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Qualitative and geometric aspects of elliptic equations with Dirichlet or Robin boundary conditions

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Chapter 1

Introduction

One of the earliest problems in mathematics was *the isoperimetric problem*, which was considered by the ancient Greeks. The problem is to find, among all closed curves of a given length, the one which encloses the maximum area. This is one of the most famous shape optimization problems, where the optimal solution is represented by the circle. These problems have been much studied in mathematics, hence we recall some of these. Saint Venant in [102], conjectured that the ideal shape of a beam's cross-section should maximize its torsional rigidity, that is the capacity to withstand torsional stress. This conjecture was ultimately solved by Pólya in 1948, who proved that the bar exhibiting maximum torsional rigidity features a circular cross-section. From a mathematical point of view, if we let Ω represent the cross-section of the bar, the torsional rigidity can be defined as the L^1-norm of the unique solution in $H_0^1(\Omega)$ to the following equation:

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Denoting this solution as u_Ω, known as the *torsion function*, then *the torsional rigidity*, is expressed as:

$$T(\Omega) = \int_\Omega u_\Omega \, dx.$$

Hence, the Saint-Venant inequality asserts that:

$$T(\Omega) \leq T(B), \tag{1.0.1}$$

where B represents any open ball in $\mathbb{R}^N$, with the same volume of Ω.

Another significant problem originating in the late 19th century is to identify the planar membrane, fixed along its boundary, that possesses the lowest principal frequency. In his renowned work, *The Theory of Sound* [101], Lord Rayleigh conjectured that, among all planar figures with a fixed area, the disk minimizes the principal frequency. This assertion can be reformulated by stating that the disk minimizes the first Dirichlet eigenvalue $\lambda_1(\Omega)$ of the Laplace operator, that is the smallest λ for which the following problem:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a nontrivial solution in $H_0^1(\Omega)$. This conjecture was proved 50 years later in two simultaneous yet independent contributions, one by Faber [56] and the other by Krahn [84] which proved the following inequality

$$\lambda_1(\Omega) \geq \lambda_1(B), \tag{1.0.2}$$

with B being any open ball in $\mathbb{R}^N$, which has the same volume of Ω. We refer the reader to the book by Pólya and Szegő [100] for further developments.
From this point of view, in this thesis work, we are concerned with studying qualitative and geometric aspects of some elliptic equations with Dirichlet or Robin boundary conditions. To facilitate the reader, in what follows we will describe the main points of the various results we obtained.

Comparison results in quantitative form for Dirichlet boundary value problems

Investigating isoperimetric inequalities needs an integration of analysis, geometry, and the theory of partial differential equations. A crucial method for tackling isoperimetric problems is *Schwarz symmetrization*, also known as the spherical symmetric and decreasing rearrangement of a function. In essence, the Schwarz symmetrization of a measurable function involves rearranging the superlevel sets of the function into a spherical configuration while preserving all L^p norms.

The introduction of symmetrization techniques, particularly in relation to the qualitative properties of solutions to second-order elliptic boundary value problems, was pioneered by Talenti. In Chapter 3 we study symmetrization techniques in the context of qualitative properties of solutions to second-order elliptic boundary value problems which have been widely studied in the last decades. In the seminal paper [109], Talenti considers an open and bounded set $\Omega \subset \mathbb{R}^N$ and the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.0.3}$$

where $f \in L^{\frac{2N}{N+2}}(\Omega)$ if $N > 2$, $f \in L^p(\Omega)$, $p > 1$ if $N = 2$. He proved that it is possible to compare the solution to (1.0.3) with the solution to the symmetrized problem

$$\begin{cases} -\Delta v = f^\sharp & \text{in } \Omega^\sharp \\ v = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{1.0.4}$$

where $\Omega^\sharp$ is the ball centered in the origin with the same measure as Ω and $f^\sharp$ is the Schwarz rearrangement of f (see Definition 2.3.4). More precisely, the following pointwise comparison holds

$$u^\sharp(x) \leq v(x) \quad \forall x \in \Omega^\sharp. \tag{1.0.5}$$

As a matter of fact, this result holds also for the solutions of a more general uniformly elliptic linear operator in divergence form. The technique employed by Talenti is a very powerful tool, as it adapts well to solving various types of variational problems. An application of the Talenti's result is Saint Venant's conjecture, which is obtained immediately integrating the inequality (1.0.5).

Among researchers, Talenti's type result has gained more and more interest. A version of the aforementioned result for nonlinear operators in divergence form is contained in [111], which includes the $p-$Laplace operator as a special case; see also [22]. Further extensions can be found, for instance, in [5] for anisotropic elliptic operators, in [7] for the parabolic case, in [16, 110] for higher-order operators, and in [9], where the Steiner symmetrization is used in place of the Schwarz one. In the recent paper [8], the authors consider problem (1.0.3) replacing the Dirichlet boundary conditions with the Robin one and they prove a comparison result involving Lorentz norms of u and v whenever f is a non-negative function in $L^2(\Omega)$. In particular, in the case $f \equiv 1$ and $N = 2$, they recover the point-wise inequality as in [109]. Generalizations of the results contained in [8] can be found for the anisotropic case in [103, 12], for mixed boundary conditions in [2], in the case of the Hermite operator in [36] and in the nonlinear case [13]. For further generalizations and general overviews of comparison results, we refer, for instance, to [3, 37, 40, 92, 80, 112, 113] and to the references therein, not claiming to give here a comprehensive list of all the results developed in this context.

Another interesting question about the topic is trying to understand if comparison inequalities are rigid, as it has been done in [6]. Indeed, the authors characterize the equality in (1.0.5) in the case $f \geq 0$, proving that if equality occurs, then Ω is a ball, u is radially symmetric and decreasing, and $f = f^\sharp$. Rigidity results for a generic linear, elliptic second-order operator can be found in [59] and [79] and for Robin boundary conditions in [90, 91].

Once a comparison result and a rigidity result hold, it is natural to ask whether the comparison result can be improved in a quantitative version. More precisely, considering the rigidity result in [6], if equality *almost* holds in (1.0.5), is it true that the set Ω is *almost* a ball, and the function u and f are *almost* radially symmetric and decreasing? The answer to this question is contained in Chapter 3, which is the main objective of the paper [11]. What we do is to study the stability of (1.0.5) and to answer the following question: if $u^\sharp$ is *close* to v in some sense, can we say that the set Ω and the functions u and f are *almost* symmetric? One of the difficulties we have to deal with is to give the most suitable notions of "closeness" for this type of problem.

More precisely, we state our main result, contained in [11], in the following way.

Theorem 1.0.1. *Let Ω be a bounded open set of $\mathbb{R}^N$ and let f be a non-negative function in $L^2(\Omega)$. Suppose that u and v are the solutions to (1.0.3) and (1.0.4), respectively. Then, there exist some positive constants $\theta_1 = \theta_1(N), \theta_2 = \theta_2(N)$ and $C_1 := C_1(N, |\Omega|, f^\sharp)$, $C_2 := C_2(N, |\Omega|, f^\sharp)$, $C_3 := C_3(N, |\Omega|, f^\sharp)$, such that*

$$C_1\alpha^3(\Omega) + C_2 \inf_{x_0\in\mathbb{R}^N} \|u - u^\sharp(\cdot + x_0)\|^{\theta_1}_{L^1(\mathbb{R}^N)} + C_3 \inf_{x_0\in\mathbb{R}^N} \|f - f^\sharp(\cdot + x_0)\|^{\theta_2}_{L^1(\mathbb{R}^N)} \leq ||v - u^\sharp||_{L^\infty(\Omega)}. \quad (1.0.6)$$

Moreover, the dependence of C_1, C_2 and C_3 from $|\Omega|$ and $f^\sharp$ is explicit.

In order to prove Theorem 1.0.1, we show that the quantity $||v - u^\sharp||_\infty$ bounds from above each term on the left-hand side: this is the aim of Theorems 1.0.2, 1.0.3 and 1.0.4.

The first step is contained in Theorem 1.0.2, which can be read in this sense: if the L^∞ distance between $u^\sharp$ and v is small, then the set Ω has a small Fraenkel asymmetry index, defined as

$$\alpha(\Omega) := \min_{x\in\mathbb{R}^N} \left\{ \frac{|\Omega \Delta B_r(x)|}{|B_r(x)|} \ , \ |B_r(x)| = |\Omega| \right\}, \quad (1.0.7)$$

where the symbol Δ stands for the symmetric difference and $|\Omega|$ is the measure of the set. This is a standard way to measure how much a set Ω differs from a ball B_r with the same measure in the L^1 norm (see Figure 1).

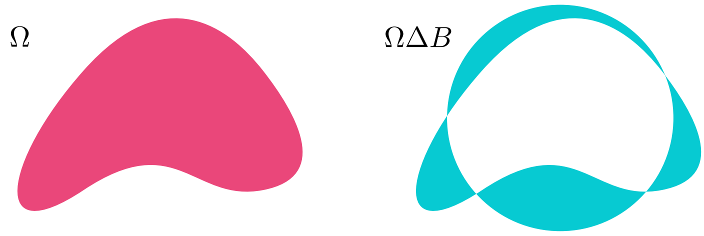

Figure 1.1: Fraenkel asymmetry of Ω

Theorem 1.0.2 (Control of the asymmetry). *Let $f \in L^{\frac{2N}{N+2}}(\Omega)$ if $N > 2$, $f \in L^p(\Omega)$, $p > 1$ if $N = 2$ be a non-negative function and let u and v be the solutions to (1.0.3) and (1.0.4) respectively, and let $u^\sharp$ be the Schwarz rearrangement of u. Then, there exists a constant $\widetilde{C}_1 = \widetilde{C}_1(N) > 0$, such that*

$$\alpha^3(\Omega) \leq \widetilde{C}_1 \frac{\|v - u^\sharp\|_\infty}{|\Omega|^{\frac{2-N}{N}} \|f\|_1}, \tag{1.0.8}$$

where

$$\widetilde{C}_1 = \max\left\{ \left(8N^2 \omega_N^{\frac{2}{N}}\right)^3, (16N^2 \omega_N^{\frac{2}{N}} \gamma_n) \right\}, \tag{1.0.9}$$

ω_N is the measure of the unit ball of $\mathbb{R}^N$ and γ_n is the constant appearing in the quantitative isoperimetric inequality (see Theorem 2.3.10).

To prove this result we make use of a clever idea introduced in [75], used for obtaining non-sharp quantitative isoperimetric inequalities for the principal frequency and the capacity. This idea is also explained and exploited in the survey paper [25] to prove a quantitative version of the Pólya-Szegő principle (see Theorem 2.3.7). Moreover, we observe that Theorem 1.0.2 implies the quantitative result contained in [82] in the case $f \equiv 1$.

In Theorem 3.2.2, we obtain a similar bound for the Fraenkel asymmetry of almost any superlevel set $U_t = \{u > t\}$ of the function u solution to (1.0.3).

The second step for proving Theorem 1.0.1 is contained in Theorem 1.0.3. We prove that the $L^1(\mathbb{R}^N)$ distance between u and $u^\sharp$ is small, assuming that $u^\sharp$ is close to v with respect to the L^∞-distance.

Theorem 1.0.3 (Estimate on the radiality of the solution). *Let $f \in L^{\frac{2N}{N+2}}(\Omega)$ if $N > 2$, $f \in L^p(\Omega)$, $p > 1$ if $N = 2$ be a non-negative function and let u and v be the solution to (1.0.3) and (1.0.4) respectively. Then, there exist two constants $\tilde{\theta}_1 = \tilde{\theta}_1(N) > 0$ and $\widetilde{C}_2 = \widetilde{C}_2(N) > 0$ such that*

$$\inf_{x_0 \in \mathbb{R}^N} ||u - u^\sharp(\cdot + x_0)||_{L^1(\mathbb{R}^N)} \leq \widetilde{C}_2 \frac{||f||_{\frac{2N}{N+2}}^{1-2\tilde{\theta}_1}}{||f||_1^{1-3\tilde{\theta}_1}} |\Omega|^{1+(\frac{2}{N}-1)\tilde{\theta}_1} ||v - u^\sharp||_{L^\infty(\Omega)}^{\tilde{\theta}_1}. \tag{1.0.10}$$

In order to prove this result, we observe in Lemma 3.3.1 that a small L^∞ distance between $u^\sharp$ and v forces the Pólya-Szegő inequality, that is

$$\int_{\Omega^\sharp} |\nabla u^\sharp|^2 \, dx \leq \int_\Omega |\nabla u|^2 \, dx,$$

to hold almost as an equality. So, at this stage, it is natural to consider the Pólya-Szegő quantitative result proved in [38]:

$$\inf_{x_0\in\mathbb{R}^N}\int_{\mathbb{R}^N}|u(x)-u^\sharp(x+x_0)|\,dx \le C\,[M_{u^\sharp}(E(u)^r)+E(u)]^s$$

where

$$E(u)=\frac{\displaystyle\int_{\mathbb{R}^N}|\nabla u|}{\displaystyle\int_{\mathbb{R}^N}|\nabla u^\sharp|}-1 \text{ and } M_{u^\sharp}(\delta)=\frac{\left|\left\{|\nabla u^\sharp|\le\delta\right\}\cap\left\{0<u^\sharp<\|u\|_\infty\right\}\right|}{|\{|u|>0\}|}.$$

Hence, to prove the closeness of u to $u^\sharp$ it remains to bound from above these quantities in terms of $||v-u||_\infty$.

Eventually, to conclude the proof of Theorem 1.0.1, we show the closeness of f and $f^\sharp$, that is the third and last step, contained in Theorem 1.0.4. Here, we prove that it is possible to bound the L^m distance between f and $f^\sharp$, for all $m<2$, provided $f\in L^2$.

Theorem 1.0.4 (Estimate on the radiality of the source term). *Let $f\in L^2(\Omega)$ be a non-negative function and let u, v be the solutions to (1.0.3) and (1.0.4) respectively. Then, for all $1\le m<2$ there exist $\tilde\theta_2=\tilde\theta_2(N,m)>0$, $\tilde\theta_3=\tilde\theta_3(N,m)>0$ and a positive constant $\widetilde C_3=\widetilde C_3(N,m)>0$ such that*

$$\inf_{x_0\in\mathbb{R}^N}\|f-f^\sharp(\cdot+x_0)\|_{L^m(\mathbb{R}^N)}\le \widetilde C_3\|f\|_2^{\tilde\theta_2}\|f\|_1^{1-\tilde\theta_3-\tilde\theta_2}|\Omega|^{\frac{\tilde\theta_2}{2}+\frac{1-m}{m}+(\frac{N-2}{N})\tilde\theta_3}||v-u^\sharp||_{L^\infty(\Omega)}^{\tilde\theta_3}.$$

We prove this result using the quantitative version of the Hardy-Littlewood inequality (see (2.3.2)) proved in [39].

Nonlinear eigenvalues problems with Robin boundary conditions

In the context of the shape optimization problem, a great interest is posed in the eigenvalues problems. In this context, we obtained some results on nonlinear problems with Robin boundary conditions. Among the nonlinear operators we consider, it is worth mentioning the p-Laplace operator, defined as $\Delta_p u=\mathrm{div}(|\nabla u|^{p-2}\nabla u)$. Examples of such problems are the following. The Dirichlet problem,

$$\begin{cases}-\Delta_p u=\lambda|u|^{p-2}u & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

the Neumann problem

$$\begin{cases}-\Delta_p u=\lambda|u|^{p-2}u & \text{in } \Omega\\ \dfrac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega\end{cases}$$

and the Robin problem,

$$\begin{cases}-\Delta_p u=\lambda|u|^{p-2}u & \text{in } \Omega\\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial\nu}+\beta|u|^{p-2}u=0 & \text{on } \partial\Omega,\end{cases}$$

with $\Omega\subset\mathbb{R}^N$ an open bounded set, sufficiently smooth, ν is the outer unit normal and β is a real parameter.

We recall some basic facts on the first Robin eigenvalue for the p-Laplace operator. First, there exists a principal eigenvalue λ_1, defined as

$$\lambda_p(\beta,\Omega)=\inf_{\varphi\in W^{1,p}(\Omega)}\frac{\int_\Omega|\nabla\varphi|^p\,dx+\beta\int_{\partial\Omega}|\varphi|^p\,d\mathcal{H}^{N-1}}{\int_\Omega|\varphi|^p\,dx}. \tag{1.0.11}$$

If Ω is connected, $\lambda_p(\beta,\Omega)$ is simple and isolated and the corresponding eigenfunctions minimize (1.0.11). Furthermore, the eigenfunctions are of constant sign. Moreover for $\beta>0$ we have that $\lambda_p(\beta,\Omega)>0$ and for $\beta<0$, $\lambda_p(\beta,\Omega)<0$. When $\beta=\infty$, we recover the Dirichlet case

$$\lambda_p^D(\Omega)=\inf_{\varphi\in W^{1,p}(\Omega)}\frac{\int_\Omega|\nabla\varphi|^p\,dx}{\int_\Omega|\varphi|^p\,dx},$$

while for $\beta=0$ we have $\lambda_p(\Omega)=0$. Moreover, the first nontrivial Neumann eigenvalue is defined as

$$\lambda_p^N(\Omega)=\inf_{\substack{\varphi\in W^{1,p}(\Omega)\setminus\{0\}\\ \int_\Omega|\varphi|^{p-2}\varphi\,dx=0}}\frac{\int_\Omega|\nabla\varphi|^p\,dx}{\int_\Omega|\varphi|^p\,dx}.$$

In Chapter 4 we discuss the results obtained in [19], in which we consider the Robin eigenvalue problem in the anisotropic case and we concern our attention on the behavior of the first eigenvalue when p goes to one. Hence, let Ω be a bounded, connected, sufficiently smooth open set, $p>1$ and $\beta\in\mathbb{R}$. In this Section, we study the Γ-convergence, as $p\to1^+$, of the functional

$$J_p(\varphi)=\frac{\displaystyle\int_\Omega F^p(\nabla\varphi)dx+\beta\int_{\partial\Omega}|\varphi|^pF(\nu)d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega|\varphi|^pdx}$$

where $\varphi\in W^{1,p}(\Omega)\setminus\{0\}$ and F is a sufficiently smooth norm on $\mathbb{R}^n$. We study the limit of the first eigenvalue $\lambda_p(\beta,\Omega)=\inf_{\substack{\varphi\in W^{1,p}(\Omega)\\ \varphi\neq0}}J_p(\varphi)$, as $p\to1^+$, that is:

$$\Lambda(\Omega,\beta)=\inf_{\substack{\varphi\in BV(\Omega)\\ \varphi\not\equiv0}}\frac{\displaystyle|Du|_F(\Omega)+\min\{\beta,1\}\int_{\partial\Omega}|\varphi|F(\nu)d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega|\varphi|dx}.$$

Furthermore, for $\beta>-1$, we obtain an isoperimetric inequality for $\Lambda(\Omega,\beta)$ depending on β.

The proof uses an interior approximation result for $BV(\Omega)$ functions by $C^\infty(\Omega)$ functions in the sense of strict convergence on $\mathbb{R}^N$ and a trace inequality in BV with respect to the anisotropic total variation. Let Ω be a bounded, connected, sufficiently smooth open set, $p>1$, and $\beta\in\mathbb{R}$. We study the asymptotic behavior, as $p\to1^+$, of the following minimum problem

$$\lambda_p(\beta,\Omega)=\inf_{\substack{\varphi\in W^{1,p}(\Omega)\\ \varphi\neq0}}J_p(\varphi) \tag{1.0.12}$$

where

$$J_p(\varphi)=\frac{\displaystyle\int_\Omega F^p(\nabla\varphi)dx+\beta\int_{\partial\Omega}|\varphi|^pF(\nu)d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega|\varphi|^pdx}, \tag{1.0.13}$$

ν is the outer normal to $\partial\Omega$ and F is a sufficiently smooth norm on $\mathbb{R}^N$. If $u \in W^{1,p}(\Omega)$ is a minimizer of (1.0.12), then it solves the following Robin eigenvalue problem

$$\begin{cases} -\mathcal{Q}_p u = \lambda_p(\beta,\Omega)|u|^{p-2}u & \text{in } \Omega \\ F^{p-1}(\nabla u)F_\xi(\nabla u)\cdot\nu + \beta F(\nu)|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathcal{Q}_p u$ is the anisotropic $p-$Laplace operator

$$\mathcal{Q}_p u := \operatorname{div}\left(\frac{1}{p}\nabla_\xi[F^p](\nabla u)\right).$$

From the point of view of finding optimal domains for $\lambda_p(\beta,\Omega)$, there is a significant difference from the case of $\beta > 0$ to the case $\beta < 0$. It is known that the optimal shape with a volume constraint for (1.0.12) depends on the sign of β. For positive values of the Robin parameter, the so-called Wulff shape is a minimizer (see [47] and for the Euclidean case see [24, 31, 43]):

$$\lambda_p(\beta,\Omega) \geq \lambda_p(\beta,\mathcal{W}) \quad \text{where } |\mathcal{W}| = |\Omega|. \tag{1.0.14}$$

If $\beta < 0$, the problem is not completely solved, even in the Euclidean case (that is when $F(\xi) = \sqrt{\sum_i \xi_i^2}$). Indeed, in 1977 Bareket [20] conjectured that the first eigenvalue is maximized by a ball in the class of the smooth bounded domains of given volume. In [58] it has been showed that it is true for domains close, in certain sense, to a ball. Subsequently, the authors in [62] (see [83] for the p-Laplacian case) have disproved the conjecture for $|\beta|$ large enough and have shown that it is true for small values of $|\beta|$ in a suitable class of domains. In the Finsler setting, this problem has been addressed in [97].

Our final aim is to obtain optimal shapes for the limiting functional of (1.0.12), as $p \to 1^+$. This type of result falls within a well-investigated research field. For example, convergence results for spectral functionals of the p-Laplacian, as p goes to 1^+, in relation also to isoperimetric inequalities, have been treated in the Euclidean case in [23, 33, 35, 49, 77, 88, 98] and in [21, 78] when $\mathbb{R}^N$ is equipped with a Finsler metric.

These results are contained in the paper [17] and we first study the limit of $\lambda_p(\beta,\Omega)$; in particular, in Theorem 4.5.2 we prove that when $\beta > -1$, the functional J_p, defined in (1.0.13), $\Gamma-$converges, as $p \to 1^+$, to

$$J(\varphi) = \frac{|D\varphi|_F(\Omega) + \min\{1,\beta\}\displaystyle\int_{\partial\Omega}|\varphi|F(\nu)d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega |\varphi|dx},$$

where $|D\varphi|_F(\Omega)$ is the anisotropic total variation of φ. This will imply that (see Proposition 4.5.3)

$$\lim_{p\to 1^+}\lambda_p(\beta,\Omega) = \Lambda(\beta,\Omega) := \inf_{\varphi\in BV(\Omega)} J(\varphi). \tag{1.0.15}$$

Then, we prove an isoperimetric inequality for $\Lambda(\Omega,\beta)$. In particular, in Proposition 4.6.3 we obtain that keeping the volume of Ω fixed, the Wulff shape minimises $\Lambda(\Omega,\beta)$ when $\beta \geq 0$, and maximises it when $-1 < \beta < 0$.

The proof of the convergence result, and then of the two isoperimetric inequalities, relies on two results on the anisotropic total variation, which are also of independent interest. The first one is a trace inequality in the BV space:

$$\int_{\partial^*\Omega}|u|F(\nu)\,d\mathcal{H}^{N-1} \leq c_1|Du|_F(\Omega) + c_2\int_\Omega |u|dx, \quad \forall u \in BV(\Omega), \tag{1.0.16}$$

where c_1 and c_2 are two constants which depend on the geometry of the domain (see Proposition 4.1.1 and Proposition 4.1.2 below). This inequality has been revealed very useful in capillarity problems, and it has been studied for example in [15, 66, 71].

The second key result is an interior approximation for BV functions by smooth functions with compact support. It is well known that if Ω is an open set, then the total variation of a function $u \in BV(\Omega)$ can be approximated with the corresponding total variation of a sequence in $C^\infty(\Omega)$. Actually, an analogous result is not true in general if one needs to approximate $|Du|$ with a sequence of C^∞ function with compact support in Ω. In order to do that, more regularity is needed on Ω. In the Euclidean setting, this problem has been addressed in [88, 105]. In this paper (Theorem 4.2.3), we show that, for any $u \in BV(\Omega) \cap L^p(\Omega)$, for some $p \in [1, \infty)$, there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq C_0^\infty(\Omega)$ such that, for any $q \in [1, p]$,

$$u_k \to u \quad \text{in} \quad L^q(\Omega) \quad \text{and} \quad |Du_k|_F(\mathbb{R}^N) \to |Du|_F(\mathbb{R}^N).$$

We finally stress that the problem we deal with is strictly related to capillarity problems. We refer the reader, for example, to [66, 71] for the Euclidean case and to [44] for the anisotropic case.

Another topic, about the first Robin eigenvalue, is afforded in Chapter 5, and it is about estimates for this eigenvalue. In this Chapter, we prove upper and lower bounds for the first eigenvalue of the p-Laplace operator with Robin boundary condition, namely

$$\lambda_p(\beta, \Omega) = \min_{w \in W^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla w|^p \, dx + \beta \int_{\partial\Omega} |w|^p \, d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega |w|^p \, dx}, \tag{1.0.17}$$

where Ω is a bounded Lipschitz open set of $\mathbb{R}^N$, $N \geq 2$, $\beta > 0$, $1 < p < +\infty$. If $u_\beta \in W^{1,p}(\Omega)$ is a minimizer of (1.0.17), then it satisfies

$$\begin{cases} -\Delta_p u_\beta = \lambda_p(\beta, \Omega)|u_\beta|^{p-2}u_\beta & \text{in } \Omega \\ |\nabla u_\beta|^{p-2}\dfrac{\partial u_\beta}{\partial \nu} + \beta |u_\beta|^{p-2}u_\beta = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.0.18}$$

It is well-known that u_β has a sign, and $u_\beta \in C^{1,\alpha}(\bar{\Omega})$ if Ω is $C^{1,\gamma}$ for some $0 < \alpha, \gamma < 1$ (see [87]).

We know that when $\beta = 0$, then $\lambda_p(\beta, \Omega) = 0$ which corresponds to the first trivial Neumann eigenvalue; in the case $\beta = +\infty$, then $\lambda_p(\beta, \Omega)$ coincides with $\lambda_p^D(\Omega)$, the first Dirichlet eigenvalue of $-\Delta_p$, that we remember is

$$\lambda_p^D(\Omega) = \min_{\varphi \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla \varphi|^p \, dx}{\displaystyle\int_\Omega |\varphi|^p \, dx}.$$

Finding estimates, perhaps the best possible ones, for $\lambda_p(\beta, \Omega)$ is a topic of great interest to researchers and several results have been given. As we have seen in (1.0.14), a Faber-Krahn inequality holds:

$$\lambda_p(\beta, \Omega) \geq \lambda_p(\beta, B)$$

where B is a ball whose Lebesgue measure coincides with that of Ω. Recently, in [53] it has been proved that if Ω is a bounded, mean convex set, then it holds that

$$\lambda_1(\beta, \Omega) \geq (p-1)\left(\frac{\pi_p}{2}\right)^p \frac{1}{\left(R(\Omega) + \frac{\pi_p}{2}\beta^{-\frac{1}{p-1}}\right)^p}, \tag{1.0.19}$$

where $R(\Omega)$ is the anisotropic inradius of Ω and $\pi_p = \frac{2\pi}{p\sin\frac{\pi}{p}}$. Moreover, in [45] an upper bound of $\lambda_p(\beta,\Omega)$ in terms of the first eigenvalue of a one-dimensional elliptic problem, depending on the perimeter and the volume of a bounded convex set Ω is given; in particular, if $p=2$ the following estimate holds:

$$\lambda_2(\beta,\Omega) \le \frac{\pi^2}{4}\frac{P^2(\Omega)}{|\Omega|^2}\frac{1}{1+\frac{2P(\Omega)}{\beta|\Omega|}}. \tag{1.0.20}$$

We refer the reader also to [86, 104] for related results. As matter of fact, the inequalities (1.0.19) and (1.0.20) generalize the well-known Hersch and Pólya inequalities for the Dirichlet-Laplace first eigenvalue (see [48, 46] and the references therein contained).

We aim to prove some new upper and lower bounds for $\lambda_p(\beta,\Omega)$. In particular, our first main result is the following:

Theorem 1.0.5. *Let $\Omega\subset\mathbb{R}^N$ be an open bounded set, with $C^{1,\gamma}$ boundary, and let β be a positive number. Then,*

$$\frac{1}{\lambda_p^{\frac{1}{p-1}}(\beta,\Omega)} \ge \frac{1}{\lambda_p^D(\Omega)^{\frac{1}{p-1}}} + \frac{\tilde{K}}{(\beta P(\Omega))^{\frac{1}{p-1}}}, \tag{1.0.21}$$

where $\tilde{K}$ is given by

$$\tilde{K} = \tilde{K}(N,p,q,r,\lambda_p^D(\Omega)) = \frac{\|v\|_r}{\|v\|_q},$$

and v is a first eigenfunction of $\lambda_p^D(B)$, defined in a ball B such that $\lambda_p^D(B)=\lambda_p^D(\Omega)$.

In the case $N=p=2$, the inequality (1.0.21) recovers a result proved in [108].

In order to prove (1.0.21), we use two main tools. The first is given by a characterization of $\lambda_p(\beta,\Omega)$ by means of a Thompson-type principle (see subsection 5.1.1). Then, we use the behavior of the eigenvalue and its eigenfunctions as $\beta\to+\infty$ (see Section 5.2).

A second result we obtain is again an upper bound for $\lambda_p(\beta,\Omega)$, but in terms of the (Dirichlet) p-torsional rigidity of Ω. More precisely, for a given open bounded domain Ω, let $u_\Omega\in W_0^{1,p}(\Omega)$ be the unique solution to

$$\begin{cases} -\Delta_p u_\Omega = 1 & \text{in } \Omega \\ u_\Omega = 0 & \text{on } \partial\Omega. \end{cases}$$

The $p-$torsional rigidity of Ω is

$$T_p(\Omega) = \left(\max_{\varphi\in W_0^{1,p}(\Omega)} \frac{\left(\int_\Omega |\varphi|dx\right)^p}{\int_\Omega |\nabla\varphi|^p dx} \right)^{\frac{1}{p-1}} = \int_\Omega u_\Omega dx = \int_\Omega |\nabla u_\Omega|^p dx.$$

Then we obtain the following.

Theorem 1.0.6. *Let $\Omega\subset\mathbb{R}^N$ be an open bounded set, with $C^{1,\gamma}$ boundary, and let β be a positive number. Then,*

$$\frac{1}{\lambda_p(\beta,\Omega)^{\frac{1}{p-1}}} \ge \frac{T_p(\Omega)}{|\Omega|} + \left(\frac{|\Omega|}{\beta P(\Omega)}\right)^{\frac{1}{p-1}}. \tag{1.0.22}$$

Again, inequality (1.0.22) generalizes a result by [108] given for $N = p = 2$; the main tool to prove (1.0.22) still relies on a Thompson-like principle.

As regards a lower bound for the eigenvalue, we get that if $\Omega \subset \mathbb{R}^N$ is a open bounded set, with $C^{1,\gamma}$ boundary, and $\beta > 0$, Then

$$\frac{1}{\lambda_p(\beta,\Omega)} \leq \frac{1}{\nu_p} + \frac{|\Omega|^{\frac{1}{p-1}}}{\beta^{\frac{1}{p-1}} P(\Omega)^{\frac{1}{p-1}}}, \tag{1.0.23}$$

where ν_p is a suitable value independent of β (see Section 5.4 for its definition). This result generalizes the case $p = N = 2$ proved in [108]. In that case, when Ω is a suitable symmetric plane domain, ν_2 coincides with the first nontrivial Neumann eigenvalue of the Laplacian.

The proof of (1.0.23) is based, in addition to the aforementioned Thompson's principle, on the precise behavior of the eigenfunctions of $\lambda_p(\beta,\Omega)$ when β goes to 0. In Theorem 5.2.3, Proposition 5.2.4 and Theorem 5.2.5 of Section 5.2 we state precisely such behavior. We show in particular that

$$\lim_{\beta \to 0} \frac{\lambda_p(\beta,\Omega)}{\beta} = \frac{P(\Omega)}{|\Omega|},$$

where, for the sake of completeness, we compute the limit also considering negative β's (see [69] for the case $p = 2$). These results are contained in [18].

Nonlinear elliptic problems related to thermal insulation

In the last part of this thesis, we also study questions related to insulation problems. This kind of topic has interested many researchers and it's a very active field of research as it is related to environmental improvement. In fact, thermal insulation may be applied to save energy and even if it may seem counterintuitive, having too much insulation can have a negative impact on the energy efficiency; furthermore, adding too much insulation is expensive and unsustainable. On this topic, there are many papers, as for instance [1, 26, 28, 29, 30, 32, 54, 55, 63].

