Maximum rank distance codes and partitions of affine vector spaces

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To the teachers who believed in me.

Contents

Introduction

Geometry – the study of shapes. Coding theory – transmission of a message in such a way that one can recover the original message even if some part of it is corrupted or lost. These diverse-looking topics are blended by the researchers to find many new results resulting in a significant improvement in both the topics.

Nowadays, it is hard to find an electronic device without some code implemented on it. Therefore, coding theory can be thought of as an application of mathematics without which the evolution of the information age could not be possible to this extend. In this thesis, we will see some aspects where these areas feed off each other, with results and techniques from one informing the other.

Rank metric codes are subsets of matrices where the distance between a pair of codewords is measured as the rank of their difference. These codes were first studied by Delsarte (Delsarte (1978)), particularly for combinatorial reasons. They arise in applications such as error correction in networks, criss-cross error correction, and in code-based cryptography. There are many equivalent ways of representing the rank metric codes. We will see some correspondence here to give an idea of how geometry comes into the picture.

The Chapter 1 is introductory in nature. It provides the essence of this dissertation to the reader. It starts with giving basics of projective spaces over finite fields moving to some special substructures which are studied in classical finite geometry, preciously spreads. In the next section, we start by dealing with linear sets which are certain point sets in projective spaces, generalizing the notion of a subgeometry.

In Section 1.3 we deal with rank metric codes in the general setting. We describe their connection with linearized polynomials and linear sets. These connections are essential, as they highlight how ideas from geometry can be applied to solve problems in coding theory, particularly in the context of rank metric codes. We also give a detailed list of all the known maximum scattered linear sets over projective line which has found connection with MRD codes having high minimum distance. The chapter also introduces bilinear forms and the polarities. Bilinear forms are used to define polar spaces, and the chapter explore the generators and properties of these spaces. The most important part of this section is that where we describe the spread of polar spaces. This part of the chapter mainly deals with a brief introduction to our final chapter which in turn addresses (affine) vector space partitions.

Chapter 2 treats two main problems (1) as per Lunardon & Polverino (2004) every linear set can be viewed as a subgeometry or as a projection of a subgeometry from a suitable vertex to a suitable axis; we want to find geometric ways to determine known maximum scattered linear sets over the projective line; (2) determine the inequivalence classes in a family of MRD codes discovered by Neri et al. (2022).

The chapter begins by listing all the known maximum scattered linear sets over projective line and describe also connection they have with MRD codes having high minimum distance. Then we give a geometric construction of the vertex for an evasive linear set of the projective space $\mathrm{PG}(r-1, q^n)$. Later in the Section 2.3 we start focusing on the projective line in order to dig deeper and establish a geometric description of the vertex for all the known maximum scattered linear sets following the same idea used by Zanella & Zullo (2020) and Csajbók & Zanella (2016*b*) in their articles. Taking into account an interesting Valentina Pepe's work (Pepe (2024)) we were able to give a characterization of the vertex for all the known maximum scattered linear sets over $\mathrm{PG}(1, q^n)$.

Moving forward in this chapter we solve the equivalence issue left as an open problem in Section 5 of Neri et al. (2022). The authors extended a class of MRD code with minimum distance $(n-1)$ and rank 2 constructed using an object known from finite geometry, called a linear set. They find out under which circumstances the corresponding MRD codes can be called 'new' for $n \geq 5$. The necessity of their main theorem (see Theorem 4.6), still holds if $t = 3, 4$. However, as underlined in the initial part of (Neri et al. 2022, Section 4), in such cases the sufficiency part, and more generally the equivalence issue for the codes require more demanding computations and is left as an open problem (see (Neri et al. 2022, Section 5)). In this chapter we deal with these problems, extending their result. We also determined the number of inequivalent codes for $t = 3, 4$. Finally, we showed that for $t = 4$, the linear sets of $\mathrm{PG}(1, q^8)$, ensuing from codes in the relevant family, are not equivalent to any scattered linear set known so far.

Vector space partitions are a powerful tool for simplifying complex problems, improving efficiency, and enabling more precise computations in a wide variety of applications. In Chapter 3, we consider (affine) vector space partitions (abbreviating them as avsps) of a special kind, preciously avsps which have properties like irreducibility and tightness. We introduce hyperbolic, parabolic or elliptic affine spread of $\mathrm{PG}(n,q)$ and show them to be equivalent to spread of $\mathcal{Q}^+(n+1,q)$, $\mathcal{Q}(n+1,q)$ and $\mathcal{Q}^-(n+1,q)$, respectively. Based on our results the extension problem formulated in (Bamberg et al. 2023, Conjecture 2) is shown to be equivalent to that of the existence of a spread of the triality quadric $\mathcal{Q}^+(7,q)$, which in turn is equivalent to establish the existence of an ovoid of $\mathcal{Q}^+(7,q)$.

Personal Contribution

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- Grimaldi et al. (2024) Grimaldi, G.G., Gupta, S., Longobardi, G., Trombetti, R. A geometric characterization of known maximum scattered linear sets of $\mathrm{PG}(1, q^n)$. Submitted. https://doi.org/10.48550/arXiv.2405.01374.

- Gupta et al. (2023) Gupta, S., Longobardi, G., Trombetti, R. On the equivalence issue of a class of 2-dimensional linear maximum rank distance codes, *Linear Algebra and its Applications*, 678, 33-56 (2023). https://doi.org/10.1016/j.laa.2023.08.012.

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This thesis is based on work done in first three articles.

CHAPTER 1

Preliminaries

1.1 Projective spaces over finite field

Let us consider an $(n+1)$-dimensional vector space V over $\mathbb{F}_q$, where $\mathbb{F}_q$ is a finite field with q elements, q is a prime power. Consider an equivalence relation on the points of $V \setminus \{0\}$ defined by the proportionality over a certain basis of V, i.e., $x, y \in V$ are equivalent if $\exists \alpha \in \mathbb{F}_q^*$ such that $x = \alpha y$. The set of these equivalence classes, called *points*, is called n-dimensional projective space over $\mathbb{F}_q$ and it is denoted by $\mathrm{PG}(n,q)$ or $\mathrm{PG}(V)$. An r-space π of $\mathrm{PG}(n,q)$ is set of points all of whose representing vectors together with zero forms an $(r+1)$-dimensional subspace of $V(n+1,q)$. Therefore, we can say the projective space $\mathrm{PG}(n,q)$ is a geometry whose points, lines, planes, $\ldots$, hyperplanes are the subspaces of V of dimension 1, 2, 3,$\ldots$,n.

The number of r-spaces of $\mathrm{PG}(n,q)$ or number of $(r+1)$-spaces of V is given by

$$\begin{bmatrix} n+1 \\ r+1 \end{bmatrix}_q = \prod_{i=1}^{r+1} \frac{q^n - q^i}{q^k - q^i}.$$

Remark 1.1.1 We can see that in the above fraction, numerator is the number of ordered sets of $(r+1)$-linearly independent vectors in V and the denominator is number of ordered sets of $(r+1)$-linearly independent vectors that generate the same $(r+1)$-dimensional subspace of V.

We will be using vector subspace and projective subspace interchangeably when the context is clear.

The number of r-dimensional subspaces of $\mathrm{PG}(n,q)$ containing a fixed s-dimensional subspace is given by $\begin{bmatrix} n-s \\ r-s \end{bmatrix}_q$.

If H_∞ is an hyperplane of $\mathrm{PG}(n-1,q)$, then $\mathrm{AG}(n-1,q) = \mathrm{PG}(n-1,q) \setminus H_\infty$ is an $(n-1)$-dimensional Desarguesian affine space over $\mathbb{F}_q$. The subspaces of $\mathrm{AG}(n-1,q)$ are $U \setminus H_\infty$, where U is a subspace of $\mathrm{PG}(n-1,q)$. We say that H_∞ is the hyperplane at infinity of affine space $\mathrm{AG}(n-1,q)$. We now define the concept of equivalence between two projective spaces.

Definition 1.1.2 Let V_1, V_2 be two n-dimensional $\mathbb{F}_q$-vector spaces. A *collineation* between $\mathrm{PG}(V_1)$ and $\mathrm{PG}(V_2)$ is a bijection between their points which preserves incidence, i.e., the bijection Φ is a collineation between $\mathrm{PG}(V_1)$ and $\mathrm{PG}(V_2)$ if for any two subspaces Π, Π' of $\mathrm{PG}(V_1)$, then $\Pi \subseteq \Pi'$ implies $\Pi^\Phi \subseteq \Pi'^\Phi$.

The set of all automorphisms forms a group, namely $\mathrm{Aut}(\mathbb{F}_q)$, where the binary operation is composition. Let $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$ and $A \in \mathrm{GL}(n+1,q)$. Then

$$F_{\sigma,A} : x \in V(n+1,q) \mapsto x^\sigma A \in V(n+1,q)$$

is a semilinear isomorphism of $V(n+1,q)$. The set of all semilinear isomorphism is denoted by $\Gamma\mathrm{L(n+1,q)}$. As $F_{\sigma,A}(\lambda x) = \lambda^\sigma F_{\sigma,A}(x)$, it defines a bijection on the points of $\mathrm{PG}(n,q)$. Therefore $F_{\sigma,A}$ defines a collineation of $\mathrm{PG}(n,q)$ called *projective semilinear map*. If $\sigma = id$, the collineation is called projectivity. The set of all projectivities of $\mathrm{PG}(n,q)$ and set of all projective semilinear map forms a group under composition called projective linear group and projective semilinear group, denoted by $\mathrm{PGL}(n+1,q)$ and $\mathrm{P\Gamma L}(n+1,q)$, respectively.

Theorem 1.1.3 *(Fundamental Theorem of Projective Geometry) Any collineation of* $\mathrm{PG}(n,q)$ *is a projective semilinear map.*

Let $\mathrm{PG}(n,q)^*$ denote the projective space whose points are the hyperplanes of $(n+1)$-dimensional vector space V over $\mathbb{F}_q$ and where, for each non-trivial r-dimensional subspace U of V, we have a subspace of $\mathrm{PG}(n,q)^*$ consisting of the hyperplanes containing U. The set system $\mathrm{PG}(n,q)$ is isomorphic to $\mathrm{PG}(n,q)^*$ (see Ball (2015)). A collineation between $\mathrm{PG}(n,q)$ and its dual $\mathrm{PG}(n,q)^*$ is called reciprocity of $\mathrm{PG}(n,q)$.

If $\sigma = id$, the reciprocity is called correlation. A correlation of $\mathrm{PG}(n,q)$ having order two is called polarity.

If ρ is a polarity of $\mathrm{PG}(n,q)$, P is a point and H is a hyperplane, then the hyperplane P^ρ is called the polar of P and the point H^ρ is called the pole of H. It follows immediately from the fact that ρ is an incidence preserving mapping that when a point Q is incident with the hyperplane P^ρ, the point P is incident with the hyperplane Q^ρ, i.e., $P \in Q^\rho \iff Q \in P^\rho$. In this case the points P and Q are conjugate points. If P is on the hyperplane P^ρ, then P is a self-conjugate or totally isotropic point. The notions of polar subspace and self-conjugate subspace can be extended to arbitrary dimensions. This means that Π and Π' are conjugate if $\Pi^\rho \subseteq \Pi'$ or $\Pi'^\rho \subseteq \Pi$ and Π is self-conjugate or totally isotropic if $\Pi \subseteq \Pi^\rho$. In this case it follows that Π is at most an $\lfloor \frac{n-1}{2} \rfloor$-space of $\mathrm{PG}(n,q)$.

We will consider $\mathrm{PG}(n,q)$ equipped with homogeneous projective coordinates $X = \langle (X_1, \cdots, X_{n+1}) \rangle_{\mathbb{F}_q}$. Hence a hyperplane H of $\mathrm{PG}(n,q)$ can be described by a homogeneous equation $u_1 X_1 + \cdots + u_n X_n = 0$ (or by a vector $u = [u_1, \ldots, u_n]$) and a k-space Π of $\mathrm{PG}(n,q)$ by a system of $(n-k)$ homogeneous linear equations. A point x belongs to Π if and only if $xu^t = 0$. Let Φ be a collineation of $\mathrm{PG}(n,q)$ induced by a matrix $A \in \mathrm{GL}(n+1,q)$ and $\sigma \in \mathrm{Aut}(\mathbb{F}_q)$, then Π^Φ is the intersections of hyperplanes $u^\sigma A^{-1}$. Indeed $xu^t = 0$ if and only if $x^\sigma A^t A^{-t} (u^\sigma)^t = 0$. A classical problem in finite geometry is the study of interesting substructures such as arcs, caps, spreads, etc. We aim here to explore spreads. We give a formal definition and then discuss how they are related to MRD codes.

Definition 1.1.4 Let r be a positive integer. An r-spread of $\mathrm{PG}(n,q)$ (resp. of $\mathbb{F}_q^{n+1}$) is a set of r-dimensional projective subspaces (resp. vector subspaces of dimension (r+1) minus 0) partitioning $\mathrm{PG}(n,q)$ (resp. of $\mathbb{F}_q^{n+1} \setminus 0$).

The number of elements in r-spreads of $\mathrm{PG}(n,q)$ is exactly $\frac{q^{n+1}-1}{q^{r+1}-1}$. Hence, if such a spread exists then $r+1$ divides $n+1$.

Construction 1.1.5 Let $n+1 = (r+1)(s+1)$. Let α be a root of an irreducible polynomial of degree $r+1$ over $\mathbb{F}_q$. Every element $x \in \mathbb{F}_{q^{r+1}}$ can be written as

$$x = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_r\alpha^r$$

were $a_i \in \mathbb{F}_q$ and let $(x_0, x_1, \ldots, x_s) \in \mathbb{F}_{q^{r+1}}^{s+1}$ and consider $\forall j = 0, 1, \ldots, s$

$$x_j = a_{j0} + a_{j1}\alpha + a_{j2}\alpha^2 + \cdots + a_{jr}\alpha^r.$$

Let $\phi : \mathbb{F}_{q^{r+1}}^{s+1} \mapsto \mathbb{F}_q^{n+1}$ be an $\mathbb{F}_q$-linear map, defined as

$$\phi(x_0, x_1, \ldots, x_s) = (a_{00}, a_{01}, \ldots a_{0r}, \ldots, a_{s0}, \ldots, a_{sr}).$$

Now, it is easy to observe that if S is an h-dimensional subspace of $\mathbb{F}_{q^{r+1}}^{s+1}$ then $\phi(S)$ is an $h(r+1)$-dimensional subspace of $\mathbb{F}_q^{n+1}$. In particular, if H is an 1-dimensional subspace $\mathbb{F}_{q^{r+1}}^{s+1}$, then $\phi(H)$ is an $(r+1)$-dimensional subspace of $\mathbb{F}_q^{n+1}$. The set of all 1-dimensional subspaces of $\mathbb{F}_{q^{r+1}}^{s+1} \setminus \{0\}$, is an 1-spread of $\mathbb{F}_{q^{r+1}}^{s+1}$ to which there corresponds an $(r+1)$-spread of $\mathbb{F}_q^{n+1}$.

The spreads constructed above are called *linear* spread of $\mathbb{F}_q^{n+1}$. The map ϕ is called field reduction and using this construction it is easy to prove the following.

Proposition 1.1.1 *Let r and n be two integer. There exists an r-spread S of* $\mathrm{PG}(n, q)$ *if and only if $r + 1$ divides $n + 1$.*

In the following sections, we will see a point set called *linear* sets constructed using these spreads and then consider equivalent way to define it. We will further elaborate their connections with linear codes.

1.2 Linear sets

We now see a projective setting of the spreads we just discussed. Let $\mathrm{PG}(n, q) = \mathrm{PG}(V, \mathbb{F}_q)$ and using above description we know that each point P of $\mathrm{PG}(s, q^{r+1})$ defines an r-dimensional subspace X_P of $\mathrm{PG}(n, q)$. The set $S = \{X_P : P \in \mathrm{PG}(s, q^{r+1})\}$ defines an r-spread called Desarguesian spread of $\mathrm{PG}(n, q)$. Also, the incidence structure (S, L) whose points are the elements of S and whose lines are the $(2r-1)$-dimensional subspaces spanned by two elements of S is isomorphic to $\mathrm{PG}(s, q^{r+1})$ and is so called linear representation of $\mathrm{PG}(s, q^{r+1})$ over $\mathbb{F}_q$. A t-dimensional $\mathbb{F}_q$-vector subspace U of V defines a $(t-1)$-dimensional projective subspace $\mathrm{PG}(U, \mathbb{F}_q)$

in $\mathrm{PG}(n,q)$ and the set L_U of points of $\mathrm{PG}(s,q^{r+1})$ defined as

$$L_U = \{P \in \mathrm{PG}(s,q^{r+1}) : X_P \cap \mathrm{PG}(U,\mathbb{F}_q) \neq \emptyset\}$$

is called a linear set. The term *linear* was introduced in Lunardon (1999) where a special kind of blocking sets of the incidence structure (S,L) was considered. In particular it was gaven a relation between normal spreads and linear k-blocking sets. Any r-spread S is said to be normal if it induces a spread in any subspace generated by two elements of S (i.e., if $T = \langle A, B\rangle$ with $A, B \in S$ then an element of S is either disjoint from T or is contained in T). In the two decades after this work, linear sets have been intensively investigated due to their applications in construction and characterization of various objects in finite geometry, such as blocking sets, complete sets, two-intersection sets, translation spreads of the Cayley generalized Hexagon, translation ovoids of polar spaces, semifield flocks, semifields and rank-metric codes. We refer to Bacsó et al. (2013), Bader et al. (2007), Bonoli & Polverino (2005*a*), Ball et al. (2003), Bartoli et al. (2018), Blokhuis & Lavrauw (2000, 2002), Bonoli & Polverino (2005*a*,*b*), Cardinali et al. (2002, 2003, 2006), Costa (2016), Cossidente et al. (2007), Ebert et al. (2009*a*,*b*), Johnson et al. (2008, 2009), Jungnickel et al. (2018), Lavrauw (2006), Lavrauw et al. (2015), Lunardon & Polverino (2000, 2001, 2003, 2004), Marino et al. (2007, 2008), Polito & Polverino (1998, 1999, 2002), Lunardon (2001, 2003), and the references therein. Next, we give an equivalent definition of linear set which is common in current research scenarios.

