Università Degli Studi Di Napoli Federico II

Scuola Politecnica e delle Scienze di Base

Dipartimento di Matematica e Applicazioni
"Renato Caccioppoli"

Tesi di Dottorato in Matematica e Applicazioni
XXXVII CICLO

Uncountable groups with some embedding properties on large subgroups

Dottoranda:
Giulia Sabatino

Supervisor:
Prof.ssa Maria De Falco
Co-supervisor:
Prof.ssa Carmela Musella

Contents

Introduction

A theoretical group property is said to be a *finiteness condition* if it is satisfied by every finite group; for instance, the classical chain conditions on systems of subgroups are natural examples of finiteness conditions. Since the 1920s, a relevant part of investigation in infinite groups has been grounded in the observation that in many cases the imposition of finiteness conditions on an infinite group forces the group to be "close" to a finite group. This theory has been developed by many prominent mathematicians such as S.N. Černikov, R. Baer, K.A. Hirsch, A.G. Kuroš, O.J. Schmidt, H. Wielandt. In these years, the masterpiece *Finiteness conditions and generalized soluble groups* [51] by D.J.S. Robinson (1972), which collects all most important classical results about finiteness conditions, has been the book reference for many group theorists; thirty years later, J.C. Lennox and D.J.S. Robinson wrote the book *The theory of infinite soluble groups* [39], which represents a picture of the landscape after the work made on finiteness conditions in the years between the two books.

More recently, a new perspective has been adopted with an increasing number of group theorists focusing on groups that are far from being finite. This thesis work is framed within this research line; in particular, throughout this dissertation, we will try to describe the structure of groups which are *large* in terms of cardinality; such groups are really far from being finite and the subgroups we are considering will be similarly far from finiteness, even though in many cases the properties imposed on them will be finiteness conditions.

In literature there have been many different approaches to the study of *large* groups; the first classical approach goes in the direction to construct groups in which it is possible to embed given groups. For instance, P.

Hall in the famous paper [31] constructed a countable locally finite group U (called *Hall's universal group*), which is uniquely characterized by the properties that every finite group G admits a monomorphism to U and all such monomorphisms are conjugate by inner automorphisms of U. Hall's construction has been one of the starting motivations for the subsequent deep investigation on certain large groups, like, for instance,the so-called *existentially closed large groups* (see [36], [42], [43]).

A group G is said to have *finite* (*Prüfer*) *rank* if there is a positive integer r such that every finitely generated subgroup of G can be generated by at most r elements; if such an r does not exist, we say that the group G has infinite rank. It is not difficult to see that in some universes of (generalized) soluble groups, if a group G has infinite rank, then it must be rich in subgroups of infinite rank. Moreover, in recent years, a series of relevant papers has been published by many authors (including M.R. Dixon, J. Evans, L.A. Kurdachenko, N.N. Semko, H. Smith) which shows that the subgroups of infinite rank of a group of infinite rank have the power to influence the structure of the whole group and to force also the behaviour of the *small* subgroups of G, i.e., the subgroups of finite rank.

Recall here the definition of *embedding property*.

Definition. *For every group G, let $\mathcal{P}(G)$ be a set of subgroups of G containing the trivial subgroup; the elements of $\mathcal{P}(G)$ are called $\mathcal{P}$-subgroups of G. We shall say that $\mathcal{P}$ is an embedding subgroup property if*

$$(\mathcal{P}(G))^{\phi} = \mathcal{P}(G^{*}),$$

for every group isomorphism $\phi : G \longrightarrow G^{*}$ and X belongs to $\mathcal{P}(Y)$, whenever $X \leq Y \leq G$ and $X \in \mathcal{P}(G)$.

Furthermore, we will call the following properties *embedding properties of normal type* or *normality conditions* (because they are the natural generalizations of normality): normality, subnormality, Neumann's properties (which consider normality with the obstruction of a finite section), (m,k)-subnormality and permutability.

An embedding subgroup property $\mathcal{P}$ is called *absolute* if it holds for each subgroup X^{*} of G^{*} such that $X^{*} \simeq X$, where $X \in \mathcal{P}(G)$ for some group G. If $\mathfrak{X}$ is a class of groups, the property of a group of belonging to $\mathfrak{X}$ is an embedding absolute property. On the other hand, normality conditions are nonabsolute embedding properties.

It has been proved that, for some choices of group theoretical properties $\mathcal{P}$, if G is a group of infinite rank in which all subgroups of infinite rank satisfy the property $\mathcal{P}$, then the same happens also to the subgroups of finite rank (see, for instance, [7], where a full reference list on this subject can be found). In particular, M.J. Evans and Y. Kim [27] proved that if G is a locally soluble group of infinite rank in which all subgroups of infinite rank are normal, then G is a Dedekind group, i.e. all subgroups of G are normal. Actually, Evans and Kim considered the more general situation of groups of infinite rank in which all subgroups of infinite rank are subnormal of bounded defect. In addition, M. De Falco, F. de Giovanni, C. Musella et al. proved that all embedding properties of normal type pass from subgroups of infinite rank to those of finite rank.

- Results of this type have suggested that the behaviour of *small* subgroups in a *large* group is negligible, at least for a right choice of the definition of largeness and within a suitable universe.

On the other hand, it is well-known that proper subgroups of countable cardinality play a key-role in determining the structure of an infinite group. Indeed, recall here that a class of groups $\mathfrak{X}$ is said to be *countably recognizable* if G is an $\mathfrak{X}$-group whenever each countable subgroup belongs to $\mathfrak{X}$. Countable recognizability was introduced by R. Baer [1] who proved that a group G is nilpotent when every countable proper subgroup of G is nilpotent. Many authors worked on this topic finding out that many group classes have this property (see, for instance, [20]). Even normality conditions can be countably detected in some cases: for instance, F. de Giovanni and M. Trombetti proved that if G is a group the pronormality of a subgroup X of G can be detected from the pronormality in G of the countable subgroups of X (see Theorem 2.3 of [22]). The same authors [[21], Corollary 2.5] proved that when in a group G every countable subgroup is closed with respect to the profinite topology, then every subgroup of G is closed.

- The role of countable subgroups in determining the structure of the whole group has suggested that subgroups of a fixed cardinality $\aleph$ might influence the behaviour of the group, at least for certain cardinals $\aleph$ with suitable properties.

Thus, a new point of view has been adopted in many recent papers where uncountable groups whose proper subgroups with the same cardinality as the whole group satisfy certain notable embedding properties. Specifically, in this research line, uncountable groups G of cardinality $\aleph$ are considered and any subgroup X of G is called *large* if it has likewise cardinality $\aleph$, and *small* otherwise.

This type of investigation, that goes from *large* to *small*, represents a *dual* approach with respect to countable recognizability, which detects information about *large* subgroups (and the whole group) starting from the *small* ones. Furthermore, it is a natural consequence of the positive results obtained for groups of infinite rank.

Papers as [18] and [19] consider uncountable groups where all large subgroups belong to a given group class while many subsequent papers focused on uncountable groups with large subgroups satisfying an embedding property of normal type (see [8], [9], [10], [11], [12], [54], [13], [14]). In this work, we will mainly focus on uncountable groups whose large subgroups verify a nonabsolute embedding property.

The results we are going to show are concerned with groups of *regular* cardinality.

Definition. *Let κ be an ordinal. A sequence of ordinals $\langle \lambda_\xi < \kappa \,|\, \xi < \beta \rangle$ is cofinal to κ if for every $\alpha < \kappa$ there is a $\xi < \beta$ such that $\lambda_\xi \geq \alpha$. The cofinality of κ, denoted by $cf(\kappa)$, is the least β for which such a sequence exists.*

Definition. *A cardinal κ is called regular when $cf(\kappa) = \kappa$; otherwise it is called singular.*

When the Axiom of Choice (AC) is assumed to be true, the above definition is equivalent to the following.

Definition. *A cardinal number κ is said to be regular if, whenever the set $S = \bigcup\limits_{i \in I} S_i$ is such that $|I| < \kappa$ and $|S_i| < \kappa$ for each $i \in I$, it follows that $|S| < \kappa$.*

Throughout this work, we will always assume AC and we will use the latter definition. Among regular cardinals, clearly there is $\aleph_0$ as the union of finitely many finite sets is still a finite set. However, in our dissertation, we will consider groups of uncountable cardinality.

The main obstacle in the study of groups of large cardinality is the existence of uncountable groups, of cardinality $\aleph$, say, in which all proper subgroups have cardinality strictly smaller than $\aleph$ (the so-called *Jónsson groups*). Considerations about the existence of Jónsson groups and how to avoid them can be found in Chapter 1 (Sec. 1.2).

As a first step, in Chapter 1 (Sec. 1.1) we give a focus on uncountable abelian groups of regular cardinality, as these groups possess certain peculiar properties that are particularly crucial for studying the behaviour of uncountable groups where all large subgroups satisfy an embedding property. Furthermore, although the Prüfer rank of an abelian group is less or equal to its cardinality, in the uncountable case they coincide. In general, we will observe that most normality conditions in uncountable groups containing a large abelian subgroup pass from the large subgroups to the small ones, as it happens for generalized soluble groups of infinite rank where the presence of an abelian subgroup of infinite rank is guaranteed. Anyway, there exist uncountable nilpotent groups in which all abelian subgroups are countable. In particular, an example of A. Ehrenfeucht and V. Faber [26] shows that a result similar to the above quoted theorem of Evans and Kim does not hold for groups in which all uncountable subgroups are normal; therefore, the presence of an abelian large subgroup is necessary to be a Dedekind group.

Clearly, among groups whose large subgroups are normal, there are the groups in which all large subgroups have the same commutator subgroup. These latter groups, which will be called here $C^1_\aleph$–groups when they have cardinality $\aleph$, will naturally occur in the description of groups in which all large subgroups are normal and also will play an important role in the proofs regarding uncountable groups without large abelian subgroup and whose large subgroups satisfy a condition generalizing normality. Note that, while the commutator subgroup of a group in which all subgroups are normal must have order at most 2, the commutator subgroup of a $C^1_\aleph$-

group can have any cardinality less than $\aleph$; in fact, Theorem 3 of [26] shows that for every cardinal $\alpha \geq \aleph_\beta$ (with $2^{\aleph_\beta} = \aleph_{\beta+1}$) there exists a nilpotent $C^1_{\aleph_{\beta+1}}$-group of class 2 whose commutator subgroup has cardinality α and either has prime exponent or is a torsion-free divisible group.

As the reader will observe, $C^1_\aleph$-groups play an analogous role with respect to abelian groups when the focus is only on the properties of large subgroups of a group of cardinality $\aleph$.

The widest part of the thesis discusses normality condition for large subgroups in uncountable groups and it consists of Chapter 2.

Definition. *A subgroup H of a group G is said to be subnormal of defect a non-negative integer k if there exists a finite series of G containing both H and G and the minimum length for such a series is k.*

A famous theorem by J.E. Roseblade [53] asserts that a group G where all subgroups are subnormal of bounded defect k is nilpotent with class of nilpotency bounded by a function of k. The above mentioned theorem by Evans and Kim ensures that the thesis of Roseblade's statement holds in a locally soluble group of infinite rank where all subgroups of infinite rank are subnormal of bounded defect. In Sec. 2.1, uncountable groups having large subgroups that are subnormal of bounded defect are analyzed. Here, an analogous version of the result by Evans and Kim is proved, provided that the group contains a large abelian subgroup. Moreover, a structure theorem for uncountable groups with normal large subgroups is shown and the commutator subgroup of a $C^1_\aleph$-group is described. Most of these results are in [8], [12] and [10].

Definition. *A subgroup H of a group G is said to be nearly normal in G if it has finite index in its normal closure H^G.*

In a famous paper of 1955, B.H. Neumann [46] proved that all subgroups of a group G are nearly normal if and only if the commutator subgroup G' of G is finite. I.D. Macdonald [41] proved a similar result with bounds on the index of any subgroup in its normal closure and stated that in such a case the order of the commutator subgroup is likewise bounded.

M. De Falco, F. de Giovanni and C. Musella [6] showed that in an infinite rank generalized radical group it suffices that any subgroup of infinite rank is nearly normal to have a finite commutator subgroup. Sec. 2.2 is devoted to the study of uncountable group whose large subgroups are nearly normal: the investigation of these groups was started in Sec. 3 of [12] where it was proved that an uncountable groups where all large subgroups are nearly normal has finite commutator subgroup if it contains a large FC-subgroup; the results obtained in [13] improved the knowledge of the structure of such groups even when there are no large FC-subgroups, especially when a bound is considered for the index of every large subgroup in its normal closure.

In [62] and in [28] some theorems of Neumann's type are proved considering a generalization of the property of being nearly normal; in particular, in [62] it has been proved that if α is an infinite cardinal and G is an FC-group, then every subgroup H of G has index strictly smaller than α in its normal closure if and only if $|G'| < \alpha$. In Sec. 2.2, even groups of regular cardinality $\aleph$ in which every large subgroup has index strictly smaller than a fixed infinite cardinal α in its normal closure are considered and the above result has been improved.

Definition. *A subgroup H of a group G is said to be almost normal if it has finitely many conjugates in G.*

In [46], Neumann proved that a group G is finite over its centre if and only if every subgroup of G has finitely many conjugates; even in this case Macdonald [41] provided a version with bounds of Neumann's theorem. Moreover in [49], Y. D. Polovickǐ proved that groups with this property are precisely those with finitely many normalizers of (abelian) subgroups, and by a celebrated theorem of Schur [57], they are also finite-by-abelian. In a group of infinite rank, in the universe of generalized radical groups, when every subgroup of infinite rank is almost normal, the centre has finite index (see [6]). In Sec. 2.3 some kind of translations of these results are presented by restricting the attention only to the set of large subgroups of an uncountable group. Most of the theorems of this section are available in [12] and [14], where a bound is imposed on the number of conjugates of any large subgroup.

It is a matter of fact that groups with finite commutator subgroup have finitely many commutators of subgroups. In [17], de Giovanni and Robinson proved that the converse is also true when the group is locally graded. In the same paper, they considered also infinite (locally graded) groups with finitely many commutator subgroups of infinite subgroups. Sec. 2.4 is devoted to uncountable groups with the same property imposed only on large subgroups and it is based on results from [11] (where groups with a large abelian subgroup are considered) and [14]. Here, we state and prove a version of Schur's theorem for large subgroups.

Clearly, *nearly normality*, *almost normality*, the property of a group to have finitely many normalizers, or finitely many commutator subgroups are all examples of *finiteness conditions*. The results of the thesis involving such embedding properties illustrate how uncountable groups can exhibit behaviour similar to that of finite groups when these conditions are imposed on large subgroups.

Definition. *Let m, k be nonnegative integers. A subgroup H of a group G is said to be (m, k)-subnormal in G if there exists a subgroup K containing H such that $|K : H| \leq m$ and K is subnormal in G with defect at most k.*

A subgroup H is $(m, 1)$-subnormal in G if and only if it has index at most m in its normal closure H^G, while H is $(1, k)$-subnormal if and only if it is subnormal of defect at most k. As we said above, groups where all subgroups satisfy one of these conditions have been already considered by Neumann [46], Macdonald [41] and Roseblade [53]. However, Lennox [37] generalized Roseblade's theorem, proving that there exists an integer function μ such that if every finitely generated subgroup of a group G is (m, k)-subnormal in G, then the term $\gamma_{\mu(m+k)}(G)$ of the lower central series of G is finite of order less than or equal to $m!$. On the other hand, in [24] it is proved that locally soluble groups G of infinite rank whose subgroups of infinite rank are (m, k)-subnormal are finite-by-nilpotent and the authors showed that the same bound from Lennox's theorem holds for the cardinality of $\gamma_{\mu(m+k)}(G)$.

Sec. 2.5 deals with uncountable groups with a large abelian subgroup and where all large subgroups are (m,k)-subnormal, for m,k fixed non-negative integers and the reference paper is [54].

Definition. *A subgroup H of a group G is said to be permutably complemented if there exists a subgroup K of G such that $G = HK$ and $H \cap K = \{1\}$.*

A group G is named a *C-group* if all its subgroups are permutably complemented. The structure of C-groups has been completely described in [56] and there are a lot of papers dealing with groups with some permutably complemented subgroups.

In particular, in [29] locally soluble groups of infinite rank in which every infinite rank subgroup is permutably complemented are studied. Chapter 3 contains a translation for uncountable groups of the result obtained for groups of infinite rank and gives a further contribution to the topic of groups in which some subgroups are permutably complemented. Most of the results of this section are in [55].

It is worth noticing that many of the results about uncountable groups are not true in the countable setting, since large subgroups of a countable group are precisely all the *infinite* subgroups. A significant body of literature exists on infinite groups where all infinite (proper) subgroups fulfill a specific property. For more in-depth information about this subject, we refer the reader to [4], [34], [16].

Another way to go from finite to infinite groups is doing inverse limits of finite groups.

Profinite groups are inverse limits of finite groups and are characterized as topological groups that are compact, Hausdorff and totally disconnected. Clearly, every finite group is profinite while every infinite profinite group G is uncountable, specifically $|G| = 2^{w_0(G)}$ where $w_0(G)$ is the cardinal of any fundamental system of neighborhoods of 1 consisting of open subgroups [[63], Theorem 4.9]. For this reason, the exploration of infinite profinite groups appears to be the natural continuation of the study of abstract uncountable groups. Nevertheless, in a profinite group a subgroup is *open* if and only if it is *closed* of finite index. It follows that open subgroups are *large* in some sense since, in the infinite case, they share the cardinality with the group. The intersection of all open (normal) subgroups of such a group is trivial, so that profinite groups are residually finite. Here, an uncountable group G of cardinality $\aleph$ will be called *residually small* if the intersection of all normal subgroups of index smaller than $\aleph$ is trivial. In particular, residually finite uncountable groups are also residually small.

As the reader can observe in Chapter 2, in residually small uncountable groups with large subgroups satisfying a normality condition, the behaviour of small subgroups can often be neglected. This suggests that the investigation of profinite groups where all open subgroups verify a specific embedding theoretical property could offer a new framework for further research and Chapter 4 outlines some initial steps in this direction.

Indeed, Sec. 4.2 is concerned with profinite groups where all open subgroups satisfy one of the previously defined normality conditions and numerous results from this line of research have proven to be encouraging.

In Sec. 4.3 the structure of *profinite C-groups*, i.e. profinite groups in which all closed subgroups are permutably complemented, is examined. It is shown that these groups are precisely the profinite groups where all open subgroups have a permutable complement, and they are inverse limits of finite C-groups. This latter result has enabled us to prove that the structure of profinite C-groups is really close to that of the C-groups described in [56], although they are two different group classes.

A *profinite SC-group* is a profinite group G where each closed subgroup H of G admits a closed sectional complement in G, that is a closed complement K to H such that if $H < X < G$ and X is closed in G, then $X \cap K$ is a complement to H in X.

In the same section, we will see that the class of profinite SC-groups coincides with that of profinite C-groups.

Most of our notation is standard and can be found in [51] for abstract groups, in general, while concepts related to subgroup lattices, such as permutable complementation, are available in [56]. For profinite groups, we refer to [50] and [65]. However, we point out that:

(1) "*cartesian product*" means "*unrestricted direct product*";

(2) "*direct product*" means "*restricted direct product*";

(3) "$H \leq_o (\trianglelefteq_o)\, G$" means "*$H$ is an open (normal) subgroup of G*";

(4) "$H \leq_c (\trianglelefteq_c)\, G$" means "*$H$ is a closed (normal) subgroup of G*".

Chapter 4 was written under the supervision of G. A. Fernandez Alcober during my stay at the University of the Basque Country (UPV/EHU). I would like to express my gratitude to him and to the University for their hospitality, as well as for the valuable discussions and meetings we had on profinite groups.

Chapter 1

Preliminaries about Large Subgroups

1.1 Uncountable abelian groups

The class of abelian groups plays a crucial role in the study of groups of uncountable regular cardinality, due to certain distinctive aspects that warrant particular attention.

Lemma 1.1.1. *Let G be an infinite group of regular cardinality $\aleph$, and let I be a set of cardinality strictly smaller than $\aleph$. If G is the direct product of a collection $(G_i)_{i \in I}$ of subgroups, then there exists $i \in I$ such that G_i has cardinality $\aleph$.*

Proof. Let $\leq$ be a well-ordering in the set I, and for each $i \in I$ put

$$G_i^* = \underset{j \leq i}{\mathrm{Dr}} G_j.$$

Then $(G_i^*)_{i \in I}$ is a chain of subgroups of G, and

$$G = \bigcup_{i \in I} G_i^*.$$

As the cardinal $\aleph$ is regular, at least one of the G_i^*'s has cardinality $\aleph$, and we may consider the smallest element k of I such that G_k^* has cardinality

$\aleph$. Then

$$\bigcup_{h<k} G_h^*$$

has cardinality strictly smaller than $\aleph$, and hence k is the successor of an element k' of I. It follows that $G_k^* = G_{k'}^* \times G_k$, and so G_k has cardinality $\aleph$. $\square$

Lemma 1.1.2. *Let A be an uncountable abelian group of regular cardinality $\aleph$. Then A contains a subgroup of the form*

$$B = \underset{i \in I}{\mathrm{Dr}} B_i,$$

where the set I has cardinality $\aleph$ and either all B_i are infinite cyclic or they all have the same prime order.

Proof. Suppose first that the subgroup T of G consisting of all elements of finite order has cardinality $\aleph$. In this case, it follows from Lemma 1.1.1 that there exists a prime number p such that also the p-component G_p of G has cardinality $\aleph$. Then the subgroup $G_p[p^k]$ has cardinality $\aleph$ for some positive integer k, and so the socle of G_p is the direct product of a collection of cardinality $\aleph$ consisting of subgroups of order p. Assume now that T has cardinality strictly smaller than $\aleph$, so that A/T has cardinality $\aleph$. Let U/T be a free abelian subgroup of A/T such that A/U is periodic. Then it is easy to prove that U/T has cardinality $\aleph$, so that it is the direct product of a collection of cardinality $\aleph$ consisting of infinite cyclic subgroups. On the other hand, there exists a subgroup B of U such that $U = B \times T$, so that $B \simeq U/T$ and the statement is proved. $\square$

Theorem 1.1.1. *Let $\aleph$ be a regular uncountable cardinal number, and let R be a principal ideal domain of cardinality strictly smaller than $\aleph$. If M is an R-module of cardinality $\aleph$, then M contains an R-submodule which is the direct sum of a collection of cardinality $\aleph$ of nontrivial R-submodules.*

Proof. The injective hull $E_R(M)$ of M can be decomposed into a direct sum

$$E_R(M) = \bigoplus_{i \in I} E_i,$$

where each E_i is an injective indecomposable R-module. In particular, each E_i has cardinality strictly smaller than $\aleph$, since it is a homomorphic image of the fraction field of R, and hence the set I has cardinality $\aleph$. On the other hand, M is an essential submodule of $E_R(M)$, so that $M \cap E_i \neq \{0\}$ for all $i \in I$. Therefore the direct sum

$$\bigoplus_{i \in I} (M \cap E_i)$$

is a submodule of M satisfying the condition of the statement. $\square$

Corollary 1.1.1. *Let $\aleph$ be a regular uncountable cardinal number, and let R be a principal ideal domain of cardinality strictly smaller than $\aleph$. If M is an R-module of cardinality $\aleph$ and N is a small R-submodule of M, there exists a large R-sumbodule L of M intersecting N trivially.*

Proof. By Theorem 1.1.1, there exists an R-submodule

$$B = \bigoplus_{i \in I} B_i$$

of M where $|I| = \aleph$ and each B_i is a nontrivial R-submodule of M. It is clear that, for each $x \in N \cap B$, we can find a finite subset I_x of I such that

$$x \in \bigoplus_{i \in I_x} B_i$$

and, since N is small, the set $J = \bigcup_{x \in N} I_x$ has cardinality strictly smaller than $\aleph$. Furthermore,

$$N \cap \bigoplus_{i \in I \setminus J} B_i = \{0\}.$$

Hence, we can take

$$L = \bigoplus_{i \in I \setminus J} B_i.$$

$\square$

Theorem 1.1.1 can, of course, be specialized by choosing as R the ring $\mathbb{Z}$ of integers. In this case, we obtain the following statement, which shows in particular that any uncountable abelian group A of cardinality $\aleph$ contains a subgroup which can be decomposed into the direct product of two subgroups of cardinality $\aleph$.