To this aim in Chapter 6 we consider a problem of this type: let Ω be a bounded connected open set of $\mathbb{R}^N$, $D \subset \overline{\Omega}$ a compact set, $\beta > 0$ a fixed constant. Let

$$\begin{aligned} I_{\beta,p}(D;\Omega) = \inf\Big\{ \int_\Omega |D\phi|^p dx + \beta \int_{\partial\Omega} |\phi|^p d\mathcal{H}^{N-1}, \\ \phi \in W^{1,p}(\Omega), \phi \geq 1 \text{ in } D \Big\}. \end{aligned} \tag{1.0.24}$$

Our aim is to study maximization and minimization problems of $I_{\beta,p}(D,\Omega)$, among domains with given geometrical constraints. In Proposition 6.1.1 we prove that, if Ω has Lipschitz boundary, then there exists a minimizer u of (1.0.24) that satisfies

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega \setminus \overline{D} \\ u = 1 & \text{in } D \\ |Du|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2} u = 0 & \text{on } \partial\Omega \end{cases}$$

and the functional $I_{\beta,p}(D,\Omega)$ assumes the following form

$$I_{\beta,p}(D,\Omega) = \beta \int_{\partial\Omega} |u|^{p-2} u \, d\mathcal{H}^{N-1}.$$

In this order of ideas, the case $p = 2$ has been treated in [50]. In this case, the minimization problem arises from a thermal insulation problem, in fact, they consider a domain of given temperature, thermally insulated by a thermal insulator of constant thickness and they ask for the best or worst choice in terms of heat dispersion.
In this Chapter we consider the general case $1 < p < +\infty$ and it is contained in the paper [17]. First, we consider the domains of the type $\Omega = D + \delta B$, with $\delta > 0$ and B the unit ball in $\mathbb{R}^N$; In particular, in the planar case, in Theorem 6.2.1 we prove that if D is an open, bounded, connected set of $\mathbb{R}^N$ with piecewise C^1 boundary, the maximum of $I_{\beta,p}(D, D + \delta B)$ is achieved at the disk having the same perimeter of D. Moreover, we consider the N-dimensional case for convex sets and we obtain in Theorem 6.3.1 that among the convex sets $\Omega = D + \delta B$, with fixed δ and of given W_{N-1} quermassintegral of D, the maximum is attained when D is a ball.
Finally, we show a counterintuitive behaviour of the functional $I_{\beta,p}(D, \Omega)$; indeed we prove in Proposition 6.4.3 that for suitable values of β, there exists a positive constant δ_0 such that for any bounded domain Ω, with $D \subset \Omega$ and $|\Omega| - |D| < \delta_0$, then $I_{\beta,p}(B_R, \Omega) > I_{\beta,p}(B_R, B_R)$.

Chapter 2

Preliminaries

2.1 Notations

- We denote by N the dimension of the Euclidean space;
- $\mathcal{H}^k$ stands for the k-dimensional Hausdorff measure in $\mathbb{R}^N$;
- ω_N is measure of the unit $N-$dimensional ball;
- $x \cdot y$ is the Euclidean scalar product for $N \geq 2$;
- we denote with $|\cdot|$ the Lebesgue measure and the Euclidean norm in $\mathbb{R}^N$.

2.2 Basic definitions

We will denote by $P(\Omega)$ the perimeter of Ω in the sense of De Giorgi (see [14]).
For convenience, we will denote by $||\cdot||_p$ the norm $||\cdot||_{L^p(\Omega^\sharp)}$ or $||\cdot||_{L^p(\Omega)}$; otherwise, we will write the space explicitly.

If Ω is an open and Lipschitz set, it holds the following coarea formula. Some references for results relative to the sets of finite perimeter and the coarea formula are, for instance, [14, 89].

Theorem 2.2.1 (Coarea formula)**.** *Let $f : \Omega \to \mathbb{R}$ be a Lipschitz function and let $u : \Omega \to \mathbb{R}$ be a measurable function. Then,*

$$\int_\Omega u|\nabla f(x)|dx = \int_{\mathbb{R}} dt \int_{(\Omega\cap f^{-1}(t))} u(y)\, d\mathcal{H}^1(y). \tag{2.2.1}$$

2.3 Rearrangement of functions

This section is dedicated to stating some useful results on the rearrangement of functions, on quantitative inequalities and rescaling properties. For a general overview of this topic, we refer, for instance, to [81].

Definition 2.3.1. Let $u : \Omega \to \mathbb{R}$ be a measurable function, the *distribution function* of u is the function $\mu : [0, +\infty[\to [0, +\infty[$ defined as the measure of the superlevel sets of u, i.e.

$$\mu(t) = |\{\, x \in \Omega \,:\, |u(x)| > t \,\}|.$$

In particular, if u is the solution to (1.0.3) and v is the solution to (1.0.4), we fix the following notations, for $t \geq 0$

$$U_t = \{x \in \Omega : u(x) > t\}, \quad \mu(t) = |U_t|$$

and

$$V_t = \left\{x \in \Omega^\sharp : v(x) > t\right\}, \quad \nu(t) = |V_t|.$$

By using the Coarea formula (2.2.1), one can deduce the following expression for μ

$$\mu(t) = |\{u > t\} \cap \{|\nabla u| = 0\}| + \int_t^{+\infty} \left(\int_{u=s} \frac{1}{|\nabla u|} \, d\,\mathcal{H}^{N-1} \right) ds,$$

as a consequence, for almost all $t \in (0, +\infty)$,

$$\infty > -\mu'(t) \geq \int_{u=t} \frac{1}{|\nabla u|} d\mathcal{H}^{N-1} \tag{2.3.1}$$

and if μ is absolutely continuous, equality holds in (2.3.1).

Definition 2.3.2. Let $u : \Omega \to \mathbb{R}$ be a measurable function, the *decreasing rearrangement* of u, denoted by $u^*(\cdot)$,is defined as

$$u^*(s) = \inf\{t \geq 0 : \mu(t) < s\}.$$

From Definitions 2.3.1 and 2.3.2, one can prove that

$$u^*(\mu(t)) \leq t, \quad \forall t \geq 0,$$

$$\mu(u^*(s)) \leq s \quad \forall s \geq 0.$$

Remark 2.3.3. Let us notice that the function $\mu(\cdot)$ is decreasing and right continuous and the function $u^*(\cdot)$ is its generalized inverse.

Definition 2.3.4. The *Schwartz rearrangement* of u is the function $u^\sharp$ whose superlevel sets are balls with the same measure as the superlevel sets of u.

We have the following relation between $u^\sharp$ and u^*:

$$u^\sharp(x) = u^*(\omega_N |x|^N),$$

where ω_N is the measure of the unit ball in $\mathbb{R}^N$. One can easily check that the functions u, u^* e $u^\sharp$ are equi-distributed, i.e. they have the same distribution function, and it holds

$$\|u\|_p = \|u^*\|_p = \|u^\sharp\|_p, \quad \text{for all } p \geq 1.$$

The following Lemma characterizes the absolute continuity of μ (we refer, for instance, to [27, 40]).

Lemma 2.3.5. *Let $u \in W^{1,p}(\mathbb{R}^N)$, with $p \in (1, +\infty)$. The distribution function μ of u is absolutely continuous if and only if*

$$|\{0 < u < ||u^\sharp||_\infty\} \cap \{|\nabla u^\sharp| = 0\}| = 0.$$

An important property of the decreasing rearrangement is the Hardy-Littlewood inequality, see [76].

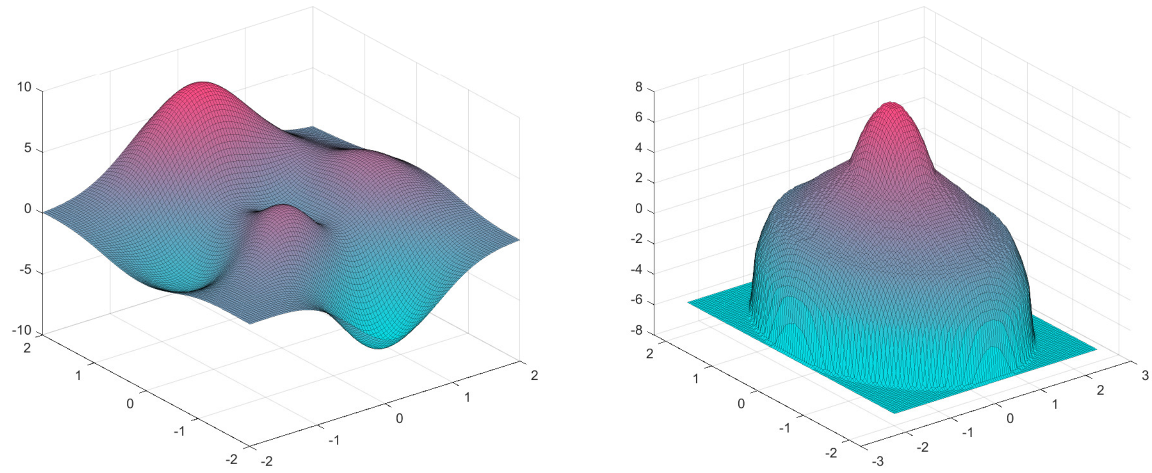

Figure 2.1: The Schwarz symmetrization

Theorem 2.3.6 (Hardy-Littlewood inequaliy)**.** *Let us consider $h \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$. Then*

$$\int_\Omega |h(x)g(x)|\,dx \le \int_0^{|\Omega|} h^*(s)g^*(s)\,ds. \tag{2.3.2}$$

By choosing $h = \chi_{\{|u|>t\}}$ in (2.3.2), one has

$$\int_{|u|>t} |h(x)|\,dx \le \int_0^{\mu(t)} h^*(s)\,ds.$$

If we require that the function u is a *Sobolev function*, i.e. $u \in W_0^{1,p}(\Omega)$, then also $u^\sharp$ is a Sobolev function and, moreover, the gradient does not increase under symmetrization, as a consequence of the Pólya-Szegő inequality (see [100]).

Theorem 2.3.7 (Pólya-Szegő inequality)**.** *Let $u \in W_0^{1,p}(\Omega)$, then $u^\sharp \in W_0^{1,p}(\Omega^\sharp)$ and*

$$\|\nabla u^\sharp\|_p \le \|\nabla u\|_p. \tag{2.3.3}$$

The following remark will be useful in the sequel.

Remark 2.3.8. We explicitly observe that the function

$$\varphi : s \to \frac{1}{s}\int_0^s f^*$$

is decreasing. Indeed, let $s_1 \le s_2$, and let us show that $\varphi(s_1) \ge \varphi(s_2)$. This is true if and only if

$$\frac{1}{s_1}\int_0^{s_1} f^* - \frac{1}{s_2}\int_0^{s_1} f^* \ge \frac{1}{s_2}\left(\int_0^{s_2} f^* - \int_0^{s_1} f^*\right)$$
$$\frac{1}{s_1}\int_0^{s_1} f^* \ge \frac{1}{s_2 - s_1}\int_{s_1}^{s_2} f^*.$$

Let us recall that f^* is a decreasing function, so

$$f^*(s) \geq f^*(s_1) \geq f^*(r)$$

if $s < s_1 < r$. Hence,

$$\fint_0^{s_1} f^*(s)\, ds \geq f^*(s_1) \geq \fint_{s_1}^{s_2} f^*(r)\, dr.$$

We recall now some properties of the symmetric solution. The explicit expression of v, solution to problem (1.0.4), is

$$v(x) = \int_{\omega_N|x|^N}^{|\Omega|} \frac{1}{N^2\omega_N^{\frac{2}{N}} t^{2-\frac{2}{N}}} \int_0^t f^*(r)\, dr\, dt, \tag{2.3.4}$$

and its gradient is

$$|\nabla v|(x) = \frac{1}{N\omega_N|x|^{N-1}} \int_0^{\omega_N|x|^N} f^*(r)\, dr. \tag{2.3.5}$$

From (2.3.5) we can deduce that the gradient of v vanishes at most at the origin and Lemma 2.3.5 ensures that the distribution function ν is absolutely continuous and it satisfies

$$N^2\omega_N^{\frac{2}{N}} \nu(t)^{2-\frac{2}{N}} = (-\nu'(t)) \left(\int_0^{\nu(t)} f^* \right). \tag{2.3.6}$$

While for the distribution function μ

$$N^2\omega_N^{\frac{2}{N}} \leq \frac{-\mu'(t)}{\mu(t)^{2-\frac{2}{N}}} \int_0^{\mu(t)} f^*(r)\, dr, \tag{2.3.7}$$

Remark 2.3.9. With the same techniques used for proving (1.0.5), it is possible to prove that

$$\int_\Omega |\nabla u|^q\, dx \leq \int_{\Omega^\sharp} |\nabla v|^q\, dx, \quad 0 < q \leq 2, \tag{2.3.8}$$

and it is possible to estimate the L^q-norms of the gradient of v in terms of L^p-norms of f via Bliss inequality (we refer to [109, Equation (23.a)]), obtaining

$$\int_{\Omega^\sharp} |\nabla v|^q\, dx = \frac{1}{N^q\omega_N^{\frac{q}{N}}} \int_0^{|\Omega|} \left(\frac{1}{s} \int_0^2 f^* \right)^q s^{\frac{q}{N}}\, ds \leq K_1(N,q) \left(\int_0^{|\Omega|} f^{\frac{qN}{q+N}} \right)^{\frac{q+N}{N}}. \tag{2.3.9}$$

2.3.1 Some quantitative inequalities

We recall now some quantitative results that we need to prove our main results Theorem 1.0.2, Theorem 1.0.3 and Theorem 1.0.4. Let us start with the quantitative isoperimetric inequality, proved in [65] (see also [73, 74, 41, 64]).

Theorem 2.3.10. *There exists a constant γ_n such that, for any measurable set Ω of finite measure*

$$P(\Omega) \geq N\omega_N^{\frac{1}{N}} |\Omega|^{\frac{N-1}{N}} \left(1 + \frac{\alpha^2(\Omega)}{\gamma_n} \right), \tag{2.3.10}$$

where $\alpha(\Omega)$ is defined in (1.0.7).

We observe that the constant γ_n is explicitly computed in [60].

In the proof of Theorem 1.0.2, we apply the quantitative isoperimetric inequality (2.3.10) to the superlevel set U_t of the solution to (1.0.3). In order to do that, it is useful to estimate the asymmetry of U_t in terms of the asymmetry of Ω, as in the following lemma (we refer to [25, Lemma 2.8]).

Lemma 2.3.11. *Let $\Omega \subset \mathbb{R}^N$ be an open set with finite measure and let $U \subset \Omega$, $|U| > 0$ be such that*

$$\frac{|\Omega \setminus U|}{|\Omega|} \leq \frac{1}{4}\alpha(\Omega). \tag{2.3.11}$$

Then, we have

$$\alpha(U) \geq \frac{1}{2}\alpha(\Omega). \tag{2.3.12}$$

As already stressed in the introduction, a key role will be played by the quantitative version of Pólya-Szegő inequality (2.3.7), proved in [38].

Theorem 2.3.12 (Quantitative Pólya-Szegő inequality)**.** *Let $u \in W^{1,2}(\mathbb{R}^N)$, $N \geq 2$. Then, there exist positive constants r, s and C, depending only on n, such that, for every $u \in W^{1,2}(\mathbb{R}^N)$, it holds*

$$\inf_{x_0 \in \mathbb{R}^N} \frac{\displaystyle\int_{\mathbb{R}^N} |u(x) \pm u^\sharp(x + x_0)|\, dx}{|\{u > 0\}|^{\frac{1}{N}+\frac{1}{2}} \|\nabla u^\sharp\|_2} \leq C(N)\left[M_{u^\sharp}(E(u)^r) + E(u)\right]^s, \tag{2.3.13}$$

where

$$E(u) = \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2}{\displaystyle\int_{\mathbb{R}^N} |\nabla u^\sharp|^2} - 1 \qquad \textit{and} \qquad M_{u^\sharp}(\delta) = \frac{\left|\left\{|\nabla u^\sharp| < \delta\right\} \cap \left\{0 < u^\sharp < ||u||_\infty\right\}\right|}{|\{|u| > 0\}|}. \tag{2.3.14}$$

We will also use a quantitative version of the Hardy-Littlewood inequality (2.3.2), proved in [39]. In order to state the result, we need to introduce the following notation:

$$\begin{aligned} \mu_h(t) &= |\{h > t\}|, \\ \mu_g(t) &= |\{g > t\}|, \\ g_h(x) &= g^*(\mu_h(h(x))), \end{aligned}$$

and we observe that if the set Ω is a ball and the function h is radial and decreasing, then the function $g_h(x) = g^\sharp(x)$. Moreover, let us introduce the following Lorentz-type norm

$$\|g\|_{\Lambda_p^q(\Omega)} = \left(\int_0^{|\Omega|} g^*(s)^q \theta_q'(s)\, ds\right)^{1/q}, \tag{2.3.15}$$

where, for $s \in [0, |\Omega|)$,

$$\theta_p(s) = \left(\int_0^s \left(-(h^*)'(\sigma)\right)^{-1/(p-1)} d\sigma\right)^{1/p'}. \tag{2.3.16}$$

Theorem 2.3.13 (Quantitative Hardy Littlewood inequality)**.** *Let h and g be two measurable functions such that $hg \in L^1(\Omega)$ and let us assume that there exist $p \in (1, +\infty)$ and $q \in [1, +\infty)$ such that $\theta_p(s) < \infty$ for every $0 \leq s < |\Omega|$ and $\|g\|_{\Lambda_p^q(\Omega)} < \infty$. Then, by setting*

$$m = \frac{qp+1}{p+1}, \tag{2.3.17}$$

it holds

$$\int_\Omega h(x)g(x)\,dx + \frac{1}{2^{p+1}eq}\|g\|^{-qp}_{\Lambda_p^q(\Omega)}\|g-g_h\|^{1+pq}_{L^m(\Omega)} \leq \int_0^{|\Omega|} h^*(s)g^*(s)\,ds. \tag{2.3.18}$$

2.3.2 Rescaling properties

For simplicity, we will often prove our results under two extra assumptions:

$$|\Omega| = 1, \qquad \|f\|_1 = 1. \tag{2.3.19}$$

If this is not the case, we recover the result in the more general setting in the following way. Let us set

$$a = \frac{|\Omega|^{1-\frac{2}{N}}}{\|f\|_1}, \qquad b = |\Omega|^{-\frac{1}{N}}.$$

If u and v are solutions to (1.0.3) and (1.0.4) respectively, we define the functions $w(x) = au\left(\frac{x}{b}\right)$ and $z(x) = av\left(\frac{x}{b}\right)$, that are solutions to

$$\begin{cases} -\Delta w = g & \text{in } \tilde\Omega \\ w = 0 & \text{on } \partial\tilde\Omega, \end{cases} \qquad \begin{cases} -\Delta z = g^\sharp & \text{in } \tilde\Omega^\sharp \\ z = 0 & \text{on } \partial\tilde\Omega^\sharp, \end{cases}$$

with $|\tilde\Omega| = 1$ and $\|g\|_1 = 1$. Furthermore, the following holds

$$\begin{aligned} \alpha(\tilde\Omega) &= \alpha(\Omega); \\ ||z - w^\sharp||_\infty &= \frac{|\Omega|^{1-\frac{2}{N}}}{\|f\|_1}||v - u^\sharp||_\infty; \\ ||w - w^\sharp||_1 &= \frac{|\Omega|^{-\frac{2}{N}}}{\|f\|_1}||u - u^\sharp||_1; \\ \|g\|_p &= |\Omega|^{1-\frac{1}{p}}\frac{\|f\|_p}{\|f\|_1} \ \ \forall p \ \text{ s.t. } \ f \in L^p(\Omega). \end{aligned}$$

Moreover, if $\sigma(t)$ and $\mu(t)$ are the distribution functions of w and u respectively, then

$$\sigma(t) = |\Omega|^{-1}\mu\left(\frac{t}{a}\right).$$

2.4 Some basic facts on convex sets

Let K be a nonempty, bounded, convex set in $\mathbb{R}^N$ and let $\delta > 0$. Then the Steiner formulas for the volume and the perimeter states that

$$
\begin{aligned}
|K+\delta B| &= \sum_{j=0}^{N} \binom{N}{j} W_j(K)\delta^j \\
&= |K| + NW_1(K)\delta + \frac{N(N-1)}{2} W_2(K)\delta^2 + \cdots + \omega_N \delta^n
\end{aligned}
$$

and

$$
\begin{aligned}
P(K+\delta B) &= N \sum_{j=0}^{N-1} \binom{N-1}{j} W_{j+1}(K)\delta^j \\
&= P(K) + N(N-1)W_2(K)\delta + \cdots + N\omega_N \delta^{N-1},
\end{aligned}
\tag{2.4.1}
$$

where B is the unit ball in $\mathbb{R}^N$ centered at the origin and $K+\delta B$ stands for the Minkowski sum. The coefficients $W_j(K)$ are the so-called quermassintegrals of K. In particular W_0 is the volume of K, $W_1 = \frac{P}{N}$ and $W_N = \omega_N$.
It follows that

$$
\lim_{\delta \to 0^+} \frac{P(K+\delta B) - P(K)}{\delta} = N(N-1)W_2(K). \tag{2.4.2}
$$

If K has C^2 boundary, with nonzero Gaussian curvature, the quermassintegrals can be rewritten in terms of the principal curvatures of ∂K. Indeed, in such a case

$$
W_i(K) = \frac{1}{N} \int_{\partial K} H_{i-1}(x) d\mathcal{H}^{N-1}, \qquad i = 1, \cdots, N. \tag{2.4.3}
$$

Meanwhile, H_j denotes the j-th normalized elementary symmetric function of the principal curvatures of ∂K, that is $H_0 = 1$, and

$$
H_j(x) = \binom{N-1}{j}^{-1} \sum_{1 \le i_1 \le \cdots \le i_j \le N-1} k_{i_1}(x) \cdots k_{i_j}(x), \qquad j = 1, \cdots, N-1,
$$

where $k_1(x) \cdots k_{N-1}(x)$ are the principal curvatures at a point $x \in \partial K$. In particular, by (2.4.2) and (2.4.3) follows also that

$$
\lim_{\delta \to 0^+} \frac{P(K+\delta B) - P(K)}{\delta} = (N-1) \int_{\partial K} H_1(x) d\mathcal{H}^{N-1},
$$

where $H_1(x)$ is the mean curvature of ∂K at a point x.
The Alexandrov-Fenchel inequalities state that

$$
\left(\frac{W_j(K)}{\omega_N} \right)^{\frac{1}{N-j}} \ge \left(\frac{W_i(K)}{\omega_N} \right)^{\frac{1}{N-i}}, \qquad 0 \le i < j \le N-1, \tag{2.4.4}
$$

where the inequality becomes an equality if and only if K is a ball.
In what follows, we will use the Alexandrov-Fenchel inequalities for particular values of i and j. When $i = 0$ and $j = 1$, we have the classical isoperimetric inequality:

$$
P(K) \ge N\omega_N^{\frac{1}{N}} |K|^{1-\frac{1}{N}}.
$$

Moreover, if $i = k-1$, and $j = k$, we have

$$W_k(K) \geq \omega_N^{\frac{1}{N-k+1}} W_{k-1}(K)^{\frac{N-k}{N-k+1}}.$$

Now we denote by K^* a ball such that $W_{N-1}(K) = W_{N-1}(K^*)$ and then by the Alexandrov-Fenchel inequalities (2.4.4), for $0 \leq i < N-1$ it holds that

$$\left(\frac{W_i(K^*)}{\omega_N}\right)^{\frac{1}{N-i}} = \frac{W_{N-1}(K^*)}{\omega_N} = \frac{W_{N-1}(K)}{\omega_N} \geq \left(\frac{W_i(K)}{\omega_N}\right)^{\frac{1}{N-i}},$$

hence

$$W_i(K) \leq W_i(K^*), \qquad 0 \leq i \leq N-1. \tag{2.4.5}$$

Now consider the two-dimensional case. Let D be an open, bounded, connected set in $\mathbb{R}^2$, we denote by D^* the disk having the same perimeter of D. If $\Omega = D + \delta B$ and $\Omega_* = D^* + \delta_* B$, where B is the disk centered at the origin, the Steiner formulae become

$$|\Omega| = |D| + P(D)\delta + \pi\delta^2, \qquad P(\Omega) = P(D) + 2\pi\delta,$$

$$|\Omega_*| = |D^*| + P(D^*)\delta_* + \pi\delta_*^2, \qquad P(\Omega_*) = P(D^*) + 2\pi\delta_*.$$

Let us observe that, in our context, if we ask that the area of the insulating material $\Omega \setminus \overline{D}$ remains constant, then

$$|\Omega| - |D| = P(D)\delta + \pi\delta^2 = |\Omega_*| - |D^*| = P(D^*)\delta_* + \pi\delta_*^2.$$

Since $P(D) = P(D^*)$, then $\delta = \delta_*$ and, as byproduct, $P(\Omega) = P(\Omega_*)$. On the contrary, if $\delta = \delta_*$, then $|\Omega| - |D| = |\Omega_*| - |D^*|$.
If D is a general bounded domain, with piecewise C^1 boundary, it holds

$$|\Omega| \leq |D| + P(D)\delta + \pi\delta^2, \qquad P(\Omega) \leq P(D) + 2\pi\delta, \tag{2.4.6}$$

hence $\delta = \delta_*$ implies

$$|\Omega| - |D| \leq |\Omega_*| - |D^*|,$$

which means that, if we fix the perimeter $P(D)$ and the thickness δ, the area of the insulating material increases.

2.5 The Finsler norm

We give several definitions and properties related to the Finsler norm (for further references see [57, 107, 106]). In particular, we recall some basic facts on the anisotropic total variation of a BV function, and on the anisotropic curvatures.
We will assume that F is a convex, even, $1-$homogeneous function

$$\xi \in \mathbb{R}^N \mapsto F(\xi) \in [0, +\infty[,$$

such that

$$F(t\xi) = |t|F(\xi), \quad t \in \mathbb{R},\, \xi \in \mathbb{R}^N, \tag{2.5.1}$$

and such that

$$a|\xi| \leq F(\xi), \quad \xi \in \mathbb{R}^N, \tag{2.5.2}$$

for some constant $a > 0$. It is easily seen that this hypothesis assures the existence of a positive constant $b \geq a$ such that

$$F(\xi) \leq b|\xi|, \quad \xi \in \mathbb{R}^N.$$

Throughout the paper, we will also assume that F belongs to $C^2(\mathbb{R}^N \setminus \{0\})$ and that

$$\nabla^2_\xi [F^p](\xi) \text{ is positive definite in } \mathbb{R}^N \setminus \{0\}. \tag{2.5.3}$$

The assumption (2.5.3) on F ensures that the operator

$$\mathcal{Q}_p u := \operatorname{div}\left(\frac{1}{p}\nabla_\xi [F^p](\nabla u)\right)$$

is elliptic, therefore there exists a positive constant γ such that

$$\sum_{i,j=1}^{N} \nabla^2_{\xi_i \xi_j}[F^p](\eta)\xi_i\xi_j \geq \gamma|\eta|^{p-2}|\xi|^2 \qquad \forall \eta \in \mathbb{R}^N \setminus \{0\},\ \forall \xi \in \mathbb{R}^N.$$

The polar function $F^o \colon \mathbb{R}^N \to [0, +\infty[$ of F is

$$F^o(v) = \sup_{\xi \neq 0} \frac{\xi \cdot v}{F(\xi)}.$$

It is easily seen that also F^o is a convex function satisfying the properties (2.5.1) and (2.5.2). Furthermore, we have

$$F(v) = \sup_{\xi \neq 0} \frac{\xi \cdot v}{F^o(\xi)},$$

and from this follows that

$$|\xi \cdot \eta| \leq F(\xi) F^o(\eta) \qquad \forall \xi, \eta \in \mathbb{R}^N. \tag{2.5.4}$$

The Wulff shape centered at the origin is the set denoted by

$$\mathcal{W} = \{\xi \in \mathbb{R}^N \colon F^o(\xi) < 1\}.$$

We denote $\kappa_N = |\mathcal{W}|$, where $|\mathcal{W}|$ is the Lebesgue measure of $\mathcal{W}$. More generally, the set $\mathcal{W}_r(x_0)$ indicates $r\mathcal{W} + x_0$, that is the Wulff shape centered at x_0 with volume $\kappa_N r^N$, while its anisotropic perimeter is $N\kappa_N r^{N-1}$. If no ambiguity occurs, we will write $\mathcal{W}_r$ instead of $\mathcal{W}_r(0)$.

The functions F and F^o enjoy the following properties:

$$F_\xi(\xi) \cdot \xi = F(\xi), \quad F^o_\xi(\xi) \cdot \xi = F^o(\xi) \qquad \forall \xi \in \mathbb{R}^N \setminus \{0\}, \tag{2.5.5}$$

$$F(F^o_\xi(\xi)) = F^o(F_\xi(\xi)) = 1 \qquad \forall \xi \in \mathbb{R}^N \setminus \{0\}, \tag{2.5.6}$$

$$F^o(\xi) F_\xi(F^o_\xi(\xi)) = F(\xi) F^o_\xi(F_\xi(\xi)) = \xi \qquad \forall \xi \in \mathbb{R}^N \setminus \{0\}, \tag{2.5.7}$$

where $F_\xi = \nabla F(\xi)$.

Given a bounded domain Ω, the anisotropic distance of $x \in \overline{\Omega}$ to $\partial\Omega$ is defined as

$$d_F(x) := \inf_{y \in \partial\Omega} F^o(x - y), \quad x \in \overline{\Omega}.$$

We highlight that, when $F(\xi) = \sqrt{\sum_i \xi_i^2}$, then $F = F^o$ and $d_F = d_{\mathcal{E}}$ is the Euclidean distance function from the boundary.

The function d_F is a uniform Lipschitz function in $\overline{\Omega}$, and

$$F(\nabla d_F(x)) = 1 \quad \text{a.e. in } \Omega.$$

We have that $d_F \in W_0^{1,\infty}(\Omega)$. Many properties of the anisotropic distance function are studied in [42].

Finally, the anisotropic inradius of Ω is

$$R^F(\Omega) = \max\{d_F(x),\ x \in \overline{\Omega}\},$$

that is the radius of the largest Wulff shape $\mathcal{W}_r(x)$ contained in Ω.

2.5.1 Anisotropic curvatures

Here we recall some properties of the anisotropic mean curvature, as well as an integration formula in anisotropic normal coordinates. We refer to [42] for further details.

If Ω has a C^2 boundary, the anisotropic outer normal to $\partial\Omega$ is defined as

$$n^F(y) = F_\xi(\nu(y)), \qquad y \in \partial\Omega,$$

where $\nu(y)$ is the Euclidean outer normal to $\partial\Omega$ at y. Moreover, by (2.5.6) it holds that

$$F^o(n^F(y)) = 1.$$

Let us denote by $T_y\partial\Omega$ the tangent space to $\partial\Omega$ at y; the anisotropic Weingarten map is defined as

$$dn^F \colon T_y\partial\Omega \to T_{n^F(y)}\mathcal{W}.$$

The eigenvalues $\kappa_1^F \le \kappa_2^F \le \ldots \le \kappa_{N-1}^F$ of this map are called the anisotropic principal curvatures at y (see also [114]). The anisotropic mean curvature of $\partial\Omega$ at a point y is defined as

$$\mathcal{H}^F(y) = \kappa_1^F(y) + \ldots \kappa_{N-1}^F(y), \quad y \in \partial\Omega.$$

The anisotropic distance d_F is a C^2 function in a tubular neighborhood of $\partial\Omega$; hence we are in position to define the matrix-valued function

$$W(y) = -F_{\xi\xi}(\nabla d_F(y))\nabla^2 d_F(y), \quad y \in \partial\Omega.$$

Based on this function, it is possible to give a different definition of the anisotropic principal curvatures [42, Remark 5.9]. Since $W(y)v \in T_y\partial\Omega$, for any $v \in \mathbb{R}^N$, it remains defined the map $\overline{W}(y) \colon T_y\partial\Omega \to T_y\partial\Omega$, as $\overline{W}(y)w = W(y)w$, $w \in T_y\partial\Omega$. The matrix $\overline{W}(y)$ (that is, in general, non-symmetric) admits the real eigenvalues $\kappa_1^F(y) \le \kappa_2^F(y) \le \ldots \le \kappa_{N-1}^F(y)$. Actually, the definition is equivalent to the preceding one. Moreover, it holds that

$$\mathcal{H}^F(y) = \operatorname{div}\left[F_\xi\left(-\nabla d_F(y)\right)\right] = \operatorname{tr}(W(y))$$

(see also [114, Sec. 3]).

To state the change of variable formula in anisotropic normal coordinates, we need some preliminary definitions.

Let
$$\phi(y,t) = y - tF_\xi(\nu(y)), \qquad y \in \partial\Omega, \quad t \in \mathbb{R}$$
and for $y \in \partial\Omega$,
$$\ell(y) = \sup\{d_F(z),\, z \in \Omega \text{ and } y \in \Pi(z)\}$$
where
$$\Pi(z) = \{\eta \in \partial\Omega\colon d_F(z) = F^o(z-\eta)\} \tag{2.5.8}$$
is the set of the anisotropic projections of a point $z \in \Omega$ on $\partial\Omega$.

Then, we recall the following

Theorem 2.5.1. *([42, Theorem 7.1]) For every $h \in L^1(\Omega)$, it holds*
$$\int_\Omega h(x)dx = \int_{\partial\Omega} F(\nu(y)) \int_0^{\ell(y)} h(\phi(y,t))J(y,t)\,dt\,d\mathcal{H}^{N-1}(y),$$
where
$$J(y,t) = \prod_{i=1}^{N-1}(1 - t\kappa_i^F(y)). \tag{2.5.9}$$

Since $1 - t\kappa_i^F(y) > 0$ for any $i = 1, \dots, N-1$ ([42, Lemma 5.4]), $J(y,t)$ is positive. Moreover, it holds that
$$\frac{-\frac{d}{dt}\left[J(y,t)\right]}{J(y,t)} = \sum_{i=1}^{N-1} \frac{\kappa_i^F(y)}{1 - t\kappa_i^F(y)}. \tag{2.5.10}$$

Finally, we conclude this subsection, by recalling that, for any $x \in \overline{\Omega}$ such that $\Pi(x) = \{y\}$, it holds that ([42, Theorem 4.16]):
$$\nabla d_F(x) = -\frac{\nu(y)}{F(\nu(y))}. \tag{2.5.11}$$

2.5.2 The anisotropic total variation

Let Ω be an open set in $\mathbb{R}^N$, a function $f \in L^1(\Omega)$ has a bounded variation in Ω if
$$\sup\left\{ \int_\Omega f \operatorname{div}\phi\, dx : \phi \in C^1_c(\Omega;\mathbb{R}^N), |\phi| \le 1 \right\} < \infty.$$
A function $f \in L^1_{loc}(\Omega)$ has a bounded variation in Ω if for each open set $U \subset\subset \Omega$,
$$\sup\left\{ \int_U f \operatorname{div}\phi\, dx : \phi \in C^1_c(U;\mathbb{R}^N), |\phi| \le 1 \right\} < \infty.$$
We write $BV(\Omega)$ and $BV_{loc}(\Omega)$ respectively, the space of such functions.
A measurable subset $E \subset \mathbb{R}^N$ has finite perimeter if $\chi_E \in BV(\Omega)$ and has locally finite perimeter if $\chi_E \in BV_{\text{loc}}(\Omega)$.
If $f = \chi_E$, and E is a set of locally finite perimeter in Ω, we can write $\|\partial E\|$ for the Radon measure μ and we observe that $\|\partial E\|$ is the perimeter measure of E, and $\|\partial E\|(\Omega)$ is the perimeter of E in Ω.

