^{\frac{i}{\hbar}P_S \otimes x_B} e^{-\frac{i}{\hbar}H_S \otimes T_B} \left|\pi_S, m_S\right\rangle_S \otimes \left|p_B\right\rangle_B \otimes \left|t_B\right\rangle_{C_B} = \\
&= \int \mathrm{d}x_B \, \mathrm{d}q_B \, \langle x_B|p_B\rangle \cdot \langle q_B|x_B\rangle \, e^{\frac{i}{\hbar}\pi_S x_B} e^{-\frac{i}{\hbar}E_S^{\pi_S} t_B} \left|\pi_S, m_S\right\rangle_S \otimes \left|q_B\right\rangle_B \otimes \left|t_B\right\rangle_{C_B} ,
\end{aligned}
\tag{4.A.4}
$$

  where $E_S^{\pi_S} := \sqrt{\pi_S^2 c^2 + m_S^2 c^4}$.

[5] $\rho_{BS}\left(p_B, p_S, p'_B, p'_S, m_S, m'_S; t, t'_B\right) = \psi_B(p_B; t_B)\phi_S(p_S, m_S)\psi_B^*\left(p'_B; t'_B\right)\phi_S^*\left(p'_S, m'_S\right)$ and $E_{S'}^{p'_S} := \sqrt{{p'_S}^2 c^2 + {m'_S}^2 c^4}$. Of course, we could start from directly with a density operator formalism and relax the hypothesis that systems B and S are pure states. In this case, ρ_{BS} would be a general mixed state.

- All transformations on S have been performed, therefore we now have to exchange the degrees of freedom of B with the degrees of freedom of A with a swap operator, $\mathscr{P}_{A\to B}$. The action of this swap operator is given by

$$\mathscr{P}_{A\to B}\left|q_B\right\rangle_B \otimes \left|t_B\right\rangle_{C_B} \otimes \left|\tau_A\right\rangle_{C_A} = \left|-\frac{m_A}{m_B}q_B\right\rangle_A \otimes \left|\gamma^{-1}\left(\frac{q_B}{m_B}\right)t_B\right\rangle_{C_B} \otimes \left|\gamma\left(\frac{q_B}{m_B}\right)\tau_A\right\rangle_{C_A}. \tag{4.A.5}$$

  Then, with a change of variables $\pi_A := -\frac{m_A}{m_B}q_B$, we have $\gamma\left(\frac{q_B}{m_B}\right) = \gamma\left(\frac{\pi_A}{m_A}\right)$. We also define $\tau_B := \gamma^{-1}\left(\frac{q_B}{m_B}\right)t_B$ and denote with $\mathscr{G}_B^{(B)}$ the new clock space.

- Finally, we need to evolve the system A at the correct time with $e^{-\frac{i}{\hbar}\left(H_A+H_{C_A}\right)\otimes T_B}$. This gives

$$\begin{aligned}
&e^{-\frac{i}{\hbar}\left(H_A+H_{C_A}\right)\otimes T_B}\left|\pi_A\right\rangle_A \otimes \left|\gamma\left(\frac{\pi_A}{m_A}\right)\tau_A\right\rangle \otimes \left|\tau_B\right\rangle_{C_B} = \\
&\qquad = e^{-\frac{i}{\hbar}E_A^{\pi_A}\tau_B}\left|\pi_A\right\rangle_A \otimes \left|\gamma\left(\frac{\pi_A}{m_A}\right)\tau_A + \tau_B\right\rangle \otimes \left|\tau_B\right\rangle_{C_B},
\end{aligned} \tag{4.A.6}$$

  where we have used the covariance condition $e^{-\frac{i}{\hbar}H_C t'}\left|t\right\rangle_C = \left|t+t'\right\rangle_C$.

Therefore, the overall transformation is given by

$$\mathscr{U} = e^{-\frac{i}{\hbar}\left(H_A+H_{C_A}\right)\otimes T_B}\mathscr{P}_{A\to B}\, e^{\frac{i}{\hbar}P_S\otimes x_B}\, e^{-\frac{i}{\hbar}H_S\otimes T_B}\, U_S^{\Lambda}\left(\frac{P_B}{m_B}\right) e^{\frac{i}{\hbar}\left(H_B+H_{C_B}\right)\otimes T_A}, \tag{4.A.7}$$

and the transformed state is obtained by means of $\rho_{AS}^{(B)} = \mathscr{U}\rho_{BS}^{(A)}\mathscr{U}^\dagger$. After some long but simple algebra this yields

$$\begin{aligned}
\rho_{AS}^{(B)} = &\int_{\mathbb{R}^2} \mathrm{d}\mu_{m_A}(\pi_A)\,\mathrm{d}\mu_{m_A}(\pi'_A)\int_{\mathscr{G}_A^{(A)2}}\mathrm{d}\tau_A\,\mathrm{d}\tau'_A\int_{\mathscr{G}_B^{(B)2}}\mathrm{d}\tau_B\,\mathrm{d}\tau'_B\int_{\mathbb{R}^2}\mathrm{d}m_S\,\mathrm{d}m'_S\int_{\mathbb{R}^2}\mathrm{d}\mu_{m_S}(\pi_S)\,\mathrm{d}\mu_{m'_S}\left(\pi'_S\right)\cdot \\
&\cdot\int_{\mathbb{R}^2}\mathrm{d}\mu_{m_B}(p_B)\,\mathrm{d}\mu_{m_B}(p'_B)\int_{\mathbb{R}}^2\frac{\mathrm{d}x_B}{2\pi\hbar}\frac{\mathrm{d}x'_B}{2\pi\hbar}\rho_{C_A}(\tau_A,\tau'_A)\rho_{C_B}(\tau_B,\tau'_B;\pi_A,\pi'_A)\gamma\left(\frac{\pi_A}{m_A}\right)\gamma\left(\frac{\pi'_A}{m_A}\right)\cdot \\
&\cdot e^{-\frac{i}{\hbar}\left[E_S^{\pi_S}\tau_B\gamma\left(\frac{\pi_A}{m_A}\right)-E_{S'}^{\pi'_S}\tau'_B\gamma\left(\frac{\pi'_A}{m_A}\right)\right]}\rho_{AS}\left(p_B,p'_B,\pi_S,\pi'_S,m_S,m'_S\right)\left\langle x_B|p_B\right\rangle\left\langle-\frac{m_B}{m_A}\pi_A\middle|x_B\right\rangle\left\langle p'_B|x'_B\right\rangle\cdot \\
&\cdot\left\langle x'_B\middle|-\frac{m_B}{m_A}\pi'_A\right\rangle e^{-\frac{i}{\hbar}\left(E_A^{\pi_A}\tau_B-E_A^{\pi'_A}\tau'_B\right)}e^{-\frac{i}{\hbar}\left(\mathscr{K}_{\pi_S}^B\tau_A-\mathscr{K}_{\pi'_S}^B\tau'_A\right)}e^{\frac{i}{\hbar}\left(x_B\pi_S-x'_B\pi'_S\right)}\left|\pi_S,m_S\right\rangle\!\left\langle\pi'_S,m'_S\right|_S\otimes \\
&\otimes\left|\tau_B\right\rangle\!\left\langle\tau'_B\right|_{C_B}\otimes\left|\pi_A\right\rangle\!\left\langle\pi'_A\right|_A\otimes\left|\gamma\left(\frac{\pi_A}{m_A}\right)\tau_A+\tau_B\right\rangle\!\left\langle\gamma\left(\frac{\pi'_A}{m_A}\right)\tau'_A+\tau'_B\right|_{C_A},
\end{aligned} \tag{4.A.8}$$

with

$$\begin{aligned}
\rho_{AS}\left(p_B,p'_B,\pi_S,\pi'_S,m_S,m'_S\right) &:= \rho_{BS}\left(p_B,\Lambda_B^{-1}\pi_S,p'_B,\Lambda_{B'}^{-1}\pi'_S,m_S,m'_S\right), \\
\rho_{C_B}\left(\tau_B,\tau'_B;\pi_A,\pi'_A\right) &:= \rho_{C_B}\left(\tau_B\gamma\left(\frac{\pi_A}{m_A}\right),\tau'_B\gamma\left(\frac{\pi'_A}{m_A}\right)\right),
\end{aligned} \tag{4.A.9}$$

where $\Lambda_B^{-1}\pi_S := \gamma\left(\frac{p_B}{m_B}\right)\left(\pi_S + \beta\left(\frac{p_B}{m_B}\right)E_S^{\pi_S}\right)$, and analogously for $\Lambda_{B'}^{-1}\pi'_S$, all the variables (but the mass of system B) being primed, and

$$\mathcal{K}_{\pi_S}^B := \gamma\left(\frac{p_B}{m_B}\right)\left[E_S^{\pi_S} + \beta\left(\frac{p_B}{m_B}\right)\pi_S\right] . \tag{4.A.10}$$

Since we used

$$\mathbb{1}_B = \int_{\mathbb{R}} \frac{\mathrm{d}x_B}{2\pi\hbar} \left|x_B\right\rangle\!\left\langle x_B\right|_B , \ \mathbb{1}_B = \int_{\mathbb{R}} \mathrm{d}\mu_{m_B}(q_B) \left|q_B\right\rangle\!\left\langle q_B\right|_B \tag{4.A.11}$$

the correct normalisation for plane waves is

$$\left\langle x_B \middle| p_B \right\rangle = \left[4(m_B^2c^4 + p_B^2c^2)\right]^{\frac{1}{4}} e^{\frac{i}{\hbar}x_B p_B} , \tag{4.A.12}$$

with the Dirac delta defined by

$$\delta(a) = \int_{\mathbb{R}} \frac{\mathrm{d}x}{2\pi\hbar} e^{\frac{i}{\hbar}ax} . \tag{4.A.13}$$

Integrating over x_B and x'_B we therefore get

$$\begin{aligned}4\delta\left(p_B + \pi_S + \frac{m_B}{m_A}\pi_A\right)\delta\left(p'_B + \pi'_S + \frac{m_B}{m_A}\pi'_A\right)m_B^2c^2\left[1 + \left(\frac{\pi_A}{m_Ac}\right)^2\right]^{\frac{1}{4}}\left[1 + \left(\frac{\pi'_A}{m_Ac}\right)^2\right]^{\frac{1}{4}} \cdot \\ \cdot\left[1 + \left(\frac{\pi_S}{m_Bc} + \frac{\pi_A}{m_Ac}\right)^2\right]^{\frac{1}{4}}\left[1 + \left(\frac{\pi'_S}{m_Bc} + \frac{\pi'_A}{m_Ac}\right)^2\right]^{\frac{1}{4}} .\end{aligned} \tag{4.A.14}$$

If we consider B to be macroscopic, we have $m_Bc \gg \pi_S$, therefore the above factor becomes

$$4\delta\left(p_B + \pi_S + \frac{m_B}{m_A}\pi_A\right)\delta\left(p'_B + \pi'_S + \frac{m_B}{m_A}\pi'_A\right)m_B^2c^2\left[1 + \left(\frac{\pi_A}{m_Ac}\right)^2\right]^{\frac{1}{2}}\left[1 + \left(\frac{\pi'_A}{m_Ac}\right)^2\right]^{\frac{1}{2}} , \tag{4.A.15}$$

and it just simplifies the measure of the integrals in $\mathrm{d}\mu_{m_B}(p_B)\,\mathrm{d}\mu_{m_B}\left(p'_B\right)$. Moreover, all relativistic factors $\gamma\left(\frac{p_B}{m_B}\right)$ and $\beta\left(\frac{p_B}{m_B}\right)$ become $\gamma\left(\frac{\pi_A}{m_A}\right)$ and $\beta\left(\frac{\pi_A}{m_A}\right)$ (same for primed variables). The state of the system then becomes

$$\begin{aligned}\rho_{AS}^{(B)} = &\int_{\mathbb{R}^2} \mathrm{d}\mu_{m_A}(\pi_A)\,\mathrm{d}\mu_{m_A}(\pi'_A)\int_{\mathcal{G}_A^{(A)2}} \mathrm{d}\tau_A\,\mathrm{d}\tau'_A\int_{\mathcal{G}_B^{(B)2}} \mathrm{d}\tau_B\,\mathrm{d}\tau'_B\int_{\mathbb{R}^2} \mathrm{d}m_S\,\mathrm{d}m'_S\int_{\mathbb{R}^2} \mathrm{d}\mu_{m_S}(\pi_S)\,\mathrm{d}\mu_{m'_S}\left(\pi'_S\right) \cdot \\ &\cdot \rho_{C_A}\left(\tau_A, \tau'_A\right)\rho_{C_B}\left(\tau_B, \tau'_B; \pi_A, \pi'_A\right)\rho_{AS}\left(\pi_S, \pi'_S, \pi_A, \pi'_A, m_S, m'_S\right)\gamma\left(\frac{\pi_A}{m_A}\right)\gamma\left(\frac{\pi'_A}{m_A}\right) \cdot \\ &\cdot e^{-\frac{i}{\hbar}\left[E_S^{\pi_S}\tau_B\gamma\left(\frac{\pi_A}{m_A}\right) - E_{S'}^{\pi'_S}\tau'_B\gamma\left(\frac{\pi'_A}{m_A}\right)\right]} e^{-\frac{i}{\hbar}\left(E_A^{\pi_A}\tau_B - E_A^{\pi'_A}\tau'_B\right)} e^{-\frac{i}{\hbar}\left(\mathcal{K}_{\pi_S}^A\tau_A - \mathcal{K}_{\pi'_S}^A\tau'_A\right)} \cdot \\ &\cdot \left|\pi_S, m_S\right\rangle\!\left\langle\pi'_S, m'_S\right|_S \otimes \left|\tau_B\right\rangle\!\left\langle\tau'_B\right|_{C_B} \otimes \left|\pi_A\right\rangle\!\left\langle\pi'_A\right|_A \otimes \left|\gamma\left(\frac{\pi_A}{m_A}\right)\tau_A + \tau_B\right\rangle\!\left\langle\gamma\left(\frac{\pi'_A}{m_A}\right)\tau'_A + \tau'_B\right|_{C_A} ,\end{aligned} \tag{4.A.16}$$

where we have now $\mathcal{K}^{A}_{\pi_S} := \gamma\left(\frac{\pi_A}{m_A}\right)\left[E^{\pi_S}_S + c\beta\left(\frac{\pi_A}{m_A}\right)\pi_S\right]$ and

$$\rho_{AS}\left(\pi_S, \pi'_S, \pi_A, \pi'_A, m_S, m'_S\right) := \rho_{BS}\left(-\pi_S - \frac{m_B}{m_A}\pi_A, \Lambda^{-1}_A \pi_S, -\pi'_S - \frac{m_B}{m_A}\pi'_A, \Lambda^{-1}_{A'}\pi'_S, m_S, m'_S\right). \tag{4.A.17}$$

By tracing over A, we get $4(m_A^2 c^2 + \pi_A^2 c^4)\delta\left(\pi_A - \pi'_A\right)$. In this way, we get rid of the integral over π'_A[6] and, changing variables $t_A := \gamma\left(\frac{\pi_A}{m_A}\right)$, $t'_A := \tau'_A \gamma\left(\frac{\pi_A}{m_A}\right)$, we have

$$\begin{aligned}\rho^{(B)}_S = &\int_{\mathbb{R}} \mathrm{d}\mu_{m_A}(\pi_A) \int_{\mathcal{G}^{(B)2}_A} \mathrm{d}t_A\, \mathrm{d}t'_A \int_{\mathcal{G}^{(B)2}_B} \mathrm{d}\tau_B\, \mathrm{d}\tau'_B \int_{\mathbb{R}^2} \mathrm{d}m_S\, \mathrm{d}m'_S \int_{\mathbb{R}^2} \mathrm{d}\mu_{m_S}(\pi_S)\, \mathrm{d}\mu_{m'_S}\left(\pi'_S\right) \cdot \\ &\cdot \rho_{C_A}\left(t_A, t'_A; \pi_A\right) \rho_{C_B}\left(\tau_B, \tau'_B; \pi_A, \pi_A\right) \rho_S\left(\pi_S, \pi'_S, m_S, m'_S; \pi_A\right) e^{-\frac{i}{\hbar}\left(E^{\pi_S}_S \tau_B - E^{\pi'_S}_{S'}\tau'_B\right)\gamma\left(\frac{\pi_A}{m_A}\right)} e^{-\frac{i}{\hbar}\left(\tau_B - \tau'_B\right)E^{\pi_A}_A} \cdot \\ &\cdot e^{-\frac{i}{\hbar}\left(\mathcal{K}^A_{\pi_S} t_A - \mathcal{K}^A_{\pi'_S} t'_A\right)\gamma^{-1}\left(\frac{\pi_A}{m_A}\right)} \left|\pi_S, m_S\right\rangle\!\left\langle \pi'_S, m'_S\right|_S \otimes \left|\tau_B\right\rangle\!\left\langle\tau'_B\right|_{C_B} \otimes \left|t_A + \tau_B\right\rangle\!\left\langle t'_A + \tau'_B\right|_{C_A},\end{aligned} \tag{4.A.18}$$

where

$$\rho_{C_A}\left(t_A, t'_A; \pi_A\right) := \rho_{C_A}\left(\tau_A\left[\gamma\left(\frac{\pi_A}{m_A}\right)\right]^{-1}, \tau'_A\left[\gamma\left(\frac{\pi_A}{m_A}\right)\right]^{-1}\right), \tag{4.A.19}$$

$$\rho_S\left(\pi_S, \pi'_S, m_S, m'_S; \pi_A\right) := \rho_{AS}\left(\pi_S, \pi'_S, \pi_A, \pi_A, m_S, m'_S\right).$$

This is the history state of the system S in the reference frame B. We can obtain the state of the system S detected at some time τ^*_B after being emitted at some time t^*_A with finite resolution time measurements by performing two POVMs as shown in section 4.2. Specifically, we have

$$\begin{aligned}\rho_S\left(\tau^*_B; t^*_A\right) &= \mathrm{Tr}_{C_A, C_B}\left\{\rho^{(B)}_S \Pi(\tau^*)\right\} = \mathrm{Tr}_C\left\{\rho_S\left[M_{C_B}(\tau^*_B; \sigma_B) \otimes M_{C_A}(t^*_A; \sigma_A)\right]\right\} = \\ &= \int_{\mathcal{G}^{(B)}_B} \mathrm{d}\tau_B \int_{\mathcal{G}^{(B)}_A} \mathrm{d}t_A\, c_{A,\sigma_A}\left(t_A - t^*_A\right) c_{B,\sigma_B}\left(\tau_B - \tau^*_B\right) \left\langle t_A\right| \otimes \left\langle \tau_B\right| \rho^{(B)}_S \left|\tau_B\right\rangle \otimes \left|t_A\right\rangle.\end{aligned} \tag{4.A.20}$$

Therefore we get

$$\begin{aligned}\rho_S\left(\tau^*_B; t^*_A\right) = &\int_{\mathbb{R}} \mathrm{d}\mu_{m_A}(\pi_A) \int_{\mathcal{G}^{(B)}_A} \mathrm{d}t_A \int_{\mathcal{G}^{(B)}_B} \mathrm{d}\tau_B \int_{\mathbb{R}^2} \mathrm{d}m_S\, \mathrm{d}m'_S \int_{\mathbb{R}^2} \mathrm{d}\mu_{m_S}(\pi_S)\, \mathrm{d}\mu_{m'_S}\left(\pi'_S\right) \cdot \\ &\cdot \rho_{C_A}\left(t_A - \tau_B, t_A - \tau_B; \pi_A\right) \rho_{C_B}\left(\tau_B, \tau_B; \pi_A, \pi_A\right) \rho_S\left(\pi_S, \pi'_S, m_S, m'_S; \pi_A\right) e^{-\frac{i}{\hbar}\left(E^{\pi_S}_S - E^{\pi'_S}_{S'}\right)\tau_B \gamma\left(\frac{\pi_A}{m_A}\right)} \cdot \\ &\cdot e^{-\frac{i}{\hbar}\left(\mathcal{K}^A_{\pi_S} - \mathcal{K}^A_{\pi'_S}\right)(t_A - \tau_B)\gamma^{-1}\left(\frac{\pi_A}{m_A}\right)} c_{B,\sigma_B}\left(\tau_B - \tau^*_B\right) c_{A,\sigma_A}\left(t_A - t^*_A\right) \left|\pi_S, m_S\right\rangle\!\left\langle \pi'_S, m'_S\right|_S,\end{aligned} \tag{4.A.21}$$

Given that

$$\mathcal{K}^A_{\pi_S} = \gamma\left(\frac{\pi_A}{m_A}\right)\left[E^{\pi_S}_S + c\beta\left(\frac{\pi_A}{m_A}\right)\pi_S\right] \tag{4.A.22}$$

[6] The trace over π_A is defined as $\mathrm{Tr}\{\cdot\} := \int_{\mathbb{R}} \mathrm{d}\mu_{m_A}(\pi_A) \left\langle \pi_A \right| \cdot \left| \pi_A \right\rangle$, therefore the measure in the integral over π'_A and the measure of the trace integral simplify the normalisation factor in front of the delta.

the state above becomes

$$\begin{aligned}\rho_S\left(\tau_B^*; t_A^*\right) = &\int_{\mathbb{R}} \mathrm{d}\mu_{m_A}(\pi_A) \int_{\mathscr{G}_A^{(B)}} \mathrm{d}t_A \int_{\mathscr{G}_B^{(B)}} \mathrm{d}\tau_B \int_{\mathbb{R}^2} \mathrm{d}m_S \, \mathrm{d}m_S' \int_{\mathbb{R}^2} \mathrm{d}\mu_{m_S}(\pi_S) \, \mathrm{d}\mu_{m_S'}\left(\pi_S'\right) \cdot \\ &\cdot c_{A,\sigma_A}\left(t_A - t_A^*\right) c_{B,\sigma_B}\left(\tau_B - \tau_B^*\right) e^{-\frac{i}{\hbar}\left\{\left(E_S^{\pi_S} - E_{S'}^{\pi_S'}\right)\left[\gamma\left(\frac{\pi_A}{m_A}\right) - 1\right] - \left(\pi_S - \pi_S'\right) c\beta\left(\frac{\pi_A}{m_A}\right)\right\}\tau_B} \cdot \\ &\cdot e^{-\frac{i}{\hbar}\left[E_S^{\pi_S} - E_{S'}^{\pi_S'} + \left(\pi_S - \pi_S'\right) c\beta\left(\frac{\pi_A}{m_A}\right)\right] t_A} \rho_S\left(\pi_S, \pi_S', m_S, m_S'; \pi_A\right) \left|\pi_S, m_S\right\rangle\left\langle \pi_S', m_S'\right|_S ,\end{aligned} \tag{4.A.23}$$

where we have further considered $\rho_{C_A} = \rho_{C_B} = 1$, namely that all the time readings of both clocks have the same probability of being observed.

4.B Special-relativistic phase factor

In this appendix, we provide the details of the derivation presented in section 4.1. In the following, we will make use of the general expression for the Baker-Campbell-Hausdorff formula (BCH) [271]

$$e^X e^Y = e^Z \, , \;\; Z = X + \int_0^1 \mathrm{d}t \, g\left(e^{\mathrm{ad}_X} e^{t\,\mathrm{ad}_Y}\right)[Y] = X + Y + \frac{1}{2}[X, Y] + \dots \, , \tag{4.B.1}$$

where $\mathrm{ad}_A[B] := [A, B]$ is the adjoint action and

$$g(z) = \frac{\log z}{1 - \frac{1}{z}} = \sum_{n=0}^{\infty} g_n (z-1)^n = 1 + \sum_{n=1}^{\infty} g_n (z-1)^n \, , \tag{4.B.2}$$

is an analytic function in the disk $\{|z-1| < 1\}$.

We will consider a more general chain of transformations with respect to the one in section 4.1, given by

$$e^{-i\boldsymbol{\zeta}\cdot\boldsymbol{N}} e^{-\frac{i}{\hbar}a^0 P_0 - \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} e^{i\boldsymbol{\zeta}\cdot\boldsymbol{N}} e^{\frac{i}{\hbar}a^0 P_0 + \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} \, , \tag{4.B.3}$$

where N_i, P_0, and P_i are the generators of Lorentz boosts, time translations, and spatial translations, respectively. The relevant commutators are

$$\left[N_i, P_j\right] = i\delta_{ij} P_0 \, , \;\; [N_i, P_0] = i P_i \, , \;\; \left[P_\mu, P_\nu\right] = 0 \, . \tag{4.B.4}$$

To perform the computations, we set

$$e^{Z_P} := e^{-i\boldsymbol{\zeta}\cdot\boldsymbol{N}} e^{-\frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} \, , \;\; e^{Z_0} := e^{-i\boldsymbol{\zeta}\cdot\boldsymbol{N}} e^{-\frac{i}{\hbar}a^0 P_0} \, , \tag{4.B.5}$$

so that

$$e^{-i\boldsymbol{\zeta}\cdot\boldsymbol{N}} e^{-\frac{i}{\hbar}a^0 P_0 - \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} = e^{Z_P + Z_0 + i\boldsymbol{\zeta}\cdot\boldsymbol{N}} \, , \tag{4.B.6}$$

since $\left[P_\mu, P_\nu\right] = 0$. We start by computing Z_P. Expanding eq. (4.B.1) we get

$$Z_P = -i\boldsymbol{\zeta}\cdot\boldsymbol{N} - \frac{i}{\hbar}\sum_{j=1}^{3} a_j \sum_{n=0}^{\infty} g_n \int_0^1 \mathrm{d}t \left(e^{-i\,\mathrm{ad}_{\boldsymbol{\zeta}\cdot\boldsymbol{N}}} e^{-\frac{i}{\hbar}t\,\mathrm{ad}_{\boldsymbol{a}\cdot\boldsymbol{P}}} - \mathbb{1}\right)^n \left[P_j\right] . \tag{4.B.7}$$

Since $\left[P_\mu, P_\nu\right] = 0$ we have $\mathrm{ad}_{\boldsymbol{a}\cdot\boldsymbol{P}}[P_j] = 0$, so that

$$\begin{aligned}\left(e^{-i\,\mathrm{ad}_{\boldsymbol{\zeta}\cdot\boldsymbol{N}}} e^{-\frac{i}{\hbar} t\,\mathrm{ad}_{\boldsymbol{a}\cdot\boldsymbol{P}}} - \mathbb{1}\right)^n [P_j] = \left(e^{-i\,\mathrm{ad}_{\boldsymbol{\zeta}\cdot\boldsymbol{N}}} - \mathbb{1}\right)^n [P_j] = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} e^{-i\,r\,\mathrm{ad}_{\boldsymbol{\zeta}\cdot\boldsymbol{N}}} [P_j] = \\ = \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} \sum_{k=0}^{\infty} \frac{r^k}{k!} (-i)^k \left(\mathrm{ad}_{\boldsymbol{\zeta}\cdot\boldsymbol{N}}\right)^k [P_j]\,.\end{aligned} \tag{4.B.8}$$

From this we get

$$\begin{aligned}Z_P = -i\boldsymbol{\zeta}\cdot\boldsymbol{N} - \frac{i}{\hbar}\sum_{j=1}^{3} a_j \sum_{n=0}^{\infty} g_n \sum_{r=0}^{n} (-1)^{n-r}\binom{n}{r} \sum_{k=0}^{\infty}\frac{r^k}{k!}(-i)^k \left(\mathrm{ad}_{\boldsymbol{\zeta}\cdot\boldsymbol{N}}\right)^k [P_j] = -i\boldsymbol{\zeta}\cdot\boldsymbol{N} + \\ - \frac{i}{\hbar}\sum_{j=1}^{3} a_j \sum_{n=0}^{\infty} g_n \sum_{r=0}^{n} (-1)^{n-r}\binom{n}{r} \left\{ \sum_{l=0}^{\infty} \frac{r^{2l}}{(2l)!} i^{2l} \left(\mathrm{ad}_{\boldsymbol{\zeta}\cdot\boldsymbol{N}}\right)^{2l}[P_j] - \sum_{l=0}^{\infty}\frac{r^{2l+1}}{(2l+1)!} i^{2l+1}\left(\mathrm{ad}_{\boldsymbol{\zeta}\cdot\boldsymbol{N}}\right)^{2l+1}[P_j]\right\}.\end{aligned} \tag{4.B.9}$$

Using (4.B.4) we see that $\left(\mathrm{ad}_{\boldsymbol{\zeta}\cdot\boldsymbol{N}}\right)^{2l}[P_j] = i^{2l}\zeta_j \boldsymbol{\zeta}\cdot\boldsymbol{P}\left(\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}\right)^{l-1}$ for $l \geq 1$ and $\left(\mathrm{ad}_{\boldsymbol{\zeta}\cdot\boldsymbol{N}}\right)^{2l}[P_j] = P_j$ for $l = 0$. Analogously, we have $\left(\mathrm{ad}_{\boldsymbol{\zeta}\cdot\boldsymbol{N}}\right)^{2l+1}[P_j] = i^{2l+1}(\boldsymbol{\zeta}\cdot\boldsymbol{\zeta})^l \zeta_j P_0$. From this we obtain

$$\begin{aligned}Z_P = -i\boldsymbol{\zeta}\cdot\boldsymbol{N} - \frac{i}{\hbar}\alpha\,\boldsymbol{a}\cdot\boldsymbol{P} - \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{\zeta}\sum_{n=1}^{\infty} g_n \sum_{r=1}^{n}(-1)^{n-r}\binom{n}{r}\left\{\sum_{l=1}^{\infty}\frac{r^{2l}}{(2l)!} i^{4l}\frac{(\boldsymbol{\zeta}\cdot\boldsymbol{\zeta})^l}{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\boldsymbol{\zeta}\cdot\boldsymbol{P} + \right. \\ \left. - \sum_{l=0}^{\infty}\frac{r^{2l+1}}{(2l+1)!} i^{4l+2}(\boldsymbol{\zeta}\cdot\boldsymbol{\zeta})^l P_0\right\} = -i\boldsymbol{\zeta}\cdot\boldsymbol{N} - \frac{i}{\hbar}\alpha\,\boldsymbol{a}\cdot\boldsymbol{P} + \\ - \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{\zeta}\sum_{n=1}^{\infty} g_n \sum_{r=1}^{n}(-1)^{n-r}\binom{n}{r}\left\{\frac{\boldsymbol{\zeta}\cdot\boldsymbol{P}}{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\left[\cosh\left(r\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\right) - 1\right] + \frac{P_0}{\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}}\sinh\left(r\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\right)\right\} = \\ = -i\boldsymbol{\zeta}\cdot\boldsymbol{N} - \frac{i}{\hbar}\alpha\,\boldsymbol{a}\cdot\boldsymbol{P} - \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{\zeta}\left\{\frac{\boldsymbol{\zeta}\cdot\boldsymbol{P}}{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\left[h_c\left(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\right) - \alpha\right] + \frac{P_0}{\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}} h_s\left(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\right)\right\},\end{aligned} \tag{4.B.10}$$

where

$$\begin{aligned}h_c(x) &:= \sum_{n=0}^{\infty} g_n \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\cosh(rx)\,, \\ h_s(x) &:= \sum_{n=0}^{\infty} g_n \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}\sinh(rx)\,, \\ \alpha &:= \sum_{n=0}^{\infty}(-1)^n g_n \sum_{r=0}^{n}(-1)^r\binom{n}{r}\,.\end{aligned} \tag{4.B.11}$$

Notice that

$$(1+x)^n = \begin{cases} 1\,, & n = 0 \\ \sum_{r=0}^{n} x^r \binom{n}{r}\,, & n > 0 \end{cases} \implies \sum_{r=0}^{n}(-1)^r\binom{n}{r} = \begin{cases} 1\,, & n=0 \\ 0\,, & n>0\end{cases}\,, \tag{4.B.12}$$

therefore $\alpha = g_0 = 1$. Analogously, for later convenience, we also want to compute the sum

$$\sum_{n=0}^{\infty} g_n \sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r} r = g_1 + \sum_{n=2}^{\infty}(-1)^n g_n \sum_{r=1}^{n}(-1)^r\binom{n}{r} r\,. \tag{4.B.13}$$

We observe that

$$nx(1+x)^{n-1} = \sum_{r=1}^{n} x^r \binom{n}{r} r \Longrightarrow \sum_{r=1}^{n} (-1)^r \binom{n}{r} r = 0 \quad \forall\, n \geq 2\,, \tag{4.B.14}$$

therefore

$$\sum_{n=0}^{\infty} g_n \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} r = g_1 = \frac{1}{2}\,. \tag{4.B.15}$$

With $\alpha = 1$ we finally have

$$Z_P = -i\boldsymbol{\zeta}\cdot\boldsymbol{N} - \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P} - \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{\zeta}\left\{\frac{\boldsymbol{\zeta}\cdot\boldsymbol{P}}{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\left[h_c\Big(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\Big) - 1\right] + \frac{P_0}{\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}} h_s\Big(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\Big)\right\}. \tag{4.B.16}$$

With similar steps it is possible to show that

$$Z_0 = -i\boldsymbol{\zeta}\cdot\boldsymbol{N} - \frac{i}{\hbar}a^0 \sum_{n=0}^{\infty} g_n \int_0^1 \mathrm{d}t \left(e^{-i\,\mathrm{ad}_{\boldsymbol{\zeta}\cdot\boldsymbol{N}}} e^{-\frac{i}{\hbar}t\,\mathrm{ad}_{a^0 P_0}} - \mathbb{1}\right)^n [P_0]\,, \tag{4.B.17}$$

reads

$$Z_0 = -i\boldsymbol{\zeta}\cdot\boldsymbol{N} - \frac{i}{\hbar}a^0\left\{h_c\Big(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\Big)P_0 + \frac{\boldsymbol{\zeta}\cdot\boldsymbol{P}}{\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}} h_s\Big(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\Big)\right\}, \tag{4.B.18}$$

so that

$$\begin{aligned} e^{-i\boldsymbol{\zeta}\cdot\boldsymbol{N}} e^{-\frac{i}{\hbar}a^0 P_0 - \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} = \exp\Bigg\{ &- i\boldsymbol{\zeta}\cdot\boldsymbol{N} - \frac{i}{\hbar}a^0 h_c\Big(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\Big)P_0 - \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P} - \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{\zeta}\frac{\boldsymbol{\zeta}\cdot\boldsymbol{P}}{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\left[h_c\Big(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\Big) - 1\right] + \\ &- i\frac{h_s\Big(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\Big)}{\hbar\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}}\Big(a^0\boldsymbol{\zeta}\cdot\boldsymbol{P} + \boldsymbol{a}\cdot\boldsymbol{\zeta}P_0\Big)\Bigg\} =: e^X\,. \end{aligned} \tag{4.B.19}$$

For the other part of the chain of transformations

$$e^{i\boldsymbol{\zeta}\cdot\boldsymbol{N}} e^{\frac{i}{\hbar}a^0 P_0 + \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}}\,, \tag{4.B.20}$$

we have (4.B.19) with $\boldsymbol{\zeta} \to -\boldsymbol{\zeta}$, $a^0 \to -a^0$, and $\boldsymbol{a} \to -\boldsymbol{a}$, thus

$$\begin{aligned} e^{i\boldsymbol{\zeta}\cdot\boldsymbol{N}} e^{\frac{i}{\hbar}a^0 P_0 + \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} = \exp\Bigg\{ & i\boldsymbol{\zeta}\cdot\boldsymbol{N} + \frac{i}{\hbar}a^0 h_c\Big(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\Big)P_0 + \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P} + \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{\zeta}\frac{\boldsymbol{\zeta}\cdot\boldsymbol{P}}{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\left[h_c\Big(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\Big) - 1\right] + \\ &- i\frac{h_s\Big(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\Big)}{\hbar\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}}\Big(a^0\boldsymbol{\zeta}\cdot\boldsymbol{P} + \boldsymbol{a}\cdot\boldsymbol{\zeta}P_0\Big)\Bigg\} =: e^Y\,. \end{aligned} \tag{4.B.21}$$

With the computations above, we now have

$$e^{-i\boldsymbol{\zeta}\cdot\boldsymbol{N}} e^{-\frac{i}{\hbar}a^0 P_0 - \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} e^{i\boldsymbol{\zeta}\cdot\boldsymbol{N}} e^{\frac{i}{\hbar}a^0 P_0 + \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} = e^X e^Y\,, \tag{4.B.22}$$

where X and Y are defined in (4.B.19) and (4.B.21), respectively. To write the product $e^X e^Y$ as e^Z we have to use once again the BCH. However, the commutator between X and Y is simply given by

$$[X, Y] = -2\frac{h_s\Big(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\Big)}{\hbar\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}}\Big[\boldsymbol{\zeta}\cdot\boldsymbol{N}, a^0\boldsymbol{\zeta}\cdot\boldsymbol{P} + \boldsymbol{a}\cdot\boldsymbol{\zeta}P_0\Big] = -2\frac{i}{\hbar}\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\, h_s\Big(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\Big)\left(a^0 P_0 + \frac{\boldsymbol{a}\cdot\boldsymbol{\zeta}}{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\boldsymbol{\zeta}\cdot\boldsymbol{P}\right). \tag{4.B.23}$$

For this reason, we have that $\left[X,\left[X,Y\right]\right]=-\left[Y,\left[X,Y\right]\right]=-i\left[\boldsymbol{\zeta}\cdot\boldsymbol{N},\left[X,Y\right]\right]$, so that all the subsequent commutators in the BCH will be of the form $i^n\left(-i\right)^r\left[\boldsymbol{\zeta}\cdot\boldsymbol{N},\left[X,Y\right]\right]_{n+r}$, where $\left[A,B\right]_l$ denotes l commutators of A with B, namely $\left[A,B\right]_l=\left[A,\left[A,\ldots\left[A,B\right]\right]\right]$. The BCH is antisymmetric in the exchange $X\leftrightarrow Y$ so that we have $e^Xe^Y=e^{X+Y+\frac{1}{2}[X,Y]+W}$, where

$$W:=\sum_{n+r=1}^{\infty}i^{n+r}\left[\boldsymbol{\zeta}\cdot\boldsymbol{N},\left[X,Y\right]\right]_{n+r}\left\{(-1)^r-(-1)^n\right\}. \tag{4.B.24}$$

The term in the curly brackets reads

$$(-1)^r-(-1)^n=\begin{cases}0\,,\ r=2l+1\,,\ n=2p+1\\ 0\,,\ r=2l\,,\ n=2p\\ -2\,,\ r=2l+1\,,\ n=2p\\ 2\,,\ r=2l\,,\ n=2p+1\end{cases}. \tag{4.B.25}$$

The last two lines above are both characterised by $r+n=2(l+p)+1$, therefore we conclude that $W=0$. With this result, we have $e^Xe^Y=e^{X+Y+\frac{1}{2}[X,Y]}$, so that the chain of transformations (4.B.3) reads

$$\exp\left\{-\frac{i}{\hbar}\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\,h_s\!\left(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\right)\left(\frac{2\,\boldsymbol{a}\cdot\boldsymbol{\zeta}}{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}+a^0\right)P_0-\frac{i}{\hbar}h_s\!\left(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\right)\left(2\,a^0+\boldsymbol{a}\cdot\boldsymbol{\zeta}\right)\frac{\boldsymbol{\zeta}\cdot\boldsymbol{P}}{\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}}\right\}. \tag{4.B.26}$$

If we take units of momentum for P_μ, we have $P_0\left|\boldsymbol{p},m\right\rangle=mc\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}}$, where $\left|\boldsymbol{p},m\right\rangle$ is a momentum eigenstate of an on-shell particle. The rapidity parameter is defined in terms of the velocity of the boost as

$$\boldsymbol{\zeta}=\frac{\boldsymbol{\zeta}}{\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}}\operatorname{arctanh}\frac{\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}}{c}\,, \tag{4.B.27}$$

so that, in the non-relativistic limit $\boldsymbol{\zeta}\approx\frac{\boldsymbol{v}}{c}$. From this we have

$$h_s\!\left(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\right)\approx\sum_{n=0}^{\infty}g_n\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}r\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\approx\frac{\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}}{2\,c}\,, \tag{4.B.28}$$

where we have used (4.B.15). Therefore by acting with (4.B.26) on a state $\left|\boldsymbol{p},m\right\rangle$, we get

$$\begin{aligned}&\exp\left\{-\frac{i}{\hbar}\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\,h_s\!\left(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\right)\left(\frac{2\,\boldsymbol{a}\cdot\boldsymbol{\zeta}}{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}+a^0\right)P_0-\frac{i}{\hbar}h_s\!\left(\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}\right)\left(2\,a^0+\boldsymbol{a}\cdot\boldsymbol{\zeta}\right)\frac{\boldsymbol{\zeta}\cdot\boldsymbol{P}}{\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}}\right\}\left|\boldsymbol{p},m\right\rangle\approx\\ &\approx\exp\left\{-i\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2\hbar}\left(\frac{2\,\boldsymbol{a}\cdot\boldsymbol{v}}{\boldsymbol{v}\cdot\boldsymbol{v}}+t_0\right)m\sqrt{1+\frac{\boldsymbol{p}^2}{m^2c^2}}-\frac{i}{\hbar}\left(t_0+\frac{\boldsymbol{a}\cdot\boldsymbol{v}}{2\,c^2}\right)\boldsymbol{v}\cdot\boldsymbol{p}\right\}\left|\boldsymbol{p},m\right\rangle\xrightarrow[c\to\infty]{}\\ &\xrightarrow[c\to\infty]{}\exp\left\{-\frac{i}{\hbar}\left(\boldsymbol{a}\cdot\boldsymbol{v}+\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}t_0\right)m-\frac{i}{\hbar}t_0\,\boldsymbol{v}\cdot\boldsymbol{p}\right\}\left|\boldsymbol{p},m\right\rangle=\exp\left\{-\frac{i}{\hbar}\left(\boldsymbol{a}\cdot\boldsymbol{v}+\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}t_0\right)M-\frac{i}{\hbar}t_0\,\boldsymbol{v}\cdot\boldsymbol{P}\right\}\left|\boldsymbol{p},m\right\rangle\,,\end{aligned} \tag{4.B.29}$$

in the non-relativistic limit, where $a^0 = ct_0$ and M is the mass operator. The last line above is exactly the result of

$$e^{-i\boldsymbol{v}\cdot\boldsymbol{G}} e^{\frac{i}{\hbar}t_0 H - \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} e^{i\boldsymbol{v}\cdot\boldsymbol{G}} e^{-\frac{i}{\hbar}t_0 H + \frac{i}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}} |\boldsymbol{p}, m\rangle \,. \tag{4.B.30}$$

By setting $ct_0 = a^0 = 0$, (4.8) is recovered. Of course, it is possible to obtain the same result from purely algebraic arguments by considering the İnönü-Wigner contraction of the extension of the Poincaré algebra $\boldsymbol{N} = c\boldsymbol{G}$, $P_0 = \frac{H}{c} + Mc$, $a^0 = ct_0$ and $\boldsymbol{\zeta} = \frac{\boldsymbol{\zeta}}{\sqrt{\boldsymbol{\zeta}\cdot\boldsymbol{\zeta}}} \operatorname{arctanh} \frac{\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}}{c}$ in (4.B.26). By taking the $c \to \infty$ limit we get

$$\exp\left\{-\frac{i}{\hbar}\left(\boldsymbol{a}\cdot\boldsymbol{v} + \frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}t_0\right)M - \frac{i}{\hbar}t_0\,\boldsymbol{v}\cdot\boldsymbol{P}\right\}, \tag{4.B.31}$$

that is the operator appearing in the last line of (4.B.29).