Let V be a vector space over $\mathbb{F}_{q^n}$ of dimension r and $\Omega = \mathrm{PG}(V,q^n) = \mathrm{PG}(r-1,q^n)$. A point set L_U of Ω is called an $\mathbb{F}_q$-*linear set* of *rank* k if it consists of the points defined by the non-zero elements of an $\mathbb{F}_q$-subspace U of V of dimension k, that is,

$$L_U = \left\{\langle \mathbf{u}\rangle_{\mathbb{F}_{q^n}} : \mathbf{u} \in U \setminus \{\mathbf{0}\}\right\}.$$

Here $\langle \mathbf{u}\rangle_{\mathbb{F}_{q^n}}$ denotes the projective point in $\mathrm{PG}(r-1,q^n)$ corresponding to the vector u, where the notation reflects the fact that all $\mathbb{F}_{q^n}$-multiples of u define the same projective point. By Grassmann's Identity, it can be easily seen that if $k > (r-1)n$, then $L_U = \mathrm{PG}(V,q^n)$. Hence L_U is a proper subset of $\mathrm{PG}(V,q^n)$ if and only if $\dim \mathrm{PG}(U,F_q) \leq rn-n-1$, i.e., the rank of the linear set is at most $rn-n$. A linear set of rank $rn - n$ is said to be of maximum rank. Note that, if $\dim U_1 = rn - n + 1$ and $\dim U_2 = \dim U_1 + 1$ then $L_{U_1} = L_{U_2} = PG(V, q^n)$. So rank of a linear set cannot be defined just by knowing the pointset of linear set L.

Remark 1.2.1 If U is $\mathbb{F}_q$- subspace of V and $\lambda \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$, then $L_U = L_{\lambda U}$.

Definition 1.2.2 Let L_U be a linear set of rank k and $\Lambda = \mathrm{PG}(W, q^n)$, where W is an $\mathbb{F}_{q^n}$-subspace of V of dimension t, then we say Λ has weight i with respect to L_U if $\dim_{\mathbb{F}_q}(W \cap U) = i$ and denote the weight of Λ in L_U by $w_{L_U}(\Lambda) = i$.

For each $i \in \{1, \cdots, n\}$, let $N_i(L_U)$ denote the number of points in $\mathrm{PG}(k-1, q^n)$ of weight i. We denote it by N_i if there is no ambiguity in definition. We state some basic properties for weight.

Proposition 1.2.1 *If L_U is an $\mathbb{F}_q$-linear set of $\Omega = \mathrm{PG}(r-1, q^n)$ of rank $k > 0$, then*

1. $|L_U| = N_1 + N_2 + \cdots + N_n$
2. $N_1 + (q+1)N_2 + \cdots + (q^{n-1} + \cdots + q + 1)N_n = q^{k-1} + q^{k-2} + \cdots + q + 1$
3. $|L_U| \leq q^{k-1} + q^{k-2} + \cdots + q + 1$,
4. $|L_U| \equiv 1 \pmod q$.

Definition 1.2.3 A canonical subgeometry Σ of $\Omega = \mathrm{PG}(V, \mathbb{F}_{q^n}) = \mathrm{PG}(r-1, q^n)$ is an $\mathbb{F}_q$-linear set of Ω defined by the non-zero vectors of an r-dimensional $\mathbb{F}_q$-vector subspace U of V such that $\langle U \rangle = V$.

Proposition 1.2.2 *Two canonical subgeometries of Ω on the same field are isomorphic.*

Any canonical subgeometry of $\mathrm{PG}(r-1, q^n)$ is equivalent to the linear set

$$\{\langle (a_0, \ldots, a_{r-1}) \rangle_{\mathbb{F}_{q^n}} : (a_0, \ldots, a_{r-1}) \in \mathbb{F}_q^r \setminus \{\mathbf{0}\}\} \cong \mathrm{PG}(r-1, q).$$

Let σ be the semilinear collineation such that $\sigma : (x_0, ..., x_{r-1}) \mapsto (x_0^q, ..., x_{r-1}^q)$. It is easy to see that σ is a collineation of order n that fixes Σ pointwise (i.e., $\mathrm{Fix}(\sigma) = \Sigma$). We will denote by $\mathrm{Aut}(\Sigma)$ the collineation group of Ω fixing Σ. In particular, it is know that $\mathrm{Aut}(\Sigma) \cong \mathrm{PGL}(r+1,q) \rtimes \mathrm{Aut}(\mathbb{F}_{q^n})$, where $\rtimes$ denotes semidirect product. We denote by $\mathrm{LAut}(\Sigma)$, the linear part of $\mathrm{Aut}(\Sigma)$ which is isomorphic to $\mathrm{PGL}(r+1,q)$. We see that the point 3 of above Proposition gives an upper bound on the number of points of a linear set L_U of rank k. If the equality holds then the linear set is said to be *scattered*. Here, we say that L_U is scattered if and only if any point of Ω has weight atmost one in L_U, i.e, for any point $P = \langle v\rangle_{\mathbb{F}_{q^n}}$ in Ω we have $\dim_{\mathbb{F}_q}(U \cap \langle v\rangle_{\mathbb{F}_{q^n}}) \leq 1$. Here it is interesting to mention a result which tells us about what can be the maximum rank of a scattered linear set. We call the linear set having maximum possible rank to be maximum linear set.

Theorem 1.2.4 *Blokhuis & Lavrauw (2000) If L_U is an $\mathbb{F}_q$-scattered linear set of* $\mathrm{PG}(r-1,q^n)$ *then rank of L_U is atmost $\lfloor \frac{rn}{2} \rfloor$.*

Furthermore, if rn is even then Csajbók et al. (2017) give an explicit construction of scattered linear set of rank $\frac{rn}{2}$. Here, we can note that if $r = 2$, then there always exists a scattered linear set of rank n. The Chapter 2 is completely dedicated to these kind of linear sets because of their major connection with the maximum rank metric codes.

Definition 1.2.5 The *maximum field of linearity* of an $\mathbb{F}_q$-linear set L is $\mathbb{F}_{q^d}$, where $d|n$ and it is the largest integer such that $L = L_U$ for some $\mathbb{F}_{q^d}$-subspace U of V.

Definition 1.2.6 Two $\mathbb{F}_q$-linear sets L and L' in Ω are said to be equivalent if there exists $\phi \in \mathrm{P\Gamma L}(r,q^n)$ such that $L^\phi = L'$.

Proposition 1.2.3 *If U and U' are two $\mathbb{F}_q$-subspaces of V in the same $\Gamma\mathrm{L}(r,q^n)$ orbit then L_U and $L_{U'}$ are equivalent.*

Remark 1.2.7 The converse of above statement is not true. For example, if $n = 2t$, $\gcd(n,s) = 1$ and $U = \{(x,x^q) : \mathbb{F}_{q^n}\}$ and $U' = \{(x,x^{q^s}) : \mathbb{F}_{q^n}\}$ then $L_U = L_{U'} = \{\langle(x,x^{q-1})\rangle_{q^n} : \mathbb{F}^*_{q^n}\}$ but there does not exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,q^n)$, and $\sigma \in \mathrm{Aut}(\mathbb{F}_{q^n})$ such that for each $y \in \mathbb{F}_{q^n}$ there exists $z \in \mathbb{F}_{q^n}$ with the property that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y \\ x^q \end{pmatrix}^{\sigma} = \begin{pmatrix} z \\ z^{q^s} \end{pmatrix}.$$

In this remark we saw that two $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^n}^2$ defining the same scattered linear set in $\mathrm{PG}(1, q^n)$ are not necessarily on the same orbit under the action of $\Gamma\mathrm{L}(2, q^n)$. In Lunardon & Polverino (2004) the authors show that every linear set is either a canonical subgeometry or the projection of a canonical subgeometry. Let $\Sigma = \mathrm{PG}(r-1, q)$ be a canonical subgeometry of $\Sigma^* = \mathrm{PG}(r-1, q^n)$. Suppose there is an $(r-m-1)$-dimensional subspace Γ of Σ^* disjoint from Σ. Let $\Lambda = \mathrm{PG}(m-1, q^t)$ be an $(m-1)$-dimensional subspace of Σ^* disjoint from Γ. Then, the set

$$p_{\Gamma,\Lambda}(\Sigma) = \{P \in \Lambda \mid \exists\, Q \in \Sigma \text{ such that } P = \langle \Gamma, Q\rangle \cap \Lambda\}$$

is called the projection of Σ from Λ to Γ. We call Γ and Λ the vertex and the axis of the projection, respectively. The following two important results were given by Lunardon and Polverino which demonstrates a geometric description of linear sets.

Theorem 1.2.8 *Lunardon & Polverino (2004) If L is a projection of Σ from Λ to Γ, then L is an $\mathbb{F}_q$-linear set of Λ of rank m and $\langle L\rangle = \Lambda$.*

The converse of the above theorem can be seen as follows

Theorem 1.2.9 *Lunardon & Polverino (2004) If L is an $\mathbb{F}_q$-linear set of $\Lambda = \mathrm{PG}(m-1, q^t)$, then either L is a canonical geometry or there is a $(r-m-1)$ dimensional subspace Γ of $\Sigma^* = \mathrm{PG}(r-1, q^t)$ disjoint from Λ and a canonical subgeometry Σ of Σ^* disjoint from Γ, such that $L = p_{\Gamma,\Lambda}(\Sigma)$.*

In the first part of Chapter 2, we provide a geometric way to construct Γ and Λ using a special point of Σ^* which we define now. Let P be a point of Σ^*, and define the subspace $\mathcal{L}_{P,\sigma} = \langle P, P^\sigma, \ldots, P^{\sigma^{n-1}}\rangle$ of Σ^*. Clearly, this subspace of Σ has dimension at most $n-1$. The point P is called an *imaginary* point of Σ^*, with respect to Σ, if $\dim \mathcal{L}_{P,\sigma} = n-1$. We now define *evasive* linear sets.

Definition 1.2.10 Bartoli, Csajbók, Marino & Trombetti (2021) A linear set $L \subset \mathrm{PG}(V, q^n)$ is said to be $(h,k)_q$-*evasive* if there exists a subspace $U \leq V$ with $L = L_U$ such that $\dim\langle L\rangle \geq h-1$, and for every h-dimensional $\mathbb{F}_{q^n}$-subspace T of V, we have that $L_{U\cap T}$ has rank at most k.

Note that if $h = k = 1$, then L_U is scattered.

Remark 1.2.11 It should be noted that if Σ is a subgeometry of Σ^* then for each subspace S^* of Σ^* we can define a set $S = \Sigma^* \cap \Sigma$, which is a subspace of Σ whose rank is at most equal to the rank of S^*. We say that a subspace S^* of Σ^* is a subspace of Σ if S and S^* has same rank.

1.3 Rank metric codes

Rank metric codes have gained a lot of attention from researchers due to there applications in network coding, post quantum cryptography and distributed storage. Rank distance codes were introduced in Delsarte (1978) and independently in Gabidulin (1985). The set of $m \times n$ matrices over $\mathbb{F}_q$ denoted by $\mathbb{F}_q^{m\times n}$ is a rank metric $\mathbb{F}_q$-space where the metric endowed is

$$d(A, B) = \mathrm{rk(A - B)}.$$

Let C be a subset of $\mathbb{F}_q^{m\times n}$, we call C to be a rank distance code (shortly called as RD code) and the minimum distance d is given by

$$d = d(C) = \min_{A,B\in C A\neq B}\mathrm{rk(A - B)}.$$

We will refer to C as to an RD code with parameter $(m, n; q, d)$. In the seminal article by Delsarte they also proved that the code must follow singleton like bound.

Theorem 1.3.1 *Let $C \subset \mathbb{F}_q^{m\times n}$ be a rank metric code with parameter $(m, n; q, d)$, then*

$$|C| \leq q^{\mathrm{max(m,n)(min(m,n)-d+1)}}.$$

If the bound is met we say C is a *maximum rank distance code*, or abbreviated as *MRD* codes. When C is an $\mathbb{F}_q$-subspace of $\mathbb{F}_q^{m\times n}$, then we call C an $\mathbb{F}_q$-linear RD code. There are many equivalent ways to represent RD codes. In this work we preciously need the connection of RD codes with linearized polynomials. Any linear RD code over $\mathbb{F}_q$ can be either seen as subspaces of $\mathbb{F}_q^{m\times n}$ or as subspaces of $\mathrm{Hom(V_n, V_m)}$. Here, $\mathrm{Hom(V_n, V_m)}$ is set of all homomorphism from V_n (vector space of dimension n) to V_m (vector space of dimension m). Next we detail how this relation is suppose to work. Let us consider two vector spaces V_m and V_n over $\mathbb{F}_q$ of dimension m and n, respectively. If $n \geq m$, we can see V_m as a subspace of V_n and we identify $\mathrm{Hom(V_n, V_m)}$ as subspaces of those homomorphisms $\varphi \in \mathrm{Hom(V_n, V_n)}$ such that $\mathrm{Im}(\varphi) \subseteq V_m$. Now let $\mathrm{Hom_q(\mathbb{F}_{q^n})} := \mathrm{Hom_q(\mathbb{F}_{q^n}, \mathbb{F}_{q^n})}$ be the set of all linear maps over $\mathbb{F}_{q^n}$; it is well know that, for any $\varphi \in \mathrm{Hom_q(\mathbb{F}_{q^n})}$ there is an unique linearised polynomial $f(x)$ associated to it; let it be

$$f(x) := \sum_{i=0}^{n-1} c_i x^{q^i},$$

where $c_i \in \mathbb{F}_{q^n}$ such that for all $x \in \mathbb{F}_{q^n}$, we have $\varphi = f(x)$. Assume C be an RD code over $\mathbb{F}_q^{m\times n}$ and $n \geq m$. Consider $M \in C$ be an $m \times n$ matrix over $\mathbb{F}_q$. Let Φ be an element of $\mathrm{Hom(V_n, V_m)}$ associated with matrix M and identify it with $\mathrm{Hom_q(\mathbb{F}_{q^n})}$ as explained above to get a linearised polynomial $f(x)$. Now, it is straightforward to see that any $\mathbb{F}_q$-linear RD-code might be regarded as a suitable $\mathbb{F}_q$-subspace of $\tilde{\mathcal{L}}_{n,q}[x]$. This approach shall be extensively used in the present work.

Let $n \in \mathbb{N}$, q a prime power and denote by $\mathcal{L}_{n,q}[x]$ the vector space of all $\mathbb{F}_q$-linearized polynomials with coefficients over $\mathbb{F}_{q^n}$ (also known as *q-polynomials*); i.e.,

$$\mathcal{L}_{n,q}[x] = \left\{ \sum_{i=0}^{r} c_i x^{q^i} \,:\, c_i \in \mathbb{F}_{q^n} \text{ and } r \geq 0 \right\}.$$

We can consider a quotient of $\mathcal{L}_{n,q}[x]$ with its two-sided ideal $(x^{q^n} - x)$ as following

$$\tilde{\mathcal{L}}_{n,q}[x] := \mathcal{L}_{n,q}[x]/(x^{q^n} - x) = \left\{ \sum_{i=0}^{n-1} c_i x^{q^i} \,:\, c_i \in \mathbb{F}_{q^n} \right\},$$

It is known that $\mathbb{F}_q$-algebra $\tilde{\mathcal{L}}_{n,q}[x]$ is isomorphic to set of all $\mathbb{F}_q$-linear endomorphisms of $\mathbb{F}_{q^n}$ (denoted as $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n})$), considering the operations to be addition and composition, as follows (see Lidl & Niederreiter (1997))

$$\theta \in \mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \mapsto \sum_{i=0}^{n-1} c_i \theta^{q^i}.$$

Let ℓ be an integer such that $\ell \mid n$. A very special kind of q-polynomial which appears very often in Galios and duality theories is called *trace* the function of $\mathbb{F}_{q^n}$ over $\mathbb{F}_{q^\ell}$ of an element $x \in \mathbb{F}_{q^n}$ given by $\mathrm{Tr}_{q^n/q^\ell}(x) = \sum_{i=0}^{n/\ell-1} x^{q^{i\ell}}$. There is a multiplicative counterpart of trace function known as *norm* function given by $\mathrm{N}_{q^n/q^\ell}(x) = x^{\frac{q^n-1}{q^\ell-1}}$.