Definition 2.5.2. Let $x \in \mathbb{R}^N$. We say $x \in \partial^*\Omega$, the reduced boundary of E, if

1. $\|\partial E\|(B(x,r)) > 0$ for all $r > 0$;

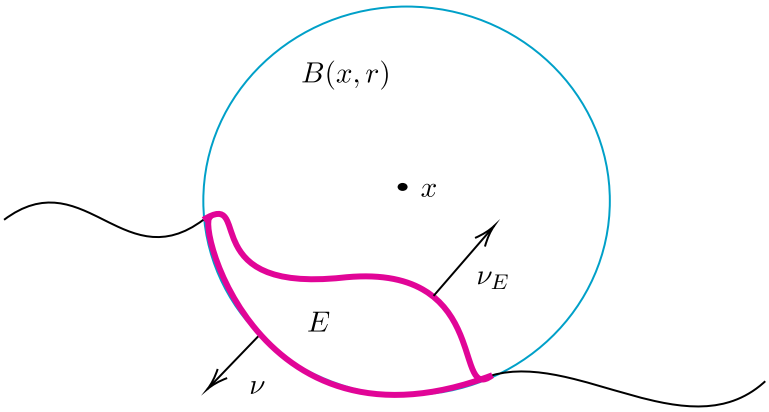

Figure 2.2: Normals to E and to $B(x, r)$.

2. $\lim_{r\to 0} \fint_{B(x,r)} \nu_E \, d\|\partial E\| = \nu_E(x)$;
3. $|\nu_E(x)| = 1$, where ν_E is the outer normal to E.

What happens in the anisotropic case?
Let Ω be an open set in $\mathbb{R}^N$ and $u \in BV(\Omega)$; the total variation of u with respect to F is defined as

$$|Du|_F(\Omega) = \sup\left\{\int_\Omega u \operatorname{div}(g)\, dx \quad : \; g \in C_0^1(\Omega; \mathbb{R}^N),\; F^o(g) \le 1\right\}.$$

It holds that $|Du|_F$ is L^1- lower semicontinuous on $BV(\Omega)$ ([10, Th. 5.1]) and, in the case u is a smooth function, we have

$$|Du|_F(\Omega) = \int_\Omega F(\nabla u) dx.$$

The perimeter of a set E in Ω with respect to F is:

$$P_F(E; \Omega) = |D\chi_E|_F(\Omega) = \sup\left\{\int_E \operatorname{div}(g)\, dx \quad : \; g \in C_0^1(\Omega; \mathbb{R}^N),\; F^o(g) \le 1\right\}.$$

Moreover,

$$P_F(E; \Omega) = \int_{\Omega \cap \partial^* E} F(\nu_E) d\mathcal{H}^{N-1}$$

where $\partial^* E$ is the reduced boundary of E and ν_E is the Euclidean normal to $\partial^* E$. Finally, we denote by $P_F(E) = P_F(E; \mathbb{R}^N)$.

If Ω is a bounded Lipschitz open set and $u \in BV(\Omega)$, assuming $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, we have $u \in BV(\mathbb{R}^N)$. Moreover, if with an abuse of notation we denote the trace of a BV function u and the function u itself with the same symbol, we have also that

$$|Du|_F(\mathbb{R}^N) = |Du|_F(\Omega) + \int_{\partial\Omega} |u| F(\nu) d\mathcal{H}^{N-1} \tag{2.5.12}$$

(see for example [34, Lemma 3.9]).

An isoperimetric inequality holds for the anisotropic perimeter. Indeed

$$P_F(E) \geq P_F(\mathcal{W}_R), \tag{2.5.13}$$

where $\mathcal{W}_R$ is the Wulff shape with the same measure of E, and the equality holds if and only if E is the Wulff shape centered at the origin (see for example [61]).

The following approximation results in BV hold (refer to [15], [70, Theorem 1.17] for the Euclidean case and to [5, Proposition 2.1] for the Finsler case).

Proposition 2.5.3. *Let $f \in BV(\Omega)$, then there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subseteq C^\infty(\Omega)$ such that:*

$$\lim_{k\to+\infty} \int_\Omega |f_k - f| dx = 0$$

and

$$\lim_{k\to+\infty} |Df_k|_F(\Omega) = |Df|_F(\Omega).$$

Proposition 2.5.4. *Let E be a set of finite perimeter in Ω. A sequence of C^∞ sets $\{E_k\}_k$ exists, such that:*

$$\lim_{k\to+\infty} \int_\Omega |\chi_{E_k} - \chi_E| dx = 0$$

and

$$\lim_{k\to+\infty} |D\chi_{E_k}|_F(\Omega) = P_F(E;\Omega).$$

To conclude, we recall the coarea formula, which holds for any $u \in BV(\Omega)$

$$\int_\Omega F(\nabla u)\, dx = \int_{\mathbb{R}^N} P_F(\{u > s\}, \Omega), \tag{2.5.14}$$

(see for example [10]).

Chapter 3

The Talenti result in a quantitative version

In this Chapter, we prove a quantitative version of the classical comparison result of Talenti for elliptic problems with Dirichlet boundary conditions. The key role is played by quantitative versions of the Pólya-Szegő inequality and of the Hardy-Littlewood inequality.

3.1 The Talenti result

As we want to obtain a quantitative version of (1.0.5), it is useful to recall briefly the outline of its proof.

Theorem 3.1.1 (Talenti, [109]). *Let $f \in L^{\frac{2N}{N+2}}(\Omega)$ if $N > 2$, $f \in L^p(\Omega)$, $p > 1$ if $N = 2$ and let u and v be the solutions to problems* (1.0.3) *and* (1.0.4) *respectively. Then,*

$$u^\sharp(x) \le v(x).$$

Proof. Let us observe that u is a weak solution to (1.0.3) if and only if

$$\int_\Omega \nabla u \nabla \varphi \, dx = \int_\Omega f \varphi \, dx, \qquad \forall \varphi \in W_0^{1,2}(\Omega). \tag{3.1.1}$$

Let us fix $t \in [0, \|u\|_\infty[$ and $h > 0$, and let us define the function φ:

$$\varphi(x) = \begin{cases} 0 & \text{if } u < t \\ u - t & \text{if } t < u < t + h \\ h & \text{if } u > t + h. \end{cases} \tag{3.1.2}$$

Using (3.1.2) as a test in the weak formulation (3.1.1), we have

$$\int_{U_t \setminus U_{t+h}} |\nabla u|^2 \, dx = \int_{U_t \setminus U_{t+h}} f(x)(u - t) \, dx + \int_{U_{t+h}} h f(x) \, dx.$$

If we divide now by h, and we send h to 0, we get

$$\int_{\partial U_t} |\nabla u| \, d\mathcal{H}^{N-1}(x) = \int_{U_t} f(x) \, dx \le \int_0^{\mu(t)} f^*(s) \, ds.$$

Applying the isoperimetric inequality to the superlevel set U_t, the Hölder inequality and the Hardy-Littlewood inequality, we get, for almost every t,

$$\begin{aligned} N^2\omega_N^{\frac{2}{N}}\mu(t)^{2-\frac{2}{N}} \leq P^2(U_t) &\leq \left(\int_{\partial U_t} d\mathcal{H}^{N-1}(x)\right)^2 \\ &\leq \left(\int_{\partial U_t} |\nabla u|\, d\mathcal{H}^{N-1}(x)\right)\left(\int_{\partial U_t} \frac{1}{|\nabla u|}\, d\mathcal{H}^{N-1}(x)\right) \\ &\leq \left(\int_0^{\mu(t)} f^*(s)\, ds\right)(-\mu'(t))\,. \end{aligned}$$

Equivalently, we can write

$$1 \leq \frac{1}{N^2\omega_N^{\frac{2}{N}}}\mu(t)^{-2+\frac{2}{N}}\left(\int_0^{\mu(t)} f^*(s)\, ds\right)(-\mu'(t))\,,$$

and, integrating from 0 to t, we obtain

$$t \leq \frac{1}{N^2\omega_N^{\frac{2}{N}}}\int_{\mu(t)}^{|\Omega|} r^{-2+\frac{2}{N}}\int_0^r f^*(s)\, ds\, dr = v^*(\mu(t)). \tag{3.1.3}$$

Since $u^*(r) = \inf\{\tau \,|\, \mu(\tau) < r\}$, (3.1.3) implies

$$u^*(r) = \inf\{\tau \,|\, \mu(\tau) < r\} \leq \inf\{v^*(\mu(\tau)) \,|\, \mu(\tau) < r\} \leq v^*(r),$$

that gives (1.0.5). $\square$

The quantitative version of The Talenti comparison result, states as follows:

Theorem 3.1.2. *Let Ω be a bounded open set of $\mathbb{R}^n$ and let f be a non-negative function in $L^2(\Omega)$. Suppose that u and v are the solutions to (1.0.3) and (1.0.4), respectively. Then, there exist some positive constants $\theta_1 = \theta_1(n), \theta_2 = \theta_2(n)$ and $C_1 := C_1(n, |\Omega|, f^\sharp)$, $C_2 := C_2(n, |\Omega|, f^\sharp)$, $C_3 := C_3(n, |\Omega|, f^\sharp)$, such that*

$$C_1\alpha^3(\Omega) + C_2 \inf_{x_0\in\mathbb{R}^n} \|u - u^\sharp(\cdot + x_0)\|^{\theta_1}_{L^1(\mathbb{R}^n)} + C_3 \inf_{x_0\in\mathbb{R}^n} \|f - f^\sharp(\cdot + x_0)\|^{\theta_2}_{L^1(\mathbb{R}^n)} \leq ||v - u^\sharp||_{L^\infty(\Omega)}. \tag{3.1.4}$$

Moreover, the dependence of C_1, C_2 and C_3 from $|\Omega|$ and $f^\sharp$ is explicit.

The choice of the L^∞-norm to bound the left-hand side in (3.1.4) is motivated by the fact that we are quantifying the pointwise inequality established by Talenti. It is possible to replace the L^∞-norm with a weaker one but in this case, we only obtain the bound of the asymmetry of the domain, see Section 3.5.
In order to prove this Theorem, we show that the quantity $||v - u^\sharp||_\infty$ bounds from above each term on the left-hand side. Hence, in the next Sections, we will prove Theorems 1.0.2, 1.0.3 and 1.0.4.

3.2 Proof of Theorem 1.0.2

The following Lemma is inspired by Lemma 2.9 (*Boosted Pólya-Szegő principle*) contained in [25].

Lemma 3.2.1. *Let $f \in L^{\frac{2N}{N+2}}(\Omega)$ if $N > 2$, $f \in L^p(\Omega)$, $p > 1$ if $N = 2$ and let u and v be the solutions to (1.0.3) and (1.0.4) respectively.*

Let s_Ω be defined as follows

$$s_\Omega = \sup\left\{ t \geq 0 : \mu(t) \geq |\Omega| \left(1 - \frac{\alpha(\Omega)}{4}\right)\right\} \in \mathbb{R}, \tag{3.2.1}$$

then, it results

$$\frac{1}{2\gamma_n} s_\Omega \alpha^2(\Omega) \leq ||v - u^\sharp||_\infty, \tag{3.2.2}$$

where γ_n is the constant appearing in the quantitative isoperimetric inequality (2.3.10).

Let us point out that $s_\Omega \in (0, +\infty)$:

- It is greater then zero whenever $\alpha(\Omega) > 0$ since $t \to \mu(t)$ is right-continuous and $\mu(0) = |\Omega|$;
- It is finite since there exists r small enough that $u^*(r) > s_\Omega$. Indeed, by contradiction let us assume that $\forall r$ $u^*(r) < s_\Omega$, then $\forall r$

  $$r \geq \mu(u^*(r)) > |\Omega| \left(1 - \frac{\alpha(\Omega)}{4}\right).$$

  This is a contradiction for r sufficiently small.
- By the definition of decreasing rearrangement 2.3.2 it follows that

  $$s_\Omega = u^* \left(|\Omega| \left(1 - \frac{\alpha(\Omega)}{4}\right)\right).$$

Proof. We assume that $\alpha(\Omega) > 0$, otherwise the result is trivial. Let us apply the quantitative isoperimetric inequality (2.3.10) to the level set U_t and, then, argue as in the proof of Theorem 3.1.1,

$$\begin{aligned} N^2 \omega_N^{\frac{2}{N}} \mu^{2-\frac{2}{N}}(t) \left(1 + \frac{2}{\gamma_n}\alpha^2(U_t)\right) &\leq N 2\omega_N \frac{2}{N} \mu^{2-\frac{2}{N}}(t) \left(1 + \frac{1}{\gamma_n}\alpha^2(U_t)\right)^2 \leq P^2(U_t) \\ &\leq \int_{u=t} |\nabla u| \int_{u=t} \frac{1}{|\nabla u|} \leq \left(\int_0^{\mu(t)} f^*\right)(-\mu'(t)). \end{aligned} \tag{3.2.3}$$

If we divide by $N2\omega_N \frac{2}{N} \mu^{2-\frac{2}{N}}(t)$ and we integrate between 0 and τ, we get

$$\tau + \frac{2}{\gamma_n} \int_0^\tau \alpha^2(U_t)\, dt \leq \int_{\mu(\tau)}^{|\Omega|} \left(\frac{1}{N^2 \omega_N^{\frac{2}{N}} t^{2-\frac{2}{N}}} \int_0^t f^*\right) dt = v^*(\mu(\tau)).$$

The last, due to the definition of decreasing rearrangement, can be written as

$$\frac{2}{\gamma_n} \int_0^{u^*(r)} \alpha^2(U_t)\, dt \leq \int_r^{|\Omega|} \left(\frac{1}{N^2 \omega_N^{\frac{2}{N}} t^{2-\frac{2}{N}}} \int_0^t f^*\right) dt - u^*(r) = v(r) - u^*(r). \tag{3.2.4}$$

We define the set

$$A := \left\{ t \geq 0 : \mu(t) \geq |\Omega| \left(1 - \frac{\alpha(\Omega)}{4}\right) \right\}, \tag{3.2.5}$$

then $0 \in A$ whenever $\alpha(\Omega) > 0$, and it is an interval since μ is a decreasing function. Moreover, for all $t \in A$, it holds

$$\frac{|\Omega \setminus U_t|}{|\Omega|} = 1 - \frac{\mu(t)}{|\Omega|} \leq \frac{1}{4}\alpha(\Omega).$$

Therefore, we can apply Lemma 2.3.11 and we get

$$\alpha(U_t) \geq \frac{\alpha(\Omega)}{2} \qquad \forall t \in A. \tag{3.2.6}$$

On the other hand, from (3.2.4) it follows

$$\min\{u^*(r), s_\Omega\} \frac{\alpha^2(\Omega)}{2\gamma_n} \leq \frac{2}{\gamma_n} \int_0^{u^*(r)} \alpha^2(U_t)\, dt \leq v(r) - u^*(r), \quad \forall r \in [0, |\Omega|], \tag{3.2.7}$$

and, choosing r sufficiently small in (3.2.7), we can conclude that

$$\frac{1}{2\gamma_n} s_\Omega \alpha^2(\Omega) \leq ||v - u^\sharp||_\infty.$$

□

We are now in a position to prove Theorem 1.0.2.

Proof Theorem 1.0.2. Firstly, let us suppose $|\Omega| = 1$, and $\|f\|_1 = 1$. Let us assume that $\alpha(\Omega) > 0$, otherwise the inequality (1.0.8) is trivial. We set

$$\varepsilon := ||v - u^\sharp||_\infty, \tag{3.2.8}$$

and we suppose that $\varepsilon < 1$. If this is not the case, inequality (1.0.8) holds, being $\alpha(\Omega) < 2$ and $K_1 \geq 8$. Definition (3.2.8) implies

$$v(x) - \varepsilon \leq u^\sharp(x) \leq v(x), \qquad \text{for almost any } x \in \Omega^\sharp$$

and it follows that

$$\nu(t+\varepsilon) \leq \mu(t) \leq \nu(t). \tag{3.2.9}$$

Moreover, as ν is absolutely continuous, we have

$$\begin{aligned} \nu(t+\varepsilon) - \nu(t) &= \int_t^{t+\varepsilon} \nu' = \int_t^{t+\varepsilon} \frac{-N^2\omega_N^{\frac{2}{N}}}{\nu^{-1+\frac{2}{N}} \fint_0^{\nu(r)} f^*} \\ &\geq \int_t^{t+\varepsilon} -\frac{N^2\omega_N^{\frac{2}{N}}}{|\Omega|^{-1+\frac{2}{N}}} \frac{|\Omega|}{\|f\|_1} = -N^2\omega_N^{\frac{2}{N}}\varepsilon, \end{aligned}$$

where the inequality follows from Remark 2.3.8. So, (3.2.9) becomes

$$\nu(t) - N^2\omega_N^{\frac{2}{N}}\varepsilon \leq \mu(t) \leq \nu(t). \tag{3.2.10}$$

By the definition of s_Ω and the property of decreasing rearrangement, we get

$$\mu(s_\Omega) = \mu\left(u^*\left(1 - \frac{\alpha(\Omega)}{4}\right)\right) \leq 1 - \frac{\alpha(\Omega)}{4}. \tag{3.2.11}$$

So, combining (3.2.10), (3.2.11) and the absolutely continuity of $\nu(t)$, we have

$$\begin{aligned}\frac{\alpha(\Omega)}{4} &\leq 1 - \mu(s_\Omega) \leq 1 - \nu(s_\Omega) + N^2\omega_N^{\frac{2}{N}}\varepsilon = \int_0^{s_\Omega} -\nu'(t)dt + N^2\omega_N^{\frac{2}{N}}\varepsilon \\ &= \int_0^{s_\Omega} \frac{N^2\omega_N^{\frac{2}{N}}}{\nu^{-1+\frac{2}{N}}\int_0^{\nu(r)} f^*}\,dr + N^2\omega_N^{\frac{2}{N}}\varepsilon \leq N^2\omega_N^{\frac{2}{N}}|\Omega|^{2-\frac{2}{N}}\frac{s_\Omega}{\|f\|_1} + N^2\omega_N^{\frac{2}{N}} = N^2\omega_N^{\frac{2}{N}}(s_\Omega + \varepsilon),\end{aligned}$$

that gives

$$s_\Omega \geq \frac{\alpha(\Omega)}{4N^2\omega_N^{\frac{2}{N}}} - \varepsilon. \tag{3.2.12}$$

We now distinguish two cases

Case 1 Let us assume that $N^2\omega_N^{\frac{2}{N}}\varepsilon \leq \dfrac{\alpha(\Omega)}{8}$, then by (3.2.12) and Lemma 3.2.1, we obtain

$$\frac{1}{16N^2\omega_N^{\frac{2}{N}}\gamma_n}\alpha^3(\Omega) \leq \varepsilon; \tag{3.2.13}$$

Case 2 Let us assume that $N^2\omega_N^{\frac{2}{N}}\varepsilon > \dfrac{\alpha(\Omega)}{8}$ then

$$\alpha(\Omega) \leq 8N^2\omega_N^{\frac{2}{N}}\varepsilon \leq 8N^2\omega_N^{\frac{2}{N}}\varepsilon^{\frac{1}{3}} \tag{3.2.14}$$

From (3.2.13) and (3.2.14), we get the claim (1.0.8) by choosing

$$\widetilde{C}_1 = \max\left\{\left(8N^2\omega_N^{\frac{2}{N}}\right)^3, (16N^2\omega_N^{\frac{2}{N}}\gamma_n)\right\},$$

in the case $|\Omega| = 1$ and $||f||_1 = 1$. In the general case, using the scaling properties contained in Section 2.3.2, we obtain the desired claim (1.0.8).

□

An equivalent result holds for almost every superlevel set $U_t = \{u > t\}$.

Theorem 3.2.2. *Let u and v be the solutions to (1.0.3) and (1.0.4) respectively, and let $u^\sharp$ be the Schwarz rearrangement of u. Then, it holds*

$$\alpha^3(U_t) \leq \widetilde{C}_1\frac{|\Omega|^3}{\mu^3(t)}\frac{|\Omega|^{1-\frac{2}{N}}}{\|f\|_1}||v - u^\sharp||_\infty, \tag{3.2.15}$$

where $\widetilde{C}_1$ is defined in (1.0.9).

Proof. Firstly, let us assume that $|\Omega| = 1$, and $\|f\|_1 = 1$.

As in the proof of Theorem 1.0.2, let us suppose that $\alpha(U_t) > 0$ and $||v - u^\sharp||_\infty < 1$, otherwise the inequality (3.2.15) is trivial. By applying the quantitative isoperimetric inequality (2.3.10) to the superlevel sets U_t, and arguing as in (3.2.3), we have

$$1 + \frac{2}{\gamma_n}\alpha^2(U_t) \leq \frac{-\mu'}{N^2\omega_N^{\frac{2}{N}}\mu(t)^{2-\frac{2}{N}}}\int_0^{\mu(t)} f^* \tag{3.2.16}$$

and, integrating from t to $\tau > t$, we obtain

$$\tau - t + \frac{2}{\gamma_n}\int_t^\tau \alpha^2(U_r)\, dr \leq \int_t^\tau \frac{-\mu'(r)}{N^2\omega_N^{\frac{2}{N}}\mu(r)^{2-\frac{2}{N}}}\int_0^{\mu(r)} f^*\, dr.$$

Performing the change of variables $\mu(r) = s$ on the right-hand side, we have

$$\tau - t + \frac{2}{\gamma_n}\int_t^\tau \alpha^2(U_r)\, dr \leq \int_{\mu(\tau)}^{\mu(t)} \frac{1}{N^2\omega_N^{\frac{2}{N}}s^{2-\frac{2}{N}}}\int_0^s f^*\, ds = v^*(\mu(\tau)) - v^*(\mu(t)),$$

that implies

$$\frac{2}{\gamma_n}\int_t^{u^*(r)} \alpha^2(U_r)\, dr \leq v^*(r) - u^*(r) \leq ||v - u^\sharp||_\infty, \tag{3.2.17}$$

obtained using (3.1.3) and the definition of decreasing rearrangement.

Arguing as in Lemma 3.2.1, we define the set

$$A_t := \left\{r \geq t : \mu(r) \geq \mu(t)\left(1 - \frac{\alpha(U_t)}{4}\right)\right\}, \tag{3.2.18}$$

that is non-empty whenever $\alpha(U_t) > 0$. Moreover, we define $s_t = \sup(A_t)$ and we observe that

- $s_t > t$, as we are assuming $\alpha(U_t) > 0$;
- s_t is finite;
- $s_t = u^*\left(\mu(t)\left(1 - \frac{\alpha(U_t)}{4}\right)\right)$;
- for all $r < s_t$, the set U_r verifies the hypothesis (2.3.11) and, so,

$$\alpha(U_r) \geq \frac{\alpha(U_t)}{2}. \tag{3.2.19}$$

Combining (3.2.17) and (3.2.19) we have

$$\frac{1}{2\gamma_n}(s_t - t)\alpha^2(U_t) \leq \varepsilon. \tag{3.2.20}$$

Let us observe that

$$\mu(s_t) = \mu\left(u^*\left(\mu(t)\left(1 - \frac{\alpha(U_t)}{4}\right)\right)\right) \leq \mu(t)\left(1 - \frac{\alpha(U_t)}{4}\right),$$

so, we have

$$\mu(t)\frac{\alpha(U_t)}{4} \leq \mu(t) - \mu(s_t) \leq \nu(t) - \nu(s_t) + N^2\omega_N^{\frac{2}{N}}\varepsilon = \int_t^{s_t} -\nu'(r)\,dr + N^2\omega_N^{\frac{2}{N}}\varepsilon$$
$$= \int_t^{s_t} \frac{N^2\omega_N^{\frac{2}{N}}}{\nu^{-1+\frac{2}{N}}\int_0^{\nu(r)} f^*}\,dr + N^2\omega_N^{\frac{2}{N}}\varepsilon \leq \frac{N^2\omega_N^{\frac{2}{N}}|\Omega|^{2-\frac{2}{N}}}{\|f\|_1}(s_t - t) + N^2\omega_N^{\frac{2}{N}}\varepsilon,$$

that gives

$$(s_t - t) \geq \mu(t)\frac{\alpha(U_t)}{4N^2\omega_N^{\frac{2}{N}}} - \varepsilon. \tag{3.2.21}$$

Now, we distinguish two cases.

Case 1 Let us assume that $N^2\omega_N^{\frac{2}{N}}\varepsilon \leq \mu(t)\dfrac{\alpha(U_t)}{8}$, then by (3.2.21) and (3.2.20), we obtain

$$\frac{\mu(t)}{16N^2\omega_N^{\frac{2}{N}}\gamma_n}\alpha^3(U_t) \leq \varepsilon; \tag{3.2.22}$$

Case 2 Let us assume that $N^2\omega_N^{\frac{2}{N}}\varepsilon > \mu(t)\dfrac{\alpha(U_t)}{8}$ then

$$\mu(t)\alpha(U_t) \leq 8N^2\omega_N^{\frac{2}{N}}\varepsilon \leq 8N^2\omega_N^{\frac{2}{N}}\varepsilon^{\frac{1}{3}} \tag{3.2.23}$$

From (3.2.22) and (3.2.23), we get the claim (3.2.15) with the same costant $\widetilde{C}_1$ defined in (1.0.9).

As in Theorem 1.0.2, we prove the Theorem in the case $|\Omega| = 1$ and $||f||_1 = 1$, and we recover the general case using the rescaling properties from Section 2.3.2. □

3.3 Proof of the Theorem 1.0.3

The aim of this Section is to prove Theorem 1.0.3, and as a first step, we observe that the following lemma holds.

Lemma 3.3.1. *Let $f \in L^{\frac{2N}{N+2}}(\Omega)$ if $N > 2$, $f \in L^p(\Omega)$, $p > 1$ if $N = 2$ and let u be the solution to (1.0.3) and let v be the solution to (1.0.4), then*

$$\int_\Omega |\nabla u|^2 - \int_{\Omega^\sharp} |\nabla u^\sharp|^2 \leq 2\|f\|_1||v - u^\sharp||_\infty. \tag{3.3.1}$$

Proof. We observe that the solution v to (1.0.4) is the unique minimizer in $W_0^{1,2}(\Omega^\sharp)$ of the functional

$$\mathcal{F}(\varphi) = \frac{1}{2}\int_{\Omega^\sharp} |\nabla\varphi|^2 - \int_{\Omega^\sharp} f^\sharp\varphi,$$

that implies

$$\frac{1}{2}\int_{\Omega^\sharp} |\nabla u^\sharp|^2 - \int_{\Omega^\sharp} f^\sharp u^\sharp \geq \frac{1}{2}\int_{\Omega^\sharp} |\nabla v|^2 - \int_{\Omega^\sharp} f^\sharp v.$$

Hence, by the Talenti's inequality (see Remark 2.3.9 with $q = 2$),

$$\int_{\Omega} |\nabla u|^2 \leq \int_{\Omega^\sharp} |\nabla v|^2,$$

and the Hölder inequality, we obtain

$$\int_{\Omega} |\nabla u|^2 - \int_{\Omega^\sharp} |\nabla u^\sharp|^2 \leq \int_{\Omega^\sharp} |\nabla v|^2 - \int_{\Omega^\sharp} |\nabla u^\sharp|^2 \leq 2 \int_{\Omega^\sharp} f^\sharp (v - u^\sharp) < 2\|f\|_1 ||v - u^\sharp||_\infty,$$

and the thesis (3.3.1) follows. □

In particular, (3.3.1) suggests that the Pólya-Szegő inequality holds true almost as an equality when $||v - u^\sharp||_\infty$ is small enough. So it is natural to consider the Pólya-Szegő quantitative inequality, recalled in Theorem 2.3.12.

At this point, to prove the closeness of u to $u^\sharp$ it is enough to show that, for every positive δ, the quantity $M_{u^\sharp}(\delta)$ can be estimated from above by some power of δ. This is not true in general for a generic Sobolev function, but it holds for u because it is the solution of (1.0.3).

The main idea to bound from above the quantity $M_{u^\sharp}(\delta)$ is to write the set $\{|\nabla u| \leq \delta\}$ as the union of suitable subsets, and to bound from above the measure of each of them. In order to do that, in Lemma 3.3.2 we define a set $I \subseteq [0, \|u\|_\infty]$ and, in (3.3.8), we define a positive number $t_{\varepsilon,\beta} \in [0, \|u\|_\infty]$, so that we have

$$\begin{aligned}\{|\nabla u^\sharp| < \delta\} =& \left(\left\{ |\nabla u^\sharp| < \delta \right\} \cap u^{\sharp-1}(I^c \cap (0, t_{\varepsilon,\beta})) \right) \\ &\cup \left(\left\{ |\nabla u^\sharp| < \delta \right\} \cap u^{\sharp-1}(I \cap (0, t_{\varepsilon,\beta})) \right) \\ &\cup \left(\left\{ |\nabla u^\sharp| < \delta \right\} \cap u^{\sharp-1}(t_{\varepsilon,\beta}, +\infty) \right).\end{aligned}$$

In Propositions 3.3.4 and 3.3.6 and in Remark 3.3.5, we study separately the above sets.

We will prove Lemma 3.3.2, Propositions 3.3.3, 3.3.4, 3.3.6, 3.3.7 under the additional assumptions

$$\|f\|_1 = 1, \quad |\Omega| = 1, \quad ||v - u^\sharp||_\infty \leq \varepsilon_0,$$

where ε_0 will be suitably chosen later.

Lemma 3.3.2. *Let us fix $\alpha > 0$, let $\varepsilon = ||v - u^\sharp||_\infty$ and let us define the set I as follows*

$$I = \left\{ t \in [0, \|u\|_\infty] : \int_{v=t} |\nabla v| - \int_{u^\sharp = t} |\nabla u^\sharp| > \varepsilon^\alpha \right\}. \tag{3.3.2}$$

Then,

$$|I| \leq 2\varepsilon^{1-\alpha}, \tag{3.3.3}$$

and, for every $t \in I^c$, it holds

$$\int_{v=t} |\nabla v| \; - \int_{u^\sharp = t} |\nabla u^\sharp| \leq \varepsilon^\alpha. \tag{3.3.4}$$

Proof. Claim (3.3.4) is a direct consequence of the definition of the set I in (3.3.2). Moreover, using the Coarea formula (2.2.1), we have

$$\begin{aligned}\varepsilon^\alpha |I| \leq \int_I \left(\int_{v=t} |\nabla v| - \int_{\{u^\sharp = t\}} |\nabla u^\sharp| \right) &\leq \int_0^{+\infty} \left(\int_{v=t} |\nabla v| - \int_{u^\sharp = t} |\nabla u^\sharp| \right) \\ &= \int_{\Omega^\sharp} |\nabla v|^2 - \int_{\Omega^\sharp} |\nabla u^\sharp|^2 < 2\varepsilon,\end{aligned}$$

and claim (3.3.3) follows. □

Proposition 3.3.3. *Let $\delta > 0$ and let v be the solution to (1.0.4). Then, there exists a positive constant $K_2 = K_2(N)$ such that*

$$|\{|\nabla v| \leq \delta\}| \leq K_2 \delta^n, \tag{3.3.5}$$

where

$$K_2 = \omega_N N^N.$$

Proof. To prove (3.3.5), it is enough to show

$$\{|\nabla v| \leq \delta\} \subseteq \{|x| < N\delta\}. \tag{3.3.6}$$

So, if we take $x \in \{|\nabla v| \leq \delta\}$, we have, using (2.3.4), (2.3.5) and Remark 2.3.8

$$\delta \geq |\nabla v|(x) = \frac{1}{N\omega_N |x|^{N-1}} \int_0^{\omega_N |x|^n} f^*(r)\, dr \geq \frac{|x|}{N|\Omega|} \int_0^{|\Omega|} f^*(r)\, dr = \frac{|x|}{N},$$

and, as a consequence, we have

$$|\nabla v|(x) \geq \frac{|x|}{N}, \tag{3.3.7}$$

and we can conclude. □

Proposition 3.3.4. *Let u be the solution to (1.0.3), let $u^\sharp$ be its Schwartz rearrangement and I as in (3.3.2). If we define the following quantity*

$$t_{\varepsilon,\beta} = \sup\left\{t > 0 \ : \ \mu(t) > \varepsilon N^{n'\beta}\right\}, \quad n' = \frac{N}{N-1}, \tag{3.3.8}$$

then, for every $\delta > 0$, it holds

$$|\left\{|\nabla u^\sharp| < \delta\right\} \cap u^{\sharp -1}(I^c \cap (0, t_{\varepsilon,\beta}))| \leq \frac{1}{N^N}\left(\delta + \frac{\varepsilon^{\alpha-\beta}}{N\omega_N^{\frac{1}{N}}}\right)^N, \tag{3.3.9}$$

for every $\alpha, \beta \in \mathbb{R}$ such that $0 < \beta < \alpha < 1$, $\beta < \dfrac{1}{n'}$.