Fifth Chapter

Quantum observers: agency-dependence of spacetime and Doubly Quantum Mechanics

General Relativity is, in Einstein's words, "a theory of rods and clocks". Observers are assumed to have a negligible effect on spacetime, which they can probe without influencing it. For all practical purposes, observers are "external" elements to spacetime: nothing they do, in this idealisation, can influence anything other than their degree of knowledge of the geometry of spacetime. In Quantum Theory, observers play a similar "external" role with respect to observed systems. This is known as the "Heisenberg cut" [14, 287]: the separation between an observed quantum system and the observer, which has to be classical. The observer's knowledge, information, and choices are described classically, and this is an essential feature of the Copenhagen interpretation of Quantum Theory. However, due to the quantum nature of observed systems, there are mutually incompatible measurements which the observers can choose to perform (complementarity). Therefore, Quantum Mechanics introduces an important novelty: *the operationally-meaningful properties of the observed systems depend on the choices made by the observer.* Through the impact of these choices the observer is already an "agent" [288] for what concerns measurements in Quantum Theory.

As discussed in chapter 1 and chapter 2, when Quantum Theory and General Relativity are combined, in any of the many approaches to Quantum Gravity, spacetime acquires quantum features, in one form or another [26]. We must then contemplate the possibility that the aforementioned agency-dependence of Quantum Mechanics, as implied by the complementarity principle, might in turn introduce a dependence of certain properties of spacetime on the choices of the observer. The notion of an objective spacetime which all observers agree upon might have to be revised. The way in which spacetime reveals itself to observers may depend on some of their choices. Furthermore, the quantum properties of spacetime will likely affect the spectrum of possible operations available to agents, possibly making them *quantum observers*. These observations suggest a picture in which spacetime and the agency of observers affect each other inextricably, so much so that the "externality" idealisation, a good working hypothesis in General Relativity and Quantum Theory, will have to be abandoned in favour of a notion of "internal" [289] and "quantum" observers that is required by the logical structure of Quantum Gravity. Moreover, in chapter 2 we also noted that to fully develop a theory of quantum observers, we should also contemplate the possibility that the measurement devices that such observers use could be themselves quantum in nature, *i.e.* measurement apparatuses that produce non-classical readings. The exact meaning of this has yet to be understood, as trying to give quantum features to the two distinctive elements of a measurement device, *i.e.* its outputs and the probabilities of observing them, defies our common experience. In the second part of this chapter, we will formally develop a proposal for such a quantum measurement device, by defining the notion of *quantum probability* in a new framework that we call *"Doubly Quantum Mechanics"*, a theory in which the geometrical configurations of physical systems, measurement apparatuses, and reference frame transformations are themselves quantised, in addition to the phase space variables of standard Quantum Mechanics.

The material presented in this chapter, which is based on [2, 4], investigates these largely unexplored issues that arise in laying the foundations for a Quantum Theory of Gravity.

$SU_q(2)$ quantum group model

Most approaches to the Quantum Gravity problem suggest that some fundamental geometric notions that are pervasively used in physics might be "quantised", so that there are fundamental limitations to the measurability of the observables that depend on them. The most studied possibility is that Quantum Gravity effects, which are supposedly characterised by a length scale of the order of the Planck length $\ell_P \sim 1.6 \times 10^{-35} m$, determine a small-scale discreteness or fuzziness in dimensionful quantities like lengths (or distances), areas, and (possibly hyper) volumes [290–298]. These arguments also suggest that the Riemannian geometry that provides the mathematical foundation of General Relativity should emerge as a coarse-grained picture of some other form of geometry [299], one that incorporates the mentioned fuzzy/discrete properties of spacetime at small scales. As discussed in chapter 1, one of the most natural settings to realise such a notion of "quantum spacetime" is provided by non-commutative geometry, where coordinates are promoted to operators satisfying non-trivial commutation relations [34, 35, 55, 300], and the group structure of empty spacetime symmetries is replaced by the notion of quantum groups, where group parameters are also promoted to non-commutative operators [24].

In recent years there has also been a growing interest in investigating putative Quantum Gravity effects in the regime of Quantum Mechanics, see chapter 1. In fact, in this context it is possible to question the logical consistency of the very fundamental assumptions of General Relativity and Quantum Theory [45, 112, 117, 131, 301–307], as well as to concretely realise table-top experiments to test these assumptions at the interface between the two theories [48, 51, 52, 119, 130, 133–138]. In the following, we focus on this physical regime by investigating the effects of quantum group deformations on quantum-mechanical systems, a scarcely explored area [1, 2, 308] since these deformations are typically studied in the context of classical relativistic mechanics [75, 87, 309–316] and quantum field theory [60, 317–320].

A convenient setting to make the first steps in this direction is the spin sector of standard Quantum Mechanics, where physical results concerning spin systems solely depend on the geometrical degrees of freedom of spin systems themselves, the Stern-Gerlach apparatuses used to prepare and/or measure it, and the relative orientation between different observers. In this context, we can harness the complexity of spacetime quantisation by ascribing quantum properties of space to the deformation of the standard $SU(2)$ rotation symmetry, which is promoted to a $SU_q(2)$ quantum symmetry [321], where the dimensionless deformation parameter $q \in \mathbb{C}$ is such that $q = 1$ reproduces the standard $SU(2)$ group. Relying on $SU_q(2)$ to explore the notion of "quantum geometry" is rather natural, since in the classical case $SU(2)$ describes the geometrical configurations of spins and Stern-Gerlach apparatuses and, through a double cover, the rotations in three-dimensional space. In the context of Quantum Gravity, the study of quantum groups that involve a dimensionless deformation scale, such as $SU_q(2)$, is motivated by the observation that General Relativity does not only describe the geometrical properties of spacetime, which describe distances and volumes, but also its *conformal* geometry, *i.e.* angles. One piece of theoretical evidence in favour of quantum/non-classical angles [2,114] in Quantum Gravity is that the introduction of a cosmological constant in different approaches, such as Loop Quantum Gravity/Spin Foam models [322–324] and Group Field Theory [325], requires the deformation of the local gauge group from $SU(2)$ to $SU_q(2)$. In these models, the deformation parameter q is a function of the dimensionless ratio between the Planck length and the Hubble length scale associated with the cosmological constant such that $q \sim 1$. In fact, it has been argued that this reflects a minimal possible resolution in angular measurements, in a universe that is characterised by a fundamental discreteness (a short-distance cut-off) and a cosmological horizon (a large-distance cut-off) [326,327].

The mathematical details of $SU_q(2)$ have been developed extensively [321,328–330]. In this chapter, we want to further explore the physical features of this group to arrive at a prediction for the novel "quantum geometrical" effects that observers would see if, in an effective regime of Quantum Gravity, rotations of reference frames were described by $SU_q(2)$ transformations. We will consider the case in which $q \in [0,1]$ and specifically $q \sim 1$, since we expect the deformation to be of quantum-gravitational origin.

5.1 Quantum protocol for the alignment of reference frames in classical space

The ultimate goal of this chapter is to investigate the non-commutative generalisation of the following setup. Two observers, Alice and Bob, do not share the same Cartesian reference frame and do not know the relation (*i.e.* the rotation matrix) between their reference frames. They want to perform an alignment protocol that allows them to measure the elements of the rotation matrix and hence synchronise their spatial orientations. We will now consider a simple example of such a protocol in which the messengers are qubits, *i.e.* $SU(2)$ spinors. We will then take a journey towards the generalisation of this protocol when the $SU(2)$ gruop is replaced by the $SU_q(2)$ quantum group and, as is often the case, the most important part will not be the end, but what we will find along the journey. In fact, we will see how such a generalisation requires a notion of agency-dependence of space and the formulation of the Doubly Quantum Mechanics framework, in which the notion of quantum measurement device is introduced.

5.1.1 Qubits as $SU(2)$ spinors in standard Quantum Theory

It is well-known that any single qubit state, represented as a density matrix ρ, can be expanded in the basis of the identity $\mathbb{1}$ and Hermitian Pauli matrices σ_i as

$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}) \,, \tag{5.1}$$

where the expectation values of the spin observables give the components of the Bloch vector:

$$r_i = \langle\sigma_i\rangle = \mathrm{Tr}\left\{\rho\,\sigma_i\right\} . \tag{5.2}$$

The latter relation holds since the Pauli matrices form an orthonormal basis

$$\mathrm{Tr}\left\{\sigma_\mu \sigma_\nu\right\} = 2\,\delta_{\mu\nu}\,, \tag{5.3}$$

where $\sigma_0 := \mathbb{1}$. As such, we can also define a 4-dimensional Bloch vector

$$r_\mu = \mathrm{Tr}\left\{\rho\,\sigma_\mu\right\}, \tag{5.4}$$

whose zero-component $r_0 = \mathrm{Tr}\,\rho = 1$ is simply the normalisation of the state.

Consider now a $SU(2)$ transformation of the quantum state. As it turns out, the density matrix (5.1) decomposes into an $SU(2)$-invariant singlet and an $SU(2)$-covariant triplet term, since for any matrix $U \in SU(2)$,

$$\rho \to U\,\rho\,U^\dagger = \frac{1}{2}(\mathbf{1} + \vec{r}\cdot U\,\vec{\sigma}\,U^\dagger) = \frac{1}{2}(\mathbf{1} + \vec{r}\,'\cdot\vec{\sigma})\,, \tag{5.5}$$

where now the transformed Bloch vector is $\vec{r}\,' = R\cdot\vec{r}$ with

$$R_{ij} = \frac{1}{2}\,\mathrm{Tr}\left\{U\sigma_j\,U^\dagger\,\sigma_i\right\} \tag{5.6}$$

an element of $PSU(2) \simeq SO(3)$. In other words, if the quantum state is transformed by a $SU(2)$ matrix, the Bloch vector $\vec{r}$ that represents it transforms like a 3-dimensional vector under spatial rotations.

Clearly, the phase of the $SU(2)$ transformations is cancelled and the adjoint action of $SU(2)$ yields a spatial rotation in its fundamental representation. Hence, the normalisation r_0 is a $SU(2)$-invariant, while $\vec{r}$ transforms in the adjoint representation of $SU(2)$ and thus under the fundamental representation of $SO(3)$.

In this manner, we can associate to every $SU(2)$ transformation of states or spin observables a $SO(3)$ spatial rotation. In particular, we can write the same qubit state relative to different bases of Pauli matrices $U\,\vec{\sigma}\,U^\dagger$ by simply rotating the corresponding Bloch vector accordingly

$$\vec{r}\,'_\rho = \mathrm{Tr}\left\{\rho\,U\,\vec{\sigma}\,U^\dagger\right\} = R^{-1}\cdot\vec{r}_\rho = R^{-1}\cdot\mathrm{Tr}\left\{\rho\,\vec{\sigma}\right\}, \tag{5.7}$$

where $\vec{r}\,'_\rho$ and $\vec{r}_\rho$ are both Bloch vectors of the *same* quantum state ρ, however, written relative to different observable bases. Interpreting $U\,\vec{\sigma}\,U^\dagger$ and $\vec{\sigma}$ as being the observables corresponding to differently orientated sets of Stern-Gerlach devices in the lab, this rewriting allows us to express how observers with different spatially orientated reference frames will "see" the same quantum state of a (possibly an ensemble of) qubit.

5.1.2 Alignment protocol between two laboratories by exchanging qubits

Suppose that Alice and Bob have never met before and thus do not know how their reference frames are orientated with respect to each other. Supposing further that they can communicate classically, they can proceed as follows in order to figure out the relation between their descriptions. Alice prepares three ensembles of qubits, each prepared so that the ensemble state corresponds to a Bloch sphere basis vector. For example, she could prepare three ensembles of N qubits each so that

$$E_{x_A} = \{\vec{r}_{x_A,n} = (1,0,0)\}_{n=1}^{N}\,,\;\; E_{y_A} = \{\vec{r}_{y_A,n} = (0,1,0)\}_{n=1}^{N}\,,\;\; E_{z_A} = \{\vec{r}_{z_A,n} = (0,0,1)\}_{n=1}^{N}\,, \tag{5.8}$$

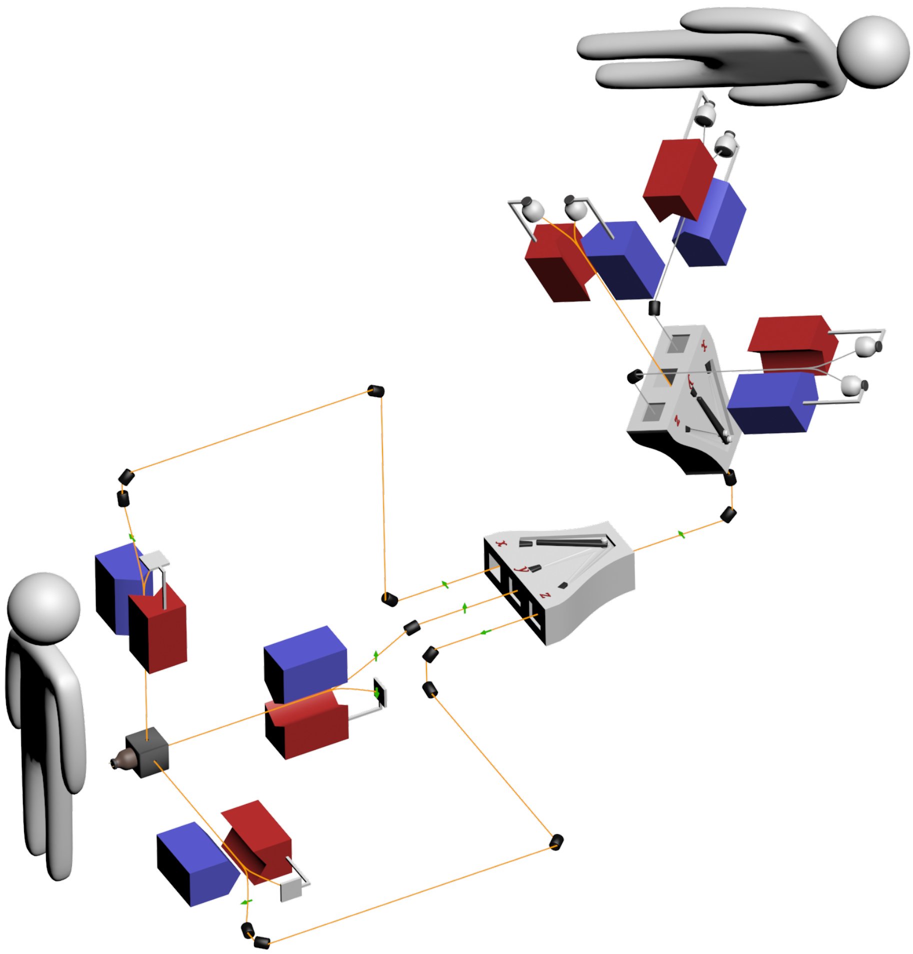

Figure 10: Alice (on the left) prepares a set of N qubits (*e.g.*, electron spins) in the spin up eigenstate of her x axis (*e.g.*, by passing unpolarised electrons through a x-orientated Stern–Gerlach apparatus, and selecting only the ones that emerge with their spins up). She then sends these electrons to Bob, whose laboratory is rotated by an unknown amount with respect to hers. Bob divides these N qubits in three groups and sends each group through a machine that measures the spin along one of his three orthogonal axes (*e.g.*, three perpendicular Stern–Gerlach apparatuses; in the picture, he is passing the electrons through a y-orientated machine). He then counts the number of spin up and spin down measurements that each machine reads, and calculates the expectation value of the corresponding observable. Repeating the experiment for a set of N qubits that Alice selected to be polarised along the $y-$, and, respectively, $z-$axes allows Bob to build a statistics of the expectation values of the nine observables associated to each pair of choices of axes made by him and Alice. In the large-N limit, these expectation values tend to the nine components of the rotation matrix R that connects Alice's reference frame and Bob's. Notice that, in this illustrative picture the electron beams (in orange) are manipulated with some “mirrors” (the black cylinders) which are assumed not to have any effect on the qubit states.

i.e., every qubit of the first ensemble is prepared to be in the "up in x_A direction" pure state, where x_A is Alice's x direction, and similarly for the other two ensembles. Alice can then send Bob these three ensembles to perform state tomography on them. Bob could proceed as follows: he divides each ensemble in three subensembles and measures all qubits of the first subensemble with a Stern-Gerlach (SG) device orientated in the x_B direction, of the second in the y_B direction, and of the third in the z_B direction. This will tell him how to write Alice's states relative to his basis and, via (5.7), what the relative $SO(3)$ rotation between their spatial frames is. For example, relative to Alice, the ensemble state of E_{x_A} appears as

$$\rho_{x_A} = \frac{1}{2}(\mathbb{1} + \vec{r}_{x_A} \cdot \vec{\sigma}_A) = \frac{1}{2}(\mathbb{1} + \sigma_{x_A}) \,. \tag{5.9}$$

Suppose now that Bob's measurement apparatuses are in the following relation with Alice's, $\vec{\sigma}_A = U \vec{\sigma}_B U^\dagger$. Then, using (5.7), we can write the same ensemble state, but now relative to Bob as

$$\rho_{x_A} = \frac{1}{2}\left(\mathbb{1} + \vec{r}_{x_A} \cdot U \vec{\sigma}_B U^\dagger\right) = \frac{1}{2}\left(\mathbb{1} + \vec{r}_{x_B} \cdot \vec{\sigma}_B\right) , \tag{5.10}$$

where $\vec{r}_{x_B} = R \cdot \vec{r}_{x_A}$ and

$$R_{ij} = \frac{1}{2} \mathrm{Tr}\left\{\left(\sigma_A\right)_i U \left(\sigma_A\right)_j U^\dagger\right\} \equiv \frac{1}{2} \mathrm{Tr}\left\{U \left(\sigma_A\right)_j U^\dagger \left(\sigma_A\right)_i\right\} , \tag{5.11}$$

which is equivalent to (5.6) if we remember that U are unitary matrices. Bob can measure $\vec{r}_{x_B} = \mathrm{Tr}\left\{\rho_{x_A} \vec{\sigma}_B\right\}$ using tomography on E_{x_A}, which is why he has to split that ensemble into three subensembles to measure its qubits relative to a basis of SG devices.

Proceeding this way also with E_{y_A}, E_{z_A}, Bob can figure out the Bloch vector basis $\vec{r}_{x_B}$, $\vec{r}_{y_B}$, $\vec{r}_{z_B}$ of Alice's ensemble states relative to his frame, and if Alice also tells him how she describes the same states, namely as $\vec{r}_{x_B}$, $\vec{r}_{y_B}$, $\vec{r}_{z_B}$, then they can simply compute R from the relation of these two sets of bases. In principle, Bob has to perform an infinite number of measurements to determine the rotation matrix connecting his reference frame with Alice's with arbitrary precision. From this, the matrix (5.A.5) admits a direct operational interpretation in terms of (5.11): it is the matrix of the asymptotic $N \to \infty$ values of the averages of the spin measurements that Bob performs on the qubit states sent to him by Alice.

5.2 Quantum Euler angles and agency-dependent spacetime

With the protocol outlined in the previous section, two observers can in principle measure the rotation matrix that relates their reference frames with arbitrary precision. In a quantum space in which $SU(2)$ is replaced by $SU_q(2)$, there can be a fundamental limit that prevents observers from doing the same, *i.e.* a fundamental limit on the precision of such a measurement can originate from the quantum properties of space. Although this is what we want to ultimately do, we shall not describe in its entirety the alignment protocol with $SU_q(2)$ transformations right away. This section reports the results in [2], which is intended as a preliminary investigation of how Quantum Gravity-induced deformations of the classical rotation symmetry, described by the quantum group $SU_q(2)$, could modify the transformation laws among reference frames in an effective regime. We interpret here the states of a representation of its algebra as describing the relative orientation between two reference frames. This leads to a quantisation of one of the Euler angles and to an aspect of *agency-dependence*: space is reconstructed as a collection of fuzzy points, exclusive to each agent, which depends on their choice of reference frame. Each agent can choose only one direction in which points can be sharp, while points in all other directions become fuzzy in a way that depends on this choice. Thus, two agents making different choices will observe the same points with different degrees of fuzziness.

5.2.1 Quantum rotation matrices in $SU_q(2)$

The quantum group $SU_q(2)$ is defined by considering the algebra of complex functions on $SU(2)$, denoted by $C(SU(2))$ and deforming it in a non-commutative way. The generators $\{a, c\}$ are regarded as functions on the group, satisfying non-trivial commutation relations

$$\begin{aligned} &ac = qca \qquad ac^* = qc^*a \qquad cc^* = c^*c \\ &c^*c + a^*a = \mathbb{1} \qquad aa^* - a^*a = (1-q^2)c^*c\,. \end{aligned} \tag{5.12}$$

Here, $\mathbb{1}$ refers to the identity element of the algebra and q is the deformation parameter assumed to be close to 1 and in particular $q \lesssim 1$. Indeed, in the $q \to 1$ limit, we obtain the commutative limit and recover the classical description of $SU(2)$. In passing, we mention that it suffices to consider $q \lesssim 1$ since, if $q > 1$, the mapping $a \mapsto a^*$, $c \mapsto qc^*$ sends the $SU_q(2)$ algebra to the $SU_{q^{-1}}(2)$ one.

To establish a first link with the classical picture, we present the generalisation of the spin-$\frac{1}{2}$ representation (5.A.1), given by

$$U_q = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \qquad a, c \in C(SU_q(2)) \quad q \in (0,1)\,. \tag{5.13}$$

We will now construct one of the key ingredients of the analysis which is a "quantisation" of the $SU(2)$ to $SO(3)$ homomorphism. To do so, we take the classical homomorphism $R_{ij} = \frac{1}{2}\mathrm{Tr}\left\{\sigma_j U^\dagger \sigma_i U\right\}$, where σ_i are the Pauli matrices, and promote U to U_q

$$\left(R_q\right)_{ij} = \frac{1}{2}\mathrm{Tr}\left\{\sigma_j U_q^\dagger \sigma_i U_q\right\}. \tag{5.14}$$

By computing these elements explicitly, we obtain

$$R_q = \begin{pmatrix} \frac{1}{2}(a^2 - qc^2 + (a^*)^2 - q(c^*)^2) & \frac{i}{2}(-a^2 + qc^2 + (a^*)^2 - q(c^*)^2) & \frac{1}{2}(1+q^2)(a^*c + c^*a) \\ \frac{i}{2}(a^2 + qc^2 - (a^*)^2 - q(c^*)^2) & \frac{1}{2}(a^2 + qc^2 + (a^*)^2 + q(c^*)^2) & -\frac{i}{2}(1+q^2)(a^*c - c^*a) \\ -(ac + c^*a^*) & i(ac - c^*a^*) & 1-(1+q^2)cc^* \end{pmatrix} \tag{5.15}$$

and one can check that in the commutative limit $q \to 1$ this coincides with the matrix obtained via the classical homomorphism. This matrix provides a (co)-representation of $SO_q(3)$ related to the (co)-representation found in [330] by a similarity transformation. We note that in (5.14) the cyclic property of the trace operation does not hold due to the non-commutative nature of a and c. This gives rise to a "quantisation ambiguity" which however does not affect our phenomenological results. The relevant technical issues are addressed in section 5.A.

The classical analogue of (5.15) describes all possible elements of $SO(3)$ when varying the complex numbers a and c with continuity. Each of these classical matrices describes a possible relative orientation between observers A and B. In the quantum case (5.15) has no physical meaning when taken alone, but only when paired with a certain state $|\psi\rangle \in \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space on which a and c act. In the classical case, a and c codify information about the alignment of two reference frames, via their angular parameterisation (see below and section 5.A). For this reason, when being promoted to operators, we interpret the states on which they act as those codifying the relative orientation between the reference frames of observers A and B. Physical information about this orientation can be extracted by computing the expectation values $\langle\psi|(R_q)_{ij}|\psi\rangle$ with their relative uncertainties resulting from the non-commutative nature of these matrix elements, given by

$$\Delta_{ij} = \sqrt{\langle\psi|(R_q)^2_{ij}|\psi\rangle - \langle\psi|(R_q)_{ij}|\psi\rangle^2}\,. \tag{5.16}$$

In this framework, the Δ_{ij} do not vanish simultaneously, in general, for a given state $|\psi\rangle$ introducing a "fuzziness" in the alignment procedure. When this occurs, the latter is affected by an intrinsic uncertainty which cannot be eliminated: A and B are no longer able to sharply align their reference frames. In light of these arguments, it is crucial to identify $\mathcal{H}$ and study the representations of operators a and c on it.

$SU_q(2)$ representations and quantum Euler angles

The representations of the algebra (5.12) have been thoroughly studied in [330]. The Hilbert space containing the two unique irreducible representations of the $SU_q(2)$ algebra, $q \in (0,1)$, is $\mathcal{H} = \mathcal{H}_\pi \oplus \mathcal{H}_\rho$ where $\mathcal{H}_\pi = \ell^2 \otimes L^2(S^1) \otimes L^2(S^1)$ and $\mathcal{H}_\rho = L^2(S^1)$. If $\chi, \phi \in [0, 2\pi[$ are coordinates on S^1 and $|n\rangle$ is the canonical basis of ℓ^2, the algebra of functions on $SU_q(2)$ is represented as

$$\rho(a)|\chi\rangle = e^{i\chi}|\chi\rangle \qquad \rho(a^*)|\chi\rangle = e^{-i\chi}|\chi\rangle \qquad \rho(c)|\chi\rangle = \rho(c^*)|\chi\rangle = 0\,, \tag{5.17}$$

$$\begin{gathered}\pi(a)|n,\phi,\chi\rangle = e^{i\chi}\sqrt{1-q^{2n}}\,|n-1,\phi,\chi\rangle \qquad \pi(c)|n,\phi,\chi\rangle = e^{i\phi}q^n|n,\phi,\chi\rangle\,,\\ \pi(a^*)|n,\phi,\chi\rangle = e^{-i\chi}\sqrt{1-q^{2n+2}}\,|n+1,\phi,\chi\rangle \qquad \pi(c^*)|n,\phi,\chi\rangle = e^{-i\phi}q^n|n,\phi,\chi\rangle\,.\end{gathered} \tag{5.18}$$

It is not coincidental that quantum number χ is common in both representations. We show in section 5.C that representation ρ can be obtained as a limit of representation π, in agreement with the fact that in the classical case we only need three real parameters to specify rotations.

Let us give a physical meaning to the quantum numbers appearing in the representations, making a comparison between these relations and the classical parameterisation of a and c for $U \in SU(2)$

$$a = e^{i\eta}\cos\frac{\theta}{2}\,,\quad c = e^{i\delta}\sin\frac{\theta}{2}\,, \tag{5.19}$$

where η, δ and θ are a linear redefinition of Euler angles, as discussed in section 5.A. We note that $\rho(a)$ and $\rho(c)$ act diagonally, therefore we can make the identifications $\chi \equiv \eta$ and $\theta \equiv 0$. For what concerns representation π, c acts diagonally, while a and a^* act as ladder operators.

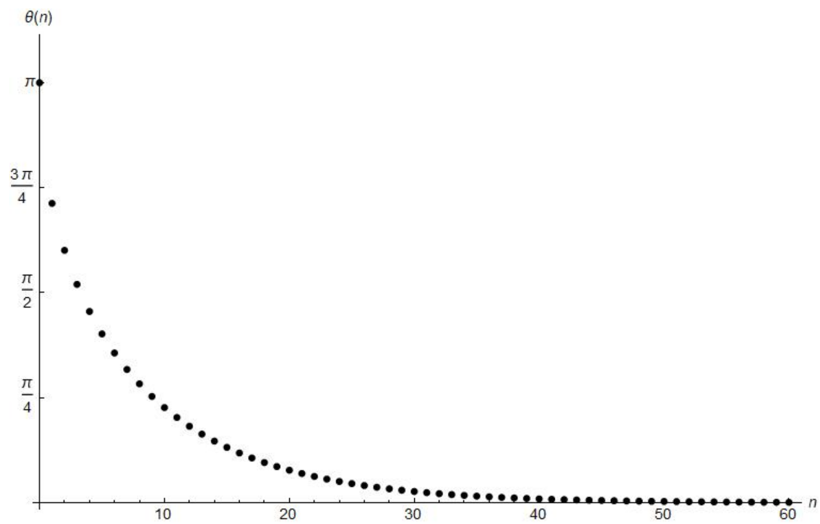

Figure 11: Discretised angles (5.20) computed with $q = 0.9$. $\theta(n)$ is decreasing with n starting from π and approaching 0 for large values of n. Interestingly, the step between consecutive angles decreases as n gets larger, and for very large n we can approximate the angular distribution as continuous. This can be easily verified analytically by computing $\Delta\theta(n) = \theta(n+1) - \theta(n)$ and taking the limit for $n \to \infty$. Another feature that gives robustness to our proposal is that the angular steps decrease as q is closer to 1.

For operator a, we added a phase that is implicitly set to 0 in the literature [321], which does not affect the commutation relations (5.12), but is needed to have a comparison of (5.18) with (5.19). From this, we identify $\chi \equiv \eta$, $\phi \equiv \delta$ and, exploiting the fact that c acts diagonally, we are led to the significant result

$$q^n = \sin\left(\frac{\theta(n)}{2}\right) \iff \theta(n) = 2\arcsin\left(q^n\right). \tag{5.20}$$

The Euler angle θ becomes quantised, a feature captured in fig. 11. The quantisation of θ can also be inferred using $\pi(a)$ (see section 5.A).

Semi-classical rotations

To gain further intuition from the classical picture, where the rotation axis and the angle by which the rotation is performed are specified by three Euler angles, we henceforth focus on "semi-classical" rotations, specified by the three quantum numbers $\theta(n), \phi, \chi$, which describe small deformations of classical rotations defined by these angles. More precisely, the states $|\psi(\theta,\phi,\chi)\rangle$ that produce such deformed rotations should satisfy

$$\begin{cases} \left\langle \psi(\theta,\phi,\chi) \middle| \left(R_q\right)_{ij} \middle| \psi(\theta,\phi,\chi) \right\rangle = R_{ij}(\theta,\phi,\chi) + O(1-q) \\ \Delta^2_{ij} = O(1-q) \end{cases} \quad \forall\, i, j\,, \tag{5.21}$$

where $R_{ij} \in SO(3)$ and where θ is one of the allowed values in (5.20). This prescription restricts the quantum rotations that we can construct to those in which the θ Euler angle can only take the allowed values in (5.20), for a fixed q. In order to clarify the meaning of this prescription, we provide some examples.

A first class of states that trivially satisfies the requirement (5.21) is the set of eigenstates $|\chi\rangle \in \mathscr{H}_\rho$. It is easy to see that the expectation value of the quantum rotation matrix (5.15) in such states describes a rotation of angle 2χ around the z axis, namely

$$\langle\chi|R_q|\chi\rangle = \begin{pmatrix} \cos(2\chi) & -\sin(2\chi) & 0 \\ \sin(2\chi) & \cos(2\chi) & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{5.22}$$

with all the Δ_{ij} being zero. Thus, rotations around the z axis are classical/continuous: two observers, A and B, whose relative orientation is described by $|\chi\rangle \in \mathscr{H}_\rho$, can align themselves sharply.

The relative simplicity of rotations around the z axis is not representative of the richness of structure of other rotations, and this is mainly due to the fact that for generic basis states of the form $|n,\phi,\chi\rangle$ the first condition in (5.21) is not satisfied, forcing us to consider superpositions of such states. Since ϕ and χ are identified with their classical counterparts, we are led to consider superpositions

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n |n,\phi,\chi\rangle \quad , \quad \sum_{n=0}^{\infty} |c_n|^2 = 1\,. \tag{5.23}$$

Among the states of this form that satisfy (5.21), we need to find those for which the uncertainties are kept under control. The best way to do this would be to demand that the coefficients $\{c_n\}$ minimise the functional

$$S\left[\{c_n\}_{n=0}^{\infty},\mu\right] = \sum_{i,j} \Delta_{ij}^2 - \mu\big(\langle\psi|\psi\rangle - 1\big)\,, \tag{5.24}$$

where μ is a Lagrange multiplier enforcing normalisation of $|\psi\rangle$. In general, solving the minimisation problem (5.24) is a daunting computational task. Therefore, in section 5.B, we invoke physical intuition to construct these states. Fixing a value for q, we build superpositions of states $|n,\phi,\chi\rangle$ centred around a certain $\bar{n} \in \mathbb{N}$. The expectation values and uncertainties relative to these states will reproduce deformations of the classical rotation matrices specified by the angles $(\theta(\bar{n}),\phi,\chi)$, in the sense of (5.21).

It is worth noticing that there is a special class of such semi-classical rotations for which some simplifications arise. This is the case of rotations with $\theta = \pi$ around the axes of the $x-y$ plane. The classical theory predicts that $\chi = 0$ and a generic ϕ select a direction in the $x-y$ plane ($\phi = \pi/2$ selects a rotation around the x axis, while $\phi = 0$ describes one around the y axis) around which we rotate of an angle θ. As emphasised before, since ϕ,χ behave classically, the quantum numbers associated to these angles specify the axis of rotation in the quantum case too. For the case $\theta(0) = \pi$, building a superposition of the basis states is not necessary to satisfy (5.21) and the basis state $|0,\phi,0\rangle$ is sufficient to describe the corresponding semi-classical rotation around an axis in the $x-y$ plane, specified by ϕ. The expectation value of R_q on such state is

$$\langle 0,\phi,0|R_q|0,\phi,0\rangle = \begin{pmatrix} -q\cos(2\phi) & -q\sin(2\phi) & 0 \\ -q\sin(2\phi) & q\cos(2\phi) & 0 \\ 0 & 0 & -1 \end{pmatrix} , \tag{5.25}$$

while the Δ_{ij} are non-zero and do not depend on the specific value of ϕ

$$\Delta R_q = \begin{pmatrix} \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{(1+q^2)}{2}\sqrt{(1-q^2)} \\ \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{(1+q^2)}{2}\sqrt{(1-q^2)} \\ \sqrt{q^2-q^4} & \sqrt{q^2-q^4} & 0 \end{pmatrix}. \tag{5.26}$$

From the above matrices, it can be verified that the state $|0,\phi,0\rangle$ satisfies (5.21).

For a generic axis of rotation, the quantum Euler angle θ will play a non-trivial role both in the determination of the axis itself and in the determination of the rotation angle.

5.2.2 Same stars, different skies

Perhaps the most striking implication of our results is that they provide a possible scenario for a novel type of relationship between observers and the spacetime they observe, such that the choices made by the observer in setting up her reference frame affect the spacetime she observes. We shall argue this by first advocating Einstein's operational notion of spacetime, whose points have physical meaning only in as much as they label an event there occurring. We use as a reference example a network of sources emitting photons ("stars"): each photon emission is a physical point of spacetime. These points of spacetime will be labelled by measured coordinates and uncertainties on those coordinates. The key observation here is that, in our model, these uncertainties are not intrinsic properties of the spacetime points but rather depend on the choices made by the observer. These arguments give rise to a framework in which *different skies may originate from the observation of the same stars.*

Agency-dependent space

We will apply our formalism to establish the connection between the q-deformation of the $SU(2)$ group and the agency-dependence of space. Before showing this with a concrete numerical example, we will discuss why this property of space is to be expected with a thought experiment.

Consider two observers, Alice and Bob, each equipped with their own set of telescopes, who want to map the starry sky. Each of them chooses their reference frame (in particular, their z axis) to assign coordinates to the stars they observe. We assume that their origins coincide, but their z axes do not. Without loss of generality, we can focus on the (y,z)-plane so that, for each observer, any telescope is mapped onto another by a rotation about the x axis. In the classical case, Alice and Bob can compare their experimental results by rotating their data with a classical rotation matrix. Formally, there exists an element of the $SO(3)$ group that sharply describes the relative orientation between Alice and Bob.

We now analyse how the situation changes in our novel non-commutative framework. We henceforth assume that Alice and Bob can choose the direction along which points can be sharp independently from one another. For definiteness, we focus on observer Alice first. In her task of mapping the starry sky, she focuses on a particular star, which she observes with one of her telescopes. This instrument is generally misaligned with respect to the z axis she has chosen. In our framework, this means that there exists a state $|\psi_A\rangle$ connecting the relative orientation between the aforementioned telescope and the z axis. Following our physical interpretation, $|\psi_A\rangle$ determines a quantum rotation matrix through the expectation value of

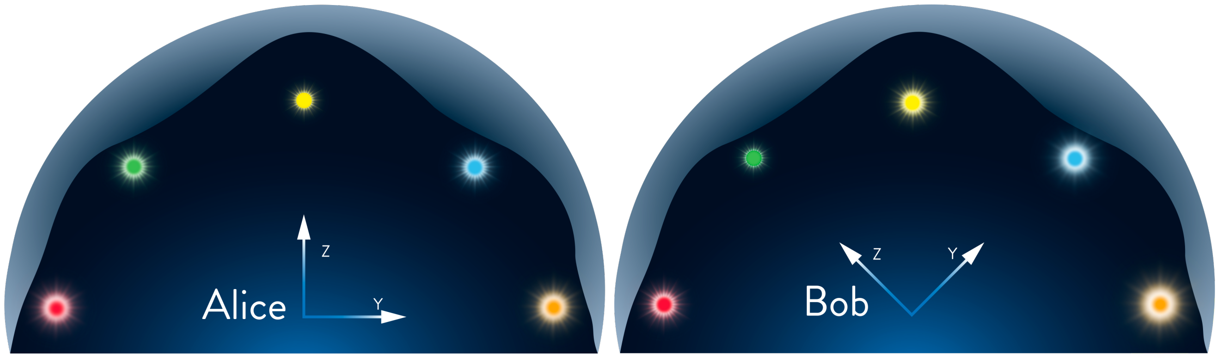

Figure 12: A semi-quantitative description of the starry skies observed by Alice and Bob. Alice's z axis is aligned with the yellow star, while Bob's z axis is aligned with the green star, so that the yellow star is sharp for Alice while the green star is sharp for Bob. For Alice (Bob) the fuzziness increases as the angular deviation from the yellow (green) star increases, and the same stars are observed with different fuzziness because of the relative orientation of the z axes of the two agents. We thank Francesco Minieri for the realisation of this image.

R_q and an uncertainty matrix defined by (5.16). This means that Alice cannot appreciate the relative orientation between the telescope and her z axis with arbitrary precision, since there will always be some intrinsic uncertainty given by the deformation of the $SU(2)$ group. As a consequence of this, she will not be able to sharply deduce the position of this star in the sky. For that same star, this line of reasoning also applies to Bob, but the state $|\psi_B\rangle$ will generally be different from $|\psi_A\rangle$, implying a different degree of fuzziness. In turn, this procedure can be applied to any star in the sky so that Alice and Bob each have their own picture of the celestial sphere, as can be seen in fig. 12. The striking result is that the uncertainty associated with each star is not an intrinsic property of spacetime but depends on the choices made by the observers. In particular, different choices of the z axis give rise to different pictures of the starry sky. The subtle observation is that there is no way for Alice and Bob to define an objective celestial sphere, meaning that the definition of space itself cannot be independent of the observer who reconstructs it. In this sense, we say that Alice and Bob are agents: we abandon the idea of objective space, replacing it with a notion of *observed space* from which we cannot subtract the choices of the observer who infers it.

As mentioned earlier, it is important to note that, even though the z axis is a preferred direction for observers (only its points can be sharp), our framework produces an isotropic description of space. In fact, spatial rotations are a (q-deformed) symmetry. There is no direction that is preferred *a priori*, and the special role played by the z axis in the reference frame of a given observer is only the result of the choices made by that observer in setting up their frame. The observer freely chooses their preferred z direction. This should be contrasted with the case of standard spatial anisotropy, in which different directions have different properties *a priori*, independently of the choices made by observers, and invariance under spatial rotations is lost.