Corollary 1.1.2. *Let A be an uncountable abelian group of regular cardinality $\aleph$. Then A contains a subgroup which is the direct product of a collection of cardinality $\aleph$ of nontrivial subgroups.*

1.2 Jónsson groups

A group G is called a Jónsson group (after Bjarni Jónsson) if every proper subgroup of G has strictly smaller cardinality than G itself.

All finite groups are Jónsson and, among countably infinite examples, we find the Prüfer groups $\mathbb{Z}(p^\infty)$ and the Tarski groups constructed by Ol'šanskiĭ. Shelah, in his work [60], provided a construction of a Jónsson group of cardinality $\aleph_1$, without relying on the continuum hypothesis, thereby providing a negative answer to a question posed by Kuroš and Černikov in their paper [35]. Jónsson groups of cardinality $\aleph_1$ were also constructed by Obraztsov in [47]. Moreover, it has been shown that there exist Jónsson groups of cardinality $\aleph_n$, for each natural number n (see [44], Theorem E).

It turns out that Jónsson groups represent the main obstacle in the study of uncountable groups whose large subgroups satisfy an embedding property. We list here a number of conditions which are enough in order to avoid Jónsson groups. In fact, it is known that if $\aleph$ is any uncountable regular cardinal, then a periodic group of cardinality $\aleph$ cannot be a Jónsson group, provided that it is either locally finite or a 2-group (see [33], Theorem 2.6), and a similar result was proved by S.P. Strunkov [61] for groups in which every 2-generator subgroup is finite. Actually, groups satisfying some weak solubility condition cannot be Jónsson groups, since it was remarked by A. Macintyre that every Jónsson group is simple over the centre; we present here the easy proof of this interesting result.

Proposition 1.2.1. *Let G be a Jónsson group of cardinality $\aleph$. Then G is perfect, and $G/Z(G)$ is a simple group of cardinality $\aleph$.*

Proof. Let x be any non-central element of G. Then the centralizer $C_G(x)$ is a proper subgroup of G, and so its cardinality is strictly smaller than $\aleph$. It follows that $|G : C_G(x)| = \aleph$, and hence the conjugacy class of x in G has cardinality $\aleph$. Therefore, also the normal closure of $\langle x \rangle$ in G has cardinality $\aleph$, and so $\langle x \rangle^G = G$. In particular, the factor group $G/Z(G)$ is simple, and of course it has cardinality $\aleph$. Obviously, G' is not contained in $Z(G)$, and if x is an element of $G' \setminus Z(G)$, it follows that $G = \langle x \rangle^G = G'$ is perfect. $\square$

Chapter 2

Normality conditions on Large Subgroups

2.1 Normality and Subnormality

A celebrated theorem of J.E. Roseblade states that if k is a positive integer and G is a group in which all subgroups are subnormal of defect at most k, then G is nilpotent; moreover, there exists a function f such that the nilpotency class of G is bounded by $f(k)$ (see [51] Part 2, p.71). In [27], Evans and Kim proved that in a locally soluble group of infinite rank, when all subgroups of infinite rank are subnormal of defect at most k, also the finite rank subgroups satisfy the same property, so that the group is nilpotent of class at most $f(k)$. As expected, Roseblade's function plays a role in our considerations.

Lemma 2.1.1. *Let k be a positive integer and let G be an uncountable group of regular cardinality $\aleph$ in which all large subgroups are subnormal of defect at most k. If G is abelian-by-cyclic, then it is nilpotent of class at most $f(k)$.*

Proof. We can write $G = A\langle x\rangle$, where A is a normal abelian subgroup of G and x is an element of G. Of course, A is large and it can be regarded as a module over the countable ring $R = \mathbb{Z}\langle x\rangle$. Moreover, the R-submodules of A are all its G-invariant subgroups. By Theorem 1.1.1, A contains the direct product

$$\underset{i\in I}{\mathrm{Dr}} M_i,$$

where I is a set of cardinality $\aleph$ and, for each $i \in I$, M_i is a G-invariant subgroup of A. It follows that there exist two subsets I_1 and I_2 of I, each of cardinality $\aleph$, such that

$$I_1 \cap I_2 = \emptyset.$$

For $j = 1, 2$, put

$$B_j = \underset{i\in I_j}{\mathrm{Dr}} M_i,$$

then B_1 and B_2 are G-invariant subgroups of A of cardinality $\aleph$ whose intersection is trivial.

Thus, Roseblade's theorem [53] ensures that $\gamma_{f(k)+1}(G/B_j)$ is also trivial, for $j = 1, 2$. Therefore,

$$\gamma_{f(k)+1}(G) \leq B_1 \cap B_2 = \{1\}$$

and G is nilpotent of class at most $f(k)$. $\square$

Theorem 2.1.1. *Let k be a positive integer, and let G be an uncountable group of regular cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ are subnormal of defect at most k.*

(a) If G contains a large abelian normal subgroup, then all subgroups of G are subnormal of defect at most k, and so G is nilpotent of class at most $f(k)$.

(b) If G contains a large abelian subgroup, then G is nilpotent and the subgroup $\gamma_{f(k)+1}(G)$ is small.

Proof.

(a) Let A be an abelian normal subgroup of G of cardinality $\aleph$. If x is any element of G, the subnormal subgroup $\langle A, x\rangle$ is nilpotent by Lemma 2.1.1, and hence G is a Baer group. Let $E = \langle x_1, \ldots, x_t\rangle$ be any finitely generated subgroup of G. Then E is subnormal and nilpotent, so that also AE is nilpotent, and so there exists a nonnegative integer m such that

$$B = [A, \underbrace{E, \ldots, E}_{\leftarrow m \rightarrow}]$$

has cardinality $\aleph$ and $C = [B, E]$ has cardinality strictly smaller than $\aleph$. For each positive integer $i \leq t$, let

$$\theta_i : B \longrightarrow C$$

be the homomorphism defined by putting $b^{\theta_i} = [b, x_i]$ for all $b \in B$. Then every $B/ker\theta_i$ has cardinality strictly smaller than $\aleph$, and so the subgroup

$$U = \bigcap_{i=1}^{t} ker\theta_i$$

has cardinality $\aleph$. Clearly, U is contained in the centre of AE, and by Corollary 1.1.1 and Corollary 1.1.2 it contains two subgroups U_1 and U_2 of cardinality $\aleph$ such that

$$U_1 \cap U_2 = E \cap \langle U_1, U_2 \rangle = \{1\}.$$

It follows that the subnormal subgroup

$$EU_1 \cap EU_2$$

has defect at most k in G. Therefore, all finitely generated subgroups of G are subnormal of defect at most k, and so even every subgroup of G is subnormal of defect at most k (see [51] Part 2, Lemma 7.41).

(b) Suppose now that A is an abelian subgroup of G of cardinality $\aleph$, but that A is not necessarily normal. However, by assumption A is at least subnormal in G, and we will first prove that G is nilpotent, by using induction on the defect m of A in G. This is certainly true if $m = 1$ by part (a). If $m > 1$, the subgroup A has defect $m - 1$ in its normal closure A^G, and so it can be assumed that A^G is nilpotent. It follows that the abelian group $A^G/(A^G)'$ has cardinality $\aleph$, and so the factor group $G/(A^G)'$ is nilpotent by (a). Therefore, G itself is nilpotent. As G is nilpotent, there exists a positive integer s such that $\gamma_s(G)$ has cardinality $\aleph$ and $\gamma_{s+1}(G)$ has cardinality strictly smaller than $\aleph$. Then the factor group $G/\gamma_{s+1}(G)$ contains the abelian normal subgroup $\gamma_s(G)/\gamma_{s+1}(G)$, which has cardinality $\aleph$, and so it follows from part (a) that $G/\gamma_{s+1}(G)$ is nilpotent of class at most $f(k)$. Therefore, $\gamma_{f(k)+1}(G)$ is contained in $\gamma_{s+1}(G)$, and hence it has cardinality strictly smaller than $\aleph$.

Recall here that a *Dedekind group* is a group whose subgroups are all normal.

Corollary 2.1.1. *Let G be an uncountable group of regular cardinality $\aleph$ in which all large subgroups are normal. If G contains a large abelian subgroup, then G is a Dedekind group.*

The structure of a Dedekind group has been fully described by R. Bear who proved that such groups are either abelian or a direct product of the form $Q_8 \times B \times D$, where Q_8 is a quaternion group of order 8, B is an elementary abelian 2-group, and D is a torsion abelian group with all elements of odd order. In particular, these groups have finite commutator subgroup of order at most 2.

Corollary 2.1.2. *Let G be an uncountable group of cardinality $\aleph$ in which all large subgroups are normal, and let X be a large subgroup of G. Then G/X' is a Dedekind group. Moreover, if the factor group G/X' is not abelian, then $Y' = X'$ for each large subgroup Y of X.*

Proof. Of course, X' is a normal subgroup of G. Suppose first that also the commutator subgroup X' of X is large; then all subgroups of G containing X' are normal, and so G/X' is obviously a Dedekind group. On the other hand, if the subgroup X' is small, we have that X/X' is an abelian large subgroup of G/X', and so G/X' is a Dedekind group by Corollary 2.1.1. Suppose that G/X' is not abelian, and let Y be any large subgroup of X. Then Y' is normal in G, and G/Y' is a Dedekind nonabelian group by the first part of the statement, so that $G' = Y'$ has order 2. As $Y' \leq X' < G'$, it follows that $Y' = X'$. $\square$

Lemma 2.1.2. *Let G be an uncountable group of regular cardinality $\aleph$ in which all large subgroups are normal. If the intersection of all large subgroups of G is trivial, then G is a Dedekind group.*

Proof. Let $\mathcal{U}$ be the set of all large subgroups of G. Obviously, G/U is a Dedekind group for each $U \in \mathcal{U}$, and we may suppose that there exists an element X of $\mathcal{U}$ such that G/X is not abelian. If U is any element of $\mathcal{U}$, the factor group $G/(X \cap U)$ can be embedded into the direct product $G/X \times G/U$ and so it contains an abelian subgroup of finite index. Thus, $G/(X \cap U)$ is a Dedekind group by Corollary 2.1.1, independently on the cardinality of $X \cap U$. It follows that $G' \cap X = G' \cap X \cap U$, and hence,

$$G' \cap X \leq \bigcap_{U \in \mathcal{U}} U = \{1\}.$$

Therefore, X is abelian, and a further application of Corollary 2.1.1 yields that G is a Dedekind group. □

Lemma 2.1.3. *Let G be an uncountable group of regular cardinality $\aleph$ in which all large subgroups are normal. If the commutator subgroup G' of G is large, then G'' is a Jónsson group of cardinality $\aleph$. In particular, if G has no Jónsson normal large subgroups, then G' is small.*

Proof. Assume that the abelian group G'/G'' has cardinality $\aleph$, so that there exists a subgroup X of G such that $G'' < X < G'$ and both the groups X/G'' and G'/X have cardinality $\aleph$ (see Corollary 1.1.2). Then X is normal in G and G/X is a Dedekind group, which is impossible as its commutator subgroup is infinite. It follows that G'/G'' has cardinality strictly smaller than $\aleph$, and hence G'' is large. On the other hand, G'' is obviously contained in every large subgroup of G, so that all its proper subgroups have cardinality strictly smaller than $\aleph$. Therefore, G'' is a Jónsson group. □

Theorem 2.1.2. *Let G be an uncountable group of regular cardinality $\aleph$ in which all large subgroups are normal. If G has no Jónsson normal subgroups, then G is nilpotent of class at most* 2.

Proof. Let N be the intersection of all large subgroups of G, so that it follows from Lemma 2.1.2 that G/N is a Dedekind group. Let g be an arbitrary element of G. Since by Lemma 2.1.3 the commutator subgroup G' of G is small, also the conjugacy class of g has cardinality strictly smaller than $\aleph$. It follows that the index in G of the centralizer $C_G(g)$ is likewise small, so that $C_G(g)$ has cardinality $\aleph$ and hence $N \leq C_G(g)$. Thus, N is contained in the centre of G, whence $G/Z(G)$ is a Dedekind group. It follows that $G/Z(G)$ is even abelian (see, for instance, [59]), and so G has nilpotency class at most 2. □

Groups whose large subgroups have all the same commutator subgroup, which in [13] have been called $C^1_\aleph$-groups when they have cardinality $\aleph$, were constructed in [26]. These groups are involved in the description of groups in which all large subgroups are normal, as the reader will be able to observe in the main theorem of this section.

For the proof of the following lemma we refer to [8] (see Theorem 3.5).

Lemma 2.1.4. *Let G be an uncountable locally finite group of regular cardinality $\aleph$ in which all large subgroups are normal. Then either G is a Dedekind group or there is a prime number p such that $G = H \times K$, where H is a large nilpotent p-subgroup and K is a small Dedekind p'-group. Moreover:*

- *(α) If $p > 2$, then the subgroup $\Omega_1(H)$ has exponent p and cardinality $\aleph$, while the groups H^p and $H/\Omega_1(H)$ have cardinality strictly smaller than $\aleph$.*
- *(β) If $p = 2$, then either G has a homomorphic image isomorphic to Q_8 and $H^2 = H'$ or G has no homomorphic images isomorphic to Q_8 and $(G')^2 = \{1\}$.*

The following result is the main theorem of this section.

Theorem 2.1.3. *Let G be an uncountable group of regular cardinality $\aleph$ which has no Jónsson large subgroups. Then all large subgroups of G are normal if and only if one of the following conditions holds:*

- *(a) G is a Dedekind group;*
- *(b) G is a $C^1_\aleph$-group;*
- *(c) $G = P \times Q$, where P is a nilpotent p-group of cardinality $\aleph$ for some prime p, Q is a periodic Dedekind p'-group, and*
    - *(i) if $p > 2$, P is a $C^1_\aleph$-group and Q is nonabelian,*

(ii) if $p = 2$ and P contains a nonabelian $C^1_{\aleph}$-subgroup N of finite index and such that P/N' is a Dedekind nonabelian group.

Proof. Assume that all large subgroups of G are normal, so that G is nilpotent by Theorem 2.1.2. Suppose that G neither is a Dedekind group nor satisfies condition (b), and let Y be a large subgroup of G such that $Y' \neq G'$. It follows from Corollary 2.1.2 that G/Y' is a Dedekind nonabelian group and so there exists a normal subgroup K of G such that G/K is isomorphic to the quaternion group of order 8. Since G/Y' is periodic and G is nilpotent, we have that G itself is periodic. Then it follows from Lemma 2.1.4 that there exists a prime number p such that $G = P \times Q$, where P is a p-group of cardinality $\aleph$ and Q is a Dedekind group of cardinality strictly smaller than $\aleph$ which has no elements of order p. Assume first that $p > 2$, so that P is contained in K and hence G/P is not abelian. It follows from Corollary 2.1.2 that $X' = P'$ for each large subgroup X of P, and so P satisfies condition (b) of the statement. Suppose now that $p = 2$, so that Q is an abelian subgroup contained in K. Put $N = K \cap P$. Thus, P/N is isomorphic to the quaternion group of order 8; in particular, N has cardinality $\aleph$ and so it cannot be abelian. Moreover, P/N' is a Dedekind nonabelian group by Corollary 2.1.2, while it follows again from Corollary 2.1.2 that $X' = N'$ for every large subgroup X of N.

Conversely, let $G = P \times Q$ be a group satisfying condition (c) of the statement, and let X be any large subgroup of G. Then

$$X = (X \cap P) \times (X \cap Q),$$

where $X \cap P$ has cardinality $\aleph$ and $X \cap Q$ is a normal subgroup of G. Moreover, we have $(X \cap P)' = P'$ if $p > 2$ and $(X \cap P)' = N'$ when $p = 2$, so that $X \cap P$ is normal in P, and so also in G. Therefore, the subgroup X is normal in G, and the proof is complete. □

The consideration of the direct product of a quaternion group of order 8 by a Prüfer 2-group shows that the statement of Theorem 2.1.3 does not hold if $\aleph = \aleph_0$.

Our aim is now to describe the behaviour of uncountable groups in which all large subgroups have the same commutator subgroup. Notice that these groups may have an infinite commutator subgroup; in fact, Ehrenfeucht and Faber exhibited a nilpotent group G of class 2 and cardinality $\aleph_1$ whose commutator subgroup is isomorphic to the additive group of rational numbers, but $X' = G'$ for each uncountable subgroup X. In the same paper, the authors constructed (for a suitable prime number p) an extraspecial p-group G of cardinality $\aleph_1$ whose abelian subgroups are countable, so that also in this case G' is the commutator subgroup of any large subgroup.

Let G be a group, and let X be a subgroup of G. Recall that the *isolator* of X in G is the set $I_G(X)$ of all elements g of G such that g^k belongs to X for some positive integer k. It is known that in a locally nilpotent group the isolator of any subgroup is likewise a subgroup. Moreover, if G is a torsion-free locally nilpotent group and X is a subgroup of G which is nilpotent of class c, then $I_G(X)$ is also nilpotent of class c (see [[39], Section 2.3]).

Lemma 2.1.5. *Let G be a torsion-free locally nilpotent group whose commutator subgroup G' is divisible abelian. Then the factor group G/G' is torsion-free.*

Proof. The isolator $I_G(G')$ of G' in G is an abelian normal subgroup of G, and of course the factor group $G/I_G(G')$ is torsion-free. On the other hand, $I_G(G')$ splits over its divisible subgroup G', and so $I_G(G') = G'$ because $I_G(G')/G'$ is periodic. Therefore, the group G/G' is torsion-free. □

Theorem 2.1.4. *Let G be an uncountable group of regular cardinality $\aleph$ which has no Jónsson large subgroups. If G is a $C^1_{\aleph}$-group, then it is nilpotent of class at most 2. Moreover, either the subgroup T consisting of all elements of finite order of G is large and G' has prime exponent or T is small and G' is divisible.*

Proof. All large subgroups of G are obviously normal, so that it follows from Theorem 2.1.2 that G is nilpotent of class at most 2 and G' is small by Lemma 2.1.3.

Assume first that T is a large subgroup of G, so that $T' = G'$. Of course, it follows from the hypotheses that all finite homomorphic images of T are abelian, and hence in this case G' has prime exponent by Lemma 2.1.4.

Suppose now that T is small and assume by contradiction that $(G')^k \neq G'$ for some positive integer k. As $\overline{G} = G/T$ is a torsion-free group of cardinality $\aleph$ in which all large subgroups are normal, its commutator subgroup $\overline{G}'$ is divisible (see [8], Theorem 3.6) and hence $\overline{G}/\overline{G}'$ is likewise torsion-free by Lemma 2.1.5. It follows that $(\overline{G}/\overline{G}')^k$ has cardinality $\aleph$, and so also G^k is large. Moreover, $[G^k, G] \leq (G')^k$ because G has class 2, so that $G^k/(G')^k$ is contained in the centre of $G/(G')^k$ and hence $Z(G/(G')^k)$ has cardinality $\aleph$. Thus, $G/(G')^k$ is a Dedekind group by Corollary 2.1.1. On the other hand, $G/(G')^k$ cannot be abelian, and so G contains a normal subgroup N such that G/N is isomorphic to the quaternion group of order 8, which is impossible, since in this case N is large and $N' \neq G'$. Therefore G' is a divisible group. $\square$

The above quoted examples due to Ehrenfeucht and Faber show that both situations described in Theorem 2.1.4 may actually occur.

2.2 Nearly Normality

A subgroup H of a group G is said to be *nearly normal* in G if it has finite index in its normal closure H^G. If $\mathcal{H}$ is a family of subgroups of a group G, we will say that the subgroups in $\mathcal{H}$ are *boundedly* nearly normal if there exists a positive integer k such that $|H^G : H| \leq k$ for each subgroup H in $\mathcal{H}$.

B.H. Neumann [46] proved that all subgroups of a group G are nearly normal if and only if the commutator subgroup G' of G is finite; Neumann's Theorem yields that if every subgroup of a group is nearly normal, then there is a bound for the index of every subgroup in its normal closure, so that every subgroup is boundedly nearly normal.

Theorem 3.6 of [12] shows that the imposition of nearly normality only on the set of all large subgroups of an uncountable group G of cardinality $\aleph$ is enough to have that G' is finite, provided that G contains a large FC-subgroup. Hence, even in such a case, the behaviour of small subgroups can be neglected and every subgroup of the group is boundedly nearly normal. This fact suggests a deeper investigation of the structure uncountable groups whose large subgroups have boundedly finite index in their normal closure. This choice links directly the study of nearly normality to that of normality for large subgroups and to the investigation of the influence of this latter property on the behaviour of the commutator subgroup.

For the purposes of our discussion, here we will state and prove a version with bounds of the above quoted result of [12].

Theorem 2.2.1. *Let G be an uncountable group of regular cardinality $\aleph$. If there exists a positive integer k such that for every large subgroup X of G, $|X^G : X| \leq k$ and G contains a large abelian subgroup, then the commutator subgroup G' of G is finite of order bounded by a function of k.*

Proof. Since G contains a large abelian subgroup, it follows from Corollary 1.1.2 that there exists in G an abelian subgroup of the form $A = A_1 \times A_2$, where both A_1 and A_2 have cardinality $\aleph$. Then all subgroups of the factor group G/A_i^G are nearly normal with the same bound as in G, and hence the commutator subgroup $G'A_i^G/A_i^G$ of G/A_i^G is finite of order at most a positive integer m depending only on k (see [41], Theorem B), for $i = 1, 2$. On the other hand, as $|A_1^G : A_1| \leq k$ and $|A_2^G : A_2| \leq k$, we have that

$$|A_1^G \cap A_2^G| = |A_1^G \cap A_2^G : A_1^G \cap A_2||A_1^G \cap A_2 : A_1 \cap A_2| \leq k^2.$$

The commutator subgroup $G'(A_1^G \cap A_2^G)/(A_1^G \cap A_2^G)$ is isomorphic to a subgroup of the direct product

$$G'/(A_1^G \cap G') \times G'/(A_2^G \cap G'),$$

so that it is finite of order at most m^2, hence G' has order at most m^2k^2. $\square$

Now we are able to show that the condition of nearly normality (with bounded index) on large subgroups is equivalent to the property that the commutator subgroup of every large subgroup coincides with the commutator subgroup of the whole group up to a finite section (of bounded order).