Proof. We observe that (3.3.4) in Lemma 3.3.2 is equivalent to

$$P(v = t)|\nabla v|(y) - P(u^\sharp = t)|\nabla u^\sharp|(x) < \varepsilon^\alpha, \tag{3.3.10}$$

for every $y \in \{v = t\}$ and $x \in \{u^\sharp = t\}$. If we fix $x \in \{u^\sharp = t\}$ and we consider $y = \ell(t)x$, where $\ell(t) = \left(\frac{\nu(t)}{\mu(t)}\right)^{\frac{1}{N}}$, we have that $y \in \{v = t\}$.

Then, for every $t \in I^c$ such that $t < t_{\varepsilon,\beta}$, (3.3.10) becomes

$$P(v = t)|\nabla v|(\ell(t)x) - P(u^\sharp = t)|\nabla u^\sharp|(x) < \varepsilon N^\alpha, \tag{3.3.11}$$

and, since

$$P(u^\sharp = t) \leq P(v = t),$$

we have

$$|\nabla v|(\ell(t)x) - |\nabla u^\sharp|(x) \leq \frac{\varepsilon N^\alpha}{P(v = t)}. \tag{3.3.12}$$

Now, from definition (3.3.8), follows

$$\omega_N \left(\frac{P(v=t)}{N\omega_N} \right)^{n'} = \nu(t) > \mu(t) > \varepsilon N^{n'\beta}, \tag{3.3.13}$$

and, combining (3.3.12) with (3.3.13), we obtain

$$|\nabla v|(\ell(t)x) - |\nabla u^\sharp|(x) \leq \frac{\varepsilon^{\alpha-\beta}}{N\omega_N^{\frac{1}{N}}}. \tag{3.3.14}$$

Moreover, combining (3.3.7) and (3.3.14) and the fact that $\ell(t) \geq 1$, we have

$$\frac{|x|}{N} - |\nabla u^\sharp|(x) \leq |\nabla v|(\ell(t)x) - |\nabla u^\sharp|(x) \leq \frac{\varepsilon^{\alpha-\beta}}{N\omega_N^{\frac{1}{N}}}, \qquad \forall x \in u^{\sharp-1}(I^c \cap (0, t_{\varepsilon,\beta})). \tag{3.3.15}$$

From (3.3.15), follows that

$$\left\{ |\nabla u^\sharp| < \delta \right\} \cap u^{\sharp-1}(I^c \cap (0, t_{\varepsilon,\beta})) \subseteq \left\{ |x| < N\delta + \frac{\varepsilon^{\alpha-\beta}}{\omega_N^{\frac{1}{N}}} \right\},$$

and, consequently,

$$|\left\{ |\nabla u^\sharp| < \delta \right\} \cap u^{\sharp-1}(I^c \cap (0, t_{\varepsilon,\beta})| \leq \omega_N \left(N\delta + \frac{\varepsilon^{\alpha-\beta}}{\omega_N^{\frac{1}{N}}} \right)^N.$$

□

Remark 3.3.5. We observe that

$$|u^{\sharp-1}(t_{\varepsilon,\beta}, +\infty)| \leq \varepsilon^{n'\beta}, \tag{3.3.16}$$

indeed

$$\mu(t_{\varepsilon,\beta}) = \lim_{t \to t^+_{\varepsilon,\beta}} \mu(t) \leq \varepsilon^{n'\beta},$$

since μ is right continuous.

Proposition 3.3.6. *Let u be the solution to (1.0.3), let $u^\sharp$ be its Schwartz rearrangement,and let I and $t_{\varepsilon,\beta}$ defined as in (3.3.2) and (3.3.8), respectevelly. Then, there exists a positive constant $K_3 = K_3(N)$ and a real number $\theta_4 = \theta_4(N)$ such that*

$$|u^{\sharp-1}(I \cap (0, t_{\varepsilon,\beta}))| < K_3 \varepsilon^{\theta_4}, \tag{3.3.17}$$

where $K_3 = 5N^N\omega_N$, for every $\alpha, \beta \in \mathbb{R}$ such that $0 < \beta < \alpha < 1$, $\beta < \dfrac{1}{n'}$.

Proof. From (2.3.6) and (2.3.7), we have

$$\frac{-\nu'(t)}{\nu(t)^{2-\frac{2}{N}}} \int_0^{\nu(t)} f^*(r)\, dr \leq \frac{-\mu'(t)}{\mu(t)^{2-\frac{2}{N}}} \int_0^{\mu(t)} f^*(r)\, dr \leq \frac{-\mu'(t)}{\mu(t)^{2-\frac{2}{N}}} \int_0^{\nu(t)} f^*(r)\, dr$$

and, consequently, using (3.2.10), for all $t \in (0, t_{\varepsilon,\beta})$, it holds

$$-\mu'(t) \geq -\nu'(t)\left(\frac{\mu(t)}{\nu(t)}\right)^{2-\frac{2}{N}} \geq -\nu'(t)\left(1 - \frac{N^2\omega_N^{\frac{2}{N}}\varepsilon}{\nu(t)}\right)^{2-\frac{2}{N}} \geq -\nu'(t)\left(1 - N^2\omega_N^{\frac{2}{N}}\varepsilon^{1-n'\beta}\right)^{2-\frac{2}{N}}. \tag{3.3.18}$$

Moreover, it is easily seen that

$$|u^{\sharp-1}(I^c \cap (0, t_{\varepsilon,\beta}))| = |u^{\sharp-1}(I^c \cap (0, t_{\varepsilon,\beta})) \cap \{|\nabla u^\sharp| = 0\}| + \int_{I^c\cap(0,t_{\varepsilon,\beta})}\int_{u^\sharp=t}\frac{1}{|\nabla u^\sharp|} \geq \int_{I^c\cap(0,t_{\varepsilon,\beta})} -\mu'.$$

Hence, from (3.3.18) and the fact that the distribution function of v is absolutely continuous, we get

$$\begin{aligned}
|u^{\sharp-1}(I^c \cap (0, t_{\varepsilon,\beta}))| &\geq \left(1 - N^2\omega_N^{\frac{2}{N}}\varepsilon^{1-n'\beta}\right)^{2-\frac{2}{N}} \int_{I^c\cap(0,t_{\varepsilon,\beta})} -\nu' \\
&= \left(1 - N^2\omega_N^{\frac{2}{N}}\varepsilon^{1-n'\beta}\right)^{2-\frac{2}{N}} |v^{-1}(I^c \cap (0, t_{\varepsilon,\beta}))| \\
&\geq |v^{-1}(I^c \cap (0, t_{\varepsilon,\beta}))| \left(1 - \left(2 - \frac{2}{N}\right) N^2\omega_N^{\frac{2}{N}}\varepsilon^{1-n'\beta}\right),
\end{aligned}$$

where the last inequality is the Bernoulli inequality, which holds true as

$$\varepsilon < \varepsilon_0 := \left(\frac{1}{N^2\omega_N^{\frac{2}{N}}}\right)^{\frac{1}{1-n'\beta}}. \tag{3.3.19}$$

So, we have that

$$\begin{aligned}
|u^{\sharp-1}(I \cap (0, t_{\varepsilon,\beta}))| &= 1 - |u^{\sharp-1}(I^c \cap (0, t_{\varepsilon,\beta}))| - |u^{\sharp-1}(t_{\varepsilon,\beta}, +\infty)| \\
&\leq 1 - |u^{\sharp-1}(I^c \cap (0, t_{\varepsilon,\beta}))| - \mu(t_{\varepsilon,\beta}) \\
&\leq 1 - |v^{-1}(I^c \cap (0, t_{\varepsilon,\beta}))| \left(1 - \left(2 - \frac{2}{N}\right) N^2\omega_N^{\frac{2}{N}}\varepsilon^{1-n'\beta}\right) - \nu(t_{\varepsilon,\beta}) + N^2\omega_N^{\frac{2}{N}}\varepsilon \\
&= |v^{-1}(I \cap (0, t_{\varepsilon,\beta}))| + N^2\omega_N^{\frac{2}{N}}\left(\left(2 - \frac{2}{N}\right)\varepsilon^{1-n'\beta} + \varepsilon\right).
\end{aligned} \tag{3.3.20}$$

Arguing as in [38, Lemma 3.2], we can deduce that

$$\begin{aligned}
|v^{-1}(I)| &= \int_I \int_{v=t} \frac{1}{|\nabla v|} \\
&= \int_I \int_{\{v=t\}\cap\{|\nabla v| \leq \delta\}} \frac{1}{|\nabla v|} + \int_{\{v=t\}\cap\{\nabla v > \delta\}} \frac{1}{|\nabla v|} \\
&\leq |\{|\nabla v| \leq \delta\}| + \frac{1}{\delta}\int_I \int_{\{v=t\}\cap\{|\nabla v|>\delta\}} \frac{1}{|\nabla v|} \\
&\leq |\{|\nabla v| \leq \delta\}| + \frac{1}{\delta}\int_I N\omega_N\left(\frac{\nu(t)}{\omega_N}\right)^{\frac{N-1}{N}} \\
&= |\{|\nabla v| \leq \delta\}| + \frac{1}{\delta}N\omega_N^{\frac{1}{N}}|I|
\end{aligned}$$

and, by Proposition 3.3.3 and Lemma 3.3.2,

$$|v^{-1}(I \cap (0, t_{\varepsilon,\beta}))| \leq |v^{-1}(I)| \leq \omega_N N^N \delta^N + \frac{N\omega_N^{\frac{1}{N}}}{\delta}\varepsilon^{1-\alpha}. \tag{3.3.21}$$

Combining (3.3.20) and (3.3.21), we obtain, setting $\delta = \varepsilon^q$, for $0 < q < 1-\alpha$

$$|u^{\sharp-1}(I \cap (0, t_{\varepsilon,\beta}))| \leq N^2 \omega_N^{\frac{2}{N}} \left(\left(2 - \frac{2}{N}\right) \varepsilon^{1-n'\beta} + \varepsilon \right) + \omega_N N^N \varepsilon^{nq} + n\omega_N^{\frac{1}{N}} \varepsilon^{1-\alpha-q}.$$

Choosing

$$\theta_4 = \min\{1 - n'\beta, 1 - \alpha - q, nq\},$$

we get the desired claim for a positive constant K_3. □

Proposition 3.3.7. *Let u be the solution to (1.0.3) and let $M_{u^\sharp}$ be the quantity defined in (2.3.14). Then, there exist $\theta_5 = \theta_5(N) > 0$ and $K_4 = K_4(N)$ such that*

$$M_{u^\sharp}(\varepsilon^r) \leq K_4 \varepsilon^{\theta_5},$$

where $K_4 = 7N^n\omega_N$.

Proof. We have

$$\begin{aligned}
M_{u^\sharp}(\delta) &\leq |\{|\nabla u^\sharp| < \delta\}| \\
&= |\Big\{|\nabla u^\sharp| < \delta\Big\} \cap u^{\sharp-1}(I^c \cap (0, t_{\varepsilon,\beta}))| \\
&+ |\Big\{|\nabla u^\sharp| < \delta\Big\} \cap u^{\sharp-1}(I \cap (0, t_{\varepsilon,\beta}))| \\
&+ |\Big\{|\nabla u^\sharp| < \delta\Big\} \cap u^{\sharp-1}(t_{\varepsilon,\beta}, +\infty)| \\
&\leq |\Big\{|\nabla u^\sharp| < \delta\Big\} \cap u^{\sharp-1}(I^c \cap (0, t_{\varepsilon,\beta})| \\
&+ |u^{\sharp-1}(I \cap (0, t_{\varepsilon,\beta}))| + \mu(t_{\varepsilon,\beta}) \\
&\leq \frac{1}{N^N}\left(\delta + \frac{\varepsilon^{\alpha-\beta}}{N\omega_N^{\frac{1}{N}}}\right)^N + K_3\varepsilon^{\theta_4} + \varepsilon^{n'\beta},
\end{aligned}$$

where in the last inequality we have used (3.3.9), (3.3.16) and (3.3.17). So, evaluating $M_{u^\sharp}(\delta)$ in ε^r, we obtain

$$M_{u^\sharp}(\varepsilon^r) \leq \frac{1}{N^N}\left(\varepsilon^r + \frac{\varepsilon^{\alpha-\beta}}{N\omega_N^{\frac{1}{N}}}\right)^N + K_3\varepsilon^{\theta_4} + \varepsilon^{n'\beta},$$

so we can conclude, by setting

$$\theta_5 = \min\{Nr, N(\alpha - \beta), \theta_4, n'\beta\},$$

obtaining the desired claim for a positive constant K_4. □

We are now in position to conclude with the proof of Theorem 1.0.3.

Proof of Theorem 1.0.3. Firstly, let us assume $|\Omega| = 1$, and $\|f\|_1 = 1$ and $\varepsilon < \varepsilon_0$. If this is not the case, i.e. $\varepsilon > \varepsilon_0$, the thesis (1.0.10) by choosing $\tilde{C}_2$ sufficiently big.

Other, from Theorem 2.3.12, we have that there exist two positive exponents r and s such that

$$\inf_{x_0 \in \mathbb{R}^N} \frac{\int_{\mathbb{R}^N} |u(x) - u^\sharp(x + x_0)|\,dx}{\|\nabla u^\sharp\|_2} \leq C(N)\left[M_{u^\sharp}(E(u)^r) + E(u)\right]^s, \tag{3.3.22}$$

where

$$E(u) = \frac{\int_\Omega |\nabla u|^2 - \int_{\Omega^\sharp} |\nabla u^\sharp|^2}{\int_{\Omega^\sharp} |\nabla u^\sharp|^2}.$$

By using Lemma 3.3.1, we have that

$$E(u) \leq 2\frac{||v - u^\sharp||_\infty}{\int_{\Omega^\sharp} |\nabla u^\sharp|^2}. \tag{3.3.23}$$

Hence, using (3.3.23) and Proposition 3.3.7, we get

$$\inf_{x_0 \in \mathbb{R}^N} \frac{||u - u^\sharp(\cdot + x_0)||_{L^1(\mathbb{R}^N)}}{\left(\int_{\Omega^\sharp} |\nabla u^\sharp|^2\right)^{\frac{1}{2}}} \leq C(N)\left[K_4\left(\frac{2||v - u^\sharp||_\infty}{\int_{\Omega^\sharp} |\nabla u^\sharp|^2}\right)^{\theta_5} + \frac{2||v - u^\sharp||_\infty}{\int_{\Omega^\sharp} |\nabla u^\sharp|^2}\right]^s.$$

So, there exists $K_5 = K_5(N)$, such that

$$\inf_{x_0 \in \mathbb{R}^N} \frac{||u - u^\sharp(\cdot + x_0)||_{L^1(\mathbb{R}^N)}}{\left(\int_{\Omega^\sharp} |\nabla u^\sharp|^2\right)^{\frac{1}{2}}} \leq K_5\left(\frac{||v - u^\sharp||_\infty}{\int_{\Omega^\sharp} |\nabla u^\sharp|^2}\right)^{\tilde{\theta}_1}, \tag{3.3.24}$$

being $\tilde{\theta}_1 = s\theta_5$. Moreover, if we apply the Talenti'result (2.3.9) with $q = 2$ there exists $K_1 = K_1(N, 2)$ such that

$$||\nabla u^\sharp||_2 \leq ||\nabla v||_2 \leq K_1 \|f\|_{\frac{2N}{N+2}}.$$

Hence, it follows that there exists a constant $\widetilde{C}_2 > 0$ such that

$$\inf_{x_0 \in \mathbb{R}^N} ||u - u^\sharp(\cdot + x_0)||_{L^1(\mathbb{R}^N)} \leq \widetilde{C}_2 ||v - u^\sharp||_\infty^{\tilde{\theta}_1} \|f\|_{\frac{2N}{N+2}}^{1 - 2\tilde{\theta}_1}. \tag{3.3.25}$$

Finally, we remove the additional assumptions $|\Omega| = 1$ and $||f||_1 = 1$ using the rescaling properties in Section 2.3.2.

$\square$

Remark 3.3.8. We can improve estimate (1.0.10) by replacing the L^1-norm with the L^q- norm, with $q \in \left[1, \frac{2N}{N-2}\right[$, by exploiting the classical Gagliardo-Nirenberg estimate (see [95, pag.125]). In particular, for $q = 2$, we obtain

$$\inf_{x_0 \in \mathbb{R}^N} ||u - u^\sharp(\cdot + x_0)||_{L^2(\mathbb{R}^N)}^{\frac{N+2}{2}} \leq K_6 \frac{\|f\|_{\frac{2N}{N+2}}^{2+N-4\tilde{\theta}_1}}{\|f\|_1^{1-3\tilde{\theta}_1}} |\Omega|^{1 + (\frac{2}{N} - 1)\tilde{\theta}_1} ||v - u^\sharp||_\infty^{\tilde{\theta}_1}. \tag{3.3.26}$$

where $K_6 = K_6(N) > 0$ is a positive constant.

3.4 Proof of Theorem 1.0.4

The aim of this Section is to prove Theorem 1.0.4. We are assuming that f is in $L^2(\mathbb{R}^N)$, $|\Omega| = 1$ and $||f||_1 = 1$.

Furthermore, without loss of generality, we can assume that $x_0 = 0$ achieves the infimum in

$$\inf_{x_0 \in \mathbb{R}^N} \|u - u^\sharp(\cdot + x_0)\|_{L^2(\mathbb{R}^N)} = \|u - u^\sharp\|_{L^2(\mathbb{R}^N)}, \tag{3.4.1}$$

and by the arguments in [38] it is possible to show that

$$|\Omega \Delta \Omega^\sharp| \leq \tilde{C}_4 \varepsilon^{\frac{1}{4}}. \tag{3.4.2}$$

Indeed, following [38, Lemma 4.5], $\Omega^\sharp$ is concentric to the ball which achieves the infimum in

$$\alpha(U_{t_0}) = \inf_{x_0 \in \mathbb{R}^N} \left\{ \frac{|U_{t_0} \Delta B_{t_0}|}{|B_{t_0}|} \,|\, |U_{t_0}| = |B_{t_0}| \right\},$$

with $t_0 \leq C\varepsilon^{\frac{1}{4}}$.

Moreover, using (3.2.9), we have

$$\mu(t_0) \geq \nu(t_0+\varepsilon) = 1 - \int_0^{t_0+\varepsilon} \frac{N^2 \omega_N^{\frac{2}{N}}}{\nu^{-1+\frac{2}{N}} \fint_0^{\nu(r)} f^*} \geq 1 - \int_t^{t+\varepsilon} \frac{N^2 \omega_N^{\frac{2}{N}}}{|\Omega|^{-1+\frac{2}{N}}} \frac{|\Omega|}{\|f\|_1} = 1 - N^2 \omega_N^{\frac{2}{N}} (t_0+\varepsilon). \tag{3.4.3}$$

Eventually, taking into account Theorem 3.2.2, formula (3.4.3) implies

$$\begin{aligned}
|\Omega \Delta \Omega^\sharp| &= 2|\Omega \setminus \Omega^\sharp| \\
&= 2\left|\Big(\{u > t_0\} \cup \{u \leq t_0\}\Big) \setminus \Omega^\sharp\right| \\
&= 2|\{u > t_0\} \setminus \Omega^\sharp| + 2|\{u \leq t_0\} \setminus \Omega^\sharp| \\
&\leq 2|\{u > t_0\} \setminus B_{t_0}| + 2|\{u \leq t_0\}| \\
&= \alpha(U_{t_0}) + (1 - \mu(t_0)) \\
&\leq \frac{\tilde{C}_1}{\mu(t_0)} \varepsilon^{\frac{1}{3}} + N^2 \omega_N^{\frac{2}{N}} (t_0 + \varepsilon) \\
&\leq 2\tilde{C}_1 \varepsilon^{\frac{1}{3}} + N^2 \omega_N^{\frac{2}{N}} (C\varepsilon^{\frac{1}{4}} + \varepsilon) \\
&\leq \tilde{C}_4 \varepsilon^{\frac{1}{4}}.
\end{aligned}$$

The main idea of this section is to apply the quantitative Hardy-Littlewood inequality (Theorem 2.3.13) to $h = v$, the solution to (1.0.4), and the function g defined as follows

$$g = f\chi_{\Omega^\sharp \cap \Omega}. \tag{3.4.4}$$

Next lemma ensures us that the functions h and g are admissible functions for Theorem 2.3.13.

Lemma 3.4.1. *Let $f \in L^2(\Omega)$ and let g be the function defined in (3.4.4). Then , there exist $p \in (1, +\infty)$ and $q \in [1, +\infty)$ such that, we have*

$$\inf_{x_0 \in \mathbb{R}^N} \|g - g^\sharp(\cdot + x_0)\|_{L^m(\mathbb{R}^N)} \leq \left(\frac{1}{2^{p+1} eq} \|g\|^{pq}_{\Lambda^q_p(\Omega)} \int_{\Omega^\sharp} (g^\sharp - g) v \right)^{\frac{1}{1+pq}}, \tag{3.4.5}$$

where

$$m = \frac{qp+1}{p+1},$$

and moreover, it holds $m < 2$.

Proof. In order to prove (3.4.5), we apply Theorem 2.3.13 to the function $h = v$ and g defined in (3.4.4). Let us check that the assumptions of the theorem are satisfied, namely

$$\theta_p(s) < \infty; \quad \|g\|_{\Lambda_p^q(\Omega)} < \infty,$$

for some $p, q \in [1, +\infty)$. Indeed,

$$\begin{aligned}
\theta_p(s) &= \left(\int_0^s \left(\frac{1}{N^2 \omega_N^{\frac{2}{N}} t^{1-\frac{2}{N}}} \fint_0^t f^*(r)\, dr \right)^{-\frac{1}{p-1}} dt \right)^{\frac{1}{p'}} \\
&\le \left(\frac{N^2 \omega_N^{\frac{2}{N}} |\Omega|}{\|f\|_1} \right)^{\frac{1}{p}} \left(\int_0^s t^{(1-\frac{2}{N})\frac{1}{p-1}} \right)^{\frac{1}{p'}} < +\infty,
\end{aligned}$$

for all $p > 1$, and

$$\begin{aligned}
\|g\|^q_{\Lambda_p^q} &= \int_0^{|\Omega|} (g^*)^q \theta_p'(s) \\
&= \frac{1}{p'} \int_0^{|\Omega|} (g^*)^q \Big(\int_0^s \Big(\frac{1}{N^2 \omega_N^{\frac{2}{N}} t^{1-\frac{2}{N}}} \fint_0^t f^* \Big)^{-\frac{1}{p-1}} \Big)^{-\frac{1}{p}} \Big(\frac{1}{N^2 \omega_N^{\frac{2}{N}} s^{1-\frac{2}{N}}} \fint_0^s f^* \Big)^{-\frac{1}{p-1}} ds \\
&\le \frac{1}{p'} \int_0^{|\Omega|} (f^*)^q \left[\frac{\Big(\frac{1}{N^2 \omega_N^{\frac{2}{N}} s^{2-\frac{2}{N}}} \int_0^s f^* \Big)^{-\frac{1}{p-1}}}{\int_0^s \Big(\frac{1}{N^2 \omega_N^{\frac{2}{N}} t^{2-\frac{2}{N}}} \int_0^t f^* \Big)^{-\frac{1}{p-1}}} \right]^{\frac{1}{p}} \Big(\frac{1}{N^2 \omega_N^{\frac{2}{N}} s^{2-\frac{2}{N}}} \int_0^s f^* \Big)^{-\frac{1}{p}} ds.
\end{aligned}$$

We observe that, if $t \le s$, we have

- $\displaystyle\int_0^t f^* \le \int_0^s f^*$, since f^* is positive;
- $f^*(t) \ge f^*(s)$, since f^* is decreasing.

It follows that

$$\begin{aligned}
\|g\|^q_{\Lambda_p^q} &\le \frac{1}{p'} \int_0^{|\Omega|} (f^*)^q \left(\frac{s^{(2-\frac{2}{N})\frac{1}{p-1}}}{\int_0^s t^{(2-\frac{2}{N})\frac{1}{p-1}}} \right)^{\frac{1}{p}} \Big(\frac{1}{N^2 \omega_N^{\frac{2}{N}} s^{2-\frac{2}{N}}} \int_0^s f^* \Big)^{-\frac{1}{p}} \\
&\le K_7(p,N) \int_0^{|\Omega|} (f^*)^{q-\frac{1}{p}} \left(\frac{s^{(2-\frac{2}{N})\frac{1}{p-1}}}{s^{(2-\frac{2}{N})\frac{1}{p-1}+1}} \right)^{\frac{1}{p}} s^{(1-\frac{2}{N})\frac{1}{p}} \\
&\le K_7(p,N) \int_0^{|\Omega|} (f^*)^{q-\frac{1}{p}} s^{-\frac{2}{Np}}\, ds = K_7(p,N) \int_0^{|\Omega|} \left(f^* s^{\frac{p-\frac{2}{N}}{qp-1}} \right)^{q-\frac{1}{p}} \frac{ds}{s},
\end{aligned}$$

where

$$K_7(p,N) = \frac{1}{p'}\left[\left(2-\frac{2}{N}\right)\frac{1}{p-1}+1\right]^{\frac{1}{p}}(N^2\omega_N^{\frac{2}{N}})^{\frac{1}{p}}.$$

So, we have that

$$\|g\|^q_{\Lambda^q_p} \leq K_7(p,N)\|f\|^l_{L^{k,l}(\mathbb{R}^N)},$$

where

$$k = \frac{qp-1}{p-\frac{2}{N}}, \qquad l = \frac{qp-1}{p}.$$

By imposing $l > 0$ and $0 < k < 2$, we obtain the following constraints on p and q:

$$p > 1, \qquad \frac{1}{p} \leq q < 2 + \left(1-\frac{4}{N}\right)\frac{1}{p}. \tag{3.4.6}$$

So, by the classical embedding Theorem for Lorentz space (see [85]), we have

$$L^{2,2} \hookrightarrow L^{k,l}.$$

Now, we are in position to use Theorem 2.3.13 and we obtain

$$\inf_{x_0\in\mathbb{R}^N}\|g-g^\sharp(\cdot+x_0)\|_{L^m(\mathbb{R}^N)} \leq \left(\frac{1}{2^{p+1}\,e\,q}\|g\|^{pq}_{\Lambda^q_p(\Omega)}\int_{\Omega^\sharp}(g^\sharp-g)v\right)^{\frac{1}{1+pq}},$$

where

$$m = \frac{qp+1}{p+1} = \frac{2N(k-l)+2kl}{2k+n(k-l)}.$$

Moreover, we observe that, from (3.4.6), it follows that $m < 2$.

□

Let us prove now Theorem 1.0.4.

Proof of Theorem 1.0.4. Firstly, let us observe that

$$\begin{aligned}\inf_{x_0\in\mathbb{R}^N}||f-f^\sharp(\cdot+x_0)||_{L^m(\mathbb{R}^N)} &\leq ||f-f^\sharp||_{L^m(\mathbb{R}^N)}\\ &\leq \|f-g\|_{L^m(\mathbb{R}^N)} + ||g-g^\sharp||_{L^m(\mathbb{R}^N)} + ||f^\sharp-g^\sharp||_{L^m(\mathbb{R}^N)}.\end{aligned}$$

Let us estimate from above the quantities $||f^\sharp-g^\sharp||_{L^m(\mathbb{R}^n)}$ and $\|f-g\|_{L^m(\mathbb{R}^n)}$:

$$\begin{aligned}||f^\sharp-g^\sharp||_{L^m(\mathbb{R}^N)} \leq \|f-g\|_{L^m(\mathbb{R}^N)} &= \|f\|_{L^m(\Omega\setminus\Omega^\sharp)}\\ &\leq |\Omega\setminus\Omega^\sharp|^{\frac{1}{m}-\frac{1}{2}}\|f\|_2\\ &\leq \tilde{C}_4\|f\|_2\varepsilon^{\frac{2-m}{8m}}\end{aligned} \tag{3.4.7}$$

where in the last inequality we have used Holder's inequality and (3.4.2).

It only remains to estimate $||g-g^\sharp||_{L^m(\mathbb{R}^N)}$, and thanks to Lemma 3.4.1 and (3.3.26), it is enough to estimate the following quantity

$$\begin{aligned}\int_{\Omega^\sharp}(g^\sharp-g)v &\leq \int_{\Omega^\sharp} f^\sharp v-\int_\Omega fu+\int_{\mathbb{R}^N} f(u-u^\sharp)+\int_{\Omega^\sharp} f(u^\sharp-v)\\ &=\int_{\Omega^\sharp}|\nabla v|^2-\int_\Omega|\nabla u|^2+\int_{\mathbb{R}^N} f(u-u^\sharp)\\ &\leq 2\varepsilon+\|f\|_2||u-u^\sharp||_{L^2(\mathbb{R}^N)}\\ &\leq 2\varepsilon+K_6\|f\|_2\left(\|f\|_{\frac{2N}{N+2}}^{2+N-4\tilde\theta_1}\varepsilon^{\tilde\theta_1}\right)^{\frac{2}{N+2}}\\ &\leq 2\varepsilon+K_6\|f\|_2^{3-\frac{8}{N+2}\tilde\theta_1}\varepsilon^{\frac{2}{N+2}\tilde\theta_1}.\end{aligned} \tag{3.4.8}$$

Hence,

$$||g-g^\sharp||_{L^m(\mathbb{R}^N)}\leq\left(\frac{1}{2^{p+1}\,e\,q}\right)^{\frac{1}{1+pq}}\|f\|_2^{\frac{pq-1}{1+pq}}\left(2\varepsilon+K_6\|f\|_2^{3-\frac{8}{N+2}\tilde\theta_1}\varepsilon^{\frac{2}{N+2}\tilde\theta_1}\right)^{\frac{1}{1+pq}}. \tag{3.4.9}$$

Combining (3.4.7) and (3.4.9), there exists a constant $\widetilde{C}_3$ depending only on m and the dimension n, such that

$$\inf_{x_0\in\mathbb{R}^N}\|f-f^\sharp(\cdot+x_0)\|_{L^m(\mathbb{R}^N)}\leq\widetilde{C}_3\|f\|_2^{\tilde\theta_2}\varepsilon^{\tilde\theta_3},$$

where

$$\tilde\theta_2=\max\left\{1,\left(2+pq-\frac{8\tilde\theta_1}{N+2}\right)\left(\frac{1}{1+pq}\right)\right\},\qquad \tilde\theta_3=\min\left\{\frac{2-m}{8m},\frac{1}{1+pq},\left(\frac{2\tilde\theta_2}{N+2}\right)\frac{1}{1+pq}\right\}.$$

□

Remark 3.4.2. We explicitly observe that in Theorem 1.0.4 we cannot obtain a comparison result for the quantity $||f-f^\sharp||_{L^2(\mathbb{R}^N)}$. Indeed, let B be the unitary ball, centered at the origin, in the plane, and let us consider the following problem and its symmetrized:

$$\begin{cases}-\Delta u_\sigma=f_\sigma & \text{in } B\\ u_\sigma=0 & \text{on } \partial B.\end{cases};\qquad\qquad \begin{cases}-\Delta v_\sigma=f_\sigma^\sharp & \text{in } B\\ v_\sigma=0 & \text{on } \partial B,\end{cases}$$

where

$$f_\sigma(x)=1+\sigma^{-1}\chi_{B_\sigma(1/2,0)}(x).$$

and, consequently,

$$f_\sigma^\sharp(x)=1+\sigma^{-1}\chi_{B_\sigma(0,0)}(x).$$

With these choices, we obtain

$$\|f_\sigma-f_\sigma^\sharp\|_{L^2(B)}=\sqrt{2\pi}$$

and

$$\|f_\sigma-f_\sigma^\sharp\|_{L^r(B)}=(2\pi)^{1/r}\,\sigma^{2/r-1}\to 0$$

as $\sigma\to 0$ and $r<2$.

In order to conclude, we refer now to [67, Theorem 8.16], that states that the following estimate holds

$$\|w\|_\infty \leq C\left(\sup_{\partial\Omega} w + \|g\|_{\frac{q}{2}}\right)$$

where w solves in Ω

$$-\Delta w = g$$

with $g \in L^{\frac{q}{2}}$, $q > 2$. So, in our case, we can choose $w = v_\sigma - u_\sigma$ and any $r > 1$, obtaining

$$\|v_\sigma - u_\sigma\|_{L^\infty(B)} \leq C\|f_\sigma^\sharp - f_\sigma\|_{L^r(B)},$$

so it is clear that, if $1 \leq r < 2$, $\|v_\sigma - u_\sigma\|_{L^\infty(B)} \to 0$.

3.5 L^p-bound of the asymmetry

In Theorem 1.0.2, we bound the Frankel asymmetry of the set Ω with the quantity $||u^\sharp - v||_\infty$. We observe that it is possible to obtain an analogous result in terms of $||v||_p - ||u||_p$ and this kind of estimate might seem more natural, as the solution to (1.0.3) and (1.0.4)are in L^∞ only if $f \in L^p$ with $p > N/2$.