We will now define rank metric codes in terms of q-polynomial. The distance between two different q-polynomial f and g, defined as

$$d(f, g) = \dim(\mathrm{im}(f - g)),$$

gives rise to a metric on $\tilde{\mathcal{L}}_{n,q}[x]$, which is called the *rank metric*. In general, if $\mathcal{C}$ and $\mathcal{C}'$ are two subsets of $\tilde{\mathcal{L}}_{n,q}[x]$, we say that they are *equivalent* if there exist $\rho \in \mathrm{Aut}(\mathbb{F}_{q^n})$ and two permutation polynomials L_1 and L_2 such that

$$\mathcal{C}' = \{(L_1 \circ f^\rho \circ L_2)(x) \,:\, f \in \mathcal{C}\},$$

where $f^\rho(x) = \left(\sum a_i x^{q^i}\right)^\rho = \sum a_i^\rho x^{q^i}$. We denote by $\tau = (L_1, L_2, \rho)$, the equivalence defined above and consequently denote by $\tau(f)$ the elements in $\mathcal{C}'$, see Morrison (2014). A *linear rank-metric* code $\mathcal{C} \leq \tilde{\mathcal{L}}_{n,q}[x]$ is an $\mathbb{F}_q$-subspace of the vector space $\tilde{\mathcal{L}}_{n,q}[x]$ endowed with the above mentioned rank metric $d(\cdot,\cdot)$. The minimum of the set of all distances between two distinct words of a linear rank-metric code $\mathcal{C} \leq \tilde{\mathcal{L}}_{n,q}[x]$, $|\mathcal{C}| \geq 2$, is denoted by $d_{\mathcal{C}}$, and one has

$$d_{\mathcal{C}} = \min\{\dim(\mathrm{im}(f)) \,:\, f \in \mathcal{C} \setminus \{0\}\}.$$

If $\dim_{\mathbb{F}_q}(\mathcal{C}) = k$, then $d_{\mathcal{C}} \leq n - \frac{k}{n} + 1$ and in the case where the equality is attained, $\mathcal{C}$ is said to be a *linear MRD code.*

Example 1.3.1 The set of linearzied polynomials defined as $G_{k,s} := \{f(x) = a_0x + a_1x^{q^s} + a_2x^{q^{2s}} + \ldots + a_{k-1}x^{q^{s(k-1)}} \,:\, a_i \in \mathbb{F}_{q^n}\}$ are called *generalised Gabidulin codes.* First observe that $|G_{k,s}| = q^{nk}$ and $|ker(f)| \leq q^{k-1}$ and therefore $d_{G_{k,s}} \geq n - k + 1$. Now, when $d_{G_{k,s}} \geq n - k + 2$ then $|G_{k,s}| \leq q^{n(k-1)}$ giving us contradiction therefore $d_{G_{k,s}} = n - k + 1$ and $G_{k,s}$ are MRD codes with dimension nk and minimum rank distance $n - k + 1$. If $s = 1$, these codes are called Gabidulin codes.

Still we have not said anything on how linear sets and MRD codes are related. In the following we focus on linear set of rank n in $\mathrm{PG}(r - 1, q^n)$. Particular attention will be paid to the case where $r = 2$, that is, linear sets of rank n on a projective line.

Lemma 1.3.2 *(Lunardon 2017, Sheekey & Voorde 2020, Lemma 7, Lemma 2.1) Sup-*

pose L is an $\mathbb{F}_q$-linear set of rank n in $\mathrm{PG}(r-1,q^n)$*. Then there exist $\mathbb{F}_q$-linear maps $f_i : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ such that*

$$L = \{\langle (f_1(x),\dots,f_r(x))\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\},$$

and $\ker(f_1) \cap \cdots \cap \ker(f_r) = \{0\}$*. If* $\dim\langle L\rangle = r-1$*, then the maps $\{f_1,\dots,f_r\}$ are linearly independent over $\mathbb{F}_{q^n}$. Vice versa, if $f_i : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ are $\mathbb{F}_q$-linear maps with* $\ker(f_1) \cap \cdots \cap \ker(f_r) = \{0\}$*, then $L = \{\langle (f_1(x),\dots,f_r(x))\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$ is an $\mathbb{F}_q$-linear set of rank n in* $\mathrm{PG}(r-1,q^n)$*.*

In the following part we will focus on linear sets of rank n of the projective line $\mathrm{PG}(1,q^n)$. Let us denote by L one of these sets; from the above result we can write $L = \{\langle f_1(x), f_2(x)\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}_{q^n}^*\}$, for suitable $\mathbb{F}_q$-linear maps maps f_1 and f_2 and because it can always be assumed that the point $\langle (0,1)\rangle_{\mathbb{F}_{q^n}}$ does not belong to L, we can always write $L = \{\langle x, g(x)\rangle : x \in \mathbb{F}_{q^n}^*\}$. From now on, we will deal with $L_U = L_g$ where U is the underlying vector space $U := U_g = \{(x,g(x)) : x \in \mathbb{F}_{q^n}\} \leq \mathbb{F}_{q^n}^2$. Further, we recall a seminal result by Sheekey (see Sheekey (2016)) which have really boosted the study of MRD code of rank 2 and with high minimum distance. This means these codes have high error correcting capabilities. Before that we give a definition of scattered polynomials.

Definition 1.3.3 Let $f \in \tilde{\mathcal{L}}_{n,q}[x]$, f is called scattered if $\forall\, x,y \in \mathbb{F}_{q^n}$ we have

$$\frac{f(x)}{x} = \frac{f(y)}{y} \Rightarrow x \text{ and } y \text{ are } \mathbb{F}_q\text{-dependent.}$$

Theorem 1.3.4 *(Sheekey (2016)) Let $f \in \tilde{\mathcal{L}}_{n,q}[x]$ then the following statements are equivalent:*

- *$U_f = \{(x,f(x)) : x \in \mathbb{F}_{q^n}\}$ is scattered subspace.*
- *f is a scattered polynomial.*
- *$\mathcal{C}_f = \{ax+bf(x) : a,b \in \mathbb{F}_{q^n}\} = \langle x, f(x)\rangle_{q^n}$ is an MRD codes minimum distance $(n-1)$.*

Proof. *i)* $\implies$ *ii)*. Suppose that $f(x)/x = f(y)/y$ for some $x,y \in \mathbb{F}_{q^n}^*$. Then $(y,f(y)) \in U_f \cap \langle (x,f(x))\rangle_{\mathbb{F}_{q^n}}$. Since U_f is scattered, $\dim_{\mathbb{F}_q}(U_f \cap \langle (x,f(x))\rangle_{\mathbb{F}_{q^n}}) \leq 1$.

Therefore $(y, f(y)) = \lambda(x, f(x))$ for some $\lambda \in \mathbb{F}_q$, giving that x and y are $\mathbb{F}_q$-linear dependent. Hence, f is scattered polynomial.

ii) $\implies$ *i)*. Let us suppose by contradiction that U_f is not scattered, i.e., there exist some $x \in \mathbb{F}_{q^n}^*$ such that $\dim_{\mathbb{F}_q}(U_f \cap \langle(x, f(x))\rangle_{\mathbb{F}_{q^n}}) > 1$. Then, if $y \in \mathbb{F}_{q^n}^*$ satisfies $(y, f(y)) \in U_f \cap \langle(x, f(x))\rangle_{\mathbb{F}_{q^n}}$, there exists $\lambda \in \mathbb{F}_{q^n} \setminus \mathbb{F}_q$ such that $y = \lambda x$ with $f(x)/x = f(y)/y$. This contradicts the assumption that f is a scattered polynomial.

ii) $\implies$ *iii)*. It is easy to observe that $|\mathcal{C}_f| = q^{2n}$. In order to be a maximum rank distance code, by Singleton bound, $d_{\mathrm{rk}}(\mathcal{C}_f)$ must be atleast $n-1$. Then, it is enough to prove that $\dim_{\mathbb{F}_q} \ker(aX + bf) \in \{0, 1\}$ for every $(a, b) \in \mathbb{F}_{q^n}$ not both zero. Clearly, if $b = 0$, then $\dim_{\mathbb{F}_q} \ker aX = 0$ for any $a \in \mathbb{F}_{q^n}^*$. If $b \neq 0$, prove that $\dim_{\mathbb{F}_q} \ker(aX + bf) \in \{0, 1\}$ for every $a, b \in \mathbb{F}_{q^n} \times \mathbb{F}_{q^n}^*$ is equivalent to show that $\dim_{\mathbb{F}_q}(mX + f) \leq 1$ for every $m \in \mathbb{F}_{q^n}$.

Let $\overline{x}, \overline{y} \in \ker(mX + f)$. Then, $m\overline{x} + f(\overline{x}) = m\overline{y} + f(\overline{y}) = 0$. This implies that $f(\overline{x})/\overline{x} = f(\overline{y})/\overline{y}$ and hence $\overline{y} = \lambda\overline{x}$ for some $\lambda \in \mathbb{F}_q^*$. This gives $\dim_{\mathbb{F}_q} \ker(mx+f) = 1$ and the result follows.

iii) $\implies$ *ii)*. This follows directly by noting that $d_{\mathrm{rk}}(\mathcal{C}_f) = n-1$ implies $\dim_{\mathbb{F}_q} \ker(mX+f)$ is either 0 or 1. Then, if $f(x)/x = f(y)/y = -m$ for some $m \in \mathbb{F}_{q^n}$ and $x, y \in \mathbb{F}_{q^n}^*$, then $y \in \langle x\rangle_{\mathbb{F}_q}$. This concludes the proof.

The only optimal rank-metric codes known were Gabidulin codes, for over three decades, until Sheekey in the same article constructed a new family of optimal rank-metric codes (MRD-codes). This boosted the study and since then there are many scattered polynomials which are studied obtaining new classes of MRD codes.

1.4 Polar Spaces

In this thesis, precisely in the last part of it, we will exhibit an interesting result regarding a kind of (affine) vector space partitions. We will make use of spreads of quadrics to obtain vector space partitions. This section is dedicated to set a background for quadrics and their spreads. Let σ be an automorphism of $\mathbb{F}_q$; a σ-sesquilinear form on V is a map $\beta : V \times V \mapsto \mathbb{F}_q$ such that

1. $\beta(x + y, z) = \beta(x, z) + \beta(y, z)$
2. $\beta(x, y + z) = \beta(x, y) + \beta(x, z)$

3. $\beta(ax, by) = a\sigma(b)\beta(x, y)$

for all $x, y, z \in V$ and $a, b \in \mathbb{F}_q$. If $\sigma = id$, then β is called bilinear. The σ-sesquilinear form β is non-degenerate if $\beta(x, y) = 0$, $\forall x \in V$ implies that $y = 0$. A σ-sesquilinear form β is said to be reflexive if $\beta(x, y) = 0 \iff \beta(y, x) = 0$, for all $x, y \in V$. A vector $u \in U$ is called isotropic if $\beta(u, u) = 0$. A totally isotropic subspace (with respect to β) is a subspace U with the property that $\beta(u, v) = 0$, $\forall u, v \in U$.

Definition 1.4.1 A maximal totally isotropic subspaces is a totally isotropic subspaces not contained in a larger totally isotropic subspaces.

Proposition 1.4.1 *Let β be a non-degenerate reflexive σ-sesquilinear form of V; then β falls in one of the following types:*

1. *β is symmetric: $\sigma = id$ and $\beta(x, y) = \beta(y, x)$,*

2. *β is alternating: $\sigma = id$ and $\beta(x, x) \equiv 0$, which implies that $\beta(x, y) = -\beta(y, x)$,*

3. *β is Hermitian: $\sigma^2 = id$, $\sigma \neq id$, and $\beta(x, y) = \sigma(\beta(y, x))$.*

Example 1.4.1 Let us consider a trace function of $\mathbb{F}_{q^n}$ over $\mathbb{F}_q$ as $\mathrm{Tr} : x \in \mathbb{F}_{q^n} \mapsto x + x^q + \cdots + x^{q^{n-1}} \in \mathbb{F}_q$, let V be a vector space defined as $V = \{(x, y), : x, y \in \mathbb{F}_{q^n}\}$, then β is the $\mathbb{F}_q$-bilinear skew-symmetric (β is alternating if $char(\mathbb{F}_q) \neq 2$ and symmetric if $char(\mathbb{F}_q) = 2$) form defined over $V \times V$ such that $\beta((a_1, b_1), (a_2, b_2)) = \mathrm{Tr}(a_1 b_2 - a_2 b_1)$.

A non-degenerate reflexive sesquilinear form β gives rise to a polarity $\perp$ on $\mathrm{PG}(V, \mathbb{F}_q)$ by mapping a vector subspace U of V to

$$U^{\perp} = \{v \in V | \beta(u, v) = 0 \forall u \in U\}.$$

Definition 1.4.2 Let β be a (non-degenerate) reflexive σ-sesquilinear form on a vector space V. In $\mathrm{PG}(V)$ the set of totally isotropic subspaces with respect to β is called (non-degenerate) finite classical polar space.

It is known that a polar space can be determined by the equation $Q'(x) := \beta(x, x) = 0$. We call any function $Q' : \mathbb{F}_{q^{n+1}} \mapsto \mathbb{F}_q$ satisfying $Q'(\alpha X) = \sigma(\alpha)Q'(X)$ for some involutary field automorphism σ, a *pseudo-quadratic form* (or quadratic form in case σ is the identity). We call Q' non-degenerate in case there does not exist a coordinate transformation in which Q' does not depend on all $n+1$ coordinates. In case q is odd, every non-degenerate quadratic form $Q'(X)$ arises from the non-degenerate symmetric bilinear form $B(X,Y) = \frac{1}{4}(Q'(X+Y) - Q'(X-Y))$. This form fails if q is even, and there exist non-degenerate quadratic forms not arising from symmetric bilinear forms. Given such a quadratic form Q', the equation $Q'(X) = 0$ still defines a point set, and the set of subspaces completely contained in it still forms a polar space. Next we describe the known categories of polar spaces defined with a non-degenerate σ-sesquilinear form or equipped with a non-degenerate quadratic form.

1.4.1 Symplectic polar spaces

Let n is odd (in case n is even all alternating forms are symmetric forms). Consider the non-degenerate alternating form

$$\beta(x,y) = x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2 + \cdots + x_{n-1}y_n - x_ny_{n-1}. \tag{1.1}$$

Let $\perp$ denote the associated polarity in $\mathrm{PG}(n,q)$. The set of totally isotropic subspaces with respect to $\perp$ is denoted by $\mathcal{W}(n,q)$ and is called a symplectic polar space. Note that every point of $\mathrm{PG}(n,q)$ is isotropic.

1.4.2 Hermitian polar spaces

Let β be a Hermitian form over $V(n+1,q^2)$, we may assume

$$\beta(x,y) = x_1y_1^q + x_2y_2^q + \cdots + x_{n+1}y_{n+1}^q. \tag{1.2}$$

This corresponds to the Hermitian polar space $\mathcal{H}(n,q^2)$ of $\mathrm{PG}(n,q^2)$ that is given by the polarity associated with $X_1^{q+1} + ... + X_{n+1}^{q+1} = 0$.

1.4.3 Quadratic forms and Quadrics

Let $Q \in \mathbb{F}_q[X_0,...,X_n]$ be a quadratic form on $\mathrm{PG}(n,q)$ such that

$$Q = \sum_{i,j=1,i\geq j}^{n} a_{ij}X_iX_j. \tag{1.3}$$

The set of points of $\mathrm{PG}(n,q)$ where Q vanishes is called a quadric, respectively. Therefore, up to isomorphism, there are three non-degenerate quadrics.

- For n odd and the quadric form is $Q(X) = X_0X_1+\cdots+X_{n-3}X_{n-2}+f(X_{n-1},X_n)$, where f a non-degenerate quadratic form in two variables, if f is irreducible we have elliptic quadric denoted by $\mathcal{Q}^-(n,q)$,

- for n odd and the quadric form is $Q(X) = X_0X_1+\cdots+X_{n-3}X_{n-2}+f(X_{n-1},X_n)$, if f is reducible we have hyperbolic quadric denoted by $\mathcal{Q}^+(n,q)$,

- for n even and the quadric form is $Q(X) = X_0^2 + X_1X_2 + \cdots + X_{n-1}X_n$ then parabolic quadric denoted by $\mathcal{Q}(n,q)$.

Definition 1.4.3 The *rank r of the polar space is one more than the dimension of its maximal totally isotropic subspace called generator.*

For our purpose we will focus on the quadrics. Let $\mathcal{Q}_{n,e}$ denote a non-degenerate quadric of $\mathrm{PG}(n,q)$ where e is exponent as indicated below:

$\mathcal{Q}_{n,e}$	$\mathcal{Q}^+(n,q)$	$\mathcal{Q}(n,q)$	$\mathcal{Q}^-(n,q)$
e	0	1	2

Consider the bilinear form

$$\beta(X,Y) = Q(X+Y) - Q(X) - Q(Y).$$

This determines a non-degenerate polarity $\perp$ in usual sense. Associated with $\mathcal{Q}$, there is a polarity $\perp$ of $\mathrm{PG}(n,q)$, which is non-degenerate except when $e=1$ and q is even. In particular, the polarity $\perp$ is symplectic if $\mathcal{Q}_{n,e} \in \{\mathcal{Q}^+(n,q), \mathcal{Q}^-(n,q)\}$, q even, and orthogonal if $\mathcal{Q}_{n,e} \in \{\mathcal{Q}(n,q), \mathcal{Q}^+(n,q), \mathcal{Q}^-(n,q)\}$, q odd. For $\mathcal{Q}_{n,1} = \mathcal{Q}(n,q)$, q even, the polarity $\perp$ is degenerate, indeed $N^\perp = \mathrm{PG}(n,q)$, if N is the nucleus of $\mathcal{Q}(n,q)$, whereas $P^\perp$ is a hyperplane of $\mathrm{PG}(n,q)$ for any other point P of $\mathrm{PG}(n,q)$. Every line through N is tangent to $\mathcal{Q}(n,q)$.