Numerical analysis

To gain insight into these conceptual novelties and to better grasp the meaning of fig. 12, we now show some numerical examples to support our claims. We start with Alice's point of view. Let her z axis be aligned with a certain star (α) and consider another star (β) she wants to observe from a telescope, whose relative orientation with respect to the first star is described by the following state:

$$|\psi_A\rangle = \left| n=0; \phi = \frac{\pi}{2}; \chi = 0 \right\rangle . \tag{5.27}$$

This state,already introduced in section section 5.2.1 and which satisfies the "classicality" conditions (5.21), can be seen as a q-deformed rotation of π around the x axis. The expectation value is obtained by substituting $\phi = \frac{\pi}{2}$ in (5.25) while the variances are the ones in (5.26)

$$\begin{gathered}\langle\psi_A|R_q|\psi_A\rangle = \begin{pmatrix} q & 0 & 0 \\ 0 & -q & 0 \\ 0 & 0 & -q^2 \end{pmatrix} , \\ \Delta R_q = \begin{pmatrix} \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{(1+q^2)}{2}\sqrt{(1-q^2)} \\ \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{1}{2}\sqrt{(1-q^2)(1-q^4)} & \frac{(1+q^2)}{2}\sqrt{(1-q^2)} \\ \sqrt{q^2-q^4} & \sqrt{q^2-q^4} & 0 \end{pmatrix} ,\end{gathered} \tag{5.28}$$

which at first order in $(1-q)$ give

$$\begin{gathered}\langle\psi_A|R_q|\psi_A\rangle = \begin{pmatrix} 1-(1-q) & 0 & 0 \\ 0 & -1+(1-q) & 0 \\ 0 & 0 & -1+2(1-q) \end{pmatrix} , \\ \Delta R_q = \begin{pmatrix} \sqrt{2}(1-q) & \sqrt{2}(1-q) & \sqrt{2(1-q)} \\ \sqrt{2}(1-q) & \sqrt{2}(1-q) & \sqrt{2(1-q)} \\ \sqrt{2(1-q)} & \sqrt{2(1-q)} & 0 \end{pmatrix} .\end{gathered} \tag{5.29}$$

Identifying Alice's z axis with the vector $v = (0,0,1)$ and applying these matrices on v, the transformed vector $v' = (v'_1, v'_2, v'_3)$ will lie in the range

$$-\sqrt{2(1-q)} \le v'_1 \le \sqrt{2(1-q)} \qquad -\sqrt{2(1-q)} \le v'_2 \le \sqrt{2(1-q)} \qquad v'_3 = -1+2(1-q) . \tag{5.30}$$

This quantitatively shows what we mean by fuzziness: the q-rotated vector v' lies in a cone with aperture given by

$$\Delta\alpha \approx 2\sqrt{2(1-q)} . \tag{5.31}$$

The agency feature of the model can be well understood if we now compare these results with those obtained if Alice chose to align her z axis with the star β. In this case she would have seen β sharply and α under a cone with aperture (5.31). This line of reasoning can be extended when considering multiple stars: different states describing the relative orientation of these stars with respect to Alice's z axis will produce different uncertainties in determining their directions. In turn, this is characterised by different cone apertures under which the stars are seen. In table 5.1 we show some numerical examples presenting this feature.

$\theta(n)$	Aperture
$0°$	0
$\pm 45.275°$	0.210
$\pm 90.560°$	0.284
$\pm 137.518°$	0.402
$\pm 180°$	0.442

Table 5.1: Dependence of the aperture on the observation angle $\theta(n)$ using $q = 0.99$. The aperture is monotonically increasing as the angles grow in absolute value. The states used to obtain these results are constructed numerically in section 5.B. For consistency, the aperture for 180° is not truncated at first order in $(1-q)$ as in (5.31), but is computed numerically with eq. (5.28) with $q = 0.99$.

From this analysis, it is worth noting that the fuzziness grows as our quantised Euler angle increases, resulting in a starry sky inferred by Alice, similar to what is depicted in fig. 12. Of course, the analysis can also be repeated for Bob, who, in general, chooses a different z axis and observes the same stars as Alice. The states describing the relative orientation between these stars and his z axis will be different from those characterising Alice's frame. The degree of fuzziness that he observes for a particular star will be different from the one assigned to it by Alice, resulting in a different inference of the starry sky, as shown in fig. 12.

5.3 Doubly Quantum Mechanics

In the previous section, we investigated some preliminary effects that arise when the rotation symmetry group $SU(2)$ is deformed in the $SU_q(2)$ quantum group. We did not delve into the formulation of a non-commutative analogue of the quantum protocol analogous presented in section 5.1, since this requires the development of a fully quantum framework, in which the geometrical degrees of freedom of spin systems and Stern-Gerlach apparatuses are affected by the quantisation of the rotation group. The aim of this section, which is based on [4], is to complete this discussion and thus to lay the foundations of a "Doubly Quantum Mechanics" (DQM) theory, where the geometrical configuration of physical systems and measurement apparatuses, as well as the relation between reference frames, are described by quantum states that are elements of the Hilbert space of the irreducible representations of the quantum group describing the (deformed) symmetries of the quantum spacetime under study. Such a framework is doubly quantum since not only the phase space of physical systems is quantised but also their geometrical configurations. In particular, we will focus on the spin sector as in the previous section. Naturally, the doubly quantisation procedure in this context is realised by promoting the standard $SU(2)$ rotation symmetry to the quantum deformation $SU_q(2)$. This is implemented as a quantisation of the complex coefficients characterising a generic spin state and a generic Pauli matrix, which in standard Quantum Mechanics are completely specified by $SU(2)$ elements, thus requiring a doubly quantum-mechanical description of the system. With the same line of reasoning, the complex coefficients of the $SU(2)$ elements describing the relation between different observers will also be affected by the same type of quantisation. We develop the formalism by generalising the axioms of Quantum Theory in a consistent way.

As a consequence of our axioms, we will show that the formalism naturally yields a quantisation of the Born rule, introducing the notion of *quantum measurement devices*, which requires measurement probabilities to be described by self-adjoint and positive semi-definite operators acting on the Hilbert space associated to the $SU_q(2)$ quantum group, introducing a novel operational meaning for probability measurements. After introducing a suitable class of semi-classical geometry states, which describe near-to-classical geometrical configurations of physical systems, we finally investigate the doubly quantum version of the alignment protocol in section 5.1 using these states. We find that probability measurements are affected, in these configurations, by intrinsic uncertainties arising from the quantum properties of $SU_q(2)$. It is exactly this feature that translates into an unavoidable fuzziness for observers who attempt to align their reference frames by exchanging qubits, even when the number of exchanged qubits approaches infinity. In the classical $SU(2)$ case, two observers can in principle sharply align their reference frames, but this is prevented in the DQM framework because the quantum properties of the geometrical configurations, as described by the quantum group $SU_q(2)$, yield a fundamental limit on the measurement of probabilities.

5.3.1 Spin measurements in Quantum Mechanics

In Quantum Mechanics, rotational symmetry is governed by the $SU(2)$ group. All the machinery required for quantum-mechanical computations with spinors can be derived starting from three basic ingredients, σ_z and its eigenstates $|\uparrow\rangle, |\downarrow\rangle$, given by

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad |\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.32}$$

Indeed, a generic spin up state $|\uparrow_{\vec{n}}\rangle$ along a direction $\vec{n} = (\sin\theta_s \cos\omega_s, \sin\theta_s \sin\omega_s, \cos\theta_s)$ where $\theta_s, \omega_s \in [0,\pi] \times [0,2\pi]$, can be written as

$$|\uparrow_{\vec{n}}\rangle = U_s |\uparrow\rangle = \begin{pmatrix} x & -y^* \\ y & x^* \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x|\uparrow\rangle + y|\downarrow\rangle \ , \ \ x, y \in \mathbb{C} \, , \ \ x^* x + y^* y = 1 \, , \tag{5.33}$$

namely by acting with an $SU(2)$ matrix on the spin up state. These complex parameters can be represented in terms of angular variables as

$$x = e^{i\chi_s} \cos\frac{\theta_s}{2} \, , \qquad y = e^{i\phi_s} \sin\frac{\theta_s}{2} \, , \qquad \chi_s, \phi_s \in [0, 2\pi] \, , \tag{5.34}$$

so that the resulting spin state can also be written as

$$|\uparrow_{\vec{n}}\rangle = e^{i\chi_s} \cos\frac{\theta_s}{2} |\uparrow\rangle + e^{i\phi_s} \sin\frac{\theta_s}{2} |\downarrow\rangle = \cos\frac{\theta_s}{2} |\uparrow\rangle + e^{i\omega_s} \sin\frac{\theta_s}{2} |\downarrow\rangle \ , \tag{5.35}$$

up to a global phase. Analogously, the generic spin down state is given by

$$|\downarrow_{\vec{n}}\rangle = -y^* |\uparrow\rangle + x^* |\downarrow\rangle = -e^{-i\omega_s} \sin\frac{\theta_s}{2} |\uparrow\rangle + \cos\frac{\theta_s}{2} |\downarrow\rangle \ . \tag{5.36}$$

Since states are defined up to a global phase, we have defined $\omega_s = \phi_s - \chi_s$ to match the angle appearing in the generic direction $\vec{n}$ by omitting the irrelevant global phase $e^{i\chi_s}$.

It is possible to measure the spin along a direction $\vec{m} = (\sin\theta_a \cos\omega_a, \sin\theta_a \sin\omega_a, \cos\theta_a)$ with a generic spin state, using a Stern-Gerlach apparatus orientated in direction $\vec{m}$. The observable associated with this measurement is described by the Pauli matrix $\sigma_{\vec{m}}$, obtained by acting on σ_z with a $SU(2)$ matrix

$$\sigma_{\vec{m}} = U_a \sigma_z U_a^\dagger = \begin{pmatrix} aa^* - c^* c & 2ac^* \\ 2ca^* & cc^* - a^* a \end{pmatrix} = \begin{pmatrix} \cos\theta_a & e^{-i\omega_a}\sin\theta_a \\ e^{i\omega_a}\sin\theta_a & -\cos\theta_a \end{pmatrix}, \tag{5.37}$$

where

$$U_a = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix}, \quad a = e^{i\chi_a}\cos\frac{\theta_a}{2}, \quad c = e^{i\phi_a}\sin\frac{\theta_a}{2}. \tag{5.38}$$

Notice that (5.37) can be rewritten in terms of projector operators $\Pi_{\uparrow_{\vec{m}}}, \Pi_{\downarrow_{\vec{m}}}$ as

$$\sigma_{\vec{m}} = \Pi_{\uparrow_{\vec{m}}} - \Pi_{\downarrow_{\vec{m}}} := \left|\uparrow_{\vec{m}}\right\rangle\!\left\langle\uparrow_{\vec{m}}\right| - \left|\downarrow_{\vec{m}}\right\rangle\!\left\langle\downarrow_{\vec{m}}\right|. \tag{5.39}$$

These projectors define the probabilities of finding spin up and down in the direction $\vec{m}$ by performing a measurement on a spin up state along a direction $\vec{n}$. These are given by

$$\begin{aligned} P_{\uparrow_{\vec{m}}}(\uparrow_{\vec{n}}) &= \langle\uparrow_{\vec{n}}|\Pi_{\uparrow_{\vec{m}}}|\uparrow_{\vec{n}}\rangle = x^* x a a^* + y^* y c c^* + x^* y a c^* + y^* x c a^* = \\ &= \frac{1}{2}\Big[1 + \cos(\theta_a)\cos(\theta_s) + \cos(\omega_a - \omega_s)\sin(\theta_a)\sin(\theta_s)\Big], \\ P_{\downarrow_{\vec{m}}}(\uparrow_{\vec{n}}) &= \langle\uparrow_{\vec{n}}|\Pi_{\downarrow_{\vec{m}}}|\uparrow_{\vec{n}}\rangle = x^* x c^* c + y y^* a^* a - x^* y c^* a - y^* x a^* c = \\ &= \frac{1}{2}\Big[1 - \cos(\theta_a)\cos(\theta_s) - \cos(\omega_a - \omega_s)\sin(\theta_a)\sin(\theta_s)\Big], \end{aligned} \tag{5.40}$$

so that the expectation value of $\sigma_{\vec{m}}$ in $\left|\uparrow_{\vec{n}}\right\rangle$ is

$$\langle\sigma_{\vec{m}}\rangle = P_{\uparrow_{\vec{m}}}(\uparrow_{\vec{n}}) - P_{\downarrow_{\vec{m}}}(\uparrow_{\vec{n}}). \tag{5.41}$$

The quantities of interest for this result are the probabilities and these depend on both couples of parameters $\{x, y\}$ and $\{a, c\}$, defining the spin state and the direction of the measurement, respectively. With the same techniques, it is possible to calculate similar expressions when the spin is in a generic down state, $\left|\downarrow_{\vec{n}}\right\rangle$.

The physical setup described above is relative to a single observer, Alice, in her reference frame. If we consider a second observer, Bob, whose axes are misaligned with Alice's, we can obtain Bob's description of Alice's physical results by applying a $SU(2)$ symmetry transformation, denoted by U_g, on both states and observables as

$$\left|\uparrow^B_{\vec{n}}\right\rangle = U_g\left|\uparrow^A_{\vec{n}}\right\rangle, \quad \sigma^B_{\vec{m}} = U_g \sigma^A_{\vec{m}} {U_g}^\dagger, \tag{5.42}$$

where $\left|\uparrow^A_{\vec{m}}\right\rangle$ and $\sigma^A_{\vec{m}}$ are the generic spin up state and Pauli matrix in Alice's description, respectively given by (5.35) and (5.37). Of course, given that U_g is unitary, the probabilities of spin measurements and the expectation value of the spin are invariant under this transformation. This matrix is given by

$$U_g = \begin{pmatrix} u & -v^* \\ v & u^* \end{pmatrix}, \quad u = e^{i\chi_g}\cos\frac{\theta_g}{2}, \quad v = e^{i\phi_g}\sin\frac{\theta_g}{2}. \tag{5.43}$$

Contrary to the definition of spins and Pauli matrices along generic directions for the single observer, both phases χ_g and ϕ_g are relevant since the relative orientation between two observers is defined by three angles, whereas a generic direction in space is specified only by two angles.

5.3.2 Mathematical preliminaries

In this section, we are want to develop a quantum-mechanical framework in which the $SU(2)$ rotational symmetry of section 5.3.1 is promoted to a quantum symmetry described by $SU_q(2)$ [321, 328–330]. Before diving into the physical construction, we present some basic features of this quantum group. It is defined by deforming the algebra of complex functions on $SU(2)$, denoted by $C\big(SU(2)\big)$, in a non-commutative way. The generators $\{\alpha,\gamma\}$ are regarded as functions on the group, satisfying non-trivial commutation relations

$$\begin{aligned} &\alpha\gamma = q\gamma\alpha \qquad \alpha\gamma^* = q\gamma^*\alpha \qquad \gamma\gamma^* = \gamma^*\gamma \\ &\gamma^*\gamma + \alpha^*\alpha = \mathbb{1} \qquad \alpha\alpha^* - \alpha^*\alpha = (1-q^2)\gamma^*\gamma\,, \end{aligned} \tag{5.44}$$

where $q \in (0,1)$ and $*$ denotes the hermitian conjugation. The deformed algebra is denoted by $C\Big(SU_q(2)\Big)$. In terms of these operators, the non-commutative counterpart of (5.38) is written as [331]

$$U^q = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}. \tag{5.45}$$

The representations of α,γ have been thoroughly studied in the literature, *e.g.* [332, 333]. The Hilbert space containing the two unique irreducible representations of the $SU_q(2)$ algebra, when $q \in (0,1)$, is $\mathcal{H}_{SU_q(2)} = \mathcal{H}_\pi \oplus \mathcal{H}_\rho$ where $\mathcal{H}_\pi = \ell^2 \otimes L^2(S^1) \otimes L^2(S^1)$ and $\mathcal{H}_\rho = L^2(S^1)$. If $\phi,\chi \in [0,2\pi[$ are coordinates on S^1 and $|n\rangle$ is the canonical basis of ℓ^2, the algebra of functions on $SU_q(2)$ is represented as

$$\rho(\alpha)\big|\chi\big\rangle = e^{i\chi}\big|\chi\big\rangle \qquad \rho(\alpha^*)\big|\chi\big\rangle = e^{-i\chi}\big|\chi\big\rangle \qquad \rho(\gamma)\big|\chi\big\rangle = \rho(\gamma^*)\big|\chi\big\rangle = 0 \tag{5.46}$$

$$\begin{aligned} &\pi(\alpha)\big|n,\phi,\chi\big\rangle = e^{i\chi}\sqrt{1-q^{2n}}\big|n-1,\phi,\chi\big\rangle \qquad \pi(\gamma)\big|n,\phi,\chi\big\rangle = e^{i\phi}q^n\big|n,\phi,\chi\big\rangle \\ &\pi(\alpha^*)\big|n,\phi,\chi\big\rangle = e^{-i\chi}\sqrt{1-q^{2n+2}}\big|n+1,\phi,\chi\big\rangle \qquad \pi(\gamma^*)\big|n,\phi,\chi\big\rangle = e^{-i\phi}q^n\big|n,\phi,\chi\big\rangle\,. \end{aligned} \tag{5.47}$$

In the following sections, we will consider multiple copies of $C(SU_q(2))$ in which we will label the quantum numbers appearing in their representations by subscripts $\{s,a,g\}$ to distinguish the role of the various $SU_q(2)$ transformations, as done in section 5.3.1.

5.3.3 Spin measurements in Doubly Quantum Mechanics

In classical mechanics, the state of a physical system at a given time is completely characterised by a point in phase space. When transitioning to Quantum Mechanics, the notion of a point in phase space is replaced by the concept of a state in a Hilbert space. Both theories share the same classical Euclidean background: the quantum properties of a spin system are encoded in the fact that its state can always be described in terms of a superposition of spin states, which, however, are always relative to some classical direction in space. When taking the quantum spacetime hypothesis at face value, it is plausible to imagine that a necessary step toward the understanding of the Quantum Gravity problem should require a transition from Quantum Mechanics to a novel theory, in which also the geometrical properties of physical systems and the transformation laws between observers are described by states in a Hilbert space. We present an example of such a Doubly Quantum Mechanics model, limiting our attention to the physics of spin measurements. Specifically, the standard rotation symmetry under the $SU(2)$ group is promoted to a deformed symmetry under the quantum group $SU_q(2)$, quantum deformation of its classical counterpart, widely employed in the study of the Quantum Gravity problem [322–325]. The classical $SU(2)$ group parameters become operators satisfying the commutation relations (5.44) and the states on which they act encode the relevant information regarding the orientation in space of a physical system.

From an operational point of view, in the commutative case, an observer can extract relevant physical information from a spin state only by performing some measurements on the latter with a Stern-Gerlach apparatus. A spin measurement is thus characterised by two independent three-vectors in space: one defining the spin orientation and the other the Stern-Gerlach orientation. To carry on this operational framework to the non-commutative case, we are led to characterise the physical information relevant to an observer by means of two independent geometry states. One of them encodes information about the spin orientation and the other contains information about the Stern-Gerlach orientation.

In this regard, a spin system becomes a doubly quantum-mechanical system, since its full description now requires two different states living in different Hilbert spaces. The Stern-Gerlach apparatus, albeit being a classical measurement device from the point of view of standard Quantum Mechanics, acquires non-classical properties in Doubly Quantum Mechanics, since its orientation in space becomes characterised by a geometry state. Specifically, we will see that the characteristic feature of such a *quantum measurement device* is that the probability of a measurement outcome is quantised by being promoted to an operator acting on the Hilbert space associated to the experimental setup.

In the following, we adopt an axiomatic approach to extend the basic principles of Quantum Mechanics when the quantisation of rotation symmetry by means of $SU_q(2)$ is taken into account. By adapting the logical steps outlined in section 5.3.1 to our Doubly Quantum Mechanics formalism, we derive the fundamental quantities describing a spin measurement in this context.

The axioms of Doubly Quantum Mechanics

Axiom 0 (Geometry) The information on the directions in space of physical systems and the relative orientation between reference frames is encoded in *geometry states*, elements of the Hilbert space $\mathscr{H}_{SU_q(2)}$.

Axiom 1 (Pre-measurement states) The states prepared by an observer, referred to as *pre-measurement states*, are spin $\frac{1}{2}$ states with coefficients that are operators acting on the Hilbert space $\mathscr{H}_{SU_q(2)}$. These states are elements of $\widehat{\mathscr{H}_{\frac{1}{2}}} := \mathscr{H}_{\frac{1}{2}} \otimes C(SU_q(2))$, where $C(SU_q(2))$ is the algebra of functions on $SU_q(2)$ and $\mathscr{H}_{\frac{1}{2}} := \mathbb{C}^2$ is the Hilbert space of a spin in standard Quantum Mechanics. In the following, to simplify the notation, we omit the tensor product between the $\mathscr{H}_{\frac{1}{2}}$ component and the operator coefficients.

The generic spin up pre-measurement state is obtained, analogously to the generic spin up state in (5.33), acting on $\left|\uparrow\right\rangle \mathbb{1}$ with an $SU_q(2)$ matrix

$$\left|\psi\right\rangle = U_s^q\Big(\left|\uparrow\right\rangle \mathbb{1}\Big) = \left|\uparrow\right\rangle x + \left|\downarrow\right\rangle y\,, \tag{5.48}$$

with

$$U_s^q := \begin{pmatrix} x & -qy^* \\ y & x^* \end{pmatrix} = \left|\uparrow\right\rangle\!\left\langle\uparrow\right| x - q\left|\uparrow\right\rangle\!\left\langle\downarrow\right| y^* + \left|\downarrow\right\rangle\!\left\langle\uparrow\right| y + \left|\downarrow\right\rangle\!\left\langle\downarrow\right| x^*\,, \tag{5.49}$$

where $\mathbb{1}$ is the identity operator in the $SU_q(2)$ algebra and the operators $x, y \in C(SU_q(2))$ and their hermitian conjugates satisfy the $SU_q(2)$ algebra relations (5.44), with $x = \alpha$ and $y = \gamma$. The bra corresponding to (5.48) is given by

$$\left\langle\psi\right| = \left\langle\uparrow\right| x^* + \left\langle\downarrow\right| y^*\,, \tag{5.50}$$

and is such that

$$\left\langle\psi\middle|\psi\right\rangle = x^* x + y^* y = \mathbb{1}\,, \tag{5.51}$$

where the $\langle\cdot|\cdot\rangle$ notation denotes the scalar product on the $\mathscr{H}_{\frac{1}{2}}$ Hilbert space and the pointwise product between the operator coefficients. The last equality follows from the commutation relations of the $SU_q(2)$ algebra (5.44). This is just the non-commutative analogue of the fact that a generic state is normalised in the commutative setting.

Analogously to (5.36), the generic spin down state can be obtained by acting with U_s^q on $\left|\downarrow\right\rangle \mathbb{1}$ as done for (5.48)

$$\left|\psi'\right\rangle = -qy^* + \left|\downarrow\right\rangle x^*\,, \tag{5.52}$$

which is also normalised since

$$\left\langle\psi'\middle|\psi'\right\rangle = q^2 x x^* + y y^* = \mathbb{1}\,. \tag{5.53}$$

Moreover, $\left|\psi\right\rangle$ and $\left|\psi'\right\rangle$ satisfy $\left\langle\psi'\middle|\psi\right\rangle = \left\langle\psi\middle|\psi'\right\rangle = 0$.

Axiom 2 (Observables) A generic Pauli matrix, representing the observable of a spin measurement, is an element of $\mathscr{D}\Big(\mathscr{H}_{\frac{1}{2}}\Big) \otimes C(SU_q(2))$, where $\mathscr{D}\Big(\mathscr{H}_{\frac{1}{2}}\Big)$ is the Hilbert space of linear operators on $\mathscr{H}_{\frac{1}{2}}$. We omit the tensor product between the $\mathscr{D}\Big(\mathscr{H}_{\frac{1}{2}}\Big)$ component and the operator coefficients as in the previous axiom. The possible outcomes of a spin measurement are given by the eigenvalues of the corresponding self-adjoint Pauli matrix.

The generic Pauli matrix is obtained, analogously to the one in (5.37), by conjugating the q-deformed Pauli z matrix with an $SU_q(2)$ matrix

$$\sigma^q = U_a^q \sigma_z^q U_a^{q\dagger} = \begin{pmatrix} q\Big(\mathbb{1} - \big(1+q^2\big)c^* c\Big) & (q+q^{-1})ac^* \\ (q+q^{-1})ca^* & -q^{-1}\Big(\mathbb{1} - \big(1+q^2\big)c^* c\Big) \end{pmatrix}, \tag{5.54}$$

where

$$U_a^q := \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} = |\uparrow\rangle\langle\uparrow|\, a - q\,|\uparrow\rangle\langle\downarrow|\, c^* + |\downarrow\rangle\langle\uparrow|\, c + |\downarrow\rangle\langle\downarrow|\, a^* \,, \tag{5.55}$$

with the operators $a, c \in C\big(SU_q(2)\big)$ and their hermitian conjugates defining a different copy of the $SU_q(2)$ algebra (5.44), with respect to axiom 1, with $a = \alpha$ and $c = \gamma$. Notice that (5.54) is the non-commutative generalisation of the generic Pauli matrix (5.37) that represents a Stern-Gerlach apparatus orientated along a generic direction and is self-adjoint in the sense that $(\sigma^q)^*_{kh} = (\sigma^q)_{hk}$.

The matrix σ_z^q, is the q-deformation of σ_z and reads [332]

$$\sigma_z^q = \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix} \mathbb{1} = q\,|\uparrow\rangle\langle\uparrow|\,\mathbb{1} - q^{-1}\,|\downarrow\rangle\langle\downarrow|\,\mathbb{1}\,. \tag{5.56}$$

We interpret this matrix as the one characterising a Stern-Gerlach apparatus orientated along the positive z direction. In the commutative case, σ_z is traceless and the generic Pauli matrix obtained by conjugation with $U \in SU(2)$ is still traceless. In the present case, σ_z^q is q-traceless [334] and the generic Pauli matrix in (5.54), obtained by conjugation wtih $U_a^q \in SU_q(2)$, is still q-traceless, where the q-trace is defined as

$$\mathrm{Tr}_q\{A\} := \sum_i q^{2i} A_{ii}\,. \tag{5.57}$$

Analogously to the classical case, (5.54) can be rewritten in terms of the non-commutative generalisation of projectors along directions specified by the generic Pauli matrix, which we denote by $\Pi_{\uparrow_\sigma}$, $\Pi_{\downarrow_\sigma}$, as

$$\sigma^q = q\,\Pi_{\uparrow_\sigma} - q^{-1}\,\Pi_{\downarrow_\sigma} := q\,|\uparrow_\sigma\rangle\langle\uparrow_\sigma| - q^{-1}\,|\downarrow_\sigma\rangle\langle\downarrow_\sigma|\,, \tag{5.58}$$

where

$$|\uparrow_\sigma\rangle = |\uparrow\rangle\, a + |\downarrow\rangle\, c, \quad |\downarrow_\sigma\rangle = -q\,|\uparrow\rangle\, c^* + |\downarrow\rangle\, a^*\,, \tag{5.59}$$

with

$$\langle\uparrow_\sigma|\uparrow_\sigma\rangle = \langle\downarrow_\sigma|\downarrow_\sigma\rangle = \mathbb{1}\,,\ \langle\uparrow_\sigma|\downarrow_\sigma\rangle = \langle\downarrow_\sigma|\uparrow_\sigma\rangle = 0\,. \tag{5.60}$$

From (5.58), it immediately follows that $|\uparrow_\sigma\rangle$ ($|\downarrow_\sigma\rangle$) is an eigenstate of σ^q with eigenvalue q ($-q^{-1}$). The explicit expressions for $|\uparrow_\sigma\rangle\langle\uparrow_\sigma|$ and $|\downarrow_\sigma\rangle\langle\downarrow_\sigma|$ are given by

$$\begin{aligned} \Pi_{\uparrow_\sigma} &= |\uparrow_\sigma\rangle\langle\uparrow_\sigma| = |\uparrow\rangle\langle\uparrow|\, aa^* + |\downarrow\rangle\langle\downarrow|\, cc^* + |\uparrow\rangle\langle\downarrow|\, ac^* + |\downarrow\rangle\langle\uparrow|\, ca^* \\ \Pi_{\downarrow_\sigma} &= |\downarrow_\sigma\rangle\langle\downarrow_\sigma| = |\uparrow\rangle\langle\uparrow|\, q^2cc^* + |\downarrow\rangle\langle\downarrow|\, a^*a - |\uparrow\rangle\langle\downarrow|\, qc^*a - |\downarrow\rangle\langle\uparrow|\, qa^*c \end{aligned}\,. \tag{5.61}$$

Therefore the projectors $\Pi_{\uparrow_\sigma}$ and $\Pi_{\downarrow_\sigma}$ satisfy the following properties

$$\Pi^2_{\uparrow_\sigma} = \Pi_{\uparrow_\sigma}\,,\ \Pi^2_{\downarrow_\sigma} = \Pi_{\downarrow_\sigma}\,,\ \Pi_{\uparrow_\sigma}\Pi_{\downarrow_\sigma} = \Pi_{\downarrow_\sigma}\Pi_{\uparrow_\sigma} = 0\,,\ \Pi_{\uparrow_\sigma} + \Pi_{\downarrow_\sigma} = \mathbb{I} \otimes \mathbb{1}\,, \tag{5.62}$$

where $\mathbb{I} := |\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|$.

This axiom introduces an important conceptual departure from standard Quantum Mechanics for what concerns the nature of measurement apparatuses as their geometrical configurations are now specified by quantum geometry states in the Hilbert space $\mathscr{H}_{SU_q(2)}$. This opens up the possibility of describing “non-classical measurement devices”, characterised by geometrical configurations that do not have a classical counterpart. We will not delve into this feature of our framework in this chapter, postponing this task to future work.

Axiom 3 (Probabilities and expectation values) The expectation value of an observable $\mathscr{O} \in \mathscr{D}\left(\mathscr{H}_{\frac{1}{2}}\right) \otimes C(SU_q(2))$ on a generic state $|\psi\rangle \in \mathscr{H}_{\frac{1}{2}} \otimes C(SU_q(2))$ is defined as a map

$$\langle\mathscr{O}\rangle : \left[\mathscr{H}_{\frac{1}{2}} \otimes C(SU_q(2))\right] \times \left[\mathscr{D}\left(\mathscr{H}_{\frac{1}{2}}\right) \otimes C(SU_q(2))\right] \ni |\psi\rangle \times \mathscr{O} \mapsto \langle\psi|\mathscr{O}|\psi\rangle \in C(SU_q(2)) \otimes C(SU_q(2))\,. \quad (5.63)$$

At the practical level, this map performs the standard expectation value in the $\mathscr{H}_{1/2}$ component, which is then multiplied by a coefficient that is given by the tensor product between the operator coefficients of $\mathscr{O}$ (on the right side of the tensor product) and the product between the operator coefficients of $\langle\psi|$ and the operator coefficients of $|\psi\rangle$ (on the left side of the tensor product). Of course, the ordering of the tensor product is just a choice that does not affect any result. Analogously, this map can be defined on a generic spin state $|\psi'\rangle$.

With this definition, the non-commutative generalisations of the probabilities of finding spin $\uparrow_\sigma$ or spin $\downarrow_\sigma$ on a generic spin up state $|\psi\rangle$ are elements of $C\left(SU_q(2)\right) \otimes C\left(SU_q(2)\right)$ and can be defined as

$$\begin{aligned} P(\uparrow_\sigma) &:= \langle\Pi_\uparrow\rangle = \langle\psi|\Pi_{\uparrow_\sigma}|\psi\rangle = \langle\psi|\uparrow_\sigma\rangle\langle\uparrow_\sigma|\psi\rangle\,, \\ P(\downarrow_\sigma) &:= \langle\Pi_\downarrow\rangle = \langle\psi|\Pi_{\downarrow_\sigma}|\psi\rangle = \langle\psi|\downarrow_\sigma\rangle\langle\downarrow_\sigma|\psi\rangle\,. \end{aligned} \quad (5.64)$$

The non-commutative generalisations of the probabilities of finding spin $\uparrow_\sigma$ or spin $\downarrow_\sigma$ on a generic spin down state $|\psi'\rangle$ are given by the same formula with $|\psi'\rangle$ replacing $|\psi\rangle$. By definition, the probabilities defined in (5.64) are self-adjoint and positive semi-definite operators, and from (5.62) it immediately follows that

$$P(\uparrow_\sigma) + P(\downarrow_\sigma) = \mathbb{1} \otimes \mathbb{1}\,. \quad (5.65)$$

Therefore, they satisfy the desirable properties that probabilities must have. The explicit expressions for $P(\uparrow_\sigma)$ and $P(\downarrow_\sigma)$ are given by

$$\begin{aligned} P(\uparrow_\sigma) &= x^* x \otimes aa^* + y^* y \otimes cc^* + x^* y \otimes ac^* + y^* x \otimes ca^* \\ P(\downarrow_\sigma) &= q^2 x^* x \otimes c^* c + y^* y \otimes a^* a - q x^* y \otimes c^* a - q y^* x \otimes a^* c \end{aligned}\,. \quad (5.66)$$

The properties listed above for these operators can also be explicitly verified using the defining rules of the $SU_q(2)$ algebra in (5.44). These expressions are the non-commutative generalisations of (5.40), with the tensor product separating the spin and the Stern-Gerlach components.

From the definitions of probabilities, we see that the expectation value of the generic Pauli matrix (5.54) in a generic spin up pre-measurement state (5.48) is given by

$$\langle\sigma^q\rangle = \langle\psi|\sigma^q|\psi\rangle = q\,P(\uparrow_\sigma) - q^{-1} P(\downarrow_\sigma)\,. \quad (5.67)$$

It is possible to repeat the same steps and calculate the analogous of (5.66) and (5.67) for $|\psi'\rangle$.

Axiom 4 (Measurements) Performing a measurement with a macroscopic Stern-Gerlach apparatus associated to (5.54) projects a pre-measurement state $|\psi\rangle$ or $|\psi'\rangle$ given by (5.48) and

(5.52) respectively, onto $|\uparrow_\sigma\rangle$ or $|\downarrow_\sigma\rangle$. As a consequence, the geometry state of the spin system is updated to the geometry state of the Stern-Gerlach apparatus, given that $|\uparrow_\sigma\rangle$ and $|\downarrow_\sigma\rangle$ are eigenstates of the generic Pauli matrix (5.54). States $|\uparrow_\sigma\rangle$ and $|\downarrow_\sigma\rangle$ are *post-measurement* states that can also be interpreted as the *pre-measurement* states of a subsequent measurement, analogously to standard Quantum Mechanics, since the operators a, c appearing in (5.59) satisfy the commutation relations of the $SU_q(2)$ algebra as also the operators x, y in (5.48) do. This means that we can make the identifications

$$\begin{aligned} a = x' \,,\ c = y' \,, \\ -qc^* = -qy'^* \,,\ a^* = x'^* \,, \end{aligned} \tag{5.68}$$

when the measurement outcome is $|\uparrow_\sigma\rangle$ or $|\downarrow_\sigma\rangle$, respectively. The operators x', y' satisfy the commutation relations in (5.44), with $x' = \alpha$ and $y' = \gamma$. This ensures that the *post-measurement* state has the same structure as a *pre-measurement* state, meaning that the relevant physical quantities can be calculated as outlined in axiom 3. Of course, the observable used for a subsequent measurement is obtained as described in axiom 2 with a different copy of $SU_q(2)$ defined by the operators a', c'.

What about Bob?

The axioms stated above govern the measurement procedures performed by a single observer. We now wish to extend this formalism to the case in which two observers connected by a $SU_q(2)$ transformation are considered, to investigate the covariance properties of the framework we are proposing.

A change of reference frame between two observers Alice and Bob is described by a $SU_q(2)$ transformation that maps $\widehat{\mathscr{H}}^{(A)}_{\frac{1}{2}}$ in $\widehat{\mathscr{H}}^{(B)}_{\frac{1}{2}} := \widehat{\mathscr{H}}^{(A)}_{\frac{1}{2}} \otimes C\Big(SU_q(2)\Big) = \mathscr{H}_{\frac{1}{2}} \otimes C\Big(SU_q(2)\Big) \otimes C\Big(SU_q(2)\Big)$. Analogously, Alice's observables are mapped into Bob's, elements of $\mathscr{D}\Big(\mathscr{H}_{\frac{1}{2}}\Big) \otimes C\Big(SU_q(2)\Big) \otimes C\Big(SU_q(2)\Big)$. The last copy of $C(SU_q(2))$ encodes the information on the relative orientation between the two observers. In the following, the quantities relative to Alice refer to those introduced in axioms 1, 2, 3, formally extended with an additional $\mathbb{1}$ on the last copy of $C\Big(SU_q(2)\Big)$, and will be denoted by a superscript or subscript A, while the quantities relative to Bob are denoted by a superscript or subscript B.

The generic spin up state in Bob's reference frame can be obtained as

$$\Big|\psi^B\Big\rangle = U^q_g \Big|\psi^A\Big\rangle \,, \tag{5.69}$$

where

$$U^q_g := \begin{pmatrix} \mathbb{1}\otimes u & -q\mathbb{1}\otimes v^* \\ \mathbb{1}\otimes v & \mathbb{1}\otimes u^* \end{pmatrix} = |\uparrow\rangle\langle\uparrow|\big(\mathbb{1}\otimes u\big) - q|\uparrow\rangle\langle\downarrow|\big(\mathbb{1}\otimes v^*\big) + |\downarrow\rangle\langle\uparrow|\big(\mathbb{1}\otimes v\big) + |\downarrow\rangle\langle\downarrow|\big(\mathbb{1}\otimes u^*\big) \,, \tag{5.70}$$

is the $SU_q(2)$ matrix connecting the two observers written in terms of a further copy of the $SU_q(2)$ algebra generators u, v, which still satisfy the commutation relations (5.44), with u and v taking the role of α and γ, respectively. It is straightforward to show that the coefficients of $|\uparrow\rangle$ and $|\downarrow\rangle$ in $\Big|\psi^B\Big\rangle$ still satisfy the commutation relations of $SU_q(2)$. With a similar argument, it is possible to derive the generic spin down state, $\left|\psi'^{B}\right\rangle = U_g^q \left|\psi'^{A}\right\rangle$. The generic Pauli matrix can be written in Bob's reference fame as

$$\sigma_B^q = U_g^q \sigma_A^q U_g^{q\dagger} \tag{5.71}$$

With these definitions, the probabilities and the expectation value in (5.66) and (5.67) are invariant, namely

$$\left\langle \sigma_B^q \right\rangle_B = \left\langle \sigma_A^q \right\rangle_A \,,\; P_B\left(\uparrow_\sigma^B\right) = P_A\left(\uparrow_\sigma^A\right),\; P_B\left(\downarrow_\sigma^B\right) = P_A\left(\downarrow_\sigma^A\right), \tag{5.72}$$

where

$$\sigma_B^q = q\left|\uparrow_\sigma^B\right\rangle\left\langle\uparrow_\sigma^B\right| - q^{-1}\left|\downarrow_\sigma^B\right\rangle\left\langle\downarrow_\sigma^B\right| := q\,U_g^q\left|\uparrow_\sigma^A\right\rangle\left\langle\uparrow_\sigma^A\right|U_g^{q\dagger} - q^{-1}\,U_g^q\left|\downarrow_\sigma^A\right\rangle\left\langle\downarrow_\sigma^A\right|U_g^{q\dagger}\,, \tag{5.73}$$

$$\left\langle \sigma_I^q \right\rangle_J = \left\langle \psi^J \middle| \sigma_I^q \middle| \psi^J \right\rangle \,,\; P_I\left(\uparrow_\sigma^J\right) = \left\langle \psi^I \middle| \uparrow_\sigma^J \right\rangle \left\langle \uparrow_\sigma^J \middle| \psi^I \right\rangle \,,\; I, J \in \{A, B\}\,, \tag{5.74}$$

and the expectation values are elements of $C\left(SU_q(2)\right) \otimes C\left(SU_q(2)\right) \otimes C\left(SU_q(2)\right)$ defined as maps analogously to (5.63), where the third component of the tensor product contains products of elements of the copy of $C(SU_q(2))$ pertaining to the relative orientation, and the terms appearing in the first two copies of the tensor product are defined as in (5.63). This implies that the framework we developed is truly covariant under $SU_q(2)$ transformations, meaning that different observers agree on the physical laws describing the theory.

Quantum probabilities

In standard Quantum Mechanics, observables are given by self-adjoint operators and measurement outcomes are described by probability distributions that depend on the form of the operator and of the state of the physical system on which measurements are performed. The probabilistic nature of measurement outcomes arises from the superposition principle. In the axiomatisation of our framework, the implementation of quantum rotational symmetry has the natural consequence that probabilities themselves are promoted from non-negative real numbers to positive semi-definite self-adjoint operators, so that the probabilities of observing given outcomes are characterised by probability distributions as well. The latter depend on the form of the probability operators and of the geometry states $|\Phi\rangle \in \mathscr{H}_{SU_q(2)} \otimes \mathscr{H}_{SU_q(2)}$, which codify information on the geometrical configuration of the spin state and of the Stern-Gerlach device. Measurement outcomes of the probability thus depend on these geometrical configurations, just as in standard Quantum Mechanics, with the additional feature that our novel non-commutative framework allows for the geometry states to be written in terms of superpositions of probability eigenstates, defined as

$$P(\uparrow_\sigma)\left|p, r\right\rangle = p\left|p, r\right\rangle \,, \tag{5.75}$$

where r denotes the possible degeneracy of the eigenvalue p. In section 5.D we present a detailed analysis of such eigenstates. As we will discuss in the next section, generic geometry states describing semi-classical scenarios are factorisable in $\mathscr{H}_{SU_q(2)} \otimes \mathscr{H}_{SU_q(2)}$. Therefore, these states will be written as $|\Phi\rangle = \left|\Phi^S\right\rangle \otimes \left|\Phi^{SG}\right\rangle$, where $\left|\Phi^S\right\rangle$ ($\left|\Phi^{SG}\right\rangle$) is the geometry state describing the spatial orientation of the spin state (Stern-Gerlach apparatus), and will be generally given by superpositions of basis states in its copy of $\mathcal{H}_{SU_q(2)}$. In section 5.D we show that, in general, these semi-classical states of the geometry are not eigenstates of the probability operator, so they must be written as a superposition thereof. The overlap between the geometry states $|\Phi\rangle$ and the probability eigenstates $|p,r\rangle$ defines the distribution of outcomes of a probability measurement, according to

$$f(p) = \sum_r \langle \Phi | p,r \rangle \langle p,r | \Phi \rangle \ , \tag{5.76}$$

which is automatically normalised. This distribution is characterised by a mean value p_0 and a variance Δ_f^2, which is in general non-zero. Of course, a classical probability is described by a distribution with vanishing variance, namely a delta function of the form $f(p) = \delta(p - p_0)$, obtained when the geometry state of the experimental setup is a probability eigenstate with eigenvalue p_0.