Theorem 2.2.2. *Let G be an uncountable group of regular cardinality $\aleph$.*

Then all large subgroups of G are nearly normal if and only if for every large subgroup X of G, the index $|G' : X'|$ is finite.

Proof. Assume that all large subgroups of G are nearly normal, and let X be any large subgroup of G; then the index $|X^G : X|$ is finite and the factor group G/X^G is finite-by-abelian, so that also the index $|G' : (G' \cap X)|$ is finite. If the commutator subgroup X' of X is large, the above argument shows that $|G' : X'|$ is finite. Assume that X' is small, and let N be the core of X in X^G; then N' is small, so that the factor group X^G/N' contains the large abelian subgroup N/N'. Therefore G'/N' is finite (see [12], Theorem 3.6), and hence $|G' : X'|$ is finite. □

Proposition 2.2.1. *Let G be an uncountable group of regular cardinality $\aleph$. If there exists a positive integer k such that for every large subgroup X of G, $|X^G : X| \leq k$, then for every large subgroup X of G, the index $|G' : X'|$ is finite and bounded by a function of k.*

Proof. Let X be any large subgroup of G. There exists a positive integer m, depending on k, such that $G'X^G/X^G$ is finite of order less or equal than m (see [41], Theorem B), so that $|G' : (G' \cap X)| \leq mk$. If the commutator subgroup X' of X is large, the above argument shows that $|G' : X'| \leq mk$. Assume that X' is small, and let N be the core of X in X^G; then N' is small, so that the factor group X^G/N' contains the large abelian subgroup N/N'. Therefore $|G'/N'| \leq m^4k^4$ by Theorem 2.2.1, and hence the index $|G' : X'|$ is likewise less or equal than m^4k^4. □

Recall first that every group G contains a maximum locally nilpotent normal subgroup $H(G)$, the so-called Hirsch-Plotkin radical, and that the upper Hirsch-Plotkin series

$$\{1\} = H_0(G) \leq H_1(G) \leq \cdots \leq H_\alpha(G) \leq H_{\alpha+1}(G) \leq \ldots$$

is defined recursively by putting

$$H_{\alpha+1}(G)/H_\alpha(G) = H(G/H_\alpha(G))$$

for each ordinal α and

$$H_\lambda(G) = \bigcup_{\alpha<\lambda} H_\alpha(G)$$

if λ is a limit ordinal. A group G is said to be *radical* if it admits an ascending (normal) series whose factors are locally nilpotent, or equivalently if its upper Hirsch-Plotkin series terminates with G. It is easy to show that in any group G the subgroup $R(G)$ generated by all radical normal subgroups is likewise radical and coincides with the last term of the upper Hirsch-Plotkin series of G.

In [12], the authors proved that the subgroup $R(G)$ of an uncountable group G of cardinality $\aleph$ in which all large subgroups have finite index has to be small (see Lemma 3.7). In the same paper, some consequences of that lemma concerning uncountable groups G in which $R(G)$ has countable index are shown (see Corollary 3.8 and Corollary 3.9). Since every ascendant radical subgroup of a group G is contained in $R(G)$ by [[51], Part I, Lemma 1.31], these results can be slightly improved as follows.

Corollary 2.2.1. *Let G be an uncountable group of regular cardinality $\aleph$. If G contains a large radical ascendant subgroup, G contains a large subgroup X such that the index $|G : X|$ is infinite.*

Corollary 2.2.2. *Let G be an uncountable group of regular cardinality $\aleph$ in which all large subgroups are nearly normal. If G contains a large radical subgroup, then the commutator subgroup G' of G is small.*

Proof. Assume for a contradiction that G' is a large subgroup of G.

Let X be a large radical subgroup of G; then if Y is the core of X in X^G, also $|X^G : Y|$ is finite, and hence Y is likewise large. Moreover, Y is contained in $R(G)$ (see [51], Part I, Lemma 1.31). It follows that $R = R(G)$ is large and $|G' : R'|$ is finite by Theorem 2.2.2, so that Corollary 2.2.1 ensures that G' contains a large subgroup X such that the index $|G' : X|$ is infinite. Then all subgroups of G/X^G are nearly normal, and hence the commutator subgroup G'/X^G is finite. On the other hand, X has finite index in X^G and so also in G', and this contradiction proves the statement. $\square$

Proposition 2.2.2. *Let G be an uncountable group of regular cardinality $\aleph$ in which all large subgroups are nearly normal. If G contains a large radical subgroup, then the commutator subgroup G' of G is small and G is locally (finite-by-nilpotent).*

Proof. As in the proof of Corollary 2.2.2, we have that $R(G)$ is large while $R(G)'$ is small; it follows that the factor group $G/R(G)'$ contains the large abelian subgroup $R(G)/R(G)'$, and hence it is finite-by-abelian (see [12], Theorem 3.6). Therefore, G' is small; hence for every element x of G, we have that $|G : C_G(x)| < \aleph$.

Let $F = \langle x_1, \dots, x_t \rangle$ be any finitely generated subgroup of G. The centralizer

$$C_G(F) = \bigcap_{i=1}^{t} C_G(x_i)$$

has index in G strictly smaller than $\aleph$, and hence it is a large subgroup, so that $C_G(F)F$ has finite index in $(C_G(F)F)^G$. It follows now from Theorem 1.1 of [3] that G is locally (finite-by-nilpotent). □

Proposition 2.2.3. *Let G be an uncountable group of regular cardinality $\aleph$ in which all large subgroups are nearly normal. If G contains a large locally nilpotent subgroup, then the commutator subgroup G' of G is finite-by-(locally nilpotent).*

Proof. Let X be a large locally nilpotent subgroup of G; then if Y is the core of X in X^G, also $|X^G : Y|$ is finite, and hence Y is likewise large. Moreover, Y is contained in the Hirsch-Plotkin radical H of G (see [51], Part I, Lemma 1.31), so that H is large, and hence G/H is finite-by-abelian. In particular, the index $|G' : (G' \cap H)|$ is finite, and so there exists a finitely generated subgroup F such that $G' = F(G' \cap H)$. As in the proof of Proposition 2.2.2, we can see that the subgroup $FC_G(F)$ is nearly normal in G, and hence $F_1 = FC_{G'}(F)$ has finite index in $F_2 = F_1^{G'}$.

It follows from Proposition 2.2.2 that F is finite-by-nilpotent, so that there exists a positive integer m such that $\gamma_m(F)$ is finite; moreover,

$$F_1/\gamma_m(F) = F/\gamma_m(F) \cdot ((F_1 \cap H)\gamma_m(F))/\gamma_m(F)$$

is locally nilpotent.

The subgroup $\gamma_m(F)$ is normalized by F_1, hence it has finitely many conjugates in F_2, so that $K = \gamma_m(F)^{F_2}$ is finite; clearly, $F_2 = F_1(F_2 \cap H)$, and hence

$$F_2/K = F_1K/K \cdot ((F_2 \cap H)K)/K$$

is locally nilpotent (see [3], Lemma 2.7). Therefore, F_2 contains a finite characteristic subgroup M such that F_2/M is locally nilpotent (see [3], Lemma 2.9) and hence

$$G'/M = F_2/M \cdot ((G' \cap H)M)/M$$

is locally nilpotent. $\square$

Proposition 2.2.4. *Let G be an uncountable group of regular cardinality $\aleph$ in which all large subgroups are nearly normal. If the commutator subgroup G' of G is small, it is an FC-group and $G'Z(G)/Z(G)$ is a residually finite group.*

Proof. Let x be any element of G, and let C be the centralizer of x in G; then the index $|G : C|$ is strictly smaller than $\aleph$, so that C is large. It follows from Theorem 2.2.2 that the index $|G' : (G' \cap C)|$ is finite. Therefore, $G'Z(G)/Z(G) \simeq G'/(G' \cap Z(G))$ is residually finite and every element of G' has finitely many conjugates in G'. $\square$

Next two theorems give a description of the structure of groups G of cardinality $\aleph$ in which all large subgroups are boundedly nearly normal; in particular, the first of them provides a necessary and sufficient condition for the bounded nearly normality of all large subgroups of G based on the fact that their commutator subgroups are constrained between G' and the commutator subgroups of some $C^1_\aleph$-subgroup of G.

The second theorem must be seen in relation with Theorem 2.1.2 and Theorem 2.1.4 which ensure that any $C^1_\aleph$-group (containing no Jònsson large subgroups) is nilpotent of class at most 2 and its commutator subgroup either is divisible or has prime exponent.

If G is a group of cardinality $\aleph$, following [11], we shall denote by $\mathcal{C}_\aleph(G)$ the set of all commutator subgroups of the large subgroups of G.

Theorem 2.2.3. *Let G be an uncountable group of regular cardinality $\aleph$ which has no Jònsson large subgroups. Then there exists a positive integer k such that $|X^G : X| \leq k$ for every large subgroup X of G if and only if there exist a positive integer h and a family $\{K_j \mid j \in J\}$ of $C^1_\aleph$-subgroups of G such that $|G' : K'_j| \leq h$ and X' contains some K'_j, for every large subgroup X of G.*

Proof. Assume that $|X^G : X| \leq k$ for every large subgroup X of G, so that the set $\mathcal{C}_\aleph(G)$ of all commutator subgroups of large subgroups satisfies the minimal condition by Proposition 2.2.1. Indeed, if

$$X'_1 \geq X'_2 \geq \dots X'_n \geq X'_{n+1} \geq \dots$$

is a descending chain of elements in $\mathcal{C}_\aleph(G)$, we have that there exists a positive integer h such that, for every n, $|G' : X'_n| \leq h$; it follows that the chain above has to be definitively constant. Therefore, for every large subgroup X of G, there exists a large subgroup K_X of G such that $K'_X \leq X'$ and K'_X is a minimal element of $\mathcal{C}_\aleph(G)$; then $Y' = K'_X$ for every large subgroup Y of K_X; finally, it follows again from Proposition 2.2.1 that $|G' : K'_X| \leq h$ for every large subgroup X of G. □

Theorem 2.2.4. *Let G be an uncountable group of regular cardinality $\aleph$ which has no Jònsson large subgroups. If there exists a positive integer k such that for every large subgroup X of G, $|X^G : X| \leq k$, then G' is small and $[G', G]$ is finite.*

Moreover, if G' is infinite, it contains a finite subgroup N containing $[G', G]$ such that either G'/N has prime exponent p or G/N is a $C^1_\aleph$-group with divisible commutator subgroup.

In the first case, the subgroup T of G consisting of all elements of finite order is large, and the p'-component of T/N is small, while in the second case T is small and G has finitely many commutator subgroups of large subgroups.

Proof. As in the proof of Theorem 2.2.3, it follows from Proposition 2.2.1 that the set $\mathcal{C}_\aleph(G)$ of all commutator subgroups of large subgroups satisfies the minimal condition; let K be a large subgroup of G whose commutator subgroup K' is a minimal element of $\mathcal{C}_\aleph(G)$; then K is clearly a $C^1_\aleph$-group.

In particular, K' is small (see Theorem 2.5 2.1.4), so that G' is likewise small since $|G' : K'|$ is finite; therefore, for every $x \in G$, $C_G(x)$ is large, and hence $|G' : C_{G'}(x)|$ is finite and bounded by a function of k again by Proposition 2.2.1. An application of Theorem 1.2 of [23] ensures that $\gamma_3(G)$ is finite.

Let N be a finite subgroup of G' containing $\gamma_3(G)$ such that

$$G'/\gamma_3(G) = (K'\gamma_3(G)/\gamma_3(G)) \cdot (N/\gamma_3(G)).$$

Clearly N is normal in G and G'/N is either an abelian group of prime exponent or a divisible abelian group by Theorem 2.1.4.

Assume first that G'/N is an abelian group of prime exponent p; then Theorem 2.1.4 ensures that the torsion subgroup of the $C^1_{\aleph}$-group K is large so that T is large.

Suppose now that G' is infinite, so that G/N does not contain any large abelian subgroup (see [12], Theorem 3.6); on the other hand, the p'-component of T/N has trivial intersection with G'/N, and hence it is contained in the centre of G/N, so that it must be small.

Otherwise, if G'/N is divisible, then G/N is a $C^1_{\aleph}$-group by Theorem 2.2.2, and T is small by Theorem 2.1.4. □

In this paragraph, we will extend the concept of (boundedly) nearly normality of a subgroup. If G is a group and $\mathcal{H}$ is a family of subgroups in G, we will say that the subgroups in $\mathcal{H}$ are *weakly* nearly normal if there exists a cardinal number $\alpha > 1$ such that $|H^G : H| < \alpha$ for each subgroup H in $\mathcal{H}$. When α is finite, weakly nearly normal subgroups (with bound α) are boundedly nearly normal.

Subsequently, we will examine the structure of uncountable groups where large subgroups are weakly nearly normal with an infinite cardinal as a bound.

Tomkinson in [62] studied FC-groups where every subgroup has index less than an infinite cardinal α in its normal closure and he proved the following theorem.

Theorem 2.2.5. *Let G be an FC-group and let α be an infinite cardinal.*

If $|U^G : U| < \alpha$ for each subgroup U of G, then $|G'| < \alpha$.

The assumptions of Theorem 2.2.5 can be weaken when we consider uncountable FC-groups of regular cardinality.

Theorem 2.2.6. *Let G be an uncountable FC-group of regular cardinality $\aleph$.*

Let α be an infinite cardinal. If $|U^G : U| < \alpha$ for each large subgroup U of G, then $|G'| < \alpha$.

Proof. There exists a large subgroup K of G whose commutator subgroup K' is finite (see [63], p.155); therefore, K/K' contains an abelian subgroup of the form $A_1/K' \times A_2/K'$ where both A_1 and A_2 have cardinality $\aleph$ by Corollary 1.1.2. Hence, for $i = 1, 2$, the commutator subgroup $G'A_i^G/A_i^G$ of G/A_i^G has cardinality less than α by Theorem 2.2.5, so that $|G' : (G' \cap A_1^G \cap A_2^G)| < \alpha$.

On the other hand, both $|A_1^G : A_1|$ and $|A_2^G : A_2|$ are strictly smaller than α and $|A_1^G \cap A_2^G : K'| = |A_1^G \cap A_2^G : A_1 \cap A_2|$, hence

$$|A_1^G \cap A_2^G : K'| = |A_1^G \cap A_2^G : A_1^G \cap A_2||A_1^G \cap A_2 : A_1 \cap A_2| < \alpha.$$

As a consequence, we have that

$$|G' : K'| = |G' : (G' \cap A_1^G \cap A_2^G)||(G' \cap A_1^G \cap A_2^G) : K'| < \alpha,$$

therefore $|G'| < \alpha$. $\square$

If β is a cardinal number, a group G is said to be a βC-group if for every element x of G, the conjugacy class of x has cardinality strictly smaller than β. In other words, G is a βC-group if $|G : C_G(x)| < \beta$, for every element x of G. When β is a finite cardinal number, a βC-group is a BFC-group. Hence, we will focus on the case of βC-groups where β is infinite.

Theorem 2.2.5 has been improved in [28], where the authors prove that if G is a βC-group for some infinite cardinal $\beta < \alpha$ and $|U^G : U| < \alpha$ for each subgroup U of G, then $|G'| < \alpha$, provided that GCH is assumed to be true.

Theorem 2.2.6 leads to the following corollary in the case that G is a βC-group, with $\beta < \alpha$, or α is a *strong limit cardinal*, i.e., $2^\gamma < \alpha$ for every

$\gamma < \alpha$.

Corollary 2.2.3. *Let G be an uncountable group of regular cardinality $\aleph$ containing a large FC-subgroup. Let α be an infinite cardinal such that $|U^G : U| < \alpha$ for each large subgroup U of G.*

If either α is a strong limit cardinal or GCH is true and G is a βC-group for some infinite cardinal $\beta < \alpha$, then $|G'| < \alpha$.

Proof. Clearly, we can assume that $\alpha \leq \aleph$. Let F be a large FC-subgroup of G; by Theorem 2.2.6, the commutator subgroup F' of F has cardinality strictly smaller than α. Therefore, F/F' contains an abelian subgroup of the form $A_1/F' \times A_2/F'$ where both A_1 and A_2 have cardinality $\aleph$ by Corollary 1.1.2.

Assume first that α is a strong limit cardinal; then for each $i = 1, 2$, for every $x \in G$, the normal closure $\langle xA_i^G \rangle^{G/A_i^G}$ has cardinality strictly smaller than α, so that the factor group G/A_i^G is an αC-group, and hence the commutator subgroup $G'A_i^G/A_i^G$ of G/A_i^G has cardinality less than α (see [28], proof of Corollary 3.3 and Theorem B). Otherwise, if GCH is true and G is a βC-group for some infinite cardinal $\beta < \alpha$, then G/A_i^G is a βC-group (see [28], Lemma 3.4), so that, also in this case, the commutator subgroup $G'A_i^G/A_i^G$ of G/A_i^G has cardinality less than α by Theorem C of [28]. Reminding that both $|A_1^G : A_1|$ and $|A_2^G : A_2|$ are strictly smaller than α (so that $|A_1^G \cap A_2^G : F'| < \alpha$), we have that $|G'| < \alpha$. $\square$

2.3 Almost Normality and Finite Number of Normalizers

A subgroup H of a group G is said to be *almost normal* in G if it has finitely many conjugates in G or, equivalently, if the normalizer $N_G(H)$ of H in G has finite index in G. If $\mathcal{H}$ is a family of subgroups of a group G, we will say that the subgroups in $\mathcal{H}$ are *boundedly* almost normal if there exists a positive integer k such that $|G : N_G(H)| \leq k$ for each subgroup H in $\mathcal{H}$.

In [46], Neumann proved that a group G is finite over its centre if and only if every subgroup of G has finitely many conjugates; moreover, in [49], Polovickǐ proved that groups with this property are precisely those with finitely many normalizers of (abelian) subgroups, and by a celebrated theorem of Schur [57], they are also finite-by-abelian. In particular, when all subgroups of a group G are almost normal, they are both boundedly almost normal and boundedly nearly normal.

De Falco, de Giovanni and Musella in [12] showed that in an uncountable group containing a large abelian subgroup, if all large subgroups are nearly normal, the group is central-by-finite. As a consequence, in such groups all subgroups are (boundedly) almost and nearly normal. In addition, the same authors proved that, when all large subgroups are almost normal, they are also nearly normal.

In the present section, we will provide two similar results with bounds since it will be useful for our aims. Indeed, the goal of this section is to explore the behaviour of uncountable groups of regular cardinality whose large subgroups are boundedly almost normal, starting from the study of uncountable groups where there are finitely many normalizers of large subgroups. Actually, it will turn out that these properties are equivalent, even when we focus only on the set of large subgroups, giving a kind of translation of the above quoted Polovickǐ's theorem.

Our first result shows that the commutator subgroup of a group G of cardinality $\aleph$ with finitely many normalizers of large subgroups must be small, provided that G has no simple sections of cardinality $\aleph$.

Proposition 2.3.1. *Let G be an uncountable group of regular cardinality $\aleph$ having no simple sections of cardinality $\aleph$. If G contains only finitely many normalizers of large subgroups, then the commutator subgroup G' of G is small.*

Proof. We will proceed by induction on the number k of the proper normalizers of large subgroups. If $k = 0$, all large subgroups are normal, and the statement follows from Lemma 2.1.3.

Assume that $k > 0$, and let $N_1, \dots, N_k$ be the proper normalizers of large subgroups; for every $i = 1, \dots, k$, the subgroup N_i has less than k proper normalizers of large subgroups, and hence, by induction hypothesis, N_i' is small. On the other hand, the subgroup N_i has finitely many conjugates in G, so that the conjugacy class of N_i' is also finite and hence the subgroup $N = \langle N_1', \dots, N_k' \rangle^G$ is small.

Let H/N any non-abelian large subgroup of G/N; then H is not contained in any N_i, so that H is normal in G. Therefore, G'/N is small (see [8], Theorem 4.3), so that G' is small. □

In [45], B. H. Neumann provided a characterization of groups with finitely many normalizers:

Theorem 2.3.1. *A group G is covered by finitely many abelian subgroups if and only if $G/Z(G)$ is finite.*

Such a result has an interesting consequence concerning groups with a finite covering of central-by-finite subgroups. In order to prove it, we require the following lemma by Neumann, which plays a key role in all arguments related to finite coverings of groups.

Lemma 2.3.1. *Let the group $G = X_1 \cup \dots \cup X_n$ be the union of finitely many subgroups $X_1, \dots, X_n$. Then any X_i of infinite index can be omitted from this decomposition; in particular, at least one of the subgroups $X_1, \dots, X_n$ has finite index in G.*

Corollary 2.3.1. *Let G be a group covered by finitely many central-by-finite subgroups. Then $G/Z(G)$ is finite.*

Proof. Assume that $G = X_1 \cup \dots \cup X_t$, where $X_1, \dots, X_t$ are central-by-finite subgroups of G. It follows from Lemma 2.3.1 that there exists a subset $\{i_1, \dots, i_m\}$ of $\{1, \dots, t\}$ such that $G = X_{i_1} \cup \dots \cup X_{i_m}$ and $|G : X_{i_j}|$ is finite for every $j = 1, \dots, m$. Thus, $|G : Z(X_{i_j})|$ is finite for each $j = 1, \dots, m$; so the centre

$$Z(G) = \bigcap_{j=1}^{m} Z(X_{i_j})$$

has likewise finite index in G. □

Proposition 2.3.2. *Let G be an uncountable group of regular cardinality $\aleph$ having no simple sections of cardinality $\aleph$ such that G contains only finitely many* *normalizers of large subgroups. If G contains a large abelian subgroup, then $G/Z(G)$ is finite.*

Proof. We will proceed by induction on the number k of the proper normalizers of large subgroups. If $k = 0$, all large subgroups are normal and the statement follows from Corollary 2.1.1.

Assume that $k > 0$, and let $N_1, \dots, N_k$ be the proper normalizers of large subgroups; for every $i = 1, \dots, k$, the subgroup N_i has less than k proper normalizers of large subgroups, and hence, by induction hypothesis, $N_i/Z(N_i)$ is finite.

If $G = N_1 \cup \dots \cup N_k$, then $G/Z(G)$ is finite by Corollary 2.3.1.

Assume that $N_1 \cup \dots \cup N_k$ is properly contained in G, and let g be an element of $G \setminus (N_1 \cup \dots \cup N_k)$. It follows from Proposition 2.3.1 that $|G : C_G(g)| < \aleph$, so that $C_G(g)$ contains a large abelian subgroup; therefore $C_G(g)$ contains a subgroup of the form $A_1 \times A_2$ where each A_i is a large abelian group by Corollary 1.1.2. Clearly, for each $i = 1, 2$, $g \in N_G(A_i)$, so that A_i is normal in G; moreover, G/A_i has finitely many normalizers, so that its centre Z_i/A_i has finite index in G/A_i. It follows that $Z_1 \cap Z_2$ has finite index in G, so that $Z(G)$ has likewise finite index in G. □

As previously stated, now we will give a version with bounds of Theorem 3.3 and Corollary 3.4 of [12]:

Theorem 2.3.2. *Let G be an uncountable group of regular cardinality $\aleph$. If there exists a positive integer k such that for every large subgroup X of G, $|G : N_G(X)| \leq k$ and G contains a large abelian subgroup, then $G/Z(G)$ is finite of order bounded by a function of k.*

Proof. Since G contains a large abelian subgroup, it follows from Corollary 1.1.2 that G contains an abelian subgroup of the form $A = A_1 \times A_2$, where both A_1 and A_2 have cardinality $\aleph$. Then the normalizer $N_G(A_i)$ has index in G bounded by k for $i = 1, 2$. Moreover, all subgroups of the group $N_G(A_i)/A_i$ are boundedly almost normal with bound (at most) k, and so its centre C_i/A_i has index at most $g(k)$ in $N_G(A_i)/A_i$, where g is an integer function (see [41], Theorem A). Thus, $|G : C_i| \leq kg(k)$ for $i = 1, 2$ and so the core $C = (C_1 \cap C_2)_G$ is an abelian normal subgroup such that $|G : C|$

is bounded by a function $h(k)$ of k. If x is any element of G, by Theorem 1.1.1 we have that C contains a subgroup $C(x)$ which is the direct product of a collection of cardinality $\aleph$ of nontrivial $\langle x\rangle$-invariant subgroups.