Proposition 1.4.2 *A* generator *of $\mathcal{Q}_{n,e}$ is a projective space of maximal dimension contained in $\mathcal{Q}_{n,e}$ and generators of $\mathcal{Q}_{n,e}$ are $\left(\frac{n-e-1}{2}\right)$-spaces.*

Proposition 1.4.3 *The number of k-spaces in $\mathcal{Q}_{n,e}$ is given by a number*

$$\begin{bmatrix} r \\ k+1 \end{bmatrix}_q \prod_{i=1}^{k+1}(q^{r+e-i}+1).$$

Proposition 1.4.4 *Let $\mathcal{Q}_{n,e}$ be a quadric of PG(n, q) and let Π_{k-1} be a $(k-1)$- space of $\mathcal{Q}_{n,e}$. Thus $\Pi_{k-1}^{\perp}$ is an $(n-k)$-space of* $\mathrm{PG}(n,q)$ *intersecting $\mathcal{Q}_{n,e}$ in a cone having as vertex Π_{k-1} and as base a quadric $\mathcal{Q}_{n-k,e}$, where of $\mathcal{Q}_{n-k,e}$ is of the same type as $\mathcal{Q}_{n,e}$.*

Proposition 1.4.5 *A line l of* $\mathrm{PG}(n,q)$ *meets the quadric $\mathcal{Q}_{n,e}$ in 0,1,2 or $q+1$ points. Moreover, a line l through $x \in \mathcal{Q}_{n,e}$ meets $x \in \mathcal{Q}_{n,e}$ in 1 or $q+1$ points if and only if $l \subseteq x^{\perp}$.*

When it comes to generator of hyperbolic quadric, they embed some more interesting property for their generators. Their generators can be partitioned into two classes. Each equivalence class is called the *system of generators*. We mention this useful result here.

Proposition 1.4.6 *Let Π_r and $\overline{\Pi}_r$ be distinct generators of $\mathcal{Q}^+(n,q)$, with $n = 2r+1$. Their possible intersection are as follows.*

1. $n = 4s+1,\ r = 2s,$

$$\dim(\Pi_r \cap \overline{\Pi}_r) = \begin{cases} 0, 2, 4, \dots, 2s-1 & \textit{same system} \\ 1, 3, \dots, 2s-1 & \textit{different system} \end{cases}$$

2. $n = 4s+3,\ r = 2s+1,$

$$\dim(\Pi_r \cap \overline{\Pi}_r) = \begin{cases} -1, 1, 3, \dots, 2s-1 & \textit{same system} \\ 0, 2, \dots, 2s & \textit{different system.} \end{cases}$$

Definition 1.4.4 A *spread* of a polar space $\mathcal{A}$ is a set of generators (i.e., maximal totally isotropic subspaces) of $\mathcal{A}$, which partitions the points of $\mathcal{A}$.

Remark 1.4.5 A spread of $\mathcal{W}(n,q)$ is also a spread of $\mathrm{PG}(n,q)$ into $\left(\frac{n-1}{2}\right)$-dimensional projective subspaces.

Remark 1.4.6 It is easily seen using Proposition (1.4.6) that the two distinct generators never have an empty intersection for $\mathcal{Q}^+(n,q)$, if $n \equiv 1 \pmod 4$. Therefore, $\mathcal{Q}^+(n,q)$, if $n \equiv 1 \pmod 4$ has no spread. Many authors investigated spreads of polar spaces, see Conway et al. (1988), Dye (1977), Kantor (1982*a*,*b*,*c*), Moorhouse (1993), Shult (1985), Thas (1992). In the following, we give a complete list indicating the parameters for spreads in polar space to exists.

Polar space	Existence of spreads
$\mathcal{W}(n,q)$, $n = 2t$ and $t \geq 1$	Yes
$\mathcal{Q}(2n,q)$, $n \geq 2$ and q even	Yes
$\mathcal{Q}(6,q)$, q odd, with q prime or $q \equiv 0$ or $2(mod 3)$	Yes
$\mathcal{Q}(4n,q)$, q odd	No
$\mathcal{Q}^+(3,q)$	Yes
$\mathcal{Q}^+(7,q)$, q odd, with q prime or $q \equiv 0$ or $2(mod 3)$	Yes
$\mathcal{Q}^+(4n+3,q)$, q even	Yes
$\mathcal{Q}^+(4n+1,q)$	No
$\mathcal{Q}^-(5,q)$	Yes
$\mathcal{Q}^-(2n+1,q)$, $n > 2$, q even	Yes
$\mathcal{H}(4,4)$	No
$\mathcal{H}(2n+1,q^2)$	No

Table 1.1: Existence of spreads

CHAPTER 2

Scattered linear set of the projective line

In this chapter, we will deal with a special class of linear sets and the MRD codes ensuing from them. As discussed in the introduction, every linear set can be seen as subgeometry or projection of subgeometry from a suitable vertex to a suitable axis.

We will start by showing a method to reconstruct the vertex Γ for a peculiar class of linear sets of rank $t = n(r-1)$ in $\mathrm{PG}(r-1, q^n)$ called *evasive* linear sets. Building on this result we will characterize several families of linear sets of $\mathrm{PG}(1, q^n)$ introduced from 2018 onward. Then we will elaborate on the equivalence issue of a quite large class of 2-dimensional linear MRD codes introduced in Neri et al. (2022) having minimum distance $d = n-1$; indeed, associated with maximum scattered linear sets of $\mathrm{PG}(1, q^n)$.

2.1 Known maximum scattered linear set over projective line

In this section, we summarise all the maximum scattered linear sets known so far over the projective line. The first type of maximum scattered linear set of *pseudoregulus* *type* was first introduced in Blokhuis & Lavrauw (2000) given by,

$$L^{1,n}_s = \{\langle (x, x^{q^s})\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}^*_{q^n}\}; \tag{2.1}$$

In Lunardon & Polverino (2001), Lunardon and Polverino presented a class of linear sets which was generalised later by Sheekey in Sheekey (2016). They are known in the literature as linear sets of *LP-type* and an element of this family is PΓL-equivalent to

$$L^{2,n}_{s,\eta} = \{\langle (x, \eta x^{q^s} + x^{q^{n-s}})\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}^*_{q^n}\}, \tag{2.2}$$

where $\mathrm{N}_{q^n/q}(\eta) \notin \{0, 1\}$, $\gcd(s, n) = 1$ for any $n \geq 3$ and $q > 2$. When $n \in \{6, 8\}$, in Csajbók, Marino, Polverino & Zanella (2018), the following family of $\mathbb{F}_q$-linear sets was presented

$$L^{3,n}_{\ell,\eta} = \{\langle (x, \eta x^{q^\ell} + x^{q^{\ell+n/2}})\rangle_{\mathbb{F}_{q^n}} : x \in \mathbb{F}^*_{q^n}\}, \tag{2.3}$$

where $\gcd(\ell, n/2) = 1$ and $\eta \in \mathbb{F}_{q^n}$ such that $\mathrm{N}_{q^n/q^{n/2}}(\eta) \notin \{0, 1\}$. Moreover in Csajbók, Marino, Polverino & Zanella (2018), it was shown that, for some choices of q and η, $L^{3,n}_{\ell,\eta}$ turns out to be scattered, see (Csajbók, Marino, Polverino & Zanella 2018, Theorem 7.1 and Theorem 7.2). Later, necessary and sufficient conditions for these examples in order to be scattered, were also provided. Precisely,

- if $n = 6$ and $q > 2$, $L^{3,n}_{2,\eta}$ is scattered if and only if the equation

  $$Y^2 - (\mathrm{Tr}_{q^3/q}(\gamma) - 1)Y + \mathrm{N}_{q^3/q}(\gamma) = 0,$$

  where $\gamma = -\frac{\eta^{q^3+1}}{1-\eta^{q^3+1}}$, admits two solutions in $\mathbb{F}_q$, see Bartoli, Csajbók & Montanucci (2021) and (Polverino & Zullo 2020, Theorem 7.3),

- if $n = 8$ and $q \leq 11$ or $q \geq 1039891$ odd, $L^{3,n}_{\ell,\eta}$ is scattered if and only $\mathrm{N}_{q^8/q^4}(\eta) = -1$, see Timpanella & Zini (2023).

In 2018, another example was found in the case when $n = 6$ and q is odd, consisting of linear sets with the following shape:

$$L^{4,6}_\delta = \{\langle (x, x^q + x^{q^3} + \delta x^{q^5})\rangle_{\mathbb{F}_{q^6}} : x \in \mathbb{F}^*_{q^6}\}, \tag{2.4}$$

where $\delta \in \mathbb{F}_{q^6}^*$ such that $\delta^2 + \delta = 1$; see Csajbók, Marino & Zullo (2018) and Marino et al. (2020). Very recently, it has been proven that for a different set of conditions on δ, $L_\delta^{4,6}$ is scattered also for q even and large enough, see Bartoli et al. (2024).
In the series of articles Bartoli et al. (2020), Gupta et al. (2023), Longobardi et al. (2023), Longobardi & Zanella (2021), Neri et al. (2022), Smaldore et al. (2024), Zanella & Zullo (2020), various authors studied the class of linear sets defined as

$$L_{m,h,s}^{5,t} = \{\langle (x, \psi_{t,m,h,s}(x)) \rangle_{\mathbb{F}_{q^{2t}}} : x \in \mathbb{F}_{q^{2t}}^*\} \tag{2.5}$$

where

$$\psi_{t,m,h,s}(x) = x^{q^{s(t-1)}} + h^{1-q^{s(2t-1)}} x^{q^{s(2t-1)}} + m\left(x^{q^s} - h^{1-q^{s(t+1)}} x^{q^{s(t+1)}}\right)$$

for q odd, $t \geq 3$, $\gcd(s, 2t) = 1$ and $m, h \in \mathbb{F}_{q^{2t}}^*$ with $\mathrm{N}_{q^{2t}/q^t}(h) = -1$.

The linear set $L_{m,h,s}^{5,t}$ turns out to be scattered under one of the following hypotheses:

(*i*) $m = 1$ and $h \in \mathbb{F}_{q^t}^*$, see Bartoli et al. (2020), Longobardi & Zanella (2021), Neri et al. (2022), Zanella & Zullo (2020).

(*ii*) $m = 1$ and $h \in \mathbb{F}_{q^{2t}} \setminus \mathbb{F}_{q^t}$, see Bartoli et al. (2020), Gupta et al. (2023), Longobardi et al. (2023), Neri et al. (2022).

(*iii*) m belongs to $\mathbb{F}_{q^t}$ and it is neither a $(q-1)$-th power nor a $(q+1)$-th power of an element belonging to $\ker \mathrm{Tr}_{q^{2t}/q^t}$ and $h = 1$, see Smaldore et al. (2024).

Note that if $h \in \mathbb{F}_{q^t}^*$, $t \geq 3$ odd and $\mathrm{N}_{q^{2t}/q^t}(h) = -1$ implies that $q \equiv 1 \pmod 4$. We will associate an MRD code with each scattered linear set and denote it just by changing the letter L to C. For example, if we have the linear set $L_{m,h,s}^{5,t}$ then the associated MRD code is $C_{m,h,s}^{5,t}$ given by

$$C_{m,h,s}^{5,t} = \langle x, \psi_{t,m,h,s}(x) \rangle_{\mathbb{F}_{q^{2t}}}.$$

Moreover, in Bartoli et al. (2020), the authors showed that $L_{1,h,1}^{5,3}$ does not lie in Classes (2.1),(2.2),(2.3) and (2.4), except when q is a power of 5, and $h \in \mathbb{F}_q^*$. In this case, the linear set is equivalent to $L_\delta^{4,6}$ for some $\delta \in \mathbb{F}_{q^6}^*$, see (Bartoli et al. 2020, Corollary

3.11). In our article (see in Gupta et al. (2023)) which we will describe in details in later part of this chapter, it was proven that $L^{5,4}_{1,h,s}$ is not PΓL-equivalent to any linear sets of $\mathrm{PG}(1,q^8)$ belonging to classes in (2.1),(2.2),(2.3), (2.4), see (Gupta et al. 2023, Theorem 4.4).

2.2 Vertex properties of evasive linear set

Up to a suitable projectivity, any linear set L of rank $n(r-1)$ in $\mathrm{PG}(r-1,q^n)$ can be written in the following form

$$L = L_F = \{\langle(\mathbf{x}, F(\mathbf{x}))\rangle_{\mathbb{F}_{q^n}} : \mathbf{x} = (x_0, x_1, ..., x_{r-2}) \in \mathbb{F}_{q^n}^{r-1}, \mathbf{x} \neq \mathbf{0}\}, \tag{2.6}$$

where

$$F(\mathbf{x}) = \sum_{j=0}^{r-2} f_j(x_j), \tag{2.7}$$

and $f_j(x_j) = \sum_{i=1}^{n-1} a_{ij} x_j^{q^{si}}$ is an $\mathbb{F}_{q^s}$-linearized polynomial with coefficients over $\mathbb{F}_{q^n}$ in the indeterminate x_j for each $j \in \{0, 1, ..., r-2\}$ and $\gcd(s, n) = 1$, see (Csajbók et al. 2023, proof of Proposition 4).

Let us set some notations that we will use throughout this chapter. If $F(\mathbf{x})$ is a multivariate polynomial as in (2.7), we will denote by

a) $m_j := \deg_{q^s} f_j(x_j) = \max\{i \in \{1, 2, \ldots, n-1\} \mid a_{ij} \neq 0\}$, i.e., the q^s-*degree* of $f_j(x_j)$, and

b) $I_j := \mathrm{supp}_{q^s} f_j(x_j) = \{i \in \{1, 2, \ldots, n-1\} : a_{ij} \neq 0\}$, i.e., the q^s-*support* of $f_j(x_j)$.

Let $\Sigma^* = \mathrm{PG}(n(r-1)-1, q^n)$ be a projective space and consider the canonical subgeometry of Σ^* defined as follows:

$$\boldsymbol{\Sigma} = \{\langle(\ \mathbf{x},\ \mathbf{x}^q, \ldots,\ \mathbf{x}^{q^{n-1}})\rangle_{\mathbb{F}_{q^n}} :\ \ \mathbf{x} \in \mathbb{F}_{q^n}^{r-1},\ \mathbf{x} \neq \mathbf{0}\} \cong \mathrm{PG}(n(r-1)-1, q), \tag{2.8}$$

where $\mathbf{x}^{q^m} = (x_0^{q^m}, x_1^{q^m}, \ldots, x_{r-2}^{q^m})$ for each $m \in \{0, 1, \ldots, n-1\}$. Clearly, $\boldsymbol{\Sigma}$ is the set of the points fixed by the collineation $\boldsymbol{\sigma}$ of Σ^* of order n, defined as

$$\boldsymbol{\sigma} : \langle(x_{00}, x_{10}, \ldots, x_{r-2,0}, x_{01}, x_{11}, \ldots, x_{r-2,1}, \ldots, x_{0,n-1}, x_{1,n-1}, \ldots, x_{r-2,n-1})\rangle_{\mathbb{F}_{q^n}} \mapsto$$

$$\langle (x^q_{0,n-1}, x^q_{1,n-1}, \ldots, x^q_{r-2,n-1}, x^q_{00}, x^q_{10}, \ldots, x^q_{r-2,0}, \ldots, x^q_{0,n-2}, x^q_{1,n-2}, \ldots, x^q_{r-2,n-2}) \rangle_{\mathbb{F}_{q^n}}. \tag{2.9}$$

Theorem 2.2.1 *Let L_F be a linear set of rank $n(r-1)$ of $\mathrm{PG}(r-1, q^n)$ with $F(\mathbf{x})$ as in (2.7). Let $\Sigma^* = \mathrm{PG}(n(r-1)-1, q^n)$. Then, there exist*

1. *a canonical subgeometry Σ, a generator σ of the subgroup of $\mathrm{P\Gamma L}(n(r-1), q^n)$ fixing pointwise Σ and $r-1$ imaginary points $P_0, P_1, \ldots, P_{r-2}$ of Σ^* (wrt Σ),*
2. *an $(n(r-1)-(r+1))$-dimensional subspace Γ of Σ^* fulfilling*
   - *i) $P_k^{\sigma^{j_k}} \in \Gamma$ for any $j_k \notin I_k$, $j_k \neq 0$ and $k \in \{0, 1, \ldots, r-2\}$,*
   - *ii) any line*
   $$\langle P_\ell^{\sigma^{j_\ell}}, P_\ell^{\sigma^{m_\ell}} \rangle$$
   with $j_\ell \in I_\ell \setminus \{m_\ell\}$ for $\ell \in \{0, 1, \ldots, r-2\}$ meets Γ,
   - *iii) if $r > 2$, any line*
   $$\langle P_i^{\sigma^{m_i}}, P_{r-2}^{\sigma^{m_{r-2}}} \rangle$$
   with $i \in \{0, 1, \ldots, r-3\}$ meets Γ,
3. *an $(r-1)$-dimensional subspace Λ of Σ^* through the points P_i, $i \in \{0, 1, \ldots, r-2\}$,*

such that $\Gamma \cap \Sigma = \emptyset = \Gamma \cap \Lambda$ and $\mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ is equivalent to L_F.