What is the meaning of a probability distribution for probabilities? Ideally, a probability measurement can be performed by measuring the frequency of some event on an infinite number of trials. An observer studying the behaviour of a beam of spin 1/2 particles passing through a Stern-Gerlach apparatus could thus measure the probability of finding spin up with arbitrary precision when the number of particles in the beam goes to infinity. Standard Quantum Mechanics dictates that such an experiment, performed several times under the same conditions, would yield the same outcome for the spin up probability. Doubly Quantum Mechanics allows for superpositions of probability eigenstates and predicts, in general, non-vanishing variance for the probability measurements described above. Therefore, an observer repeating the experiment in the exact same geometrical configuration, both for the spin system and the Stern-Gerlach apparatus, would find a statistical distribution for the probability measurements.

A tentative interpretation for this feature is the following. When setting up the Stern-Gerlach apparatuses (one for preparing the spin states and another to actually perform the spin measurement), the observer fixes the geometrical configurations of the devices. Two experiments in which the Stern-Gerlach apparatuses are prepared in the exact same geometrical configuration will share the same geometry state, which is a superposition of probability eigenstates, in general. When performing a probability measurement, the geometry state collapses in a probability eigenstate and yields one of the probability values in the spectrum of the probability operator $P(\uparrow_\sigma)$ in (5.64). Due to quantum superposition of probability eigenstates, the ensuing statistical distribution for this observable will be characterised by a non-vanishing variance.

5.3.4 Semi-classical states

In the previous section, we have defined the non-commutative generalisation of physical quantities related to the outcome of spin measurements and expressed them in terms of operators that act on geometry states. The effective outcome of actual measurements is described by a distribution with mean value and variance that are obtained by computing the average values and variances of these operators in the spinor and Stern-Gerlach geometry states. We will denote the average value in the geometry states with a bar. For instance, the average value of σ^q in the geometry state $|\Phi\rangle$ is denoted by $\overline{\sigma^q} := \langle\Phi|\sigma^q|\Phi\rangle$ and the average value in the geometry of the expectation value of σ^q in a spin state is $\overline{\langle\sigma^q\rangle}$. The uncertainty on the measurement of a given observable, defined as the square root of the variance of the observable in the geometry states, is denoted by $\overline{\Delta}$. For instance, the uncertainty in the geometry states of the expectation value of the spin will be indicated by $\overline{\Delta}[\langle\sigma^q\rangle] = \sqrt{\overline{\Delta}^2[\langle\sigma^q\rangle]} := \sqrt{\overline{\langle\sigma^q\rangle^2} - \overline{\langle\sigma^q\rangle}^2}$.

Semi-classical conditions

In this section, we focus on *semi-classical* geometry states that yield small deviations, which vanish in the limit $q \to 1$, with respect to the results of standard Quantum Mechanics. This is done by requiring the average values of the relevant physical quantities in the geometry states to differ by $\mathcal{O}(1-q)$ with respect to the standard quantum-mechanical counterparts and their variances to be $\mathcal{O}(1-q)$. These requirements are consistent with the fact that the parameter q is expected to be very close to 1, $q \sim 1$, if the $SU_q(2)$ deformation is taken to be quantum-gravitational in origin, as in our case.

According to axiom 0, the states encoding information on the spin, Stern-Gerlach apparatuses, and relative orientation are elements of the Hilbert spaces of the representations of the $SU_q(2)$ algebras defined by $\{x, y\}$, $\{a, c\}$, and $\{u, v\}$ respectively. The Hilbert space is always given by $\mathcal{H}_{SU_q(2)} = \mathcal{H}_\pi \oplus \mathcal{H}_\rho$ defined in section 5.3.2. The representations are given by (5.46) and (5.47) where the quantum numbers $\{n, \phi, \chi\}$ are replaced by $\{n_s, \phi_s, \chi_s\}$, $\{n_a, \phi_a, \chi_a\}$, and $\{n_g, \phi_g, \chi_g\}$ for the representations of $\{x, y\}$, $\{a, c\}$, and $\{u, v\}$, respectively.

As pointed out in [2], these representations can be interpreted as giving a quantum description of the angles that define directions and relative orientations in space. Specifically, the angles ϕ, χ in the classical representations (5.34) and (5.38) retain their classical nature, while the angle θ becomes discretised in the range $[0, \pi]$, according to

$$\theta(n) = \begin{cases} 2\arcsin q^n \,, & n \in \mathbb{N}_0 \\ 0 \,, & n = \infty \end{cases} \,, \tag{5.77}$$

where we formally set $\theta(\infty) = 0$. Specifically, the values in $]0, \pi]$ corresponding to $n \in \mathbb{N}_0$ are derived from the $\mathcal{H}_\pi$ component of $\mathcal{H}_{SU_q(2)}$, while $\theta(\infty) = 0$ arises from the $\mathcal{H}_\rho$ component.

We start by identifying a class of semi-classical states describing the directions of spin systems and Stern-Gerlach apparatuses in terms of these quantum numbers. Since we want to describe a physical setup in which spin states and Stern-Gerlach apparatuses can be prepared independently, we require that the semi-classical geometry state describing the experimental setup is separable, so that it can be written as the tensor product of a geometry state characterising the spin direction and the geometry state describing the Stern-Gerlach direction. Each of these states is an element of $\mathcal{H}_{SU_q(2)}$ and will be specified by two angles $\theta = \theta(n)$ and $\omega = \phi - \chi$, indicating the classical counterpart of the direction along which spins and Stern-Gerlach apparatuses are aligned, where $\theta(n)$ is one of the allowed values in (5.77). These states will be denoted by $\left|\Phi^S(\theta_s, \omega_s)\right\rangle$ and $\left|\Phi^{SG}(\theta_a, \omega_a)\right\rangle$ and the full geometry state of the experimental setup will then be given by the tensor product $\left|\Phi(\theta_s, \omega_s, \theta_a, \omega_a)\right\rangle = \left|\Phi^S(\theta_s, \omega_s)\right\rangle \otimes \left|\Phi^{SG}(\theta_a, \omega_a)\right\rangle \in \mathcal{H}_{SU_q(2)} \otimes \mathcal{H}_{SU_q(2)}$. Of course, since the angle $\theta(n)$ assumes discrete values, for a given classical direction specified by angles (θ, ω) it is only possible to find, in general, a semi-classical state that describes a spin or a Stern-Gerlach aligned along a direction $(\theta(n_\theta),\omega)$ that is as close as possible to (θ,ω), where n_θ is the value of n such that $\left|\theta(n)-\theta\right|$ is minimal. For values of q closer to 1, the gap between two consecutive angles becomes smaller, so that the values $\theta(n)$ are more dense in any given angular range.

The semi-classicality conditions for the geometry states $\left|\Phi^S(\theta_s,\omega_s)\right\rangle$, $\left|\Phi^{SG}(\theta_a,\omega_a)\right\rangle$ read

$$\begin{cases} \overline{|\psi\rangle}_{\theta_s,\omega_s} = \left(\cos\frac{\theta_s}{2}+\mathcal{O}(1-q)\right)|\uparrow\rangle + \left(e^{i\omega_s}\sin\frac{\theta_s}{2}+\mathcal{O}(1-q)\right)|\downarrow\rangle \\ \overline{\Delta}^2\left[\,|\psi\rangle\,\right]_{\theta_s,\omega_s} = \mathcal{O}(1-q)\,|\uparrow\rangle + \mathcal{O}(1-q)\,|\downarrow\rangle \\ \overline{\sigma^q}_{\theta_a,\omega_a} = \begin{pmatrix} \cos\theta_a+\mathcal{O}(1-q) & e^{-i\omega_a}\sin\theta_a+\mathcal{O}(1-q) \\ e^{i\omega_a}\sin\theta_a+\mathcal{O}(1-q) & -\cos\theta_a+\mathcal{O}(1-q)\end{pmatrix} \\ \overline{\Delta}^2\left[\sigma^q\right]_{\theta_a,\omega_a} = \begin{pmatrix} \mathcal{O}(1-q) & \mathcal{O}(1-q) \\ \mathcal{O}(1-q) & \mathcal{O}(1-q)\end{pmatrix} \end{cases}, \tag{5.78}$$

where the average value and variances on the geometry are taken in the full geometry state $\left|\Phi(\theta_s,\omega_s,\theta_a,\omega_a)\right\rangle$ and the subscripts indicate which of the two components of the full geometry state enters non-trivially in the computation. The variance of a non-Hermitian operator O is defined as $\overline{\Delta}^2[O] = \langle\Phi|O^\dagger O|\Phi\rangle - \langle\Phi|O^\dagger|\Phi\rangle\,\langle\Phi|O|\Phi\rangle$ following [335]. Additionally, the full state of the geometry $\left|\Phi(\theta_s,\omega_s,\theta_a,\omega_a)\right\rangle$ has to satisfy

$$\begin{cases} \overline{P(\uparrow_\sigma)}_{\theta_s,\omega_s,\theta_a,\omega_a} = \frac{1}{2}\Big[1+\cos(\theta_a)\cos(\theta_s)+\cos(\omega_a-\omega_s)\sin(\theta_a)\sin(\theta_s)\Big] + \mathcal{O}(1-q) \\ \overline{\Delta}^2\left[P(\uparrow_\sigma)\right]_{\theta_s,\omega_s,\theta_a,\omega_a} = \mathcal{O}(1-q) \end{cases}. \tag{5.79}$$

We emphasise that the quantities in (5.78) only serve as guidelines for identifying the states that describe semi-classical experimental setups, but do not enter directly in the physical predictions of the theory. As discussed in section 5.3.3, the average value and variance of the probability are the observable quantities and are linked to the expectation value of the spin. The condition (5.79) guarantees that they are affected by small corrections of order $(1-q)$ when the geometry states describe a semi-classical scenario.

When two observers are involved, we consider an additional copy of $C(SU_q(2))$ and the corresponding additional copy of $\mathcal{H}_{SU_q(2)}$ contains states describing the relative orientation between the observers (see section 5.3.3). The discussion of semi-classical states in this context is a simple extension of the one presented above. Specifically, the semi-classical conditions can be generalised by replacing the operators appearing in (5.78), (5.79) by the corresponding ones for Bob, defined in section 5.3.3. The full state of the geometry now involves three states and is of the form $\left|\Phi\left(\theta_s,\omega_s,\theta_a,\omega_a,\theta_g,\chi_g,\phi_g\right)\right\rangle =$ $= \left|\Phi^S(\theta_s,\omega_s)\right\rangle \otimes \left|\Phi^{SG}(\theta_a,\omega_a)\right\rangle \otimes \left|\Phi^{RO}\left(\theta_g,\chi_g,\phi_g\right)\right\rangle$, *i.e.* the full geometry state is factorisable. In section 5.3.5 and section 5.E we show that states of the form $\left|\Phi^{RO}\left(\theta_g,\chi_g,\phi_g\right)\right\rangle$ are the semi-classical counterpart of generic rotations $R_z(\alpha)R_x(\theta_g)R_z(\gamma)$, where $\chi_g = \frac{\alpha+\gamma}{2}$, $\phi_g = \frac{3}{2}\pi - \frac{\alpha-\gamma}{2}$, and $\theta_g = \theta(n)$ is again one of the allowed values in eq. (5.77). The discretised nature of $\theta(n)$ in the context of two observers implies that for a given classical rotation specified by angles (θ, ϕ, χ) it is only possible to find, in general, a semi-classical state that describes a rotation defined by $(\theta(n_\theta), \phi, \chi)$, where n_θ has the same meaning as before. Notice that the states $\left|\Phi^{RO}\big(\theta_g, \chi_g, \phi_g\big)\right\rangle$ depend on both angles χ_g and ϕ_g and not only on their difference, as is the case for semi-classical states describing the direction of physical systems, since a rotation is specified by three angles, while a direction requires only two.

Semi-classical states and probability eigenstates

We now comment on the connection between the semi-classical geometry states and probability eigenstates. As shown in section 5.D, the only factorisable eigenstates of the probability operator are of the form $|\chi_s\rangle \otimes |\chi_a\rangle$, $|\chi_s\rangle \otimes |n_a, \phi_a, \chi_a\rangle$, $|n_s, \phi_s, \chi_s\rangle \otimes |\chi_a\rangle$, $|\mu_s, \omega_s\rangle_s \otimes |\mu_a, \omega_a\rangle_a$, with

$$\begin{aligned}
|\mu_s, \omega_s\rangle_s &= N_s \sum_{n_s=0}^{\infty} f(\mu_s, n_s)\, |n_s, \phi_s, \chi_s\rangle = N_s \sum_{n_s=0}^{\infty} \frac{q^{\frac{n_s}{2}(n_s-1)} \sqrt{(1-q^2)^{n_s-1}}}{\sqrt{(q^4; q^2)_{n_s-1}}}\, \mu_s^{n_s}\, |n_s, \phi_s, \chi_s\rangle \,, \\
|\mu_a, \omega_a\rangle_a &= N_a \sum_{n_a=0}^{\infty} q^{n_a} f(\mu_a, n_a)\, |n_a, \phi_a, \chi_a\rangle = \\
&= N_a \sum_{n_a=0}^{\infty} \frac{q^{\frac{n_a}{2}(n_a+1)} \sqrt{(1-q^2)^{n_a-1}}}{\sqrt{(q^4; q^2)_{n_a-1}}}\, \mu_a^{n_a}\, |n_a, \phi_a, \chi_a\rangle \,,
\end{aligned} \tag{5.80}$$

where $\omega_{s,a} = \phi_{s,a} - \chi_{s,a}$, $N_{s,a}$ are normalisation constants, and the q-Pochhammer symbol $(a; q)_n$ is defined in section 5.D. As discussed in section 5.E, the only separable probability eigenstates that do not satisfy the requirements (5.78), (5.79) are of the form $|\chi_s\rangle \otimes |n_a, \phi_a, \chi_a\rangle$ and $|n_s, \phi_s, \chi_s\rangle \otimes |\chi_a\rangle$ for $n_s, n_a \geq 1$. We also show that states of the form $|\chi_{s,a}\rangle$ and $|\mu_{s,a}, \omega_{s,a}\rangle$ describe spin systems and Stern-Gerlach apparatuses aligned along directions $\big(\theta_{s,a}(n), \omega_{s,a}\big)$. In particular, $\left|\Phi^{S,SG}\big(0, \omega_{s,a}\big)\right\rangle = |\chi_{s,a} = \omega_{s,a}\rangle$ describe spin and Stern-Gerlach apparatuses that are aligned in the positive z direction, while for a classical direction (θ, ω) with $\theta \neq 0$, the states that semi-classically describe physical systems aligned along the direction $\big(\theta(n_\theta), \omega\big)$ which is the closest to (θ, ω) are given by $\left|\Phi^{S,SG}\big(\theta_{s,a}(n_\theta), \omega_{s,a}\big)\right\rangle = \left|\mu_{s,a}(\theta), \omega_{s,a}\right\rangle_{s,a}$. In the previous relations, n_θ is the value of n such that $|\theta(n) - \theta|$ is minimal, where $\theta(n)$ is given by (5.77), and $\mu_{s,a}(\theta)$ is such that the distributions $\left|f(\mu_{s,a}(\theta), n_{s,a})\right|^2$ have maximum in n_θ. In general, a semi-classical full geometry state $\left|\Phi^S(\theta_s, \omega_s)\right\rangle \otimes \left|\Phi^{SG}(\theta_a, \omega_a)\right\rangle$ will not be a probability eigenstate, of course. Probability measurements involving these geometry states will exhibit doubly quantum-mechanical behaviour, as discussed in section 5.3.3. For what concerns the relative orientation, in section 5.E we show that the states of the same form as those considered for the single observer satisfy the generalised semi-classical conditions, where states of the form (5.80) are denoted as $\left|\mu_g, \phi_g, \chi_g\right\rangle_g$ to make the dependence on both angles ϕ_g, χ_g explicit. The states $\left|\Phi^{RO}\big(\theta_g(\theta), \phi_g, \chi_g\big)\right\rangle$ semi-classically describe the relative orientation between two

	$(0,\omega)$ direction	(θ,ω) direction, $\theta \neq 0$
$\left\|\Phi^S\left(\theta_s(n_\theta),\omega_s\right)\right\rangle$	$\|\chi_s = \omega_s\rangle$	$\left\|\mu_s(\theta),\omega_s\right\rangle_s$
$\left\|\Phi^{SG}\left(\theta_a(n_\theta),\omega_a\right)\right\rangle$	$\|\chi_a = \omega_a\rangle$	$\left\|\mu_a(\theta),\omega_a\right\rangle_a$
	$R_z\left(2\chi_g\right)$	$R_z(\alpha)R_x(\theta)R_z(\gamma)$, $\theta \neq 0$
$\left\|\Phi^{RO}\left(\theta_g(n_\theta),\phi_g,\chi_g\right)\right\rangle$	$\left\|\chi_g\right\rangle$	$\left\|\mu_g(\theta),\phi_g,\chi_g\right\rangle_g$

Table 5.2: Semi-classical states used in the rest of the section. Further details about the notation adopted for these states can be found in section 5.3.4.

observers. Classically, the latter is specified by the rotation matrix that relates the two observers, parameterised as $R_z(\alpha)R_x(\theta)R_z(\gamma)$. In our quantum setting, for a given classical rotation specified by three angles (θ,α,γ), the state that describes the rotation that is closest to $R_z(\alpha)R_x(\theta)R_z(\gamma)$ is $\left|\Phi^{RO}\left(\theta_g(n_\theta),\phi_g,\chi_g\right)\right\rangle = \left|\mu_g(\theta),\phi_g,\chi_g\right\rangle_g$, where $\chi_g = \frac{\alpha+\gamma}{2}$, $\phi_g = \frac{3}{2}\pi - \frac{\alpha-\gamma}{2}$, and $\theta_g\left(n_\theta\right) \neq 0$ is such that $\left|f(\mu_g,n_g)\right|^2$, which replaces $\left|f(\mu_{s,a},n_{s,a})\right|^2$ in (5.80), is peaked around n_θ. States of the form $\left|\Phi^{RO}\left(0,\phi_g,\chi_g\right)\right\rangle = \left|\chi_g\right\rangle_g$ semi-classically describe rotations $R_z(\alpha)R_x(0)R_z(\gamma) = R_z(\alpha+\gamma) = R(2\chi_g)$. As for the case of the single observer, a semi-classical full geometry state $\left|\Phi^S(\theta_s,\omega_s)\right\rangle \otimes \left|\Phi^{SG}(\theta_a,\omega_a)\right\rangle \otimes \left|\Phi^{RO}\left(\theta_g,\phi_g,\chi_g\right)\right\rangle$ will not be a probability eigenstate, in general. In table 5.2 we summarise all the semi-classical states that we are going to use in the following. Further details, as well as some numerical examples involving these states can be found in section 5.E.

5.3.5 (Non)-Alignment protocol between two observers

In this section, we analyse a protocol in which two observers attempt to align themselves by exchanging spin systems, emphasising the novelties introduced by the Doubly Quantum Mechanics framework and providing also some numerical examples using the semi-classical states introduced in the previous section.

In the classical setting, two observers can reconstruct the rotation matrix that relates their (in general misaligned) reference frames according to the following protocol. Suppose that Alice and Bob are two observers, each equipped with their own set of Stern-Gerlach apparatuses defining their Cartesian axes. Alice prepares $3N$ spins orientated along the positive direction of each one of her axes, x_A, y_A, z_A and sends them to Bob. Then Bob divides each group into three sets of N spins, and with each set he performs spin measurements along each of his axes, x_B, y_B, z_B, by using the corresponding Stern-Gerlach apparatuses. This physical setup is depicted in fig. 10 (and, schematically, in fig. 13). The spin expectation values along Bob's axes computed with the spin states prepared by Alice are the elements of the rotation matrix

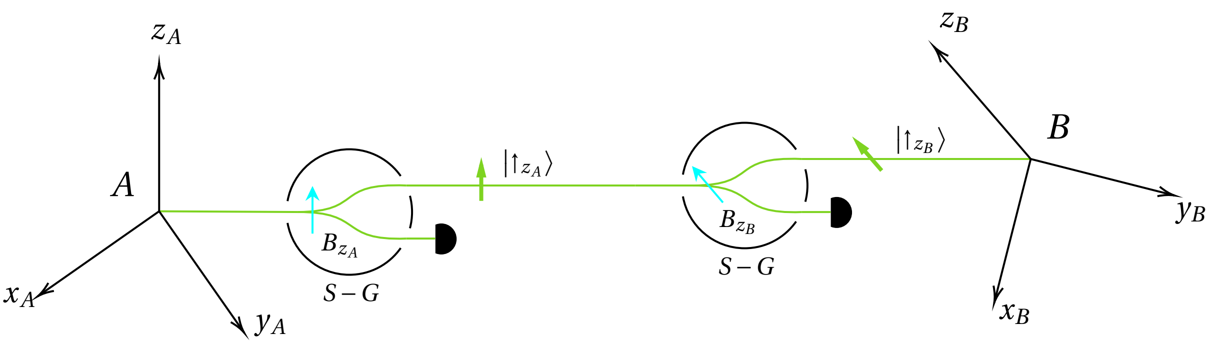

Figure 13: Schematic description of the alignment protocol in fig. 10. Alice sends spin systems aligned along her axes and Bob performs spin measurements on these spins with Stern-Gerlach apparata aligned along his axes. Notice that Alice prepares her spin states in the $|\uparrow_{z_A}\rangle$ state by blocking the lower path of her Stern-Gerlach apparatus, and Bob counts the number of spin up results by doing the same on his Stern-Gerlach apparatus. The picture describes the measurement of the R_{33} component of the rotation matrix, since the Stern-Gerlach apparatuses (*i.e.* the magnetic fields B_i) used by Alice and Bob are aligned along their z axes, z_A and z_B respectively. By preparing spin systems aligned along the other directions i and performing spin measurement along other directions j, all the R_{ij} matrix elements can be measured.

R that relates the two reference frames [2] according to

$$R_{ij} = \left\langle \uparrow^A_{\vec{i}} \middle| \sigma^B_{\vec{j}} \middle| \uparrow^A_{\vec{i}} \right\rangle , \quad i,j = x,y,z \,. \tag{5.81}$$

These expectation values are given by the difference between the probability of obtaining spin up and the probability of obtaining spin down, according to (5.41). To compute these probabilities, Bob will measure the frequencies of spin up (down) outcomes for every i, j pair. For any of these pairs, the number of spin up results, k, on a total of N measurements, will be distributed according to a binomial distribution of the form

$$P(k,N,p) = \binom{N}{k} p^k (1-p)^{N-k} \,. \tag{5.82}$$

where p is the probability of obtaining spin up, and depends on the spin state used to perform the measurement, the direction along which the Stern -Gerlach is orientated and the relative orientation between the two observers. The expectation value of k is given by $\mathbb{E}[k] = Np$ and its uncertainty is given by the square root of the variance $\Delta[k] = \sqrt{\mathbb{E}[k^2] - \mathbb{E}[k]^2} = \sqrt{Np(1-p)}$. Therefore, the expectation value for the frequency is given by p with uncertainty $N^{-\frac{1}{2}}\sqrt{p(1-p)}$. For any given i, j pair, denoting the probability of obtaining spin up by p_{ij}, Bob will then measure the expectation value of the spin to be distributed according to

$$\left\langle \uparrow^A_{\vec{i}} \middle| \sigma^B_{\vec{j}} \middle| \uparrow^A_{\vec{i}} \right\rangle^{\text{meas}} := \frac{\left(2\mathbb{E}[k] - N\right) \pm 2\Delta[k]}{N} = \left(2p_{ij} - 1\right) \pm 2\sqrt{\frac{p_{ij}(1-p_{ij})}{N}} \xrightarrow[N\to\infty]{} 2p_{ij} - 1 \,. \tag{5.83}$$

By means of this protocol, Bob can measure the elements of the rotation matrix with arbitrary precision, since the uncertainty scales as $N^{-\frac{1}{2}}$, hence the two observers can sharply align their reference frames when exchanging an infinite number of spin systems.

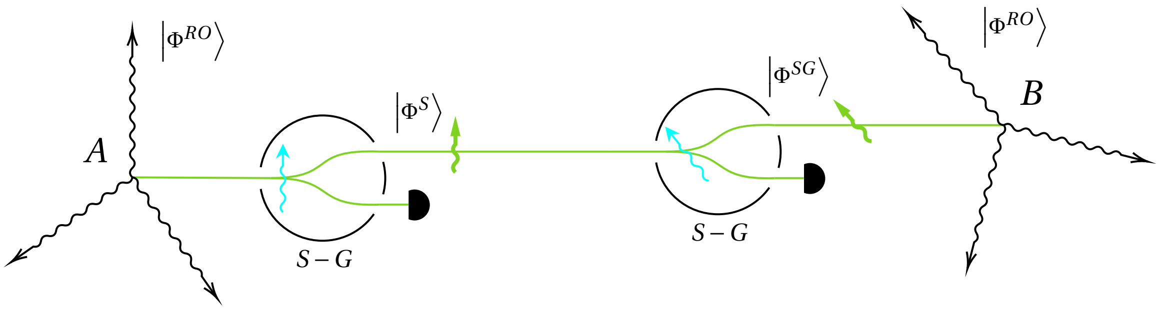

Figure 14: The doubly quantum version of the protocol in fig. 13. The schematic setup is similar to the classic case, the difference being that the the geometrical degrees of freedom are now described by quantum states in the Hilbert space $\mathscr{H}_{SU_q(2)}$. Specifically, the states $\left|\Phi^S\right\rangle$ and $\left|\Phi^{SG}\right\rangle$ replace the notion of alignment of spins and Stern-Gerlach apparata along the observers' axes, while the state $\left|\Phi^{RO}\right\rangle$ replaces the notion of coordinate axes and describes the relative orientation between Alice and Bob, whose reference frames will not be sharp anymore. The quantum nature of these geometrical degrees of freedom is represented by curly lines for the reference frames, the magnetic fields inside the Stern-Gerlach apparata, and the spin systems in place of the straight lines of fig. 13.

The doubly quantum protocol

We now investigate how the protocol outlined above changes in our non-commutative framework, as a consequence of the quantisation of the geometry of the spin states, Stern-Gerlach apparatuses, and relative orientation. In doing so, we have to describe how Bob performs measurements with his apparatuses on the states that Alice prepares, according to (5.81), which we want to generalise to the non-commutative setting. This means that measurements are performed on the states $\left|\psi^A\right\rangle \otimes \mathbb{1}$, where $\left|\psi^A\right\rangle$ is given by (5.48), and the observable is σ_B^q, given by (5.71), which describes a rotation of Alice's observable.

In light of the properties of quantum probabilities discussed in section 5.3.3, the probability that Bob measures spin up or down in a measurement is itself described by a probability distribution $f(p)$. This means that the number of spin up outcomes k in N spin measurements is distributed according to the joint probability density

$$\mathscr{P}(k,N,p) = \binom{N}{k} p^k (1-p)^{N-k} f(p)\,, \tag{5.84}$$

obtained by multiplying the binomial distribution $P(k,N,p)$ with the probability density $f(p)$[1], where p is the probability of obtaining spin up in a measurement and $f(p)$ is given by

$$f(p) = \sum_r \langle\Phi|p,r\rangle\langle p,r|\Phi\rangle\ ,\ |\Phi\rangle = \left|\Phi^S\right\rangle \otimes \left|\Phi^{SG}\right\rangle \otimes \left|\Phi^{RO}\right\rangle\,, \tag{5.85}$$

[1]This is nothing more than the probability density of measuring the value p for the observable $P_A\left(\uparrow_\sigma^B\right)$ and observing k spin up outcomes within N spin measurements, which is given by the product between the conditional probability for the latter event given that the value measured for the probability is p, $P(k,N,p)$, and the probability density of measuring the value p, $f(p)$. This is discussed in more detail in section 5.F.3.

which generalises (5.76) to the case in which also the relative orientation between observers is taken into account, and $|p,r\rangle$ are now the eigenstates of the probability operator $P_A\left(\uparrow^B_\sigma\right)$ defined in (5.74). We will focus on the case in which these three states are semi-classical (in the sense of section 5.3.4) and can be assigned independently.

With these ingredients, we can generalise the classical measurement procedure. The experimental setup is the same as in the commutative case (see fig. 14), but the results obtained by Bob will be different because the number of spin up outcomes is distributed according to (5.84). For this reason, the expectation value of k and its uncertainty, as shown in section 5.F.1, read

$$\mathbb{E}[k] = Np_0\,,\;\; \Delta[k] = \sqrt{N(N-1)\Delta_f^2 + Np_0(1-p_0)} \tag{5.86}$$

where p_0 and Δ_f^2 are the mean value and variance of $f(p)$, respectively. Recalling that $\left\langle \sigma_B^q \right\rangle = qP_A\left(\uparrow^B_\sigma\right) - q^{-1}P_A\left(\downarrow^B_\sigma\right) = \left(q+q^{-1}\right)P_A\left(\uparrow^B_\sigma\right) - q^{-1}\mathbb{1}$, the measurement outcomes of the expectation value of the spin will be distributed, up to constant factors, according to $f(p)$, as shown in section 5.F.2. Therefore, analogously to (5.83), Bob will measure the expectation value of the spin $\left\langle \sigma_B^q \right\rangle^{\text{meas}} := \overline{\left\langle \sigma_B^q \right\rangle} \pm \overline{\Delta}\left[\left\langle \sigma_B^q \right\rangle\right]$ to be distributed as

$$\begin{aligned} \left\langle \sigma_B^q \right\rangle^{\text{meas}} &= \frac{\left[\left(q+q^{-1}\right)\mathbb{E}[k] - Nq^{-1}\right] \pm \left(q+q^{-1}\right)\Delta[k]}{N} \\ \Longrightarrow \left\langle \sigma_B^q \right\rangle^{\text{meas}} &\xrightarrow[N\to\infty]{} \left[\left(q+q^{-1}\right)p_0 - q^{-1}\right] \pm \left(q+q^{-1}\right)\Delta_f\,. \end{aligned} \tag{5.87}$$

We thus see that in the non-commutative case, Bob cannot measure the elements of the rotation matrix with arbitrary precision. Indeed, a fundamental uncertainty on these matrix elements, that depends on the states $\left|\Phi^S\right\rangle$, $\left|\Phi^{SG}\right\rangle$ and $\left|\Phi^{RO}\right\rangle$, arises in our framework. Of course, the more the distribution function $f(p)$ is peaked around the value p_0, the more precise the measurements performed by Bob will be. In standard Quantum Mechanics, this function becomes $f(p) = \delta\left(p-p_0\right)$, and we recover the classical result (5.83), since $\Delta_f = 0$. However, in our setting, this uncertainty is typically non-vanishing, since geometry states describing semi-classical experimental configurations are not probability eigenstates, in general. This yields a fundamental uncertainty on the measured expectation value of the spin that does not vanish even when performing an infinite number of measurements, contrary to the standard result in (5.83). In the following subsection, we will consider concrete examples of the theoretical result outlined above to numerically quantify the fundamental uncertainty appearing in (5.87).

Numerical analysis of the doubly quantum protocol

In the classical protocol, the directions along which spins and Stern-Gerlach apparatuses are aligned coincide with the Cartesian axes of the two observers involved, which define the rows and columns indexes of the rotation matrix that connects them, see (5.81). In the non-commutative setting, these directions are quantised, so that the rows and column indexes of the rotation matrix are replaced by geometry states that semi-classically describe the directions x_q, y_q, z_q that are closest to three Cartesian axes x, y, z, in the spirit of section 5.3.4.

These states are given by

$$
\begin{aligned}
x_q \,:\, \left|x_q^S\right\rangle &:= \left|\Phi^S\left(\theta\left(n_{\frac{\pi}{2}}\right),0\right)\right\rangle = \left|\mu_s\left(\frac{\pi}{2}\right),0\right\rangle_s \,,\, \left|x_q^{SG}\right\rangle := \left|\Phi^{SG}\left(\theta\left(n_{\frac{\pi}{2}}\right),0\right)\right\rangle = \left|\mu_a\left(\frac{\pi}{2}\right),0\right\rangle_a \,,\\
y_q \,:\, \left|y_q^S\right\rangle &:= \left|\Phi^S\left(\theta\left(n_{\frac{\pi}{2}}\right),\frac{\pi}{2}\right)\right\rangle = \left|\mu_s\left(\frac{\pi}{2}\right),\frac{\pi}{2}\right\rangle_s \,,\, \left|y_q^{SG}\right\rangle := \left|\Phi^{SG}\left(\theta\left(n_{\frac{\pi}{2}}\right),\frac{\pi}{2}\right)\right\rangle = \left|\mu_a\left(\frac{\pi}{2}\right),\frac{\pi}{2}\right\rangle_a \,,\\
z_q \,:\, \left|z_q^S\right\rangle &:= \left|\Phi^S(0,\omega_s)\right\rangle = \left|\chi_s = \omega_s\right\rangle \,,\, \left|z_q^{SG}\right\rangle := \left|\Phi^{SG}(0,\omega_a)\right\rangle = \left|\chi_a = \omega_a\right\rangle \,,
\end{aligned}
\tag{5.88}
$$

For the subsequent numerical examples, we take $q = 0.99$ so that $\theta\left(n_{\frac{\pi}{2}}\right) = \theta(34) = 1.006\,\frac{\pi}{2}$ is the $\theta(n)$ that is closest to $\frac{\pi}{2}$, corresponding to $\mu\left(\frac{\pi}{2}\right) = 7$, with the prescriptions described in section 5.E and section 5.3.4.

The building blocks needed to construct the non-commutative analogue of the matrix elements describing the relative orientation between Alice and Bob are

$$
\left\langle \sigma_B^q \right\rangle_A = \left\langle \psi^A \middle| \sigma_B^q \middle| \psi^A \right\rangle = \left\langle \psi^A \middle| U_g^q \sigma_A^q U_g^{q\dagger} \middle| \psi^A \right\rangle \,, \tag{5.89}
$$

as is also the case in the classical protocol. Expectation values and variances of this operator in the states of the geometry yield the mean value and uncertainty of the distribution associated to the rotation matrix connecting the two observers. In particular, we define

$$
\begin{aligned}
\overline{R_{ij}^q}\left(\theta_g,\phi_g,\chi_g\right) &:= \left\langle \Phi_{ij}\left(\theta_g,\phi_g,\chi_g\right) \middle| \left\langle \sigma_B^q \right\rangle_A \middle| \Phi_{ij}\left(\theta_g,\phi_g,\chi_g\right) \right\rangle \,,\\
\overline{\Delta}^2 [R_{ij}^q]\left(\theta_g,\phi_g,\chi_g\right) &:= \left\langle \Phi_{ij}\left(\theta_g,\phi_g,\chi_g\right) \middle| \left\langle \sigma_B^q \right\rangle_A^2 \middle| \Phi_{ij}\left(\theta_g,\phi_g,\chi_g\right) \right\rangle - \left[\overline{R_{ij}^q}\left(\theta_g,\phi_g,\chi_g\right)\right]^2 ,
\end{aligned}
\tag{5.90}
$$

where

$$
\left|\Phi_{ij}\left(\theta_g,\phi_g,\chi_g\right)\right\rangle = \left|i_q^S\right\rangle \otimes \left|j_q^{SG}\right\rangle \otimes \left|\Phi^{RO}\left(\theta_g,\phi_g,\chi_g\right)\right\rangle . \tag{5.91}
$$

and $i, j = x, y, z$.

As we will concretely show in the following examples, the doubly quantum nature of the alignment protocol generally yields a deformed rotation matrix, with elements specified by (5.90), that replaces the standard rotation matrix. The elements in (5.90) are characterised, in general, by a non-vanishing Δ_f that limits the possibility of the two observers to learn about their relative orientation with infinite precision. We will consider three examples given by the states

$$
\begin{aligned}
\left|\Phi^{RO}\left(\theta_g,\phi_g,\chi_g\right)\right\rangle &= \left|\chi_g = 0\right\rangle \,,\\
\left|\Phi^{RO}\left(\theta_g,\phi_g,\chi_g\right)\right\rangle &= \left|\mu_g\left(\frac{\pi}{2}\right),\phi_g = \frac{3}{2}\pi,\chi_g = 0\right\rangle_g \,,\\
\left|\Phi^{RO}\left(\theta_g,\phi_g,\chi_g\right)\right\rangle &= \left|\mu_g(\pi),\phi_g = \frac{3}{2}\pi,\chi_g = 0\right\rangle_g = \left|n_g = 0,\phi_g = \frac{3}{2}\pi,\chi_g = 0\right\rangle .
\end{aligned}
\tag{5.92}
$$

We denote the measurement outcomes of the alignment protocol as deformed rotation matrix elements $R_{ij}^q\left(\theta_g,\phi_g,\chi_g\right)$, characterised by their mean values and uncertainties, collected in a

3×3 matrix denoted by $R^q\left(\theta_g, \phi_g, \chi_g\right)$. Using the geometry states listed above, with the same numerical approximations outlined in section 5.E and up to two decimal places, we obtain

$$R^q\left(0,0,0\right) = \begin{pmatrix} 0.99 & 0.00 & -0.03 \\ 0.00 & 0.99 & -0.03 \\ 0.00 & 0.00 & 0.99 \end{pmatrix} \pm \begin{pmatrix} 0.00 & 0.14 & 0.10 \\ 0.14 & 0.00 & 0.10 \\ 0.10 & 0.10 & 0.00 \end{pmatrix} \tag{5.93}$$

$$R^q\left(1.006\frac{\pi}{2}, \frac{3}{2}\pi, 0\right) = \begin{pmatrix} 0.98 & 0.02 & 0.00 \\ 0.00 & 0.01 & -0.99 \\ 0.00 & 0.97 & 0.00 \end{pmatrix} \pm \begin{pmatrix} 0.03 & 0.17 & 0.10 \\ 0.17 & 0.17 & 0.02 \\ 0.14 & 0.02 & 0.10 \end{pmatrix} \tag{5.94}$$

$$R^q\left(\pi, \frac{3}{2}\pi, 0\right) = \begin{pmatrix} 0.98 & 0.00 & 0.02 \\ 0.00 & -0.98 & 0.02 \\ 0.00 & 0.00 & -0.97 \end{pmatrix} \pm \begin{pmatrix} 0.03 & 0.14 & 0.17 \\ 0.14 & 0.03 & 0.17 \\ 0.17 & 0.17 & 0.00 \end{pmatrix} \tag{5.95}$$

The values in these matrices differ by at most $\mathcal{O}[(1-q)]$ from their classical counterparts $R_z(\alpha)R_x(\theta_g)R_z(\gamma)$, with $\chi_g = \frac{\alpha+\gamma}{2}$ and $\phi_g = \frac{3}{2}\pi - \frac{\alpha-\gamma}{2}$. The uncertainties are the square roots of the elements $\overline{\Delta}^2[R_{ij}]$ appearing in eq. (5.90) and are at most $\mathcal{O}[(1-q)^{1/2}]$. Analogously to the examples for the single observer presented in section 5.E, one can numerically check that as q increases, the uncertainties become closer to 0. Notice that the result obtained for the first example, where the relative orientation state effectively describes the identity transformation, still predicts an intrinsic uncertainty in the alignment procedure. This is in agreement with the fact that, in our framework, the geometrical degrees of freedom describing the spin and Stern-Gerlach orientations and the relative orientation between reference frames acquire a quantum nature. Therefore, the directions associated with the very messengers of the protocol and the measurement apparatuses are quantum, so that an intrinsic uncertainty arises even when the state describing the relative orientation between the two observers corresponds to the identity transformation.

The numerical examples analysed in this subsection confirm what was expected from the general theory presented in section 5.3.5. We have chosen a particular class of semi-classical states and other choices are possible, which do not affect significantly the numerical results. Indeed, in section 5.D we have shown that there are no separable eigenstates of the probability operator that describe measurements of spin systems misaligned with the Stern-Gerlach apparatuses. For this reason, some of the elements of the quantum rotation matrix are inevitably affected by a non-vanishing uncertainty, for any choice of semi-classical states. This yields a fundamental limit on the observers' knowledge regarding other observers' orientation in space when executing the alignment protocol described above, preventing them from sharply aligning their reference frames.

5.4 Discussion

In this chapter, we deepened the physical interpretation of the quantum group $SU_q(2)$. With the goal of investigating the emergence of fundamental limits on the precision of spatial synchronisation in an alignment protocol in a quantum spacetime, we first proposed a model of agency-dependence of spacetime and then developed the full fledged formalism of Doubly

Quantum Mechanics, in which the geometrical degrees of freedom of all physical systems, including measurement devices, exhibit non-classical features.

In the first part of the chapter, based on [2], we derived a framework in which one of the Euler angles becomes quantised and reference frames exhibit fuzzy properties. The nature of this fuzziness is distinct from the one usually associated with quantum reference frames (cf. chapter 2) [51, 102–126, 147, 148, 336, 337], which arises by describing the frame itself as an ordinary quantum system. The key difference is the source of the frame fuzziness: for QRFs it is governed by Planck's constant $\hbar$ and follows from the standard laws of Quantum Mechanics, while in our scenario it is the deformation of spatial rotations, governed by the parameter q, that produces the fuzziness. Accordingly, the states encoding the fuzziness are associated with physical systems (including the frame) in the context of QRFs, while here they are abstract geometry states that describe the relation between two frames. QRFs can be in relative superposition, however, the transformations between them act on the states of the physical systems directly, without being described by a quantum state of their own.