Arguing as in the proof of Theorem (1.0.2), the following estimate holds

$$\alpha^{2+p}(\Omega) \leq C(\|v\|_p^p - \|u\|_p^p).$$

Indeed, let us observe that, if we multiply (3.2.3), i.e.

$$N^2\omega_N^{\frac{2}{N}}\mu^{2-\frac{2}{N}}(t)\left(1 + \frac{2}{\gamma_n}\alpha^2(U_t)\right) \leq \left(\int_0^{\mu(t)} f^*\right)(-\mu'(t)),$$

by the quantity

$$\frac{pt^{p-1}\mu(t)^{1-2+\frac{2}{N}}}{N^2\omega_N^{\frac{2}{N}}}$$

and integrate between 0 and $||u||_\infty$, we obtain

$$\begin{aligned}\int_0^{||u||_\infty} pt^{p-1}\mu(t) + c_n\int_0^{||u||_\infty} pt^{p-1}\mu(t)\alpha^2(U_t)\,dt \leq \\ \frac{1}{N^2\omega_N^{2/N}}\int_0^{||u||_\infty} pt^{p-1}t\mu(t)^{\frac{2}{N}-1}(-\mu'(t))\left(\int_0^{\mu(t)} f^*(s)ds\right)dt.\end{aligned} \tag{3.5.1}$$

We define now

$$F(l) = \int_0^l w^{\frac{2}{N}-1}\left(\int_0^w f^*(s)ds\right)dw$$

and, since $F(\cdot)$ is an increasing function and the Talenti comparison in Theorem 3.1.1 holds, we have

$$F(\mu(t)) < F(\nu(t)).$$

So, (3.5.1), becomes

$$\int_0^{||u||_\infty} pt^{p-1}\mu(t)+\frac{2}{\gamma_n}\int_0^{||u||_\infty} pt^{p-1}\mu(t)\alpha^2(U_t)dt \leq -\frac{1}{N^2\omega_N^{2/N}}\int_0^{\|u\|_\infty} -pt^{p-1}\frac{d}{dt}(F(\mu(t)))\,dt$$
$$= \frac{1}{N^2\omega_N^{2/N}}\int_0^{\|u\|_\infty} p(p-1)t^{p-2}F(\mu(t)) \leq \frac{1}{N^2\omega_N^{2/N}}\int_0^{\|v\|_\infty} p(p-1)t^{p-2}F(\nu(t))$$
$$= \int_0^{\|v\|_\infty} pt^{p-1}\nu(t)\,dt.$$

Then,

$$\frac{2}{\gamma_n}\int_0^{s_\Omega} pt^{p-1}\mu(t)\alpha^2(U_t) \leq \frac{2}{\gamma_n}\int_0^{\|u\|_\infty} pt^{p-1}\mu(t)\alpha^2(U_t) \leq \|v\|_p{}^p - \|u\|_p{}^p$$

and by definition of s_Ω in (3.2.1),

$$\frac{2}{\gamma_n}\int_0^{s_\Omega} pt^{p-1}\mu(t)\alpha^2(U_t) \geq \frac{2}{\gamma_n}|\Omega|\left(1-\frac{\alpha(U_t)}{4}\right)\frac{\alpha^2(\Omega)}{4}s_\Omega{}^p \geq \frac{|\Omega|}{4\gamma_n}\alpha^2(\Omega)s_\Omega^p.$$

Hence,

$$\frac{|\Omega|}{4\gamma_n}s_\Omega^p\alpha^2(\Omega) \leq \|v\|_p{}^p - \|u\|_p{}^p. \tag{3.5.2}$$

In order to conclude, we have to define t_1 such that

$$\nu(2t_1) = |\Omega|\left(1-\frac{1}{8}\alpha(\Omega)\right),$$

for which it holds the following bound

$$\frac{|\Omega|}{8}\alpha(\Omega) = |\Omega| - |\Omega|\left(1-\frac{1}{8}\alpha(\Omega)\right) = \nu(0)-\nu(2t_1) = -\int_0^{2t_1}\nu'(s)\,ds$$
$$= \int_0^{2t_1}\frac{N^2\omega_N^{\frac{2}{N}}}{\nu^{-1+\frac{2}{N}}(t)\int_0^{\nu(t)} f^*} \leq \frac{N^2\omega_N^{\frac{2}{N}}}{|\Omega|^{-1+\frac{2}{N}}\|f\|_1}2t_1.$$

Now let us distinguish two cases:

- $s_\Omega \geq t_1 \geq C\alpha(\Omega)$ then from (3.5.2), we obtain

$$\|v\|_p{}^p - \|u\|_p{}^p \geq \frac{|\Omega|}{4\gamma_n}C\alpha^{2+p}(\Omega);$$

- $s_\Omega < t_1$ we have

$$\|v\|_p{}^p - \|u\|_p{}^p = \int_0^{\|v\|_\infty} pt^{p-1}(\nu(t)-\mu(t))\,dt \geq \int_{t_1}^{2t_1} pt^{p-1}(\nu(t)-\mu(t))\,dt. \tag{3.5.3}$$

Since $s_\Omega < t_1 \leq t \leq 2t_1$, we have both

$$\nu(t) \geq \nu(2t_1) = |\Omega|\left(1-\frac{1}{8}\alpha(\Omega)\right), \tag{3.5.4}$$

and

$$\mu(t) \leq |\Omega| \left(1 - \frac{1}{4}\alpha(\Omega)\right). \tag{3.5.5}$$

Consequently, combining (3.5.3), (3.5.4) and (3.5.5) we have

$$\begin{aligned}\|v\|_p{}^p - \|u\|_p{}^p &\geq \int_{t_1}^{2t_1} pt^{p-1}(\nu(t) - \mu(t))\, dt \geq \frac{|\Omega|}{8}\alpha(\Omega) \int_{t_1}^{2t_1} pt^{p-1}\, dt \\ &\geq \frac{|\Omega|}{8}\alpha(\Omega)(2^p - 1)t_1^p \geq C(2^p - 1)\frac{|\Omega|}{8}\alpha^{p+1}(\Omega) \geq C(2^p - 1)\frac{|\Omega|}{8}\alpha^{2+p}(\Omega),\end{aligned}$$

since $\alpha(\Omega) < 1$.

3.6 Conclusions and Open Problems

Proof of Theorem 1.0.1. The proof of the main Theorem follows directly by combining Theorem 1.0.2, Theorem 1.0.3, and Theorem 1.0.4, by choosing

$$\theta_1 = \frac{1}{\tilde\theta_1}, \quad \theta_2 = \frac{1}{\tilde\theta_2}$$

and

$$C_1 = \frac{1}{\widetilde{C}_1}|\Omega|^{\frac{2-n}{N}}\|f\|_1, \quad C_2 = \left(\frac{1}{\widetilde{C}_2}\frac{\|f\|_1^{1-3\tilde\theta_1}}{\|f\|_{\frac{2N}{N+2}}^{1-2\tilde\theta_1}}|\Omega|^{-1-\left(\frac{2}{N}-1\right)\tilde\theta_1}\right)^{\frac{1}{\tilde\theta_1}},$$

$$C_3 = \left(\widetilde{C}_3\|f\|_2^{\tilde\theta_2}\|f\|_1^{1-\tilde\theta_3-\tilde\theta_2}|\Omega|^{\frac{\tilde\theta_2}{2}+\frac{1-m}{m}+\left(\frac{N-2}{N}\right)\tilde\theta_3}\right)^{-\frac{1}{\tilde\theta_3}},$$

where $\widetilde{C}_1$, $\widetilde{C}_2$ and $\widetilde{C}_3$ are the constants appearing, respectively, in Theorem 1.0.2, Theorem 1.0.3, and Theorem 1.0.4, $\tilde\theta_1$ is the exponent in Theorem 1.0.3 and $\tilde\theta_2$ and $\tilde\theta_3$ are the exponents appearing in Theorem 1.0.4. □

We conclude with a list of open problems.

Open problems

- It remains open the issue regarding the sharpness of the exponents of the quantities $\alpha(\Omega)$, $||u - u^\sharp||_{L^1(\mathbb{R}^N)}$ and $||f - f^\sharp||_{L^1(\mathbb{R}^N)}$in Theorem 1.0.1. As far as the exponent of $\alpha(\Omega)$, it may depend on the function f and it remains open to establish if there is a function f for which it is sharp.
  We already know that in [82], it is conjectured that for $f = 1$ the sharp exponent is two. The method we use is known to produce nonsharp inequalities in the case of asymmetry. For the others, the exponent is not explicitly known.

- It could be interesting to prove the quantitative result of the Talenti comparison result in the non-linear setting. We recall that the comparison result for the $p-$Laplace operator is proved in [111] and the relative rigidity result can be found in [91]

- In [8] it is proved that a Talenti comparison result holds, when replacing the Dirichlet boundary conditions with the Robin boundary conditions with a positive boundary parameter. It could be interesting to study a quantitative version of this result, as it has already been proved in [91] that the estimates are rigid. This last point may be more challenging, as the solution to the Poisson equation with Robin boundary condition is not constant on the boundary of Ω. This last feature does not allow one to use a Pólya- Szegö principle either its quantitative counterpart.

Chapter 4

On the first Robin eigenvalue of the Finsler p-Laplace operator as $p \to 1$

Let Ω be a bounded, connected, sufficiently smooth open set, $p > 1$ and $\beta \in \mathbb{R}$. In this Chapter, we study the Γ-convergence, as $p \to 1^+$, of the functional

$$J_p(\varphi) = \frac{\displaystyle\int_\Omega F^p(\nabla\varphi)dx + \beta \int_{\partial\Omega} |\varphi|^p F(\nu) d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega |\varphi|^p dx}$$

where $\varphi \in W^{1,p}(\Omega) \setminus \{0\}$ and F is a sufficiently smooth norm on $\mathbb{R}^N$. We study the limit of the first eigenvalue $\lambda_p(\beta, \Omega) = \inf_{\substack{\varphi \in W^{1,p}(\Omega) \\ \varphi \neq 0}} J_p(\varphi)$, as $p \to 1^+$, that is:

$$\Lambda(\Omega, \beta) = \inf_{\substack{\varphi \in BV(\Omega) \\ \varphi \not\equiv 0}} \frac{\displaystyle |Du|_F(\Omega) + \min\{\beta, 1\} \int_{\partial\Omega} |\varphi| F(\nu) d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega |\varphi| dx}. \tag{4.0.1}$$

Furthermore, for $\beta > -1$, we obtain an isoperimetric inequality for $\Lambda(\Omega, \beta)$ depending on β.

The proof uses an interior approximation result for $BV(\Omega)$ functions by $C^\infty(\Omega)$ functions in the sense of strict convergence on $\mathbb{R}^N$ and a trace inequality in BV with respect to the anisotropic total variation.

4.1 An anisotropic trace inequality

To study the behavior of the first Robin eigenvalue for the Finsler p-Laplace operator, as p goes to one, we have to prove the independent result. In this section, we prove a trace inequality in BV with respect to the anisotropic total variation. We first give the result in a general case (Proposition 4.1.1), then we refine the constants involved in the inequality by requiring more regularity on the boundary of Ω (Proposition 4.1.2).

Firstly, let Ω be an open set with $\mathcal{H}^{N-1}(\partial\Omega) < +\infty$ and let us set, for every $y \in \partial\Omega$,

$$q(y) = \lim_{\rho \to 0^+} \sup \left\{ \frac{\int_{\partial^*\Omega} \chi_A F(\nu) d\mathcal{H}^{N-1}}{|D\chi_A|_F(\Omega)} : A \subset \Omega \cap B_\rho(y), |A| > 0, P_F(A; \Omega) < +\infty \right\}$$

and $Q = \sup_{y \in \partial\Omega} q(y)$. The following inequality generalizes the trace inequality given in [15, Theorem 4].

Proposition 4.1.1. *Let Ω be a bounded open set with $\mathcal{H}^{N-1}(\partial\Omega) < +\infty$, $Q < +\infty$, and let u be a function in $BV(\Omega)$. Then for any $\varepsilon > 0$, it holds*

$$\int_{\partial^*\Omega} |u| F(\nu) d\mathcal{H}^{N-1} \leq (Q + \varepsilon)|Du|_F(\Omega) + c(\Omega, \varepsilon) \int_\Omega |u| dx, \tag{4.1.1}$$

where $c(\Omega, \varepsilon)$ does not depend on u.

Proof. Let us fix $y \in \partial\Omega$ and $\rho(y) > 0$ such that

$$\int_{\partial^*\Omega} \chi_B F(\nu) d\mathcal{H}^{N-1} \leq (Q + \varepsilon)|D\chi_B|_F(\Omega),$$

for any $B \subset \Omega \cap B_{\rho(y)}(y)$ with $P_F(B; \Omega) < +\infty$.

If $\operatorname{spt}(u) \subset B_{\rho(y)}(y)$, then

$$\begin{aligned}
\int_{\partial^*\Omega} |u| F(\nu) d\mathcal{H}^{N-1} &= \int_{\partial^*\Omega} F(\nu) \left(\int_0^{+\infty} \chi_{\{|u|>t\}}(y) dt \right) d\mathcal{H}^{N-1} \\
&= \int_0^{+\infty} \left(\int_{\partial^*\Omega} F(\nu) \chi_{\{|u|>t\}}(y) d\mathcal{H}^{N-1} \right) dt \leq (Q + \varepsilon) \int_0^{+\infty} |D\chi_{\{|u|>t\}}|_F(\Omega) dt.
\end{aligned}$$

By using the coarea formula, we have

$$\int_{\partial^*\Omega} |u| F(\nu) d\mathcal{H}^{N-1} \leq (Q + \varepsilon)|D|u||_F(\Omega) \leq (Q + \varepsilon)|Du|_F(\Omega).$$

Let $\{B_\rho(y)\}_{y \in \partial\Omega}$ be a cover of $\partial\Omega$ and let us extract a finite sub-cover $B_1, \ldots, B_k$. Now, considering a partition of unity $\varphi_1, \ldots, \varphi_k$ such that

$$0 \leq \varphi_i \leq 1, \qquad \varphi_i \in C_0^1(B_i), \qquad \sum_{i=1}^k \varphi_i(y) = 1 \qquad \text{if } y \in \partial\Omega.$$

If $f \in BV(\Omega)$, then

$$\begin{aligned}
\int_{\partial^*\Omega} |u| F(\nu)\, d\mathcal{H}^{N-1} &\leq (Q + \varepsilon) \left| D \left(\sum_{i=1}^k \varphi_i u \right) \right|_F (\Omega) \\
&\leq (Q + \varepsilon) \sum_{i=1}^k (|\varphi_i Du|_F(\Omega) + |u D\varphi_i|_F(\Omega)) \\
&= (Q + \varepsilon) \sum_{i=1}^k \left(\int_\Omega \varphi_i d|Du|_F + \int_\Omega u d|D\varphi_i|_F(\Omega) \right) \\
&\leq (Q + \varepsilon)|Du|_F(\Omega) + c(\Omega, \varepsilon) \int_\Omega |u| dx.
\end{aligned}$$

□

If the boundary of Ω is sufficiently smooth, we can show that Q can be taken equal to 1 and $\varepsilon = 0$. More precisely, we have the following (refer to [66, Lemma 1] for the Euclidean case).

Proposition 4.1.2. *Let Ω be a bounded open connected set of class C^2. Then there exists a positive constant c such that*

$$\int_{\partial\Omega} |u| F(\nu)\, d\mathcal{H}^{N-1} \leq |Du|_F(\Omega) + c\int_\Omega |u| dx, \quad \forall u \in BV(\Omega). \tag{4.1.2}$$

Proof. Since Ω is C^2, then a uniform sphere condition of radius $r > 0$ holds, in the sense that for every point $y \in \partial\Omega$ there exists $z \in \Omega$ such that $y \in \overline{B_r(z)} \subset \overline{\Omega}$. Let $R \in]0, +\infty[$ be the maximum of the principal radii of curvature of $\partial\mathcal{W}$. Such maximum exists being F (and F^o) strongly convex. If $\kappa_1^F, \dots, \kappa_{N-1}^F$ are the anisotropic principal curvatures, we have that

$$\kappa_i{}^F(y) \leq \frac{1}{\mu}, \quad i = 1, \dots, N-1,$$

with $\mu = \frac{r}{R}$ ([42, Lemma 5.4]). Therefore, in the set $\Omega_{\frac{\mu}{2}} := \{x \in \Omega \ : \ d_F(x) < \frac{\mu}{2}\}$, it holds that d_F is C^2 ([42, Lemma 4.1 and Theorem 4.16]) and

$$\frac{\kappa_i^F(y)}{1 - \kappa_i^F(y) d_F(x)} \leq \begin{cases} 0 & \text{if } \kappa_i^F \leq 0 \\ \dfrac{2}{\mu} & \text{if } 0 < \kappa_i^F \leq \dfrac{1}{\mu}, \end{cases}$$

where $y \in \partial\Omega$ is the anisotropic projection of $x \in \Omega$ on $\partial\Omega$. Then

$$\sum_{i=1}^{N-1} \frac{\kappa_i{}^F(y)}{1 - \kappa_i{}^F(y) d_F(x)} \leq 2\frac{N-1}{\mu} \tag{4.1.3}$$

for $x \in \Omega_{\frac{\mu}{2}}$. We may restrict ourselves to the case where u is nonnegative (being $|D|u||_F(\Omega) \leq |Du|_F(\Omega)$). Moreover, Proposition 2.5.3 assures that a sequence $u_n \in C^\infty(\Omega)$ which strongly converges to u in $L^1(\Omega)$ and such that $\|F(\nabla u_n)\|_{L^1(\Omega)}$ converges to $|Du|_F(\Omega)$, as $n \to +\infty$, exists. Hence we can assume also that u is smooth.

Integrating by parts and recalling that $F(\nabla d_F(x)) = 1$ in Ω, it holds that

$$\begin{aligned}
&\int_\Omega -\mathcal{Q}_2 d_F(x)\, u(x) \left(\frac{\mu}{2} - d_F(x)\right)^+ dx \\
&\qquad = \int_\Omega F_\xi(\nabla d_F(x)) \cdot \nabla u(x) \left(\frac{\mu}{2} - d_F(x)\right)^+ dx \\
&\qquad - \int_{\Omega_{\frac{\mu}{2}}} u(x) F_\xi(\nabla d_F(x)) \cdot \nabla d_F(x)\, dx - \frac{\mu}{2}\int_{\partial\Omega} u(x) F_\xi(\nabla d_F(x)) \cdot \nu\, d\mathcal{H}^{N-1} \\
&= \int_\Omega F_\xi(\nabla d_F(x)) \cdot \nabla u(x) \left(\frac{\mu}{2} - d_F(x)\right)^+ dx - \int_{\Omega_{\frac{\mu}{2}}} u(x)\, dx + \frac{\mu}{2}\int_{\partial\Omega} u(y)\, F(\nu) d\mathcal{H}^{N-1},
\end{aligned}$$

where in the last equality we have used (2.5.11).

At this stage, in the Euclidean context (as in [66, Lemma 1]), the proof follows by directly estimating from above the Laplacian of the distance function in terms of the curvatures of $\partial\Omega$. Here we use a different strategy. In particular, we estimate the term

$$\int_\Omega F_\xi(\nabla d_F(x)) \cdot \nabla u(x) \left(\frac{\mu}{2} - d_F(x)\right)^+ dx.$$

Hence, by (2.5.4) and (2.5.6) it holds that

$$\int_\Omega F_\xi(\nabla d_F(x))\cdot\nabla u(x)\left(\frac{\mu}{2}-d_F(x)\right)^+dx \geq -\frac{\mu}{2}\int_\Omega F(\nabla u(x))\,dx = -\frac{\mu}{2}|Du|_F(\Omega). \tag{4.1.4}$$

On the other hand, the change of variable formula of Theorem 2.5.1 gives that

$$\begin{aligned}&\int_\Omega F_\xi(\nabla d_F(x))\cdot\nabla u(x)\left(\frac{\mu}{2}-d_F(x)\right)^+dx\\ &=\int_{\partial\Omega}F(\nu)\int_0^{\frac{\mu}{2}}\left(\frac{\mu}{2}-t\right)\frac{d}{dt}[u(\phi(y,t))]J(y,t)dt\,d\mathcal{H}^{N-1}.\end{aligned}$$

Integrating by parts and using the fact that $J(y,0)=1$, the above integral becomes

$$\begin{aligned}&-\frac{\mu}{2}\int_{\partial\Omega}u(y)F(\nu)d\mathcal{H}^{N-1}(y)-\int_{\partial\Omega}F(\nu)\int_0^{\frac{\mu}{2}}u(\phi(y,t))\left(\frac{\mu}{2}-t\right)\frac{dJ}{dt}dt\,d\mathcal{H}^{N-1}(y)\\ &\qquad\qquad+\int_{\partial\Omega}F(\nu)\int_0^{\frac{\mu}{2}}u(\phi(y,t))J(y,t)dt\,d\mathcal{H}^{N-1}(y)\\ &\leq -\frac{\mu}{2}\int_{\partial\Omega}u(y)F(\nu)d\mathcal{H}^{N-1}(y)+(N-1)\int_{\partial\Omega}F(\nu)\int_0^{\frac{\mu}{2}}u(\phi(y,t))J(y,t)\,dt\,d\mathcal{H}^{N-1}(y)\\ &\qquad\qquad+\int_{\partial\Omega}F(\nu)\int_0^{\frac{\mu}{2}}u(\phi(y,t))J(y,t)dt\,d\mathcal{H}^{N-1}(y)\\ &\qquad\qquad= -\frac{\mu}{2}\int_{\partial\Omega}u(y)F(\nu)d\mathcal{H}^{N-1}(y)+N\int_\Omega u\,dx\end{aligned}$$

where in the inequality we have used (2.5.10) and the bound (4.1.3). Hence, joining with (4.1.4) it holds that

$$\int_{\partial\Omega}u(y)F(\nu)d\mathcal{H}^{N-1}\leq|Du|_F(\Omega)+\frac{2N}{\mu}\int_\Omega u(x)\,dx.$$

□

Remark 4.1.3. Using the notation of the above theorem, we explicitly observe that the constant in (4.1.2) is

$$c=\frac{2N}{\mu}.$$

4.2 Interior approximation

Now we provide an approximation result for BV-functions by smooth functions with compact support in Ω.

We preliminary state two useful lemmas. Firstly, we recall from [88, Lemma 3.2] the following result on diffeomorphic perturbations of sets Ω with Lipschitz boundary. We denote by ι and I the identical vector and matrix function, respectively.

Lemma 4.2.1. *Let $\Omega\subset\mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Then there exists $\tau_0>0$ and, for $0\leq\tau\leq\tau_0$, a family of $C^\infty-$diffeomorphisms $\Phi^\tau:\mathbb{R}^N\to\mathbb{R}^N$ with inverses Ψ^τ such that*

- $\Phi^0 = \Psi^0 = \iota$;
- $\Phi^\tau \to \iota$ *and* $\Psi^\tau \to \iota$ *as* $\tau \to 0$ *uniformly on* $\mathbb{R}^N$;
- $\nabla\Phi^\tau(x) \to I$ *and* $\nabla\Psi^\tau(x) \to I$ *as* $\tau \to 0$ *uniformly with respect to* x *on* $\mathbb{R}^N$;
- $\Phi^\tau(\overline{\Omega}) \Subset \Omega$ *for all* $\tau \in (0, \tau_0]$.

Now, we give the anisotropic version of the change of coordinates formula for BV-functions, stated in [70, Lemma 10.1].

Lemma 4.2.2. *Let u be a function in $BV_{loc}(\Omega)$, $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be a diffeomorphism and $A \Subset \Omega$. Then*

$$|D(u \circ \Phi^{-1})|_F(\Phi(A)) = |HDu|_F(A), \tag{4.2.1}$$

where $H = |\det \nabla\Phi|[\nabla\Phi]^{-1}$ *and* $|HDu|_F(A) := \sup_{F^o(g) \le 1} \int_A g \cdot H \; dDu$.

Proof. Let us consider $u \in C^1(\Omega)$ and $g \in C_0^1(A; \mathbb{R}^N)$, then the following change of area formula holds

$$\begin{aligned} \int_{\Phi(A)} (g \circ \Phi^{-1}) \cdot \nabla(u \circ \Phi^{-1}) dx &= \int_{\Phi(A)} (g \circ \Phi^{-1}) \cdot ((\nabla u \circ \Phi^{-1}) \nabla\Phi^{-1}) dx \\ &= \int_A g \cdot (\nabla u (\nabla\Phi^{-1} \circ \Phi)) |\det \nabla\Phi| dz \\ &= \int_A g \cdot (H \nabla u) dz. \end{aligned} \tag{4.2.2}$$

Thus, the thesis (4.2.1) holds for u in $C^1(\Omega)$, that is

$$\int_{\Phi(A)} F(\nabla(u \circ \Phi^{-1})) dx = \int_A F(H\nabla u) dx.$$

Suppose now that $u \in BV_{\mathrm{loc}}(\Omega)$. By Proposition 2.5.3, in $A \Subset \Omega$ we can approximate u by a sequence $\{u_i\} \subset C^\infty$. Moreover, the corresponding functions $u_i \circ \Phi^{-1}$ converge to $u \circ \Phi^{-1}$ in $L^1(\Phi(A))$.

Hence, we can pass to the limit in (4.2.2), obtaining

$$\int_{\Phi(A)} (g \circ \Phi^{-1}) \cdot dD(u \circ \Phi^{-1}) = \int_A g \cdot H \, dDu = \int_A g \cdot (H\nu) d|Du| \tag{4.2.3}$$

where ν is obtained by differentiating Du with respect to $|Du|$.

If $F^o(g) \le 1$, then also $F^o(g \circ \Phi^{-1}) \le 1$ and $\mathrm{spt}(g \circ \Phi^{-1})) \subseteq \Phi(A)$. Therefore, by definition of total variation with respect to F, we have

$$\int_A g \cdot (H\nu) d|Du| \le |D(u \circ \Phi^{-1})|_F(\Phi(A)), \tag{4.2.4}$$

We have

$$|HDu|_F(A) = \sup_{F^o(g) \le 1} \int_A g \cdot H \; dDu \overset{(4.3)}{=} \sup_{F^o(g) \le 1} \int_A g \cdot (H\nu) \; d|Du|.$$

Hence, taking the supremum on the left hand side in (4.2.4), it holds

$$|HDu|_F(A) \leq |D(u\circ\Phi^{-1})|_F(\Phi(A)).$$

For the reverse inequality, let $\gamma \in C^1_0(\Phi(A);\mathbb{R}^N)$ with $F^o(\gamma)\leq 1$. By (4.2.3) applied with $g=\gamma\circ\Phi \in C^1_0(A;\mathbb{R}^N)$, and $F^o(\gamma)\leq 1$, we have

$$\begin{aligned}\int_{\Phi(A)} \gamma\cdot dD(u\circ\Phi^{-1}) &\overset{(4.3)}{=} \int_A (\gamma\circ\Phi)\cdot(H\nu)d|Du| \\ &\overset{F^o(\gamma\circ\Phi)\leq 1}{\leq} \sup_{F^o(g)\leq 1}\int_A g\cdot(H\nu)d|Du| = |HDu|_F(A).\end{aligned}$$

Hence, since γ is arbitrary, taking the supremum on γ at the left hand side, we have

$$|D(u\circ\Phi^{-1})|_F(\Phi(A)) \leq |HDu|_F(A).$$

□

At this stage, we are in position to state the main approximation result.

Theorem 4.2.3. *Let $\Omega\subset\mathbb{R}^N$ be an open bounded set with Lipschitz boundary and let $u\in BV(\Omega)\cap L^p(\Omega)$ for some $p\in[1,\infty)$. Then there exists a sequence $\{u_k\}_{k\in\mathbb{N}}\subseteq C^\infty_0(\Omega)$ such that, for any $q\in[1,p]$,*

$$u_k\to u \quad \textit{in} \quad L^q(\Omega) \quad \textit{and} \quad |Du_k|_F(\mathbb{R}^N)\to|Du|_F(\mathbb{R}^N).$$

Proof. Let us fix a family $(\Phi^\tau)_{0\leq\tau\leq\tau_0}$ of diffeomorphisms fron $\mathbb{R}^N$ to $\mathbb{R}^N$ with inverses $(\Psi^\tau)_{0\leq\tau\leq\tau_0}$ according to Lemma 4.2.1, and consider

$$u^\tau := u\circ\Psi^\tau \quad \text{for} \quad \tau\in[0,\tau_0].$$

By construction $u^\tau=0$ a.e. outside a compact subset of Ω. By [88, Theorem 3.1], we know that $u^\tau\in L^p(\Omega)$ for all $\tau\in[0,\tau_0]$, $u^\tau\to u$ in $L^p(\Omega)$ and also in $L^p(\mathbb{R}^N)$.
The change of coordinates formula in Lemma 4.2.2 implies that

$$|Du^\tau|_F(\mathbb{R}^N) = \int_{\mathbb{R}^N} |(\nabla\Psi^\tau)^T|\ |\det(\nabla\Phi^\tau)|\ d|Du|_F.$$

Thus $u^\tau\in BV(\mathbb{R}^N)$ and, since the integrand on the right hand side uniformly converges to 1,

$$\lim_{\tau\to0^+}|Du^\tau|_F(\mathbb{R}^N) = |Du|_F(\mathbb{R}^N)$$

It remains only to prove that there exists $v^\tau\in C^\infty_0(\Omega)$ such that

$$||u^\tau-v^\tau||_p<\tau \quad \text{and} \quad ||Du^\tau|_F(\mathbb{R}^N)-|Dv^\tau|_F(\mathbb{R}^N)|<\tau.$$

The first convergence (in L^p) has been proved in [88, Theorem 3.1]; meanwhile the convergence of the total variation is based on the following argument.

For any $\varepsilon>0$, let us consider the mollification $u_\varepsilon := u^\tau * \eta_\varepsilon$, where $\eta_\varepsilon(x) =: \varepsilon^{-n}\eta(\varepsilon^{-1}x)$, for the standard mollifier η. Hence $u_\varepsilon\to u^\tau$ in $L^p(\Omega)$ and for the zero extensions, in $L^1(\mathbb{R}^N)$ [14, Proposition 3.2.c].

Then, by the lower semicontinuity of the anisotropic total variation, we have

$$|Du^\tau|_F(\mathbb{R}^N) \le \lim_{\varepsilon\to 0^+} \inf |Du_\varepsilon|_F(\mathbb{R}^N).$$

Therefore, it remains to prove the opposite inequality

$$\lim_{\varepsilon\to 0^+} \sup |Du_\varepsilon|_F(\mathbb{R}^N) \le |Du^\tau|_F(\mathbb{R}^N). \tag{4.2.5}$$

Let us choose $\varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^N)$ with $F^o(\varphi) \le 1$ and calculate

$$\int_{\mathbb{R}^N} u_\varepsilon \operatorname{div}(\varphi) dx = \int_{\mathbb{R}^N} u^\tau (\eta_\varepsilon * \operatorname{div}\varphi) dx = \int_{\mathbb{R}^N} u^\tau \operatorname{div}(\eta_\varepsilon * \varphi) dx \le |Du^\tau|_F(\mathbb{R}^N), \tag{4.2.6}$$

where the inequality in the last term holds since $F^o(\eta_\varepsilon * \varphi) \le 1$. Indeed, by using the 1-homogeneity of F^o and Jensen's Inequality with the probability measure $\eta_\varepsilon(x-\cdot)\mathcal{L}^N$ (see, for instance, [94, Lemma 1.8.2]), we gain that

$$F^o\left(\int_{\mathbb{R}^N} \eta_\varepsilon(x-y)\varphi(y)dy\right) \le \int_{\mathbb{R}^N} \eta_\varepsilon(x-y) F^o(\varphi(y))dy \le 1.$$

Hence, by passing to the limit in (4.2.5), we reach the inequality (4.2.6) by the arbitrariness of φ. $\square$

4.3 The first Robin eigenvalue of the Finsler p-Laplacian as $p \to 1$

In this Section, we give an application of the results proved above to a Robin eigenvalue problem. More precisely, our aim is to analyze the Γ-limit of the functional

$$J_p(\varphi) = \frac{\displaystyle\int_\Omega F^p(\nabla\varphi)dx + \beta \int_{\partial\Omega} |\varphi|^p F(\nu) d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega |\varphi|^p dx}, \quad \varphi \in W^{1,p}(\Omega) \setminus \{0\}, \tag{4.3.1}$$

where Ω is a bounded, connected, sufficiently smooth open set, $p > 1$ and $\beta \in \mathbb{R}$, and prove an isoperimetric inequality for the limit, as $p \to 1^+$, of the first eigenvalue

$$\lambda_p(\beta, \Omega) = \inf_{\substack{\varphi \in W^{1,p}(\Omega)\\ \varphi \ne 0}} J_p(\varphi), \tag{4.3.2}$$

depending on the value of the parameter β. The analogous results for the Euclidean case have been proved in [49].

A key point for proving this result is the convergence of the functional J_p.

We first recall the following existence result for (4.3.2) holds.

Theorem 4.3.1 ([47, 53]). *Let $p > 1$, $\beta \in \mathbb{R}$ and Ω bounded Lipschitz domain. Then there exists a minimum $u \in C^{1,\alpha}(\Omega) \cap C(\overline{\Omega})$ of (4.3.2) that satisfies*

$$\begin{cases} -\mathcal{Q}_p u = \lambda(\Omega, p, \beta)|u|^{p-2}u & \text{in } \Omega \\ F^{p-1}(\nabla u) F_\xi(\nabla u) \cdot \nu + \beta F(\nu)|u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3.3}$$

Moreover, u does not change sign in Ω. Finally, $\lambda_p(\beta, \Omega)$ is positive if $\beta > 0$, while is negative if $\beta < 0$.

4.4 The case $p = 1$

In order to study the limit case of J_p as p goes to 1, we consider the functional

$$J(\varphi) = \frac{|D\varphi|_F(\Omega) + \min\{\beta, 1\} \displaystyle\int_{\partial\Omega} |\varphi| F(\nu) d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega |\varphi| dx}, \tag{4.4.1}$$

where $\varphi \in BV(\Omega)$ and $\varphi \not\equiv 0$. Hence, we study the associated minimum problem

$$\Lambda(\Omega, \beta) = \inf_{\substack{\varphi \in BV(\Omega) \\ \varphi \not\equiv 0}} J(\varphi). \tag{4.4.2}$$

Depending on β, we will impose different assumptions on the regularity of the domain. Indeed:

- if $\beta \geq 0$, we will suppose that $\partial\Omega$ is Lipschitz;
- if $-1 < \beta < 0$, we will assume that $\partial\Omega$ is C^2.