Proof. Let $F(\mathbf{x})$ be the multivariate polynomial with coefficients over $\mathbb{F}_{q^n}$ defined as in (2.7) with $\gcd(s, n) = 1$. Let us assume that $\Sigma = \boldsymbol{\Sigma}$ and hence that $\sigma = \boldsymbol{\sigma}^s$ (as defined in (2.8) and (2.9), respectively) and let Γ be the $(n(r-1)-(r+1))$-dimensional subspace of Σ^* with equations

$$\Gamma : \begin{cases} x_{00} = x_{10} = \cdots = x_{r-2,0} = 0 \\ \sum_{i \in I_0} a_{0i} x_{0,is} + \sum_{i \in I_1} a_{1i} x_{1,is} + \cdots + \sum_{i \in I_{r-2}} a_{r-2,i} x_{r-2,is} = 0 \end{cases} \tag{2.10}$$

and P_k, $k \in \{0, 1, \ldots, r-2\}$, be the points having k-th coordinate equals to 1 and all the others equal to zero. Clearly, each of these points is imaginary with respect to Σ. Moreover, it is straightforward to see that *i*) for any $j_k \notin I_k$ and $j_k \neq 0$, $P_k^{\sigma^{j_k}} \in \Gamma$. Indeed, observe that $P_k^{\sigma^{j_k}}$ has the $(sj_k(r-1)+k+1)$-th $(\bmod\, n(r-1))$ coordinate 1 and all others equal to zero;

ii) any line

$$\langle P_\ell^{\sigma^{j_\ell}}, P_\ell^{\sigma^{m_\ell}} \rangle$$

with $j_\ell \in I_\ell \setminus \{m_\ell\}$ for $\ell \in \{0,1,\dots,r-2\}$ meets Γ in the point Q_{j_ℓ,m_ℓ}, with all zero coordinates except the $(sj_\ell(r-1)+\ell+1)$-th $(\bmod\, n(r-1))$ and the $(sm_\ell(r-1)+\ell+1)$-th $(\bmod\, n(r-1))$ ones, which are equal to a_{ℓ,m_ℓ} and $-a_{\ell,j_\ell}$, respectively; and

iii) if $r > 2$, any line

$$\langle P_i^{\sigma^{m_i}}, P_{r-2}^{\sigma^{m_{r-2}}} \rangle$$

with $i \in \{0,1,\dots,r-3\}$ meets Γ in the point R_i with all zero coordinates except the $(sm_i(r-1)+i+1)$-th $(\bmod\, n(r-1))$ and the $(sm_{r-2}(r-1)+r-1)$-th $(\bmod\, n(r-1))$ ones, which are equal to $a_{r-2,m_{r-2}}$ and $-a_{i,m_i}$, respectively.

Finally, let $\Lambda = \langle P_0, P_1, \dots, P_{r-2}, P_{r-2}^{\sigma^{m_{r-2}}} \rangle$. Easy computations show that Λ is an $(r-1)$-dimensional subspace of Σ^* such that $\Gamma \cap \Sigma = \emptyset = \Lambda \cap \Gamma$ and $\mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ is equivalent to L_F.

Using the notation introduced in Theorem 2.2.1 and taking into account that the points $P_i^{\sigma^j}$'s are independent, it is straightforward to check that the points

(*a*) $P_k^{\sigma^{j_k}}, j_k \notin I_k, j_k \neq 0$ and $k \in \{0,1,\dots,r-2\}$,

(*b*) $Q_{j_\ell,m_\ell} \in \langle P_\ell^{\sigma^{j_\ell}}, P_\ell^{\sigma^{m_\ell}} \rangle, j_\ell \in I_\ell \setminus \{m_\ell\}$ and $\ell \in \{0,1,\dots,r-2\}$,

(*c*) $R_i \in \langle P_i^{\sigma^{m_i}}, P_{r-2}^{\sigma^{m_{r-2}}} \rangle, i \in \{0,1,\dots,r-3\}$ for $r > 2$,

are independent and they span the subspace Γ. Next, we prove the following result.

Lemma 2.2.2 *Let Σ be a subgeometry of $\Sigma^* = \mathrm{PG}(n(r-1)-1, q^n)$ and let σ denote a collineation of order n of Σ^* such that $\mathrm{Fix}(\sigma) = \Sigma$. Let $P_0, P_1, \dots, P_{r-2}$ and $P'_0, P'_1, \dots, P'_{r-2}$ be imaginary points (wrt Σ) such that*

$$\mathcal{L}_{P_i} = \mathcal{L}_{P'_i} \text{ and } \langle \mathcal{L}_{P_0}, \mathcal{L}_{P_1}, \dots, \mathcal{L}_{P_{r-2}} \rangle = \Sigma^*.$$

Then, there exists a collineation $\varphi \in \mathrm{LAut}(\Sigma)$ *such that* $P_i^{\varphi} = P_i'$ *for* $i \in \{0, 1, \ldots, r-2\}$.

Proof. Set $\mathcal{L}_i := \mathcal{L}_{P_i} = \mathcal{L}_{P_i'}$. By hypothesis, for each $i \in \{0, 1, \ldots, r-2\}$,

$$\Sigma_i := \mathcal{L}_i \cap \Sigma = \mathrm{PG}(V_i) \cong \mathrm{PG}(n-1, q)$$

and $\mathcal{L}_i = \mathrm{PG}(W_i) \cong \mathrm{PG}(n-1, q^n)$ where $W_i = \langle V_i \rangle_{\mathbb{F}_{q^n}}$. Moreover, note that $\mathrm{Fix}(\sigma|_{\mathcal{L}_i}) = \Sigma_i$. By applying (Bonoli & Polverino 2005*a*, Proposition 3.1), we have that there exists a collineation $\varphi_i \in \mathrm{LAut}(\Sigma_i)$ of $\mathcal{L}_i$ fixing Σ_i and mapping P_i into P_i'. In the remaining part of the proof we will use the same symbol φ_i to denote both the collineation and its underlying linear map; i.e., we will write $\varphi_i : W_i \longrightarrow W_i$ for any $i \in \{0, 1, \ldots, r-2\}$.

Since $\Sigma^* = \mathrm{PG}(\oplus_{i=0}^{r-2} W_i)$ and $\Sigma = \mathrm{PG}(\oplus_{i=0}^{r-2} V_i)$, we may consider the map $\varphi \in \mathrm{PGL}(n(r-1), q^n)$ with the following associated linear map

$$w_0 + w_1 + w_2 + \ldots + w_{r-2} \in \oplus_{i=0}^{r-2} W_i \longrightarrow \varphi_0(w_0) + \varphi_1(w_1) + \varphi_2(w_2) + \ldots + \varphi_{r-2}(w_{r-2}) \in \oplus_{i=0}^{r-2} W_i.$$

First, we prove that $\varphi \in \mathrm{LAut}(\Sigma)$, i.e., $\sigma\varphi = \varphi\sigma$. Let $P = \langle w \rangle_{\mathbb{F}_{q^n}}$ with $w = w_0 + w_1 + \ldots + w_{r-2}$ where $w_i \in W_i$, then

$$\begin{aligned}
\sigma(\varphi(\langle w \rangle_{\mathbb{F}_{q^n}})) &= \sigma(\langle \varphi_0(w_0) + \varphi_1(w_1) + \varphi_2(w_2) + \ldots + \varphi_{r-2}(w_{r-2}) \rangle_{\mathbb{F}_{q^n}}) \\
&= \langle \sigma(\varphi_0(w_0)) + \sigma(\varphi_1(w_1)) + \sigma(\varphi_2(w_2)) + \ldots + \sigma(\varphi_{r-2}(w_{r-2})) \rangle_{\mathbb{F}_{q^n}} \\
&= \langle \sigma_{|W_0}(\varphi_0(w_0)) + \sigma_{|W_1}(\varphi_1(w_1)) + \sigma_{|W_2}(\varphi_2(w_2)) + \ldots + \sigma_{|W_{r-2}}(\varphi_{r-2}(w_{r-2})) \rangle_{\mathbb{F}_{q^n}} \\
&= \langle \varphi_0(\sigma(w_0)) + \varphi_1(\sigma(w_1)) + \varphi_2(\sigma(w_2)) + \ldots + \varphi_{r-2}(\sigma(w_{r-2})) \rangle_{\mathbb{F}_{q^n}} \\
&= \varphi(\langle \sigma(w_0) + \sigma(w_1) + \sigma(w_2) + \ldots + \sigma(w_{r-2}) \rangle_{\mathbb{F}_{q^n}}) \\
&= \varphi(\sigma(\langle w \rangle_{\mathbb{F}_{q^n}}))
\end{aligned}$$

where again we have denoted the semilinear map associated with σ and φ by the same letters, respectively. Finally, it is routine to check that $P_i^{\varphi} = P_i'$, $i \in \{0, 1, \ldots, r-2\}$. Thus, the result follows.

We now prove a generalization of (Bonoli & Polverino 2005*a*, Proposition 3.1) which will play an important role in the proof of Theorem 2.2.4.

Proposition 2.2.3 *Let Σ be a subgeometry of $\Sigma^* = \mathrm{PG}(n(r-1)-1, q^n)$. Let σ denote a collineation of order n of Σ^* such that* $\mathrm{Fix}(\sigma) = \Sigma$. *Then, the group* $\mathrm{LAut}(\Sigma)$ *acts $(r-1)$-transitively on sets of $r-1$ independent points $P_0, P_1, \ldots, P_{r-2}$ such that $\langle \mathcal{L}_{P_0}, \mathcal{L}_{P_1}, \ldots, \mathcal{L}_{P_{r-2}} \rangle = \Sigma^*$.*

Proof. Let $P_0, P_1, \ldots, P_{r-2}$ be the independent points such that $\langle \mathcal{L}_{P_0}, \mathcal{L}_{P_1}, \ldots, \mathcal{L}_{P_{r-2}} \rangle = \Sigma^*$. Therefore, any point P_i, $i \in \{0, 1, \ldots, r-2\}$, is an imaginary point of Σ^* (wrt Σ). So, for any $i \in \{0, 1, \ldots, r-2\}$, there exist n points in $\Sigma \cap \mathcal{L}_{P_i}$, say $P_{i0}, P_{i1}, \ldots, P_{i(n-1)}$, such that:

$$\{P_{ij} \in \Sigma : i \in \{0, 1, \ldots, r-2\} \text{ and } j \in \{0, 1, \ldots, n-1\}\}$$

are in general position in Σ. Let $Q_0, Q_1, \ldots, Q_{r-2}$ be $(r-1)$ points such that $\langle \mathcal{L}_{Q_i} : i = 0, 1, \ldots, r-2 \rangle = \Sigma^*$. So, in the same way, we get a set

$$\{Q_{ij} \in \Sigma : i \in \{0, 1, \ldots, r-2\} \text{ and } j \in \{0, 1, \ldots, n-1\}\}$$

of points of Σ in general position, as well. Since $\mathrm{LAut}(\Sigma)$ acts transitively on subset of Σ of order $n(r-1)$ which are in general position, there exists a collineation $\psi \in \mathrm{LAut}(\Sigma)$ such that $P_{ij}^{\psi} = Q_{ij}$ for any $i \in \{0, 1, \ldots, r-2\}$ and $j \in \{0, 1, \ldots, n-1\}$. Moreover, it holds that

$$\begin{aligned}\mathcal{L}_{P_i}^{\psi} &= (\langle P_{i0}, P_{i1}, \ldots, P_{i(n-1)} \rangle)^{\psi} = \langle P_{i0}^{\psi}, P_{i1}^{\psi}, \ldots, P_{i(n-1)}^{\psi} \rangle \\ &= \langle Q_{i0}, Q_{i1}, \ldots, Q_{i(n-1)} \rangle = \mathcal{L}_{Q_i}.\end{aligned}$$

Now, if $P_i^{\psi} = Q_i$, we get the result. Otherwise, assume $P_i^{\psi} = R_i$, for some $R_0, R_1, \ldots, R_{r-2}$. Since $\mathcal{L}_{R_i} = \mathcal{L}_{Q_i}$, by Lemma 2.2.2 there exists $\varphi \in \mathrm{LAut}(\Sigma)$ such that $R_i^{\varphi} = Q_i$ and so $P_i^{\psi\varphi} := \varphi(\psi(P_i)) = Q_i$. This completes the proof. We are now in the position to prove the following inverse of Theorem 2.2.1.

Theorem 2.2.4 *Let $r, n \geq 2$, $I_j \subseteq \{1, \ldots, n-1\}$, with $j \in \{0, 1, \ldots, r-2\}$. Let Σ be a canonical subgeometry of $\Sigma^* = \mathrm{PG}(n(r-1)-1, q^n)$ and consider Γ and Λ subspaces of Σ^* with dimensions $(n(r-1)-(r+1))$ and $(r-1)$, respectively, such that $\Gamma \cap \Sigma = \emptyset = \Lambda \cap \Gamma$.*
Assume $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ is an $(r-1, n(r-1)-2)_q$-evasive linear set of rank $n(r-1)$.

If there exist $r-1$ points $P_0, P_1, \dots, P_{r-2}$ such that Γ is spanned by points defined as in (a), (b) and (c) where σ is a generator of the subgroup of $\mathrm{P\Gamma L}(n(r-1), q^n)$ fixing Σ pointwise. Then,

(1) $P_0, P_1, \dots, P_{r-2}$ are imaginary points (wrt Σ), and

(2) $L \cong L_F$ as in (2.6) where $\mathrm{supp}_{q^s} f_j(x_j) = I_j$, for some integer $1 \le s \le n-1$ with $\gcd(s,n) = 1$ and $j = 0, \dots, r-2$.

Proof. Proving point *1.* of the statement, is equivalent to show that $\dim \mathcal{L} = n(r-1)-1$, where $\mathcal{L} = \langle \mathcal{L}_{P_0}, \mathcal{L}_{P_1}, \dots, \mathcal{L}_{P_{r-2}} \rangle$. By our hypothesis, $\Gamma \subset \mathcal{L}$ and hence

$$n(r-1) - (r+1) < \dim(\mathcal{L}) \le n(r-1) - 1.$$

Assume that $\dim \mathcal{L} < n(r-1)-1$. Precisely, let $\dim \mathcal{L} = n(r-1)-(r+1)+t$, where $t \in \{1, 2, \dots, r-1\}$. Then, by Grassman's Formula $\dim(\Lambda \cap \mathcal{L}) = t-1$. Since $\mathcal{L}$ is fixed by σ, we get

$$\mathrm{wt}_L(\Lambda \cap \mathcal{L}) = \mathrm{rk}(\mathcal{L} \cap \Sigma) = \dim \mathcal{L} + 1 = n(r-1) - (r+1) + t + 1.$$

This means that we have a t-dimensional $\mathbb{F}_{q^n}$-subspace of $\mathbb{F}_{q^n}^r$, which meets an $\mathbb{F}_q$-linear subspace in an $\mathbb{F}_q$-linear subspace with dimension $(n(r-1)-(r+1)+t+1)$ over $\mathbb{F}_q$. Since by (Bartoli, Csajbók, Marino & Trombetti 2021, Proposition 2.6) an $(r-1, n(r-1)-2)_q$-evasive is also $(t, n(r-1)-(r+1)+t)_q$-evasive, we contradict the assumption. Therefore, $\dim(\mathcal{L}) = n(r-1)-1$. By Proposition 2.2.3, let Σ as in (2.8), then there exists an integer $1 \le s \le n-1$ with $\gcd(s,n) = 1$ such that $\sigma = \boldsymbol{\sigma}^s$. Then, we may assume that any P_k, $k = 0, 1, \dots, r-2$, is a point with homogeneous coordinates equal to 1 in the k-th position and 0 in all others. Furthermore, since Γ is spanned by points defined as in (a), (b) and (c), it turns out to have equations

$$\Gamma : \begin{cases} x_{00} = x_{10} = \cdots = x_{r-2,0} = 0 \\ \sum_{i \in I_0} a_{0i} x_{0,is} + \sum_{i \in I_1} a_{1i} x_{1,is} + \cdots + \sum_{i \in I_{r-2}} a_{r-2,i} x_{r-2,is} = 0 \end{cases}$$

with some $a_{ji} \in \mathbb{F}_{q^n}^*$, $j = 0, \dots, r-2$ and $i \in I_j$. Consequently, $L \cong L_F$, where $F(\mathbf{x}) = \sum_{j=0}^{r-2} \sum_{i \in I_j} a_{ji} x^{q^{si}}$.

2.3 Maximum scattered linear sets of $\mathrm{PG}(1,q^n)$

In this section, we will focus on maximum scattered linear sets of the projective line $\mathrm{PG}(1,q^n)$. These are covered as a special case in the results of the previous section providing $r = 2$. As intended, we aim to characterize the families of these linear sets which were introduced from 2018 onwards (see Csajbók, Marino & Zullo (2018), Csajbók, Marino, Polverino & Zanella (2018), Longobardi et al. (2023), Longobardi & Zanella (2021), Neri et al. (2022), Smaldore et al. (2024)) using certain properties of their projection vertices.

For this purpose, we will start by recalling some characterization results obtained in this direction, regarding the two classical and up-to-date most investigated examples: pseudoregulus and Lunardon-Polverino type scattered linear sets of the line.

As noted in the introduction, two $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^n}^2$ defining the same scattered linear set in $\mathrm{PG}(1,q^n)$ are not necessarily on the same orbit under the action of $\Gamma\mathrm{L}(2,q^n)$. This leads us to Csajbók, Marino & Polverino (2018) where the authors introduce the notion of $\mathcal{Z}(\Gamma\mathrm{L})$-class and $\Gamma\mathrm{L}$-class of a linear set of rank n in $\mathrm{PG}(1,q^n)$. Precisely,

Definition 2.3.1 (Csajbók, Marino & Polverino 2018, Definition 2.4) Let L_U be an $\mathbb{F}_q$-linear set of $\mathrm{PG}(1,q^n)$ of rank n with maximum field of linearity $\mathbb{F}_q$, i.e., L_U is not an $\mathbb{F}_{q^u}$-linear set for any $u > 1$. Then L_U has $\mathcal{Z}(\Gamma\mathrm{L})$-*class* z, if z is the largest integer such that there exist $\mathbb{F}_q$-subspaces $U_1, U_2, \dots, U_z$ of $\mathbb{F}_{q^n}^2$ with

1. $L_{U_i} = L_U$ for $i \in \{1, 2, \dots, z\}$, and
2. there is no $\lambda \in \mathbb{F}_{q^n}^*$ such that $U_i = \lambda U_j$ for each $i \neq j$ and $i, j \in \{1, 2, ..., z\}$.