The most striking consequence of our work is that, contrary to the classical case in which all observers can objectively reconstruct space(time), in our work, they are agents: using quantum rotations, they reconstruct their own versions of the space(time) that depend on their choices of the z axis, and there is no operational procedure they can use to reconstruct it objectively. In fact, even if they were to exchange their data, they would need to q-rotate it in their coordinates, inevitably introducing the intrinsic uncertainty predicted by the $SU_q(2)$ model.

In [2] we focused on a semi-classical regime, in which the expectation values and the variances of the q-deformed rotation matrix act on classical vectors, to derive some novel qualitative properties. This is a first step toward a description of a q-deformed version of the alignment procedure described in section 5.1. The next step in this programme is to develop the full quantum regime, with a more developed characterisation of physical objects such as spinors and Stern-Gerlach apparatuses, as well as a consistent interpretational framework for such "Quantum Mechanics on a quantum space(time)". This is exactly what we did in the second part of the chapter, based on [4]. We moved the first steps toward the formulation of a DQM theory, namely a theory of Quantum Mechanics in which the geometrical degrees of freedom of physical systems are themselves quantised, formulating its axioms by specifying them for spin measurements. This is achieved by deforming the rotation symmetry group $SU(2)$ into the quantum group $SU_q(2)$. The striking prediction of this framework is the notion of quantum probabilities, namely probabilities that are operator-valued. The probability distribution of a given measurement depends on the geometrical configuration of the experimental setup, specified by its geometry states, and exhibits, in general, non-classical features, such as a non-vanishing variance. We have focused on semi-classical states, which present small deviations from classical geometrical configurations, and used them to implement the doubly quantum-mechanical analogue of the protocol in section 5.1. While in the classical setting this protocol allows them to sharply measure their relative orientation (the rotation matrix relating their reference frames), the quantumness of probabilities deriving from the DQM framework prevents them to do so: an intrinsic, in general non-vanishing, variance affects the matrix elements of the rotation matrix relating their reference frames. This feature could be a hint that the amount of information that two observers can exchange when they do not know the relation between their reference frames is limited, even if the number of exchanged messages is infinite. In the standard $SU(2)$ case, there is no such limit, as the number of logical bits (and qubits) encoded per physical qubit approaches unity for large numbers of exchanged physical qubits [273, 338], and the two observers can then align their reference frames with infinite precision [141]. This is prevented when $SU(2)$ is replaced by $SU_q(2)$ in our DQM framework, and this could signal the emergence of an intrinsic limit on the amount of information that two observers can exchange.

At the practical level, the axiomatic approach outlined in our work promotes complex numbers, appearing in linear combinations of spin states, in the expression of the generic Pauli matrix, and in the matrix defining the relative orientation between two observers, to operators satisfying the commutation relations of $SU_q(2)$. This is the only technical difficulty introduced in our framework, as relevant physical quantities are then computed using formal prescriptions analogous to those adopted in Quantum Mechanics. Nevertheless, this is already enough to introduce, for the first time, the notion of quantum probability. Typically, the geometric configurations of experimental setups are given by superpositions of probability eigenstates, hence the probability distribution for a given measurement is non-classical, *i.e.* it has a non-vanishing variance. This opens up the possibility of defining non-classical measurement apparatuses in a formal way, but also would require a rethinking of the operational meaning of probability itself.

Our work shares some features with [114], in which quantum reference frames for spin systems are introduced. The main analogy lies in the fact that both frameworks allow for non-classical notions of directions in space and rotation angles, even though the origin and the nature of these two types of fuzziness are different in the two contexts. Moreover, in [114] the physical regime of "unlimited resources" is considered, which allows for the definition of sharp classical directions. This is done by considering spin coherent states with large spin to define axes in space. In our framework, directions of physical systems (spins and Stern-Gerlach apparatuses), as well as the relative orientation between different observers, are specified by quantum states in Hilbert spaces that do not describe any physical system, rather they characterise the intrinsic (quantum) structure of space. These states are such that, in general, classical directions cannot be defined. It would be interesting to investigate the possible connections between our work and the "limited resources" regime of the framework presented in [114]. This would offer fertile ground for investigating the possible connections between the quantum reference frames [102, 103, 117] and the quantum groups approaches to quantum spacetime a topic that has attracted growing interest in recent years with preliminary studies that are focusing on trying to find a (quantum) group structure for quantum reference frames transformations [109, 149].

The formalism developed in [4] can be regarded as a template for incorporating quantum spacetime effects in Quantum Mechanics in a consistent way, enforcing the relativistic invariance of the theory and building up from first principles. Interestingly, one of the axioms we have formulated attributes quantum properties to macroscopic measuring devices. While they are still "classical" in the sense of standard Quantum Mechanics, they are described by geometry states in a Hilbert space in the sense of quantum spacetime. This suggests the possibility of realising "non-classical measurement devices", intended as measurement apparatuses that are characterised by fully quantum geometrical configurations, such as superpositions of semi-classical geometry states. We did not consider such possibilities in our work, postponing this task to future works to investigate the consequences of this property that, as of current knowledge, is exclusive to the DQM framework. The latter could also be generalised to accommodate other quantum symmetries and to characterise observables for quantum systems in a quantum spacetime invariant under such symmetries. Some preliminary studies have analysed geometry states of some non-commutative spacetime models and their relative quantum group transformations [59, 339, 340] and it would be interesting to incorporate these results in a DQM model, in order to also explore the boost and translation sectors and define a DQM with general spacetime (quantum) symmetries. It could even be conceivable to implement a general formalism of DQM valid for any quantum group, analogously to [113] in which a framework for general symmetry groups has been developed formally for quantum reference frames. A DQM framework for general quantum symmetries of spacetime would also allow to formalise dynamical models compatible with the quantum symmetries [308].

Finally, the DQM framework allows for a deeper discussion about the fundamental principles underlying the formulation of a Quantum Gravity theory, particularly concerning the nature of probabilities that acquire quantum features in DQM. At the effective level, it is conceivable to try to encompass such a prediction in the broader area of generalised probabilistic theories [341], possibly signaling that the DQM framework describes a model beyond Quantum Theory. To investigate this latter possibility, the DQM for spin measurements that we developed in our work could already be sufficient. Indeed, it might be possible to implement a doubly quantum version of the CHSH game [342] to test if the Tsirelson's bound [343], which sets the maximum violation of the CHSH inequality in Quantum Theory, is violated. This would lead to the violation of some fundamental principles of Quantum Theory, such as Information Causality [344] (a generalisation of the no-signaling principle), and provide an insight for going beyond them, in laying down the foundations of a Quantum Gravity theory.

All of the future perspectives presented above offer an opportunity for enriching both phenomenological and theoretical aspects of studies in Quantum Gravity, a much needed effort in light of the recent phenomenological opportunities apt to test [48, 51, 52, 119, 130, 133–138] the quantum nature of spacetime and probe the limits of the assumptions underlying General Relativity and Quantum Mechanics at the interface between them.

5.A Classical $SU(2)$ parameterisation and quantisation ambiguity

The $SU(2)$ group is defined as the group of 2×2 unitary matrices with determinant equal to 1. A generic element $U \in SU(2)$ can be parameterised as

$$U = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix}, \tag{5.A.1}$$

where a and c are two complex numbers satisfying

$$aa^* + cc^* = 1\,, \tag{5.A.2}$$

constraining the degrees of freedom to three real numbers. A commonly used parameterisation for a and c in terms of three real numbers is

$$a = e^{i\eta}\cos\frac{\theta}{2}\,,\ \ c = e^{i\delta}\sin\frac{\theta}{2}\,, \tag{5.A.3}$$

with $\eta, \delta \in [0, 2\pi)$ and $\theta \in [0, \pi)$. These three angles encode all the information needed to uniquely identify a $SU(2)$ element and are useful in identifying the connection of $SU(2)$ to the rotation group $SO(3)$. Indeed, the canonical homomorphism between the two groups is realised via

$$R_{ij} = \frac{1}{2}\mathrm{Tr}\left\{\sigma_i U \sigma_j U^\dagger\right\} \equiv \frac{1}{2}\mathrm{Tr}\left\{U \sigma_j U^\dagger \sigma_i\right\}, \tag{5.A.4}$$

where R_{ij} is a rotation matrix, U is parameterised as in (5.A.1) and σ_i are the Hermitian Pauli matrices. Writing (5.A.4) explicitly, we have

$$R = \begin{pmatrix} \frac{1}{2}(a^2 - c^2 + (a^*)^2 - (c^*)^2) & \frac{i}{2}(-a^2 + c^2 + (a^*)^2 - (c^*)^2) & (a^* c + c^* a) \\ \frac{i}{2}(a^2 + c^2 - (a^*)^2 - (c^*)^2) & \frac{1}{2}(a^2 + c^2 + (a^*)^2 + (c^*)^2) & -i(a^* c - c^* a) \\ -(ac + c^* a^*) & i(ac - c^* a^*) & 1 - 2cc^* \end{pmatrix}. \tag{5.A.5}$$

As anticipated in the results section, matrix (5.15) indeed coincides with this classical one in the $q \to 1$ limit.

Inserting the parameterisation (5.A.3) in (5.A.5) we obtain the rotation matrix in terms of trigonometric functions of the three real angles η, δ and θ. These are simply a redefinition of the well known Euler angles (α, β, γ)

$$\theta = \beta \qquad \eta = \frac{\alpha+\gamma}{2} \qquad \delta = \frac{\pi}{2} - \frac{\alpha-\gamma}{2}\,. \tag{5.A.6}$$

In terms of these, a generic rotation matrix is written as $R(\alpha, \beta, \gamma) = R_z(\alpha) R_x(\beta) R_z(\gamma)$, where R_z and R_x are rotations around the z and x axis, respectively.

In the quantisation procedure, we replace the classical U with the deformed U_q in (5.A.5), thus replacing the complex numbers a and c with the operators satisfying (5.12), and leave the Pauli matrices untouched. It may be argued that a deformation of the Pauli matrices, such as the one studied in [345], should be considered here. However, doing so leads to non-Hermitian operators for the entries of the resulting matrix. With our choice, not only do we get hermitian operators, but we reconstruct a matrix that matches the one obtained in [330] upon doing a change of basis, and this guarantees that we obtain a representation of $SO_q(3)$ as explained shortly.

To discuss our results, we used the matrix (5.15) that is obtained with the first ordering in (5.A.4). Due to the non-commutative nature of the operators a and c, the cyclic property of the trace, expressed in (5.A.4) is no longer valid. This leads to two possible definitions of the quantum rotation matrix. The alternative to (5.15), obtained using the second ordering in (5.A.4), is

$$P_q = \begin{pmatrix} \frac{1}{2}(a^2 - qc^2 + (a^*)^2 - q(c^*)^2) & \frac{i}{2}(-a^2 + qc^2 + (a^*)^2 - q(c^*)^2) & q(a^* c + c^* a) \\ \frac{i}{2}(a^2 + qc^2 - (a^*)^2 - q(c^*)^2) & \frac{1}{2}(a^2 + qc^2 + (a^*)^2 + q(c^*)^2) & -iq(a^* c - c^* a) \\ -\frac{1+q^2}{2q}(ac + c^* a^*) & i\frac{1+q^2}{2q}(ac - c^* a^*) & 1 - (1+q^2)cc^* \end{pmatrix}. \tag{5.A.7}$$

It turns out that P_q and R_q are actually similar matrices: $R_q = MP_qM^{-1}$, with

$$M = \begin{pmatrix} 2^{\frac{1}{3}}(\frac{q}{1+q^2})^{\frac{1}{3}} & 0 & 0 \\ 0 & 2^{\frac{1}{3}}(\frac{q}{1+q^2})^{\frac{1}{3}} & 0 \\ 0 & 0 & \frac{(\frac{1}{q}+q)^{\frac{2}{3}}}{2^{\frac{2}{3}}} \end{pmatrix}. \tag{5.A.8}$$

This matrix (with its inverse) reduces to the identity at first order in $(1-q)$ so that the two matrices are equivalent from a phenomenological point of view. Indeed, since the Quantum Gravity expectation is that q is extremely close to 1, the search for experimental manifestations of our q-deformation is not likely to ever go beyond the leading $(1-q)$-order effects [346]. Notice that the two matrices come from an ambiguity of the "quantisation procedure", namely the ordering of the matrices in (5.A.4). This is similar to what happens in standard Quantum Mechanics where different orderings in the classical observables give rise to different quantum operators that give the same classical limits.

Furthermore, the link with the quantum group $SO_q(3)$ is established as follows. The vector (co)-representation of $SU_q(2)$ is linked to the (co)-representation of the quantum group $SO_q(3)$ through the isomorphism $C(SO_q(3)) := C(SU_q(2)/\mathbb{Z}_2)$; the representation reads [330]

$$d_1 = \begin{pmatrix} (a^*)^2 & -(1+q^2)a^* c & -qc^2 \\ c^* a^* & 1-(1+q^2)c^* c & ac \\ -q(c^*)^2 & -(1+q^2)c^* a & a^2 \end{pmatrix}. \tag{5.A.9}$$

In the commutative limit, this matrix approaches a generic rotation matrix written in the so-called complex basis, and can be rewritten in the Cartesian basis by a similarity transformation, with

$$N = \begin{pmatrix} -1 & -i & 0 \\ 0 & 0 & 1 \\ -1 & i & 0 \end{pmatrix} \tag{5.A.10}$$

acting as the change of basis. By applying this transformation in the q-deformed case too, one can easily check that $N^{-1}d_1N$ is identically equal to (5.15). This clarifies the link to the $SO_q(3)$ quantum group. The matrices P_q and R_q are (co)-representations of $SO_q(3)$ since they can be obtained by a similarity transformation from (5.A.9). Therefore, our ansatz (5.14) realises a "quantisation" of the $SU(2)$ to $SO(3)$ homomorphism.

In closing this section, we offer some comments on another quantisation ambiguity, concerning our definition of the angle $\theta(n)$ in (5.20). There we obtained the definition of $\sin\left(\frac{\theta(n)}{2}\right)$ by comparing the eigenvalues of $\pi(c)$ to the classical parameterisation of c in (5.19). We find that this is consistent with the definition of $\cos\left(\frac{\theta(n)}{2}\right)$ in terms of $\pi(a^* a)$, which also acts diagonally. Indeed, on a generic eigenstate $|n,\phi,\chi\rangle$ (cf. (5.18)) one has that

$$\pi(a^* a)\,|n,\phi,\chi\rangle = (1-q^{2n})\,|n,\phi,\chi\rangle = \cos^2\left(\frac{\theta(n)}{2}\right)|n,\phi,\chi\rangle\,. \tag{5.A.11}$$

Furthermore, this is in agreement with the fact that $a^* a + c^* c = \mathbb{1}$, as can be inferred from (5.12). However, it should be noted that since a and a^* do not commute (see (5.12)), another result could be obtained for $\theta(n)$ by defining its cosine in terms of $\pi(aa^*)$, which also acts diagonally (cf. (5.18)), rather than $\pi(a^* a)$. As can be easily checked, this would lead to allowed values of θ which are the same as in our $\pi(a^* a)$ case, with the only difference that $\theta = \pi$ would be missing from the spectrum of $\theta(n)$. With this ordering, we would have $\sin\left(\frac{\theta(n)}{2}\right) = q^{n+1}$, which is simply a shift of n with respect to the ordering we adopted. Evidently, the main results and the main aspects of physical interpretation of our analysis are unaffected by this quantisation ambiguity.

5.B Numerical construction of semi-classical Gaussian states

In order to construct the semi-classical states, we have resorted to numerical computations, focusing on the case $q = 0.99$ as an illustrative example. However, by increasing the value of q, we have checked that the following states satisfy condition (5.21), thus approximating classical rotations with increasing precision.

We will focus on the (y,z) plane setting $\chi = 0$, $\phi = \frac{\pi}{2}$, as can be seen from (5.A.6). We want to construct states that semi-classically describe a deformed rotation of a certain angle in this plane. Since n defines the only angle left, θ, we choose to build superpositions (5.23) centred on particular values of n, denoted as $\bar{n}$, with coefficients c_n multiplying the states $|n,\phi,\chi\rangle$ rapidly decreasing as $\theta(n)$ deviates from $\theta(\bar{n})$. Our ansatz is that these coefficients have the form of a discretised Gaussian distribution.

The variance is then chosen as follows. Recalling that

$$\theta(n) = 2\arcsin q^n\,, \tag{5.B.1}$$

we can define $\Delta\bar{n}$ as

$$\Delta\bar{n} := \left.\frac{dn}{d\theta}\right|_{\theta(\bar{n})}\Delta\theta\,. \tag{5.B.2}$$

We take $\Delta\theta = \frac{\pi}{2} - \arcsin q$, which is just half the value of the maximum angular deviation, and weigh it with the rate of change of n with respect to θ, approximated as the derivative. Therefore, the value of the variance depends on the central value $\bar{n}$. This approximation becomes more and more accurate with increasing values of $\bar{n}$ for which $\theta(\bar{n})$ becomes quasi-continuous. Moreover, the higher the value of $\bar{n}$, the larger the variance is.

We then define our superposition coefficients c_n as

$$c_n = \frac{e^{-\frac{(\bar{n}-n)^2}{2\Delta\bar{n}^2}}}{\sum_{n=0}^{\infty} e^{-\frac{(\bar{n}-n)^2}{2\Delta\bar{n}^2}}} \,. \tag{5.B.3}$$

For computational reasons, we do not consider the full superposition going from $n = 0$ to $n = \infty$, but we truncate the series considering the 3-σ range of our Gaussian. That is, the sum goes from $n_{\min}$ to $n_{\max}$, where

$$n_{\min} - \bar{n} = -3\Delta\bar{n} \qquad n_{\max} - \bar{n} = 3\Delta\bar{n} \,. \tag{5.B.4}$$

For relatively small values of $\bar{n}$, the value of $n_{\min}$ could be negative when considering the 3-σ range. When this happens, we simply truncate the Gaussian and start the series from $n = 0$. We have explicitly verified that this does not affect our results in a significant way.

We have used this algorithm to construct the states used for the numerical analysis of table 5.1. In what follows, we show the results for the computation of the expectation values of the matrix elements $\left(R_q\right)_{ij}$ with their relative uncertainties for states centred around $n = 95, 34, 7, 0$, corresponding to angles $\theta = 45.275°, 90.560°, 137.518°, \theta = 180°$.

- $n = 95$, $\theta = 45.275°$

$$\begin{aligned} \left\langle \psi(45.275°) \middle| R_q \middle| \psi(45.275°) \right\rangle &= \begin{pmatrix} 0.997 & 0.000 & 0.000 \\ 0.000 & 0.695 & 0.708 \\ 0.000 & -0.708 & 0.698 \end{pmatrix} , \\ \Delta R_q &= \begin{pmatrix} 0.005 & 0.071 & 0.030 \\ 0.071 & 0.073 & 0.069 \\ 0.030 & 0.069 & 0.073 \end{pmatrix} . \end{aligned} \tag{5.B.5}$$

- $n = 34$, $\theta = 90.560°$

$$\begin{aligned} \left\langle \psi(90.560°) \middle| R_q \middle| \psi(90.560°) \right\rangle &= \begin{pmatrix} 0.990 & 0.000 & 0.000 \\ 0.000 & -0.015 & 0.990 \\ 0.000 & -0.990 & -0.005 \end{pmatrix} , \\ \Delta R_q &= \begin{pmatrix} 0.015 & 0.100 & 0.100 \\ 0.100 & 0.099 & 0.004 \\ 0.100 & 0.004 & 0.100 \end{pmatrix} . \end{aligned} \tag{5.B.6}$$

- $n = 7$, $\theta = 137.518°$

$$\begin{aligned} \left\langle \psi(137.518°) \middle| R_q \middle| \psi(137.518°) \right\rangle &= \begin{pmatrix} 0.982 & 0.000 & 0.000 \\ 0.000 & -0.739 & 0.663 \\ 0.000 & -0.663 & -0.721 \end{pmatrix} , \\ \Delta R_q &= \begin{pmatrix} 0.025 & 0.067 & 0.173 \\ 0.067 & 0.067 & 0.074 \\ 0.173 & 0.074 & 0.067 \end{pmatrix} . \end{aligned} \tag{5.B.7}$$

- n=0, $\theta = 180°$

$$
\begin{aligned}
\langle \psi(180°) | R_q | \psi(180°) \rangle &= \begin{pmatrix} 0.99 & 0 & 0 \\ 0 & -0.99 & 0 \\ 0 & 0 & -0.99 \end{pmatrix}, \\
\Delta R_q &= \begin{pmatrix} 0.014 & 0.014 & 0.140 \\ 0.014 & 0.014 & 0.140 \\ 0.140 & 0.140 & 0 \end{pmatrix}.
\end{aligned} \tag{5.B.8}
$$

To obtain the states that describe the q-deformations of rotations of negative angles $\zeta < 0$, we only have to consider the same states with $\phi = -\frac{\pi}{2}$. It is straightforward to show that for states of the form (5.23) with $\chi = 0$ having real superposition coefficients like (5.B.3), the map $\phi \mapsto -\phi$ exchanges $\left(R_q\right)_{23}$ with $\left(R_q\right)_{32}$, leaving all the other elements unchanged, and yields the same uncertainty matrix. Therefore, the aperture of the cone in table 5.1 is the same for opposite angles.

For the $\theta = 0$ case, we use the state in the representation ρ that gives the identity matrix when computing the expectation values. This state is simply given by $|\chi = 0\rangle \in \mathscr{H}_\rho$. From (5.22) we thus have

$$
\langle \psi(0°) | R_q | \psi(0°) \rangle = \mathbb{1}_{3\times 3} \qquad \Delta R_q = \mathbf{0}_{3\times 3}\,. \tag{5.B.9}
$$

5.C ρ as limit of π

By adding a phase to the representation π discussed in the mathematical literature [321] we have

$$
\rho(a)|\chi\rangle = e^{i\chi}|\chi\rangle \qquad \rho(a^*)|\chi\rangle = e^{-i\chi}|\chi\rangle \qquad \rho(c)|\chi\rangle = \rho(c^*)|\chi\rangle = 0\,, \tag{5.C.1}
$$

$$
\begin{aligned}
\pi(a)|n,\phi,\epsilon\rangle = e^{i\epsilon}\cos\left(\frac{\theta(n)}{2}\right)|n-1,\phi,\epsilon\rangle \qquad \pi(c)|n,\phi,\epsilon\rangle = e^{i\phi}\sin\left(\frac{\theta(n)}{2}\right)|n,\phi,\epsilon\rangle\,, \\
\pi(a^*)|n,\phi,\epsilon\rangle = e^{-i\epsilon}\sqrt{1-q^2\cos^2\left(\frac{\theta(n)}{2}\right)}\,|n+1,\phi,\epsilon\rangle \\
\pi(c^*)|n,\phi,\epsilon\rangle = e^{-i\phi}\sin\left(\frac{\theta(n)}{2}\right)|n,\phi,\epsilon\rangle\,,
\end{aligned} \tag{5.C.2}
$$

with our definition of quantised angle, $\theta(n) = 2\arcsin q^n$. Looking at (5.C.1) and (5.C.2), it may seem that we have four Euler angles in the representations of $SU_q(2)$, namely the aforementioned $(\chi, \epsilon, \phi, \theta(n))$. This is a subtlety solved by observing that, with the additional ϵ phase that we introduced in the π-representation, the representation ρ can be obtained as a somewhat trivial limit of representation π. Therefore, we have three physical degrees of freedom, with the angles χ and ϵ linked by this limit operation, so that we can ultimately set $\epsilon = \chi$.

To see this, we need to find a class of states such that

$$
\pi(a)|\psi(\chi)\rangle \to e^{i\chi}|\psi(\chi)\rangle \quad , \quad \pi(c)|\psi(\chi)\rangle \to 0 \tag{5.C.3}
$$

in this limit. We can show that states of the type (with the identification $\epsilon = \chi$)

$$\left|\psi_n\right\rangle = \frac{1}{\sqrt{n+1}} \sum_{k=n}^{2n} \left|k, \phi, \chi\right\rangle \tag{5.C.4}$$

from the π-representation satisfy requirement (5.C.3) for $n \to \infty$.

In fact, for the operator c we can immediately see that

$$\pi(c)\left|\psi_n\right\rangle = e^{i\phi} \frac{1}{\sqrt{n+1}} \sum_{k=n}^{2n} q^k \left|k, \phi, \chi\right\rangle , \tag{5.C.5}$$

and, therefore, since q is slightly less than one

$$\left\langle\psi_n\middle|\pi(c^*)\pi(c)\middle|\psi_n\right\rangle = \frac{1}{n+1} \frac{q^{2n}\left(1-q^{2n+2}\right)}{1-q^2} \to 0, \qquad n \to \infty . \tag{5.C.6}$$

Hence, $\pi(c)\left|\psi_n\right\rangle \xrightarrow{n\to\infty} 0$. The limit for the operator a is more subtle; in this case we have that

$$\pi(a)\left|\psi_n\right\rangle = \frac{e^{i\chi}}{\sqrt{n+1}} \sum_{k=n}^{2n} \sqrt{1-q^{2k}} \left|k-1, \phi, \chi\right\rangle , \tag{5.C.7}$$

and we consider the difference

$$\left|\psi'_n\right\rangle = \pi(a)\left|\psi_n\right\rangle - e^{i\chi}\left|\psi_n\right\rangle . \tag{5.C.8}$$

To show that the limit $\pi(a)\left|\psi_n\right\rangle \to e^{i\chi}\left|\psi\right\rangle$ holds we will show that the succession of the norms of the states in (5.C.8) has 0 limit. We have that

$$\begin{aligned}
0 \leq \left\langle\psi'_n\middle|\psi'_n\right\rangle &= \left\langle\psi_n\middle|\pi(a^*)\pi(a)\middle|\psi_n\right\rangle + \left\langle\psi_n\middle|\psi_n\right\rangle - 2\,\mathrm{Re}\left\{e^{i\chi}\left\langle\psi_n\middle|\pi(a^*)\middle|\psi_n\right\rangle\right\} = \\
&= 1 + \frac{1-q^{2n}}{n+1} - \frac{1}{n+1}\sum_{k=n+1}^{2n}\left[2\sqrt{1-q^{2k}} - \left(1-q^{2k}\right)\right] \leq 1 + \frac{1}{n+1} - \frac{1}{n+1}\sum_{k=n+1}^{2n}\left(1-q^{2k}\right) = \\
&= \frac{1}{n+1}\left[2 - q^{2n+2}\frac{1-q^{2n}}{1-q^2}\right] \xrightarrow{n\to\infty} 0 .
\end{aligned} \tag{5.C.9}$$

This observation that representation π converges for certain states to representation ρ can be extended to generic functions of operators. In general, such functions can be written as the sum of powers of the operators $\pi(a)$, $\pi(a^*)$, $\pi(c)$ and $\pi(c^*)$. We now prove that these functions acting on a class of states in representation π give, in a certain limit, the value that the same function gives in the limit of these states with $\pi(a)$ replaced by $\rho(a)$ and so on. To do so, we consider a slightly different class of states than the one in (5.C.4). In particular, we consider

$$\left|\Psi_n\right\rangle = \frac{1}{\sqrt{n!+1}} \sum_{k=n!}^{2n!} \left|k, \phi, \chi\right\rangle . \tag{5.C.10}$$

The reason why we make this choice with the factorial will be clear in the following.

Consider first the case in which we have an operator of the form $\pi(c)^m\pi(a)^l$. If we act with this operator on the state (5.C.10) and we compute the norm of the state obtained in this way, we get

$$0 \leq \langle \Psi_n | \pi(a^*)^l \pi(c^*)^m \pi(c)^m \pi(a)^l | \Psi_n \rangle \leq \frac{q^{2m(n!-l)}\left[1-q^{2m(n!+1)}\right]}{\left(1-q^{2m}\right)(n!+1)} \xrightarrow[m\neq 0]{n\to\infty} 0\,. \tag{5.C.11}$$

The same argument holds if we have only powers of the operator $\pi(c)$ (*i.e*, $l=0$) or other combinations involving $\pi(a^*)$, $\pi(c^*)$ and different orderings. The case with $m=0$, in which the limit above is 1, corresponds to a power of $\pi(a)$ that will be handled slightly differently in the following.

Now we turn to the powers of the operator $\pi(a)$, *i.e* $\pi(a)^m$. We do something analogous to the argument used for $\pi(a)$, namely we define the difference state

$$\left|\Psi'_n\right\rangle = \pi(a)^m \left|\Psi_n\right\rangle - e^{im\chi}\left|\Psi_n\right\rangle\,, \tag{5.C.12}$$

and show its norm goes to zero as $n\to\infty$. After some algebraic manipulations, this norm is found to satisfy

$$0 \leq \langle \Psi'_n | \Psi'_n \rangle \leq \frac{2m}{n!+1} + \frac{1}{n!+1}\sum_{k=n!+m}^{2n!}\left[q^{l(l+1)}q^{2kl}(-1)^l\left(q^{-2k};q^2\right)_l - 1\right] \xrightarrow{n\to\infty} 0\,, \tag{5.C.13}$$

where $\left(q^{-2k};q^2\right)_l := \prod_{s=0}^{l-1}\left(1-q^{-2k+2s}\right)$ is the q-Pochhammer symbol. The result of the limit comes from the fact that the term in the square brackets above is essentially just a sum of powers of q.

We have thus shown that a generic function of $\pi(a)$ and $\pi(c)$ operators applied to (a class of) states in the representation π converges to the same function applied to a state in the representation ρ (a similar argument holds for functions of $\pi(a^*)$ and $\pi(c^*)$ operators). The reason why we considered a state of the form (5.C.10) is now clear: it is necessary to have a number of states in the superposition that goes to infinity faster than m to properly carry out the limit when considering generic functions, in which case we have a series in m.

The states (5.C.4) can also be used to show directly for the vector (co)-representation that it is possible to obtain the rotations about the z axis as a limit in the representation π. In the same way, it is possible to obtain the identity, *i.e.* the null rotation, in the same representation.

To obtain a rotation about the z axis, we must have $\phi=0$, and we have to consider the large n limit. In this limit, setting $\phi=0$, we can approximate our π representation as

$$\pi(a)\left|n,\phi,\chi\right\rangle = e^{i\chi}\left|n-1,\phi,\chi\right\rangle \qquad \pi(c)=\pi(c^*)=0 \qquad \pi(a^*)\left|n,\phi,\chi\right\rangle = e^{-i\chi}\left|n+1,\phi,\chi\right\rangle\,. \tag{5.C.14}$$

Therefore, using the states (5.C.4), we obtain

$$\langle\psi|R_q|\psi\rangle = \begin{pmatrix} \frac{-1+n}{1+n}\cos\chi & \frac{-1+n}{1+n}\sin\chi & 0 \\ -\frac{-1+n}{1+n}\sin\chi & \frac{-1+n}{1+n}\cos\chi & 0 \\ 0 & 0 & 1 \end{pmatrix}\,. \tag{5.C.15}$$

In the large n limit, we see that (5.C.15) approximates a rotation matrix about the z axis with greater and greater precision and it can be shown that the variances are equal to 0. Moreover, $\chi=0$ in (5.C.15) gives the identity matrix in the large n limit.

5.D Eigenstates of the probability operator

With explicit computations involving the probability operator (5.64), we see that the states $\{|\chi_s\rangle \otimes |\chi_a\rangle, |\chi_s\rangle \otimes |n_a, \phi_a, \chi_a\rangle, |n_s, \phi_s, \chi_s\rangle \otimes |\chi_a\rangle\}$ are eigenstates of $P(\uparrow_\sigma)$ with eigenvalues given by $\{1, 1-q^{2n_a+2}, 1-q^{2n_s}\}$, respectively. Recalling that $n_a, n_s \in [0,\infty)$, these probability eigenvalues are discrete and take values in $[0,1]$. These states span the $\mathscr{H}_\rho \otimes \mathscr{H}_\rho$, $\mathscr{H}_\rho \otimes \mathscr{H}_\pi$, and $\mathscr{H}_\pi \otimes \mathscr{H}_\rho$ components of the Hilbert space $\mathscr{H}_{SU_q(2)} \otimes \mathscr{H}_{SU_q(2)}$, given that they are basis states. Using (5.75), we show that there is only one other class of separable eigenstates in $\mathscr{H}_\pi \otimes \mathscr{H}_\pi$. To show this, consider a generic factorisable probability eigenstate independent from the ones written above, given by

$$|p,r\rangle = \sum_{n_s,n_a=0}^{\infty} h_{n_s} k_{n_a} |n_s, \phi_s, \chi_s\rangle \otimes |n_a, \phi_a, \chi_a\rangle \tag{5.D.1}$$

The eigenstate equation $P(\uparrow_\sigma)|p,r\rangle = p|p,r\rangle$ yields

$$\begin{aligned} &h_{n_s} k_{n_a}\Big[(1-q^{2n_s})(1-q^{2n_a+2}) + q^{2(n_s+n_a)} - p\Big] + \\ &+ h_{n_s-1} k_{n_a+1} e^{i(\omega_s-\omega_a)} q^{n_s+n_a}\sqrt{(1-q^{2n_s})(1-q^{2n_a+2})} + \\ &+ h_{n_s+1} k_{n_a-1} e^{-i(\omega_s-\omega_a)} q^{n_s+n_a}\sqrt{(1-q^{2n_a})(1-q^{2n_s+2})} = 0\,, \end{aligned} \tag{5.D.2}$$

where $\omega_{s,a} = \phi_{s,a} - \chi_{s,a}$.

Setting $n_s = 0$ and $n_a = 0$ in (5.D.2), we obtain

$$h_0 k_{n_a}(q^{2n_a} - p) + h_1 k_{n_a-1} e^{-i(\omega_s-\omega_a)} q^{n_a}\sqrt{(1-q^{2n_a})(1-q^2)} = 0\,, \tag{5.D.3}$$

$$h_{n_s} k_0\Big(1 + q^{2n_s+2} - q^2 - p\Big) + h_{n_s-1} k_0 e^{i(\omega_s-\omega_a)} q^{n_s}\sqrt{(1-q^{2n_s})(1-q^2)} = 0\,. \tag{5.D.4}$$

Inspecting (5.D.3), consider the case in which $h_0 = 0$, so that either $h_1 = 0$ or $k_{n_a} = 0\ \forall\, n_a$. In the latter case, the state (5.D.1) is a null vector. In the former, we consider (5.D.2) for $n_s = 1$, which yields $h_2 k_{n_a-1} = 0$. Therefore, either $h_2 = 0$ or $k_{n_a} = 0\ \forall\, n_a$. Again, in the latter case, (5.D.1) is the null vector, while in the former we move on to consider (5.D.2) for $n_s = 3$. This argument repeats in the same way for all the other values of n_s, ultimately yielding that either $h_{n_s} = 0\ \forall\, n_s$ or $k_{n_a} = 0\ \forall\, n_a$. Therefore if $h_0 = 0$ the vector in (5.D.1) is the null vector and an analogous reasoning applies also for the case in which $k_0 = 0$. This means that if a separable state of the form (5.D.1) exist, necessarily $h_0 \neq 0$ and $k_0 \neq 0$.

Among the eigenstates with $h_0, k_0 \neq 0$, let us first focus on the cases in which $p \neq 1$. If we set $n_a = 0$ in (5.D.3) we get $h_0 k_{n_a} = 0$, since $p \neq 1$. Therefore, since $h_0 \neq 0$, we have $k_{n_a} = 0\ \forall\, n_a$, in which case the state in (5.D.1) is the null vector. Let us move on to the case in which $h_0, k_0 \neq 0$ and $p = 1$. Equations (5.D.3) and (5.D.4) are solved by

$$\begin{aligned} &h_{n_s} = N_s f(\mu_s, n_s) = N_s \frac{q^{\frac{n_s}{2}(n_s-1)}\sqrt{(1-q^2)^{n_s-1}}}{\sqrt{(q^4;q^2)_{n_s-1}}}\mu_s^{n_s}\,, \\ &k_{n_a} = N_a q^{n_a} f(\mu_a, n_a) = N_a \frac{q^{\frac{n_a}{2}(n_a+1)}\sqrt{(1-q^2)^{n_a-1}}}{\sqrt{(q^4;q^2)_{n_a-1}}}\mu_a^{n_a}\,, \\ &\mu_s = e^{i(\omega_s-\omega_a)}\mu_a\,, \end{aligned} \tag{5.D.5}$$

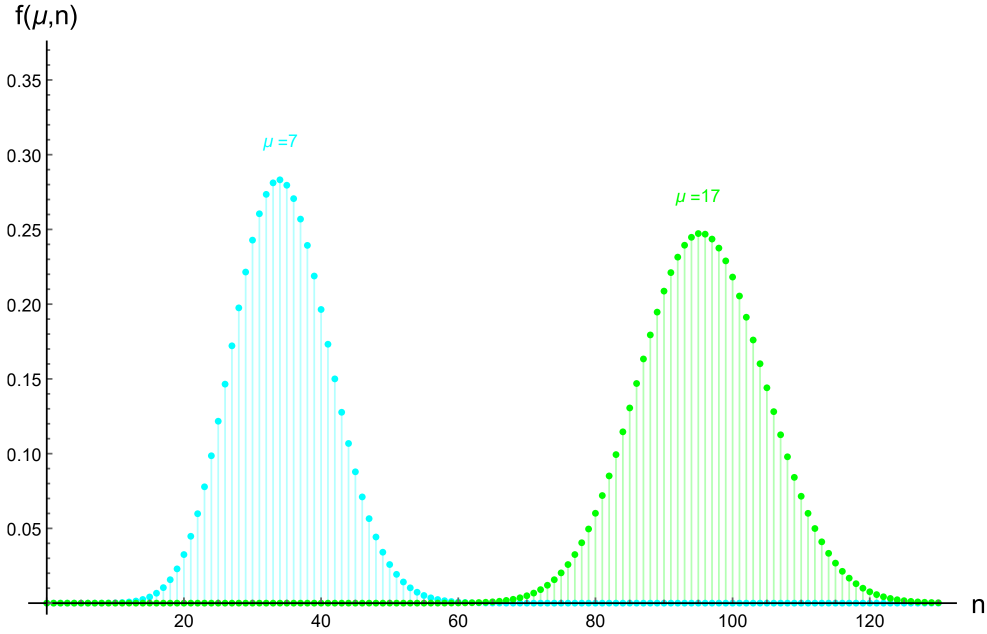

Figure 15: Distribution $f(\mu,n)$ with $n\in[0,130]\cap\mathbb{N}_0$ for two different values of μ, $\mu=7$ (cyan plot) and $\mu=17$ (green plot), corresponding to distributions centred in $n=34$ and $n=96$ (or $\theta(n)\approx\frac{\pi}{2}$ and $\theta(n)\approx\frac{\pi}{4}$), respectively, for $q=0.99$. As we see, these distributions are well approximated by the Gaussians defined in section 5.B that we used in [2] and in the first part of this chapter. Notice also that the variance increases as the value of n on which they are peaked increases, in the same way as the Gaussian states in section 5.B do. Therefore, these states, derived by searching for the semi-classical eigenstates of the probability operator, are an improvement of those that we used in the first part of this chapter and somewhat formally justify the numerical construction of Gaussian states that we proposed in [2], since the characteristic behaviour of the physical quantities we consider there is not altered by the small differences between the states that we found here and the Gaussian states.

where $(a;q)_n$ is the q-Pochhammer symbol defined as $(a;q)_n:=\prod_{k=0}^{n-1}(1-aq^k)$, $n>0$ and $(a;q)_0=1$, $\mu_s:=h_1h_0^{-1}$ ($\mu_a:=k_1k_0^{-1}$) determines the value of n_s (n_a) on which the distribution of $|f(\mu_s,n_s)|^2$($|f(\mu_a,n_a)|^2$) is centred as shown in fig. 15, and $N_s=h_0$ ($N_a=k_0$) is a normalisation constant. We will denote with $|\mu_s,\omega_s\rangle$ and $|\mu_a,\omega_a\rangle$ states of the form

$$\begin{aligned} |\mu_s,\omega_s\rangle_s &:= \sum_{n_s=0}^{\infty} h_{n_s}|n_s,\phi_s,\chi_s\rangle\,, \\ |\mu_a,\omega_a\rangle_a &:= \sum_{n_a=0}^{\infty} k_{n_a}|n_a,\phi_a,\chi_a\rangle\,. \end{aligned} \tag{5.D.6}$$

In the case where μ_s and μ_a are complex, $\mu_s=|\mu_s|e^{i\beta_s}$, $\mu_a=|\mu_a|e^{i\beta_a}$, it is easy to show that the additional phase can be absorbed into the angles ω_s and ω_a, meaning that the computation of all quantities of interest (expectation values in the geometry state of spin states, Pauli matrices and probabilities) gives the same result when performed with either $\left||\mu_s|,\omega_s-\beta_s\right\rangle\otimes\left||\mu_a|,\omega_a-\beta_a\right\rangle$ or $\left||\mu_s|e^{i\beta_s},\omega_s\right\rangle\otimes\left||\mu_a|e^{i\beta_a},\omega_a\right\rangle$. Given the generality of the angles ω_s and ω_a, we consider $\mu_s,\mu_a\in\mathbb{R}^+$. Therefore, the probability eigenstates in (5.D.1) with eigenvalue

$p = 1$ can be written as $|\mu,\omega\rangle_s \otimes |\mu,\omega\rangle_a$, and describe spin systems and Stern-Gerlach apprata aligned along the same semi-classical directions.