In particular, this difference depends on the fact that in the case $\beta < 0$ we use the trace inequality, studied in Section 4.1.

Finally, if $\beta \leq -1$ the problem is not well posed; indeed if $\beta < -1$, then $\Lambda(\Omega, \beta) = -\infty$ while if $\beta = -1$, Λ is finite but can be not achieved, also in the case of smooth domains. For further details, we refer the reader to the Euclidean case treated in [49].

Let us discuss the presence of the term $\min\{\beta, 1\}$ in (4.4.1). For any value of β, it could seem more natural to study the problem

$$\lambda(\Omega, 1, \beta) = \inf_{\substack{\varphi \in BV(\Omega) \\ \varphi \not\equiv 0}} \frac{|D\varphi|_F(\Omega) + \beta \displaystyle\int_{\partial\Omega} |\varphi| F(\nu) d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega |\varphi| dx}. \tag{4.4.3}$$

Actually, we have that for $\beta \geq 1$ it holds

$$\lambda(\Omega, 1, \beta) = \Lambda(\Omega, \beta) = h_F(\Omega),$$

where $h_F(\Omega)$ is the first Cheeger constant of Ω in the Finsler setting (see e.g. [34]):

$$h_F(\Omega) = \inf_{\substack{\varphi \in BV(\Omega) \\ \varphi \not\equiv 0}} \frac{|D\varphi|_F(\mathbb{R}^N)}{\displaystyle\int_\Omega |\varphi| dx} = \inf_{E \subseteq \Omega} \frac{P_F(E; \Omega)}{|E|}. \tag{4.4.4}$$

Indeed, in this case, it is immediate to see that

$$\lambda(\Omega, 1, \beta) \geq h_F(\Omega).$$

On the other hand, if u is a minimizer of (4.4.4), then, by Theorem 4.2.3, there exists $u_k \in C_0^\infty(\Omega)$ such that

$$u_k \xrightarrow{L^q} u, \qquad |Du_k|_F(\mathbb{R}^N) \xrightarrow{L^1} |Du|_F(\mathbb{R}^N),$$

for any $q \leq \dfrac{N}{N-1}$. Therefore

$$\lambda(\Omega, 1, \beta) \leq \lim_{k\to+\infty} \frac{\displaystyle\int_\Omega F(\nabla u_k)dx}{\displaystyle\int_\Omega |u_k|dx} = h_F(\Omega).$$

Now we focus on the possibility of studying the minimization problem (4.4.2) restricting our analysis to characteristic functions. Hence if $E \subseteq \Omega$, we have

$$J(\chi_E) = \frac{P_F(E;\Omega) + \min\{1,\beta\} \displaystyle\int_{\partial\Omega\cap\partial E} F(\nu_E) d\mathcal{H}^{N-1}}{|E|}.$$

By denoting

$$R(E,\beta) := J(\chi_E),$$

we consider the minimization problem

$$\ell(\Omega,\beta) = \inf_{E\subseteq\Omega} R(E,\beta). \tag{4.4.5}$$

Before proving the equivalence between problems (4.4.2) and (4.4.5), we need the following result on the lower semicontinuity of the numerator of the functional J.

Lemma 4.4.1. *Let $\beta \geq -1$. The functional*

$$G(u) = |Du|_F(\Omega) + \min\{1,\beta\} \int_{\partial\Omega} |u| F(\nu) d\mathcal{H}^{N-1}$$

is lower semicontinuous on $BV(\Omega)$ with respect to the topology of $L^1(\Omega)$.

Proof. If $\beta \geq 0$, the lower semicontinuity (with Ω Lipschitz) follows immediately by the lower semicontinuity of each term. Then we assume $\beta < 0$ (and Ω in C^2). The proof is an adaptation of [93, Proposition 1.2] to the Finsler case.

Let $u \in BV(\Omega)$, and let us consider a sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq BV(\Omega)$ converging to u in $L^1(\Omega)$; we have the following estimate

$$G(u) - G(u_k) \leq |Du|_F(\Omega) - |Du_k|_F(\Omega) + \int_{\partial\Omega} |u-u_k| F(\nu) d\mathcal{H}^{N-1}. \tag{4.4.6}$$

Now, for a fixed $\delta > 0$, let us define $\Omega_\delta = \{x \in \Omega : d_{\mathcal{E}}(x) < \delta\}$, where $d_{\mathcal{E}}$ is the standard Euclidean distance to the boundary of Ω ; moreover let us consider $v_\delta = (1-\chi_\delta)(u-u_k)$, where χ_δ is a cut-off function such that $\chi_\delta = 1$ in $\Omega \setminus \Omega_\delta$ and $|\nabla\chi_\delta| \leq \frac{2}{\delta}$ in Ω. The trace inequality (4.1.2) applied to v_δ gives

$$\int_{\partial\Omega} |u-u_k| F(\nu) d\mathcal{H}^{N-1} \leq |D(u-u_k)|_F(\Omega_\delta) + \frac{2b}{\delta} \int_{\Omega_\delta} |u-u_k| dx + c \int_{\Omega_\delta} |u-u_k| dx, \tag{4.4.7}$$

where $b > 0$ is independent of k and c is the same constant as in (4.1.2).
Moreover, we have

$$|D(u-u_k)|_F(\Omega_\delta) \leq |Du|_F(\Omega_\delta) + |Du_k|_F(\Omega_\delta) + |D(u-u_k)|_F(\partial(\Omega\setminus\Omega_\delta)), \tag{4.4.8}$$

but last term is zero on a set of δ's of positive measure because $u - u_k \in BV(\Omega)$, for all $k \in \mathbb{N}$. Hence, by (4.4.6)-(4.4.7)-(4.4.8), we gain:

$$G(u) - G(u_k) \leq |Du|_F(\Omega) + |Du|_F(\Omega_\delta) - |Du_k|_F(\Omega \setminus \Omega_\delta) + \left(\frac{2b}{\delta} + c\right) \int_{\Omega_\delta} |u - u_k| dx.$$

By the lower semicontinuity of the functional $|Du_k|_F(\Omega \setminus \Omega_\delta)$ in $L^1(\Omega \setminus \Omega_\delta)$, we have that

$$\limsup_{k \to +\infty} [G(u) - G(u_k)] \leq 2|Du|_F(\Omega_\delta).$$

The conclusion follows by sending $\delta \to 0^+$. □

At this stage, we state the main existence result of the minimum problem (4.4.2).

Theorem 4.4.2. *For any $\beta > -1$, there exists a minimum to problem (4.4.2). In particular, it holds*

$$\Lambda(\Omega, \beta) = \ell(\Omega, \beta).$$

Moreover, if $u \in BV(\Omega)$ is a minimum of (4.4.2), then

$$\Lambda(\Omega, \beta) = R(\{u > t\}, \beta),$$

for some $t \in \mathbb{R}$.

Proof. Let u_n be a minimizing sequence in $BV(\Omega)$ such that $\|u_n\|_{L^1(\Omega)} = 1$.

Let $\beta > 0$. Then u_n is bounded in $BV(\Omega)$, therefore u_n weakly* converges in BV and strongly in $L^1(\Omega)$ to $u \in BV(\Omega)$.

When $\beta \geq 1$, since both u_n and u are assumed to be identically zero outside Ω, we have $J(u_n) = |Du_n|_F(\mathbb{R}^N)$ and the lower semicontinuity of the anisotropic total variation gives that

$$J(u) \leq \liminf_{n \to +\infty} J(u_n) \tag{4.4.9}$$

and u is a minimum of the functional J.

On the other hand, when $0 < \beta < 1$, let us set $\Omega^\delta := \{x \in \Omega \ : \ d_{\mathcal{E}}(x) > \delta\}$, we have

$$J(u_n) \geq |Du_n|_F(\Omega^\delta) + \beta \left[|Du_n|_F(\Omega \setminus \Omega^\delta) + \int_{\partial\Omega} |u_n| F(\nu) d\mathcal{H}^{N-1} \right].$$

By the lower semicontinuity, we have

$$\liminf_{n \to +\infty} J(u_n) \geq |Du|_F(\Omega^\delta) + \beta |Du|_F(\mathbb{R}^N \setminus \Omega^\delta).$$

For $\delta \to 0^+$, since $u \in BV(\Omega)$ we obtain (4.4.9) providing that u is the minimum for J.

Finally, when $-1 < \beta < 0$, it is easily seen that $J(u_n) \leq C$. Indeed, using the trace inequality (4.1.2), we obtain

$$J(u_n) \geq (1 + \beta)|Du_n|_F(\Omega) + c\beta \geq c\beta$$

and

$$|Du_n|_F(\Omega) \leq \frac{C}{1 + \beta} - \frac{\beta c}{1 + \beta}.$$

Being $u_n \in BV(\Omega)$ and by the fact that the functional J is lower semicontinuous (proved in Lemma 4.4.1), we have that u is a minimum of J.

Now, we want to prove the second part of the Theorem. Obviously, we have

$$\Lambda(\Omega,\beta) \leq \ell(\Omega,\beta).$$

To prove the reverse inequality, we take $u \in BV(\Omega)$ a minimizer of (4.4.2). By using the coarea formula (2.5.14)

$$|Du|_F(\Omega) = \int_{-\infty}^{+\infty} P_F(\{u>t\},\Omega)dt,$$

we have

$$\begin{aligned}
\Lambda(\Omega,\beta) &= \frac{\displaystyle\int_{-\infty}^{+\infty} P_F(\{u>t\},\Omega)dt + \min\{\beta,1\}\int_{-\infty}^{+\infty} \mathcal{H}^{N-1}(\partial\Omega\cap\partial\{u>t\})F(\nu)dt}{\displaystyle\int_{-\infty}^{+\infty} |\{u>t\}|dt} \\
&= \frac{\displaystyle\int_{-\infty}^{+\infty} R(\{u>t\},\beta)|\{u>t\}|dt}{\displaystyle\int_{-\infty}^{+\infty} |\{u>t\}|dt} \\
&\geq \inf_{E\subseteq\Omega} R(E,\beta) \\
&= \ell(\Omega,\beta).
\end{aligned}$$

This shows that $\Lambda(\Omega,\beta) = \ell(\Omega,\beta)$ and, in particular, we have that

$$\int_{-\infty}^{+\infty} \{R(\{u>t\},\beta) - \ell(\Omega,\beta)\}|\{u>t\}|dt = 0$$

and using the definition of $\ell(\Omega,\beta)$ we observe that the integrand is nonnegative. In particular, $u \not\equiv 0$ and we have that $\Lambda(\Omega,\beta) = R(\{u>t\},\beta)$. □

4.5 Γ-convergence of J_p

Following the same strategy of [49, Sec. 4], we will prove that the functional J_p $\Gamma-$converges to the functional J, as $p \to 1^+$.

Definition 4.5.1. A functional J_p Γ-converges to J as $p \to 1^+$ in the weak* topology of $BV(\Omega)$ if, for any $u \in BV(\Omega)$, the following hold:

(i) For any sequence $u_p \in BV(\Omega)$ which converges to u weak* in $BV(\Omega)$ as $p \to 1^+$, then

$$\liminf_{p\to 1^+} J_p(u_p) \geq J(u). \tag{4.5.1}$$

(ii) There exists a sequence $u_p \in W^{1,p}(\Omega)$ which converges to u weak* in $BV(\Omega)$ as $p \to 1^+$, such that

$$\limsup_{p\to 1^+} J_p(u_p) \leq J(u). \tag{4.5.2}$$

Now, we are in position to prove the convergence theorem for the functional J_p (see [49, Theorem 4.2] for the Euclidean case).

Theorem 4.5.2. *Let $\beta > -1$, then J_p Γ-converges to J as $p \to 1^+$.*

Proof. Let us suppose $u_p \in W^{1,p}(\Omega)$ and $||u_p||_{L^p(\Omega)} = 1$. We give the proof by distinguishing the possible values of β. In any cases, we will have to prove (4.5.1) and (4.5.2).

The case $\beta \geq 1$. Let us fix a sequence $u_p \in W^{1,p}(\Omega)$ weak* converging to u in $BV(\Omega)$, as $p \to 1^+$. By using the Hölder-type inequality contained for example in [51, Proposition A.1], we have:

$$\left(\int_\Omega F(\nabla u_p)dx + \int_{\partial\Omega} |u_p| F(\nu) d\mathcal{H}^{N-1}\right)^p \\ \leq \left(\int_\Omega F^p(\nabla u_p)dx + \int_{\partial\Omega} |u_p|^p F(\nu) d\mathcal{H}^{N-1}\right) (|\Omega| + P_F(\Omega))^{p-1}.$$

Hence we have

$$\liminf_{p\to 1^+} J_p(u_p) \geq |Du|_F(\Omega) + \int_{\partial\Omega} |u| F(\nu) d\mathcal{H}^{N-1} = |Du|_F(\mathbb{R}^N) = J(u).$$

This proves (4.5.1).

To give the proof of (4.5.2), we observe that, by Theorem 4.2.3, there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq C_0^\infty(\Omega)$ such that, for any $q \in [1,p]$, u_k converges to u in $L^q(\Omega)$ and $||F(\nabla u_k)||_{L^1(\Omega)}$ converges to $|Du|_F(\mathbb{R}^N)$ as $k \to +\infty$. Moreover, it easily seen that for $p \to 1^+$, $||F(\nabla u_k)||_{L^p(\Omega)}$ converges to $||F(\nabla u_k)||_{L^1(\Omega)}$ and hence we have that there exists a subsequence $p_k \to 1^+$, as $k \to +\infty$, such that $||F(\nabla u_k)||^{p_k}_{L^{p_k}(\Omega)}$ converges to $|Du|_F(\mathbb{R}^N)$ as $k \to +\infty$. This implies that $\limsup_{k\to+\infty} J_{p_k}(u_k) \geq J(u)$, that concludes the proof of (4.5.2).

The case $0 \leq \beta < 1$. Let us consider a sequence u_p weak* converging to u in $BV(\Omega)$, as $p \to 1^+$. A simple application of the Young inequality $a^p \geq pab - (p-1)b^{\frac{p}{p-1}}$ with $b = \frac{1}{p}$, yields to

$$J_p(u_p) = \int_\Omega F^p(\nabla u_p)dx + \beta \int_{\partial\Omega} |u_p|^p F(\nu) d\mathcal{H}^{N-1} \\ \geq \int_\Omega F(\nabla u_p)dx + \beta \int_{\partial\Omega} |u_p| F(\nu) d\mathcal{H}^{N-1} - \frac{p-1}{p}(|\Omega| + P_F(\Omega)),$$

Therefore, the conclusion (4.5.1) follows by applying the Proposition 4.4.1.

In order to get the second claim, Proposition 2.5.3 assures the existence of a sequence $u_k \in C^\infty(\Omega)$ strongly converging to u in $L^1(\Omega)$ and $\|F(\nabla u_k)\|_{L^1(\Omega)}$ converges to $|Du|_F(\Omega)$, as $k \to +\infty$. Moreover $u_k F(\nu)$ converges to $uF(\nu)$ in $L^1(\partial\Omega, \mathcal{H}^{N-1})$. An argument similar to the previous case leads us to say that $\|F(\nabla u_k)\|_{L^{p_k}(\Omega)}$ converges to $|Du|_F(\Omega)$ and $\int_{\partial\Omega} |u_k|^p F(\nu) d\mathcal{H}^{N-1}$ converges to $\int_{\partial\Omega} |u| F(\nu) d\mathcal{H}^{N-1}$, as $k \to +\infty$. Hence the sequence $\{u_k\}_{k\in\mathbb{N}}$ satisfies (4.5.2).

The case $-1 < \beta < 0$. Let us consider a sequence $u_p \in W^{1,p}(\Omega)$ weak* converging to u in $BV(\Omega)$, as $p \to 1^+$.

For any $\delta > 0$, let us set $\Omega_\delta = \{x \in \Omega : d_\mathcal{E}(x) < \delta\}$ and consider a smooth function ψ equal to zero in $\Omega \setminus \Omega_\delta$ and to one on $\partial\Omega$, such that $|\nabla\psi| \leq \frac{2}{\delta}$.

The trace inequality (4.1.2) applied to the function $v = (u - |u_p|^{p-1}u_p)\psi$ gives

$$\begin{aligned}\int_{\partial\Omega} &|u - |u_p|^{p-1}u_p|F(\nu)d\mathcal{H}^{N-1}\\ &\le |D(u - |u_p|^{p-1}u_p)|_F(\Omega_\delta) + \left(\frac{2b}{\delta} + c\right)\int_{\Omega_\delta} |u - |u_p|^{p-1}u_p|dx\\ &\le |Du|_F(\Omega_\delta) + \int_{\Omega_\delta} F(\nabla(|u_p|^{p-1}u_p))dx + \left(\frac{2b}{\delta} + c\right)\int_{\Omega_\delta} |u - |u_p|^{p-1}u_p|dx,\end{aligned} \tag{4.5.3}$$

where we have used that $|D(u - |u_p|^{p-1}u_p)|_F(\partial\Omega_\delta) = 0$ for a set of δ's of positive measure because $u - |u_p|^{p-1}u_p \in BV(\Omega)$.

By using $||a| - |b|| \le |a - b|$ in (4.5.3), we have

$$\begin{aligned}J(u) - J_p(u_p) &= |Du|_F(\Omega) - \int_\Omega F^p(\nabla u_p)dx + \beta\int_{\partial\Omega}(|u| - |u_p|^p)F(\nu)d\mathcal{H}^{N-1}\\ &\le |Du|_F(\Omega) - \int_\Omega F^p(\nabla u_p)dx + |\beta||Du|_F(\Omega_\delta)\\ &\quad + |\beta|\int_{\Omega_\delta} F(\nabla(|u_p|^{p-1}u_p))dx + |\beta|\left(\frac{2b}{\delta} + c\right)\int_{\Omega_\delta} |u - |u_p|^{p-1}u_p|dx := A.\end{aligned} \tag{4.5.4}$$

Multiplying by $\frac{1}{|\beta|} > 1$, we have

$$\begin{aligned}A &\le 2|Du|_F(\Omega_\delta) + |Du|_F(\Omega \setminus \Omega_\delta) - \int_\Omega F^p(\nabla u_p)dx - \int_{\Omega\setminus\Omega_\delta} F(\nabla(|u_p|^{p-1}u_p))dx\\ &\quad + \int_\Omega F(\nabla(|u_p|^{p-1}u_p))dx + \left(\frac{K}{\delta} + c\right)\int_{\Omega_\delta} |u - |u_p|^{p-1}u_p|dx.\end{aligned} \tag{4.5.5}$$

The Young inequality gives that

$$\begin{aligned}\int_\Omega F(\nabla(|u_p|^{p-1}u_p))dx &= \int_\Omega p|u_p|^{p-1}F(\nabla u_p)dx\\ &\le \int_\Omega F^p(\nabla u_p)dx + (p-1)\int_\Omega |u_p|^p dx.\end{aligned} \tag{4.5.6}$$

Furthermore by (4.5.4), (4.5.5) and (4.5.6), we have that

$$\begin{aligned}J(u) - J_p(u_p) &\le 2|Du|_F(\Omega_\delta) + |Du|_F(\Omega \setminus \Omega_\delta) - \int_{\Omega\setminus\Omega_\delta} F(\nabla(|u_p|^{p-1}u_p))dx\\ &\quad + (p-1)\int_\Omega |u_p|^p dx + \left(\frac{K}{\delta} + c\right)\int_{\Omega_\delta} |u - |u_p|^{p-1}u_p|dx.\end{aligned}$$

Since u_p converges to u in $L^q(\Omega)$, then $|u_p|^{p-1}u_p$ converges to u in $L^1(\Omega)$, as $p \to 1^+$. Hence, by taking $p \to 1^+$, we have

$$\limsup_{p\to 1^+}[J(u) - J_p(u_p)] \le 2|Du|_F(\Omega_\delta).$$

By sending $\delta \to 0^+$, we obtain (4.5.1).

Finally, the inequality (4.5.2) is obtained as in the previous case. □

The proof of the Γ-convergence of the functional J_p is useful to prove the convergence of the eigenvalues and eigenfunction, as $p \to 1^+$. To do this we follow the technique used in the Euclidean case [49, Corollary 4.3].

Proposition 4.5.3. *For any $\beta > -1$, it holds*

$$\lim_{p\to 1^+} \lambda_p(\beta,\Omega) = \Lambda(\Omega,\beta).$$

Moreover, the minimizers $u_p \in W^{1,p}(\Omega)$ of (4.3.1), with $\|u_p\|_{L^p(\Omega)} = 1$, weak converge to a minimizer $u \in BV(\Omega)$ of (4.4.1) as $p \to 1^+$.*

Proof. Theorem 4.5.2 assures the existence of a sequence w_p converging to a fixed minimizer $\bar{u}$ of (4.4.1). Let us consider the sequence of minimizers u_p of (4.3.1), we have:

$$\limsup_{p\to 1^+} J_p(u_p) \le \limsup_{p\to 1^+} J_p(w_p) \le J(\bar{u}) = \Lambda(\Omega,\beta). \tag{4.5.7}$$

This means that $J_p(u_p)$ is upper bounded uniformly. If $\beta < 0$, by the trace inequality (4.1.2), we have that

$$\int_\Omega F^p(\nabla u_p)dx \le \Lambda(\Omega,\beta) - \beta|D(u_p^p)|_F(\Omega) - \beta c,$$

and, by (4.5.6) and (2.5.1), that

$$(1+\beta)a^p \int_\Omega |\nabla u_p|^p dx \le \Lambda(\Omega,\beta) - \beta c - \beta(p-1).$$

Hence, by the compactness, we have that u_p is upper bounded in $BV(\Omega)$. If $\beta \ge 0$, this directly follows from (4.5.7). Therefore u_p weak* converges to u in $BV(\Omega)$.

Finally, by (4.5.1) and (4.5.7), we have that

$$\Lambda(\Omega,\beta) = J(\bar{u}) \le J(u) \le \liminf_{p\to 1^+} J_p(u_p) \le \limsup_{p\to 1^+} J_p(u_p) \le \Lambda(\Omega,\beta),$$

and hence the conclusion by observing that $\lambda_p(\beta,\Omega) = J_p(u_p)$. □

4.6 An isoperimetric inequality

At this stage, we treat the shape optimization problem for $\Lambda(\Omega,\beta)$. To this aim, we briefly recall the properties of the eigenvalue problem $\lambda_p(\beta,\mathcal{W}_R)$ and then we prove an explicit computation for $\Lambda(\mathcal{W}_R,\beta)$. By Theorem 4.3.1, a minimizer of (4.3.2) solves the following problem:

$$\begin{cases} -\mathcal{Q}_p u = \lambda_p(\beta,\mathcal{W}_R)|u|^{p-2}u & \text{in } \mathcal{W}_R \\ (F(\nabla u))^{p-1}F_\xi(\nabla u)\cdot\nu + \beta F(\nu)|u|^{p-2}u = 0 & \text{on } \partial\mathcal{W}_R. \end{cases} \tag{4.6.1}$$

In particular, the following result holds.

Theorem 4.6.1. *If $u_p \in C^{1,\alpha}(\mathcal{W}_R) \cap C(\overline{\mathcal{W}_R})$ is a positive solution of (4.6.1), then there exists a monotone function $\varphi_p = \varphi_p(r)$, $r \in [0,R]$, such that $\varphi_p \in C^\infty(0,R) \cap C^1([0,R])$, and*

$$\begin{cases} u_p(x) = \varphi_p(F^o(x)) & \text{in } \overline{\mathcal{W}}_R \\ \varphi_p'(0) = 0 \\ |\varphi_p'(R)|^{p-2}\varphi_p'(R) + \beta\varphi_p(R)^{p-1} = 0. \end{cases} \tag{4.6.2}$$

Moreover, φ_p is decreasing if $\beta > 0$, while is increasing if $\beta < 0$.

For a proof, we refer the reader to [47] for the positive values of the Robin parameter; the case of negative values of the parameter can be treated similarly (see for example [96]).

We first compute $\Lambda(\mathcal{W}_R, \beta)$, observing that we will make use of the Γ- convergence result only for the case $-1 < \beta < 0$ (refer also to [49, Proposition 5.2]).

Proposition 4.6.2. *If $\beta > -1$, then*

$$\Lambda(\mathcal{W}_R, \beta) = \hat{\beta} h_F(\mathcal{W}_R) = \hat{\beta}\frac{N}{R}, \tag{4.6.3}$$

where $\hat{\beta} = \min\{\beta, 1\}$.

Proof. If $\beta \geq 0$, we recall that

$$\Lambda(\mathcal{W}_R, \beta) = \inf_{E \subseteq \mathcal{W}_R} J(\chi_E).$$

By using Theorem 4.4.2 and the isoperimetric inequality, denoting by $\mathcal{W}_r$ the Wulff shape of radius r such that $|\mathcal{W}_r| = |E|$ we have

$$\begin{aligned} R(E, \beta) &= J(\chi_E) \\ &= \frac{P_F(E, \mathcal{W}_R) + \hat{\beta}\displaystyle\int_{\partial \mathcal{W}_R \cap \partial E} F(\nu_E) d\mathcal{H}^{N-1}}{|E|} \\ &\geq \hat{\beta}\frac{P_F(E)}{|E|} \geq \hat{\beta}\frac{P_F(\mathcal{W}_r)}{|\mathcal{W}_r|} \geq \hat{\beta}\frac{P_F(\mathcal{W}_R)}{|\mathcal{W}_R|} \\ &= \hat{\beta}\frac{N}{R}, \end{aligned}$$

where we have used also that $P_F(\mathcal{W}_r) = N\kappa_N r^{N-1}$, $|\mathcal{W}_r| = \kappa_N r^N$.

This proves $\Lambda(\mathcal{W}_R, \beta) \geq \hat{\beta}\frac{N}{R}$. For the reverse inequality we take $E = \mathcal{W}_R$, hence

$$\Lambda(\mathcal{W}_R, \beta) = \ell(\mathcal{W}_R, \beta) \leq R(\mathcal{W}_R, \beta) = \hat{\beta}\frac{N}{R}.$$

Now, we study the case $-1 < \beta < 0$ and we will make use of the Γ-convergence.
Hence, let $u_p \in W^{1,p}(\mathcal{W}_R)$ a minimizer of (4.3.2). We know, thanks to Proposition 4.5.3, that

$$\lim_{p \to 1^+} \lambda_1(\mathcal{W}_R, p, \beta) = \Lambda(\mathcal{W}_R, \beta)$$

and we take $u_p(\cdot) = \varphi_p(F^o(\cdot))$ as in (4.6.2). So, the minimizer converges strongly in $L^1(\mathcal{W}_R)$ to $u \in BV(\mathcal{W}_R)$, almost everywhere in $\mathcal{W}_R$ and $u_p \overset{*}{\rightharpoonup} u$ in $BV(\Omega)$ for $p \to 1^+$. Moreover, u_p is radially increasing, hence u is radially nondecreasing and this implies that its superlevel sets $\{u > t\}$ are concentric Wulff shapes $\{r < F^o(x) < R\}$, where r and R are the radius of the Wulff shapes $\mathcal{W}_r$ and $\mathcal{W}_R$ respectively, and, by Theorem 4.4.2, it holds that

$$\Lambda(\mathcal{W}_R, \beta) = \frac{N}{R}\frac{(\frac{r}{R})^{N-1} + \beta}{1 - (\frac{r}{R})^{N-1}}$$

for some $r \in [0, R[$. Therefore, by minimizing the function

$$f(\xi) = \frac{\xi^{N-1} + \beta}{1 - \xi^N} \qquad \xi \in [0, 1[,$$

we observe that the minimum is attained at $\xi = 0$. Hence, the thesis follows. $\square$

Finally, we prove an isoperimetric inequality for $\Lambda(\Omega,\beta)$ when a volume constraint holds: if $\beta \geq 0$, the Wulff shape is a minimizer and, if $\beta < 0$, it is a maximizer for $\Lambda(\Omega,\beta)$ (see [49, Theorem 5.3] for the Euclidean case).

Proposition 4.6.3. *If $\beta \geq 0$ and $\mathcal{W}_R$ is the Wulff shape of radius R and $|\mathcal{W}_R| = |\Omega|$, then*

$$\Lambda(\mathcal{W}_R,\beta) \leq \Lambda(\Omega,\beta).$$

If $-1 < \beta < 0$, then

$$\Lambda(\mathcal{W}_R,\beta) \geq \Lambda(\Omega,\beta).$$

Equality holds if and only if Ω is the Wulff shape of radius R.

Proof. If $\beta \geq 0$, by using the same argument of the Proposition 4.6.2, we have that then

$$R(E,\beta) = J(\chi_E) \geq \hat{\beta}\frac{P_F(\mathcal{W}_R)}{|\mathcal{W}_R|} = \hat{\beta}\frac{N}{R} = \Lambda(\mathcal{W}_R,\beta),$$

for any $E \subseteq \Omega$. The conclusion follows by passing to the infimum on the set $E \subseteq \Omega$ and using Theorem 4.4.2.

If $-1 < \beta < 0$, then using the isoperimetric inequality [61] and (4.6.3), we have that

$$\Lambda(\Omega,\beta) \leq \beta\frac{P_F(\Omega)}{|\Omega|} \leq \beta\frac{P_F(\mathcal{W}_R)}{|\mathcal{W}_R|} = \Lambda(\mathcal{W}_R,\beta).$$

The equality case follows immediately by the equality case of the Wulff inequality (2.5.13).

□

4.7 Open Problems

We conclude with a couple of open problems. Indeed, it would be interesting to study:

1. the Γ-convergence of the functional and the behavior of the minimizers in the case of β not constant; the main problem is to find which satisfies the second part of the definition of the $\Gamma-$ convergence. Indeed, this result is known for β bounded on the boundary(see [51]);

2. the behavior of the second Robin eigenvalue as p goes to 1. We expect that for the anisotropic case, we can generalize the results proved in [99].

Chapter 5

Upper and lower bounds for the first Robin eigenvalue of nonlinear elliptic operators

Let Ω be a bounded, smooth domain of $\mathbb{R}^N$, $N \geq 2$. In this Chapter, we prove some inequalities involving the first Robin eigenvalue of the p-Laplacian operator. In particular, we prove an upper bound for the first Robin eigenvalue of nonlinear elliptic operators in terms of the first Dirichlet eigenvalue.

5.1 Preliminary results

To prove the two inequalities involving the first Robin eigenvalue for the p-laplacian operator and in particular, to prove an upper bound for the first Robin eigenvalue of nonlinear elliptic operators in terms of the first Dirichlet eigenvalue, and a lower bound, in terms of a Neumann-type eigenvalue, in this section we introduce some preliminary results.
We first recall the following version of the Hölder inequality proved in [52, Proposition A.1].

Proposition 5.1.1. *Let $1 < p, p' < \infty$ be such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Assume that $f_1, f_2 : \Omega \to \mathbb{R}$ and $g_1, g_2 : \partial\Omega \to \mathbb{R}$ are measurable functions satisfying $f_1 \in L^p(\Omega)$, $f_2 \in L^{p'}(\Omega)$, $g_1 \in L^p(\partial\Omega, \lambda)$ and $g_2 \in L^{p'}(\partial\Omega, \lambda)$. Then $f_1 f_2 \in L^1(\Omega)$, $g_1 g_2 \in L^1(\partial\Omega, \lambda)$ and*

$$\int_\Omega |f_1 f_2|\, dx + \int_{\partial\Omega} \lambda(x) |g_1 g_2|\, d\mathcal{H}^{N-1} \leq \left[\int_\Omega |f_1|^p\, dx + \int_{\partial\Omega} \lambda(x) |g_1|^p\, d\mathcal{H}^{N-1} \right]^{\frac{1}{p}} \left[\int_\Omega |f_2|^{p'}\, dx + \int_{\partial\Omega} \lambda(x) |g_2|^{p'}\, d\mathcal{H}^{N-1} \right]^{\frac{1}{p'}},$$

where λ is a bounded function defined on $\partial\Omega$, with $\lambda \geq 0$, not identically null.

Second, we recall the following Poincaré-Wirtinger inequality (see [115, Theorem 4.4.6]).

Theorem 5.1.2. *Let Ω be a bounded Lipschitz open set in $\mathbb{R}^N$ and $p > 1$. Then there exists a positive constant $\tilde{C}$ such that for any $u \in W^{1,p}(\Omega)$, with $\int_{\partial\Omega} u d\mathcal{H}^{N-1} = 0$ it holds that*

$$\int_\Omega |u|^p dx \leq \tilde{C} \int_\Omega |\nabla u|^p dx. \tag{5.1.1}$$

5.1.1 A characterization for $\lambda_p(\beta,\Omega)$

Let Ω be a bounded domain of $\mathbb{R}^N$, $N\geq 2$, $\beta>0$, and let $f\in L^\infty(\Omega)$ a given function. We consider the following Robin boundary value problem

$$\begin{cases} -\Delta_p u_f = f & \text{in } \Omega \\ |\nabla u_f|^{p-2}\dfrac{\partial u_f}{\partial \nu} + \beta |u_f|^{p-2}u_f = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1.2}$$

where, we recall that $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator with $1<p$ and β is a positive parameter. By standard existence results, there exists a unique function $u_f \in W^{1,p}(\Omega)$ which solves (5.1.2), in the sense that

$$\int_\Omega |\nabla u_f|^{p-2}\nabla u_f \nabla\varphi\, dx + \beta\int_{\partial\Omega} |u_f|^{p-2}u_f\varphi\, d\mathcal{H}^{N-1} = \int_\Omega f\varphi\, dx \quad \forall \varphi\in W^{1,p}(\Omega). \tag{5.1.3}$$

Moreover, $u_f\geq 0$ if $f\geq 0$, and by classical regularity results, if Ω has $C^{1,\gamma}$ boundary, then $u_f\in C^{1,\gamma}(\overline{\Omega})$ ([87]).