Definition 2.3.2 (Csajbók, Marino & Polverino 2018, Definition 2.5). Let L_U be an $\mathbb{F}_q$-linear set of $\mathrm{PG}(1,q^n)$ of rank n with maximum field of linearity $\mathbb{F}_q$. Then L_U is of $\Gamma\mathrm{L}$-*class* c, if c is the largest integer such that there exist $\mathbb{F}_q$-subspaces $U_1, U_2, \dots, U_c$ of $\mathbb{F}_{q^n}^2$ with

1. $L_{U_i} = L_U$ for $i \in \{1, 2, \dots, c\}$, and
2. there is no $\kappa \in \Gamma\mathrm{L}(2,q^n)$ such that $U_i = U_j^\kappa$ for each $i \neq j$, $i, j \in \{1, 2, \dots, c\}$.

The classes $\mathcal{Z}(\Gamma\mathrm{L})$-class and $\Gamma\mathrm{L}$-class are invariant under the action of the group $\mathrm{P}\Gamma\mathrm{L}(2, q^n)$ on L, see (Csajbók, Marino & Polverino 2018, Proposition 2.6), therefore these integers are useful for characterizing the linear sets. In the following, we will denote the $\mathcal{Z}(\Gamma\mathrm{L})$-class and the $\Gamma\mathrm{L}$-class of a linear set $L = L_U$ by $c_{\mathcal{Z}(\Gamma\mathrm{L})}(L)$ and $c_{\Gamma\mathrm{L}}(L)$, respectively. These integers are invariant under the action of the group $\mathrm{P}\Gamma\mathrm{L}(2, q^n)$ on L, see (Csajbók, Marino & Polverino 2018, Proposition 2.6), and it is clear that $c_{\Gamma\mathrm{L}}(L) \leq c_{\mathcal{Z}(\Gamma\mathrm{L})}(L)$. The linear set L is called *simple* if its $\Gamma\mathrm{L}$-class is one. Also, in Zanella & Zullo (2020), the $\Gamma\mathrm{L}$-class of a linear set was rephrased via a group action property on its projection vertices, see (Csajbók, Marino & Polverino 2018, Section 5.2), (Csajbók & Zanella 2016*c*, Theorem 6 and 7) and (Zanella & Zullo 2020, Theorem 3.4). More precisely, the following was proved.

Theorem 2.3.3 *Let* Σ *be a canonical subgeometry and* Λ *be a line of* $\Sigma^* = \mathrm{PG}(n-1, q^n)$*. The* $\Gamma\mathrm{L}$*-class of a linear set* $L \subset \Lambda$ *is the number of orbits, under the action of* $\mathrm{Aut}(\Sigma)$ *of* $(n-3)$*-subspaces* Γ *of* Σ^* *disjoint from* Σ *and from the line* Λ *such that* $\mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ *is equivalent to* L*.*

Let L_U be a linear set of $\mathrm{PG}(1, q^n)$ and let τ be a polarity of the projective line. Then, it is always possible to construct another linear set, which is called the dual linear set of L_U with respect to the polarity τ, see Polverino (2009). In particular, if $\beta : \mathbb{F}_{q^n}^2 \times \mathbb{F}_{q^n}^2 \longrightarrow \mathbb{F}_{q^n}$ is a non-degenerate reflexive sesquilinear form on $\mathbb{F}_{q^n}^2$ determining the polarity τ, then the map $\beta' = \mathrm{Tr}_{q^n/q} \circ \beta$ is a non-degenerate reflexive sesquilinear form on $\mathbb{F}_{q^n}^2$, when this is regarded as a $2n$-dimensional vector space over $\mathbb{F}_q$.

Let $\perp_\beta$ and $\perp_{\beta'}$ be the orthogonal complement maps defined by β and β' on the lattices of the $\mathbb{F}_{q^n}$-subspaces and $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^n}^2$, respectively. The *dual linear set* L_U^τ *of* L_U *with respect to the polarity* τ is the $\mathbb{F}_q$-linear set of rank n of $\mathrm{PG}(1, q^n)$ defined by the orthogonal complement $U^{\perp_{\beta'}}$ and, up to projective equivalence, such a linear set does not depend on τ, see (Polverino 2009, Proposition 2.5). As a direct consequence of (Pepe 2024, Theorem 1), if L is a maximum scattered linear set of $\mathrm{PG}(1, q^n)$ and $L = L_U = L_W$ where U and W are two n-dimensional $\mathbb{F}_q$-vector spaces of $\mathbb{F}_{q^n}^2$, then one of the following possibilities may occur:

1. $L_U = L_W$ is a linear set of pseudoregulas type as in 2.1.

2. $W = \lambda U$ with $\lambda \in \mathbb{F}_{q^n}^*$;

3. $W = \lambda U^{\perp_{\beta'}}$ with $\lambda \in \mathbb{F}_{q^n}^*$;

Using these argument and (Pepe 2024, Theorem 1), we state the following result:

Theorem 2.3.4 *Let L_f be a maximum scattered linear set of $\mathrm{PG}(1,q^n)$. If W is an n-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^n}^2$ such that $L_W = L_f$ with $f(x) = \sum_{i=1}^{n-1} a_i x^{q^i}$, then one of the following possibilities may occur:*

1. *L_f is equivalent to a linear set of pseudoregulus type $L_s^{1,n}$ (see equation 2.1);*
2. *$W = \lambda U_f$ for some $\lambda \in \mathbb{F}_{q^n}^*$;*
3. *$W = \lambda U_{\hat{f}}$ for some $\lambda \in \mathbb{F}_{q^n}^*$, where*

$$\hat{f}(x) = \sum_{i=1}^{n-1} a_{n-i}^{q^i} x^{q^i},$$

called the adjoint polynomial of $f(x)$. In particular, if L_f is of pseudoregulus type $c_{\Gamma\mathrm{L}}(L_f) = \phi(n)/2$, where ϕ is the totient function, otherwise $c_{\Gamma\mathrm{L}}(L_f) \leq 2$.

In the following, we will state two theorems containing characterization results for the linear sets of pseudoregulus and LP-type by means of some geometric properties of their vertices.

Theorem 2.3.5 *(Csajbók & Zanella 2016a, Theorem 2.3) Let Σ be a canonical subgeometry of $\mathrm{PG}(n-1,q^n)$, $q > 2$, $n \geq 3$. Assume that Γ and Λ are $(n-3)$-subspace and a line of $\mathrm{PG}(n-1,q^n)$, respectively, such that $\Sigma \cap \Gamma = \Lambda \cap \Gamma = \emptyset$. Then the following assertions are equivalent:*

1. *The set $\mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ is a scattered $\mathbb{F}_q$-linear set of pseudoregulus type;*
2. *there exists a generator σ of the subgroup of $\mathrm{P\Gamma L}(n,q^n)$ fixing Σ pointwise such that $\dim(\Gamma \cap \Gamma^\sigma) = n-4$; furthermore Γ is not contained in the span of any hyperplane of Σ;*
3. *There exists a point P and a generator σ of the subgroup of $\mathrm{P\Gamma L}(n,q^n)$ fixing Σ pointwise, such that $\langle P, P^\sigma, \ldots, P^{\sigma^{n-1}} \rangle = \mathrm{PG}(n-1,q^n)$, and*

$$\Gamma = \langle P, P^\sigma, \ldots, P^{\sigma^{n-3}} \rangle.$$

In the same spirit, later in Zanella & Zullo (2020), the authors characterized linear sets of LP-type. To do this, they introduced the notion of the intersection number of a subspace with respect to a collineation fixing pointwise a canonical subgeometry. More precisely, let Γ be a non-empty k-dimensional subspace of $\mathrm{PG}(n-1, q^n)$, the *intersection number of* Γ *with respect to* σ, denoted by $\mathrm{intn}_\sigma(\Gamma)$, is defined as the least positive integer γ satisfying

$$\dim(\Gamma \cap \Gamma^\sigma \cap \ldots \cap \Gamma^{\sigma^\gamma}) > k - 2\gamma.$$

It is easy to see that $\mathrm{intn}_\sigma(\Gamma)$ is invariant under the action of $\mathrm{Aut}(\Sigma)$. By Theorem 2.3.5, a maximum scattered linear set is of pseudoregulus type if and only if $\mathrm{intn}_\sigma(\Gamma) = 1$ for some σ fixing Σ pointwise. Also in Zanella & Zullo (2020) the following result was proven.

Theorem 2.3.6 *(Zanella & Zullo 2020, Theorem 3.2) Let Σ be a canonical subgeometry of $\mathrm{PG}(n-1, q^n)$, $q > 2$ and $n \geq 4$. Let L be a scattered linear set in a line Λ of $\mathrm{PG}(n-1, q^n)$. Then L is a linear set of LP-type if and only if*

1. *there exists an $(n-3)$-subspace Γ of $\mathrm{PG}(n-1, q^n)$ such that $\Gamma \cap \Sigma = \Gamma \cap \Lambda = \emptyset$ and $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$;*

2. *there exists a generator σ of the subgroup of $\mathrm{P\Gamma L}(n, q^n)$ fixing Σ pointwise such that $\mathrm{intn}_\sigma(\Gamma) = 2$;*

3. *there exists a unique point $P \in \mathrm{PG}(n-1, q^n)$ and some point Q such that*

$$\Gamma = \langle P, P^\sigma, \ldots, P^{\sigma^{n-4}}, Q\rangle;$$

4. *the line $\langle P^{\sigma^{n-1}}, P^{\sigma^{n-3}}\rangle$ meets Γ.*

Moreover, as a consequence of Theorem 2.3.4 we get a slight strengthening of (Zanella & Zullo 2020, Theorem 3.5), indeed removing the assumptions $n \leq 6$ and $n = 8$. Precisely, we may state the following result.

Theorem 2.3.7 *Let L be a maximum scattered linear set in a line Λ of $\mathrm{PG}(n-1, q^n)$ for any $n \geq 4$ and $q > 2$. Then L is a linear set of LP-type if and only if for each $(n-3)$-subspace Γ of $\mathrm{PG}(n-1, q^n)$ such that $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$, the following holds:*

i) there exists a generator σ of the subgroup of $\mathrm{P\Gamma L}(n,q^n)$ fixing Σ pointwise such that $\mathrm{intn}_\sigma(\Gamma)=2$*;*

(ii) if P is the unique point of $\mathrm{PG}(n-1,q^n)$ such that

$$\Gamma=\langle P,P^\sigma,\dots,P^{\sigma^{n-4}},Q\rangle$$

then the line $\langle P^{\sigma^{n-1}},P^{\sigma^{n-3}}\rangle$ meets Γ.

In the last decade, many new constructions of maximum scattered linear sets of the projective line were exhibited. In the following, in the light of what was done for the examples belonging to classes (2.1) and (2.2), we will provide characterization results similar to Theorems 2.3.5 and 2.3.6, for all the other constructions mentioned above. As we have seen before, in Csajbók & Zanella (2016*a*), Zanella & Zullo (2020), linear sets of pseudoregulus and LP-type were characterized by exploiting the intersection number of their vertex of projection. This approach does not work for scattered linear sets belonging to the other classes described above. Indeed in the following, we show that there exist projection vertices of inequivalent linear sets having the same intersection number. To this aim, let us consider

$$\Sigma=\{\langle(x,x^q,\dots,x^{q^5})\rangle_{\mathbb{F}_{q^6}}:x\in\mathbb{F}^*_{q^6}\},$$

the canonical subgeometry of $\mathrm{PG}(5,q^6)$ fixed by the collineation

$$\sigma:\langle(x_0,x_1,\dots,x_5)\rangle_{\mathbb{F}_{q^6}}\in\mathrm{PG}(5,q^6)\longrightarrow\langle(x_5^q,x_0^q,x_1^q,\dots,x_4^q)\rangle_{\mathbb{F}_{q^6}}\in\mathrm{PG}(5,q^6).$$

Let $L^{3,6}_{1,\eta}$ be defined as in (2.3). It is easy to see that this example is equivalent to a linear set obtained as the projection of Σ from the solid Γ_η to the line Λ_η having the following equations:

$$\Gamma_\eta:\begin{cases}x_0=0\\ \eta x_1+x_4=0\end{cases}\qquad\qquad \Lambda_\eta:x_1=x_2=x_3=x_5=0.\tag{2.11}$$

We have $\Gamma_\eta\cap\Sigma=\emptyset=\Gamma_\eta\cap\Lambda_\eta$. Moreover, $\dim(\Gamma_\eta\cap\Gamma_\eta^\sigma)=1$ and $\dim(\Gamma_\eta\cap\Gamma_\eta^\sigma\cap\Gamma_\eta^{\sigma^2})=-1$, hence $\mathrm{intn}_\sigma(\Gamma_\eta)=3$. Similarly, the linear set $L^{5,3}_{1,h,1}$ is equivalent to the projection of Σ from the vertex Γ_h on the line Λ_h with equations

$$\Gamma_h : \begin{cases} x_0 = 0 \\ h^{q-1}x_1 - h^{q^2-1}x_2 + x_4 + x_5 = 0 \end{cases} \qquad \Lambda_h : x_1 = x_2 = x_3 = x_4 = 0 \tag{2.12}$$

and as shown in (Bartoli et al. 2020, Section 3.2), $\mathrm{intn}_\sigma(\Gamma_h) = 3$ as well.

Since, Theorem 2.2.1 will be crucial in the subsequent part, we will state it when $r = 2$, i.e., for linear sets of the projective line $\mathrm{PG}(1,q^n)$.

Proposition 2.3.8 *Let L_f be a linear set of rank n of the projective line $\mathrm{PG}(1,q^n)$ as defined in (2.7) with $f(x) = \sum_{i=1}^{n-1} a_i x^{q^{si}}$, $m = \deg_{q^s} f(x)$ and $I = \mathrm{supp}_{q^s} f(x)$. Then, there exist a canonical subgeometry Σ of $\Sigma^* = \mathrm{PG}(n-1,q^n)$ and an imaginary point P of Σ^* (wrt Σ) such that L_f is equivalent to $\mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ with*

$$\Gamma_f = \langle P^{\sigma^j}, Q_i \colon j \notin I, j \neq 0 \,\text{and}\, i \in I \setminus \{m\}\rangle, \tag{2.13}$$

where $Q_i \in \langle P^{\sigma^i}, P^{\sigma^m}\rangle$, $i \in I \setminus \{m\}$ and $\Lambda = \langle P, P^{\sigma^m}\rangle$.

We observe that in light of Theorems 2.3.3, 2.3.4 and Proposition 2.3.8, the result above can be further specialized for maximum scattered linear sets not of pseudoregulus type as follows.

Theorem 2.3.9 *Let L_f be a maximum scattered linear set of the projective line $\mathrm{PG}(1,q^n)$ not of pseudoregulus type with $f(x) = \sum_{i=1}^{n-1} a_i x^{q^{si}}$, $m = \deg_{q^s} f(x)$ and $I = \mathrm{supp}_{q^s} f(x)$. Then, there exist a canonical subgeometry Σ of $\Sigma^* = \mathrm{PG}(n-1,q^n)$ and an imaginary point P of Σ^* (wrt Σ) such that*

1. *L_f is equivalent to $\mathrm{p}_{\Gamma_f,\Lambda_f}(\Sigma)$ with*

$$\Gamma_f = \langle P^{\sigma^j}, Q_i \colon j \notin I, j \neq 0 \,\text{and}\, i \in I \setminus \{m\}\rangle \quad \text{and } \Lambda_f = \langle P, P^{\sigma^m}\rangle \tag{2.14}$$

   where $Q_i \in \langle P^{\sigma^i}, P^{\sigma^m}\rangle$ and $i \in I \setminus \{m\}$,

2. *L_f is equivalent to $\mathrm{p}_{\Gamma_{\hat f},\Lambda_{\hat f}}(\Sigma)$ with*

$$\Gamma_{\hat f} = \langle P^{\sigma^j}, Q_i \colon j \notin \hat I, j \neq 0 \,\text{and}\, i \in \hat I \setminus \{\hat m\}\rangle \quad \text{and } \Lambda_{\hat f} = \langle P, P^{\sigma^{\hat m}}\rangle \tag{2.15}$$

where $Q_i \in \langle P^{\sigma^i}, P^{\sigma^{\hat{m}}}\rangle$, $i \in \hat{I} \setminus \{\hat{m}\}$, $\hat{I} = \mathrm{supp}_{q^s} \hat{f}(x)$, *and* $\hat{m} = \deg_{q^s} \hat{f}(x)$.

In particular, if $c_{\Gamma\mathrm{L}}(L_f) = 2$ *and* $\Lambda_f = \Lambda_{\hat{f}}$ *then* Γ_f *and* $\Gamma_{\hat{f}}$ *do not belong to the same orbit under the action of* $\mathrm{Aut}(\Sigma)$, *otherwise they do.*

Remark 2.3.10 We stress here the fact that in previous statement, the vertex $\Gamma_{\hat{f}}$ can be spanned by the same P^{σ^j}'s and the points $T_i \in \Gamma_{\hat{f}} \cap \langle P^{\sigma^i}, P^{\sigma^{\tilde{m}}}\rangle$, where $i \in \hat{I} \setminus \{\tilde{m}\}$ and $\tilde{m} = \min \hat{I}$, as well.

Putting together Theorem 2.3.9 above with (Csajbók & Zanella 2016*c*, Theorem 2) and (Lavrauw & Voorde 2010, Theorem 3), we derive the following result.