Let X be any small subgroup of $\langle C, x\rangle$. Then, by Corollary 1.1.1 and Corollary 1.1.2, there exist two $\langle x\rangle$-invariant large subgroups U_1 and U_2 of $C(x)$ such that

$$U_1 \cap U_2 = X \cap U_1U_2 = \{1\}.$$

As the subgroups XU_1 and XU_2 are boundedly almost normal with bound k, their intersection

$$X = XU_1 \cap XU_2$$

is boundedly almost normal with $|G : N_G(X)| \leq k^2$. It follows that all subgroups of $\langle C, x\rangle$ are boundedly almost normal with bound (at most) k^2, so that the centre $Z(\langle C, x\rangle)$ has index at most $g(k^2)$ in $\langle C, x\rangle$ and hence it has index at most $l(k) = h(k)g(k^2)$ in G.

Consider a transversal $\{x_1, \dots, x_t\}$ to C in G, so that $G = \langle C, x_1, \dots, x_t\rangle$ and the intersection

$$\bigcap_{i=1}^{t} Z(\langle C, x_i\rangle)$$

is a subgroup of $Z(G)$ of index in G less or equal to $l(k)t \leq l(k)h(k) = h^2(k)g(k^2)$. Therefore, $G/Z(G)$ is finite of order bounded by an integer function of k. □

Lemma 2.3.2. *Let G be a group and let H be a subgroup of G. If H is both boundedly nearly normal and boundedly almost normal, then H^G/H_G is finite of bounded order.*

Proof. Let k and h be positive integers such that $|H^G : H| \leq k$ and $|G : N_G(H)| \leq h$. It follows that H has at most h conjugates, say $H^{g_1}, \dots, H^{g_t}$ with $t \leq h$. Of course, $|H^G : H^{g_i}| \leq k$ for every $i \in \{1, \dots, t\}$, so that

$$|H^G : H_G| = |H^G : \bigcap_{i=1}^{t} H^{g_i}| \leq |H^G : H^{g_1}| \dots |H^G : H^{g_t}| \leq hk.$$

□

Corollary 2.3.2. *Let G be an uncountable group of regular cardinality $\aleph$ in which all large subgroups are boundedly almost normal. Then X^G/X_G is finite of bounded order for each large subgroup X of G.*

Proof. Let X_G have cardinality $\aleph$, so that G/X_G has bounded-index centre by Theorem 2.3.2. In particular, $G'X_G/X_G$ is finite of bounded order (see, for instance, Theorem 1 of [64]) and every subgroup in G/X_G is boundedly nearly normal. It follows that $|X^G : X|$ is bounded. Lemma 2.3.2 ensures that X^G/X_G is finite of bounded order.

We can assume now that X_G is small. Since X is a large subgroup, its normalizer $N_G(X)$ has bounded index in G and $N_G(X)/X$ has bounded-index centre C/X. It follows that there is a bound even on the index $|G : C_G|$ where C_G is the normal core of C in G. Of course $(C_G)' \leq C' \leq X$, thus $(C_G)' \leq X_G$ and $(C_G)'$ is small. Therefore, we can apply Theorem 2.3.2 to the group $G/(C_G)'$ whose centre tuns out to have bounded index. Since $G'/(C_G)'$ is finite of bounded order and $(C_G)' \leq X_G$, we have that the index $|X^G : X|$ is bounded too, so the statement is a consequence of Lemma 2.3.2. □

If $\mathfrak{X}$ is a class of groups, a group G is called *residually* $\mathfrak{X}$ if it has a collection of normal subgroups $(N_i)_{i\in I}$ such that G/N_i belongs to $\mathfrak{X}$ for each i and

$$\bigcap_{i\in I} N_i = \{1\}.$$

A group G is *residually finite* if the intersection of all its (normal) subgroups of finite index is trivial.

Let $\aleph$ be any infinite cardinal number. Here, a group of cardinality $\aleph$ will be called *residually small* if it has a collection of normal subgroups of index strictly smaller than $\aleph$ whose intersection is trivial.

Furthermore, we will call a group G of cardinality $\aleph$ *weakly* residually small if it contains a family of subgroups (not necessarily normal in G) of index less than $\aleph$ with trivial intersection.

It is clear that, in the universe of infinite countable groups, the notions of being residually finite, residually small, and weakly residually small coincide. Moreover, if G is a group of cardinality $\aleph$ whose large subgroups are all normal, G is weakly residually small if and only if it is residually small.

Recall here that in the above quoted example of Ehrenfucht and Faber, all large subgroups are both boundedly almost normal and boundedly nearly normal; on the other hand, as a of consequence Lemma 2.1.2 we have that the behaviour of small subgroups can be neglected in the case of (weakly) residually small uncountable groups where all large subgroups are normal.

Theorem 2.3.3. *Let G be a (weakly) residually small uncountable group of regular cardinality $\aleph$ in which all large are normal, then all subgroups of G are normal.*

Proposition 2.3.3. *Let G be a weakly residually small uncountable group of regular cardinality $\aleph$ having no simple sections of cardinality $\aleph$.*

(1) If all large subgroups of G are boundedly nearly normal, then G contains an abelian large subgroup and is finite-by-abelian with commutator subgroup of bounded order.

(2) If all large subgroups of G are boundedly almost normal, then $G/Z(G)$ is finite of bounded order.

Proof.

(1) Let K be a $C^1_\aleph$-subgroup of G (its existence is guaranteed by Theorem 2.2.3); K is residually small and it follows from Theorem 2.3.3 that K is a Dedekind group. Therefore, G contains an abelian large subgroup and it is finite-by-abelian with commutator subgroup of bounded order by Theorem 2.2.1.

(2) Corollary 2.3.2 ensures that all large subgroups of G are boundedly nearly normal, thus G contains a large abelian subgroup. Therefore, $G/Z(G)$ is finite of bounded order by Theorem 2.3.2. □

Corollary 2.3.3. *Let G be an uncountable group of regular cardinality $\aleph$ having no simple sections of cardinality $\aleph$.*

(1) *If all large subgroups of G are boundedly nearly normal, then $G/Z_3(G)$ is finite.*

(2) *If all large subgroups of G are boundedly almost normal, then $G/Z_2(G)$ is finite.*

Proof.

(1) The commutator subgroup G' is small by Theorem 2.2.4, so that $|G : C_G(g)| < \aleph$ for every $g \in G$. It follows now from Proposition 2.3.3 that $G/Z(G)$ is finite-by-abelian, and hence $Z_3(G)$ has finite index in G.

(2) As above, it follows from Proposition 2.3.3 that $G/Z(G)$ is central-by-finite, so that $Z_2(G)$ has finite index in G. □

Theorem 2.3.4. *Let G be an uncountable group of regular cardinality $\aleph$ having no simple sections of cardinality $\aleph$. Then the following conditions are equivalent:*

(i) G contains only finitely many normalizers of large subgroups.

(ii) All large subgroups are boundedly almost normal.

(iii) G contains a subgroup N of finite index in G such that N normalizes all large subgroups of G and all large subgroups of N are normal in G.

Proof.

(i) ⇒ (ii) We will proceed by induction on the number k of the proper normalizers of large subgroups. If $k = 0$, all large subgroups are normal.

Assume that $k > 0$, and let $N_1, \dots, N_k$ be the proper normalizers of large subgroups; for every $i = 1, \dots, k$, the subgroup N_i has less than k proper normalizers of large subgroups, and hence, by induction hypothesis, all large subgroups of N_i are bounedly almost normal in N_i.

If $G = N_1 \cup \cdots \cup N_k$ then, by Lemma 2.3.1, there exists $\{i_1, \ldots, i_t\} \subseteq \{1, \ldots, k\}$ such that $G = N_{i_1} \cup \cdots \cup N_{i_t}$ and $|G : N_{i_j}| < \infty$ for every $j = 1, \ldots, t$, so that if H is a large subgroup of G,

$$H = (H \cap N_{i_1}) \cup \cdots \cup (H \cap N_{i_t})$$

where each $H \cap N_{i_j}$ is almost normal in N_{i_j} for every $j = 1, \ldots, t$. It follows that $|G : N_{N_{i_j}}(H \cap N_{i_j})| < \infty$ for every $j = 1, \ldots, t$ and consequently

$$\left| G : \bigcap_{j=1}^{t} N_{N_{i_j}}(H \cap N_{i_j}) \right| < \infty.$$

Since $\bigcap_{j=1}^{t} N_{N_{i_j}}(H \cap N_{i_j}) \leq N_G(H)$, we have that H is almost normal.

Assume that $N_1 \cup \cdots \cup N_k$ is properly contained in G, and let H be any large subgroup of G. Let g be an element of $G \setminus (N_1 \cup \cdots \cup N_k)$; it follows from Proposition 2.3.1 that $|G : C_G(g)| < \aleph$, so that $L = H \cap C_G(g)$ is large. Clearly, $g \in N_G(L)$, so that L is normal in G; moreover, G/L has finitely many normalizers, so that it is central-by-finite, and hence H is almost normal in G.

In any case, the normalizers of all large subgroups of G have finite index, so that all large subgroups are boundedly almost normal.

(ii) $\Rightarrow$ (iii) All large subgroups of G are boundedly nearly normal by Corollary 2.3.2, so that the set $\mathcal{C}_{\aleph}(G)$ satisfies the minimal condition and G' is small (see Theorem 2.2.3 and Theorem 2.2.4); it follows that $G/Z(G)$ is weakly residually small, and hence $Z_2(G)$ is large by Proposition 2.3.3. Let K be a large subgroup such that K' is minimal element of $\mathcal{C}_{\aleph}(G)$. We have that for each $g \in G$, $K \cap C_G(g)$ is large, thus $K' \leq C_G(g)$. It follows that $K' \leq Z(G)$ and we can consider the factor group G/K'; such a section contains the large abelian subgroup K/K' and hence its centre N/K' has bounded finite index by Theorem 2.3.2.

If Y is any large subgroup of N, then $Y' \leq N' \leq K'$, so that $Y' = K'$ by the minimality of K'; in particular $K' \leq Y \leq N$, so that Y is normal in G. Finally, let H be any large subgroup of G. Then $H \cap N$ is large, so that, as above, $(H \cap N)' = K'$, and hence the subgroup H/K' is normalized by N/K'; therefore, H is normalized by N.

(iii) $\Rightarrow$ (i) This claim is obvious. $\square$

We can notice that the same result as Theorem 2.3.1 holds if a group G is covered by finitely many Dedekind subgroups. Indeed, G is central-by-finite by Corollary 2.3.1. In such a frame, the following result describes the behaviour of groups with a finite covering consisting of subgroups with normal large subgroups.

Theorem 2.3.5. *Let G be an uncountable group of regular cardinality $\aleph$ having no simple sections of cardinality $\aleph$. If $G = X_1 \cup \cdots \cup X_n$ is the union of finitely many subgroups $X_1 \ldots, X_n$ where each X_i has all large subgroups that are normal, then G has finitely many normalizers of large subgroups.*

In particular, when G contains a large abelian subgroup, $G/Z(G)$ is finite.

Proof. It follows from Lemma 2.3.1 that there exists a subset $\{i_1, \ldots, i_m\}$ of $\{1, \ldots, n\}$ such that $G = X_{i_1} \cup X_{i_2} \cup ... \cup X_{i_m}$ and $|G : X_{i_j}| < \infty$ for every $j = 1, \ldots, m$. Put

$$N = \bigcap_{j=1}^{m} X_{i_j};$$

then $|G : N|$ is finite.

Let H be any large subgroup of G. Then, for every $j = 1, \ldots, m$, $H \cap X_{i_j}$ is large and hence it is normalized by N; therefore,

$$H = (H \cap X_{i_1}) \cup (H \cap X_{i_2}) \cup ... \cup (H \cap X_{i_m})$$

is normalized by N. Theorem 2.3.4 ensures that G has finitely many normalizers of large subgroups while the last part of the statement follows from Proposition 2.3.2. $\square$

2.4 Finite Number of Commutator Subgroups

If G is a group with finite commutator subgroup G', it is clear that it admits only finitely many commutator subgroups. A sufficient condition for a group to have a finite commutator subgroup was found by de Giovanni and Robinson in [17], who proved that a locally graded group G is finite-by-abelian if and only if the set

$$\{X' | X \leq G\}$$

is finite. Recall here that a group G is *locally graded* if every finitely generated nontrivial subgroup of G contains a proper subgroup of finite index. The requirement that the group is locally graded cannot be removed from the above statement, since Tarski groups (i.e., infinite simple groups whose proper non-trivial subgroups have prime order) have precisely two commutator subgroups. On the other hand, the class of locally graded groups is very large, and contains in particular all locally (soluble-by-finite) groups. In the same paper [17], the authors studied also groups with finitely many commutator subgroups of infinite subgroups, and proved that locally graded groups with this property either have a finite commutator subgroup or are Černikov groups (and in particular they are countable in the latter case).

In the universe of uncountable groups of cardinality $\aleph$, we have already underlined the great importance of groups where there is only one possibility for the commutator subgroup of any large subgroup (that is, abelian groups and $C^1_{\aleph}$-groups). For this reason, a natural question arises about the behaviour of uncountable groups in which there are only finitely many commutator subgroups of large subgroups. In [11] and [14], the structure of such groups has been investigated and all the results of this section come from these two papers.

Lemma 2.4.1. *Let G be an uncountable locally graded group of regular cardinality $\aleph$ such that the set $\mathcal{C}_{\aleph}(G)$ is finite. If G contains a large abelian normal subgroup A, then the commutator subgroup G' of G is finite.*

Proof. Let g be any element of G. It follows from Corollary 1.1.1 and Corollary 1.1.2 that A contains two $\langle g\rangle$-invariant subgroups B_1 and B_2 of cardinality $\aleph$ such that $B_1 \cap B_2 = \{1\}$. For each $i = 1, 2$, the factor group $A\langle g\rangle/B_i$ is locally graded (see [40]) and has only finitely many derived subgroups, so that its commutator subgroup is finite (see [17]). Thus, also the commutator subgroup $(A\langle g\rangle)'$ of $A\langle g\rangle$ is finite. Moreover, the set

$$\{(A\langle g\rangle)' \,|\, g \in G\}$$

is finite, because it is contained in $\mathcal{C}_\aleph(G)$, and each subgroup $(A\langle g\rangle)'$ obviously has only finitely many conjugates in G. Therefore,

$$[A, G] = \langle [A, g] \,|\, g \in G\rangle = \langle (A\langle g\rangle)' \,|\, g \in G\rangle$$

is a finite normal subgroup of G. Put $\overline{G} = G/[A, G]$. Then $\overline{A}$ is contained in the centre of $\overline{G}$, and so $Z(\overline{G})$ is large. If $\overline{X}$ is any subgroup of $\overline{G}$, the product $\overline{X}Z(\overline{G})$ is a large subgroup and $(\overline{X}Z(\overline{G}))' = \overline{X}'$. It follows that the set

$$\{\overline{X}' \,|\, \overline{X} \leq \overline{G}\}$$

is finite, so that the locally graded group $\overline{G}$ has a finite commutator subgroup (see [17]), and hence G' itself is finite. □

Let G be a group, and let $\aleph$ be any cardinal number. Let us denote with $L_\aleph(G)$ the set of all large subgroups of G. As the set $\mathcal{C}_\aleph(G)$ is obviously fixed by all automorphisms of G, the intersection

$$D_\aleph(G) = \bigcap_{X \in L_\aleph(G)} (N_G(X')_G)$$

is a characteristic subgroup of G. Moreover, if $\mathcal{C}_\aleph(G)$ is finite, the subgroup X' has only finitely many conjugates in G for each $X \in L_\aleph(G)$, so that its normalizer has finite index in G and hence also the index $|G : D_\aleph(G)|$ is finite.

The following result is Lemma 3 of [11].

Lemma 2.4.2. *Let G be an uncountable locally graded group of regular cardinality $\aleph$ which has no large simple sections. If $X' = G'$ for each non-abelian large* *subgroup X of G, then G' is small. Moreover, if G is infinite, then G is nilpotent of class 2 and it has no abelian large subgroups.*

Lemma 2.4.3. *Let G be an uncountable locally graded group of regular cardinality $\aleph$ such that the set $\mathcal{C}_\aleph(G)$ is finite, and suppose that G has no large simple sections. Then G is soluble-by-finite and its commutator subgroup G' is small. Moreover, if G contains an abelian large subgroup, then G' is finite.*

Proof. Obviously, it can be assumed that G is not abelian. Let m be the number of nontrivial elements in the finite set $\mathcal{C}_\aleph(G)$, and suppose first that $m = 1$. Then $X' = G'$ for each non-abelian large subgroup X of G, so that it follows from Lemma 2.4.2 that G' is small, and it is even finite, provided that G contains an abelian large subgroup.

Assume now that $m > 1$. As the characteristic subgroup $D = D_\aleph(G)$ has finite index in G, it has cardinality $\aleph$, and so D' belongs to $\mathcal{C}_\aleph(G)$. If D is abelian, the commutator subgroup G' of G is finite by Lemma 2.4.1 and the statement holds. Suppose that D is not abelian, and let M be a non-abelian subgroup of cardinality $\aleph$ of D such that M' is minimal among the non-trivial elements of $\mathcal{C}_\aleph(D)$. Then $Y' = M'$ for every non-abelian large subgroup Y of M, and so it follows from Lemma 2.4.2 that either M' is finite or M is nilpotent and M' is small. As D is contained in the normalizer $N_G(M')$, the subgroup M' is normal in D, and in any case we obtain that D/M' is a locally graded group of cardinality $\aleph$. Moreover, the set $\mathcal{C}_\aleph(D/M')$ has order strictly smaller than m, and so it can be assumed by induction that D/M' is soluble-by-finite and D'/M' has cardinality strictly smaller than $\aleph$. Thus D is soluble-by-finite and D' is small, so that G itself is soluble-by-finite and D/D' is an abelian normal subgroup of G/D' of cardinality $\aleph$. As the set $\mathcal{C}_\aleph(G/D')$ is obviously finite, it follows from Lemma 2.4.1 that G'/D' is finite, and hence G' has cardinality strictly smaller than $\aleph$. Suppose now that G contains an abelian large subgroup A, and let $E = \langle x_1, \dots, x_k \rangle$ be any finitely generated subgroup of G. For each $i = 1, \dots, k$ the conjugacy class of x_i in G has cardinality strictly smaller than $\aleph$, so that $|G : C_G(x_i)| < \aleph$, and in particular $|A : C_A(x_i)| < \aleph$. It follows that also the index of the centralizer

$$C_A(E) = \bigcap_{i=1}^{k} C_A(x_i)$$

in A is strictly smaller than $\aleph$, so that $C_A(E)$ is large hence by Corollary 1.1.2 it contains two large subgroups B_1 and B_2 such that $B_1 \cap B_2 = \{1\}$. Clearly, the groups EA/B_1 and EA/B_2 have finitely many derived subgroups, so that they have a finite commutator subgroup, and it follows that also E' is finite. Therefore the commutator subgroup G' of G is locally finite and every finitely generated subgroup of G satisfies the maximal condition on subgroups. Since G'/D' is finite, it is enough to show that D' is finite, whence the replacement of G by D allows to suppose that X' is normal in G for each element X of $\mathcal{L}(G)$. If X is any non-abelian large subgroup of G of cardinality, we have that X/X' is an abelian group of cardinality $\aleph$, and the set $\mathcal{C}_\aleph(G/X')$ has less than m non-trivial elements, so that by induction on m it can be assumed that G'/X' is finite. Thus each non-abelian large subgroup of G has finite index in its normal closure. Let u and v be elements of G such that $uv \neq vu$, and put $U = \langle u, v \rangle$. As above, the index $|A : C_A(U)| < \aleph$, so that the centralizer $C_A(U)$ is large and Corollary 1.1.1 and Corollary 1.1.2 show that it contains two large subgroups C_1 and C_2 such that

$$C_1 \cap C_2 = C_1C_2 \cap U = \{1\}.$$

Then $U = UC_1 \cap UC_2$, and hence U has finite index in its normal closure U^G. Moreover, each large subgroup of G/U^G has finite index in its normal closure, and the group G/U^G contains an abelian large subgroup, so that Theorem 3.6 of [12] yields that G/U^G has a finite commutator subgroup. Therefore the index $|G' : G' \cap U|$ is finite, so that G' is finitely generated and hence even finite. $\square$

Let us denote by $I_\aleph(G)$ the intersection of all members of the set $\mathcal{C}_\aleph(G)$.

Theorem 2.4.1. *Let G be an uncountable locally graded group of regular cardinality $\aleph$ which has no large simple sections. Then the set $\mathcal{C}_\aleph(G)$ is finite if* *and only if the subgroup $I_\aleph(G)$ has finite index in G'. Moreover, in this case, either $I_\aleph(G)$ is an abelian group of prime exponent or it is divisible abelian and $I_\aleph(G) = M'$ for a suitable large subgroup M of G.*

Proof. If the index $|G' : I_\aleph(G)|$ is finite, it is clear that the set $\mathcal{C}_\aleph(G)$ is finite. Conversely, suppose that $\mathcal{C}_\aleph(G)$ is finite. In order to prove that $I_\aleph(G)$ has finite index in G', we may obviously assume that G' is infinite, so it follows from Lemma 2.4.3 that G has no abelian large subgroups. As the normal subgroup $D = D_\aleph(G)$ is large, also the normal subgroup D/D' of G/D' has cardinality $\aleph$, and hence G'/D' is finite by Lemma 2.4.1. Let X be any large subgroup of G, and put $Y = X \cap D$. Clearly, Y has cardinality $\aleph$, so that Y' is normalized by D and $Y/Y'$is a large abelian subgroup of D/Y'. Thus D'/Y' is finite by Lemma 2.4.3, and hence also the index $|G' : X'|$ is finite. As the set $\mathcal{C}_\aleph(G)$ is finite, it follows that the subgroup $I_\aleph(G)$ has finite index in G'. Let M be a large subgroup of G such that M' is a minimal element of the set $\mathcal{C}_\aleph(G)$. Then $U' = M'$ for each large subgroup U of M, and hence M' is either abelian of prime exponent or divisible abelian by Theorem 2.1.4. In the first case, we obtain of course that also $I_\aleph(G)$ is abelian of prime exponent. On the other hand, if M' is divisible abelian, then $I_\aleph(G) = M'$ because the index $|M' : I_\aleph(G)|$ is finite. The statement is proved. $\square$

Next lemma is an easy consequence of [[56], Theorem 3.2.5], however, for the aims of this section it is enough to give a direct proof of it, without using the structure of C-groups (i.e., groups where each subgroup admits a permutable complement).