5.E Numerical examples with semi-classical states

We consider some examples involving explicit computations of average values in the geometry states of the generic spin state (5.48) and of the generic Pauli matrix (5.54), as well as average values and uncertainties in the geometry states of the probability operator $P(\uparrow_\sigma)$ in (5.66). These examples will give an indication on which geometry states may be regarded as those semi-classically describing the directions of spin systems and Stern-Gerlach apparatuses, in the sense of conditions (5.78), (5.79). The results concerning the average value in the geometry of the generic spin states and of the generic Pauli matrices should not be regarded as the starting points for the computation of quantum-mechanical predictions for a spin system. We remind the reader that the physical predictions of our model are mainly encoded in the measurement outcomes of the probability $P(\uparrow_\sigma)$ (and consequently of $\langle\sigma^q\rangle$), according to the axioms outlined in section 5.3.3. Let us begin by exploring geometry states inspired by the probability eigenstates derived in section 5.D. Once we have obtained a class of semi-classical geometry states, we will conclude with an example showing that these are also suitable for semi-classically describing the relative orientation between two observers, satisfying the requirements outlined at the end of section 5.3.4. Some of the following examples and computations in section 5.3.5 involve states of the form (5.D.6) that require a numerical analysis, where we set $q = 0.99$ for definiteness. For practical purposes, we truncate the infinite series originating from the computations involving these states so that we have control up to the second decimal place.

1. Consider the state $\left|\Phi^S\right\rangle \otimes \left|\Phi^{SG}\right\rangle = |\chi_s\rangle \otimes |\chi_a\rangle \in \mathscr{H}_\rho \otimes \mathscr{H}_\rho$, which is an eigenstate of $P(\uparrow_\sigma)$ with eigenvalue 1. The corresponding geometry states for the spin state and the Stern-Gerlach apparatus are $\left|\Phi^S\right\rangle = |\chi_s\rangle$ and $\left|\Phi^{SG}\right\rangle = |\chi_a\rangle$, respectively. The average values in these states of the generic spin state (5.48) and of the generic Pauli matrix (5.54) are given by
$$\overline{|\psi\rangle} = |\uparrow\rangle \,, \quad \overline{\Delta}[\,|\psi\rangle\,] = 0 \,,$$
$$\overline{\sigma^q} = \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix} , \quad \overline{\Delta}[\sigma^q] = 0_{2\times 2} \,. \tag{5.E.1}$$
This semi-classically describes a spin and a Stern-Gerlach both orientated along the positive z direction, in the sense of requirements (5.78) and (5.79). We thus make the identification $\left|\Phi^S(0,\omega_s)\right\rangle \equiv |\chi_s = \omega_s\rangle$, $\left|\Phi^{SG}(0,\omega_a)\right\rangle \equiv |\chi_a = \omega_a\rangle$.

2. Consider now $\left|\Phi^S\right\rangle \otimes \left|\Phi^{SG}\right\rangle = |\chi_s\rangle \otimes |n_a,\phi_a,\chi_a\rangle$ in $\mathscr{H}_\rho \otimes \mathscr{H}_\pi$, the eigenstate of probability with eigenvalue $1 - q^{2n_a+2}$. It can be explicitly verified that $|n_a,\phi_a,\chi_a\rangle$ does not satisfy requirements (5.78) and (5.79) for $n_a \geq 1$. When $n_a = 0$, we obtain
$$\overline{|\psi\rangle} = |\uparrow\rangle \,, \quad \overline{\Delta}[\,|\psi\rangle\,] = 0 \,,$$
$$\overline{\sigma^q} = \begin{pmatrix} -q^3 & 0 \\ 0 & q \end{pmatrix} , \quad \overline{\Delta}[\sigma^q] = \begin{pmatrix} 0 & 0 \\ (1+q^2)^2(1-q^2) & 0 \end{pmatrix} . \tag{5.E.2}$$

The state $|0,\phi_a,\chi_a\rangle$ semi-classically describes a Stern-Gerlach apparatus along the negative z direction, so we make the identification $\left|\Phi^{SG}(\pi,\omega_a)\right\rangle \equiv |0,\phi_a,\chi_a\rangle$.

Analogously, we may consider $\left|\Phi^S\right\rangle \otimes \left|\Phi^{SG}\right\rangle = |n_s,\phi_s,\chi_s\rangle \otimes |\chi_s\rangle$, eigenstate of the probability with eigenvalue $1-q^{2n_s}$. Similarly to the previous case, it may be explicitly verified that for $n_s \geq 1$, $|n_s,\phi_s,\chi_s\rangle$ does not satisfy requirements (5.78) and (5.79). When $n_s = 0$, we obtain

$$\begin{aligned} &\overline{|\psi\rangle} = |{\downarrow}\rangle\ ,\ \overline{\Delta}[\,|\psi\rangle\,] = 0\ ,\\ &\overline{\sigma^q} = \begin{pmatrix} q & 0 \\ 0 & -q^{-1}\end{pmatrix}\ ,\ \overline{\Delta}[\sigma^q] = 0_{2\times 2}\ . \end{aligned} \tag{5.E.3}$$

The state $|0,\phi_s,\chi_s\rangle$ thus semi-classically describes a spin along the negative z direction, and we make the identification $\left|\Phi^S(\pi,\omega_s)\right\rangle \equiv |0,\phi_s,\chi_s\rangle$.

3. We now consider states of the form $\left|\Phi^S\right\rangle \otimes \left|\Phi^{SG}\right\rangle = |\mu_s,\omega_s\rangle_s \otimes |\mu_a,\omega_a\rangle_a$, which are eigenstates of the probability in $\mathscr{H}_\pi \otimes \mathscr{H}_\pi$ with eigenvalue 1. To exhibit a specific example involving these states, we resort to numerical computations by setting $q = 0.99$, for definiteness. We choose $\mu = 7$, so that the $|f(\mu,n)|^2$ function is peaked around $n_{\frac{\pi}{2}} = 34$, corresponding to $\theta\left(n_{\frac{\pi}{2}}\right) \approx \frac{\pi}{2}$, in the spirit of section 5.3.4. The average values and uncertainties in the geometry states for the generic spin state and generic Pauli matrix read

$$\begin{aligned} &\overline{|\psi\rangle} = 0.71\,|{\uparrow}\rangle + 0.71 e^{i\omega_s}\,|{\downarrow}\rangle\ ,\ \overline{\Delta}[\,|\psi\rangle\,] = 0.04\Big(|{\uparrow}\rangle + |{\downarrow}\rangle\Big)\ ,\\ &\overline{\sigma^q} = \begin{pmatrix} 0.00 & 0.99e^{-i\omega_a} \\ 0.99e^{i\omega_a} & 0.00\end{pmatrix}\ ,\ \overline{\Delta}[\sigma^q] = \begin{pmatrix} 0.10 & 0.10 \\ 0.10 & 0.10\end{pmatrix}\ . \end{aligned} \tag{5.E.4}$$

corresponding to acceptable values for $|7,\omega_s\rangle_s$ and $|7,\omega_s\rangle_a$ to semi-classically describe directions that differ from those in the $x-y$ plane by quantities of order $\mathscr{O}(1-q)$, both for the spin system and the Stern-Gerlach apparatus, respectively. We therefore make the identification $\left|\Phi^S\left(\frac{1.006\pi}{2},\omega_s\right)\right\rangle \equiv |7,\omega_s\rangle_s$, $\left|\Phi^{SG}\left(\frac{1.006\pi}{2},\omega_a\right)\right\rangle \equiv |7,\omega_a\rangle_a$. In particular, setting $\omega = 0,\pi/2$, we obtain a semi-classical description of the axes closest to the x,y axes, respectively, which will be employed in the deformed alignment protocol in section 5.3.5.

By varying the value of μ, so that $|f(\mu_{s,a},n_{s,a})|^2$ is peaked around a certain n_θ, it can be shown that the state $|\mu_{s,a},\omega_{s,a}\rangle_{s,a}$ is the semi-classical counterpart of other directions corresponding to $\Big(\theta\big(n_\theta\big),\omega\Big)$.

So far, we have focused on the eigenstates of probability. With similar computations, one can show that by combining geometry states representative of semi-classical directions of spin systems and Stern-Gerlach apparatuses defined in the three examples above, it is possible to construct states in $\mathscr{H}_{SU_q(2)} \otimes \mathscr{H}_{SU_q(2)}$ which are not necessarily probability eigenstates but still satisfy the semi-classical conditions (5.78), (5.79). We close the discussion concerning a single observer with such an example.

4. We consider $\left|\Phi^S\right\rangle = \left|\chi_s\right\rangle$ and $\left|\Phi^{SG}\right\rangle = |7,0\rangle_a$, describing a semi-classical scenario in which the spin is orientated along the z axis and the Stern-Gerlach apparatus is orientated along the x axis, as indicated by the computations in (5.E.1) and (5.E.4). The probability distribution function will therefore be characterised by

$$\overline{P(\uparrow_\sigma)} = 0.50\,, \;\; \overline{\Delta}\left[P(\uparrow_\sigma)\right] = 0.05\,. \tag{5.E.5}$$

As expected, the variance assumes a non-zero value, given that this particular geometry state is a superposition of probability eigenstates.

We have provided some examples concerning the physical quantities of interest for a single observer and this allowed us to identify some states as suitable semi-classical descriptions of directions in space for both spin states and the Stern-Gerlach apparatuses. We now discuss how the same states can semi-classically describe relative orientations in space between two observers, approximating classical rotations parameterised as

$$R_z(\alpha)R_x(\theta)R_z(\gamma)\,, \;\; \chi_g = \frac{\alpha+\gamma}{2}\,, \;\; \phi_g = \frac{3}{2}\pi - \frac{\alpha-\gamma}{2}\,. \tag{5.E.6}$$

Taking into account the generalised requirements at the end of section 5.3.4 and (5.E.6), one can show that states $\left|\chi_g\right\rangle$ semi-classically describe rotations around the z axis of an angle $2\chi_g$, so we make the identification $\left|\Phi^{RO}(0,0,\chi_g)\right\rangle \equiv \left|\chi_g\right\rangle$. On the other hand, states of the form

$$\left|\mu_g,\phi_g,\chi_g\right\rangle_g := \sum_{n_g=0}^{\infty} h_{n_g}\left|n_g,\phi_g,\chi_g\right\rangle\,, \tag{5.E.7}$$

semi-classically describe rotations (5.E.6) with $\theta \in\,]0,\pi]$, where the value of μ_g is determined by θ according to table 5.2. We therefore make the identification

$$\left|\mu_g,\phi_g,\chi_g\right\rangle_g \equiv \left|\Phi^{RO}(\theta_g = \theta(n_\theta),\phi_g,\chi_g)\right\rangle\,, \tag{5.E.8}$$

where n_θ has the same meaning as in the final part of example 3. In particular, the state $\left|\mu_g = 0,\phi_g,\chi_g\right\rangle_g$ coincides with $\left|n_g = 0,\phi_g,\chi_g\right\rangle$, and it can be shown that the state $\left|0,\frac{3\pi}{2},0\right\rangle$ semi-classically describes a rotation around the x axis of an angle π. We conclude this appendix with an example involving a generic rotation around the x axis.

5. Consider $\left|\Phi^S\right\rangle = \left|7,\omega_s\right\rangle_s$, $\left|\Phi^{SG}\right\rangle = \left|\chi_a\right\rangle$ and $\left|\Phi^{RO}\right\rangle = \left|7,\phi_g,\chi_g\right\rangle_g$ (as in examples 3 and 4 we consider $q = 0.99$ for definiteness). The average values and uncertainties in the geometry state for the generic spin state and the generic Pauli matrix read

$$\begin{gathered}
\overline{|\psi\rangle} = 0.50\left(1 - e^{i\left(\phi_g-\chi_g-\omega_s\right)}\right)|\uparrow\rangle + e^{i\left(\phi_g-\chi_g\right)}0.50\left(1 + e^{-i\left(\phi_g+\chi_g-\omega_s\right)}\right)|\downarrow\rangle\,, \\
\overline{\Delta}[\,|\psi\rangle\,] = 0.1\left|\sin\left(\frac{\phi_g+\chi_g-\omega_s}{2}\right)\right|\left(|\uparrow\rangle + |\downarrow\rangle\right), \\
\overline{\sigma^q} = \begin{pmatrix} 0.00 & 0.99e^{-i\left(\phi_g-\chi_g\right)} \\ 0.99e^{i\left(\phi_g-\chi_g\right)} & 0.00 \end{pmatrix}, \;\; \overline{\Delta}[\sigma^q] = \begin{pmatrix} 0.10 & 0.10 \\ 0.10 & 0.10 \end{pmatrix}
\end{gathered} \tag{5.E.9}$$

In the above, if we set $\phi_g = \frac{3}{2}\pi, \chi_g = 0$, we conclude that $\left|\Phi^{RO}\right\rangle = \left|7, \frac{3}{2}\pi, 0\right\rangle_g$ is a suitable state to semi-classically describe a counterclockwise rotation of $\theta(34) = 1.006\frac{\pi}{2}$ around the x axis. This is particularly evident when also setting $\omega_s = 0$. Thus, we make the identification $\left|7, \frac{3}{2}\pi, 0\right\rangle_g \equiv \left|\Phi^{RO}(\frac{1.006\pi}{2}, \frac{3}{2}\pi, 0)\right\rangle$. In this case, the probability is distributed according to

$$\overline{P_A\left(\uparrow_\sigma^B\right)} = 0.51\,, \;\; \overline{\Delta}[P_A\left(\uparrow_\sigma^B\right)] = 0.04\,. \tag{5.E.10}$$

The relative orientation state discussed in this example is also employed in the protocol described in section 5.3.5, where in one of the examples Alice and Bob are connected by a rotation of $1.006\frac{\pi}{2}$ around the x axis.

5.F Further discussions on the joint probability distribution

In this appendix, we provide two (simple) proofs. First, we derive (5.86). Then, we show that the uncertainties written in (5.90) are those appearing in (5.87). Finally, in the last part, we also discuss some interpretational aspects of the joint probability distribution (5.84).

5.F.1 Expectation value and variance of joint probability density

We want to compute the mean value and variance of a joint probability density given by

$$\mathscr{P}(m, N, p) = \binom{N}{m} p^m (1-p)^{N-m} F(p)\,, \tag{5.F.1}$$

where $F(p)$ is the probability distribution of p with mean value and variance given by

$$p_0 = \int_0^1 \mathrm{d}p\; pF(p)\,, \;\; \Delta_F^2 = \int_0^1 \mathrm{d}p\; p^2 F(p) - p_0^2\,. \tag{5.F.2}$$

Starting with $\mathbb{E}[k]$, applying the law of total expectation, we have

$$\begin{aligned}\mathbb{E}[k] &= \int_0^1 \mathrm{d}p \sum_{m=0}^{N} m\mathscr{P}(m, N, p) = \int_0^1 \mathrm{d}p \sum_m \binom{N}{m} m\, p^m (1-p)^{N-m} F(p) = N\int_0^1 \mathrm{d}p\; pF(p) = \\ &= Np_0\,,\end{aligned} \tag{5.F.3}$$

where we have used the definition of p_0 and

$$\sum_m \binom{N}{m} m\, p^m (1-p)^{N-m} = Np\,. \tag{5.F.4}$$

The proof for the variance takes similar steps. Again, by the law of total expectation we have

$$
\begin{aligned}
\Delta_k^2 &= \mathbb{E}\left[k^2\right] - \mathbb{E}\left[k\right]^2 = \int_0^1 \mathrm{d}p \sum_{m=0}^{N} m^2 \mathscr{P}(m,N,p) - N^2 p_0^2 = \\
&= \int_0^1 \mathrm{d}p \sum_m \binom{N}{m} m^2 p^m (1-p)^{N-m} F(p) - N^2 p_0^2 = \\
&= N \int_0^1 \mathrm{d}p \; p(1-p) F(p) - N^2 p_0^2 + N^2 \int_0^1 \mathrm{d}p \; p^2 F(p) = \\
&= N^2 \Delta_F^2 + N p_0 - N \int_0^1 \mathrm{d}p \, p^2 F(p) = \\
&= N(N-1)\Delta_F^2 + N p_0 (1-p_0) \, ,
\end{aligned}
\tag{5.F.5}
$$

where p_0 and Δ_F^2 are the mean value and the variance of F defined above and we have used

$$
\sum_m \binom{N}{m} m^2 p^m (1-p)^{N-m} = N p(1-p) + n^2 p^2 \, . \tag{5.F.6}
$$

Notice that the probability operator $P_A\left(\uparrow_\sigma^B\right)$ defined in (5.74) has a spectrum that has both a discrete and a continuous part, with degenerate eigenvalues. Therefore, the above integrals should actually be performed on the spectrum of $P_A\left(\uparrow_\sigma^B\right)$, denoted as $\Lambda(P)$. For this reason, we set the probability distribution $F(p)$ to be given by

$$
F(p) = \begin{cases} f(p)\,, & p \in \Lambda(P) \\ 0 \quad, & p \notin \Lambda(P) \end{cases} \, , \tag{5.F.7}
$$

where $f(p) = \sum_r \langle \Phi | p,r \rangle \langle p,r | \Phi \rangle$ is the probability distribution of the probability $P_A\left(\uparrow_\sigma^B\right)$ in the geometry state $|\Phi\rangle$ and r denotes the possible degeneracy of p. Of course, with this definition, the variance Δ_F^2 is the same as the variance Δ_f^2 used in the main text.

5.F.2 Equivalence between the variances of the joint probability distributions and the variance of $\left\langle \sigma_B^q \right\rangle$

Starting from the variances in (5.90), we now provide the second proof. The variances in (5.90) in a generic state of the geometry $|\Phi\rangle$ are explicitly given by

$$
\begin{aligned}
\overline{\Delta}^2\left[\left\langle \sigma_B^q \right\rangle\right] &= (q+q^{-1})^2 \langle \Phi | P_B^2\left(\uparrow_\sigma^A\right) | \Phi \rangle + q^{-2} - 2(1+q^{-2}) \langle \Phi | P_B\left(\uparrow_\sigma^A\right) | \Phi \rangle + \\
&- \left[(q+q^{-1}) \langle \Phi | P_B\left(\uparrow_\sigma^A\right) | \Phi \rangle - q^{-1} \right]^2 .
\end{aligned}
\tag{5.F.8}
$$

From the definition of $f(p)$, $f(p) = \sum_r \langle \Phi | p,r \rangle \langle p,r | \Phi \rangle$, by decomposing the identity as

$$
\mathbb{1} = \int_{\Lambda(P)} \mathrm{d}p \sum_r |p,r\rangle\langle p,r| \, , \tag{5.F.9}
$$

we have

$$\langle\Phi|P_B^2\left(\uparrow^A_\sigma\right)|\Phi\rangle = \int_{\Lambda(P)} \mathrm{d}p\, p^2 f(p)\,, \;\; \langle\Phi|P_B\left(\uparrow^A_\sigma\right)|\Phi\rangle = \int_{\Lambda(P)} \mathrm{d}p\, p f(p)\,. \tag{5.F.10}$$

Therefore, recalling the definitions of Δ_f and p_0 in (5.F.2), we get

$$\overline{\Delta}^2\left[\left\langle\sigma_B^q\right\rangle\right] = \left(q+q^{-1}\right)^2\Delta_f^2 + q^{-2} - 2\left(1+q^{-2}\right)p_0 - q^{-2} + 2\left(1+q^{-2}\right)p_0 = \left(q+q^{-1}\right)^2\Delta_f^2\,. \tag{5.F.11}$$

Of course, the integral over $\Lambda(P)$ is intended to be an integral (discrete sum) over the continuous (discrete) part of the spectrum.

5.F.3 The interpretation of the joint probability distribution

We now want to delve into some interpretational points regarding the joint probability distribution in (5.84). As extensively discussed in section 5.3.5, the quantity to be measured in the doubly quantum protocol is the expectation value in the spin state of the Pauli matrix operator, given by

$$\left\langle\sigma_B^q\right\rangle_A = \left(q+q^{-1}\right)P_A\left(\uparrow^B_\sigma\right) - q^{-1}\mathbb{1}\,. \tag{5.F.12}$$

Therefore, the measurement of this quantity effectively amounts to a measurement of the quantum probability observable $P_A\left(\uparrow^B_\sigma\right)$. As discussed in section 5.3.4, in our framework, the effective result of actual measurements is described by the average value and the variance in the geometry states of the operators describing the observable that is being measured. As noted above, in the present case, the quantity measured in the alignment protocol is the probability of observing spin up in a spin measurement $P_A\left(\uparrow^B_\sigma\right)$. Contrary to typical observables in standard QM, the measurement of the probability observable in (5.74) requires N separate spin measurements, since it is itself defined by an expectation value in the spin states. This means that the experimental setup for the measurement of a single component of the rotation matrix in the alignment protocol is made of the Stern-Gerlach apparata used to prepare and measure the spins and the N spins used to perform the measurements. At the end of the measurement procedure, a given value p, drawn from the spectrum of $P_A\left(\uparrow^B_\sigma\right)$, will be observed, with a probability density $f(p)$ that depends on the geometry states. Therefore, the probability of observing k spin up results out of N measurements will be conditioned on the observation of the value p as a result of the probability measurement. This conditional probability will be given by the binomial distribution $P(k,N,p) = \binom{N}{k}p^k(1-p)^{N-k}$, hence the probability density of observing k spin up outcomes in N measurements and observing p as a result of the probability measurement is given, from the definition of conditional probability, by the product between this binomial and the distribution $f(p)$, which is exactly (5.84). Notice also that the two proofs provided above show that the expectation value and the variance of this joint distribution coincide, in the large N limit, with the expectation value and variance of $P_A\left(\uparrow^B_\sigma\right)$ in the geometry states.

Epilogue

In this thesis, we explored several topics at the interface between Quantum Theory and Gravity, focusing on some foundational aspects of the Quantum Gravity realm. Specifically, the questions advanced in the abstract that have opened the thesis and guided the research work presented in it revolve around the expected non-classicality of symmetries, reference frames, and observers in a Quantum Gravity theory. This is analysed mainly in the two related areas of non-commutative spacetime/deformation of symmetries and quantum reference frames. The relevant results obtained in this research work are highlighted in the following and are discussed in depth at the end of every chapter.

In chapter 3, we showed that quantum systems undergo a *decoherence process* during evolution in a quantum spacetime characterised by deformed symmetries. The corresponding decoherence time sets the Planck mass to be an upper limit to the mass of quantum systems, making it a *breakdown scale* of Quantum Theory. This possibility was heuristically discussed in the Quantum Gravity literature, but we rigorously derived it from symmetry arguments. We investigated the possibility of testing some Quantum Gravity-induced decoherence processes using neutrino oscillations. We found that the decoherence induced by stochastic metric perturbations can be strongly constrained and effectively *ruled out* using data from KamLAND and Super-Kamiokande collaborations.

In the context of quantum reference frames, in chapter 4 we offered new insights into the problem of *mass superpositions* for fundamental particles. From the lack of knowledge of the relation between the quantum reference frame of the physical system producing a fundamental particle and the quantum reference frame of the detector that detects it, we showed that a particle produced in a mass superposition is observed to be *superselected in mass* if either of the resolutions of the clocks of the two quantum reference frames is smaller than the ratio between the mass-squared difference in the superposition and the mean energy of the particle. This can be seen as the *emergence* of a fundamental selective superselection rule for the mass that singles out flavour neutrinos as the only fundamental particles that can be observed in mass superpositions, and opens up the possibility of investigating the emergence of other superselection rules in Quantum Theory from the properties of quantum reference frame transformations.

In chapter 5, we studied "quantum observers", namely observers whose reference frames and measurement apparatuses have non-classical features. We focused on a *quantum group* deformation of the rotation sector of spacetime symmetries, which introduces an intrinsic fuzziness to the notion of direction in space. This, in turn, produces a fuzziness in the alignment of measurement apparatuses (*e.g.* Stern-Gerlach devices) and physical systems (*e.g.* spins) along spatial directions. The relations between observers become non-classical as well. In this framework, observers cannot reconstruct an objective notion of space(time), which becomes *agency-dependent,* and measurement devices exhibit non-classical features represented by a quantisation of the Born rule that promotes probabilities to operators, so that the outcome of a measurement is characterised by a *non-classical probability* distribution. Because of this, two observers who do not share the same reference frame cannot sharply determine their relative orientation by performing a quantum information protocol. An *intrinsic uncertainty* arises even in the limit of infinite measurements, contrary to standard Quantum Theory.

These results shed some light on the behaviour of quantum systems in the presence of quantum reference frames and non-classical spacetimes and could help to further investigate the foundational principles in the quest for a theory of Quantum Gravity.

Appendix A

Representation theory of Galilei group in Quantum Mechanics

In this appendix, we give a review of the representation theory of the Galilei group in Quantum Mechanics.

A.1 Basics of group theory

We shall here give some useful definitions of group theory that we will use in the following.

A.1.1 Direct and semidirect products (and sums)

Consider two groups, G and G' with elements $a, b, \dots$ and $a', b', \dots$ respectively. We can compose the two groups with the direct product $G \otimes G'$ whose elements will be (a, a'). The inverse elements will be given by $\left(a^{-1}, a'^{-1}\right)$ while the product will be $(a, a')(b, b') = (a\,b, a'\,b')$. This is, of course, a rather simple structure. A more involved group can be obtained if one of the groups acts by means of a group of automorphisms on the second one. An automorphism of a group G is a map $\tau : G \to G$ such that

$$\tau(e) = e\ ,\ \ \tau(a)^{-1} = \tau\left(a^{-1}\right)\ ,\ \ \tau(a)\tau(b) = \tau(a\,b)\,, \tag{A.1}$$

where e is the identity of G. If we have two groups H (with elements $h, h', \dots$) and A (with elements $a, a', \dots$) such that for each element $h \in H$ there exists an automorphism of A τ_h obeying the composition law

$$\tau_h \tau_{h'} = \tau_{h\,h'}\,, \tag{A.2}$$

then we can define the semidirect product of H and A, $H \ltimes A$, as the group with elements (h, a) and composition law

$$(h', a')(h, a) = (h'h, a'\tau_{h'}(a))\,, \tag{A.3}$$

while the inverse law is $(h, a)^{-1} = (h^{-1}, \tau_{h^{-1}}(a^{-1}))$. In the case of the Galilei group, the group H is $\mathscr{R} \ltimes \mathscr{B}$, and the group A is $\mathscr{T}_4$. The automorphisms are given by $\tau_{\boldsymbol{v},R}(b, \boldsymbol{a}) = (b, R \cdot \boldsymbol{a} + b\,\boldsymbol{v})$. Analogously, the automorphisms for the semidirect product $\mathscr{R} \ltimes \mathscr{B}$ are $\tau_R(\boldsymbol{v}) = R \cdot \boldsymbol{v}$. The composition law for $\mathscr{T}_4$ and $\mathscr{B}$ is the sum in both cases. Finally, another operation on groups is the quotient. Consider a group G which has an invariant subgroup G_0[1]. Let us call

[1]Given a group G and a subgroup G_0 we say that G_0 is an invariant subgroup (or normal subgroup) if $x \in G_0$ then $a\,x\,a^{-1} \in G_0\ \forall\, a \in G$.

$a, b, \ldots$ the elements of G and $x, y, \ldots$ the elements of G_0. Two elements a and b in G will be called equivalent if there exists an element $x \in G_0$ such that $a = xb$. G can therefore be split in disjoint equivalence classes and make a group of these, and this group will be the factor group of G by G_0, namely ${}^{G}\!/_{G_0}$. In the case where we have $G = H \ltimes A$ we find that the subgroup $\widetilde{H}$ made up of elements (h, e_A) is isomorphic to H and the subgroup $\widetilde{A}$ with elements (e_H, a) is isomorphic to A but only the latter is invariant. In the case of a semidirect product +, ${}^{(H \ltimes A)}\!/_{\widetilde{A}}$ is isomorphic to H.

The same definitions can be given for a Lie algebra corresponding to a Lie group. The Lie algebra corresponding to a direct product of Lie groups $G \otimes G'$ is the direct sum of Lie algebras $L \oplus L'$. As a vector space, it is given by the direct sum of the two vector spaces. If $u, v, \ldots$ are vectors in the Lie algebra L and $u', v', \ldots$ are vectors in the Lie algebra L', the Lie bracket is defined by

$$\left[u \oplus u', v \oplus\right]_{L \oplus L'} = \left[u, v\right]_L \oplus \left[u', v'\right]_{L'} . \tag{A.4}$$

For the analog of the semidirect product we have to deal with algebra automorphisms and derivations. An automorphism τ of a Lie algebra L is a linear map $\tau : L \to L$ such that

$$\tau(\lambda\, u + \mu\, v) = \lambda\, \tau(u) + \mu\, \tau(v) \ , \ \left[\tau(u), \tau(v)\right] = \tau\Big(\left[u, v\right]\Big) \ , \ \lambda, \mu \in \mathbb{R} \ , \ u, v \in L \, . \tag{A.5}$$

Derivations are infinitesimal automorphisms and are defined by the properties

$$D(\lambda\, u + \mu\, v) = \lambda\, D(u) + \mu\, D(v) \ , \ D\Big(\left[u, v\right]\Big) = \left[D(u), v\right] + \left[u, D(v)\right] \ , \ \lambda, \mu \in \mathbb{R} \ , \ u, v \in L \, . \tag{A.6}$$

If we denote the Lie algebra of $H \ltimes A$ by L, this will be as a vector space the direct sum of the two algebras once again, L_H and L_A. If we denote by $u, u', \ldots$ the elements of L_H and by $\alpha, \alpha', \ldots$ we now have to understand what happens to the Lie brackets $\left[u, \alpha\right]$. L_H and L_A will be subalgebras of L and in particular L_A will be an invariant subalgebra, meaning that $\left[u, \alpha\right]$ will be an element of L_A for all $u \in L_H$ and $\alpha \in L_A$. This means that the Lie bracket of an element $u \in L_H$ acts as a derivation on L_A, $\left[u, \alpha\right] = D_u(\alpha)$ and these derivations are just infinitesimal forms of the automorphisms τ_h. The Lie bracket will be given by

$$\left[u \oplus \alpha, u' \oplus \alpha'\right]_{L_H \dot{\oplus} L_A} = \left[u, u'\right]_{L_H} \oplus \left[\alpha, \alpha'\right]_{L_A} \oplus D_u\big(\alpha'\big) \oplus \Big(-D_{u'}(\alpha)\Big) \, , \tag{A.7}$$

where $L_H \dot{\oplus} L_A$ denotes the semidirect sum. Finally, the Lie algebra corresponding to a factor group ${}^{G}\!/_{G_0}$ is the factor algebra. Since G_0 is invariant, the Lie algebra L_0 corresponding to it is also invariant, and the factor algebra is obtained by projecting L onto L_0, namely simply ignoring the elements of L_0.

A.1.2 Representations of Lie groups

Given a group G and a vector space V, a representation of G on V is a map $\Pi : G \ni g \mapsto \Pi(g) \in Lin(V)$ such that the composition law and the inversion of the group are preserved, namely

$$\Pi(g_1)\Pi(g_2) = \Pi(g_1 g_2) \ , \ \Pi(g^{-1}) = \Pi^{-1}(g) \, . \tag{A.8}$$

A representation on a complex vector space is unitary if the representatives $\Pi(g)$ are unitary. A representation is projective if the composition property is satisfied up to a constant

$$\Pi(g_1)\Pi(g_2) = \omega(g_1, g_2)\Pi(g_1 g_2) \, . \tag{A.9}$$

For a projective unitary representation, the constant ω must be a phase. A representation is reducible if V admits an invariant non-trivial subspace W (*i.e.* $W \neq V$ and $V \neq \emptyset$). If this is the case, we can write V as $V = (V \backslash W) \oplus W$ and there exists an operator Θ such that all the operators of the representation can be written as

$$\Pi'(g) = \Theta^{-1}\Pi(g)\Theta = \begin{pmatrix} \Pi^{(11)}(g) & \Pi^{(12)}(g) \\ 0 & \Pi^{(22)}(g) \end{pmatrix}, \tag{A.10}$$

from which the invariance of W is evident. If there is no non-trivial invariant subspace of V the representation is irreducible. For an Abelian group, unitary irreducible representations are 1-dimensional. This follows from the Schür's Lemma

Lemma 1. *(Schür's Lemma) Given a group G and an irreducible representation of G over a complex vector space V, if $A \in Lin(V)$ is such that $\left[A, U(g)\right] = 0$ $\forall\, g \in G$ then $A = \lambda \mathbb{1}$ with $\lambda \in \mathbb{C}$.*

For a representation of an Abelian group we have $U(g)U(h) = U(gh) = U(hg) = U(h)U(g)$ $\forall\, g, h \in G$. Therefore, from the previous lemma, we have that all the representatives must be proportional to the identity $U(g) = \lambda_g \mathbb{1}$, thus any subspace of V is invariant, in particular every 1-dimensional subspace of V is invariant. Hence, if $\dim V \neq 1$ the representation cannot be irreducible. From this we conclude that $\dim V = 1$. For the representation to be unitary, we have $\lambda_g = e^{i\phi(g)}$ with $\phi(g) \in \mathbb{R}$.

A.2 The Galilei group

The Galilei group $\mathscr{G}$ is the relativistic group of transformations for a theory involving particles with velocities much smaller than the velocity of light in vacuum c. The group is composed of spacetime translations, spatial rotations, and Galilean boosts, accordingly it is a ten parameters group.

Given an event of coordinates $(\boldsymbol{x}_P, t)$ in a given reference frame S with origin O and coordinate axes $\boldsymbol{e}_i$, if we consider an active transformation to another reference frame S', with origin O' and coordinate axes $\boldsymbol{e}'_i$, which is Galilean transformed with respect to S, we have

$$\begin{aligned} \boldsymbol{x}'_P &= \boldsymbol{x}_P\big(O'\big) + \boldsymbol{x}_{O'}\,, \\ t' &= t + b\,, \end{aligned} \tag{A.11}$$

where $\boldsymbol{x}'_P$ are the coordinates of the transformed point in S, $\boldsymbol{x}_{O'} = \boldsymbol{a} + \boldsymbol{v}\,t$ are the coordinates of O' in S (*i.e.* its law of motion) and $\boldsymbol{x}_P\big(O'\big)$ are the coordinates of the transformed point in the reference frame S', *i.e.* $\boldsymbol{x}_P\big(O'\big) = x^i_P\big(O'\big)\boldsymbol{e'}_i = x^i\boldsymbol{e'}_i = R^j{}_i x^i \boldsymbol{e}_j$, where the second-last equality comes from the fact that the transformed point in S' has the same coordinates of the starting point in S. In the above, O' is translated with respect to O by $\boldsymbol{a}$, the axes of S' are rotated with respect to the axes of S by the matrix $R^i{}_j$, the time coordinate t is translated by an amount b and the relative velocity between O' and O is $\boldsymbol{v}$, in the reference frame S. From these arguments, we get, dropping the subscript P

$$x'^i = R^i{}_j x^j + a^i + v^i\,t\,, \tag{A.12}$$

namely

$$\begin{aligned} \boldsymbol{x}' &= R\cdot\boldsymbol{x} + \boldsymbol{a} + \boldsymbol{v}\,t\,, \\ t' &= t + b\,. \end{aligned} \tag{A.13}$$

A.2.1 The group structure

We label the element of the Galilei group that results in the transformation law (A.13) as $g = \left(b, \boldsymbol{a}, \boldsymbol{v}, R\right)$. From (A.13) we immediately get the composition law of the group

$$\begin{aligned} \boldsymbol{x}'' &= R' \cdot \boldsymbol{x}' + \boldsymbol{a}' + \boldsymbol{v}'\, t' = R' \cdot \left(R \cdot \boldsymbol{x} + \boldsymbol{a} + \boldsymbol{v}\, t\right) + \boldsymbol{a}' + \boldsymbol{v}'\,(t+b) = r \\ &= R'R \cdot \boldsymbol{x} + \boldsymbol{a}' + R' \cdot \boldsymbol{a} + b\, \boldsymbol{v}' + \left(\boldsymbol{v}' + R' \cdot \boldsymbol{v}\right) t\,, \\ t'' &= t' + b' = t + b + b'\,, \end{aligned} \tag{A.14}$$

thus we get

$$g'\, g = \left(b', \boldsymbol{a}', \boldsymbol{v}', R'\right)\left(b, \boldsymbol{a}, \boldsymbol{v}, R\right) = \left(b + b', \boldsymbol{a}' + R' \cdot \boldsymbol{a} + b\, \boldsymbol{v}', \boldsymbol{v}' + R' \cdot \boldsymbol{v}, R'R\right), \tag{A.15}$$

with $g, g' \in \mathscr{G}$. From this we also obtain the inversion law of the group, by requiring that

$$\left(b + b', \boldsymbol{a}' + R' \cdot \boldsymbol{a} + b\, \boldsymbol{v}', \boldsymbol{v}' + R' \cdot \boldsymbol{v}, R'R\right) = (0, \boldsymbol{0}, \boldsymbol{0}, \mathbb{1})\,, \tag{A.16}$$

i.e. the identity of the group. Solving the equation with respect to the primed variables gives

$$g^{-1} = \left(-b, -R^{-1} \cdot (\boldsymbol{a} - b\, \boldsymbol{v}), -R^{-1} \cdot \boldsymbol{v}, R^{-1}\right). \tag{A.17}$$

Writing an element $g \in \mathscr{G}$ as $g = (b, \boldsymbol{a}, \boldsymbol{v}, R)$ gives a parameterisation of the group in terms of ten coordinates, namely the transformation parameters. With respect to these coordinates, (A.15) and (A.17) are smooth functions, thus we see that $\mathscr{G}$ has the structure of a 10 dimensional Lie group. In this regard, (A.13) can be understood as a simple 5 dimensional matrix representation of the abstract group $\mathscr{G}$, indeed considering 5 dimensional vectors $(\boldsymbol{x}, t, 1)$, an element $g(b, \boldsymbol{a}, \boldsymbol{v}, R) \in \mathscr{G}$ can be represented by a 5×5 matrix $Q(g)$ as

$$\begin{pmatrix} \boldsymbol{x}' \\ t' \\ 1 \end{pmatrix} = \begin{pmatrix} R & \boldsymbol{v} & \boldsymbol{a} \\ \boldsymbol{0}^T & 1 & b \\ \boldsymbol{0}^T & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{x} \\ t \\ 1 \end{pmatrix}, \tag{A.18}$$

which is simply (A.13) in disguise, with $\boldsymbol{a}$, $\boldsymbol{v}$, $\boldsymbol{x}$ and $\boldsymbol{x}'$ being coloumn vectors, while $\boldsymbol{0}^T$ is a row vector.

From the composition law (A.15) we can also observe that $\mathscr{G}$ is not a simple group, indeed it has a 6 parameters invariant Abelian subgroup $\mathscr{U}$ composed of spatial translations and Galilean boosts, which is the maximal Abelian invariant subgroup. Indeed, given $u = \left(0, \boldsymbol{a}, \boldsymbol{v}, \mathbb{1}\right) \in \mathscr{U}$ and $g = \left(b, \boldsymbol{a}', \boldsymbol{v}', R\right) \in \mathscr{G}$ we have by direct computation

$$g\, u\, g^{-1} = \left(0, \boldsymbol{a}' + R \cdot (a - b\, \boldsymbol{v}) - b\boldsymbol{v}', R \cdot \boldsymbol{v}, \mathbb{1}\right) \in \mathscr{U}\,, \tag{A.19}$$

namely $\mathscr{U}$ is a normal subgroup of $\mathscr{G}$. It is Abelian since

$$\left(0, \boldsymbol{a}, \boldsymbol{v}, \mathbb{1}\right)\left(0, \boldsymbol{a}', \boldsymbol{v}', \mathbb{1}\right) = \left(0, \boldsymbol{a} + \boldsymbol{a}', \boldsymbol{v} + \boldsymbol{v}', \mathbb{1}\right) \tag{A.20}$$

is a commutative multiplication law.

By the same argument, the factor group ${}^{\mathscr{G}}\!/_{\mathscr{U}}$ is not a simple group since it has a 1 parameter invariant subgroup, the time translation subgroup $\mathscr{T}$. It is only the factor group $\left({}^{\mathscr{G}}\!/_{\mathscr{U}}\right)\!/_{\mathscr{T}}$, namely the rotation group $\mathscr{R}$, which is a simple group. We can therefore write the group $\mathscr{G}$ as the semidirect product by $\mathscr{U}$ of the semidirect product by $\mathscr{T}$ of $\mathscr{R}$

$$\mathscr{G} = \left(\mathscr{R} \ltimes \mathscr{T}\right) \ltimes \mathscr{U} \,. \tag{A.21}$$

This is the standard decomposition of the Galilei group in terms of its maximal normal subgroup, *i.e.* the group $\mathscr{U}$. For what concerns the study of the representations of $\mathscr{G}$, however, another decomposition is more interesting. Indeed, we can see that the group of spacetime translations $\mathscr{T}_4$ is an invariant (Abelian) subgroup of $\mathscr{G}$. This is given by the direct product of the time translation subgroup and the spatial translations subgroup, $\mathscr{T}_4 = \mathscr{T} \otimes \mathscr{T}_3$. The quotient group ${}^{\mathscr{G}}\!/_{\mathscr{T}_4}$ admits another invariant subgroup, namely the group of pure Galilean boosts $\mathscr{B}$, and the quotient of ${}^{\mathscr{G}}\!/_{\mathscr{T}_4}$ with respect to $\mathscr{B}$ gives the simple group $\mathscr{R}$. Thus we have

$$\mathscr{G} = \left(\mathscr{R} \ltimes \mathscr{B}\right) \ltimes \mathscr{T}_4 = \left(\mathscr{R} \ltimes \mathscr{B}\right) \ltimes \left(\mathscr{T} \otimes \mathscr{T}_3\right) . \tag{A.22}$$

This decomposition will be useful in the following.