It is known that the solution to this equation is the unique minimum of the functional

$$F(v) := \frac{1}{p}\int_\Omega |\nabla v|^p\, dx + \frac{\beta}{p}\int_{\partial\Omega}|v|^p\, d\mathcal{H}^{N-1}(x) - \int_\Omega fv\, dx.$$

Indeed, it is possible to prove the existence and the uniqueness of the minimum of the functional thanks to the direct method of calculus of variations.

In the limiting cases $\beta=0$ and $\beta=\infty$, we recover the Neumann and the Dirichlet boundary conditions respectively. Now we are in a position to state and prove a Thompson principle for the Robin $p-$Laplacian.
The idea of the Thompson principle is to prove that there exists a function that minimizes the energy functional.
For a given $f\in L^\infty(\Omega)$, let u_f be the solution to (5.1.2), and define the quantity

$$J_f(\beta) := \int_\Omega |\nabla u_f|^p\, dx + \beta\int_{\partial\Omega}|u_f|^p\, d\mathcal{H}^{N-1} = \int_\Omega f u_f dx$$

where the last equality follows from the equation, using u_f as test function.

Given $f\in L^\infty(\Omega)$, let us consider the set $\mathcal{V}_f$ of all the continuous vector functions $V\in C(\overline{\Omega})$ such that $-\operatorname{div} V = f$, in the sense that

$$\int_\Omega V\cdot\nabla\varphi\, dx + \int_{\partial\Omega}\varphi V\cdot\nu\, d\mathcal{H}^{N-1} = \int_\Omega f\varphi\, dx, \quad \forall\varphi\in C^1(\Omega). \tag{5.1.4}$$

Lemma 5.1.3. *Let Ω be a bounded, $C^{1,\gamma}$ domain of $\mathbb{R}^N$, $N\geq 2$, and let β be a positive parameter. If $f\in L^\infty(\Omega)$ is a nonnegative function, we have that*

$$J_f(\beta) = \min_{V\in\mathcal{V}_f}\left\{\int_\Omega |V|^{p'}\, dx + \beta^{-\frac{1}{p-1}}\int_{\partial\Omega}|V\cdot\nu|^{p'}\, d\mathcal{H}^{N-1}\right\}. \tag{5.1.5}$$

Proof. Let u_f be the solution to the problem (5.1.2). Then by definition of J_f and (5.1.4) with $\varphi = u_f$ it holds that

$$J_f(\beta) = \int_\Omega f u_f dx = \int_\Omega V\cdot\nabla u_f\, dx + \int_{\partial\Omega} u_f V\cdot\nu\, d\mathcal{H}^{N-1},$$

for any $V \in \mathcal{V}_f$. Hence, applying Proposition 5.1.1 with $f_1 = |\nabla u_f|$, $f_2 = |V|$, $g_1 = \beta^{\frac{1}{p}} u_f$ and $g_2 = \beta^{-\frac{1}{p}} |V \cdot \nu|$, we have

$$J_f(\beta) \leq \left(\int_\Omega |V|^{\frac{p}{p-1}} \, dx + \beta^{-\frac{1}{p-1}} \int_{\partial\Omega} |V \cdot \nu|^{\frac{p}{p-1}} \, d\mathcal{H}^{N-1} \right)^{\frac{p-1}{p}} \times \\ \times \left(\int_\Omega |\nabla u_f|^p \, dx + \beta \int_{\partial\Omega} |u_f|^p \, d\mathcal{H}^{N-1} \right)^{\frac{1}{p}},$$

hence

$$J_f(\beta) \leq \int_\Omega |V|^{\frac{p}{p-1}} \, dx + \beta^{-\frac{1}{p-1}} \int_{\partial\Omega} |V \cdot \nu|^{\frac{p}{p-1}} \, d\mathcal{H}^{N-1}.$$

Finally, if $V = |\nabla u_f|^{p-2} \nabla u_f$, then $V \in \mathcal{V}_f$ and the equality in the above inequality holds, hence the thesis follows.

□

Now, we want to prove that the first Robin eigenvalue of the p-Laplace operator can be written in terms of the functional $J_f(\beta)$.

Proposition 5.1.4. *It holds that*

$$\frac{1}{\lambda_p^{\frac{1}{p-1}}(\beta)} = \max \left\{ \frac{J_f(\beta)}{\displaystyle\int_\Omega f^{p'} \, dx}, \; f \in L^\infty(\Omega), \; f \geq 0, f \not\equiv 0 \right\}. \tag{5.1.6}$$

Proof. Let $f \geq 0$, $f \in L^\infty(\Omega)$, $f \not\equiv 0$ and u_f be a solution to (5.1.2). Then by the weak formulation of the solution u_f and the Hölder inequality it holds that

$$J_f(\beta) = \int_\Omega u_f f \, dx \leq \left(\int_\Omega u_f^p \, dx \right)^{\frac{1}{p}} \left(\int_\Omega f^{p'} \, dx \right)^{\frac{1}{p'}} \tag{5.1.7}$$

On the other hand, using u_f as test function in the variational formulation (1.0.17) of $\lambda_p(\beta, \Omega)$ it holds that

$$\int_\Omega u_f^p dx \leq \frac{J_f(\beta, \Omega)}{\lambda_p(\beta)} \tag{5.1.8}$$

Using (5.1.8) in (5.1.7) and rearranging the terms we get

$$\frac{J_f(\beta)}{\displaystyle\int_\Omega f^{p'} \, dx} \leq \frac{1}{\lambda_p^{\frac{1}{p-1}}(\beta, \Omega)}. \tag{5.1.9}$$

By observing that if $f = \lambda_p(\beta) u_\beta^{p-1}$, where u_β is a first positive eigenfunction of (1.0.18), the equality in (5.1.9) holds and the thesis follows. □

A useful property of $J_f(\beta)$ is a convexity property. More precisely:

Lemma 5.1.5. *Let Ω be a bounded open set in $\mathbb{R}^N$, with $C^{1,\gamma}$ boundary. Then for any $f \in L^\infty(\Omega)$, $f \geq 0$, the function J_f satisfies*

$$J_f(\beta) \leq J_f(\alpha) + \left(\frac{1}{\beta^{\frac{1}{p-1}}} - \frac{1}{\alpha^{\frac{1}{p-1}}} \right) H(\alpha), \tag{5.1.10}$$

for any $\alpha, \beta > 0$ and for some function $H(\alpha) \geq 0$.

Proof. To prove (5.1.10), let f be fixed and let us denote with $u_{f,\alpha} \in C^{1,\gamma}(\bar{\Omega})$ be the solution to (5.1.2) where the coefficient in boundary condition is given by α. By the Thompson principle (5.1.5), if we choose $V = |\nabla u_{f,\alpha}|^{p-2} \nabla u_{f,\alpha}$, we have that

$$\begin{aligned} J_f(\beta) &\leq \int_\Omega |\nabla u_{f,\alpha}|^p \, dx + \beta^{-\frac{1}{p-1}} \int_{\partial\Omega} ||\nabla u_{f,\alpha}|^{p-2} \nabla u_{f,\alpha} \cdot \nu|^{\frac{p}{p-1}} \, d\mathcal{H}^{N-1} \\ &= \int_\Omega |\nabla u_{f,\alpha}|^p + \alpha^{-\frac{1}{p-1}} \int_{\partial\Omega} ||\nabla u_{f,\alpha}|^{p-2} \nabla u_{f,\alpha} \cdot \nu|^{\frac{p}{p-1}} d\mathcal{H}^{N-1} \\ &+ \left(\frac{1}{\beta^{\frac{1}{p-1}}} - \frac{1}{\alpha^{\frac{1}{p-1}}} \right) \int_{\partial\Omega} ||\nabla u_{f,\alpha}|^{p-1} \nabla u_{f,\alpha} \cdot \nu|^{\frac{p}{p-1}} d\mathcal{H}^{N-1} \\ &\leq J_f(\alpha) + \left(\frac{1}{\beta^{\frac{1}{p-1}}} - \frac{1}{\alpha^{\frac{1}{p-1}}} \right) \int_{\partial\Omega} ||\nabla u_{f,\alpha}|^{p-2} \nabla u_{f,\alpha} \cdot \nu|^{\frac{p}{p-1}} d\mathcal{H}^{N-1}. \end{aligned}$$

Denoting by $H(\alpha)$ the function $\int_{\partial\Omega} ||\nabla u_{f,\alpha}|^{p-2} \nabla u_{f,\alpha} \cdot \nu|^{\frac{p}{p-1}} d\mathcal{H}^{N-1}$, the thesis follows. □

5.2 Limit properties when $\beta \to 0$ or $\beta \to +\infty$

The following proposition is well-known. For the sake of completeness, we give also a proof.

Proposition 5.2.1. *Let Ω be a bounded, Lipschitz open set in $\mathbb{R}^N$. Then*

$$\lim_{\beta \to +\infty} \lambda_p(\beta, \Omega) = \lambda_p^D(\Omega). \tag{5.2.1}$$

Proof. If u_β is a first positive eigenfunction for $\lambda_p(\beta, \Omega)$ such that $\|u_\beta\|_{L^p(\Omega)} = 1$, it holds that

$$\lambda_p(\beta, \Omega) = \int_\Omega |\nabla u_\beta|^p dx + \beta \int_\Omega u_\beta^p d\mathcal{H}^{N-1} \leq \lambda^D(\Omega), \quad \forall \beta \in \mathbb{R}.$$

Then u_β is bounded in $W^{1,p}(\Omega)$. Then weakly converges in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ and in $L^p(\partial\Omega)$ to a function $u_\infty \in W_0^{1,p}(\Omega)$, and $\|u_\infty\|_{L^p(\Omega)} = 1$. By the definition of $\lambda^D(\Omega)$ and semicontinuity, it holds that

$$\begin{aligned} \lambda^D(\Omega) \leq \int_\Omega |\nabla u_\infty|^p dx \leq \liminf_{\beta \to +\infty} \left[\int_\Omega |\nabla u_\beta|^p dx + \beta \int_\Omega u_\beta^p d\mathcal{H}^{N-1} \right] \leq \\ \leq \limsup_{\beta \to +\infty} \left[\int_\Omega |\nabla u_\beta|^p dx + \beta \int_\Omega u_\beta^p d\mathcal{H}^{N-1} \right] \leq \lambda^D(\Omega), \end{aligned}$$

and the proof is complete. □

Proposition 5.2.2. *Let Ω be a bounded, Lipschitz open set in $\mathbb{R}^N$, and let $u_{f,\beta} \in W^{1,p}(\Omega)$ be a solution to the problem (5.1.2), with $f \in L^{p'}(\Omega)$. Then there exists a function $u_{f,\infty} \in W_0^{1,p}(\Omega)$ that satisfies*

$$\begin{cases} -\Delta_p u_{f,\infty} = f & \text{in } \Omega, \\ u_{f,\infty} = 0 & \text{on } \partial\Omega \end{cases} \tag{5.2.2}$$

and it holds that

$$\lim_{\beta\to+\infty} J_f(\beta) = \lim_{\beta\to+\infty} \int_\Omega |\nabla u_{f,\beta}|^p dx = \int_\Omega |\nabla u_{f,\infty}|^p dx. \tag{5.2.3}$$

Proof. Using $u_{f,\beta}$ as test function in (5.1.3), by definition of $\lambda_p(\beta,\Omega)$ and the Hölder inequality, we get

$$\lambda_p(\beta,\Omega)\int_\Omega |u_{f,\beta}|^p\,dx \le \int_\Omega |\nabla u_{f,\beta}|^p\,dx + \beta\int_{\partial\Omega} |u_{f,\beta}|^p\,d\mathcal{H}^{N-1} = \int_\Omega f u_{f,\beta}\,dx \le$$
$$\le \|f\|_{L^{p'}(\Omega)}\|u_{f,\beta}\|_{L^p(\Omega)}. \tag{5.2.4}$$

Then, recalling (5.2.1), (5.2.4) implies

$$\int_\Omega |u_{f,\beta}|^p dx \le C \quad \text{as } \beta \to +\infty,$$

and using again (5.2.4) it holds that $u_{f,\beta}$ is bounded in $W^{1,p}(\Omega)$ as $\beta \to +\infty$. Hence $u_{f,\beta}$ weakly converges in $W^{1,p}(\Omega)$ and strongly converges in $L^p(\Omega)$ and $L^p(\partial\Omega)$ to a function $u_{f,\infty} \in W^{1,p}(\Omega)$. Moreover, by (5.2.4) again it holds that $u_{f,\infty} \in W_0^{1,p}(\Omega)$. Finally, we can pass to the limit in (5.1.3) in order to get that $u_{f,\infty}$ solves (5.2.2), and

$$\lim_{\beta\to+\infty} J_f(\beta) = \lim_{\beta\to+\infty}\left[\int_\Omega |\nabla u_{f,\beta}|^p dx + \beta\int_{\partial\Omega} |u_{f,\beta}|^p d\mathcal{H}^{N-1}\right]$$
$$= \lim_{\beta\to+\infty}\int_\Omega f u_{f,\beta} dx = \int_\Omega f u_{f,\infty} dx = \int_\Omega |\nabla u_{f,\infty}|^p dx.$$

Hence (5.2.3) is in force. □

Now, we show the behavior of $\lambda_p(\beta,\Omega)$ as $\beta \to 0^+$.

Theorem 5.2.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and $\lambda_p(\beta,\Omega)$ be the first Robin eigenvalue, with $\beta > 0$. Then*

$$\lim_{\beta\to 0^+} \frac{\lambda_p(\beta,\Omega)}{\beta} = \frac{P(\Omega)}{|\Omega|}. \tag{5.2.5}$$

Proof. Let u_β be a first positive eigenfunction of $\lambda_p(\beta,\Omega)$. By using as test function $\varphi \equiv 1$ in the equation solved by u_β, we have that

$$\lambda_p(\beta,\Omega)\int_\Omega u_\beta^{p-1} dx = \beta\int_{\partial\Omega} u_\beta^{p-1} dx.$$

For $\beta \to 0^+$, $\lambda_p(\beta,\Omega) \to 0$. We claim that $u_\beta \to c$, where c is a constant, for $\beta \to 0^+$. To prove the claim, let us fix $||u_\beta||_{L^p(\Omega)} = 1$. Hence

$$\int_\Omega |\nabla u_\beta|^p dx \le \int_\Omega |\nabla u_\beta|^p dx + \beta\int_{\partial\Omega} {u_\beta}^p d\mathcal{H}^{N-1} = \lambda_p(\beta,\Omega),$$

then $||\nabla u_\beta||_{L^p(\Omega)}$ is bounded, and u_β is bounded in $W^{1,p}(\Omega)$. By compactness, there exists a subsequence, still denoted by u_β, and a function $u_0 \in W^{1,p}(\Omega)$ such that

$$\begin{array}{ll} \nabla u_\beta \rightharpoonup \nabla u_0 & \text{weakly in } L^p(\Omega), \\ u_\beta \to u_0 & \text{strongly in } L^q(\Omega), \quad 1 \le q \le p \\ u_\beta \to u_0 & \text{strongly in } L^p(\partial\Omega), \quad 1 \le q \le p. \end{array}$$

Hence

$$\int_\Omega |\nabla u_0|^p dx \le \liminf_{\beta\to 0^+} \left(\int_\Omega |\nabla u_\beta|^p dx + \int_{\partial\Omega} \beta |u_\beta|^p d\mathcal{H}^{N-1} \right)$$
$$= \liminf_{\beta\to 0^+} \lambda_p(\beta,\Omega) = 0.$$

Then, $||\nabla u_0||_{L^p(\Omega)} = 0$ and $u_0 = c$ is constant in $\overline{\Omega}$. Using in particular that

$$\frac{\lambda_p(\beta,\Omega)}{\beta} = \frac{\displaystyle\int_{\partial\Omega} u_\beta^{p-1} d\mathcal{H}^{N-1}}{\displaystyle\int_\Omega u_\beta^{p-1} dx},$$

passing to the limit we have that

$$\lim_{\beta\to 0^+} \frac{\lambda_p(\beta,\Omega)}{\beta} = \frac{P(\Omega)}{|\Omega|},$$

that is the thesis. □

Now, we show the behavior of the solutions of (5.1.2) as $\beta \to 0^+$.

Proposition 5.2.4. *Let Ω be a bounded, Lipschitz domain, $0 \le f \in L^\infty(\Omega)$ and $\beta > 0$, and consider $u_{f,\beta}$ a solution to (5.1.2). Then the sequence $\psi_\beta = u_{f,\beta} - K\beta^{-\frac{1}{p-1}} \in W^{1,p}(\Omega)$, with $K = \left(\frac{1}{P(\Omega)} \int_\Omega f\, dx \right)^{\frac{1}{p-1}}$, weakly converges in $W^{1,p}(\Omega)$, as β goes to 0, to a function $v \in W^{1,p}(\Omega)$ which satisfies*

$$\begin{cases} -\Delta_p v = f & \text{in } \Omega \\ |\nabla v|^{p-2} \dfrac{\partial v}{\partial \nu} = -\dfrac{1}{P(\Omega)} \displaystyle\int_\Omega f\, dx & \text{on } \partial\Omega. \end{cases} \tag{5.2.6}$$

Moreover,

$$\lim_{\beta\to 0^+} \int_\Omega |\nabla u_{f,\beta}|^p\, dx = \int_\Omega |\nabla v|^p dx.$$

Proof. By the weak formulation (5.1.3) and the Hölder inequality, we have that

$$\int_\Omega |\nabla u_{f,\beta}|^p dx + \beta \int_{\partial\Omega} |\varphi_\beta|^p d\mathcal{H}^{N-1} = \int_\Omega f \varphi_\beta dx \le \|f\|_{L^{p'}(\Omega)} \|u_{f,\beta}\|_{L^p(\Omega)}.$$

Then by using the definition of the first Robin eigenvalue of the p-Laplace operator, we can write

$$\int_\Omega |\nabla u_{f,\beta}|^p dx + \beta \int_{\partial\Omega} |u_{f,\beta}|^p d\mathcal{H}^{N-1} \le \left(\frac{1}{\lambda_p(\beta,\Omega)} \right)^{\frac{1}{p-1}} \int_\Omega f^{p'} dx.$$

Hence recalling (5.2.5),

$$\frac{1}{\beta}\int_\Omega |\nabla[\beta^{\frac{1}{p-1}} u_{f,\beta}]|^p dx + \int_{\partial\Omega} [\beta^{\frac{1}{p-1}} u_{f,\beta}]^p d\mathcal{H}^{N-1} \leq C,$$

for some positive constant C independent of β. Hence, for $\beta \to 0^+$, we have that $\nabla[\beta^{\frac{1}{p-1}} u_{f,\beta}] \rightharpoonup 0$ in $L^p(\Omega)$ and $\beta^{\frac{1}{p-1}} u_{f,\beta}$ converges in $L^p(\partial\Omega)$ to a constant K. Hence from the equation, using $\psi \equiv 1$ as test function,

$$\int_\Omega f = \int_{\partial\Omega} [\beta^{\frac{1}{p-1}} u_{f,\beta}]^{p-1} d\mathcal{H}^{N-1},$$

then for $\beta \to 0^+$, we obtain $K = \left(\dfrac{1}{P(\Omega)}\displaystyle\int_\Omega f dx\right)^{\frac{1}{p-1}}$.

Now, let

$$\psi_\beta = u_{f,\beta} - \frac{K}{\beta^{\frac{1}{p-1}}}$$

be a test function for (5.1.3). Then it holds that

$$\int_\Omega |\nabla\psi_\beta|^p dx + \beta\int_{\partial\Omega} u_{f,\beta}^{p-1}\psi_\beta d\mathcal{H}^{N-1} \leq \int_\Omega f\psi_\beta dx. \tag{5.2.7}$$

Now, being

$$\int_{\partial\Omega} \psi_\beta d\mathcal{H}^{N-1} = 0, \tag{5.2.8}$$

we have

$$\int_{\partial\Omega} u_{f,\beta}^{p-1}\psi_\beta d\mathcal{H}^{N-1} = \int_{\partial\Omega} u_{f,\beta}^p d\mathcal{H}^{N-1} \geq 0,$$

and as a consequence in (5.2.7),

$$\int_\Omega |\nabla\psi_\beta|^p dx \leq \int_\Omega f\psi_\beta dx \leq \|f\|_{L^{p'}(\Omega)}\|\psi_\beta\|_{L^p(\Omega)}.$$

Now, recalling (5.2.8) and using the Poincaré-Wirtinger inequality (5.1.1) we get

$$\int_\Omega |\nabla\psi_\beta|^p dx \leq C, \qquad \int_\Omega |\psi_\beta|^p dx \leq C.$$

This implies that ψ_β weakly converges in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$, up to a subsequence, to a function $v \in W^{1,p}(\Omega)$. Moreover, using also the convergence of $\beta^{\frac{1}{p-1}} u_{f,\beta}$ to K in $L^p(\partial\Omega)$, we obtain that the function v is a solution to (5.2.6). By the weak formulation (5.1.3), if we use as test function ψ_β, we have

$$\int_\Omega |\nabla u_{f,\beta}|^p\, dx + \beta\int_{\partial\Omega} |u_{f,\beta}|^{p-1}\left(u_{f,\beta} - \frac{K}{\beta^{\frac{1}{p-1}}}\right) d\mathcal{H}^{N-1} = \int_\Omega f\left(u_{f,\beta} - \frac{K}{\beta^{\frac{1}{p-1}}}\right) dx,$$

and then finally

$$\lim_{\beta\to 0^+}\int_\Omega |\nabla u_{f,\beta}|^p\, dx = \int_\Omega fv - K^{p-1}\int_{\partial\Omega} v = \int_\Omega |\nabla v|^p\, dx.$$

$\square$

5.2.1 The case $\beta < 0$

For the sake of completeness, we devote the last part of this section to study the behavior of $\lambda_p(\beta,\Omega)$ as $\beta \to 0^-$. First, we prove a trace inequality (see [68] for the case $p = 2$).

Theorem 5.2.5. *Let $\Omega \subset \mathbb{R}^N$ an open, bounded domain with Lipschitz boundary, there exists $\varepsilon_\Omega > 0$ such that for every $\varepsilon > \varepsilon_\Omega$ there exists $C(\varepsilon) > 0$ for which*

$$\int_{\partial\Omega} |u|^p \, d\mathcal{H}^{N-1} \le \varepsilon \int_\Omega |\nabla u|^p \, dx + \frac{P(\Omega)}{|\Omega|}(1 + C(\varepsilon)) \int_\Omega |u|^p \, dx. \tag{5.2.9}$$

Moreover, we can choose $C(\varepsilon) \to 0$ as $\varepsilon \to +\infty$.

Proof. We proceed by contradiction, hence there exists $\delta > 0$ and a sequence $\{u_n\} \subset W^{1,p}(\Omega)$ such that

$$\int_{\partial\Omega} |u_n|^p d\mathcal{H}^{N-1} \ge n \int_\Omega |\nabla u_n|^p dx + \frac{P(\Omega)}{|\Omega|}(1 + \delta) \int_\Omega |u_n|^p dx.$$

If $\int_\Omega |\nabla u_n|^p \, dx = 0$ for some n, then u_n is constant and this is a contradiction, being $\delta > 0$. Now, we can assume that $\int_\Omega |\nabla u_n|^p \, dx \neq 0$ for every n. Let us define

$$v_n = \frac{u_n}{\left(\int_\Omega |\nabla u_n|^p \, dx\right)^{\frac{1}{p}}},$$

we have that $\int_\Omega |\nabla v_n|^p \, dx = 1$ and

$$\int_{\partial\Omega} |v_n|^p \, d\mathcal{H}^{N-1} \ge n + \frac{P(\Omega)}{|\Omega|}(1 + \delta) \int_\Omega |v_n|^p \, dx. \tag{5.2.10}$$

Now we want to prove that $\{\int_\Omega |v_n|^p dx\}$ is bounded. If this is not true, there exists $\{n_k\}$ such that $\lim_k \int_\Omega |v_{n_k}|^p dx = \infty$. Let us define

$$w_k = \frac{v_{n_k}}{\left(\int_\Omega |v_{n_k}|^p dx\right)^{\frac{1}{p}}},$$

we have that

$$\lim_{k\to+\infty} \int_\Omega |\nabla w_k|^p dx = 0 \quad \text{and} \int_\Omega |w_k|^p dx = 1,$$

then $\{w_k\}$ is bounded in $W^{1,p}(\Omega)$ and there exists $\{w_{k_j}\}$ weakly converging in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ to a function $w \in W^{1,p}(\Omega)$. By uniqueness of the limit $\nabla w = 0$ and $w_{k_j} \to \dfrac{1}{|\Omega|^{\frac{1}{p}}}$ strongly in $W^{1,p}(\Omega)$, hence

$$\frac{P(\Omega)}{|\Omega|}(1 + \delta) \le \int_{\partial\Omega} |w_{k_j}|^p \, d\mathcal{H}^{N-1},$$

and this is a contradiction, being $\lim_{j\to+\infty} \int_{\partial\Omega} |w_{k_j}|^p \, d\mathcal{H}^{N-1} = \dfrac{|\partial\Omega|}{|\Omega|}$. So, there exists a subsequence $\{v_{n_k}\}$ such that $\lim_{k\to+\infty} \int_\Omega |v_{n_k}|^p dx = c$, for $c \ge 0$. Hence, by using [72, Theorem 1.5.1.10], we have that

$$\int_{\partial\Omega} |v_{n_k}|^p d\mathcal{H}^{N-1} \le \varepsilon^{1-\frac{1}{p}} \int_\Omega |\nabla v_{n_k}|^p dx + K(\varepsilon) \int_\Omega |v_{n_k}|^p dx,$$

for $\varepsilon \in (0,1)$. Putting together with (5.2.10), we have that

$$n_k + \frac{P(\Omega)}{|\Omega|}(1+\delta)\int_\Omega |v_{n_k}|^p dx \le \varepsilon^{1-\frac{1}{p}} + K(\varepsilon)\int_\Omega |v_{n_k}|^p dx,$$

and this is a contradiction.

$\square$

The trace inequality allows us to prove that the limit (5.2.5) holds true also when $\beta \to 0^-$.

Theorem 5.2.6. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and $\lambda_p(\beta,\Omega)$ be the first Robin eigenvalue, with $\beta<0$. Then*

$$\lim_{\beta\to 0^-}\frac{\lambda_p(\beta,\Omega)}{\beta} = \frac{P(\Omega)}{|\Omega|}.$$

Proof. Let u_β be a positive eigenfunction of $\lambda_p(\beta,\Omega)$ with $\|u_\beta\|_{L^p(\Omega)}=1$. Then

$$\lambda_p(\beta,\Omega) = \int_\Omega |\nabla u_\beta|^p dx + \beta\int_{\partial\Omega}|u_\beta|^p d\mathcal{H}^{N-1}.$$

Using the trace inequality (5.2.9) with $\varepsilon = -\frac{1}{\beta}$ (as $|\beta|$ is small) we get

$$\frac{P(\Omega)}{|\Omega|} \le \frac{\lambda_p(\beta,\Omega)}{\beta} \le \frac{P(\Omega)}{|\Omega|}\left(1 + C\left(\frac{1}{-\beta}\right)\right).$$

The thesis follows, being $C(-1/\beta)\to 0$ as $\beta\to 0^-$. $\square$

5.3 Upper bounds for $\lambda_p(\beta,\Omega)$

In this Section, we want to get upper bounds for $\lambda_p(\beta,\Omega)$. The first general result is the following.

Theorem 5.3.1. *Let $\Omega\subset\mathbb{R}^N$ be an open bounded set, with $C^{1,\gamma}$ boundary, β be a positive number, and $f\in L^\infty(\Omega)$, with $f\ge 0$. then*

$$\frac{1}{\lambda_p(\beta,\Omega)^{\frac{1}{p-1}}} \ge \frac{\displaystyle\int_\Omega |\nabla u_{f,\infty}|^p\, dx}{\displaystyle\int_\Omega f^{\frac{p}{p-1}}\, dx} + \frac{\beta^{-\frac{1}{p-1}}}{P(\Omega)^{\frac{1}{p-1}}}\frac{\left(\displaystyle\int_\Omega f\, dx\right)^{\frac{p}{p-1}}}{\displaystyle\int_\Omega f^{\frac{p}{p-1}}\, dx}, \tag{5.3.1}$$

where $P(\Omega)=\mathcal{H}^{N-1}(\partial\Omega)$ denotes the perimeter of Ω, and $u_{f,\infty}$ satisfies (5.2.2).

Proof. Let f be fixed. Then by using the convexity of $J_f(\beta)$,

$$J_f(\alpha) \le J_f(\beta) + \left(\frac{1}{\alpha^{\frac{1}{p-1}}} - \frac{1}{\beta^{\frac{1}{p-1}}}\right)\int_{\partial\Omega} ||\nabla u_{f,\beta}|^{p-2}\left(\frac{\partial u_{f,\beta}}{\partial\nu}\right)|^{\frac{p}{p-1}} d\mathcal{H}^{N-1}.$$

If we multiply for $\beta^{\frac{1}{p-1}}$,

$$\beta^{\frac{1}{p-1}}J_f(\alpha) \le \beta^{\frac{1}{p-1}}J_f(\beta) + \left(-1 + \left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}\right)\int_{\partial\Omega} ||\nabla u_{f,\beta}|^{p-2}\left(\frac{\partial u_{f,\beta}}{\partial\nu}\right)|^{\frac{p}{p-1}} d\mathcal{H}^{N-1},$$

then for α that goes to infinity

$$\beta^{\frac{1}{p-1}} J_f(\beta) \geq \beta^{\frac{1}{p-1}} J_f(\infty) + \int_{\partial\Omega} ||\nabla u_{f,\beta}|^{p-2} \left(\frac{\partial u_{f,\beta}}{\partial \nu}\right)|^{\frac{p}{p-1}} d\mathcal{H}^{N-1}.$$

We know by Proposition 5.2.2 that $J_f(\infty) = \displaystyle\int_\Omega |\nabla u_{f,\infty}|^p dx$ and using the Hölder inequality and the equation solved by $u_{f,\beta}$

$$\begin{aligned}\int_{\partial\Omega} &||\nabla u_{f,\beta}|^{p-2} \left(\frac{\partial u_{f,\beta}}{\partial \nu}\right)|^{\frac{p}{p-1}} d\mathcal{H}^{N-1} \\ &\geq \frac{1}{P(\Omega)^{\frac{1}{p-1}}} \left(\int_{\partial\Omega} ||\nabla u_{f,\beta}|^{p-2} \left(\frac{\partial u_{f,\beta}}{\partial \nu}\right)| d\mathcal{H}^{N-1}\right)^{\frac{p}{p-1}} = \frac{1}{P(\Omega)^{\frac{1}{p-1}}} \left(\int_\Omega f\, dx\right)^{\frac{p}{p-1}}.\end{aligned}$$

Then, putting all together, we obtain

$$J_f(\beta) \geq \int_\Omega |\nabla u_{f,\infty}|^p dx + \frac{\beta^{-\frac{1}{p-1}}}{P(\Omega)^{\frac{1}{p-1}}} \left(\int_\Omega f\, dx\right)^{\frac{p}{p-1}}.$$

So finally, by using (5.1.6) we get that

$$\frac{1}{\lambda_p(\beta,\Omega)^{\frac{1}{p-1}}} \geq \frac{\displaystyle\int_\Omega |\nabla u_{f,\infty}|^p\, dx}{\displaystyle\int_\Omega f^{\frac{p}{p-1}}\, dx} + \frac{\beta^{-\frac{1}{p-1}}}{P(\Omega)^{\frac{1}{p-1}}} \frac{\left(\displaystyle\int_\Omega f\, dx\right)^{\frac{p}{p-1}}}{\displaystyle\int_\Omega f^{\frac{p}{p-1}}\, dx}$$

that is the result. □

Now we are in position to prove the main bounds for $\lambda_p(\beta,\Omega)$. So, let u_∞ be a first positive Dirichlet eigenfunction of $-\Delta_p$, that is a positive function such that

$$\begin{cases} -\Delta_p u_\infty = \lambda_p^D(\Omega) u_\infty^{p-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3.2}$$

We recall that following reverse Hölder inequality holds:

Theorem 5.3.2 ([4]). *For $0 < q < r \leq \infty$ we have*

$$\|u_\infty\|_r \leq \tilde{K} \|u_\infty\|_q,$$

with

$$\tilde{K} = \tilde{K}(N,p,q,r,\lambda_p^D(\Omega)) = \frac{\|v\|_r}{\|v\|_q}, \tag{5.3.3}$$

and v is a first eigenfunction of (5.3.2) in a ball B such that $\lambda_p^D(B) = \lambda_p^D(\Omega)$.

We can now prove the following main results.

Theorem 5.3.3. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, with $C^{1,\gamma}$ boundary, and let β be a positive number. Then,*

$$\frac{1}{\lambda_p^{\frac{1}{p-1}}(\beta,\Omega)} \geq \frac{1}{\lambda_p^D(\Omega)^{\frac{1}{p-1}}} + \frac{\tilde{K}}{(\beta P(\Omega))^{\frac{1}{p-1}}}, \tag{5.3.4}$$

where $\tilde{K}$ is given in (5.3.3).