Lemma 2.4.4. *Let p be a positive prime number. If G is an elementary abelian p-group, for each subgroup H of G there exists a subgroup K of G such that $G = H \times K$.*

Proof. Let us consider the non-empty set $\mathcal{L} = \{K \leq G \,|\, K \cap H = \{1\}\}$ ordered by inclusion. It is clear that $(\mathcal{L}, \subseteq)$ is inductive, so that Zorn's lemma ensures the existence of a maximal element K in $\mathcal{L}$. Assume by contradiction that $H \times K$ is a proper subgroup of G and pick an element $a \in G \setminus (H \times K)$. Since a has order p, $\langle H \times K, a\rangle = (H \times K) \times \langle a \rangle = H \times$

$(K \times \langle a \rangle)$. In particular, $H \cap (K \times \langle a \rangle) = \{1\}$, contradicting the maximality of K. Hence $G = H \times K$. □

Proposition 2.4.1. *Let G be an uncountable group of regular cardinality $\aleph$ having no simple sections of cardinality $\aleph$.*

(1) If G has finitely many commutator subgroups of large subgroups, then G is finite-by-$C^1_\aleph$.

(2) If G is finite-by-$C^1_\aleph$, then all large subgroups are boundedly nearly normal.

Proof.

(1) Since $G'/I_\aleph(G)$ is finite by Theorem 2.4.1, for every large subgroup H we have that $|G' : H'| \leq |G'/I_\aleph(G)|$, and hence

$$|H^G : H| \leq |HG' : H| = |G' : G' \cap H| \leq |G' : H'| \leq |G'/I_\aleph(G)|;$$

therefore we can assume that G is nilpotent of class 2 and G' has prime exponent (see Theorem 2.2.4). Lemma 2.4.4 guarantees that there exists a finite subgroup E of G such that $G' = I_\aleph(G) \times E$; clearly G/E is a $C^1_\aleph$-group.

(2) Let N be a finite normal subgroup such that G/N is a $C^1_\aleph$-group; for every large subgroup H we have that $H'N = G'N$, so that

$$|G' : H'| \leq |H'N : H'| \leq |N|,$$

and hence, by the same argument as above, we have $|H^G : H| \leq |N|$.

□

Lemma 2.4.5. *Let G be an uncountable group of regular cardinality $\aleph$ which has no Jónsson large subgroups. If all large subgroups of G are normal, the section $G/I_\aleph(G)$ contains a large abelian subgroup.*

Proof. If G is a Dedekind group, it contains an abelian subgroup of finite index, so the statement is obvious. In any case, by Theorem 2.1.3 G contains a $C^1_\aleph$-subgroup of small index, say M, and for each large subgroup

H of G, $H \cap M$ is a large subgroup of M, so that $M' = (H \cap M)' \leq H'$. It follows that $I_{\aleph}(G) = M'$ and $M/I_{\aleph}(G)$ is a large abelian subgroup of $G/I_{\aleph}(G)$. $\square$

This lemma, together with Theorem 2.1.1, ensures that for groups of cardinality $\aleph$ with no Jónsson large subgroups, $G/I_{\aleph}(G)$ is a Dedekind group whenever all large subgroups of G are normal.
It follows that $G'/I_{\aleph}(G)$ is finite and G has finitely many commutator subgroups of large subgroups by Theorem 2.4.1. Actually, in order to have finitely many commutator subgroups of large subgroups it suffices that all large subgroups are boundedly almost normal or, equivalently, that there are finitely many normalizers of large subgroups.

Taking into account that groups with finitely many normalizers of subgroups are central-by-finite and that (locally graded) groups with finitely many commutator subgroups are finite-by-abelian, the following theorem can be regarded as a sort of translation of Schur's theorem.

Theorem 2.4.2. *Let G be an uncountable group of regular cardinality $\aleph$ having no large simple sections. If G contains only finitely many normalizers of large subgroups, then G contains only finitely many commutator subgroups of large subgroups.*

Proof. It follows from Theorem 2.3.4 that G contains a normal subgroup N of finite index whose large subgroups are normal in G. Then $N/I_{\aleph}(N)$ contains a large abelian subgroup by Lemma 2.4.5, so that it follows from Proposition 2.3.2 that $G/I_{\aleph}(N)$ is central-by-finite; in particular, $G'/I_{\aleph}(N)$ is finite.

Finally, if H is any large subgroup of G, then $H \cap N$ is large, so that $(H \cap N)' \geq I_{\aleph}(N)$ and so $I_{\aleph}(N) \leq H' \leq G'$. $\square$

2.5 (m,k)-subnormality

If G is a group, H is a subgroup of G and (m,k) is a pair of nonnegative integers, H is said to be (m,k)-subnormal in G if there is a subgroup H_0 containing H such that $|H_0 : H| \leq m$ and H_0 is subnormal in G with defect at most k. The pairs (m,k) are ordered lexicographically; with this ordering, if H is (m,k)-subnormal for some pair (m,k), then the least such pair is called the *near defect* of H in G (see [37]).

Lennox ([37], [38]) called *almost subnormal* every (m,k)-subnormal subgroup for each pair of nonnegative integers (m,k). Clearly, a subgroup H is $(m,1)$-subnormal in G if and only if it has index at most m in its normal closure H^G, while H is $(1,k)$-subnormal if and only if it is subnormal of defect at most k. It follows from theorems of Neumann [46] and Macdonald [41] that if a group admits only $(m,1)$-subnormal subgroups for a fixed positive integer m, then its commutator subgroup is finite of order bounded by a function of m. On the other hand, the well-known theorem of Roseblade [53] asserts that there exists a function f such that if G is a group in which every subgroup is $(1,k)$-subnormal for a fixed positive integer k, then G is nilpotent of class bounded by $f(k)$. Lennox [37] generalized Roseblade's theorem, proving that there exists a function μ such that if every finitely generated subgroup of a group G is (m,k)-subnormal in G, then the term $\gamma_{\mu(m+k)}(G)$ of the lower central series of G is finite of order less than or equal to $m!$. An analogous result was proven in [24] where locally soluble groups of infinite rank whose infinite rank subgroups are all (m,k)-subnormal for some m and k are studied. In this section, we will consider uncountable groups whose large subgroups are (m,k)-subnormal for a fixed pair (m,k). For the convenience of the reader, for each pair of nonnegative integers (m,k), we define the group class $\mathfrak{X}(m,k)$ to be the class of all uncountable groups of regular cardinality $\aleph$ (and the trivial groups) whose large subgroups are (m,k)-subnormal in G. We begin this section by showing that the Baer radical of an $\mathfrak{X}(m,k)$-group has all large subgroups subnormal with bounded defect.

Lemma 2.5.1. *Let G be a locally nilpotent group. Then every (m,k)-subnormal subgroup of G is subnormal in G of defect at most $m+k$.*

Proof. Let H be an (m,k)-subnormal subgroup of G, so that there exists a subnormal subgroup X of G of defect at most k such that $|X : H| \leq m$. If $N = H_X$ is the normal core of H in X, then X/N is a finite nilpotent group, so H/N is subnormal in X/N with defect at most m. Hence, H is a subnormal subgroup of G of defect at most $m+k$. $\square$

Corollary 2.5.1. *Let G be a locally nilpotent $\mathfrak{X}(m,k)$-group. If G contains a large abelian subgroup, then G is nilpotent of class at most $f(m+k)$.*

Proof. The assertion is a consequence of Lemma 2.5.1 and of Theorem 2.1.1. $\square$

Corollary 2.5.2. *Let G be an $\mathfrak{X}(m,k)$-group containing a large abelian subgroup. Then the Baer radical of G is large and nilpotent of class at most $f(m+k)$.*

Proof. Let A be a large abelian subgroup; then A is (m,k)-subnormal in G, so that there exists a subnormal subgroup C of defect at most k such that $|C:A| \leq m$. It follows that A_C, the normal core of A in C, is a large abelian subnormal subgroup of G. Thus, B is a locally nilpotent $\mathfrak{X}(m,k)$-group containing a large abelian subgroup. The statement is now a consequence of Corollary 2.5.1. $\square$

Proposition 2.5.1. *Let G be a group of cardinality $\aleph$. If there exists a positive integer r such that $\gamma_r(G)$ is small, then for every finitely generated subgroup F of G, the centralizer $C_G(F)$ is a large subgroup of G.*

Proof. Let s be the least positive integer such that $\gamma_{s+1}(G)$ is small, and let g be any element of G; if we put

$$g^{\gamma_s(G)} = \{g^h | h \in \gamma_s(G)\},$$

the function

$$g^h \in g^{\gamma_s(G)} \longrightarrow [g,h] \in [g,\gamma_s(G)] \leq \gamma_{s+1}(G)$$

is a bijection and, therefore, the set $g^{\gamma_s(G)}$ has cardinality strictly smaller than $\aleph$. However,

$$|g^{\gamma_s(G)}| = |\gamma_s(G) : C_{\gamma_s(G)}(g)|$$

where $C_{\gamma_s(G)}(g)$ is the centralizer in $\gamma_s(G)$ of g.

Put $F = \langle g_1, \ldots, g_t \rangle$; then $|\gamma_s(G) : C_{\gamma_s(G)}(g_i)|$ is strictly smaller than $\aleph$, for each $i \in \{1, \ldots, t\}$. Moreover

$$C_G(F) = \bigcap_{i=1}^{t} C_G(g_i),$$

so that $|\gamma_s(G) : C_{\gamma_s(G)}(F)|$ is strictly smaller than $\aleph$, and hence $C_G(F)$ is a large subgroup of G. $\square$

R.E. Phillips in [48] introduced the following generalization of subnormality for infinite groups: a subgroup H of a group G is called f-subnormal in G if there exists an f-series from H to G, that is a finite series

$$H = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

such that either H_i is normal in H_{i+1} or $|H_{i+1} : H_i| < \infty$ for every $i \in \{0, \ldots, n-1\}$.

Clearly, every (m,k)-subnormal subgroup is f-subnormal.

Corollary 2.5.3. *Let G be an $\mathfrak{X}(m,k)$-group. If there exists a positive integer r such that $\gamma_r(G)$ is small, then every finitely generated subgroup F of G is f-subnormal in G.*

Proof. It follows from Proposition 2.5.1 that $C_G(F)F$ is an (m,k)-subnormal subgroup of G in which F is normal. $\square$

Lemma 2.5.2. *Let G be an $\mathfrak{X}(m,k)$-group. If G is abelian-by-cyclic, then it is finite-by-nilpotent and $|\gamma_{\mu(m+k)}(G)| \leq (m!)^2$.*

Proof. We can write $G = A\langle x \rangle$, where A is a normal abelian subgroup of G and x is an element of G. Of course, A is large and it can be regarded as a module over the countable ring $R = \mathbb{Z}\langle x \rangle$. Moreover, the R-submodules of A are all its G-invariant subgroups. By Theorem 1.1.1, A contains the direct product

$$\mathop{\mathrm{Dr}}_{i \in I} M_i,$$

where I is a set of cardinality $\aleph$ and, for each $i \in I$, M_i is a G-invariant subgroup of A. It follows that there exist two subsets I_1 and I_2 of I, each of cardinality $\aleph$, such that

$$I_1 \cap I_2 = \emptyset.$$

For $j = 1, 2$, put

$$B_j = \mathop{\mathrm{Dr}}_{i \in I_j} M_i,$$

then B_1 and B_2 are G-invariant subgroups of A of cardinality $\aleph$ whose intersection is trivial. Thus, [[37], Theorem A] ensures that, for $j = 1, 2$,

$$|\gamma_{\mu(m+k)}(G/B_j)| \leq m!.$$

Therefore, $|\gamma_{\mu(m+k)}(G)| \leq (m!)^2$ and G is finite-by-nilpotent. $\square$

Lemma 2.5.3. *Let G be an $\mathfrak{X}(m,k)$-group containing a large abelian normal subgroup.*

Then G is locally finite-by-nilpotent, and if B is the Baer radical of G, then G/B has finite exponent.

Proof. Let x be any element of G, and let A be a large abelian normal subgroup of G; the subgroup $\langle A, x\rangle = A\langle x\rangle$ is an abelian-by-cyclic $\mathfrak{X}(m,k)$-group so, by Lemma 2.5.2, $|\gamma_{\mu(m+k)}(A\langle x\rangle)| \leq (m!)^2$. Therefore, $\langle x\rangle$ has index at most $(m!)^2$ in $\langle x\rangle\gamma_{\mu(m+k)}(A\langle x\rangle)$. Since $A\langle x\rangle/\gamma_{\mu(m+k)}(A\langle x\rangle)$ is nilpotent of class at most $\mu(m+k)-1$, the subgroup $\langle x\rangle\gamma_{\mu(m+k)}(A\langle x\rangle)$ is subnormal in $A\langle x\rangle$ with defect at most $\mu(m+k)-1$. Finally, $A\langle x\rangle$ is (m,k)-subnormal in G and so it has index at most m in $(A\langle x\rangle)^{G,k}$ which is subnormal in G with defect at most k.

By an application of [[37], Theorem B], G is locally finite-by-nilpotent.

Let X be a cyclic subgroup of G. By the above argument, there exists a chain

$$X = X_n \leq \cdots \leq X_0 = G$$

such that either X_{i+1} is subnormal in X_i with defect at most $\mu(m+k)-1$ or X_{i+1} has index at most $(m!)^2$ in X_i; we shall prove by induction on the length of the chain that for every cyclic subgroup X of G, there is a positive integer s dividing $\beta(m) = ((m!)^2)!$ such that X^s is subnormal in G.

We can assume that X^s is subnormal in X_1, for some positive integer s dividing $\beta(m)$. If X_1 is subnormal in G, then X^s is subnormal in G. So we can assume that the index $|G : X_1| \leq (m!)^2$; then the core Y of X_1 in G has index at most $((m!)^2)! = \beta(m)$ in G. Hence, $X^{\beta(m)}$ is a normal subgroup of $Y \cap X^s$, which is subnormal in Y. It follows that $X^{\beta(m)}$ is subnormal in G, and hence $X^{\beta(m)} \leq B$.

Therefore, G/B has finite exponent. $\square$

The following result depends on [[2], Theorem 1′]. Recalling that an abelian normal subgroup A of a group G can be regarded as a module over the group ring $\mathbb{Z}(G/C_G(A))$, we obtain the following interesting lemma.

Lemma 2.5.4. *Let G be a group containing an uncountable abelian normal subgroup A. If $G/C_G(A)$ is finite, then A contains a subgroup of the form*

$$\underset{i\in I}{\mathrm{Dr}}\langle a_i\rangle^G,$$

where the set I has the same cardinality as A.

Lemma 2.5.5. *Let G be an $\mathfrak{X}(m,k)$-group. If G contains a large nilpotent normal subgroup N such that G/N is locally finite, then $\gamma_{\mu(m^2+k)}(G)$ is small.*

Proof. Since N is nilpotent of cardinality $\aleph$, there exists a nonnegative integer r such that $|Z_r(N)| < \aleph$ and $|Z_{r+1}(N)| = \aleph$. So, the section $\overline{A} = Z_{r+1}(N)/Z_r(N)$ is an abelian normal subgroup of $\overline{G} = G/Z_r(N)$. Let $\overline{E}$ be any finitely generated subgroup of $\overline{G}$ and put $\overline{L} = \overline{E}\cdot\overline{A}$. Since $\overline{N}\cap\overline{L} \leq C_{\overline{L}}(\overline{A})$, then $\overline{L}/C_{\overline{L}}(\overline{A})$ is finite, so that Lemma 2.5.4 ensures that $\overline{A}$ contains a subgroup $\overline{H}$ of the form

$$\overline{H} = \underset{i\in I}{\mathrm{Dr}}\overline{A}_i$$

where I is a set of cardinality $\aleph$ and, for each $i \in I$, $\overline{A}_i$ is a nontrivial normal subgroup of $\overline{L}$. So we can find two $\overline{L}$-invariant subgroups $\overline{H}_1$ and $\overline{H}_2$ of $\overline{H}$, both of cardinality $\aleph$ and such that

$$\overline{H}_1\cap\overline{H}_2 = \overline{E}\cap\overline{H}_1\cdot\overline{H}_2 = \{1\}.$$

Therefore,

$$\overline{E} = \overline{E}\cdot\overline{H}_1\cap\overline{E}\cdot\overline{H}_2$$

is the intersection of two (m,k)-subnormal subgroups of $\overline{G}$ and, by [[24], Lemma 2.5], $\overline{E}$ is (m^2,k)-subnormal. An application of [[37], Theorem A] completes the proof. □

Now we are able to state and prove the first main theorem of this section.

Theorem 2.5.1. *Let G be an $\mathfrak{X}(m,k)$-group. If G contains a large abelian normal subgroup, then G is finite-by-nilpotent and $|\gamma_{\mu(m^2+k)}(G)| \leq (m^2)!$.*

Proof. Let B be the Baer radical of G; it follows from Corollary 2.5.2 and Lemma 2.5.3 that B is nilpotent and G/B is a locally finite group, so that $\gamma_{\mu(m+k)}(G)$ is small by Lemma 2.5.5. Let F be any finitely generated subgroup of G. Applying Proposition 2.5.1 to the group AF, we can say that $C_{AF}(F)$ is large, so that also $C_{AF}(F) \cap A$ is a large subgroup. It follows that $Z(AF)$ is large, and hence it contains two large subgroups U_1 and U_2 such that

$$U_1 \cap U_2 = F \cap U_1U_2 = \{1\}.$$

Therefore,

$$F = FU_1 \cap FU_2$$

is the intersection of two (m,k)-subnormal subgroups of G and, by [[24], Lemma 2.5], F is (m^2,k)-subnormal. The statement follows from [[37], Theorem A]. $\square$

The last part of this section is devoted to its second main theorem concerning $\mathfrak{X}(m,k)$-groups which contain a large abelian subgroup not necessarily normal.

Lemma 2.5.6. *Let G be an $\mathfrak{X}(m,k)$-group. If G contains a large abelian subgroup, then the Baer radical of G is a finite-index subgroup.*

Proof. Let B be the Baer radical of G. By Corollary 2.5.2, B is a large nilpotent subgroup of G and, by Robinson's theorem, B/B' has cardinality $\aleph$. Since G/B' is an $\mathfrak{X}(m,k)$-group, as a consequence of Theorem 2.5.1, we have that

$$|\gamma_{\mu(m^2+k)}(G/B')| \leq (m^2)!.$$

By Hall's theorem, $|G/B' : Z_{2\mu(m^2+k)}(G/B')|$ is finite so, with

$$Z/B' = Z_{2\mu(m^2+k)}(G/B'),$$

also $|G : ZB|$ is finite. Clearly, ZB/B' is nilpotent by Fitting's theorem, and so the nilpotency of the normal subgroup B of ZB ensures that ZB is a nilpotent normal subgroup of G. It follows that $Z \leq B$ and the statement is proved. $\square$

Theorem 2.5.2. *Let G be an $\mathfrak{X}(m,k)$-group. If G contains a large abelian subgroup A, then G is finite-by-nilpotent and $\gamma_{\mu(m^2+k)}(G)$ is small.*

Proof. If B is the Baer radical of G, then B is a large and nilpotent subgroup of G by Corollary 2.5.2. Since G/B is finite by Lemma 2.5.6, as a consequence of Lemma 2.5.5, we have that $\gamma_{\mu(m^2+k)}(G)$ is small.

In addition, Lemma 2.5.6 ensures the existence of a finitely generated subgroup F such that $G = FB$. By Corollary 2.5.3, F is f-subnormal in G, so that there exists an f-series of the form

$$F = F_0 < F_1 < \cdots < F_n = G,$$

where either $|F_{i+1} : F_i|$ is finite or $F_i \trianglelefteq F_{i+1}$, for each $i = 0, \ldots, n-1$.

We will prove that G is finite-by-nilpotent by induction on the length n of the f-series above. If $n = 1$, since G is an uncountable group, F is normal in G. On the other hand, [[3], Theorem 1.1] tells us that F is also finite-by-nilpotent so we can find a positive integer r such that the G-invariant subgroup $\gamma_r(F)$ is finite, and

$$G/\gamma_r(F) = BF/\gamma_r(F) = (F/\gamma_r(F)) \cdot (B\gamma_r(F)/\gamma_r(F))$$

is nilpotent by Fitting's theorem.

Assume now that $n > 1$. Then $|F_{n-1} : F_{n-1} \cap B|$ is finite and $B \cap F_{n-1}$ is contained in the Baer radical of F_{n-1}, so we can apply the inductive hypothesis to F_{n-1} which has to be a finite-by-nilpotent group. It follows that there exists a finite characteristic subgroup M of F_{n-1} such that F_{n-1}/M is nilpotent (see [[3], Lemma 2.9]). If F_{n-1} is normal in G, also M is G-invariant, and

$$G/M = (F_{n-1}/M) \cdot (BM/M)$$

is nilpotent by Fitting's theorem.

Assume that the index $|G : F_{n-1}|$ is finite; then M has finitely many conjugates in G, so that M^G is finite by Dietzmann's lemma.

On the other hand,

$$G/M^G = (F_{n-1}M^G/M^G) \cdot (BM^G/M^G)$$

is finite-by-(locally nilpotent) [[3], Lemma 2.7], so that G itself is finite-by-(locally nilpotent), and hence actually finite-by-nilpotent by Corollary 2.5.1. $\square$

Theorem 2.5.3. *Let G be a residually small $\mathfrak{X}(m,k)$-group. Then G is finite-by-nilpotent and $|\gamma_{\mu(m+k)}(G)| \leq m!$.*

Proof. Let N be any normal subgroup of G of index less than $\aleph$. Theorem A of [37] ensures that

$$|\gamma_{\mu(m+k)}(G)N/N| \leq m!.$$

Assume by contradiction that $\gamma_{\mu(m+k)}(G)$ contains more than $s = m!$ elements, say $x_0, \ldots, x_s$. For each $i, j \in \{0, \ldots, s\}$ and $i \neq j$, there exists a normal subgroup $N_{i,j}$ of G with small index such that $x_i^{-1}x_j \notin N_{i,j}$. Consider

$$N = \bigcap_{\{i,j \,|\, i \neq j\}} N_{i,j},$$

we have that N has small index and $x_i^{-1}x_j \notin N$ for every $i, j \in \{0, \ldots, s\}$ and $i \neq j$, so that

$$|\gamma_{\mu(m+k)}(G/N)| > s = m!,$$

a contradiction. $\square$

Chapter 3

Permutably complemented Large Subgroups

A subgroup H of a group G is *complemented* if it admits a complement, that is, a subgroup K of G such that $G = \langle H, K \rangle$ and $H \cap K = \{1\}$. If H is a complemented subgroup of a group G and there exists a complement K to H such that $HK = KH$, then H is said to be *permutably complemented*. A group G is called a C-group if all its subgroups are permutably complemented. The structure of C-groups has been completely described. Indeed, Theorem 3.2.5 of [56] states that a C-group is the semidirect product

$$\left(\underset{i\in I}{\mathrm{Dr}} A_i\right) \rtimes \left(\underset{j\in J}{\mathrm{Dr}} B_j\right),$$

where all factors A_i and B_j are finite of prime order and, for each $i \in I$, the group A_i is G-invariant. In particular, an abelian C-group is the direct product of elementary abelian groups.