A.2.2 The Galilei Lie algebra

Finally, from (A.15) we can also derive the Lie algebra of the group [65]. Let us denote with τ, π_i, β_i and λ_i the basis elements of the Lie algebra generating time translations, spatial translations, Galilean boosts and spatial rotations, respectively. We can obtain an element of the group using the exponential map. Specifically, since

$$g(b, \boldsymbol{a}, \boldsymbol{v}, R) = \left(b, \mathbf{0}, \mathbf{0}, \mathbb{1}\right)\left(0, \boldsymbol{a}, \mathbf{0}, \mathbb{1}\right)\left(0, \mathbf{0}, \boldsymbol{v}, \mathbb{1}\right)\left(0, \mathbf{0}, \mathbf{0}, R\right) , \tag{A.23}$$

we have

$$g(b, \boldsymbol{a}, \boldsymbol{v}, R) = \exp\left\{b\,\tau\right\} \exp\left\{a^i\,\pi_i\right\} \exp\left\{v^i\,\beta_i\right\} \exp\left\{r^i\,\lambda_i\right\} , \tag{A.24}$$

where $\boldsymbol{r}$ is a vector specifying the three parameters needed to determine a rotation. The four exponential maps describe therefore time translations, spatial translations, Galilean boosts, and spatial rotations, respectively. To derive the Lie algebra, we now consider such transformations with infinitesimal parameters. For instance, considering a time translation and a spatial translation

$$\begin{aligned} 0 &= g\left(\delta b, \mathbf{0}, \mathbf{0}, \mathbb{1}\right) g\left(0, \boldsymbol{\delta} a, \mathbf{0}, \mathbb{1}\right) - g\left(0, \boldsymbol{\delta} a, \mathbf{0}, \mathbb{1}\right) g\left(\delta b, \mathbf{0}, \mathbf{0}, \mathbb{1}\right) = \\ &= \exp\left\{\delta b\,\tau\right\} \exp\left\{\delta a^i\,\pi_i\right\} - \exp\left\{\delta a^i\,\pi_i\right\} \exp\left\{\delta b\,\tau\right\} \sim \delta b\,\delta a^i (\tau\,\pi_i - \pi_i\,\tau) = \delta b\,\delta a^i \left[\tau, \pi_i\right] \end{aligned} \tag{A.25}$$

where the first equality comes from direct computation with the composition law (A.15). From this we get the Lie brackets between the time translations generator and the spatial translations generators. Let us now consider a non-trivial example. For instance, take a Galilean boost and a spatial rotation. From the composition law we have

$$\begin{aligned} &\exp\left\{\delta r^i \lambda_i\right\} \exp\left\{\delta v^i \beta_i\right\} - \exp\left\{\delta v^i \beta_i\right\} \exp\left\{\delta r^i \lambda_i\right\} = \\ &= g\left(0, \mathbf{0}, \mathbf{0}, R\right) g\left(0, \mathbf{0}, \boldsymbol{v}, \mathbb{1}\right) - g\left(0, \mathbf{0}, \boldsymbol{v}, \mathbb{1}\right) g\left(0, \mathbf{0}, \mathbf{0}, R\right) = g(0, \mathbf{0}, R \cdot \boldsymbol{v}, R) - g(0, \mathbf{0}, \boldsymbol{v}, R) = \\ &= \exp\left\{\delta v'^i \beta_i + \delta r^i \lambda_i\right\} - \exp\left\{\delta v^i \beta_i + \delta r^i \lambda_i\right\} \end{aligned} \tag{A.26}$$

$$\delta \boldsymbol{v}' := R(\delta \boldsymbol{r}) \cdot \delta \boldsymbol{v} \sim \delta \boldsymbol{v} + \delta \boldsymbol{r} \wedge \delta \boldsymbol{v} \,, \tag{A.27}$$

and considering infinitesimal transformations

$$g(0, \mathbf{0}, R(\delta \boldsymbol{r}) \cdot \delta \boldsymbol{v}, R(\delta \boldsymbol{r})) - g(0, \mathbf{0}, \delta \boldsymbol{v}, R(\delta \boldsymbol{r})) \sim \epsilon^i{}_{jk} \delta r^j \delta v^k \beta_i \,. \tag{A.28}$$

Expanding the first line in (A.26) we get $\delta v^k \delta r^j \left[\lambda_j, \beta_k\right]$ therefore

$$\left[\lambda_i, \beta_j\right] = \epsilon_{ij}{}^k \beta_k \tag{A.29}$$

With the same procedure all the Lie brackets can be obtained and one ends up with the Lie algebra of the Galilei group

$$\begin{aligned} &[\tau, \pi_i] = [\tau, \lambda_i] = 0 \;,\; \left[\beta_i, \beta_j\right] = \left[\beta_i, \pi_j\right] = \left[\pi_i, \pi_j\right] = 0 \;,\; [\beta_i, \tau] = \pi_i \,, \\ &\left[\lambda_i, \beta_j\right] = \epsilon_{ij}{}^k \beta_k \;,\; \left[\lambda_i, \pi_j\right] = \epsilon_{ij}{}^k \pi_k \;,\; \left[\lambda_i, \lambda_j\right] = \epsilon_{ij}{}^k \lambda_k \,. \end{aligned} \tag{A.30}$$

A.3 Physical representations of the Galilei group in Quantum Mechanics

Having dealt with the structure of the Galilei group in the previous section, we are now ready to study the physical representations of $\mathscr{G}$ in Quantum Mechanics. We know by Wigner theorem that when we act with a symmetry transformation on a reference frame, if the description of quantum-mechanical physical laws of a system has to be unchanged[2] (as it should be for a symmetry transformation), then the elements of the symmetry group must be represented by unitary or anti-unitary operators on the Hilbert space of the system. We now focus on physical systems given by elementary particles.

An elementary particle is completely determined once some physical parameters are fixed. For example, these could be the mass and spin of the particle, and, since these parameters determine what the particle *is*, all observers should agree on their values. This means that there should exist some observables, in our example some mass and spin observables, M and S, such that, if $U(g)$ is the representation of g on the Hilbert space of the particle, $U(g)\, M\, U^\dagger(g) = M$ and $U(g)\, S\, U^\dagger(g) = S$, namely these observables should be the same for all observers. Moreover, they should have fixed values since they define the particle whatever its quantum state is, hence $M = m\,\mathbb{1}$ and $S = s\,\mathbb{1}$. Wigner showed that a sufficient condition for this to occur is that the representation is irreducible [347].

One can also argue the following: while an elementary particle can be in whatever state in a given Hilbert space, the Hilbert space itself defines what the particle is. This means that it should be invariant under the action of the representation of the symmetry group, U; of course, the subset $\{0\}$ is invariant under the action of the representation and the latter is irreducible if $\{0\}$ and $\mathscr{H}$ are the only invariant spaces under the action of U. If this is not the case, then it means that there is a non-trivial invariant subspace of $\mathscr{H}$, say $\mathscr{H}'$. However, in

[2] This means that the eigenvalues of the physical observables, namely the possible results for measurements, and the probability to observe such values are the same in the two reference frames.

this case, it means that in the Hilbert space relative to the first particle, $\mathscr{H}$, there is a smaller Hilbert space invariant under the action of the symmetry group, *i.e* a “more fundamental” particle. This can be regarded as a heuristic/logical argument in support of the requirement that elementary particles be described by irreducible representations, which is anyway backed up by the mathematical arguments of Wigner.

Therefore, from now on, we will assume that we can describe elementary quantum particles by means of (anti-)unitary irreducible representations of symmetry groups, and we will focus on the case of unitary representations of symmetry groups, since this will be the case for the Galilei group.

A.3.1 To describe particles we need *rays*

All of this is almost true. Since Quantum Mechanics is a theory of probabilities, a physical system is not described by a state $|\psi\rangle \in \mathscr{H}$, but rather by a *ray* in the Hilbert space, namely an equivalence class of states

$$[\Psi] := \left\{e^{i\theta}\left|\psi\right\rangle \mid \theta \in \mathbb{R}, \left|\psi\right\rangle \in \mathbb{S}_{\mathscr{H}}\right\}, \tag{A.31}$$

where $\mathbb{S}_{\mathscr{H}} := \{|\psi\rangle \in \mathscr{H} \mid \langle\psi|\psi\rangle = 1\}$, *i.e.* the unit sphere in $\mathscr{H}$. Therefore, to describe probabilities, which are projections of states onto other states $\left|\langle\phi|\psi\rangle\right|$, we can define a scalar product between rays

$$[\Phi]\cdot[\Psi] := \left|\langle\phi|\psi\rangle\right|, \tag{A.32}$$

as the mathematical object describing probabilities (it is clearly independent of the choice of representatives in the equivalence class). Therefore, the proper requirement for a symmetry transformation is the following.

Symmetry requirement: *Two descriptions of a quantum-mechanical system are isomorphic if there exists a one-to-one correspondence* $[\Psi] \mapsto [\Psi']$ *between rays which preserves transition probabilities, namely* $[\Phi]\cdot[\Psi] = [\Phi']\cdot[\Psi']$.

As far as states are concerned, every ray correspondence can be extended to a state correspondence $|\psi'\rangle \mapsto U|\psi\rangle$, where U is a unitary operator. However, the ray correspondence determines U only up to a phase, therefore we can define an operator ray $[U]$ as

$$[U] := \left\{e^{i\theta}U \mid \theta \in \mathbb{R}, U \in \mathscr{U}_{\mathscr{H}}\right\}, \tag{A.33}$$

$\mathscr{U}_{\mathscr{H}}$ being the set of unitary operators on $\mathscr{H}$, and argue that any isomorphic ray correspondence defines a unitary operator ray and vice versa. The product of two such rays is defined as

$$[U][V] := \left\{e^{i\theta}UV \mid \theta \in \mathbb{R}, U, V \in \mathscr{U}_{\mathscr{H}}\right\} \tag{A.34}$$

Now, from the symmetry requirement we understand that the proper statement about the representation of a symmetry group on the Hilbert space of a quantum-mechanical system is that there should exist an isomorphic ray correspondence $[U(g)]\ \forall\, g \in \mathscr{G}$ such that the usual representation relation

$$[U(g)][U(h)] = [U(gh)] \tag{A.35}$$

holds. However, since ray elements are defined up to a phase, this relation written in terms of representatives of the rays reads

$$U(g)U(h) = \omega(g,h)U(gh)\,, \tag{A.36}$$

where $\omega(g,h)$ is called *local factor* and is a complex number of modulus 1,

$$\omega(g,h) = e^{i\zeta(g,h)} \;\;, \zeta \in \mathbb{R}\,. \tag{A.37}$$

$\zeta(g,h)$ is often referred to as *local exponent* of the group. (A.36) means that U is a unitary projective representation of $\mathscr{G}$.

For elementary particles, we have to add the irreducibility condition. This implies that, in general, elementary particles in Quantum Mechanics should be described by unitary projective irreducible representations of the symmetry group. The possibility of having a description in terms of true representations, *i.e.* $\omega = 1$ in (A.36), depends on the symmetry group.

Properties of local factors and exponents

In general, $\omega(g,h)$ can only be defined in a neighbourhood of the identity of the group, $e \in \mathscr{G}$. The same applies to the local exponent $\zeta(g,h)$. For the identity, we impose $U(e) = 1$, therefore, from (A.36) and (A.37), it follows that $\omega(e,e) = 1$ and $\zeta(e,e) = 0$. If ω and ζ can be defined on the whole group, then they are called factor and exponent of the group, respectively. In what follows, we shall consider that all the relations hold in a neighbourhood of e, $\mathscr{A} \subset \mathscr{G}$. However, this is more a technicality dictated by mathematicians rather than an interesting physical fact since we phycists are almost always interested in Lie groups obtained by exponentiation of a Lie algebra, hence everything is always defined on the whole group. In the following we will, however, maintain the notation "local" to refer to the exponent and the factor of the group.

Equivalent local factors and exponents Given two different sets of representatives $U(g)$ and $U'(g) = \alpha(g)U(g)$ ($\left|\alpha(g)\right| = 1$, $\alpha \in \mathbb{C}$) we have $U'(g)U'(h) = \omega'(g,h)U'(gh)$ where

$$\omega'(g,h) = \frac{\alpha(g)\alpha(h)}{\alpha(gh)}\omega(g,h)\,. \tag{A.38}$$

Two local factors linked by the relation (A.38) are dubbed equivalent. Given that $\omega = e^{i\zeta}$, (A.38) immediately translates into a relation for the local exponent

$$\zeta'(g,h) = \zeta(g,h) - i\Big\{\log\big[\alpha(g)\big] + \log\big[\alpha(h)\big] - \log\big[\alpha(gh)\big]\Big\}\,. \tag{A.39}$$

Two local exponents related by (A.39) are said to be equivalent.

Associativity From the associative law of the group multiplication $\big[U(g)U(h)\big]U(t) = $ $= U(g)\big[U(h)U(t)\big]$ we get

$$\omega(g,h)\omega(gh,t) = \omega(g,ht)\omega(h,t)\,. \tag{A.40}$$

This translates, for the local exponent, into

$$\zeta(g,h) + \zeta(gh,t) = \zeta(g,ht) + \zeta(h,t)\,, \tag{A.41}$$

and from this, it immediately follows that

$$\begin{aligned} \zeta(e,g) &= \zeta(g,e) = 0\,, \\ \zeta(g,g^{-1}) &= \zeta(g^{-1},g)\,. \end{aligned} \tag{A.42}$$

The local group

The operators contained in all the operator rays of a ray representation of a group $\mathscr{G}$ form a group themselves. Let $[U(g)]$ be an operator ray. All the operators in this ray have the form $e^{i\theta}U(g)$, for some U. Therefore, the ray multiplication gives

$$\left(e^{i\theta}U(g)\right)\left(e^{i\theta'}U(h)\right) = e^{i(\theta+\theta'+\zeta(g,h))}U(gh)\,. \tag{A.43}$$

The elements of the local group $\widetilde{\mathscr{G}}$ are pairs (θ, g) endowed with the multiplication law

$$(\theta, g)(\theta', h) = (\theta + \theta' + \zeta(gh), gh)\,. \tag{A.44}$$

The associativity of this multiplication law is ensured by (A.41). The inverse element of (θ, g) is derived from this as

$$(\theta, g)^{-1} = \left(-\theta - \zeta\left(g, g^{-1}\right), g^{-1}\right), \tag{A.45}$$

since $(\theta, g)(\theta, g)^{-1} = (0, e)$. The elements of $\widetilde{\mathscr{G}}$ can be also written as

$$(\theta, g) = (\theta, e)(0, g)\,, \tag{A.46}$$

where $\Theta := \{(\theta, e) \mid \theta \in \mathbb{R}\}$ is an Abelian subgroup of $\widetilde{\mathscr{G}}$ which is the center of $\mathscr{G}$, in that

$$(\theta, e)(0, g) = (0, g)(\theta, e) = (\theta, g)\,. \tag{A.47}$$

Of course, the factor group $\widetilde{\mathscr{G}}/\Theta$ is isomorphic to $\mathscr{G}$.

On $\widetilde{\mathscr{G}}$ we can define a representation $\widetilde{U}$ which is an extension of the representation U in the sense that

$$\widetilde{U}\left((\theta, g)\right) = e^{i\theta}U(g) \tag{A.48}$$

Given that the composition law of $\widetilde{\mathscr{G}}$ is (A.44), for this representation we have

$$\widetilde{U}\left((\theta, g)\right)\widetilde{U}\left((\theta', h)\right) = \widetilde{U}\left(\theta + \theta' + \zeta(g, h), gh\right) = e^{i\theta + i\theta' + i\zeta(g,h)}U(gh)\,, \tag{A.49}$$

namely $\widetilde{U}$ is a true representation of $\widetilde{\mathscr{G}}$. With this, the problem of studying a unitary projective (irreducible) representation of $\mathscr{G}$ is replaced by the problem of studying a *true* unitary (irreducible) representation of the central extension $\widetilde{\mathscr{G}}$.

The local factor for the Galilei group

For Lie groups, the local exponent, hence the local factor, can be derived from the Lie algebra. For the Galilei group, this procedure gives [66] for (A.36)

$$U(g)U(h) = \exp\left\{\frac{i}{2}m\left(\boldsymbol{a}' \cdot R' \cdot \boldsymbol{v} - \boldsymbol{v}' \cdot R' \cdot \boldsymbol{a} + b\,\boldsymbol{v}' \cdot R' \cdot \boldsymbol{v}\right)\right\}U(gh)\,, \tag{A.50}$$

where $g = g(b, \boldsymbol{a}, \boldsymbol{v}, R)$ and $h = h(b', \boldsymbol{a}', \boldsymbol{v}', R')$ and m is the mass of the particle. In the following, we will assume $m \neq 0$ since the Galilei group is the symmetry group of non-relativistic Quantum Mechanics.

The local group of the Galilei group is then an 11 parameter group with elements parameterised by $(\theta, b, \boldsymbol{a}, \boldsymbol{v}, R)$. The true representations of this group follow the composition law

$$\widetilde{U}\Big((\theta,g)\Big)\widetilde{U}\Big((\theta',h)\Big) = \widetilde{U}\Big(\theta+\theta'+\zeta(g,h), gh\Big) = e^{i(\theta+\theta'+\zeta(g,h))}U(gh) \tag{A.51}$$

where $\zeta(g,h) = \frac{m}{2}\Big(\boldsymbol{a}'\cdot R'\cdot \boldsymbol{v} - \boldsymbol{v}'\cdot R'\cdot \boldsymbol{a} + b\,\boldsymbol{v}'\cdot R'\cdot \boldsymbol{v}\Big)$ is the local exponent of the group. From this composition law we see immediately that the maximal invariant Abelian subgroup is not $\mathscr{U}$ anymore, rather it is $\widetilde{\mathscr{T}_4}$, namely the five dimensional subgroup of spacetime translations and the central extension, whose elements are parameterised by $(\theta, b, \boldsymbol{a}, \boldsymbol{0}, R)$. The factor group $\widetilde{\mathscr{G}}/\widetilde{\mathscr{T}_4}$ is not a simple group since it has an invariant Abelian subgroup $\mathscr{B}$, namely the subgroup of Galilean boosts. Only the factor group $\left(\widetilde{\mathscr{G}}/\widetilde{\mathscr{T}_4}\right)/\mathscr{B}$, namely the rotation group, is a simple group. Therefore, we have

$$\widetilde{\mathscr{G}} = \Big(\mathscr{R}\ltimes\mathscr{B}\Big)\ltimes\widetilde{\mathscr{T}_4}\,, \tag{A.52}$$

which resembles the decomposition (A.22), which will be the one that will be used to construct the representations of the Galilei group in Quantum Mechanics.

Finally, from (A.51) we can also get the Lie brackets involving the new generator by which we extend the Lie algebra. Indeed, we see immediately that since

$$\widetilde{U}\Big((\theta,\mathbb{1})\Big)\widetilde{U}\Big((0,g)\Big) = \widetilde{U}\Big((\theta,g)\Big) = \widetilde{U}\Big((0,g)\Big)\widetilde{U}\Big((\theta,\mathbb{1})\Big) \tag{A.53}$$

then the new generator will commute with all the generators of the Galilei algebra. By explicit computation, it can be shown that the extension is not trivial, namely that the new generator appears in some Lie brackets. For the specific case of the Galilei group, there is only one Lie bracket that gets modified. Indeed we have that, through the exponential map

$$\begin{aligned} &U\big(0,\boldsymbol{a},\boldsymbol{0},\mathbb{1}\big)U\big(0,\boldsymbol{0},\boldsymbol{v},\mathbb{1}\big) - \big(0,\boldsymbol{0},\boldsymbol{v},\mathbb{1}\big)\big(0,\boldsymbol{a},\boldsymbol{0},\mathbb{1}\big) = e^{i\boldsymbol{a}\cdot U(\boldsymbol{\pi})}e^{i\boldsymbol{v}\cdot U(\boldsymbol{\beta})} - e^{i\boldsymbol{v}\cdot U(\boldsymbol{\beta})}e^{i\boldsymbol{a}\cdot U(\boldsymbol{\pi})} = \\ &= \Big(e^{\frac{i}{2}m\boldsymbol{a}\cdot\boldsymbol{v}} - e^{-\frac{i}{2}m\boldsymbol{a}\cdot\boldsymbol{v}}\Big)e^{i\,\boldsymbol{a}\cdot U(\boldsymbol{\pi})+i\boldsymbol{v}\cdot U(\boldsymbol{\beta})}\,, \end{aligned} \tag{A.54}$$

which yields

$$i\,a^i v^j\Big[U(\pi_i), U(\beta_j)\Big] \sim i\,m\,a^i v^j\delta_{ij}\,. \tag{A.55}$$

From this relation between the representatives of the generators, we can infer that

$$\Big[\pi_i,\beta_j\Big] = \mu\,\delta_{ij} \tag{A.56}$$

is the corresponding Lie bracket in the abstract Lie algebra, represented in (A.55) by means of $U(\mu) = m$. With similar calculations it can be shown that the other Lie brackets are not modified.

A.3.2 The little group method

We are now ready to construct the representations of the Galilei group [67, 348, 349]. To do so, we take into account (A.22) and that $\mathscr{T}_4$ is Abelian. In general, if we have a group $\mathscr{G}$ which can be written as[3] $\mathscr{G} = H \ltimes N$, where N is Abelian, the unitary irreducible representations of N will be 1-dimensional, since this is the case for all the Abelian groups, and they can be written as

$$N \ni n \mapsto e^{i\eta(n)} \in \mathbb{S}_1 , \tag{A.57}$$

with $\eta(n) \in \mathbb{R}$. In our case, N is also simply connected and, up to isomorphisms, $\mathbb{R}^n$ is the only Abelian simply connected group, thus N is a vector space. In this case we therefore have

$$N \ni n \mapsto e^{i\eta\cdot n} \in \mathbb{S}_1 , \tag{A.58}$$

with $\eta \in \hat{N}$, $\hat{N}$ being the dual vector space. Considering the natural action of H on $\hat{N}$ defined by

$$h(\eta)\cdot n = \eta \cdot h^{-1}(n) , \tag{A.59}$$

we can define the orbit of $\eta \ni \hat{N}$ as the set

$$\mathscr{O}_\eta := \left\{ h(\eta) \in \hat{N} \mid h \in H \right\} , \tag{A.60}$$

for a fixed η. Thus, $\hat{N}$ is divided into disjoint orbits $\mathscr{O}_\eta$ and to each orbit a Hilbert space is associated $\mathscr{H}_{\mathscr{O}_\eta}$ with scalar product

$$\langle \phi | \psi \rangle := \int_{\mathscr{O}_\eta} \mathrm{d}\mu(\eta)\, \phi^*(\eta)\psi(\eta) , \tag{A.61}$$

where $\mathrm{d}\mu(\eta)$ is a measure invariant under the action of H. Given $\eta \in \hat{N}$ the subgroup of H leaving η invariant is the little group of η, $\ell_\eta \subset H$. Since the points lying on a given orbit are all linked by an element of H, all the little groups corresponding to different points on the same orbit are all equivalent. Therefore we can state that

The unitary irreducible representations of $\mathscr{G} = H \ltimes N$, where N is Abelian, are obtained by choosing an orbit $\mathscr{O}$ of H in the dual group $\hat{N}$ and a unitary irreducible representation Δ of the little group ℓ_{η_0} associated to a reference point η_0 in the orbit $\mathscr{O}$. The Hilbert space associated to the representation $(\mathscr{O}, \Delta)$ is $\mathscr{H}_{\mathscr{O}} \otimes \mathscr{H}_\Delta$.

From (A.22) we see that $H = \mathscr{R} \ltimes \mathscr{B}$ and $N = \mathscr{T}_4$. The unitary irreducible representations of $\mathscr{T}_4$ are 1-dimensional and are labelled by the elements η of the dual group $\hat{N}$. To understand the physical meaning of these labels, we recall that in Quantum Mechanics we can decompose the state of a free particle in position $\boldsymbol{x}$ at time t as

$$|t, \boldsymbol{x}\rangle = \int \mathrm{d}\mu(E, \boldsymbol{p})\, e^{-iEt} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \left| E, \boldsymbol{p} \right\rangle , \tag{A.62}$$

where E is the energy, $\boldsymbol{p}$ is the momentum and $\mathrm{d}\mu(E, \boldsymbol{p}) = \mathrm{d}^3 p\, \mathrm{d}E\, \delta\left(E - \frac{\boldsymbol{p}^2}{2m} - u \right)$ enforces that the total energy is given by the kinetic energy energy plus the internal energy of the system u. Considering a spacetime translation to a new reference frame

$$\left| t', \boldsymbol{x}' \right\rangle = U(b, \boldsymbol{a}, \boldsymbol{0}, \mathbb{1})\, |t, \boldsymbol{x}\rangle = \int \mathrm{d}\mu(E, \boldsymbol{p})\, e^{-iEt} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} e^{-ibE} e^{i\boldsymbol{a}\cdot\boldsymbol{p}} \left| E, \boldsymbol{p} \right\rangle . \tag{A.63}$$

[3] The composition law for such a structure is given by $(n', h')(n, h) = (n' + h'(n), h'h)$, where $h'(n)$ denotes the homomorphism of H into the group of automorphism of N defining the semidirect product structure.

From this we see that

$$U(b,\boldsymbol{a},\boldsymbol{0},\mathbb{1})\left|E,\boldsymbol{p}\right\rangle = e^{-ibE}e^{i\boldsymbol{a}\cdot\boldsymbol{p}}\left|E,\boldsymbol{p}\right\rangle . \tag{A.64}$$

By these arguments and from (A.58) we understand that the elements η for the translation sector of the Galilei group are given by the energy and momentum $\eta \equiv (E,\boldsymbol{p})$, namely the irreducible representations of the spacetime translation sector are labelled by $(E,\boldsymbol{p})$. Moreover, from (A.62) we get the feeling that the invariant measure under the action of $\mathscr{R}\ltimes\mathscr{B}$ is $\mathrm{d}\mu(E,\boldsymbol{p})$, namely the orbits of elements $(E,\boldsymbol{p})$ under the action of the factor group are paraboloids $E-\frac{\boldsymbol{p}^2}{2m}=u$, parameterised by u. It will be shown that this is actually the case in the following.

The next step is to find the orbits of $(E,\boldsymbol{p})$ under the action of the factor group. Once we find the orbits, we will fix one of them, choose a reference point $(E_0,\boldsymbol{p}_0)$ and study the representation of the little group of this element. The total Hilbert space will then be given by the tensor product of the Hilbert space associated to the orbit and the Hilbert space of the representation of the little group. The basis vectors of this Hilbert space will be denoted with $\left|E,\boldsymbol{p};\alpha\right\rangle := \left|E,\boldsymbol{p}\right\rangle\otimes|\alpha\rangle$, where α collectively denotes the labels on which the representations of the elements of the little group will act.

To find the orbits of an element $(E,\boldsymbol{p})$ under the action of ${}^{\mathscr{G}}\!/_{\mathscr{T}_4}$, we observe that from (A.15) and (A.50) we get

$$\begin{aligned} U(b,\boldsymbol{a},\boldsymbol{0},\mathbb{1})U(0,\boldsymbol{0},\boldsymbol{v},R) &= e^{\frac{i}{2}m\boldsymbol{a}\cdot\boldsymbol{v}}U(b,\boldsymbol{a},\boldsymbol{v},R)\,,\\ U(0,\boldsymbol{0},\boldsymbol{v},R)U(b,R^{-1}\cdot\big(\boldsymbol{a}-b\,\boldsymbol{v}\big),\boldsymbol{0},\mathbb{1}) &= e^{-\frac{i}{2}m(\boldsymbol{a}-b\boldsymbol{v})\cdot\boldsymbol{v}}U(b,\boldsymbol{a},\boldsymbol{v},R)\,, \end{aligned} \tag{A.65}$$

namely

$$U(b,\boldsymbol{a},\boldsymbol{0},\mathbb{1})U(0,\boldsymbol{0},\boldsymbol{v},R) = e^{im\left(\boldsymbol{a}-\frac{1}{2}b\boldsymbol{v}\right)\cdot\boldsymbol{v}}U(0,\boldsymbol{0},\boldsymbol{v},R)U(b,R^{-1}\cdot\big(\boldsymbol{a}-b\,\boldsymbol{v}\big),\boldsymbol{0},\mathbb{1})\,. \tag{A.66}$$

This property is useful since $U(0,\boldsymbol{0},\boldsymbol{v},R)$ is exactly the generic element of the factor group ${}^{\mathscr{G}}\!/_{\mathscr{T}_4}$, therefore this equation lets us understand how this group acts on the representations of $\mathscr{T}_4$. Specifically, setting $\left|E,\boldsymbol{p};\alpha\right\rangle_{\boldsymbol{v},R} := U(0,\boldsymbol{0},\boldsymbol{v},R)\left|E,\boldsymbol{p};\alpha\right\rangle$ we get

$$\begin{aligned} U(b,\boldsymbol{a},\boldsymbol{0},\mathbb{1})\left|E,\boldsymbol{p};\alpha\right\rangle_{\boldsymbol{v},R} &= e^{im\left(\boldsymbol{a}-\frac{1}{2}b\boldsymbol{v}\right)\cdot\boldsymbol{v}}e^{-ibE+i\boldsymbol{p}\cdot R^{-1}\cdot(\boldsymbol{a}-b\boldsymbol{v})}\left|E,\boldsymbol{p};\alpha\right\rangle_{\boldsymbol{v},R} =\\ &= e^{i\boldsymbol{a}\cdot(R\cdot\boldsymbol{p}+m\boldsymbol{v})-ib\left(E+\frac{1}{2}m\boldsymbol{v}^2+\boldsymbol{v}\cdot R\cdot\boldsymbol{p}\right)}\left|E,\boldsymbol{p};\alpha\right\rangle_{\boldsymbol{v},R}\,, \end{aligned} \tag{A.67}$$

where we have used the property $\boldsymbol{p}\cdot R^{-1}\cdot\big(a-b\boldsymbol{v}\big) = p_i(R^{-1})^i{}_j(a^j-b\,v^j) = (a^j-b\,v^j)R_j{}^i p_i = (\boldsymbol{a}-b\,\boldsymbol{v})\cdot R\cdot\boldsymbol{p}$. This means that $U(0,\boldsymbol{0},\boldsymbol{v},R)\left|E,\boldsymbol{p};\alpha\right\rangle$ transforms according to the representation $(E',\boldsymbol{p}')$, namely

$$\left|E,\boldsymbol{p};\alpha\right\rangle_{\boldsymbol{v},R} \propto \left|E',\boldsymbol{p}';\alpha\right\rangle , \tag{A.68}$$

where

$$\begin{aligned} \boldsymbol{p}' &= R\cdot\boldsymbol{p}+m\,\boldsymbol{v}\\ E' &= E+\boldsymbol{v}\cdot R\cdot\boldsymbol{p}+\frac{1}{2}m\,\boldsymbol{v}^2\,. \end{aligned} \tag{A.69}$$

This means that if a representation of the Galilei group contains a physical representation $(E,\boldsymbol{p})$ of the translations group, it contains all the representations $(E',\boldsymbol{p}')$ given by (A.69).

With this we can find the orbit of the element $(E,\boldsymbol{p})$ under the action of ${}^{\mathscr{G}}\!/_{\mathscr{T}_4}$. Indeed, from (A.69) we immediately get that the two elements $(E,\boldsymbol{p})$ and $(E',\boldsymbol{p}')$ are connected by the relation

$$E' - \frac{\boldsymbol{p}'^2}{2m} = E - \frac{\boldsymbol{p}^2}{2m}\,. \tag{A.70}$$

This means that, as we correctly guessed before from (A.62), the orbit of an element $(E,\boldsymbol{p})$ is given by the paraboloid

$$E - \frac{\boldsymbol{p}^2}{2m} = u\,. \tag{A.71}$$

Each point of this paraboloid corresponds to an irreducible representations of the translations group contained in a physical representation of the Galilei group. If $\mathscr{H}(E,\boldsymbol{p})$ is the Hilbert space of representation $(E,\boldsymbol{p})$, we can write the Hilbert space associated to the orbit of $(E,\boldsymbol{p})$, $\mathscr{H}_{m,u}$ as the direct integral

$$\mathscr{H}_{m,u} = \oint \mathrm{d}\mu(E,\boldsymbol{p})\,\mathscr{H}(E,\boldsymbol{p}) \equiv \oint \mathrm{d}^3p\,\mathrm{d}E\,\delta\left(E - \frac{\boldsymbol{p}^2}{2m} - u\right)\mathscr{H}(E,\boldsymbol{p}) \tag{A.72}$$

The little group

We now have to study the little group of a given element $(E,\boldsymbol{p})$ to complete the construction of the representations. The little group of such an element is defined as the subgroup of ${}^{\mathscr{G}}\!/_{\mathscr{T}_4}$ made up of transformations $(0,\boldsymbol{0},\boldsymbol{v},R)$ such that

$$\boldsymbol{p} = R\cdot\boldsymbol{p} + m\,\boldsymbol{v} \tag{A.73}$$

$$E = E + \boldsymbol{v}\cdot R\cdot\boldsymbol{p} + \frac{1}{2}m\,\boldsymbol{v}^2\,, \tag{A.74}$$

namely $(E,\boldsymbol{p})$ is left invariant. It can be seen that the two conditions are not independent since (A.73) implies (A.74). This means that an element of the little group is specified by a rotation, since (A.73) uniquely defines the velocity $\boldsymbol{v}$ after the choice of a rotation. This correspondence is actually an isomorphism between $\ell_{E,\boldsymbol{p}}$ and $\mathscr{R}$, since

$$\boldsymbol{p} = R\cdot\boldsymbol{p} + m\,\boldsymbol{v}\,,\;\; \boldsymbol{p} = R'\cdot\boldsymbol{p} + m\,\boldsymbol{v}'\,, \tag{A.75}$$

imply

$$\boldsymbol{p} = R'R\cdot\boldsymbol{p} + m\big(R'\cdot\boldsymbol{v} + \boldsymbol{v}'\big)\,, \tag{A.76}$$

therefore the product of two elements of the little group $\ell_{E,\boldsymbol{p}}$ corresponds to the product of two rotations $R'R$. This shows that the irreducible representations of the little group are already known: they are the irreducible representations of the rotation group labelled by an integer (or half-integer) number s, whose action is described by $2s+1$-dimensional matrices, D^s. This means that the basis states $\left|E,\boldsymbol{p}\right\rangle$ are $2s+1$ degenerate states labelled by an index ranging from $-s$ to s, which we collectively denoted as α. Therefore, the basis states of the total Hilbert space $\mathscr{H}^s_{m,u} := \mathscr{H}_{m,u}\otimes\mathscr{V}^s$ are

$$\left|E,\boldsymbol{p};\alpha\right\rangle = \left|E,\boldsymbol{p}\right\rangle \otimes \Big\{ |s,s\rangle\,, |s,s-1\rangle\,, \dots\,, |s,-s+1\rangle\,, |s,-s\rangle \Big\} \tag{A.77}$$

meaning that any state $|\psi\rangle \in \mathscr{H}^s_{m,u}$ can be decomposed as

$$|\psi\rangle = \sum_{n=-s}^{s} \int d\mu(E,\boldsymbol{p})\, \psi_{s,n}(E,\boldsymbol{p})\, |E,\boldsymbol{p}\rangle \otimes |s,n\rangle \ , \tag{A.78}$$

where

$$\psi_{s,n}(E,\boldsymbol{p}) := \Big(\langle E,\boldsymbol{p}| \otimes \langle s,n|\Big)|\psi\rangle \ . \tag{A.79}$$

We now have to show that choosing a paraboloid and a representation $\mathscr{V}^s$ determines a representation of the whole Galilei group.

We first fix a paraboloid $E - \frac{\boldsymbol{p}^2}{2m} = u$ and a reference point on the paraboloid $(E_0, \boldsymbol{p}_0)$. All other points can be obtained as $V_{E,\boldsymbol{p}} = (0, \boldsymbol{0}, \boldsymbol{v}_{E,\boldsymbol{p}}, E_{E,\boldsymbol{p}}) \in {}^{\mathscr{G}}\!/_{\mathscr{T}_4}$ so that $V_{E,\boldsymbol{p}}(E_0,\boldsymbol{p}_0) = (E,\boldsymbol{p})$ by means of (A.69). Considering another point $(E',\boldsymbol{p}')$, we will have the same property with another element $V_{E',\boldsymbol{p}'}$ and from (A.69) there exits an element V such that $V(E,\boldsymbol{p}) = (E',\boldsymbol{p}')$. Hence

$$V\, V_{E,\boldsymbol{p}}(E_0,\boldsymbol{p}_0) = V_{E',\boldsymbol{p}'}(E_0,\boldsymbol{p}_0) \ . \tag{A.80}$$

From this we see that

$$V_0^{(E,\boldsymbol{p};V)} := V^{-1}_{E',\boldsymbol{p}'}\, V\, V_{E,\boldsymbol{p}} \in \ell_{E_0,\boldsymbol{p}_0} \tag{A.81}$$

and, conversely, that any $V \in {}^{\mathscr{G}}\!/_{\mathscr{T}_4}$ can be written as $V = V_{E',\boldsymbol{p}'} V_0^{(E,\boldsymbol{p};V)} V^{-1}_{E,\boldsymbol{p}}$. We now choose to define $U(V_{E,\boldsymbol{p}})$ such that the proportionality constant in (A.68) is 1, so that

$$U(V_{E,\boldsymbol{p}})\, |E_0,\boldsymbol{p}_0\rangle = |E,\boldsymbol{p}\rangle \ . \tag{A.82}$$

For $U(V_0)$ we have that

$$U(V_0)\,|s,n\rangle = \sum_{h=-s}^{s} \Big[D^s(V_0)\Big]_n^{\ h} |s,h\rangle \ , \tag{A.83}$$

thus for any $V \in {}^{\mathscr{G}}\!/_{\mathscr{T}_4}$ we have that

$$\begin{aligned} U(V)\Big(|E,\boldsymbol{p}\rangle \otimes |s,n\rangle\Big) &= U(V_{E',\boldsymbol{p}'})U\Big(V_0^{(E,\boldsymbol{p};V)}\Big)U(V_{E,\boldsymbol{p}})^{-1}\Big(|E,\boldsymbol{p}\rangle \otimes |s,n\rangle\Big) = \\ &= U(V_{E',\boldsymbol{p}'})U\Big(V_0^{(E,\boldsymbol{p};V)}\Big)\Big(|E_0,\boldsymbol{p}_0\rangle \otimes |s,n\rangle\Big) = \\ &= |E',\boldsymbol{p}'\rangle \otimes \sum_{h=-s}^{s} \Big[D^s\Big(V_0^{(E,\boldsymbol{p};V)}\Big)\Big]_n^{\ h} |s,h\rangle \ . \end{aligned} \tag{A.84}$$

From (A.15)and (A.23) we see that

$$(b,\boldsymbol{a},\boldsymbol{v},R) = (b,\boldsymbol{a},\boldsymbol{0},\mathbb{1})(0,\boldsymbol{0},\boldsymbol{v},R) \ , \tag{A.85}$$

thus from (A.50) we get

$$U(b,\boldsymbol{a},\boldsymbol{v},R) = e^{-\frac{i}{2} m \boldsymbol{a}\cdot\boldsymbol{v}}\, U(b,\boldsymbol{a},\boldsymbol{0},\mathbb{1})U(0,\boldsymbol{0},\boldsymbol{v},R) \ . \tag{A.86}$$

Hence, the action of the representative of a generic element of the Galilei group on the basis states $|E,\boldsymbol{p}\rangle \otimes |s,n\rangle$ yields

$$U(b,\boldsymbol{a},\boldsymbol{v},R)\Big(|E,\boldsymbol{p}\rangle \otimes |s,n\rangle\Big) = e^{-\frac{i}{2} m \boldsymbol{a}\cdot\boldsymbol{v} + i\,\boldsymbol{a}\cdot\boldsymbol{p}' - i\, b E'}\, |E',\boldsymbol{p}'\rangle \otimes \sum_{h=-s}^{s} \Big[D^s\Big(V_0^{(E,\boldsymbol{p};\boldsymbol{v},R)}\Big)\Big]_n^{\ h} |s,h\rangle \ , \tag{A.87}$$

with definitions (A.69) and (A.81), with $V=(0,\mathbf{0},\boldsymbol{v},R)$. With $\langle E',\boldsymbol{p}'|E,\boldsymbol{p}\rangle=\delta(E-E')\delta^{(3)}(\boldsymbol{p}-\boldsymbol{p}')$ and $\langle s,l|s,n\rangle=\delta_{l,n}$, from (A.78) the scalar product on the total Hilbert space is found to be

$$\langle\phi|\psi\rangle=\sum_{n=-s}^{s}\int \mathrm{d}\mu(E,\boldsymbol{p})\,\phi^*_{s,n}(E,\boldsymbol{p})\psi_{s,n}(E,\boldsymbol{p})\,, \tag{A.88}$$

meaning that the functional space of the representation is the space of square integrable functions on the paraboloid $E-\frac{\boldsymbol{p}^2}{2m}=u$. Therefore, the physical representations of the Galilei group, denoted by $[m\mid u,s]$, are labelled by two real numbers m,u and a half-integer or integer number s. These are regarded as particles with mass m, internal energy u, and spin s. To understand this correspondence, a more physical discussion is in order.