Proof. If we choose in (5.3.1)

$$f = u_\infty^{p-1},$$

where u_∞ is a first positive Dirichlet eigenfunction, then

$$u_f = \frac{1}{\lambda_p^D(\Omega)^{\frac{1}{p-1}}} u_\infty$$

and we obtain

$$\frac{1}{\lambda_p(\beta,\Omega)^{\frac{1}{p-1}}} \geq \frac{1}{\lambda_p^D(\Omega)^{\frac{p}{p-1}}} \frac{\displaystyle\int_\Omega |\nabla u_\infty|^p \, dx}{\displaystyle\int_\Omega u_\infty^p \, dx} + \frac{\beta^{-\frac{1}{p-1}}}{P(\Omega)^{\frac{1}{p-1}}} \frac{\left(\displaystyle\int_\Omega u_\infty^{p-1} \, dx\right)^{\frac{p}{p-1}}}{\displaystyle\int_\Omega u_\infty^p \, dx}.$$

By the Reverse Hölder inequality (5.3.2),

$$\left(\int_\Omega u_\infty^{p-1}\right)^{\frac{p}{p-1}} dx \geq \frac{1}{\tilde{K}} \int_\Omega u_\infty^p,$$

then

$$\frac{1}{\lambda_p^{\frac{1}{p-1}}(\beta,\Omega)} \geq \frac{1}{\lambda^D{}_p(\Omega)^{\frac{1}{p-1}}} + \frac{\beta^{-\frac{1}{p-1}}}{P(\Omega)^{\frac{1}{p-1}}} \tilde{K}.$$

□

For a given open bounded domain Ω, let $u_\Omega \in W_0^{1,p}(\Omega)$ be the unique solution to

$$\begin{cases} -\Delta_p u_\Omega = 1 & \text{in } \Omega \\ u_\Omega = 0 & \text{on } \partial\Omega. \end{cases}$$

The $p-$torsional rigidity is

$$T_p(\Omega) = \left(\max_{\varphi \in W_0^{1,p}(\Omega)} \frac{\left(\displaystyle\int_\Omega |\varphi| dx\right)^p}{\displaystyle\int_\Omega |\nabla\varphi|^p dx} \right)^{\frac{1}{p-1}} = \int_\Omega u_\Omega dx = \int_\Omega |\nabla u_\Omega|^p dx.$$

Theorem 5.3.4. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, with $C^{1,\gamma}$ boundary, and let β be a positive number. Then,*

$$\frac{1}{\lambda_p(\beta,\Omega)^{\frac{1}{p-1}}} \geq \frac{T_p(\Omega)}{|\Omega|} + \left(\frac{|\Omega|}{\beta P(\Omega)}\right)^{\frac{1}{p-1}}. \tag{5.3.5}$$

Proof. It is immediate by choosing $f \equiv 1$ in (5.3.1). □

Remark 5.3.5. Using $w = 1$ or $w = u_\infty$ as test function in the variational characterization (1.0.17) of $\lambda_p(\beta, \Omega)$, we immediately get

$$\lambda_p(\beta, \Omega) \le \min\left\{\lambda_p^D(\Omega), \beta\frac{P(\Omega)}{|\Omega|}\right\}.$$

The two inequalities (5.3.4) and (5.3.5) then give a stronger bound on $\lambda_p(\beta, \Omega)$.

Remark 5.3.6. For $\beta \to \infty$, by using that the first Robin eigenvalue of the p-Laplace operator goes to the first Dirichlet eigenvalue of the p-Laplace operator, the inequality (5.3.5) becomes

$$\frac{T_p(\Omega)\lambda_p^D(\Omega)^{\frac{1}{p-1}}}{|\Omega|} \le 1,$$

which corresponds to a well-known inequality due to Pólya in the case of the Laplace operator.

5.4 A lower bound for $\lambda_p(\beta, \Omega)$

In this Section, we want to get a lower bound for $\lambda_p(\beta, \Omega)$. To this aim, let us define

$$\nu_p = \inf_{|\nabla v|^{p-2}\frac{\partial v}{\partial \nu} = K} \frac{\displaystyle\int_\Omega f^{p'}\,dx}{\displaystyle\int_\Omega |\nabla v|^p dx},$$

where $K = \frac{1}{P(\Omega)}\int_\Omega f dx$, and the infimum is computed among all the functions $f \ge 0$, $f \in L^\infty(\Omega)$ and where v is a solution to the problem (5.2.6).

Remark 5.4.1. It is easily seen that $\nu_p > 0$. Indeed, if we denote with $v_{\partial\Omega} = \fint_{\partial\Omega} v d\mathcal{H}^{N-1}$, we have $\int_{\partial\Omega}(v - v_{\partial\Omega})d\mathcal{H}^{N-1} = 0$; by using the weak formulation of (5.2.6), the Hölder inequality and the Poincaré-Wirtinger inequality (5.1.1), we have

$$\int_\Omega |\nabla v|^p dx = \int_\Omega f(v - v_{\partial\Omega})dx \le \left(\int_\Omega f^{p'}dx\right)^{\frac{1}{p'}} \left(\int_\Omega |v - v_{\partial\Omega}|^p dx\right)^{\frac{1}{p}} \le$$
$$\le \tilde{C}\left(\int_\Omega |\nabla v|^p dx\right)^{\frac{1}{p}} \left(\int_\Omega f^{p'}dx\right)^{\frac{1}{p'}},$$

then

$$\frac{\displaystyle\int_\Omega f^{p'}dx}{\displaystyle\int_\Omega |\nabla v|^p dx} \ge \frac{1}{\tilde{C}^{p'}} > 0.$$

Theorem 5.4.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, with $C^{1,\gamma}$ boundary. For any $\beta > 0$,*

$$\frac{1}{\lambda_p(\beta, \Omega)} \le \frac{1}{\nu_p} + \frac{|\Omega|^{\frac{1}{p-1}}}{\beta^{\frac{1}{p-1}}P(\Omega)^{\frac{1}{p-1}}}.$$

Proof. Let $f\geq 0$, $f\in L^\infty(\Omega)$. By using again the convexity of $J_f(\beta)$, we have

$$J_f(\beta)\leq J_f(\alpha)+\left(\frac{1}{\beta^{\frac{1}{p-1}}}-\frac{1}{\alpha^{\frac{1}{p-1}}}\right)\int_{\partial\Omega}||\nabla u_{f,\alpha}|^{p-2}\left(\frac{\partial u_{f,\alpha}}{\partial\nu}\right)|^{\frac{p}{p-1}}d\mathcal{H}^{N-1}.$$

If we multiply for $\beta^{\frac{1}{p-1}}$,

$$\beta^{\frac{1}{p-1}}J_f(\beta)\leq\beta^{\frac{1}{p-1}}J_f(\alpha)+\left(1-\left(\frac{\beta}{\alpha}\right)^{\frac{1}{p-1}}\right)\int_{\partial\Omega}||\nabla u_{f,\alpha}|^{p-2}\left(\frac{\partial u_{f,\alpha}}{\partial\nu}\right)|^{\frac{p}{p-1}}d\mathcal{H}^{N-1},$$

and using the definition of $J_f(\alpha)$ and the boundary condition

$$J_f(\beta)\leq\int_\Omega|\nabla u_{f,\alpha}|^p\,dx+\frac{1}{\beta^{\frac{1}{p-1}}}\int_{\partial\Omega}||\nabla u_{f,\alpha}|^{p-2}\frac{\partial u_{f,\alpha}}{\partial\nu}|^{p'}d\mathcal{H}^{N-1}.$$

For $\alpha\to 0^+$, we have that

$$\frac{1}{\lambda_p(\beta,\Omega)}\leq\frac{1}{\nu_p}+\frac{|\Omega|^{\frac{1}{p-1}}}{\beta^{\frac{1}{p-1}}P(\Omega)^{\frac{1}{p-1}}}.$$

Indeed, from Proposition 5.2.4 we have that

$$\lim_{\alpha\to 0^+}\int_\Omega|\nabla u_{f,\alpha}|^p\,dx=\int_\Omega|\nabla v|^p\,dx,$$

where v is a solution to (5.2.6), and

$$\lim_{\alpha\to 0^+}\int_{\partial\Omega}||\nabla u_{f,\alpha}|^{p-2}\frac{\partial u_{f,\alpha}}{\partial\nu}|^{\frac{p}{p-1}}\,d\mathcal{H}^{N-1}=\lim_{\alpha\to 0}\int_{\partial\Omega}|\alpha u_{f,\alpha}|^p d\mathcal{H}^{N-1}=\frac{1}{P(\Omega)}\left(\int_\Omega f\right)^{p'}.$$

Hence, from (5.1.6)

$$\frac{1}{\lambda_p(\beta,\Omega)^{\frac{1}{p-1}}}\leq\sup_{|\nabla u_f|^{p-2}\frac{\partial u_f}{\partial\nu}=K}\frac{\int_\Omega|\nabla v|^p}{\int_\Omega f^{p'}}+\frac{1}{\beta^{\frac{1}{p-1}}P(\Omega)^{\frac{1}{p-1}}}\sup_{\int_\Omega f^{p'}<\infty}\frac{\left(\int_\Omega f\right)^{p'}}{\int_\Omega f^{p'}}.$$

Then

$$\frac{1}{\lambda_p(\beta,\Omega)^{\frac{1}{p-1}}}\leq\frac{1}{\nu_p}+\frac{1}{\beta^{\frac{1}{p-1}}P(\Omega)^{\frac{1}{p-1}}}|\Omega|^{\frac{1}{p-1}},$$

that is the thesis. $\square$

Chapter 6

Shape optimization for a nonlinear elliptic problem related to thermal insulation

In this Chapter, we consider a minimization problem of a nonlinear functional $I_{\beta,p}(D,\Omega)$ related to a thermal insulation problem with a convection term, where Ω is a bounded connected open set in $\mathbb{R}^n$ and $D \subset \bar{\Omega}$ is a compact set. The Euler-Lagrange equation relative to $I_{\beta,p}$ is a p-Laplace equation, $1 < p < \infty$, with a Robin boundary condition with parameter $\beta > 0$.
The main aim is to study extremum problems for $I_{\beta,p}(D,\Omega)$, among domains D with given geometrical constraints and $\Omega \setminus D$ of fixed thickness. In the planar case, we show that under perimeter constraint the disk maximizes $I_{\beta,p}$.
In the N-dimensional case we restrict our analysis to convex sets showing that the same is true for the ball but under different geometrical constraints.

6.1 The variational problem

Given a compact set D in $\mathbb{R}^N$ and a bounded connected open set Ω of $\mathbb{R}^N$ with $D \subset \bar{\Omega}$, we are interested to study

$$\begin{aligned} I_{\beta,p}(D;\Omega) = \inf\Big\{ \int_\Omega |D\phi|^p dx + \beta \int_{\partial^*\Omega} |\phi|^p d\mathcal{H}^{N-1}, \\ \phi \in W^{1,p}(\Omega), \phi \geq 1 \text{ in } \overline{D} \Big\}. \end{aligned} \tag{6.1.1}$$

with $\beta > 0$. The following result holds.

Proposition 6.1.1. *If Ω is bounded connected open set of $\mathbb{R}^N$ with Lipschitz boundary, and D is a compact set with $D \subset \Omega$, then there exists a unique positive minimizer $u \in W^{1,p}(\Omega)$ of (6.1.1) such that satisfies*

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega \setminus \overline{D} \\ u = 1 & \text{in } D \\ |Du|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1.2}$$

Moreover the functional can be written as

$$I_{\beta,p}(D,\Omega)=\beta\int_{\partial\Omega}u^{p-1}\,d\mathcal{H}^{N-1}, \tag{6.1.3}$$

where u is the solution to (6.1.2).

Proof. The proof follows by a standard application of a calculus of variation's argument. We sketch the proof for completeness.
Let u_n be a minimizing sequence for $I_{\beta,p}(\Omega,D)$. Then u_n is bounded in $W^{1,p}$, $\exists u_{n_k}\to u$ weakly in $W^{1,p}(\Omega)$, strongly in L^p and a.e in Ω. Then by semicontinuity, u is a minimizer of $I_{\beta,p}(\Omega,D)$. Moreover, being $|u|$ still a minimizer, we can assume $u\geq 0$. By Harnack inequality, $u>0$ in Ω. Furthermore, by convexity, u is the unique minimizer and it satisfies

$$\begin{cases}\Delta_p u=0 & \text{in } \Omega\\ u=1 & \text{in } \bar{D}\\ |Du|^{p-2}\dfrac{\partial u}{\partial\nu}+\beta u^{p-1}=0 & \text{on } \partial\Omega.\end{cases}$$

Let $W^{1,p}_{\overline{D}}(\Omega)$ the closure in $W^{1,p}(\Omega)$ of $\{\phi|_\Omega : \phi\in C^\infty(\mathbb{R}^N) \text{ with } \bar{D}\cap\operatorname{supp}\phi=\emptyset\}$. Hence, the minimizer u is such that $u-1\in W^{1,p}_{\overline{D}}(\Omega)$ and it is a solution to (6.1.2). In particular

$$\int_{\Omega\setminus\overline{D}}|Du|^{p-2}DuD\phi\,dx+\beta\int_{\partial\Omega}u^{p-1}\phi\,d\mathcal{H}^{N-1}=0$$

for any $\phi\in W^{1,p}_{\overline{D}}(\Omega)$.
So, if we take $\phi=u-1\in W^{1,p}_{\overline{D}}(\Omega)$,

$$\int_\Omega|Du|^{p-2}Du(Du)\,dx+\beta\int_{\partial\Omega}u^{p-1}u(u-1)\,d\mathcal{H}^{N-1}=0,$$

that is

$$\int_\Omega|Du|^p\,dx+\beta\int_{\partial\Omega}(u^p-u^{p-1})\,d\mathcal{H}^{N-1}=0$$

and this implies (6.1.3). □

6.2 The planar case

In this subsection we consider the case where Ω is the Minkowski sum

$$\Omega=D+\delta B,$$

B is the unit ball centered at the origin and δ is a positive constant, representing the thickness of $\Omega\setminus D$. Then we set

$$I_{\beta,p,\delta}(D)=I_{\beta,p}(D,D+\delta B).$$

Theorem 6.2.1. *Let D be an open, bounded connected set of $\mathbb{R}^2$ with piecewise C^1 boundary. Then*

$$I_{\beta,p,\delta}(D)\leq I_{\beta,p,\delta}(D^*),$$

where D^* is the disk having the same perimeter of D.

Proof. Let v be the radial minimizer of $I_{\beta,p,\delta}(D^*)$ and $\Omega_* = D^* + \delta B$. Given $R > 0$ the radius of D^*, we denote by

$$v_m = v(R+\delta) = \min_{\Omega_*} v$$

and

$$\max_{\Omega_*} v = v(R) = 1.$$

As v is radial, the modulus of the gradient of v is constant on the level lines of v.
So we can consider the function

$$g(t) = |Dv|_{v=t}, \qquad v_m < t \leq 1$$

and

$$w(x) = G(R + d(x)), \ \ x \in \Omega, \qquad \text{where } \ G^{-1}(t) = R + \int_t^1 \frac{1}{g(s)} ds$$

and $d(x)$ is the distance of a point x from D. In particular, v is decreasing and the function g can be zero only for $v = 1$. This means that G^{-1} is decreasing, therefore so is G and then $w \in W^{1,p}(\Omega)$. So we get:

$$\begin{aligned}
&\max_{\Omega} w = w|_{\partial D} = 1 = G(R),\\
&w_m = \min_{\Omega} w = w|_{\partial\Omega} = G(R+\delta) = v_m,\\
&|Dw|_{w=t} = |Dv|_{v=t} = g(t), \ \ w_m \leq t \leq 1.
\end{aligned}$$

Hence w is a test function. Then

$$I_{\beta,p,\delta}(D) \leq \int_{\Omega\setminus D} |Dw|^p \, dx + \beta \int_{\partial\Omega} |w|^p \, d\mathcal{H}^1.$$

Let

$$E_t = \{x \in \Omega : w(x) > t\} = \{x \in \Omega : d(x) < G^{-1}(t)\} = D + G^{-1}(t)B$$

and let

$$B_t = \{x \in \Omega_* : v(x) > t\}.$$

By Steiner formula (2.4.6) we get

$$P(\Omega) \leq P(D) + 2\pi\delta,$$

so

$$P(E_t) \leq P(D) + 2\pi G^{-1}(t) = P(D^*) + 2\pi G^{-1}(t) = 2\pi(R + G^{-1}(t)) = P(B_t)$$

for every $t \in]w_m, 1]$. Hence,

$$\begin{aligned}
\int_{w=t} |Dw| d\mathcal{H}^1 = \int_{w=t} g(t) d\mathcal{H}^1 = g(t)P(E_t) \leq g(t)P(B_t)&\\
= \int_{v=t} |Dv| d\mathcal{H}^1, \ \ w_m < t \leq 1.&
\end{aligned}$$

Then, using the co-area formula

$$\begin{aligned}\int_{\Omega\setminus\overline{D}}|Dw|^p dx &= \int_{w_m}^1 dt\int_{w=t}|Dw|^{p-1}d\mathcal{H}^1\\ &= \int_{w_m}^1 [g(t)]^{p-1}P(E_t)dt \le \int_{w_m}^1 [g(t)]^{p-1}P(B_t)dt\\ &= \int_{\Omega\setminus\overline{D}_*}|Dv|^p dx.\end{aligned} \tag{6.2.1}$$

Since by construction $w = w_m = v_m$ on $\partial\Omega$ and $P(\Omega) = P(\Omega_*)$, we have

$$\int_{\partial\Omega}|w|^p d\mathcal{H}^1 = |w_m|^p P(\Omega) = |v_m|^p P(\Omega_*) = \int_{\partial\Omega_*}|v|^p d\mathcal{H}^1. \tag{6.2.2}$$

Hence, by (6.2.1) and (6.2.2) it holds that

$$\begin{aligned}I_{\beta,p,\delta}(D) &\le \int_{\Omega\setminus D}|Dw|^p dx + \beta\int_{\partial\Omega}|w|^p d\mathcal{H}^1\\ &\le \int_{\Omega_*\setminus D^*}|Dv|^p dx + \beta\int_{\partial\Omega_*}|v|^p d\mathcal{H}^1 = I_{\beta,p,\delta}(D^*).\end{aligned}$$

□

6.3 The n-dimensional case

Now we prove that in higher dimension ($n \ge 3$) balls still maximize $I_{\beta,p,\delta}$, but our result finds its natural generalization in the class of convex domains D.

Theorem 6.3.1. *Let D be an open, bounded, convex set of $\mathbb{R}^N$. Then*

$$I_{\beta,p,\delta}(D) \le I_{\beta,p,\delta}(D^*),$$

where D^* is the ball having the same W_{N-1} quermassintegral of D, that is $W_{N-1}(D) = W_{N-1}(D^*)$.

Proof. Let v be the radial minimizer of $I_{\beta,p,\delta}(D^*)$ and let $\Omega_* = D^* + \delta B$. Since $\Omega = D + \delta B$ and Steiner formula for the perimeter (2.4.1) and Aleksandrov-Fenchel inequalities (2.4.5) imply $P(\Omega) \le P(\Omega^*)$.
Let us denote by $v_m = v(R+\delta) = \min_{\Omega^*} v$ and by $\max_{\Omega_*} v = v(R) = 1$. As v is radial, the modulus of the gradient of v is constant on the level lines of v.
Now we consider the function

$$g(t) = |Dv|_{v=t}, \qquad v_m < t \le 1$$

and

$$w(x) = G(R + d(x)),\ \ x\in\Omega, \qquad \text{where } \ G^{-1}(t) = R + \int_t^1 \frac{1}{g(s)}ds,$$

and $d(x)$ is the distance of a point x from D.
By construction $w \in W^{1,p}(\Omega)$ and being G decreasing one has:

$$\begin{aligned}
&\max_{\Omega} w = w|_{\partial D} = 1 = G(R),\\
&w_m = \min_{\Omega} w = w|_{\partial\Omega} = G(R+\delta) = v_m,\\
&|Dw|_{w=t} = |Dv|_{v=t} = g(t), \quad w_m \le t \le 1.
\end{aligned}$$

Then

$$I_{\beta,p,\delta}(D) \le \int_{\Omega\setminus D} |Dw|^p \, dx + \beta \int_{\partial\Omega} |w|^p \, d\mathcal{H}^{N-1}.$$

Let

$$E_t = \{x \in \Omega : w(x) > t\} = \{x \in \Omega : d(x) < G^{-1}(t)\} = D + G^{-1}(t)B$$

and

$$B_t = \{x \in \Omega_* : v(x) > t\}.$$

Being $W_{N-1}(D) = W_{N-1}(D^*)$, using the Steiner formula and (2.4.5), we get for $t \in]w_m, 1]$ and $\rho = G^{-1}(t)$ that

$$\begin{aligned}
P(E_t) = P(D+\rho B) &= n \sum_{N=0}^{N-1} \binom{N-1}{k} W_{k+1}(D)\rho^k \\
&\le \sum_{N=0}^{N-1} \binom{N-1}{k} W_{k+1}(D^*)\rho^k = P(D^*+\rho B) = P(B_t).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{w=t} |Dw| d\mathcal{H}^{N-1} = g(t)P(E_t) &\le g(t)P(B_t)\\
&= \int_{v=t} |Dv| d\mathcal{H}^{N-1}, \qquad w_m < t \le 1
\end{aligned}$$

then, using co-area formula

$$\begin{aligned}
\int_{\Omega\setminus D} |Dw|^p dx &= \int_{w_m}^1 dt \int_{w=t} |Dw|^{p-1} d\mathcal{H}^{N-1}\\
&= \int_{w_m}^1 [g(t)]^{p-1} P(E_t) dt \le \int_{w_m}^1 [g(t)]^{p-1} P(B_t) dt\\
&= \int_{\Omega\setminus D_*} |Dv|^p dx.
\end{aligned}$$

Since by construction $w = w_m = v_m$ on $\partial\Omega$ and $P(\Omega) \le P(\Omega_*)$, we have

$$\int_{\partial\Omega} |w|^p d\mathcal{H}^{N-1} = |w_m|^p P(\Omega) = |v_m|^p P(\Omega_*) = \int_{\partial\Omega_*} |v|^p d\mathcal{H}^{N-1}.$$

So,

$$\begin{aligned}
I_{\beta,p,\delta}(D) &\le \int_{\Omega\setminus D} |Dw|^p dx + \beta \int_{\partial\Omega} |w|^p d\mathcal{H}^{N-1}\\
&\le \int_{\Omega_*\setminus D^*} |Dv|^p dx + \beta \int_{\partial\Omega_*} |v|^p d\mathcal{H}^{N-1} = I_{\beta,p,\delta}(D^*).
\end{aligned}$$

□

6.4 Remarks

There is a counterintuitive behavior of the functional $I_{\beta,p,\delta}(D,\Omega)$ when Ω and D are concentric balls, which can be seen in the next two propositions.

Proposition 6.4.1. *Let B_R be the ball of radius $R>0$ centered at the origin. If $\beta \geq \left[\frac{N-1}{R(p-1)}\right]^{p-1}$, then $I_{\beta,p,\delta}(B_R)$ is decreasing in δ. When $\beta < \left[\frac{N-1}{R(p-1)}\right]^{p-1}$, then $I_{\beta,p,\delta}(B_R)$ is increasing for $\delta < \left(\frac{N-1}{p-1}\right)\frac{1}{\beta^{\frac{1}{p-1}}} - R$ and decreasing for $\delta > \left(\frac{N-1}{p-1}\right)\frac{1}{\beta^{\frac{1}{p-1}}} - R$.*

Proof. If $D = B_R$ is the ball of radius $R>0$ centered at the origin, then obviously $\Omega = B_R + \delta B = B_{R+\delta}$ and the minimum of (6.1.1), is the radial function

$$u(r) = \begin{cases} 1 - \gamma_1^{\frac{1}{p-1}}\left(\dfrac{p-1}{p-N}\right)R^{\frac{p-N}{p-1}}\left[\left(\dfrac{r}{R}\right)^{\frac{p-N}{p-1}} - 1\right] & p \neq N \\ 1 - \gamma_1^{\frac{1}{N-1}} \log\frac{r}{R} & p = N \end{cases} \tag{6.4.1}$$

for a suitable constant γ_1. This follows from the fact that

$$r^{N-1}\Delta_p u = \frac{d}{dr}(r^{N-1}|u'(r)|^{p-2}u'(r)) = 0,$$

but u is decreasing and positive, then

$$u'(r) = -\frac{\gamma_1^{\frac{1}{p-1}}}{r^{\frac{N-1}{p-1}}}, \qquad r \in [R, R+\delta].$$

If $p \neq N$, then integrating by parts and keeping in mind that $u(R) = 1$ it holds

$$u(r) = -\gamma_1^{\frac{1}{p-1}}\left(\frac{p-1}{p-N}\right)\left[r^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}\right] + 1.$$

If $p = N$,

$$u(r) = -\gamma_1^{\frac{1}{N-1}} \log\frac{r}{R} + 1.$$

Now, we can find γ_1 using the boundary conditions.
If $p \neq N$, then using Robin condition on $\partial\Omega$ it holds that

$$-(-u'(R+\delta))^{p-1} + \beta(u(R+\delta))^{p-1} = 0.$$

By using the explicit expression of u,

$$\gamma_1^{\frac{1}{p-1}}(R+\delta)^{-\frac{N-1}{p-1}} = \beta^{\frac{1}{p-1}}\left(-\gamma_1^{\frac{1}{p-1}}\left(\frac{p-1}{p-N}\right)R^{\frac{p-N}{p-1}}\left(1 - \left(\frac{R+\delta}{R}\right)^{\frac{p-N}{p-1}}\right) + 1\right)$$

and we have that

$$\gamma_1 = \frac{\beta}{\left[(R+\delta)^{-\frac{N-1}{p-1}} + \left(\dfrac{p-1}{p-n}\right)\beta^{\frac{1}{p-1}}R^{\frac{p-N}{p-1}}\left(\left(\dfrac{R+\delta}{R} - 1\right)^{\frac{p-N}{p-1}}\right)\right]^{p-1}}.$$

In particular,

$$\int_{\partial(\Omega\setminus D)} (|Du|^{p-2}Du \cdot \nu)\, d\mathcal{H}^{N-1} = 0$$

and using the divergence theorem

$$-\int_{\Omega\setminus D} |Du|^p + \int_{\partial(\Omega\setminus D)} u(|Du|^{p-2}Du \cdot \nu) = 0$$

$$-\int_{\Omega\setminus D} |Du|^p - \int_{\partial\Omega} \beta u^p + \int_{\partial D} u(|Du|^{p-2}Du \cdot \nu) = 0.$$

This implies that

$$I_{\beta,p,\delta}(B_R) = \int_{\partial B_R} |Du|^{p-2}\frac{\partial u}{\partial \nu}\, d\mathcal{H}^{N-1}.$$

Let us observe that

$$I_{\beta,p,\delta}(B_R) = N\omega_N\gamma_1$$

and we have

$$\partial_\delta[I_{\beta,p,\delta}(B_R)] < 0$$

if

$$\partial_\delta\Big[(R+\delta)^{\frac{1-n}{p-1}} + \beta^{\frac{1}{p-1}}\Big(\frac{p-1}{p-n}\Big)R^{\frac{p-n}{p-1}}\Big(\Big(\frac{R+\delta}{R}\Big)^{\frac{p-n}{p-1}} -1\Big)\Big] > 0$$

and this is true if

$$\delta > \Big(\frac{N-1}{p-1}\Big)\frac{1}{\beta^{\frac{1}{p-1}}} - R.$$

On the other hand, when $p = N$ we have

$$\gamma_1 = \frac{\beta}{\left[\frac{1}{(R+\delta)} + \beta^{\frac{1}{N-1}} \log \frac{R+\delta}{R}\right]^{N-1}}$$

and

$$I_{\beta,N,\delta}(B_R) = \frac{N\omega_N\beta}{\left[\frac{1}{(R+\delta)} + \beta^{\frac{1}{N-1}} \log \frac{R+\delta}{R}\right]^{N-1}}.$$

So

$$\partial_\delta[I_{\beta,N,\delta}(B_R)] < 0$$

if

$$\partial_\delta\Big[\frac{1}{(R+\delta)} + \beta^{\frac{1}{N-1}} \log\Big(1 + \frac{\delta}{R}\Big)\Big] > 0$$

and this is true if and only if

$$\delta > \frac{1}{\beta^{\frac{1}{N-1}}} - R,$$

and the proposition is proved. □

To show the next result, we first need the following Lemma, using the following notation. Let $D = B_R \subset \Omega$, and denote by

$$P = P(B_R) = n\omega_N R^{N-1},\ V = |B_R| \\ = \omega_N R^n,\ \Delta P = P(\Omega) - P(B_R),\ \Delta V = |\Omega| - |B_R|.$$

Lemma 6.4.2. *Let $D = B_R$ be the ball of radius $R > 0$ centered at the origin such that $D \subset \Omega$. For any $\delta_0 > 0$, there exists a constant*

$$C = \frac{n\omega_N R^{N-1}}{\delta_0}\left[\left(1 + \frac{\delta_0}{\omega_N R^n}\right)^{1-\frac{1}{N}} -1\right]$$

such that if $\Delta V \le \delta_0$ it holds that

$$\Delta P \ge C\Delta V.$$

We refer to [50] for the proof.
Now, we want to prove that, in the regime β "small", if the thickness δ is below a certain threshold value, $I_{\beta,p}(B_R,\Omega)$ is greater then $I_{\beta,p}(B_R,B_R)$.

Proposition 6.4.3. *Let $D = B_R$ be the ball of radius $R > 0$ centered at the origin and $\beta < \left[\dfrac{N-1}{R(p-1)}\right]^{p-1}$. Then there exists a positive constant δ_0 such that for any bounded domain Ω, with $D \subset \Omega$ and $|\Omega| - |D| < \delta_0$, then*

$$I_{\beta,p}(B_R,\Omega) > I_{\beta,p}(B_R,B_R).$$

Proof. Let u be the minimizer of $I_{\beta,p}(B_R,\Omega)$. Consider

$$\Sigma = \Omega \setminus B_R,\ \ \Gamma_m = \partial\Omega \setminus \partial B_R,\ \ \Gamma_t = \partial\{u > t\} \setminus \partial B_R,\ \ \Gamma_1 = \partial B_R \cap \Omega$$

and

$$p(t) = P(\{u > t\} \cap \Sigma), \qquad \text{for a.e. } t > 0.$$

Our aim is to prove that

$$I_{\beta,p}(B_R;B_R) = \beta P(B_R) < I_{\beta,p}(B_R;\Omega) = \int_\Omega |Du|^p dx + \beta \int_{\partial\Omega} |u|^p d\mathcal{H}^{N-1}$$

or equivalently

$$\mathcal{H}^{N-1}(\Gamma_1) < \frac{1}{\beta}\int_\Omega |Du|^p dx + \int_{\Gamma_0} |u|^p d\mathcal{H}^{N-1}.$$

Indeed by co-area formula and Fubini theorem, we have

$$\int_0^1 t^{p-1}p(t)dt = \int_0^1 t^{p-1}P(\{u>t\}\cap\Sigma)dt = \int_0^1 \left(\int_{\Gamma_1} t^{p-1}d\mathcal{H}^{N-1}\right)dt \\ + \int_0^1 \left(\int_{\Gamma_t\cap\Omega} t^{p-1}d\mathcal{H}^{N-1}\right)dt + \int_0^1 \left(\int_{\Gamma_t\cap\partial\Omega} t^{p-1}d\mathcal{H}^{N-1}\right)dt \tag{6.4.2}$$

$$= \frac{\mathcal{H}^{N-1}(\Gamma_1)}{p} + \frac{1}{p}\int_{\Gamma_0} u^p d\mathcal{H}^{N-1} + \int_\Omega u^{p-1}|Du|dx.$$

From the Lemma 6.4.2 we know that for $|\Sigma| < \delta_0$, with δ_0 fixed,

$$p(t) - 2\mathcal{H}^{N-1}(\Gamma_1) \geq C\mu(t)$$

and then

$$\begin{aligned}\int_0^1 t^{p-1}p(t)dt \geq & 2\int_0^1 t^{p-1}\mathcal{H}^{N-1}(\Gamma_1)dt + C\int_0^1 t^{p-1}\mu(t)dt \\ = & \frac{2}{p}\mathcal{H}^{N-1}(\Gamma_1) + \frac{C}{p}\int_\Omega u^p dx,\end{aligned}$$

where C is the constant of the Lemma 6.4.2. Combining this with (6.4.2), we have

$$\frac{2}{p}\mathcal{H}^{N-1}(\Gamma_1) + \frac{C}{p}\int_\Omega u^p dx \leq \frac{\mathcal{H}^{N-1}(\Gamma_1)}{p} + \frac{1}{p}\int_{\Gamma_0} u^p d\mathcal{H}^{N-1} + \int_\Omega u^{p-1}|Du|dx.$$

On the other hand, by the Young inequality

$$\int_\Omega u^{p-1}|Du|dx \leq \frac{p-1}{p\varepsilon^{\frac{1}{p-1}}}\int_\Omega u^p dx + \frac{\varepsilon}{p}\int_\Omega |Du|^p dx,$$

we get the following estimate

$$\mathcal{H}^{N-1}(\Gamma_1) + C\int_\Omega u^p dx \leq \int_{\Gamma_0} u^p d\mathcal{H}^{N-1} + \frac{p-1}{\varepsilon^{\frac{1}{p-1}}}\int_\Omega u^p dx + \varepsilon\int_\Omega |Du|^p dx =$$

$$= \int_{\Gamma_0} u^p d\mathcal{H}^{N-1} + \frac{p}{\varepsilon^{\frac{1}{p-1}}}\int_\Omega u^p dx - \frac{1}{\varepsilon^{\frac{1}{p-1}}}\int_\Omega u^p dx + \varepsilon\int_\Omega |Du|^p dx$$

and choosing $\varepsilon = \dfrac{1}{\beta}$ it holds that

$$\mathcal{H}^{N-1}(\Gamma_1) + [C - \beta^{\frac{1}{p-1}}(p-1)]\int_\Omega u^p dx \leq \int_{\Gamma_0} u^p d\mathcal{H}^{N-1} + \frac{1}{\beta}\int_\Omega |Du|^p dx.$$

Therefore, as $R < \dfrac{N-1}{(p-1)\beta^{\frac{1}{p-1}}}$, for δ_0 sufficiently small, the constant C is larger then $\beta^{\frac{1}{p-1}}(p-1)$ and thesis follows. $\square$

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