Many papers address groups that admit a particular subset of permutably complemented subgroups (see, for instance, [25] and [15]), while the survey [29] by Ferrara and Trombetti provides an overview of these results, often generalizing them. In addition, in [29], the authors analyze infinite groups whose infinite subgroups are permutably complemented as well as periodic locally soluble groups of infinite rank, in which all subgroups of infinite rank are permutably complemented. Although it remains unclear whether these infinite rank groups are always C-groups, Ferrara and Trombetti proved that when the commutator subgroup has infinite rank, all subgroups are permutably complemented (see Theorem 7.1).

In this section, we will explore the behaviour of uncountable groups of regular cardinality $\aleph$ in which all subgroups of cardinality $\aleph$ (or a subset of them) are permutably complemented. For the convenience of the reader, the class of all groups of cardinality $\aleph$ whose large subgroups are permutably complemented (and the trivial groups) will be denoted with $\mathcal{C}(\aleph)$.

3.1 Large Subgroups

The first part of this section is devoted to the study of $\mathcal{C}(\aleph)$-groups under the specific condition of containing a large abelian normal subgroup. Here, the structure of K-groups, i.e., groups in which every subgroup is complemented, plays a key role. Most of the results regarding this class of groups can be found in [56].

Subsequently, we prove a result for uncountable groups that is analogous to the one presented in [29] and that we cited above. An essential step towards this objective will consist of examining the case of $\mathcal{C}(\aleph)$-groups with small commutator subgroup.

In the last part of the section, $\mathcal{C}(\aleph)$-groups with a large Hirsch-Plotkin radical are considered. In particular, we will show that such groups are actually C-groups since they have all subgroups permutably complemented.

Lemma 3.1.1. *Let G be an abelian $\mathcal{C}(\aleph)$-group. Then G is a C-group.*

Proof. Since G contains the direct product of large subgroups $A_1 \times A_2$, it is isomorphic to a subgroup of $G/A_1 \times G/A_2$. The statement follows from Lemma 3.2.1. of [56]. $\square$

Lemma 3.1.2. *Let G be a group of cardinality $\aleph$ and let A be a large abelian normal subgroup of G whose large subgroups admit a permutable complement in G. Then A is generated by cyclic G-invariant groups of prime order.*

Proof. Since each large subgroup of A is permutably complemented in G, it has a complement in A so that Lemma 3.1.1 ensures that A is the direct product of elementary abelian groups. Let H be a subgroup of A which is maximal with respect to being generated by G-invariant cyclic subgroups. Suppose that $H < A$. Then there is a proper subgroup U of A containing H and having prime index in A. If V is a permutable complement of U in G, then $V \cap A$ is a cyclic G-invariant subgroup of prime order, so that $H \times (V \cap A)$ is generated by G-invariant cyclic subgroups, contradicting the maximality of H. So A is generated by cyclic G-invariant subgroups. If D is a cyclic G-invariant subgroup of A, we have that D is a direct product of G-invariant subgroups of prime order, so we have the statement. $\square$

Now we are in the position to prove that any $\mathcal{C}(\aleph)$-group is actually a C-group, provided that it contains a large abelian normal subgroup.

Lemma 3.1.3. *Let G be a $\mathcal{C}(\aleph)$-group. If G contains a large abelian normal subgroup, then G is a C-group.*

Proof. Let A be a large abelian G-invariant subgroup. It is clear that G/A is a C-group. Moreover, Lemma 2.5.2 yields that A is generated by G-invariant cyclic subgroups of prime order. It follows from Lemma 3.1.9 of [56] that G is a K-group. Since G/A is hypercyclic by Theorem 3.2.5 of [56], the same result ensures that G is a C-group. $\square$

A group is said to be a (Hall) FK-group if every finite subgroup is (permutably) complemented. In [15], these types of groups have been thoroughly studied and the same paper provides several characterizations of periodic Hall FK-groups (cf. Theorem 3.4 and Theorem 3.5). In the following lemma, it is proved that a $\mathcal{C}(\aleph)$-group whose commutator subgroup is small is a periodic Hall FK-group and therefore it is residually a finite C-group (see Theorem 3.5 of [15]).

Proposition 3.1.1. *Let G be a $\mathcal{C}(\aleph)$-group with small commutator subgroup G'. Then,*

(a) G is locally a C-group. In particular, G is a periodic Hall FK-group;

(b) G is residually finite;

(c) *all locally nilpotent subgroups of G are (abelian) C-groups. Specifically, the Fitting radical $F(G)$ of G is abelian;*

(d) *all subgroups of $Z(G)$ are (permutably) complemented in G.*

Proof. Since G' is small, we have that for every $x \in G$, $|G : C_G(x)| < \aleph$. Let $E = \langle x_1, x_2, \dots, x_n \rangle$ be a finitely generated subgroup of G, then

$$C_G(E) = \bigcap_{i=1}^{n} C_G(x_i)$$

and $|G : C_G(E)| < \aleph$. Furthermore, G/G' contains a subgroup of the form $A_1/G' \times A_2/G'$ where each A_i is a G-invariant subgroup of cardinality $\aleph$. If D is a permutable complement to A_1, for instance, then D contains a large abelian subgroup B of G and $|B : C_B(E)| < \aleph$. It follows that $C_B(E)$ admits a large subgroup U such that $U \cap E = \{1\}$ and UE/U is isomorphic to E. Thus, E is a C-group and G is a locally finite metabelian group. Corollary 5.12 of [29] guarantees that all finite subgroups of G are permutably complemented and G is residually finite, so that (a) and (b) are proved. Reminding that, for any positive prime p, a p-group with all subgroups permutably complemented is abelian, we have (c). Finally, (d) stems from (a) and Theorem 5.28 of [29]. □

Theorem 3.1.1. *Let G be a $\mathcal{C}(\aleph)$-group with large commutator subgroup G'. Then, if G'' has no large simple sections, G is a C-group.*

Proof. Assume that G'' is large. We have that $G/G^{(3)}$ is a C-group, in particular it is metabelian and G'' is perfect. Let H be a large normal subgroup of G'', G''/H is metabelian and so $G'' = H$. It follows that G'' has no large proper normal subgroups. Let us consider the set $\mathcal{L}$ of all proper normal subgroups of G'' ordered by inclusion and let $\mathcal{E}$ be a totally ordered part of $\mathcal{L}$. Put

$$\overline{N} = \bigcup_{N \in \mathcal{E}} N.$$

Since, $\overline{N}$ is G''-invariant, we have that either $\overline{N} = G''$ or $\overline{N}$ is small. If $\overline{N} = G''$, for every $x \in G''$, there exists $N_x \in \mathcal{E}$ such that $x \in N_x$. It follows that $\langle x\rangle^{G''}$ is small an so also the index $|G'' : G_{G''}(x)|$ is strictly smaller than $\aleph$. Thus, for every finitely generated subgroup E of G'', a similar argument to the one used in Proposition 3.1.1 ensures that $|G'' : C_{G''}(E)| < \aleph$. Furthermore, the C-group $C_{G''}(E)E/C_{G''}(E)$ is isomorphic to $E/Z(E)$, so that $E^{(3)}$ is trivial and G is soluble of derived length at most 3, although G'' is perfect. Such a contradiction tells us that $\overline{N}$ is a small subgroup and the ordered set $\mathcal{L}$ is inductive. Now, if M is a maximal element of $\mathcal{L}$, then G''/M is a large simple section of G, contradicting the assumption on G''. As a consequence, we have that G'' must be small. By [Proposition 3.1.1, (a)], G is locally finite. Let F be a finite subgroup of G, and let us consider $\overline{F} = G'F$ in which G' is a finite index subgroup with small commutator subgroup. It follows from [Proposition 3.1.1, (b)] that $\overline{F}$ is residually finite, so that $\overline{F}''$ is trivial and, in particular, F' is abelian. Thus, G' is abelian and so G is a C-group by Lemma 3.1.3. □

Reminding that in a locally soluble group all simple sections are finite of prime order, Theorem 3.1.1 has the following consequences.

Corollary 3.1.1. *Let G be a locally soluble $\mathcal{C}(\aleph)$-group with large commutator subgroup G'. Then, G is a C-group.*

Corollary 3.1.2. *Let G be a locally nilpotent $\mathcal{C}(\aleph)$-group. Then G is an abelian C-group.*

Proof. Theorem 3.1.1 tells us that G' must be small so that we can apply [Proposition 3.1.1, (c)] yielding that G is an abelian C-group. □

Theorem 3.1.2. *Let G be a $\mathcal{C}(\aleph)$-group with large Hirsch-Plotkin radical.*

Then G is a C-group. In particular, in a $\mathcal{C}(\aleph)$-group containing a large ascendant finite-by-(locally nilpotent) subgroup, all subgroups are permutably complemented.

Proof. The first part of the statement is a direct consequence of Corollary 3.1.2 and Lemma 3.1.3. Now, if N is a large finite-by-(locally nilpotent) ascendant subgroup of G, actually it is finite-by-abelian by Corollary 3.1.2, so N' is finite. By Hall's theorem, also the index $|N : Z_2(N)|$ is finite, so that

Corollary 3.1.2 ensures that $Z_2(N)$ is a large abelian ascendant subgroup of G. Thus, the Hirsch-Plotkin radical of G is large. $\square$

Corollary 3.1.3. *Let G be a $\mathcal{C}(\aleph)$-group with finite commutator subgroup G', then G is a C-group.*

Recall that a group G is called *locally normal* if every finitely generated subgroup is contained in a finite normal subgroup.

Proposition 3.1.2. *Let G be a $\mathcal{C}(\aleph)$-group. If G is an FC-group, then G is a C-group.*

Proof. Let T be the torsion subgroup of G, we have that G/T is torsion-free and abelian, so that G/T has to be both torsion-free and a C-group. It follows that G is periodic, in particular it is locally normal. Assume first that G' is small, thus G is residually a C-group by [Proposition 3.1.1, (a)], so that Theorem 2.1 of [29] ensures that G is a C-group. Let G' be large. If G'' is large, we have that it is also perfect by a similar argument to the one used in Theorem 3.1.1 and, for each finite normal subgroup N of G, $G'' = C_G(N)''$. It follows that G'' is a central subgroup, which is a contradiction. So G'' has to be small and the statement stems from an application of Theorem 3.1.1. $\square$

3.2 Subnormal Large Subgroups

In the last part of the present section, it will turn out that, as for soluble groups with all subgroups permutably complemented (see, for instance, [56], Theorem 3.2.6, due to Kochendörfer), to prove that a soluble group G of cardinality $\aleph$ admitting a large abelian normal subgroup is a C-group, it is not necessary to require that every large subgroup of G is permutably complemented; it suffices to consider all large subnormal subgroups of defect at most 2.

Theorem 3.2.1. *Let G be an uncountable soluble group of regular cardinality $\aleph$ such that every large subnormal subgroup of defect at most 2 is permutably complemented. If G contains a large abelian normal subgroup, then it is a C-group.*

Proof. Let A be a large abelian G-invariant subgroup. It is clear that G/A is a C-group. Since each large subgroup of A is subnormal in G with defect at most 2, Lemma 3.1.2 yields that A is generated by G-invariant cyclic subgroups of prime order. It follows from Lemma 3.1.9 of [56] that G is a K-group. Since G/A is hypercyclic by Theorem 3.2.5 of [56], the same result ensures that G is a C-group. $\square$

In the universe of nilpotent groups, we will show that in order to prove that an uncountable group of cardinality $\aleph$ is a C-group, it is enough to look at large normal subgroups, even without the assumption of containing a large abelian normal subgroup.

Theorem 3.2.2. *Let G be an uncountable nilpotent group of regular cardinality $\aleph$ such that every large normal subgroup is complemented. Then G is a C-group. In particular, G is abelian.*

Proof. Assuming first that the centre $Z(G)$ is large, we have that there exists a subgroup K of G such that $G = Z(G) \rtimes K$ and K is a C-group. Of course K is normalized by $Z(G)$, so that, actually $G = Z(G) \times K$. Every large subgroup of $Z(G)$ is G-invariant so it is complemented both in G and in $Z(G)$. It follows that $Z(G)$ is a C-group by Lemma 3.1.1 and then G has all subgroups permutably complemented by [[56], Lemma 3.2.1, (b)]. Now, let $Z(G)$ be small and, by induction on the nilpotency class of G, assume that $G/Z(G)$ is a C-group. We have that G is nilpotent of class at most 2 and the commutator subgroup G' of G is small. [Proposition 3.1.1, (d)] ensures that $Z(G)$ is complemented and that it is a C-group, thus by the same argument as above, there exists a subgroup K of G such that $G = Z(G) \times K$ and K is a C-group. The statement is a consequence of [[56], Lemma 3.2.1, (b)]. $\square$

Corollary 3.2.1. *Let G be an uncountable soluble group of regular cardinality $\aleph$ such that every large subnormal subgroup of defect at most* 2 *is permutably complemented. If G contains a large nilpotent normal subgroup, then it is a C-group.*

Chapter 4

Profinite Groups with some Embedding Properties on Open Subgroups

4.1 Index of a Closed Subgroup

A profinite group is an inverse limit of finite groups or, equivalently, a topological group that is compact, Hausdorff and totally disconnected. Any finite group is profinite and infinite profinite groups can only have cardinality $2^{\aleph_\alpha}$ for some infinite cardinal number $\aleph_\alpha$. In a profinite group, open subgroups are precisely all closed subgroups of finite index. Moreover, we can always find a fundamental system of neighborhoods of the identity consisting of open normal subgroups. In particular, the set of all open normal subgroups of a profinite group forms a fundamental system of neighborhoods of the identity.

Let H be a closed normal subgroup of a profinite group G. Then G/H is also a profinite group, which implies that it is either finite or has cardinality $2^{\aleph_\alpha}$, for some infinite cardinal number $\aleph_\alpha$. The aim of this section is to show that the same restrictions hold for the index of any closed subgroup of a profinite group. An immediate consequence of the Baire Category Theorem is that such an index cannot be countably infinite.

Lemma 4.1.1. *Let G be a profinite group. Then every closed subgroup of G has either finite or uncountable index in G.*

Proof. Assume by contradiction that H is a closed subgroup of G of countably infinite index. Let T be a left transversal of H in G, we have that the set $\{gH \mid g \in T\}$ is a countably infinite covering of G. By Baire's Category Theorem at least one of the sets gH with $g \in T$ has to have non-empty interior.

On the other hand, G is a topological group and hence, by definition, the map $\mu : (x, y) \in G \times G \longrightarrow xy \in G$ is continuous. Since the identity map $id : g \in G \longrightarrow g \in G$ and the constant map $f_x : g \in G \longrightarrow x \in G$ are continuous, also the function $\psi_x : g \in G \longrightarrow (x, g) \in G \times G$ is continuous. Thus, the composition $l_x = \mu \circ \psi_x$ is continuous, and it sends g to xg. As above, it is easy to prove that $l_{x^{-1}} : g \in G \longrightarrow x^{-1}g \in G$, which is the inverse function of l_x, is continuous. Hence, l_x is a homeomorphism for any $x \in G$. It follows that if gH is an open left coset of H, then $l_{g^{-1}}(gH) = H$ is open. Hence, H has finite index in G. $\square$

Recall here that if G is a topological group, its *local weight* $w_0(G)$ is defined as the smallest cardinal of a fundamental system of open neighborhoods of 1 in G. When G is an infinite profinite group, it follows from Theorem 2.1.3 of [50] that $w_0(G)$ is the cardinal of any fundamental system of neighborhoods of 1 consisting of open subgroups. Actually, for an infinite profinite group G, Theorem 4.9 of [63] guarantees that $|G| = 2^{w_0(G)}$. Note that for a profinite group G, $w_0(G)$ is finite only if G is finite; and in that case $w_0(G) = 1$.

More generally, if H is a closed subgroup of the topological (profinite) group G, we will denote by G/H the topological (profinite) space of left cosets of H in G and we define the *local weight* of G/H to be the smallest cardinal of a fundamental system of open neighborhoods of a point of G/H. Since for any two points of the quotient space G/H, there is a homeomorphism of G/H that maps one of those points to the other, this definition is independent of the point used.

In the following, if H is a subgroup of a group G, we will say that a subgroup K of G is a *supersubgroup* of H if $H \leq K$.

Recall here that, if X is a topological space and σ is an equivalence relation on X, the open sets of the quotient space X/σ are all the subsets of X/σ whose preimage under the canonical projection map $\pi : X \longrightarrow X/\sigma$ is open in X.

If G is a profinite group and H is a closed subgroup of G, we denote by π_H the canonical projection from G to G/H. We have that a set $\mathcal{L}$ of cosets of H is open in G/H if and only if $\pi_H^{-1}(\mathcal{L})$ is an open subset of G.

Lemma 4.1.2. *Let G be a profinite group and let H be a closed subgroup of G. For every open set $\mathcal{L}$ of G/H, $\pi_H^{-1}(\mathcal{L})$ is an open set which is a union of left cosets of H. Moreover, if $\mathcal{L}$ is an open neighborhood of the left coset H, the subgroup H is contained in $\pi_H^{-1}(\mathcal{L})$.*

Proof. We have $\pi_H^{-1}(\mathcal{L}) = \{x \in G \mid xH \in \mathcal{L}\}$. In particular, for each $xH \in \mathcal{L}$, $xH \subseteq \pi_H^{-1}(\mathcal{L})$. On the other hand, for any element g in $\pi_H^{-1}(\mathcal{L})$, the coset gH is in $\mathcal{L}$ and $g \in gH \subseteq \bigcup_{xH\in\mathcal{L}} xH$. It follows that

$$\pi_H^{-1}(\mathcal{L}) = \bigcup_{xH\in\mathcal{L}} xH.$$

The last part of the statement follows as a consequence of the above equality. $\square$

If H is a closed normal subgroup of profinite group G and $\mathcal{N}$ is a fundamental system of neighborhoods of 1 consisting of open normal subgroups of G, it is easy to see that the set $\mathcal{N}^* = \{NH/H = \pi_H(NH) \mid N \in \mathcal{N}\}$ is a fundamental system of neighborhoods of H in the profinite group G/H. When H is a closed subgroup, not necessarily normal, we have the following generalization.

Proposition 4.1.1. *Let G be a profinite group and let H be a closed subgroup of G. Then, if $\mathcal{N}$ is a fundamental system of neighborhoods of 1 consisting of open normal subgroups of G, the set $\mathcal{N}^* = \{\pi_H(NH) \mid N \in \mathcal{N}\}$ is a fundamental system of neighborhoods of the left coset H in the profinite space G/H.*

Proof. Let $\mathcal{L}$ be an open subset of G/H containing the left coset H. By Lemma 4.1.2, $K = \pi_H^{-1}(\mathcal{L})$ is an open subset of G containing 1, so there exists an open normal subgroup $N \in \mathcal{N}$ such that $N \subseteq K$.

Now, consider $yh \in NH$ with $y \in N$ and $h \in H$. We have that $yh \in yH$. However, $y \in N \subseteq K$ and Lemma 4.1.2 ensures that $K = \bigcup_{xH \in \mathcal{L}} xH$; in particular, $yH \in \mathcal{L}$, so that $yh \in yH \subseteq K$. It follows that $NH \subseteq K$ and so $\pi_H(NH) \subseteq \mathcal{L}$. □

If G is a profinite group and H is a closed subgroup of G, let $\mathcal{O}_G(H)$ denote the set of all open supersubgroups of H in G.

When we take the fundamental system of neighborhoods $\mathcal{N}$ of 1 consisting of all open normal subgroups, Proposition 4.1.1 ensures that the set

$$\mathcal{N}^* = \{\pi_H(U) \mid U \in \mathcal{O}_G(H)\}$$

is a fundamental system of neighborhoods of the left coset H. Moreover, for each $U \in \mathcal{O}_G(H)$, $\pi_H^{-1}(\pi_H(U)) = U$, which is open, hence $\pi_H(U)$ is open.

Lemma 4.1.3. *Let G be an infinite profinite group and H be a closed subgroup of G of infinite index, then $w_0(G/H)$ is infinite.*

Proof. By way of contradiction, assume that there exists a finite fundamental system of open neighborhoods $\mathcal{N}_H$ of the left coset H. Let U be an open subgroup of G containing H. Since $\pi_H(U)$ is an open set in G/H, it must contain some element $\mathcal{L} \in \mathcal{N}_H$. It follows that $\pi_H^{-1}(\mathcal{L})$ is an open subset of G containing 1. Therefore, we can find an open normal subgroup N of G such that $N \subseteq \pi_H^{-1}(\mathcal{L}) \subseteq U$. This implies that there are only finitely many open supersubgroups of H.

On the other hand, H, being a closed subgroup of a profinite group, is the intersection of all its open supersubgroups. Hence, H is the intersection of finitely many open subgroups hence H itself is open, which is a contradiction since H is of infinite index. □

Lemma 4.1.4. *Let G be an infinite profinite group, then the set of all open subgroups of G has cardinality $w_0(G)$.*

Proof. It is clear that the cardinality of $w_0(G)$ is at most the cardinality of the set of all open subgroups of G. Now, by Theorem 2.1.3 of [50] we can fix a fundamental system of neighborhoods $\mathcal{N}$ of 1 having cardinality $w_0(G)$ and consisting of open subgroups. Then, for each open subgroup U of G, there exists $V \in \mathcal{N}$ such that V is a subgroup of U. However, for every $V \in \mathcal{N}$, there are only finitely many supersubgroups of the type U, since $|G : V|$ is finite. If $\mathcal{O}(G)$ is the set of all open subgroups of G, we have that

$$\mathcal{O}(G) = \bigcup_{V \in \mathcal{N}} \{U \mid U \geq V\}.$$

The statement is now a consequence of the fact that $w_0(G)$ is infinite. $\square$

An analogous result is true when we focus on the profinite space G/H of the left cosets of a (not-open) closed subgroup H of a profinite group G.

Corollary 4.1.1. *Let G be an infinite profinite group and let H be a closed subgroup of G of infinite index. Then, the set $\mathcal{O}_G(H)$ has cardinality $w_0(G/H)$.*

Proof. Firstly, observe that there is a one to one correspondence between $\mathcal{O}_G(H)$ and the fundamental system of open neighborhoods

$$\mathcal{N}^* = \{\pi_H(U) \mid U \in \mathcal{O}_G(H)\}$$

of the left coset H in G/H. Indeed, if U_1 and U_2 are elements of $\mathcal{O}_G(H)$, $\pi_H(U_1) = \pi_H(U_2)$ if and only if $U_1 = U_2$. It follows that $|\mathcal{O}_G(H)| = |\mathcal{N}^*|$ so we have to prove that $|\mathcal{N}^*| = w_0(G/H)$.

Now, let $\mathcal{N}_H$ be a fundamental system of neighborhoods of open subsets of the left coset H and such that $|\mathcal{N}_H| = w_0(G/H)$. Pick $U \in \mathcal{O}_G(H)$. Since $\pi_H(U)$ is an open neighborhood of the coset H, there exists an element $\mathcal{L}$ of $\mathcal{N}_H$ such that $\mathcal{L} \subseteq \pi_H(U)$. Then $K = \pi_H^{-1}(\mathcal{L})$ is an open subset of G containing 1 by Lemma 4.1.2. It follows that we can find an open normal subgroup N of G such that $N \subseteq K \subseteq U$. Thus, for every $\mathcal{L} \in \mathcal{N}_H$, there are finitely many $\pi_H(U)$ with $U \in \mathcal{O}_G(H)$ and, since $w_0(G/H)$ is infinite, $|\mathcal{N}^*| = |\mathcal{N}_H| = w_0(G/H)$.