Theorem 2.3.11 *Let* $\Sigma \cong \mathrm{PG}(n-1,q)$ *be a canonical subgeometry of* $\Sigma^* = \mathrm{PG}(n-1,q^n)$. *Let* L *be a linear set contained in a line* Λ *of* Σ^* *such that* $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ *where* Γ *is an* $(n-3)$*-subspace of* Σ^* *such that* $\Gamma \cap \Sigma = \emptyset = \Gamma \cap \Lambda$. *Let us consider* L_f *be a maximum scattered linear set of* $\mathrm{PG}(1,q^n)$ *not of pseudoregulus type and* Γ_f, $\Gamma_{\hat{f}}$, Λ_f *and* $\Lambda_{\hat{f}}$ *as defined in* (2.14) *and* (2.15). *If* L_f *is simple or* $\Lambda_f = \Lambda_{\hat{f}}$, *the following statements are equivalent:*

(*i*) $L_f \cong L$;

(*ii*) *there exists a collineation* β *of* Σ^* *such that* $\boldsymbol{\Sigma}^\beta = \Sigma$ *(cf.* (2.8)*) and either* $\Gamma = \Gamma_f^\beta$ *or* $\Gamma = \Gamma_{\hat{f}}^\beta$.

Proof. (*i*) $\Rightarrow$ (*ii*) Let us suppose that $L_f \cong L$. If the ΓL-class of L_f, and hence of L, is one then the result follows from (Csajbók & Zanella 2016*c*, Theorem 6). Suppose that $c_{\Gamma\mathrm{L}}(L_f) = c_{\Gamma\mathrm{L}}(L) = 2$ and $\Lambda_f = \Lambda_{\hat{f}}$. Let us put $\Gamma_1 = \Gamma_f$, $\Gamma_2 = \Gamma_{\hat{f}}$ and $\boldsymbol{\Lambda} = \Lambda_f = \Lambda_{\hat{f}}$ and consider $L_1 = \mathrm{p}_{\Gamma_1,\boldsymbol{\Lambda}}(\boldsymbol{\Sigma})$ and $L_2 = \mathrm{p}_{\Gamma_2,\boldsymbol{\Lambda}}(\boldsymbol{\Sigma})$. By Theorem 2.3.9, there exists $\alpha_i : \boldsymbol{\Lambda} \longrightarrow \Lambda$ such that $L_i^{\alpha_i} = L$, $i = 1,2$. Then α_i can be extended to an element, which we indicate with the same symbol α_i, belonging to $\mathrm{P\Gamma L}(n,q^n)$ such that $\Gamma_i^{\alpha_i} = \Gamma$, $i = 1,2$. Thus, $L = L_i^{\alpha_i} = \mathrm{p}_{\Gamma,\Lambda}(\boldsymbol{\Sigma}^{\alpha_i})$, $i = 1,2$.
In order to get (*ii*), it is enough to prove the existence of either a map $\phi \in \mathrm{P\Gamma L}(n,q^n)$ such that $\Gamma^\phi = \Gamma$, $\boldsymbol{\Sigma}^{\alpha_1\phi} = \Sigma$ or of a map $\psi \in \mathrm{P\Gamma L}(n,q^n)$ such that $\Gamma^\psi = \Gamma$, $\boldsymbol{\Sigma}^{\alpha_2\psi} = \Sigma$. Indeed, if such a ϕ or ψ did exist, then we would get the result by putting $\beta = \alpha_1\phi$ or $\beta = \alpha_2\psi$, respectively. In other words, it is enough to prove the assertion when

$\mathbf{\Lambda} = \Lambda$ and either $\Gamma_1 = \Gamma$ and $L_1 = L$ or $\Gamma_2 = \Gamma$ and $L_2 = L$. Let $\mathbf{\Sigma} = L_{\boldsymbol{V}}$ with

$$\boldsymbol{V} = \{(x, x^q, \ldots, x^{q^{n-1}}) \colon x \in \mathbb{F}_{q^n}\}$$

and V be an n-dimensional $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^n}^n$ such that $\Sigma = L_V$. Since $\mathbf{\Sigma}$ and Σ are canonical subgeometries, we have that $\langle \boldsymbol{V} \rangle_{\mathbb{F}_{q^n}} = \langle V \rangle_{\mathbb{F}_{q^n}} = \mathbb{F}_{q^n}^n$. Moreover, let $\boldsymbol{V} = \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle_{\mathbb{F}_q}$ and $V = \langle v_1, v_2, \ldots, v_n \rangle_{\mathbb{F}_q}$. Also let $\Gamma_i = \mathrm{PG}(Z_i, q^n)$, $\mathbf{\Lambda} = \mathrm{PG}(R, q^n)$.

Note that $\boldsymbol{W}_i = (\boldsymbol{V} + Z_i) \cap R$ and $W = (V + Z) \cap R$ are two n-dimensional $\mathbb{F}_q$-vector spaces in R. As either $\Gamma_1 = \Gamma$ and $L_1 = \mathrm{p}_{\Gamma_1, \mathbf{\Lambda}}(\mathbf{\Sigma}) = \mathrm{p}_{\Gamma, \mathbf{\Lambda}}(\Sigma) = L$ or $\Gamma_2 = \Gamma$ and $L_2 = \mathrm{p}_{\Gamma_2, \mathbf{\Lambda}}(\mathbf{\Sigma}) = \mathrm{p}_{\Gamma, \mathbf{\Lambda}}(\Sigma) = L$, by Theorem 2.3.4, it follows that there exists a non-singular $\mathbb{F}_{q^n}$-semilinear map $\gamma : R \to R$ such that either $\boldsymbol{W}_1^\gamma = W$ or $\boldsymbol{W}_2^\gamma = (\boldsymbol{W}_1^{\perp_{\beta'}})^\gamma = W$ where $\perp_{\beta'}$ is the orthogonal complement map defined by a sesquilinear form β' on the lattices of the $\mathbb{F}_q$-subspaces of $R \cong \mathbb{F}_{q^n}^2$ seen as a $2n$-dimensional space over $\mathbb{F}_q$. Moreover, $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are not on the same orbit under the action of the group of semilinear maps of R in itself, otherwise L_f, and hence also L, would not have ΓL-class two.

Assume first that $\boldsymbol{W}_1^\gamma = W$ and let $\Gamma \in \{\Gamma_1, \Gamma_2\}$ and, hence, $Z \in \{Z_1, Z_2\}$. We show that the case $\boldsymbol{W}_1^\gamma = W$ and $\Gamma_2 = \Gamma$ cannot occur. Indeed, if $\boldsymbol{W}_1^\gamma = W$ and $Z = Z_2$, then the map γ can be extended to non-singular semilinear map, saying γ again, from $\mathbb{F}_{q^n}^n$ in itself such that $Z_2^\gamma = Z_2$. Then for each vector $v \in \mathbb{F}_{q^n}^n$ and $\mu \in \mathbb{F}_{q^n}$, we have $(\mu v)^\gamma = \mu^{p^\ell} v^\gamma$ for some integer ℓ. Since $R + Z$ and $\boldsymbol{W}_2 + Z$ are direct sums, for any $\mathbf{v} \in \boldsymbol{V} \setminus \{\mathbf{0}\}$ and $z \in Z$, the intersection $(\langle \mathbf{v} \rangle_{\mathbb{F}_q} + Z) \cap R$ is a one-dimensional $\mathbb{F}_q$-subspace, say $\langle \mathbf{w} \rangle_{\mathbb{F}_q}$ with $\mathbf{w} \in \boldsymbol{W}_2$. This implies $h\mathbf{w} = \mathbf{v} + z'$ for some $z' \in Z$ and $h \in \mathbb{F}_q$ and hence $\mathbf{v} + z = h\mathbf{w} + (z - z') \in \boldsymbol{W}_2 + Z$. Then, $\boldsymbol{W}_2 + Z = \boldsymbol{V} + Z$and hence $\boldsymbol{W}_1 \subseteq \boldsymbol{V} + Z_2 = \boldsymbol{W}_2 + Z_2$. Since $\boldsymbol{W}_1 \subseteq R$ we get that $\boldsymbol{W}_1 \subset (\boldsymbol{V} + Z_2) \cap R = \boldsymbol{W}_2$ and hence $\boldsymbol{W}_1 = \boldsymbol{W}_2$, a contradiction.

Hence, let us suppose that $\boldsymbol{W}_1^\gamma = W$ and $Z = Z_1$. Similar argument shows that $\boldsymbol{W}_1 + Z = \boldsymbol{V} + Z$ and $W + Z = V + Z$. Then

$$V + Z = W + Z = \boldsymbol{W}_1^\gamma + Z^\gamma = (\boldsymbol{W}_1 + Z)^\gamma = (\boldsymbol{V} + Z)^\gamma \qquad (2.16)$$

and hence for any $i = 1, \ldots, n$, $\mathbf{v}_i^\gamma = v_i' + z_i$ for some $v_i' \in V$ and $z_i \in Z$. We will show that $v_1',\dots,v_n'$ is an $\mathbb{F}_q$ basis of V. Indeed, suppose that $\sum_{i=1}^n \lambda_i v_i' = 0$ for $\lambda_i \in \mathbb{F}_q$. Therefore, $\sum_{i=1}^n \lambda_i \mathbf{v}_i^\gamma = \sum_{i=0}^n \lambda_i z_i$. As on the left-hand since $\mathbf{v}_1^\gamma,\mathbf{v}_2^\gamma,\dots,\mathbf{v}_n^\gamma$ are $\mathbb{F}_q$ independent, it follows that either $\lambda_i = 0$ or, saying $z = \sum_{i=1}^n \lambda_i \mathbf{v}_i^\gamma = \sum_{i=0}^n \lambda_i z_i$, then $z \in \boldsymbol{V}^\gamma \cap Z$. Hence in the latter case $z^{\gamma^{-1}} \in \boldsymbol{V} \cap Z$, a contradiction. Since $\Sigma = L_V$ is a subgeometry, $v_1', v_2', \dots, v_n'$ are linearly independent over $\mathbb{F}_{q^n}$. Let ρ be the $\mathbb{F}_{q^n}$-semilinear map of $\mathbb{F}_{q^n}^n$ such that $\mathbf{v}_i^\rho = v_i'$ for each $i = 1,2,\dots,n$ and $(\mu\mathbf{v})^\rho = \mu^{p^\ell}\mathbf{v}^\rho$ for any $\mu \in \mathbb{F}_{q^n}$. If $P = \langle \mathbf{z}\rangle_{\mathbb{F}_{q^n}} \in \Gamma$, then we have $\mathbf{z} = \sum_{i=1}^n a_i \mathbf{v}_i$ for some $a_i \in \mathbb{F}_{q^n}$. Then

$$\mathbf{z}^\rho = \sum_{i=1}^n a_i^{p^\ell} v_i' = \sum_{i=1}^n a_i^{p^\ell}(\mathbf{v}_i^\gamma - z_i) = \mathbf{z}^\gamma - \sum_{i=1}^n a_i^{p^\ell} z_i \in Z$$

and hence the collineation induced by ρ maps Γ in Γ and $\boldsymbol{\Sigma}$ into Σ. Finally, a similar argument applies to the case $\boldsymbol{W}_2^\gamma = W$.

$(ii) \Rightarrow (i)$. It follows (Csajbók & Zanella 2016c, Theorem 2) and (Lavrauw & Voorde 2010, Theorem 3). Let Σ_1 and Σ_2 be two canonical subgeometries of Σ^* and let us consider $\mathbb{G}_1 = \langle\sigma_1\rangle$ and $\mathbb{G}_2$ the subgroups of $\mathrm{P\Gamma L}(n,q^n)$ fixing pointwise the canonical subgeometries Σ_1 and Σ_2 of Σ^*, respectively. If β is a collineation of Σ^* such that $\Sigma_1^\beta = \Sigma_2$, then $\mathbb{G}_2 = \mathbb{G}_1^\beta$, i.e. they are conjugate under β. Indeed, for any point $P \in \Sigma_2$ we have

$$P^{\beta^{-1}\sigma_1\beta} = ((P^{\beta^{-1}})^{\sigma_1})^\beta = P^{\beta^{-1}\beta} = P$$

Analogously, it is straightforward to see that $\mathrm{Aut}(\Sigma_2) = \mathrm{Aut}(\Sigma_1)^\beta$. Hence, as a byproduct of Theorem 2.3.9 and Theorem 2.3.11, we can state the following.

Proposition 2.3.12 *Let $\Sigma \cong \mathrm{PG}(n-1,q)$ be a canonical subgeometry of $\Sigma^* = \mathrm{PG}(n-1,q^n)$. Let L be a linear set contained in a line Λ of Σ^* such that $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$, where Γ is an $(n-3)$-subspace of Σ^* such that $\Gamma\cap\Sigma = \emptyset = \Gamma\cap\Lambda$. If L is equivalent to a maximum scattered linear set of $\mathrm{PG}(1,q^n)$ not of pseudoregulus type L_f, with $f(x) = \sum_{i\in I} a_i x^{q^{si}}$, $m = \deg_{q^s} f(x)$ and $I = \mathrm{supp}_{q^s} f(x)$ such that either L_f is simple or $\Lambda_f = \Lambda_{\hat f}$ (cf. (2.14) and (2.15)), then there exists a collineation σ of Σ^* fixing Σ pointwise and an imaginary point P of Σ^* (wrt Σ) such that either*

$$\Gamma = \langle P^{\sigma^j}, Q_i \colon j \notin I, j \neq 0 \,\mathrm{and}\, i \in I\setminus\{m\}\rangle, \tag{2.17}$$

where $Q_i \in \langle P^{\sigma^i}, P^{\sigma^m}\rangle$, $i \in I \setminus \{m\}$, *or*

$$\Gamma = \langle P^{\sigma^j}, Q_i \colon j \notin \hat{I}, j \neq 0 \,\mathrm{and}\, i \in \hat{I} \setminus \{\hat{m}\}\rangle, \tag{2.18}$$

where $Q_i \in \langle P^{\sigma^i}, P^{\sigma^{\hat{m}}}\rangle$, $i \in \hat{I} \setminus \{\hat{m}\}$ *and* $\hat{I} = \mathrm{supp}_{q^s}\hat{f}(x)$, $\hat{m} = \deg_{q^s}\hat{f}(x)$.

Proof. By Theorem 2.3.11, there exists a collineation β of Σ^* such that $\mathbf{\Sigma}^\beta = \Sigma$ and either $\Gamma = \Gamma_f^\beta$ or $\Gamma = \Gamma_{\hat{f}}^\beta$ where Γ_f and $\Gamma_{\hat{f}}$ are defined as in (2.14) or (2.15). If $\Gamma = \Gamma_f^\beta$, the result follows putting $\sigma = \beta^{-1}\boldsymbol{\sigma}^s\beta$ (cf. (2.9)), $P = \boldsymbol{P}^\beta$ and $Q_i = \boldsymbol{Q}_i^\beta$ where $\boldsymbol{P}$ and $\boldsymbol{Q}_i$ are the points as in (2.14) of the statement of Theorem 2.3.9, and noting that $P^{\sigma^j} = \boldsymbol{P}^{\beta\sigma^j} = \boldsymbol{P}^{\boldsymbol{\sigma}^{sj}\beta}$, $j = 1,\dots,n-1$. If otherwise $\Gamma = \Gamma_{\hat{f}}^\beta$ the result instead follows by putting $\sigma = \beta^{-1}\boldsymbol{\sigma}^{s(n-1)}\beta$, $P = \boldsymbol{P}^\beta$, $\boldsymbol{T}_i^\beta = Q_i$ and taking into account Remark 2.3.10.

In particular, if $L \subset \Lambda$ is simple, then $\mathrm{P\Gamma L}(n,q^n)$ acts transitively on the vertices of any linear set equivalent to L, while the group $\mathrm{Aut}(\Sigma)$ acts transitively on the subspaces Γ such that $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$. Finally, we will state Theorem 2.2.4 in the case of a linear set contained in a projective line of $\mathrm{PG}(n-1,q^n)$.

Theorem 2.3.13 *Let* $I \subseteq \{1,\dots,n-1\}$ *and* m *denotes the maximum integer in* I. *Let* Σ *be a canonical subgeometry of* $\Sigma^* = \mathrm{PG}(n-1,q^n)$ *and consider an* $(n-3)$*-dimensional subspace* Γ *and a line* Λ *of* Σ^* *such that* $\Gamma \cap \Sigma = \emptyset = \Lambda \cap \Gamma$.
Assume $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ *is a linear set* $(1,n-2)$*-evasive of rank* n. *If there exists a point* P *with the property that*

$$\Gamma = \langle P^{\sigma^j}, Q_i \mid j \notin I,\, j \neq 0 \;\; \text{and}\; i \in I \setminus \{m\}\rangle$$

where $Q_i \in \langle P^{\sigma^m}, P^{\sigma^i}\rangle$ *and* σ *is a generator of the subgroup of* $\mathrm{P\Gamma L}(n,q^n)$ *fixing* Σ *pointwise, then* P *is an imaginary point (wrt* Σ*) and* $L \cong L_f$ *where* $f(x) = \sum_{i\in I} a_i x^{q^{si}}$ *and* $\gcd(s,n) = 1$ *for some integer* $1 \le s \le n-1$.

Remark 2.3.14 (i) Note that if L_f is equivalent to $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ with Γ and Λ as Proposition 2.3.8, f is a permutation polynomial if and only if $\Sigma \cap \langle \Gamma, P\rangle = \emptyset$.