A.3.3 Galilean invariance of the Schrödinger equation

In position space, the wave function of a particle with mass m is the Schrödinger equation

$$i\,\partial_t\psi(\boldsymbol{x},t)=-\frac{1}{2m}\Delta\psi(\boldsymbol{x},t)+u\,\psi(\boldsymbol{x},t)\,, \tag{A.89}$$

where, in the free particle case, u is a constant. Now considering a (passive) Galilean transformation we have that the same point in space(time) that is described by coordinates $(\boldsymbol{x},t)$ in the transformed frame S' has coordinates $(\boldsymbol{x}',t')$ in the frame S, according to (A.13). The wave function in the S' is indicated with ψ' and if the physical predictions of the two wave fuctions have to be the same we get that they can differ at most by a phase

$$\psi'(\boldsymbol{x},t)=e^{-i\,f(\boldsymbol{x}',t')}\psi(\boldsymbol{x}',t')\,, \tag{A.90}$$

and both wave functions should satisfy the Schrödinger equation with respect to their coordinates

$$i\,\partial_t\psi'(\boldsymbol{x},t)=-\frac{1}{2m}\Delta\psi'(\boldsymbol{x},t)+u\,\psi'(\boldsymbol{x},t)\,, \tag{A.91}$$

$$i\,\partial_{t'}\psi(\boldsymbol{x}',t')=-\frac{1}{2m}\Delta'\psi(\boldsymbol{x}',t')+u\,\psi(\boldsymbol{x}',t')\,, \tag{A.92}$$

where

$$\partial_t=\frac{\partial t'}{\partial t}\partial_{t'}+\frac{\partial\boldsymbol{x}'}{\partial t}\cdot\boldsymbol{\nabla}'=\partial_{t'}+\boldsymbol{v}\cdot\boldsymbol{\nabla}'\,, \tag{A.93}$$

$$\partial_i=\frac{\partial x^{j'}}{\partial x^i}\partial'_j=R_i{}^j\partial'_j\Rightarrow\boldsymbol{\nabla}=R\cdot\boldsymbol{\nabla}'\,. \tag{A.94}$$

Therefore, from (A.91) we get

$$\left[i\,\partial_{t'}+i\boldsymbol{v}\cdot\boldsymbol{\nabla}'+\frac{1}{2m}\Delta'-u\right]e^{-i\,f(\boldsymbol{x}',t')}\psi(\boldsymbol{x}',t')=0\,, \tag{A.95}$$

and enforcing (A.92) we end up with the system of equations

$$\begin{cases}\boldsymbol{\nabla}'f-m\,\boldsymbol{v}=\mathbf{0}\\[2ex] \partial_{t'}f+\boldsymbol{v}\cdot\boldsymbol{\nabla}'f-\frac{1}{2m}\boldsymbol{\nabla}'f\cdot\boldsymbol{\nabla}'f-\frac{i}{2m}\Delta'f=0\end{cases} \tag{A.96}$$

whose solution is

$$f = m\boldsymbol{v}\cdot\boldsymbol{x}' - \frac{1}{2}m\boldsymbol{v}^2 t' + c\,, \tag{A.97}$$

where c is a constant. Thus we have

$$\psi'(\boldsymbol{x},t) = e^{-i\,m\,\boldsymbol{v}\cdot\boldsymbol{x}'+\frac{i}{2}m\,\boldsymbol{v}^2 t'-i\,c}\,\psi(\boldsymbol{x}',t')\,. \tag{A.98}$$

Via a Fourier transform we get in momentum space

$$\psi(E,\boldsymbol{p}) = \int \mathrm{d}^3x\,\mathrm{d}t\,e^{-i\,\boldsymbol{p}\cdot\boldsymbol{x}+i\,E\,t}\,\psi(\boldsymbol{x},t) \tag{A.99}$$

and the Schrödinger equation becomes

$$\left[E-\frac{\boldsymbol{p}^2}{2m}-u\right]\psi(E,\boldsymbol{p}) = 0\,, \tag{A.100}$$

namely the support of $\psi(E,\boldsymbol{p})$ is on the paraboloid $E-\frac{\boldsymbol{p}^2}{2m}=u$. If we now consider a Galilean transformation we get

$$\begin{aligned}\psi'(E,\boldsymbol{p}) &= \int \mathrm{d}^3x\,\mathrm{d}t\,e^{-i\,\boldsymbol{p}\cdot\boldsymbol{x}+i\,E\,t}\,\psi'(\boldsymbol{x},t) = \\ &= \int \mathrm{d}^3x'\,\mathrm{d}t'\,e^{-i\,\boldsymbol{p}\cdot R^{-1}\cdot(\boldsymbol{x}'-\boldsymbol{v}\,t'+\boldsymbol{v}\,b-\boldsymbol{a}+i\,E(t'-b))}\,e^{-i\,m\,\boldsymbol{v}\cdot\boldsymbol{x}'+\frac{i}{2}m\,\boldsymbol{v}^2\,t'-i\,c}\,\psi(\boldsymbol{x}',t')\,.\end{aligned} \tag{A.101}$$

By choosing $c=\frac{1}{2}m\,b\,\boldsymbol{v}^2-\frac{1}{2}m\,\boldsymbol{v}\cdot\boldsymbol{a}$, after some algebra we end up with

$$\psi'(E,\boldsymbol{p}) = e^{-\frac{i}{2}\,m\,\boldsymbol{v}\cdot\boldsymbol{a}+i\,\boldsymbol{a}\cdot\boldsymbol{p}'-i\,b\,E'}\,\psi(E',\boldsymbol{p}')\,, \tag{A.102}$$

with E' and $\boldsymbol{p}'$ defined in (A.69). We see that this is equivalent to (A.87). Indeed from (A.87) we have

$$\begin{aligned}\psi'(E,\boldsymbol{p}) &= \langle E,\boldsymbol{p}|U|\psi\rangle = \langle\psi|U^\dagger|E,\boldsymbol{p}\rangle^* = \langle\psi|E',\boldsymbol{p}'\rangle^*\,e^{-\frac{i}{2}\,m\,\boldsymbol{v}\cdot\boldsymbol{a}+i\,\boldsymbol{a}\cdot\boldsymbol{p}'-i\,b\,E'} = \\ &= e^{-\frac{i}{2}\,m\,\boldsymbol{v}\cdot\boldsymbol{a}+i\,\boldsymbol{a}\cdot\boldsymbol{p}'-i\,b\,E'}\,\psi(E',\boldsymbol{p}')\,.\end{aligned} \tag{A.103}$$

From these arguments we see that the solution of the Schrödinger equation for a free spinless particle of mass m and internal energy u belongs to the representation $[m\,|\,u,0]$ of the Galilei group. Of course, any other choice of the constant c would have led us to a different, but equivalent[4], projective representation.

A.3.4 Spin and internal energy

We now discuss the physical meaning of u and s. Starting with u, taking a basis state of the representation $[m\,|\,u,s]$, we define an operator $\mathfrak{V}_u$ such that [5]

$$\mathfrak{V}_u\,|E,\boldsymbol{p}\rangle := |E+u,\boldsymbol{p}\rangle\,. \tag{A.104}$$

[4]Two unitary operators U and U' are said to be equivalent if there exists a third unitary operator such that $U'=V\,U\,V^{-1}$. For operator rays, the notion of equivalence is somewhat generalised. Indeed, two operator rays are said to be equivalent if $[U']=[V][U][V^{-1}]$ which means, for the operators in the ray, $U'=e^{i\,\phi}V\,U\,V^{-1}$ with $\phi\in\mathbb{R}$.

[5]In this equation and in all the subsequent discussions we will neglect the spin factors since they play no role as far as the internal energy is concerned.

We call $\mathscr{H}_u$ the Hilbert space of functions $\psi(E,\boldsymbol{p})$ with scalar product

$$\langle\phi|\psi\rangle = \int \mathrm{d}^3 p\,\mathrm{d}E\,\delta\left(E-\frac{\boldsymbol{p}^2}{2m}-u\right)\phi^*(E,\boldsymbol{p})\psi(E,\boldsymbol{p})\,. \tag{A.105}$$

If we define $|\widetilde{\psi}\rangle := \mathfrak{V}_u|\psi\rangle$ we have that

$$\widetilde{\psi}(E,\boldsymbol{p}) = \langle E,\boldsymbol{p}|\mathfrak{V}_u|\psi\rangle = \langle\psi|\mathfrak{V}_u^\dagger|E,\boldsymbol{p}\rangle^* = \psi(E+u,\boldsymbol{p})\,, \tag{A.106}$$

therefore, if $|\psi\rangle\in\mathscr{H}_u$, $|\widetilde{\psi}\rangle$ will be a state in $\mathscr{H}_0$. This means that

$$\begin{aligned}\left\langle\widetilde{\phi}\middle|\widetilde{\psi}\right\rangle_{\mathscr{H}_0} &= \int \mathrm{d}^3 p\,\mathrm{d}E\,\delta\left(E-\frac{\boldsymbol{p}^2}{2m}\right)\widetilde{\phi}^*(E,\boldsymbol{p})\widetilde{\psi}(E,\boldsymbol{p}) = \\ &= \int \mathrm{d}^3 p\,\mathrm{d}E'\,\delta\left(E'-\frac{\boldsymbol{p}^2}{2m}-u\right)\widetilde{\phi}^*(E',\boldsymbol{p})\widetilde{\psi}(E',\boldsymbol{p}) \equiv \langle\phi|\psi\rangle_{\mathscr{H}_u}\,.\end{aligned} \tag{A.107}$$

This means that $\mathfrak{V}_u$ is an isomorphism between $\mathscr{H}_u$ and $\mathscr{H}_0$. Moreover, we can easily check that the operator $U_u(b,\boldsymbol{a},\boldsymbol{v},R)$ in the $[m\mid u,s]$ representation and the operator $U_0(b,\boldsymbol{a},\boldsymbol{v},R)$ in the $[m\mid 0,s]$ representation are equivalent. To show this, we observe that (A.87) implies

$$\mathfrak{V}_u U_0(b,\boldsymbol{a},\boldsymbol{v},R)\mathfrak{V}_u^{-1}|E,\boldsymbol{p}\rangle_{\mathscr{H}_u} = e^{i\,b\,u}U_u(b,\boldsymbol{a},\boldsymbol{v},R)|E,\boldsymbol{p}\rangle_{\mathscr{H}_u}\,, \tag{A.108}$$

thus we conclude that $U_u \equiv e^{i\,b\,u}\mathfrak{V}_u U_0\mathfrak{V}_u^{-1}$, namely that the two representations $[m\mid u,s]$ and $[m\mid 0,s]$ are equivalent. This means that for an isolated free particle the internal energy u has no physical meaning. This reflects the fact that for an isolated particle we can choose the origin of energy arbitrarily.

Moving on to the label s, we want now to interpret it as the spin of the particle. To do so, we focus on the little group of a given point $(E_0,\boldsymbol{p}_0)$ of the paraboloid $E-\frac{\boldsymbol{p}^2}{2m}=u$. In particular, we choose $(E_0,\boldsymbol{p}_0)=(u,\boldsymbol{0})$. Any other point on the paraboloid can be obtained from this by means of

$$V_{E,\boldsymbol{p}} = \left(0,\boldsymbol{0},m^{-1}\boldsymbol{p},\mathbb{1}\right). \tag{A.109}$$

With this choice, we see that for any

$$V=(0,\boldsymbol{0},\boldsymbol{v},R)\,,\;\;(E',\boldsymbol{p}\prime) = \left(E+\boldsymbol{v}\cdot R\cdot\boldsymbol{p}+\frac{1}{2}m\boldsymbol{p}^2, R\cdot\boldsymbol{p}+m\boldsymbol{v}\right) \tag{A.110}$$

we obtain for $V_0^{(E,\boldsymbol{p};\boldsymbol{v},R)}$ in (A.87) that

$$\begin{aligned}V_0^{(E,\boldsymbol{p};\boldsymbol{v},R)} &= \left(0,\boldsymbol{0},-m^{-1}\boldsymbol{p}',\mathbb{1}\right)(0,\boldsymbol{0},\boldsymbol{v},R)\left(0,\boldsymbol{0},m^{-1}\boldsymbol{p},\mathbb{1}\right) = \\ &= \left(0,\boldsymbol{0},-m^{-1}R\cdot\boldsymbol{p},R\right)\left(0,\boldsymbol{0},m^{-1}\boldsymbol{p},\mathbb{1}\right) = \left(0,\boldsymbol{0},\boldsymbol{0},R\right).\end{aligned} \tag{A.111}$$

Therefore we can write (A.87) in the simpler form

$$U(b,\boldsymbol{a},\boldsymbol{v},R)\Big(|E,\boldsymbol{p}\rangle\otimes|s,n\rangle\Big) = e^{-\frac{i}{2}m\boldsymbol{a}\cdot\boldsymbol{v}+i\boldsymbol{a}\cdot\boldsymbol{p}'-i\,b\,E'}|E',\boldsymbol{p}'\rangle\otimes\sum_{h=-s}^{s}\left[D^s(R)\right]_n^{\;h}|s,h\rangle\,, \tag{A.112}$$

where relations (A.69) hold. We see that with this choice the label s truly characterises the rotational properties of the particle, *i.e.* it represents its spin. This is possible since the choice (A.109) corresponds to a change of reference frame to the rest frame of the particle.

A.4 The representations of the Galilei Lie algebra

The last step to conclude the discussion of the representations of the Galilei group is to derive the infinitesimal generators of the group, *i.e.* its Lie algebra. To do so, we will consider the transformation properties of wave functions in the momentum space under the action of the Galilei group. In general, we have that

$$\begin{aligned}\psi'_{s,n}(E,\boldsymbol{p}) &= \Big(\langle E,\boldsymbol{p}\big|\otimes\langle s,n|\Big)U(g)\,|\psi\rangle = \\ &= e^{-\frac{i}{2}m\boldsymbol{a}\cdot\boldsymbol{v}+i\,\boldsymbol{a}\cdot\boldsymbol{p}'-i\,bE'}\sum_{h=-s}^{s}\Big[D^s(R)\Big]_n{}^h\Big[\langle\psi\big|\big(\big|E',\boldsymbol{p}\prime\rangle\otimes\big|s,h\rangle\big)\Big]^*,\end{aligned} \tag{A.113}$$

namely

$$\psi'_{s,n}(E,\boldsymbol{p}) = U(g)\psi_{s,n}(E,\boldsymbol{p}) = e^{-\frac{i}{2}m\boldsymbol{a}\cdot\boldsymbol{v}+i\,\boldsymbol{a}\cdot\boldsymbol{p}'-i\,bE'}\sum_{h=-s}^{s}\Big[D^s(R)\Big]_n{}^h\psi_{s,h}(E',\boldsymbol{p}')\,, \tag{A.114}$$

where $g=(b,\boldsymbol{a},\boldsymbol{v},R)$ and $U(g)=\exp\Big\{i\big(b\,T+\boldsymbol{a}\cdot\boldsymbol{P}+\boldsymbol{v}\cdot\boldsymbol{B}+\boldsymbol{\omega}\cdot\boldsymbol{J}\big)\Big\}$, with T, $\boldsymbol{P}$, $\boldsymbol{B}$ and $\boldsymbol{J}$ being the generators of time translations, spatial translations, Galilean boosts and spatial rotations respectively. In the following, the parameters b, $\boldsymbol{a}$, $\boldsymbol{v}$ and $\boldsymbol{\omega}$ will be infinitesimal.
For spatial translations we have

$$\begin{aligned}&U(0,\boldsymbol{a},\boldsymbol{0},\,\mathbb{1})\psi_{s,n}(E,\boldsymbol{p}) = e^{i\,\boldsymbol{a}\cdot\boldsymbol{P}}\psi_{s,n}(E,\boldsymbol{p}) \sim \big(\mathbb{1}+i\,\boldsymbol{a}\cdot\boldsymbol{P}\big)\psi_{s,n}(E,\boldsymbol{p})\\ &\parallel\\ &e^{i\,\boldsymbol{a}\cdot\boldsymbol{p}}\psi_{s,n}(E,\boldsymbol{p}) \sim (1+i\,\boldsymbol{a}\cdot\boldsymbol{p})\psi_{s,n}(E,\boldsymbol{p})\,,\end{aligned} \tag{A.115}$$

therefore

$$P_i \equiv p_i\,. \tag{A.116}$$

Galilean boosts yields

$$\begin{aligned}&U(0,\boldsymbol{0},\boldsymbol{v},\,\mathbb{1})\psi_{s,n}(E,\boldsymbol{p}) = e^{i\,\boldsymbol{v}\cdot\boldsymbol{B}}\psi_{s,n}(E,\boldsymbol{p}) \sim \big(\mathbb{1}+i\,\boldsymbol{v}\cdot\boldsymbol{B}\big)\psi_{s,n}(E,\boldsymbol{p})\\ &\parallel\\ &\psi_{s,n}(E',\boldsymbol{p}') \sim \psi_{s,n}(E,\boldsymbol{p})+m\,\boldsymbol{v}\cdot\boldsymbol{\nabla}_p\psi_{s,n}+\left(\frac{1}{2}m\,\boldsymbol{v}^2+\boldsymbol{v}\cdot\boldsymbol{p}\right)\partial_E\psi_{s,n}\,,\end{aligned} \tag{A.117}$$

hence[6]

$$\boldsymbol{B} = -i\,m\boldsymbol{\nabla}_p - i\,\boldsymbol{p}\,\partial_E\,. \tag{A.118}$$

Time translations are treated in a similar way

$$\begin{aligned}&U(b,\boldsymbol{0},\boldsymbol{0},\,\mathbb{1})\psi_{s,n}(E,\boldsymbol{p}) = e^{i\,bT}\psi_{s,n}(E,\boldsymbol{p}) \sim \big(\mathbb{1}+i\,b\,T\big)\psi_{s,n}(E,\boldsymbol{p})\\ &\parallel\\ &e^{-i\,bE}\psi_{s,n}(E,\boldsymbol{p}) \sim (1-i\,bE)\psi_{s,n}(E,\boldsymbol{p})\,,\end{aligned} \tag{A.119}$$

which implies

$$T = -E \tag{A.120}$$

[6] $\boldsymbol{v}^2 \sim 0$.

Finally, we have to deal with rotations. In general we see that (A.114) implies that the infinitesimal generators should act on the spin variables in a non-trivial way. In particular, we have a linear combination of components with different spin labels. Therefore, to be more precise, the previous generators (along with the generated group elements) are matrices that act in a trivial way on the spin labels, namely

$$\begin{aligned}
\boldsymbol{P}_n{}^h &= \boldsymbol{p}\,\delta_n{}^h \,, \\
\boldsymbol{B}_n{}^h &= -\delta_n{}^h\Big(i\,m\boldsymbol{\nabla}_p - i\,\boldsymbol{p}\,\partial_E\Big)\,, \\
T_n{}^h &= -E\,\delta_n{}^h \,.
\end{aligned} \tag{A.121}$$

This is left implicit to simplify notation. However, when it comes to rotations, this is not the case anymore, since rotations act in a non-trivial way on the spin variables. In this case we have

$$\begin{aligned}
&U(0,\boldsymbol{0},\boldsymbol{0},R)\psi_{s,n}(E,\boldsymbol{p}) = e^{i\boldsymbol{\omega}\cdot\boldsymbol{J}}\psi_{s,n}(E,\boldsymbol{p}) \sim \big(\mathbb{1} + i\boldsymbol{\omega}\cdot\boldsymbol{J}\big)\psi_{s,n}(E,\boldsymbol{p}) \\
&\parallel \\
&\sum_{h=-s}^{s}\Big[D^s(R)\Big]_n{}^h\psi_{s,n}(E,\boldsymbol{p}') \sim \sum_{h=-s}^{s}\Big[\delta_n{}^h + i\boldsymbol{\omega}\cdot\boldsymbol{S}_n{}^h\Big]\psi_{s,n}(E,\boldsymbol{p}+\omega\wedge\boldsymbol{p}) \sim \\
&\sim \sum_{h=-s}^{s}\Big[\delta_n{}^h + i\boldsymbol{\omega}\cdot\boldsymbol{S}_n{}^h\Big]\Big\{\psi_{s,n}(E,\boldsymbol{p}) + \boldsymbol{\omega}\cdot\Big(\boldsymbol{p}\wedge\boldsymbol{\nabla}_p\Big)\psi_{s,n}(E,\boldsymbol{p})\Big\}\,,
\end{aligned} \tag{A.122}$$

thus we conclude that

$$\boldsymbol{J}_n{}^h = -i\,\delta_n{}^h\,\boldsymbol{p}\wedge\boldsymbol{\nabla}_p + \boldsymbol{S}_n{}^h\,, \tag{A.123}$$

where $D(R) = e^{i\boldsymbol{\omega}\cdot\boldsymbol{S}}$. With this we obtain the standard splitting of the total angular momentum into its orbital and intrinsic components. Of course the operator $\boldsymbol{S}$ acts on different variables with respect to all the differential operators that appear in this representation. Therefore $\boldsymbol{S}$ commutes with every combination of $\boldsymbol{p}$ and E and their derivatives. This, alongside the fact we should have a representation of the algebra (A.30), leads us to the commutation relation

$$\Big[S_i, S_j\Big] = i\,\epsilon_{ij}{}^k S_k \tag{A.124}$$

From this, by direct computation we easily get the algebra

$$\begin{aligned}
&[T,P_i] = [T,J_i] = 0 \;\;,\;\; \Big[B_i,B_j\Big] = \Big[P_i,P_j\Big] = 0 \;\;,\;\; \Big[B_i,P_j\Big] = i\,\mu\,\delta_{i,j} \;\;,\;\; [B_i,T] = i\,P_i\,, \\
&\Big[J_i,B_j\Big] = i\,\epsilon_{ij}{}^k B_k \;\;,\;\; \Big[J_i,P_j\Big] = i\,\epsilon_{ij}{}^k P_k \;\;,\;\; [J_n,J_m] = i\,\epsilon_{nm}{}^k J_k\,,
\end{aligned} \tag{A.125}$$

which is indeed a representation of (A.30). The only difference between the two algebras is in the commutator between boosts and spatial translations. Here we get a representation of the central extension of the Galilei algebra where the central charge is given by the mass $\mu \equiv -m$.

A.4.1 Casimirs of the algebra

The algebra (A.125) admits three Casimirs. The first one is trivial, indeed, the element μ trivially commutes with all the other generators, therefore it is a Casimir of the algebra. For the second Casimir we know that $\boldsymbol{S}$ commutes with all generators but $\boldsymbol{J}$, for which we have $\Big[J_i, S_j\Big] = \epsilon_{ij}{}^k S_k$. Therefore, $\boldsymbol{S}^2$ commutes with all generators and is in fact the second Casimir.

Finally, we know that the quantity $2mE - \boldsymbol{p}^2 = 2mu$ is invariant under boosts and rotations. In terms of generators, we have $2\mu T - \boldsymbol{P}^2$, and this quantity trivially commutes with the generators of time and spatial translations. Therefore the three Casimirs of the algebra are

$$\begin{aligned} &2\mu T - \boldsymbol{P}^2 = 2mu\,, \\ &(\mu \boldsymbol{J} + \boldsymbol{P}\wedge\boldsymbol{B})^2 \equiv m^2\boldsymbol{S}^2 = m^2 s(s+1)\,, \\ &\mu = -m\,. \end{aligned} \tag{A.126}$$

We see that the characterisation of the physical representations in terms of the three labels m, u and s is recovered.

A.4.2 Mass superselection rule

The fact that we have a representation of the central extension of the Galilei group yields an important physical consequence. Consider a state in the representation $[m_1 \mid u, s] \oplus [m_2 \mid u, s]$, which can be written as $|\psi\rangle = |\psi_1\rangle \oplus |\psi_2\rangle$, with $|\psi_k\rangle \in [m_k \mid u, s]$ and $m_1 \neq m_2$. We now consider the following succession of transformations: a spatial translation, a Galilean boost, the inverse translation, and the inverse boost. The overall transformation is the identity, of course, as can be immediately seen from (A.15) (or from (A.30) since the generators commute). This means that when we represent this transformation on the Hilbert space of a quantum system, it should be given by at most a phase. When we represent the generators on the Hilbert space, the generators do not commute anymore and we have, from the Baker–Campbell–Hausdorff formula[7],

$$e^{-i\boldsymbol{v}\cdot\boldsymbol{B}}e^{-i\boldsymbol{a}\cdot\boldsymbol{P}}e^{i\boldsymbol{v}\cdot\boldsymbol{B}}e^{i\boldsymbol{a}\cdot\boldsymbol{P}} = e^{-i\boldsymbol{v}\cdot\boldsymbol{B}-i\boldsymbol{a}\cdot\boldsymbol{P}-\frac{1}{2}v^i a^j\left[B_i,P_j\right]}e^{i\boldsymbol{v}\cdot\boldsymbol{B}+i\boldsymbol{a}\cdot\boldsymbol{P}-\frac{1}{2}v^i a^j\left[B_i,P_j\right]} = e^{-i\mu\boldsymbol{a}\cdot\boldsymbol{v}}\,. \tag{A.127}$$

Alternatively, this could be derived directly from (A.50). From this we see that if we act with our transformation, which is basically the identity, on the superposition $|\psi\rangle$ we end up with

$$|\psi\rangle \mapsto |\psi'\rangle = e^{i m_1 \boldsymbol{a}\cdot\boldsymbol{v}}|\psi_1\rangle \oplus e^{i m_2 \boldsymbol{a}\cdot\boldsymbol{v}}|\psi_2\rangle\,, \tag{A.128}$$

meaning that the state is changed after an identical transformation. This led Bargmann [66] to formulate a superselection rule for the mass, the *Bargmann superselection rule*, which does not allow coherent superpositions of different masses in a Galilean-invariant Quantum Theory. However, as we discussed in chapter 1 and chapter 4, this superselection rule is not justified since this effect is just a remnant of special-relativistic features that survives in the non-relativistic limit. ù

A.5 Tensor product of two representations

Until now we have described how the Hilbert space of a single free particle is described in terms of group-theoretic arguments. A many-particle system is, of course, described by means

[7]Given a Lie group G and the exponential map from the Lie algebra to the Lie group $e: L \to G$, the Baker-Campbell-Hausdorff formula gives the solution to the equation $e^X e^Y = e^Z$ for $X, Y \in L$. Specifically, $Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\big[X,[X,Y]\big] - \frac{1}{12}\big[Y,[X,Y]\big] + \ldots$.

of tensor products of the Hilbert spaces of the single particles, *i.e.* the states of such a system transform according to the tensor product of the single-particle representations. To describe such a system, we have to introduce some collective observables describing its transformation properties and some other collective observables describing its internal structure, which does not depend on the choice of reference frame. To do so, the tensor product representation should be decomposed into a direct sum or irreducible representations.

We will focus on a two-particle state [350] that transforms according to the representation $[m_1 \mid u_1, s_1] \otimes [m_2 \mid u_2, s_2]$. A basis state in the Hilbert space $\mathscr{H}^{s_1}_{m_1,u_1} \otimes \mathscr{H}^{s_2}_{m_2,u_2}$ is given by the tensor products $\left|E_1, \boldsymbol{p}_1\,; E_2, \boldsymbol{p}_2\right\rangle \otimes \left|s_1, n_1\,; s_2, n_2\right\rangle$ on which the representation acts as

$$U(g)\Big(\left|E_1, \boldsymbol{p}_1 \;\; E_2, \boldsymbol{p}_2\right\rangle \otimes \left|s_1, n_1\,; s_2, n_2\right\rangle\Big) = e^{-\frac{i}{2} m\, \boldsymbol{a}\cdot\boldsymbol{v} + i\, \boldsymbol{a}\cdot\left(\boldsymbol{p}'_1 + \boldsymbol{p}'_2\right) - i\, b\left(E'_1 + E'_2\right)} \left|E'_1, \boldsymbol{p}'_2\,; E'_2, \boldsymbol{p}'_1\right\rangle \otimes$$
$$\otimes \textstyle\sum_{h_1=-s_1}^{s_1} \sum_{h_2=-s_2}^{s_2} \left[D^{s_1}(R)\right]_{n_1}{}^{h_1} \left[D^{s_2}(R)\right]_{n_2}{}^{h_2} \left|s_1, h_1\,; s_2, h_2\right\rangle\,, \tag{A.129}$$

where $\boldsymbol{p}'_i$ and E'_i are defined by (A.69). The scalar product on this Hilbert space is simply given by

$$\langle \phi | \psi \rangle = \textstyle\sum_{n_1=-s_1}^{s_1} \sum_{n_2=-s_2}^{s_2} \int \mathrm{d}^3 p_1\, \mathrm{d}E_1\, \mathrm{d}^3 p_2\, \mathrm{d}E_2\, \delta\left(E_1 - \frac{\boldsymbol{p}_1^2}{2m_1} - u_1\right) \delta\left(E_2 - \frac{\boldsymbol{p}_2^2}{2m_2} - u_2\right) \times$$
$$\times \phi^*_{s_1,n_1;s_2,n_2}(E_1, \boldsymbol{p}_1; E_2, \boldsymbol{p}_2)\, \psi_{s_1,n_1;s_2,n_2}(E_1, \boldsymbol{p}_1; E_2, \boldsymbol{p}_2)\,. \tag{A.130}$$

To decompose this Hilbert space into a direct sum of invariant subspaces, we first define the collective observables

$$\boldsymbol{Q} := \boldsymbol{p}_1 + \boldsymbol{p}_2\;,\;\; E = E_1 + E_2\,, \tag{A.131}$$

namely the total momentum and total energy of the two particle system. We next introduce the relative momentum

$$\boldsymbol{q} := \frac{m_1\, m_2}{m_1 + m_2}(\boldsymbol{v}_1 - \boldsymbol{v}_2) = \frac{m_2\, \boldsymbol{p}_1 - m_1\, \boldsymbol{p}_2}{m_1 + m_2}\,, \tag{A.132}$$

and we introduce a similar variable for the energy

$$\epsilon := \frac{m_2\, E_1 - m_1\, E_2}{m_1 + m_2}\,. \tag{A.133}$$

From these relations we have the change of variables

$$E_i = (-1)^{i-1}\epsilon + \frac{m_i}{M} E\;,\;\; \boldsymbol{p}_i = (-1)^{i-1}\boldsymbol{q} + \frac{m_i}{M}\boldsymbol{Q}\,, \tag{A.134}$$

whose Jacobian is 1. We introduced the total mass $M = m_1 + m_2$ which, together with the reduced mass $m = \frac{m_1 m_2}{M}$, will be useful for simplifying the equations. In addition to these variables, we can also define the total internal energy as the sum of the internal energies and the kinetic energy of the relative motion

$$u = u_1 + u_2 + \frac{\boldsymbol{q}^2}{2M} \equiv E - \frac{\boldsymbol{Q}^2}{2M}\,. \tag{A.135}$$

This allows us to write the variables relative to the two particles (including the internal energies) in terms of the collective observables E, ϵ, u, $\boldsymbol{q}$, and $\boldsymbol{Q}$. However, in doing so, we start with 8 variables and end up with 9 new ones. In order to keep the number of degrees of freedom unchanged, we need to constrain one of the new variables with the introduction of a new integration with a relative Dirac delta. In rewriting the Dirac deltas we have

$$\begin{aligned} E_1 - \frac{\boldsymbol{p}_1^2}{2m_1} - u_1 &= \epsilon + \frac{m_1}{M}E - \frac{\boldsymbol{p}_1^2}{2m_1} - u_1 = \epsilon + \frac{m_1}{M}u_1 + \frac{m_1}{M}u_2 - u_1 + \frac{\boldsymbol{p}_1^2}{2m_1} + \frac{m_1}{M}\frac{\boldsymbol{p}_2^2}{2m_2} \\ &= \epsilon - \frac{\boldsymbol{Q}\cdot\boldsymbol{q}}{M} - \frac{m_2 - m_1}{2m_1 m_2}\boldsymbol{q}^2 - \frac{m_2 u_1 - m_1 u_2}{M} \end{aligned} \tag{A.136}$$

and

$$E_2 - \frac{\boldsymbol{p}_2^2}{2m_2} - u_2 = u - E_1 + \frac{\boldsymbol{Q}^2}{2M} - \frac{\boldsymbol{p}_2^2}{2m_1} - u_2 = u - u_1 - u_2 - \frac{\boldsymbol{q}^2}{2m}\,, \tag{A.137}$$

where we have used the relation $u = E - \frac{Q^2}{2M}$ which will become the constraint to enforce with a Dirac delta in a new integration in the variable u, which will run from the minimal value $u_1 + u_2$ to $+\infty$. Therefore, the scalar product becomes

$$\begin{aligned} \langle\phi|\psi\rangle = &\sum_{n_1=-s_1}^{s_1}\sum_{n_2=-s_2}^{s_2}\int_{u_1+u_2}^{+\infty} \mathrm{d}u\,\delta\left(E - \frac{\boldsymbol{Q}^2}{2M} - u\right)\int \mathrm{d}^3P\,\mathrm{d}E\,\mathrm{d}^3q\,\mathrm{d}\epsilon\,\delta\left(u - u_1 - u_2 - \frac{\boldsymbol{q}^2}{2m}\right)\times \\ &\times\delta\left(\epsilon - \frac{\boldsymbol{Q}\cdot\boldsymbol{q}}{M} - \frac{m_2 - m_1}{2m_1 m_2}\boldsymbol{q}^2 - \frac{m_2 u_1 - m_1 u_2}{M}\right)\phi^*_{s_1,n_1;s_2,n_2}(E,\boldsymbol{Q};\epsilon,\boldsymbol{q})\psi_{s_1,n_1;s_2,n_2}(E,\boldsymbol{Q};\epsilon,\boldsymbol{q})\,. \end{aligned} \tag{A.138}$$

From this we see that the total Hilbert space $\mathscr{H}^{s_1}_{m_1,u_1}\otimes\mathscr{H}^{s_2}_{m_2,u_2}$ is written as a direct integral of Hilbert spaces $\mathscr{H}_u$

$$\mathscr{H}^{s_1}_{m_1,u_1}\otimes\mathscr{H}^{s_2}_{m_2,u_2} = \int^{\oplus\,\infty}_{u_1+u_2}\mathrm{d}u\,\mathscr{H}_u\,, \tag{A.139}$$

each associated to a paraboloid

$$E - \frac{\boldsymbol{Q}^2}{2M} = u\,. \tag{A.140}$$

From section A.3.2 we know that $\mathscr{H}_u$ will be written as a direct product of an Hilbert space that fixes a representation of the translation subgroup and the Hilbert space defining the representation of the little group of a given point on the paraboloid. From (A.140) we see that the first of these two Hilbert spaces is given by $\mathscr{H}_{M,u}$. For the second component of the tensor product, we have to consider the subspaces in $\mathscr{H}_u$ invariant under the action of the little group, which once again can be identified with the rotation group. We see from (A.138) and the change of variables that, when a point on the paraboloid $(E,\boldsymbol{Q})$ is fixed, the wave functions depend only on the vector $\boldsymbol{q}$, which has a constant norm when a paraboloid is chosen, $\boldsymbol{q}^2 = 2m(u - u_1 - u_2)$. Hence, once a point on the paraboloid is chosen, wave functions are functions on the unit sphere and can be expanded in spherical harmonics of the unit vector $\hat{\boldsymbol{q}} := |\boldsymbol{q}|^{-1}\boldsymbol{q}$, namely

$$\psi_{s_1,n_1;s_2,n_2}(E,\boldsymbol{Q};\epsilon,\boldsymbol{q}) = \sum_{\lambda=0}^{\infty}\sum_{\mu=-\lambda}^{\lambda}\psi^{\lambda,\mu}_{s_1,n_1;s_2,n_2}(E,\boldsymbol{Q})Y_{\lambda,\mu}(\hat{\boldsymbol{q}})\,, \tag{A.141}$$

where $Y_{\lambda,\mu}(\hat{\boldsymbol{q}})$ are the spherical harmonics, eigenfunctions of the operators

$$\boldsymbol{\ell}^2 = \left(-i\,\boldsymbol{q}\wedge\nabla_q\right)^2,\ \ \ell_z = -i\,q_x\nabla_{q_y}\,, \tag{A.142}$$

with eigenvalues $\lambda(\lambda+1)$ and μ respectively, where $\lambda \in \mathbb{N}_0$ and $\mu \in [-\lambda, \lambda] \cap \mathbb{Z}$. We see that

$$\boldsymbol{L}_1 + \boldsymbol{L}_2 - \boldsymbol{L} = -i\,\boldsymbol{p}_1 \wedge \boldsymbol{\nabla}_{p_1} - i\,\boldsymbol{p}_2 \wedge \boldsymbol{\nabla}_{p_2} + i\,\boldsymbol{Q} \wedge \boldsymbol{\nabla}_Q \equiv \boldsymbol{\ell}\,, \tag{A.143}$$

thus $\boldsymbol{\ell}$ is really the relative orbital angular momentum. From these arguments we understand that rotations (*i.e.* the little group) will act on the labels n_1, n_2 and μ and will be represented by

$$D^{s_1} \otimes D^{s_2} \otimes \left(\bigoplus_{\lambda=0}^{\infty} D^{\lambda} \right). \tag{A.144}$$

The Hilbert space on which the representation will act is $\bigoplus_{\lambda=0}^{\infty} \left(\mathscr{V}^{s_1} \otimes \mathscr{V}^{s_2} \otimes \mathscr{V}^{\lambda}\right)$. A basis of this Hilbert space is given by

$$\bigoplus_{\lambda=0}^{\infty} \left\{ \left|s_1, n_1\right\rangle \otimes \left|s_2, n_2\right\rangle \otimes \left|\lambda, \mu\right\rangle \mid n_i \in [-s_i, s_i] \cap \left(\mathbb{Z} \cup \frac{\mathbb{Z}}{2}\right),\, \mu \in [-\lambda, \lambda] \cap \mathbb{Z} \right\}. \tag{A.145}$$

This tensor product of representations can be decomposed in a direct sum of irreducible representation $\mathscr{V}^J$ by means of a standard addition of angular momenta

$$\bigoplus_{\lambda=0}^{\infty} \left(\mathscr{V}^{s_1} \otimes \mathscr{V}^{s_2} \otimes \mathscr{V}^{\lambda}\right) = \bigoplus_{\lambda=0}^{\infty} \bigoplus_{s=|s_1-s_2|}^{s_1+s_2} \left(\mathscr{V}^{s} \otimes \mathscr{V}^{\lambda}\right) = \bigoplus_{\lambda=0}^{\infty} \bigoplus_{s=|s_1-s_2|}^{s_1+s_2} \bigoplus_{J=|\lambda-s|}^{\lambda+s} \mathscr{V}^J\,, \tag{A.146}$$

a basis of this Hilbert space being

$$\bigoplus_{\lambda=0}^{\infty} \bigoplus_{s=|s_1-s_2|}^{s_1+s_2} \bigoplus_{J=|\lambda-s|}^{\lambda+s} \left\{ |J, \Lambda\rangle \mid \Lambda \in [-J, J] \cap \left(\mathbb{Z} \cup \frac{\mathbb{Z}}{2}\right) \right\}. \tag{A.147}$$

This is the representation of the little group of a given point on the paraboloid, therefore we have now that $\mathscr{H}_u = \mathscr{H}_{M,u} \otimes \left(\bigoplus_{\lambda=0}^{\infty} \bigoplus_{s=|s_1-s_2|}^{s_1+s_2} \bigoplus_{J=|\lambda-s|}^{\lambda+s} \mathscr{V}^J\right)$, hence

$$\mathscr{H}^{s_1}_{m_1,u_1} \otimes \mathscr{H}^{s_2}_{m_2,u_2} = \int^{\oplus\,\infty}_{u_1+u_2} \mathrm{d}u \left(\bigoplus_{\lambda=0}^{\infty} \bigoplus_{s=|s_1-s_2|}^{s_1+s_2} \bigoplus_{J=|\lambda-s|}^{\lambda+s} \mathscr{H}_{M,u} \otimes \mathscr{V}^J \right), \tag{A.148}$$

where $\mathscr{H}_{M,u} \otimes \mathscr{V}^J$ is the Hilbert space of a particle of internal energy u, mass M and total angular momentum J, namely it is the Hilbert space associated to a representation $[M \mid u, J]$ of the Galilei group, hence the previous direct integral can be written in terms of representations as

$$[m_1 \mid u_1, s_1] \otimes [m_2 \mid u_2, s_2] = \int^{\oplus\,\infty}_{u_1+u_2} \mathrm{d}u \left(\bigoplus_{\lambda=0}^{\infty} \bigoplus_{s=|s_1-s_2|}^{s_1+s_2} \bigoplus_{J=|\lambda-s|}^{\lambda+s} [m_1 + m_2 \mid u, J] \right). \tag{A.149}$$

A final remark on this decomposition is that the internal energy u is no longer an arbitrary constant. For a single particle, we observed that the internal energy is an arbitrary constant since there is an equivalence between the two representations $[m \mid u_1, s]$ and $[m \mid u_2, s]$, namely the operators $U_{u_1}(b,\boldsymbol{a},\boldsymbol{v},R)$ and $U_{u_2}(b,\boldsymbol{a},\boldsymbol{v},R)$ of the representations differ by a phase $e^{i\,b(u_1-u_2)}$. We indicate this equivalence condition as

$$[m \mid u_1, s] \approx [m \mid u_2, s] \,. \tag{A.150}$$

If we consider, for instance, the direct sum of two representations

$$[m \mid u_1, s_1] \oplus [m \mid u_2, s_2] \,, \tag{A.151}$$

then we can still define the internal energies up to a *common* constant, but we cannot redefine them arbitrarily. For instance we have that

$$[m \mid u_1, s_1] \approx [m \mid 0, s_1] \ , \ [m \mid u_2, s_2] \approx [m \mid 0, s_2] \,, \tag{A.152}$$

but

$$[m \mid u_1, s_1] \oplus [m \mid u_2, s_2] \approx [m \mid 0, s_1] \oplus [m \mid u_2 - u_1, s_2] \not\approx [m \mid 0, s_1] \oplus [m \mid 0, s_2] \,. \tag{A.153}$$

Therefore, in (A.149), which is a direct integral of the representations $[M \mid u,\lambda]$ *i.e.* a direct sum in disguise, we can still choose an arbitrary zero value for the internal energies, but we cannot arbitrarily choose the values of u in the direct integral for all the representations $[M \mid u,\lambda]$. This is also because now the internal energy of the compound system has information about the kinetic energy of the relative motion, which is a physical information, since $u = u_1 + u_2 + \frac{\boldsymbol{q}^2}{2m}$. We can, for instance, scale the internal energy with an equivalence by an amount $u_1 + u_2$, such that the direct integral is now from 0 to ∞, making the internal energy of the compound system a direct measure of the kinetic energy of the relative motion $u = \frac{\boldsymbol{q}^2}{2m}$.

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