$\square$

If α is an ordinal number, it is the set of all ordinals less than α, i.e. $\alpha = \{\beta \mid \beta < \alpha\}$, and the symbol $\alpha + 1$ denotes the successor of α, that is the set $\alpha \cup \{\alpha\}$.

Proposition 4.1.2. *Let α be an ordinal number and let $\{X_\beta\}_{\beta<\alpha}$ be a family of non-empty sets. If $\{X_\beta, \phi_{\beta,\gamma}, \alpha\}$ is an inverse system of surjective maps such that for every $\beta < \alpha$ and $x_\beta \in X_\beta$, $|\phi^{-1}_{\beta+1,\beta}(x_\beta)| \geq 2$, then*

$$\left| \varprojlim_{\beta<\alpha} X_\beta \right| \geq 2^{|\alpha|}.$$

When α is infinite and each X_β is finite for $\beta < \alpha$, the equality holds.

Proof. The first step of this proof is to construct an injective map from $\{0,1\}^\alpha$ to $\varprojlim_{\beta<\alpha} X_\beta$. Consider an arbitrary map $f : \alpha \longrightarrow \{0,1\}$. We have that for any $\beta < \alpha$ and $x_\beta \in X_\beta$, $|\phi^{-1}_{\beta+1,\beta}(x_\beta)| \geq 2$, thus there exist at least two elements of $X_{\beta+1}$ whose image is x_β, so that we can define the following map $\tilde{x_\beta} : \{0,1\} \longrightarrow X_{\beta+1}$ such that $\tilde{x_\beta}(0) = x^0_{\beta+1}$ and $\tilde{x_\beta}(1) = x^1_{\beta+1}$ are two distinct preimages of x_β in $X_{\beta+1}$ by $\phi_{\beta+1,\beta}$.

Pick an element $x_0 \in X_0$, and by transfinite induction, we define

$$x_{\beta+1} = \tilde{x_\beta}(f(\beta)) \in X_{\beta+1}.$$

If λ is a limit ordinal less than α, and x_β is defined for every $\beta < \lambda$, define x_λ as an element of the non-empty set $\bigcap_{\beta<\lambda} \phi^{-1}_{\lambda,\beta}(x_\beta)$.

Then, for every $\alpha > \beta \geq \gamma$, $\phi_{\beta,\gamma}(x_\beta) = x_\gamma$ so that $(x_\beta)_{\beta<\alpha}$ is a coherent sequence. Therefore, the map

$$\psi : \{0,1\}^\alpha \longrightarrow \varprojlim_{\beta<\alpha} X_\beta$$

$$f \longrightarrow (x_\beta)_{\beta<\alpha}$$

is well-defined. It is injective because, if $f, g \in \{0,1\}^\alpha$ are such that $\psi(f) = \psi(g)$, we have $\tilde{x_\beta}(f(\beta)) = \tilde{x_\beta}(g(\beta))$ for every $\beta < \alpha$, and so also $f(\beta) = g(\beta)$ for each $\alpha < \beta$. It follows that $f = g$. In this way, we have proved that

$$2^{|\alpha|} = |\{0,1\}^\alpha| \leq \left| \varprojlim_{\beta<\alpha} X_\beta \right|.$$

On the other hand, $\varprojlim_{\beta<\alpha} X_\beta \leq \prod_{\beta<\alpha} X_\beta$ thus, when each X_β is finite we have:

$$\left| \varprojlim_{\beta<\alpha} X_\beta \right| \leq \prod_{\beta<\alpha} |X_\beta| \leq \prod_{\beta<\alpha} \aleph_0 = \aleph_0^{|\alpha|}.$$

If α is infinite, it follows that $2^{|\alpha|} = \aleph_0^{|\alpha|}$ by Lemma 5.6 of [32]. □

Corollary 4.1.2. *Let I be a well-ordered set and let $\{X_i\}_{i\in I}$ be a family of non-empty sets. If $\{X_i, \phi_{i,j}, I\}$ is an inverse system of surjective maps such that for every $i, j \in I$ such that $i \geq j$ and $x_j \in X_j$, $|\phi_{i,j}^{-1}(x_j)| \geq 2$, then*

$$\left| \varprojlim_{i\in I} X_i \right| \geq 2^{|I|}.$$

When I is infinite and each X_i is finite for $i \in I$, the equality holds.

Lemma 4.1.5. *Let G be an infinite profinite group and let H be a closed subgroup of G. Then G/H is homeomorphic to the inverse limit*

$$\varprojlim_{U\in\mathcal{O}_G(H)} G/U.$$

Proof. Since for every $U_1, U_2 \in \mathcal{O}_G(H)$, $U_1 \cap U_2$ is still an open subgroup of G containing H, the set $\mathcal{O}_G(H)$ is a directed set and we can consider the inverse limit $\varprojlim_{U\in\mathcal{O}_G(H)} G/U$ with compatible continuous mappings

$$\phi_{U,V} : gU \in G/U \longrightarrow gV \in G/V,$$

for every $U, V \in \mathcal{O}_G(H)$ such that $U \leq V$.

For each $U \in \mathcal{O}_G(H)$, define the map

$$\pi_{H,U} : gH \in G/H \longrightarrow gU \in G/U.$$

Clearly, $\pi_{H,U}$ is surjective and continuous for every $U \in \mathcal{O}_G(H)$, so that the map

$$\phi : G/H \longrightarrow \varprojlim_{U\in\mathcal{O}_G(H)} G/U$$

induced by the $\pi_{H,U}$ with $U \in \mathcal{O}_G(H)$ is a continuous surjection by Corollary 1.1.6 of [50]. Now, let xH and yH be two distinct cosets of H in G. It follows that $x^{-1}y \notin H$. Since $H = \bigcap_{U\in\mathcal{O}_G(H)} U$, we have that $x^{-1}y \notin U$ for some $U \in \mathcal{O}_G(H)$. Hence $\pi_{H,U}(xH) \neq \pi_{H,U}(yH)$ and so also $\phi(xH) \neq \phi(yH)$. $\square$

Theorem 4.1.1. *Let G be an infinite profinite group and let H be a closed subgroup of G of infinite index. Then, $|G : H| = 2^{w_0(G/H)}$.*

Proof. Since $\mathcal{O}_G(H)$ is the set of all open subgroups of G containing H,

$$H = \bigcap_{U\in\mathcal{O}_G(H)} U.$$

Let μ be the smallest ordinal with cardinality $w_0(G/H)$ and let us well-order $\mathcal{O}_G(H)$ by setting $U_0 = G$ and $\mathcal{O}_G(H) = \{U_\lambda \mid \lambda < \mu\}$. For each $\lambda < \mu$, we put $G_0 = U_0 = G$ and

$$G_\lambda = \bigcap_{\alpha<\lambda} U_\alpha.$$

Clearly, the set $\{G_\lambda \mid \lambda < \mu\}$ is a strictly descending chain of closed subgroups including G and whose intersection is H. Even if G_λ is not necessarily normal, we have that the space of the left cosets G/G_λ is a profinite topological space. The inverse limit of the finite topological spaces $G_\lambda/G_{\lambda+1}$ with the continuous mappings

$$\phi_{\lambda,\gamma} : gG_\lambda \in G/G_\lambda \longrightarrow gG_\gamma \in G/G_\gamma$$

for every $\gamma \leq \lambda < \mu$, is a profinite topological space. We want to show that $\varprojlim_{\lambda<\mu} G/G_\lambda$ has the same cardinality as G/H. Since, for a fixed $g \in G$,

$(gG_\lambda)_{\lambda<\mu}$ is a coherent sequence, we can consider

$$\pi : gH \in G/H \longrightarrow (gG_\lambda)_{\lambda<\mu} \in \varprojlim_{\lambda<\mu} G/G_\lambda$$

which is well-defined since, for every $g_1, g_2 \in G$, $g_1H = g_2H$ if and only if $g_1^{-1}g_2 \in H$, which is true if and only if $g_1^{-1}g_2 \in G_\lambda$, for every $\lambda < \mu$. Clearly, π is injective. If we consider the continuous surjective compatible maps $\phi_\lambda : gH \in G/H \longrightarrow gG_\lambda \in G/G_\lambda$, for each $\lambda < \mu$, we have that π is the corresponding induced map, so it is surjective by Corollary 1.1.6 of [50], being each G/G_λ Hausdorff compact spaces. In particular, we have that

$$\left|\varprojlim_{\lambda<\alpha} G/G_\lambda\right| = |G : H|.$$

On the other hand, since $\{G_\lambda \mid \lambda < \mu\}$ is a strictly descending chain, for each $\lambda < \mu$, $G_\lambda > G_{\lambda+1}$ and each coset of G_λ is a union of finitely many cosets of $G_{\lambda+1}$. Indeed, for every $\lambda < \mu$, $|G_\lambda : G_{\lambda+1}| = |G_\lambda : G_\lambda \cap U_\lambda|$ is finite so, if $G_\lambda = x_1G_{\lambda+1} \cup x_2G_{\lambda+1} \cup \cdots \cup x_tG_{\lambda+1}$ and $g \in G$, we have $gG_\lambda = gx_1G_{\lambda+1} \cup gx_2G_{\lambda+1} \cup \cdots \cup gx_tG_{\lambda+1}$. Therefore,

$$\phi^{-1}_{\lambda+1,\lambda}(gG_\lambda) = \{gx_1G_\lambda, gx_2G_\lambda, \ldots, gx_tG_\lambda\}$$

where $t \geq 2$. It follows that, for each $\lambda < \mu$, every element in G/G_λ has at least two preimages by $\phi_{\lambda+1,\lambda}$. By Proposition 4.1.2, we have that $|G : H| \geq 2^{w_0(G/H)}$. Lemma 4.1.5 and Corollary 4.1.1 ensure that

$$|G/H| = \left|\varprojlim_{U\in\mathcal{O}_G(H)} G/U\right| \leq \prod_{U\in\mathcal{O}_G(H)} |G/U| \leq \prod_{U\in\mathcal{O}_G(H)} \aleph_0 = \aleph_0^{w_0(G/H)}.$$

Since $|G : H|$ is infinite, $w_0(G/H)$ is infinite by Lemma 4.1.3 and it follows that $2^{w_0(G/H)} = \aleph_0^{w_0(G/H)}$ by Lemma 5.6 of [32]. $\square$

4.2 Normality Conditions on Open Subgroups

In a profinite group, each open subgroup has finite index, so it is *large* in some sense. In this section we will explore how the behaviour of open subgroups with respect to one of the normality conditions mentioned in Chapter 2 influences the structure of a profinite group. For the convenience of the reader, it is worth noticing that for every open subgroup H of a profinite group G, all the supersubgroups of H are still open since they are union of finitely many cosets of H.

Our first two results show that Neumann's Theorems (of [46] and [45] respectively) are true in a profinite group whenever we consider only the open subgroups of G and there are bounds on the indices involved. In particular, the behaviour of subgroups that are not open is negligible and heavily affected by that of the open ones.

Theorem 4.2.1. *Let G be a profinite group and k a positive integer. If, for any open subgroup $H \leq_o G$, we have $|H^G : H| \leq k$, then the commutator subgroup G' of G is finite of order bounded by a function of k.*

Proof. Let us consider a normal open subgroup $N \trianglelefteq_o G$. We have that every subgroup $H/N \leq G/N$ has finite index bounded by k in its normal closure. Indeed, $(H/N)^G = H^G/N$ so that

$$|(H/N)^G : H/N| = |H^G/N : H/N| = |H^G : H| \leq k.$$

By Neumann's theorem, it follows that $G'N/N$ is finite of order bounded by a function $h = f(k)$ of k. Assume by contradiction that $|G'| > h$ and pick $x_0, x_1, \dots, x_h \in G'$ that are pairwise distinct. Since the intersection of all open normal subgroups of G is trivial, for every $i, j \in \{0, \dots, h\}$ and $i \neq j$, there exists an open normal subgroup $N_{i,j}$ of G such that $x_i^{-1}x_j \notin N_{i,j}$. Set

$$N = \bigcap_{\{i,j \,|\, i \neq j\}} N_{i,j}.$$

Since N is the intersection of finitely many normal subgroups, N is an open normal subgroup of G. Furthermore, for each $i, j \in \{0, \dots, h\}$ and $i \neq j$, $x_i^{-1}x_j \notin N$. It follows that $|G'N/N| > h$, which is a contradiction. $\square$

Theorem 4.2.2. *Let G be a profinite group in which for every open subgroup H of G, we have that $|G : N_G(H)| \leq k$, for a fixed positive integer k. Then $G/Z(G)$ is finite of order bounded by a function of k. In particular G' is finite of bounded order.*

Proof. Let $N \trianglelefteq_o G$, then all supersubgroups H of N in G are open, so that $|G : N_G(H)| \leq k$. Since $N_G(H)/N = N_{G/N}(H/N)$, if we set $Z(G/N) = Z_N/N$, we have that $|G/Z_N| \leq h$, where $h = f(k)$ is a function of k. Assume by contradiction that $|G/Z(G)| > h$, so there exist at least $h+1$ pairwise distinct elements $x_0, \ldots, x_h$ in a fixed transversal T of $Z(G)$. Since the intersection of all open normal subgroups of G is trivial, it follows that

$$Z(G) = \bigcap_{N \trianglelefteq_o G} Z_N.$$

Indeed, if $g \in \bigcap_{N \trianglelefteq_o G} Z_N$, then $[gN, xN] = N$ for every $x \in G$ and $N \trianglelefteq_o G$, therefore $[x, g] \in \bigcap_{N \trianglelefteq_o G} N = \{1\}$ and $g \in Z(G)$. Thus, for each $i, j \in \{0 \ldots, h\}$ and $i \neq j$, we can find an open normal subgroup $N_{i,j}$ such that $x_i^{-1}x_j$ does not belong to $Z_{N_{i,j}}$. Set $M = \bigcap_{\{i,j \,|\, i \neq j\}} N_{i,j}$. It is easy to prove that $Z_M = \bigcap_{\{i,j \,|\, i \neq j\}} Z_{N_{i,j}}$, so it follows that for each $i, j \in \{0, \ldots, h\}$ and $i \neq j$, $x_i^{-1}x_j$ does not belong to Z_M but that is a contradiction. Indeed, M turns out to be an open normal subgroup of G such that $|G/Z_M| > h$. $\square$

In particular, when in a profinite group G there are finitely many normalizers of open subgroups we have that the group is central-by-finite. Actually, such a condition is equivalent to many interesting properties, as it is stated in the following theorem of [30]. Here, we define the pro-norm $N_c(G)$ of a profinite group G as the intersection of all normalizers of closed subgroups of G; of course, $N_c(G)$ is a normal subgroup of G and it also coincides with the intersection of the normalizers of all open subgroups of G.

Theorem 4.2.3. *Let G be a profinite group. The following conditions are equivalent.*

(i) G has finitely many normalizers of closed subgroups.

(ii) G has finitely many normalizers of open subgroups.

(iii) G' is finite.

(iv) $G/Z(G)$ is finite.

(v) $G/N_c(G)$ is finite.

More precisely, for a profinite group, the converse of Schur's theorem holds; moreover, every profinite FC-group is both finite-by-abelian and central-by-finite (see also Lemma 2.6 of [58]).

The argument employed for the above results is based on the fact that, in a profinite group, the intersection of all open normal subgroups is trivial. This property can be easily applied to analyze profinite groups where open subgroups are subnormal of bounded defect. As in Subsection 3.1, f will denote Roseblade's function defined in [53] and it will play a central role in our next result.

Theorem 4.2.4. *Let k be a positive integer and let G be a profinite group whose open subgroups are all subnormal of defect at most k. Then G is nilpotent of class at most $f(k)$.*

Proof. For every open normal subgroup $N \trianglelefteq_o G$, we have that

$$\gamma_{f(k)+1}(G/N) = \gamma_{f(k)+1}(G)N/N = N/N.$$

It follows that

$$\gamma_{f(k)+1}(G) \leq \bigcap_{N \trianglelefteq_o G} N = \{1\}.$$

$\square$

Lennox [37] generalised Roseblade's theorem, proving that there exists a function μ such that if every finitely generated subgroup of a group G is (m,k)-subnormal in G, then the term $\gamma_{\mu(m+k)}(G)$ of the lower central series of G is finite of order less than or equal to $m!$. An analogous generalization can be proved for profinite groups.

Theorem 4.2.5. *Let m and k be positive integers and let G be a profinite group whose open subgroups are (m, k)-subnormal. Then, $|\gamma_{\mu(m+k)}(G)| \leq m!$.*

Proof. For each open normal subgroup N of G, Theorem A of [37] yields that

$$|\gamma_{\mu(m+k)}(G)N/N| \leq m!.$$

Put $s = m!$ and assume by contradiction that $|\gamma_{\mu(m+k)}(G)| > s$. Let $x_0, \ldots, x_s$ be pairwise distinct elements of $\gamma_{\mu(m+k)}(G)$. Since the intersection of all open normal subgroups of G is trivial, for each $i, j \in \{0, \ldots, s\}$ and $i \neq j$, there exists an open normal subgroup $N_{i,j}$ of G such that $x_i^{-1}x_j \notin N_{i,j}$. Set

$$N = \bigcap_{\{i,j \,|\, i \neq j\}} N_{i,j}.$$

We have that N is an open normal subgroup of G and $x_i^{-1}x_j$ does not belong to N for every $i, j \in \{0, \ldots, s\}$ and $i \neq j$, so that

$$|\gamma_{\mu(m+k)}(G/N)| > s,$$

which is a contradiction. □

4.3 Permutable Complementation of Closed Subgroups

4.3.1 Complemented subgroups

Let us consider the full cartesian product $G = \underset{n\in\mathbb{N}}{Cr}\, G_n$ where G_n is a cyclic group of prime order p, for every $n \in \mathbb{N}$. Then G is profinite and an elementary abelian p-group, hence every subgroup in G is permutably complemented (i.e., G is a C-group) by Theorem 3.2.5 of [56].

Furthermore, G is countably based since $\mathcal{N} = \{\underset{n\in\mathbb{N}}{\mathrm{Cr}}\, H_n | H_n \leq G_n \forall n \in \mathbb{N}$ and $H_n = \{1\}$ for finitely many $n \in \mathbb{N}\}$ is a fundamental system of neighborhoods of the identity. Now, if $U \trianglelefteq_o G$, then it contains an element $N \in \mathcal{N}$. Since G/N is finite, N can be contained in finitely many (open) subgroups as U. Thus, in G we can find only countably many open subgroups.

On the other hand, G can be regarded as a $\mathbb{F}_p$-vector space and clearly a base $\mathfrak{B}$ of G has to have cardinality $2^{\aleph_0}$. If we pick an element $x \in \mathfrak{B}$, then $\langle \mathfrak{B} \setminus \{x\} \rangle$ is a maximal subgroup of G, furthermore if x and y are distinct elements of $\mathfrak{B}$, we have that $\langle \mathfrak{B} \setminus \{x\} \rangle \neq \langle \mathfrak{B} \setminus \{y\} \rangle$, so that maximal subgroups are not countably many. It follows that there are subgroups of finite index which are not open. If H is such a subgroup, it has a finite (of prime order) permutable complement K. It follows that K is a closed subgroup with a non-closed complement. Now, a natural question arises: are there any closed complements to K? In the following we will investigate such a problem.

Let us call *profinite C-group* a profinite group G where each closed subgroup H of G admits a closed permutable complement in G, that is a closed subgroup K of G such that $G = HK$ and $H \cap K = \{1\}$.

Lemma 4.3.1. *Let G be a profinite C-group. Then, if C is a closed subgroup of G, C is a profinite C-group.*

Proof. Let H be a closed subgroup of C, then H is closed even in G and there exists a closed subgroup K such that $G = HK$ and $H \cap K = \{1\}$. We have that $C = HK \cap C = H(K \cap C)$ and $H \cap K \cap C = \{1\}$, so $K \cap C$ is a closed permutable complement of H in C. $\square$

Lemma 4.3.2. *Let G be a profinite C-group. Then, if C is a closed normal subgroup of G, G/C is a profinite C-group.*

Proof. Let H/C be a closed subgroup of G/C, then H is closed and permutably complemented in G by a closed subgroup K, so that $G = HK$ and $H \cap K = \{1\}$. It follows that $G/C = HK/C = (H/C)(KC/C)$ and $H \cap KC = C(H \cap K) = C$, thus KC/C is a closed permutable complement to H/C in G/C. $\square$

Lemma 4.3.3. *Let G be a (profinite) C-group and let H be a (closed) subgroup of G. If S is a (closed) supplement to H in G, then there is a (closed) permutable complement of H in G that is contained in S.*

Proof. Observe that S is a C-group (profinite C-group by Lemma 4.3.1). Hence $H \cap S \leq_{(c)} S$ is complemented in S, say K the (closed) permutable complement in S of $H \cap S$. We have that $S = (H \cap S)K$ and $H \cap S \cap K = H \cap K = \{1\}$. Furthermore $G = HS = H(H \cap S)K = HK$, so K is a (closed in G) permutable complement of H contained in S. $\square$

Corollary 4.3.1. *Let G be a (profinite) C-group with H (closed) subgroup of G and N (closed) normal subgroup of G. If S is a subgroup of G containing N and such that S is a (closed) permutable complement of H modulo N, then there exists a (closed) complement K of H in G that projects onto S, that is $S/N = KN/N$.*

Proof. It is easy to notice that S is a (closed) supplement of H in G such that $S \cap H \leq N$. By Lemma 4.3.3, there exists a (closed) complement K to H in G with $K \leq S$. By construction, $S = KN(H \cap S) = KN$. $\square$

Lemma 4.3.4. *Let G be a profinite group and let H and K closed subgroups of G. Then the product $HK = \{hk \mid h \in H, k \in K\}$ is a closed set of G.*

Proof. Both H and K are compact, so that the cartesian product $H \times K$ is also compact. Being G a topological group, the map $\mu : (x, y) \in G \times G \longrightarrow xy \in G$ is continuous and therefore $\mu(H \times K) = HK$ is compact. Since G is Hausdorff, HK is even closed. $\square$

Lemma 4.3.5. *Let A and B be profinite $\mathcal{C}$-groups. Then $A \times B$ is a profinite $\mathcal{C}$-group.*

Proof. Let H be a closed subgroup of $A \times B$, then $H \cap A$ is closed in A, so that it admits a complement in A. In particular, there exists a closed subgroup C of A such that $A = (H \cap A)C$ and $H \cap A \cap C = \{1\}$. Since H and A are closed, Lemma 4.3.4 ensures that HA is closed in $A \times B$. It follows that $HA \cap B$ is a closed subgroup of B, hence there exists a closed subgroup D of B such that $B = (HA \cap B)D$ and $HA \cap B \cap D = \{1\}$. Since $D \leq B$, we have that $HA \cap D = HA \cap D \cap B = \{1\}$.

Now, we can notice that CD is a closed permutable complement of H in $A \times B$:

$$HCD = H(H \cap A)CD = HAD = HA(HA \cap B)D = HAB = AB$$

and

$$H \cap CD \leq HA \cap CD = C(HA \cap D) = C,$$

hence

$$H \cap CD \leq H \cap C = H \cap A \cap C = \{1\}.$$

□

Theorem 4.3.1. *Let G be a profinite group. The following are equivalent:*

(i) *Every closed subgroup of G has a closed permutable complement.*

(ii) *Every closed subgroup of G has a permutable complement.*

(iii) *Every open subgroup of G has a permutable complement.*

(iv) *Every open subgroup of G has a closed permutable complement.*

(v) *Every G/N is a $\mathcal{C}$-group for each open normal subgroup N of G.*

Proof. It is clear that (i) implies (ii), (iii) follows from (ii) and that any permutable complement of an open subgroup of G is finite and so it is closed, hence (iv) is equivalent to (iii). Furthermore, if $N \trianglelefteq_o G$, every subgroup H/N of G/N is open in G/N and H is open in G. Hence, when (iv) is true, in G/N all subgroups are permutably complemented, so whenever we choose $N \trianglelefteq_o G$, G/N is a C-group and we have (v). To conclude, it suffices to prove that (i) stems from (v).

Let us assume that every G/N is a C-group whenever $N \trianglelefteq_o G$ and let $\mathcal{N}$ be the set of all open normal subgroups of G. Then $\mathcal{N}$ can be well-ordered so suppose that $\mathcal{N} = \{N_\mu \mid \mu < \lambda\}$, for some ordinal number λ. We set $G_0 = N_0$ and put

$$G_\mu = \bigcap_{\alpha<\mu} N_\alpha$$

for each $0 < \mu \leq \lambda$. In that way, we get a family $\{G_\mu \mid \mu \leq \lambda\}$ of closed normal subgroups of G and it is possible to show that each section G/G_μ has all closed subgroups permutably complemented. Indeed, by transfinite induction, we have that G/G_0 is a C-group; let $\mu + 1$ be a successor ordinal and assume that G/G_μ is a profinite C-group, then $G/G_{\mu+1}$ embeds into the cartesian product $G/G_\mu \times G/N_\mu$ via a continuous homomorphism

$$\iota : G/G_{\mu+1} \hookrightarrow G/G_\mu \times G/N_\mu,$$

such that $\iota(gG_{\mu+1}) = (gG_\mu, gN_\mu)$. The subgroup $\iota(G/G_{\mu+1})$ is closed in $G/G_\mu \times G/N_\mu$ and so it is a profinite C-group by Lemmas 4.3.5 and 4.3.1. It follows that every closed subgroup of $G/G_{\mu+1}$ has a closed permutable complement.

Furthermore, for every closed subgroup H of G, $HG_{\mu+1}/G_{\mu+1}$ is closed in $G/G_{\mu+1}$ and we can construct a closed complement $K_{\mu+1}/G_{\mu+1}$ starting from a closed complement K_μ/G_μ of HG_μ/G_μ in G/G_μ such that $K_{\mu+1} \leq K_\mu$. Indeed, we have $G/G_{\mu+1} = HK_\mu/G_{\mu+1}$ and Corollary 4.3.1 ensures that there exists a closed subgroup $K_{\mu+1} \leq K_\mu$ containing $G_{\mu+1}$ such that $K_{\mu+1}G_\mu = K_\mu$ and $K_{\mu+1}/G_{\mu+1}$ is a closed complement to $HG_{\mu+1}/G_{\mu+1}$ in $G/G_{\mu+1}$.

If $\gamma \leq \lambda$ is a limit ordinal, since H is closed in G, HG_γ/G_γ is a closed subgroup of G/G_γ. Clearly, for each $\mu < \lambda$, HG_μ/G_μ is closed in G/G_μ so, by inductive hypothesis, it admits a permutable complement K_μ/G_μ in G/G_μ. Moreover, we can assume that the set $\{K_\mu \mid \mu < \lambda\}$ is a chain so that it is filtered from below.

In particular, we have that

$$G = HK_\mu$$

and

$$H \cap K_\mu \leq G_\mu$$

for every $\mu < \gamma$.

It follows that $G = \bigcap\limits_{\mu<\gamma} HK_\mu$ and, since $\{K_\mu \mid \mu < \gamma\}$ is a totally ordered, an application of [[50], Proposition 2.1.4 (a)] tells us that

$$G = H\Big(\bigcap_{\mu<\gamma} K_\mu\Big).$$

If we set $K_\gamma = \bigcap\limits_{\mu<\gamma} K_\mu$, we have that $G_\gamma \leq K_\gamma$ and $H \cap K_\gamma \leq G_\gamma$ so that K_γ/G_γ is a closed complement to HG_γ/G_γ in G/G_γ. In particular for λ, we will find a closed subgroup K_λ such that $G = HK_\lambda$ and $H \cap K_\lambda \leq G_\lambda = \bigcap\limits_{\mu<\lambda} N_\mu = \{1\}$. So H has a closed permutable complement in G. $\square$

Corollary 4.3.2. *Let G be a C-group, then the profinite completion of G is a profinite C-group.*

Proof. Since G is a C-group, it is residually finite so G embeds in its profinite completion. So, if $\mathcal{N}$ is the family of all finite-index normal subgroups of G, the profinite completion

$$\hat{G} = \varprojlim_{N \in \mathcal{N}} G/N$$

is an inverse limit of finite C-groups, thus it is a profinite C-group by Theorem 4.3.1. $\square$

In general, profinite C-groups are not periodic. If we consider the full cartesian product

$$G = \underset{p \in \mathbb{P}}{\mathrm{Cr}}\, C_p$$

where C_p is cyclic of order $p \in \mathbb{P}$, it is a profinite C-group since it is an inverse limit of finite C-groups. However, the element $x = (x_p)_{p \in \mathbb{P}}$ such that every $x_p \neq 1$ is aperiodic. In particular, G is not a C-group.

On the other hand, if G is a non-torsion profinite C-group and $x \in G$ is an aperiodic element, $\langle x \rangle$ does not have any closed complement since every closed subgroup of G has either finite or uncountable index (see Theorem 4.1.1).

In a profinite C-group, the Frattini subgroup is always torsion-free since for every periodic element $x \in G$, the subgroup $\langle x \rangle$ is permutably complemented by a proper open subgroup K of G such that $G = \langle x \rangle K$. If $x \in Frat(G)$, then $G = K$, which is a contradiction as $\langle x \rangle \cap K = \{1\}$.

Proposition 4.3.1. *Let G be a profinite C-group and let A be a non-trivial closed abelian normal subgroup of G. Then A contains minimal G-invariant subgroups of G and they are finite of prime order.*

Proof. Pick an element $a \in A \setminus \{1\}$. Since G is a profinite group, we can always find an open normal subgroup M of G such that $a \notin M \cap A$. In particular, consider an open normal subgroup N of G with $a \notin N \cap A$ and of minimal index in A. This implies that $A \cap N$ is maximal as an open G-invariant subgroup of A not containing a. Clearly, $A \cap N$ admits a closed complement in G, say K and $C = A \cap K$ is a closed normal subgroup of A since A is closed and abelian. Moreover, $C = A \cap K \trianglelefteq K$ because $A \trianglelefteq G$, so $C \trianglelefteq AK = G$, and thus $A = C \times (A \cap N)$. Assume that C is not a minimal normal subgroup of G. By the above argument, $C = C_1 \times C_2$ where C_1 and C_2 are closed, G-invariant, non-trivial subgroups. For $i = 1, 2$, we have that $(A \cap N) \times C_i$ is open in A and closed in G, and moreover, $|A : (A \cap N) \times C_i| < |A : (A \cap N)|$. The minimality of this index ensures that $a \in (A \cap N) \times C_i$ for $i = 1, 2$, hence $a \in A \cap N$, a contradiction. It follows that C is a minimal normal subgroup of G.

Let D be any minimal G-invariant subgroup of A. In particular, D is non trivial. For every $N \trianglelefteq_o G$, we have that $D \cap N$ is G-invariant so, either $D \cap N = \{1\}$ or $D \leq N$ by the minimality of D. Since the intersection of all open normal subgroups of G is trivial, there exists an open normal subgroup M of G such that $D \cap M = \{1\}$. It follows that D is isomorphic to a subgroup of the finite group G/M, hence D is finite. Pick an element

$x \in D$ of prime order and consider a closed complement K to $\langle x \rangle$ in G. Since $G = \langle x \rangle K$, we have that $D = \langle x \rangle (K \cap D)$. Now, $K \cap D \trianglelefteq K$ since $D \trianglelefteq G$ and $K \cap D \trianglelefteq D$ as D is abelian, hence $K \cap D \trianglelefteq KD = G$, so $K \cap D = \{1\}$ and $D = \langle x \rangle$. $\square$

Lemma 4.3.6. *Let G be a profinite C-group and let A be an abelian closed normal subgroup of G. Then, either A is a minimal normal subgroup or the intersection $\Phi_c(A)$ of all closed G-invariant maximal subgroups of A is trivial.*

Proof. If A is not a minimal normal subgroup of G, Proposition 4.3.1 ensures that it contains a proper minimal G-invariant subgroup D which is finite of prime order. As D is closed, it has an open complement L in G and $L \cap A$ is G-invariant, closed in G and $A = D \times (L \cap A)$. In particular, $L \cap A$ is a maximal element of the set $\{K \leq_c A \mid K \trianglelefteq G\}$ ordered by inclusion and $\Phi_c(A) \leq A \cap L$.

Assuming that $\Phi_c(A)$ is non-trivial, we have that $\Phi_c(A)$ contains a G-invariant subgroup of prime order $\langle x \rangle$ by Proposition 4.3.1. Now, if C is a closed complement to $\langle x \rangle$ in G, we have that $C \cap A$ is a G-invariant maximal subgroup of A such that $x \notin C \cap A$, which is a contradiction. It follows that $\Phi_c(A) = \{1\}$. $\square$

Corollary 4.3.3. *Let G be a profinite C-group and let A be an abelian closed normal subgroup of G. Then A is isomorphic to the cartesian product of G-invariant subgroups of prime order.*

Proof. If A is a minimal normal subgroup of G, the statement is obvious. Define $\Phi(A)$ as the intersection of all G-invariant maximal subgroups of A. If A is not a minimal normal subgroup of G, Lemma 4.3.6 ensures that $\Phi(A) = \{1\}$. Since the G-invariant subgroups of A are precisely the G-submodules of A regarded as a G-module, Proposition 7.4.5 of [65] guarantees that A is isomorphic to the cartesian product of simple G-modules that, in our case, are G-invariant subgroups of prime order by Proposition 4.3.1. $\square$

Theorem 4.3.2. *Let G be a profinite C-group. Then $G = \overline{G'} \rtimes K$ where $\overline{G'}$ is isomorphic to the cartesian product $\underset{i \in I}{\mathrm{Cr}} A_i$ whose factors A_i are G-invariant sub-*

groups of prime order and K is a closed subgroup of G isomorphic to the cartesian product of prime order subgroups $\mathrm{Cr}_{j \in J} B_j$.

Proof. If $\mathcal{N}$ is the set of all open normal subgroups of G,

$$\overline{G'} = \varprojlim_{N \in \mathcal{N}} G'N/N,$$

which is isomorphic to a subgroup of the (abelian) cartesian product

$$\mathrm{Cr}_{N \in \mathcal{N}} G'N/N,$$

so that $\overline{G'}$ is an abelian G-invariant closed subgroup of G and its structure is provided by Corollary 4.3.3. If K is a closed complement to $\overline{G'}$, we have that K is an abelian profinite C-group, thus a further application of Corollary 4.3.3 completes the proof. □

4.3.2 Sectional complemented subgroups

We will call *profinite SC-group* a profinite group G where each closed subgroup H of G admits a closed sectional complement, or S-complement for short, in G, that is a closed complement K to H such that if $H < X <_c G$, then $X \cap K$ is a (closed) complement to H in X.

It is easy to see that every profinite C-group is a profinite SC-group. Indeed, if H is a closed subgroup of a profinite C-group G and K is a closed complement to H in G, for every closed subgroup X containing H we have that

$$X = X \cap HK = H(K \cap X),$$

so that $X \cap K$ is a closed complement to H in X.

Proposition 4.3.2. *Let G be a profinite SC-group and let $N \trianglelefteq_c G$, then G/N is a profinite SC-group.*

Proof. Let $H/N \leq_c G/N$. Since $H \leq_c G$, there exists a closed S-complement K to H in G. Let $X/N \leq_c G/N$ containing H/N. We have that $G/N = (KN/N)(H/N)$ and $KN \cap H = N$ so that KN/N is a complement to H/N

in G/N. Moreover, it is easy to see that $X/N = ((KN \cap X)/N)(H/N)$ where $(KN \cap X)/N = KN/N \cap X/N$ and $KN \cap X \cap H = N$. So KN/N is a closed S-complement to H/N in G/N. $\square$

In particular, if N is an open normal subgroup of a profinite SC-group, G/N is a finite SC-group and then it is a C-group by Theorem 3.2.4 of [56].

It follows that the inverse limit of finite SC-groups is a profinite C-group (so it is also a profinite SC-group).

A profinite SC-group is a group where all open subgroups have a closed S-complement. Hence, a profinite group G is a profinite SC-group if and only if every open subgroup admits a (sectional) complement. Furthermore, the profinite completion of a residually finite SC-group is a profinite C-group.

Bibliography

[1] R. Baer, *Abzählbar erkennbare gruppentheoretische Eigenschaften*, Math. Z. 79 (1962), 344—363.

[2] R. G. Burns, F. Okoh, H. Smith and J. Wiegold, *On the number of normal subgroups of an uncountable soluble group*, Arch. Math., Vol. 42 (1984), 289–295.

[3] C. Casolo and M. Mainardis, *Groups in which every subgroup is f-subnormal*, J. Group Theory 4 (2001), 341—365.

[4] S. N. Černikov, *Groups with given properties of their systems of infinite subgroups* Ukrain Math. J. 19 (1967), number 6, 715 — 731.

[5] S. M. Corson, *Jónsson groups of various cardinalities*, Proc. Amer. Math. Soc. 150 (2022), 2771–2775.

[6] M. De Falco, F. de Giovanni and C. Musella, *Groups with normality conditions for subgroups of infinite rank*, Publ. Mat. 58 (2) (2104), 331–340.

[7] M. De Falco, F. de Giovanni and C. Musella, *Large soluble groups and the control of embedding properties*, Ricerche Mat. 63 (2014), suppl.1, 117—130.

[8] M. De Falco, F. de Giovanni, H. Heineken and C. Musella, *Normality in uncountable groups*, Adv. Group Theory Appl. 3 (2017), 13–29.

[9] M. De Falco, F. de Giovanni, M. J. Evans and C. Musella, *Permutability in uncountable groups*, Ann. Mat. Pura Appl. 197 (2018), 1417–1427.

[10] M. De Falco, F. de Giovanni, C. Musella and Y. P. Sysak, *Uncountable groups in which normality is a transitive relation*, Internat. J. Algebra Comput. 29 (2019), 627–638.

[11] M. De Falco, F. de Giovanni and C. Musella, *A note on commutator subgroups in groups of large cardinality*, Monatsh. Math. 191 (2020), 249–256.

[12] M. De Falco, F. de Giovanni and C. Musella, *Groups with normality conditions for uncountable subgroups*, J. Aust. Math. Soc. 111 (2021), 268–277.

[13] M. De Falco, C. Musella and G. Sabatino, *Uncountable groups with small commutator subgroup*, J.Algebra and Appl. (2024).

[14] M. De Falco, C. Musella, and G. Sabatino, *Uncountable groups with finitely many normalizers of large subgroups*. Arch. Math. 122 (2024), 369—376.

[15] F. de Giovanni and S. Franciosi, *Sui gruppi nei quali ogni sottogruppo finito è dotato di complemento*, Matematiche (Catania) 38 (1984), 201–220.

[16] F. de Giovanni and S. Franciosi, *Groups satisfying the minimal condition on certain non-normal subgroups*, In: Kim A, Johnson D (ed.) Groups - Korea 94: Proceedings of the International Conference held at Pusan National University, Pusan, Korea, August 18-25, 1994. Berlin, New York: De Gruyter; 1995. p.107–118.

[17] F. de Giovanni and D. J. S. Robinson, *Groups with finitely many derived subgroups*, J. London Math. Soc. 71 (2005), 658–668.

[18] F. de Giovanni and M. Trombetti, *Uncountable groups with restrictions on subgroups of large cardinality*, J. Algebra 447 (2016), 383—396.

[19] F. de Giovanni and M. Trombetti, *Nilpotency in uncountable groups*, J. Austral. Math. Soc. 103(1) (2017), 59–69.

[20] F. de Giovanni and M. Trombetti, *Countable recognizability and nilpotency properties of groups*, Rend. Circ. Mat. Palermo, II. Ser 66 (2017), 399—412.

[21] F. de Giovanni and M. Trombetti, *Countable recognizability and residual properties of groups*, Rend. Sem. Mat. Univ. Padova 140 (2018), pp. 69—80.

[22] F. de Giovanni and M. Trombetti, *Pronormality in group theory*, Adv. Group Theory Appl. 9 (2020), 123–149.

[23] E. Detomi, G. Donadze, M. Morigi and P. Shumyatsky, *On finite-by-nilpotent groups*, Glasg. Math. J. 63 (2021), 54–58.

[24] M. R. Dixon, M. Ferrara and M. Trombetti, *Groups in which all subgroups of infinite special rank have bounded near defect*, Comm. Algebra 46 (2018), 5416—5426.

[25] M. Emaldi, *Sugli IK-gruppi risolubili*, Rend. Sem. Mat. Univ. Padova 46 (1971), 9–13.

[26] A. Ehrenfeucht and V. Faber, *Do infinite nilpotent groups always have equipotent abelian subgroups?*, Kon. Nederl. Akad. Wet. A 75 (1972), 202–209.

[27] M. J. Evans and Y. Kim, *On groups in which every subgroup of infinite rank is subnormal of bounded defect*, Comm. Algebra 32 (2004), 2547—2557.

[28] V. Faber and M. J. Tomkinson, *On theorems of B.H. Neumann concerning FC-groups, II*, The Rocky Mountain Journal of Mathematics, 13(3) (1983), 495–506.

[29] M. Ferrara and M. Trombetti, *Infinite groups with many complemented subgroups*, Beitr. Algebra Geom. 63 (2022), 775—808.

[30] M. Ferrara and M. Trombetti, *The pro-norm of a profinite group*, Isr. J. Math. 254 (2023), 399—429.

[31] P. Hall, *Some Constructions for Locally Finite Groups*, J. London Math. Soc. (1959), 305–319.

[32] T. Jech, *Set Theory*, Springer-Verlag, Berlin Heidelberg (2003).

[33] O. H. Kegel and B. A. F. Wehrfritz, *Locally Finite Groups*, North-Holland, Amsterdam (1973).

[34] L. A. Kurdachenko, S. S. Levishchenko and N. N. Semko, *Groups with Almost Normal Infinite Subgroups* Soviet Mathematics (Izvestiya VUZ. Matematika), 27 (1983), 73–81.

[35] A. G. Kuroš and S. N. Černikov, *Solvable and nilpotent groups*, Uspekhi Mat. Nauk (N.S.) 2 (1947), 18—59; Amer. Math. Soc. Translations 80 (1953).

[36] F. Leinen, *Existentially closed* $L\mathfrak{X}$*-groups*, Rend. Sem. Mat. Univ. Padova 75 (1986), 191–226.

[37] J. C. Lennox, *On groups in which every subgroup is almost subnormal*, J. London Math. Soc. s2 15 (1977), 221—231.

[38] J. C. Lennox, *Joins of almost subnormal subgroups*, Edinburgh Math. Z. 63 (1979), 76–96.

[39] J. C. Lennox and D.J.S. Robinson, *The Theory of Infinite Soluble Groups*, 2004.

[40] P. Longobardi, M. Maj and H. Smith, *A note on locally graded groups*, Rend. Sem. Mat. Univ. Padova 94 (1995), 275—277.

[41] I. D. Macdonald, *Some Explicit Bounds in Groups with Finite Derived Groups*, Proc. London Math. Soc., Volume s3-11, Issue 1 (1961), Pages 23–56.

[42] A. Macintyre and S. Shelah, *Uncountable universal locally finite groups*, J. Algebra 43 (1976), 168–175.

[43] B. Maier, *Existenziell abgeschlossene lokal endlichep-Gruppen*, Arch. Math. 37 (1981), 113—128.

[44] S. A. Morris and V. N. Obraztsov, *Nondiscrete topological groups with many discrete subgroups*, Topology and its Applications 84 (1998), 105–120.

[45] B. H. Neumann, *Groups covered by permutable subsets*, J. London Math. Soc. 29 (1954), 236–248.

[46] B. H. Neumann, *Groups with finite classes of conjugate subgroups*, Math. Z. 63 (1955), 76–96.

[47] V. N. Obraztsov, *An embedding theorem for groups and its corollaries*, Math. USSR-Sb. 66 (1990), 541–553.

[48] R. E. Phillips, *Some generalizations of normal series in infinite groups*, J. Austral. Math. Soc. 14 (1972), 496–502.

[49] Y. D. Polovickǐ, *Groups with finite classes of conjugate infinite abelian subgroups*, Soviet Math. (Izv. VUZ) 24 (1980), 52–59.

[50] L. Ribes and P. Zalesski, *Profinite Groups*, Springer Berlin Heidelberg (2010).

[51] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups*, Springer, Berlin (1972).

[52] D. J. S. Robinson, *A course in the theory of groups*, Springer-Verlag (1982).

[53] J. E. Roseblade, *On groups in which every subgroup is subnormal*, J. Algebra. 2 (1965), 402-–412.

[54] G. Sabatino, *Groups in which all large subgroups have bounded near defect*, Bull. Austral. Math. Soc. 107(3) (2023), 456–463.

[55] G. Sabatino, *Uncountable groups in which all large subgroups are permutably complemented*, submitted.

[56] R. Schmidt, *Subgroup Lattices of Groups*, De Gruyter, Berlin (1994).

[57] I. Schur, *Neuer Beweis eines Satzes über endliche Gruppen*, Proc. Sympos. Appl. Math. (Sitzber. Akad. Wiss., Berlin, 1902), 1013–1019.

[58] A. Shalev, *Profinite Groups with Restricted Centralizers*, Proceedings of the American Mathematical Society 122, no. 4 (1994), 1279—84.

[59] S. Shariari, *On normal subgroups of capable groups*. Arch. Math 48 (1987), 193—198.

[60] S. Shelah, *On a problem of Kurosh, Jónsson groups and applications*, in Word Problems II - the Oxford Book, North-Holland, Amsterdam (1980), 373-–394.

[61] S. P. Strunkov, *Normalizers and abelian subgroups of some classes of groups*, Izv. Akad. Nauk SSSR Ser. Mat. 31 (1967), 657–670.

[62] M. J. Tomkinson, (1981). *On theorems of B.H. Neumann concerning FC-groups*, The Rocky Mountain Journal of Mathematics, 11, 47–58.

[63] M. J. Tomkinson, *FC-groups*, Pitman, Boston (1984).

[64] B. A. F. Wehrfritz, *Schur's theorem and Wiegold's bound*, J. Algebra 504 (2018), 440–444.

[65] J. S. Wilson, *Profinite Groups*, London Math. Soc. Monographs, New Series 19 (Clarendon Press, Oxford, 1998).