(ii) The imaginary point P (wrt Σ) in Theorem 2.3.13 is not unique. For instance, let $L_\delta^{4,6}$ be as in (2.4), $\delta^2 + \delta = 1$, then $\delta \in \mathbb{F}_{q^2}$, and let us consider the point A with homogeneous coordinates $(0,-1,0,-1,0,1+\delta)$ in the frame $(P,P^\sigma,\ldots,P^{\sigma^5},U)$ with $P=\langle v\rangle_{\mathbb{F}_{q^6}}$ and $U=\langle v+v^\sigma+\ldots+v^{\sigma^{n-1}}\rangle_{\mathbb{F}_{q^6}}$. Then,

$$\begin{vmatrix} 0 & -1 & 0 & -1 & 0 & 1+\delta \\ 1+\delta^q & 0 & -1 & 0 & -1 & 0 \\ 0 & 1+\delta & 0 & -1 & 0 & -1 \\ -1 & 0 & 1+\delta^q & 0 & -1 & 0 \\ 0 & -1 & 0 & 1+\delta & 0 & -1 \\ -1 & 0 & -1 & 0 & 1+\delta^q & 0 \end{vmatrix} = (1-\delta)(2+\delta)^2(\delta^q-1)(2+\delta^q)^2 \neq 0.$$

and hence $\dim \mathcal{L}_A=5$ and the solid Γ as in (2.13) is spanned by the points $A^{\sigma^2}, A^{\sigma^4}$, $Q_1'=\langle(0,0,-2-\delta^q,0,\delta^q+1,0)\rangle_{\mathbb{F}_{q^6}}$ and $Q_2'=\langle(0,0,2+\delta^q,0,-2-\delta^q,0)\rangle_{\mathbb{F}_{q^6}}$, where $Q_1'\in\langle A^\sigma,A^{\sigma^5}\rangle$ and $Q_2'\in\langle A^{\sigma^3},A^{\sigma^5}\rangle$.

2.4 Maximum scattered linear sets which are neither of pseudoregulus nor of LP type

As we have seen in the previous section if $L=\mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ for a given subgeometry Σ and suitable subspaces Γ and Λ of $\mathrm{PG}(n-1,q^n)$, then, in general, the intersection number of Γ does not characterize the linear set L by its own.

In this section, we will deepen the study of projection vertices of linear sets of $\mathrm{PG}(1,q^n)$ known to date, which are neither of pseudoregulus nor of LP type, to provide characterization results for all of them in term of certain properties of relevant vertices.

In this regard, we will use a notion from classical projective geometry: the cross-ratio of four points of the projective line.

Let A,B,C,D be four points of $\mathrm{PG}(1,q^n)$ such that A,B,C are distinct; the *cross-ratio* $(A;B;C;D)$ of these points is defined as

$$(A;B;C;D)=\frac{\begin{vmatrix} a_0 & c_0 \\ a_1 & c_1\end{vmatrix}\begin{vmatrix} b_0 & d_0 \\ b_1 & d_1\end{vmatrix}}{\begin{vmatrix} a_0 & d_0 \\ a_1 & d_1\end{vmatrix}\begin{vmatrix} b_0 & c_0 \\ b_1 & c_1\end{vmatrix}},$$

where $A = \langle(a_0, a_1)\rangle_{\mathbb{F}_{q^n}}$, $B = \langle(b_0, b_1)\rangle_{\mathbb{F}_{q^n}}$, $C = \langle(c_0, c_1)\rangle_{\mathbb{F}_{q^n}}$, $D = \langle(d_0, d_1)\rangle_{\mathbb{F}_{q^n}}$, respectively. Setting $\infty := \frac{1}{0}$, we have

$$(A; B; C; A) = \infty,\ (A; B; C; B) = 0,\ (A; B; C; C) = 1$$

Moreover, by the definition we have

$$(A; B; C; D) = (B; A; D; C) = (C; D; A; B) = (D; C; B; A);$$

and if $(A; B; C; D) = \kappa$, all the possible values of the cross-ratio are

$$(A; B; D; C) = 1/\kappa, \qquad (A; C; B; D) = 1 - \kappa, \qquad (A; D; B; C) = (\kappa - 1)/\kappa,$$

$$(A; D; C; B) = \kappa/(\kappa - 1), \quad (A; C; D; B) = 1/(1 - \kappa).$$

We will say that the points pair $\{A, B\}$ *harmonically separates* the points pair $\{C, D\}$ if the cross-ratio $(A; B; C; D) = -1$, see (Casse 2006, Chapter 4) and Hirschfeld (1998). Finally, it is straightforward to see that if $\phi \in \mathrm{P\Gamma L}(2, q^n)$ with companion automorphism τ, then

$$(A^\phi; B^\phi; C^\phi; D^\phi) = (A; B; C; D)^\tau.$$

2.4.1 A characterization of $L^{3,n}_{\ell,\eta}$, $n \in \{6, 8\}$

Let $L^{3,n}_{\ell,\eta}$ be the linear set defined as in (2.3) and denote by $U^{3,n}_{\ell,\eta} \subseteq \mathbb{F}_{q^n} \times \mathbb{F}_{q^n}$ its underlying vector space. According to Csajbók, Marino, Polverino & Zanella (2018), $U^{3,n}_{\ell,\eta}$ is $\mathrm{GL}(2, q^n)$-equivalent to $U^{3,n}_{n-\ell,\eta^{q^{n-s}}}$ and to $U^{3,n}_{\ell+n/2,\eta^{-1}}$ as well. Thus, as indicated in (Csajbók, Marino & Zullo 2018, Section 4), we may assume $\ell < n/4$, and $\gcd(\ell, n/2) = 1$ and hence reduce the study to linear sets $L^{3,n}_{\ell,\eta}$ with $\ell = 1$. Moreover, in (Csajbók, Marino & Zullo 2018, Proposition 4.1 and 4.2), it is proven that in both cases, the relevant linear set is simple.

Theorem 2.4.1 *Let L be a $(1, 4)_q$-evasive linear set of rank 6 contained in a line Λ of $\mathrm{PG}(5, q^6)$ and let Σ be a canonical subgeometry of $\mathrm{PG}(5, q^6)$. Then L is equivalent to a maximum scattered linear set of type $L^{3,6}_{\ell,\eta}$, for some $\eta \in \mathbb{F}_{q^6}$ and $\gcd(\ell, 6) = 1$, if* *and only if for each solid* Γ *of* $\mathrm{PG}(5,q^6)$ *such that* $L=\mathrm{p}_{\Gamma,\Lambda}(\Sigma)$*, the following holds:*

1. *there exist a generator* σ *of the subgroup of* $\mathrm{P\Gamma L}(6,q^6)$ *fixing* Σ *pointwise and a point* $P\in\mathrm{PG}(5,q^6)$ *such that*

$$\Gamma=\langle P^{\sigma^i},Q : i\in\{2,3,5\}\rangle$$

   with $Q\in\langle P^{\sigma},P^{\sigma^4}\rangle$*,*

2. *the points* $P^{\sigma},P^{\sigma^4},Q,Q^{\sigma^3}$ *are collinear, and the equation*

$$Y^2-(\mathrm{Tr}_{q^3/q}(\gamma)-1)Y+\mathrm{N}_{q^3/q}(\gamma)=0, \tag{2.19}$$

   where $\gamma=(P^{\sigma};Q;Q^{\sigma^3};P^{\sigma^4})^{\tau}$ *for some* $\tau\in\mathrm{Aut}(\mathbb{F}_{q^6})$*, admits two solutions in* $\mathbb{F}_q$*.*

Proof. Let us suppose that L is equivalent to a maximum scattered linear set of type $L^{3,6}_{\ell,\eta}$. Since L is simple, we can suppose $\ell=1$ and by Proposition 2.3.12, Point *1* holds true. By Theorem 2.3.9, let $\Gamma_f=\langle \boldsymbol{P}^{\boldsymbol{\sigma}^i},\boldsymbol{Q} : i\in\{2,3,5\}\rangle$, where $f(x)=\eta x^q+x^{q^4}$, $\boldsymbol{P}$ an imaginary point of $\mathrm{PG}(5,q^6)$ (wrt $\boldsymbol{\Sigma}$), $\boldsymbol{Q}\in\langle \boldsymbol{P}^{\boldsymbol{\sigma}},\boldsymbol{P}^{\boldsymbol{\sigma}^4}\rangle$, $P=\boldsymbol{P}^{\beta}$ and $Q=\boldsymbol{Q}^{\beta}$ where β is the collineation of $\mathrm{PG}(5,q^6)$ such that $\Gamma_f^{\beta}=\Gamma$ and $\boldsymbol{\Sigma}^{\beta}=\Sigma$.
Clearly since $L^{3,6}_{1,\eta}$ is simple, all possible projection vertices for such a linear set are equivalent to Γ_f under the action of $\mathrm{P\Gamma L}(5,q^6)$. Since β preserves the collinearity of points, in order to show Point *2*, it is enough to prove that $\boldsymbol{P}^{\boldsymbol{\sigma}},\boldsymbol{P}^{\boldsymbol{\sigma}^4},\boldsymbol{Q},\boldsymbol{Q}^{\boldsymbol{\sigma}^3}$ are collinear. Indeed, note that $\boldsymbol{\sigma}^3$ fixes the line $\langle \boldsymbol{P}^{\boldsymbol{\sigma}},\boldsymbol{P}^{\boldsymbol{\sigma}^4}\rangle$, therefore $\boldsymbol{Q}^{\boldsymbol{\sigma}^3}\in\langle \boldsymbol{P}^{\boldsymbol{\sigma}},\boldsymbol{P}^{\boldsymbol{\sigma}^4}\rangle$. Let us suppose that $\boldsymbol{P}=\langle \mathbf{v}\rangle_{\mathbb{F}_{q^6}}$ with $\mathbf{v}$ a non-zero vector belonging to $\mathbb{F}^6_{q^6}$. If $\boldsymbol{Q}=\langle \mathbf{v}^{\boldsymbol{\sigma}}-\eta\mathbf{v}^{\boldsymbol{\sigma}^4}\rangle_{\mathbb{F}_{q^6}}$, then $\boldsymbol{Q}^{\boldsymbol{\sigma}^3}=\langle \mathbf{v}^{\boldsymbol{\sigma}^4}-\eta^{q^3}\mathbf{v}^{\boldsymbol{\sigma}}\rangle_{\mathbb{F}_{q^6}}$. Hence,

$$(\boldsymbol{P}^{\boldsymbol{\sigma}};\boldsymbol{P}^{\boldsymbol{\sigma}^4};\boldsymbol{Q};\boldsymbol{Q}^{\boldsymbol{\sigma}^3})=\frac{\begin{vmatrix}1 & 1\\ 0 & -\eta\end{vmatrix}\begin{vmatrix}0 & -\eta^{q^3}\\ 1 & 1\end{vmatrix}}{\begin{vmatrix}1 & -\eta^{q^3}\\ 0 & 1\end{vmatrix}\begin{vmatrix}0 & 1\\ 1 & -\eta\end{vmatrix}}=\eta^{q^3+1}.$$

Now, the linear set $L\cong L^{3,6}_{1,\eta}$ is equivalent to $L^{3,6}_{2,\eta^{-q^5}}$ which is scattered as well. Then by (Polverino & Zullo 2020, Theorem 7.3), the equation

$$Y^2 - (\mathrm{Tr}_{q^3/q}(\alpha) - 1)Y + \mathrm{N}_{q^3/q}(\alpha) = 0$$

admits two solutions in $\mathbb{F}_q$, where $\alpha = -\frac{(\eta^{-q^5})^{q^3+1}}{1-(\eta^{-q^5})^{q^3+1}} = \frac{1}{1-\eta^{q^2+q^5}}$. Moreover, we have $\alpha^q = (\boldsymbol{P}^{\boldsymbol{\sigma}}; \boldsymbol{Q}; \boldsymbol{Q}^{\boldsymbol{\sigma}^3}; \boldsymbol{P}^{\boldsymbol{\sigma}^4})$. Also, if ρ is the companion automorphism of β, then

$$(P^\sigma; Q; Q^{\sigma^3}; P^{\sigma^4}) = (\boldsymbol{P}^{\beta\sigma}; \boldsymbol{Q}^{\beta}; \boldsymbol{Q}^{\beta\sigma^3}; \boldsymbol{P}^{\beta\sigma^4}) = (\boldsymbol{P}^{\boldsymbol{\sigma}\beta}; \boldsymbol{Q}^{\beta}; \boldsymbol{Q}^{\boldsymbol{\sigma}^3\beta}; \boldsymbol{P}^{\boldsymbol{\sigma}^4\beta}) = (\boldsymbol{P}^{\boldsymbol{\sigma}}; \boldsymbol{Q}; \boldsymbol{Q}^{\boldsymbol{\sigma}^3}; \boldsymbol{P}^{\boldsymbol{\sigma}^4})^{\rho}.$$

Finally, since $\mathrm{Tr}_{q^3/q}(\alpha) = \mathrm{Tr}_{q^3/q}(\alpha^q)$ and $\mathrm{N}_{q^3/q}(\alpha) = \mathrm{N}_{q^3/q}(\alpha^q)$, Point *2* follows with $\gamma = \alpha^q$ and $\tau = \rho^{-1}$.

Suppose now that Points *1* and *2* hold true. Theorem 2.3.13 implies that $P = \langle v\rangle_{\mathbb{F}_{q^6}}$ is an imaginary point (wrt Σ), $Q = \langle v^\sigma - \eta v^{\sigma^4}\rangle_{\mathbb{F}_{q^6}}$ and $L \cong L_f$, $f(x) = \eta x^{q^s} + x^{q^{4s}}$ for some $\eta \in \mathbb{F}_{q^6}^*$ and $s \in \{1,5\}$. Since $P, P^\sigma, Q, Q^{\sigma^3}$ are distinct, $\mathrm{N}_{q^6/q^3}(\eta) \neq 1$. If $s = 1$, since $L_f = L_{\hat{f}}$ is equivalent to L_{g_1} with $g_1(x) = \eta^{-q^5}x^{q^2} + x^{q^5}$. Point *2* implies that the equation

$$Y^2 - (\mathrm{Tr}_{q^3/q}(\alpha_1) - 1)Y + \mathrm{N}_{q^3/q}(\alpha_1) = 0$$

admits two solutions in $\mathbb{F}_q$, where $\alpha_1 = -\frac{(\eta^{-q^5})^{q^3+1}}{1-(\eta^{-q^5})^{q^3+1}}$. Hence, by (Polverino & Zullo 2020, Theorem 7.3), L_{g_1} is scattered and L is equivalent to a scattered linear set of type $L^{3,6}_{1,\eta}$. If $s = 5$ and hence $f(x) = \eta x^{q^5} + x^{q^2}$, $L_f \cong L_{g_2}$ with $g_2(x) = x^{q^5} + \eta^{-1}x^{q^2}$. Point *2* implies that the equation

$$Y^2 - (\mathrm{Tr}_{q^3/q}(\alpha_2) - 1)Y + \mathrm{N}_{q^3/q}(\alpha_2) = 0$$

admits two roots in $\mathbb{F}_q$ with $\alpha_2 = -\frac{\eta^{-(q^3+1)}}{1-\eta^{-(q^3+1)}}$. Then, by (Polverino & Zullo 2020, Theorem 7.3), $g_2(x)$ is scattered and hence L is scattered as well, and it is of the type $L^{3,6}_{2,\eta^{-1}} \cong L^{3,6}_{1,\eta^q}$.

In the same way, we get

Theorem 2.4.2 *Let L be an $(1,6)_q$-evasive linear set of rank 8 contained in a line Λ of $\mathrm{PG}(7,q^8)$, with $q \leq 11$ or $q \geq 1039891$ odd. Then, L is equivalent to a maximum scattered linear set of type $L^{3,8}_{\ell,\eta}$, for some $\eta \in \mathbb{F}_{q^8}$ and $\gcd(\ell,8) = 1$, if and only if for each 5-dimensional subspace Γ of $\mathrm{PG}(7,q^8)$ such that $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$, the following hold:*

1. *there exists a generator σ of the subgroup of* $\mathrm{P\Gamma L}(8,q^8)$ *fixing Σ pointwise and a point $P \in \mathrm{PG}(7,q^8)$ such that*

$$\Gamma = \langle P^{\sigma^i}, Q : i \in \{2,3,4,6,7\}\rangle$$

with $Q \in \langle P^{\sigma}, P^{\sigma^5}\rangle$,

2. *the points $P^{\sigma}, P^{\sigma^5}, Q, Q^{\sigma^4}$ are collinear and pair $\{Q, Q^{\sigma^4}\}$ harmonically separates the pair $\{P^{\sigma}, P^{\sigma^5}\}$.*

Proof. Let L be equivalent to a maximum scattered linear set of type $L^{3,8}_{\ell,\eta}$. Since L is simple, it is sufficient to suppose that $\ell = 1$ and by Proposition 2.3.12, Point *1* holds. By Theorem 2.3.9, let $\Gamma_f = \langle \boldsymbol{P}^{\boldsymbol{\sigma}^i}, \boldsymbol{Q} : i \in \{2,3,4,6,7\}\rangle$, $f(x) = \eta x^q + x^{q^5}$, with $\boldsymbol{P}$ an imaginary point of $\mathrm{PG}(7,q^8)$ (wrt $\boldsymbol{\Sigma}$), $\boldsymbol{Q} \in \langle \boldsymbol{P}^{\boldsymbol{\sigma}}, \boldsymbol{P}^{\boldsymbol{\sigma}^5}\rangle$, $P = \boldsymbol{P}^{\beta}$ and $Q = \boldsymbol{Q}^{\beta}$ where β is the collineation of $\mathrm{PG}(7,q^8)$ such that $\boldsymbol{\Sigma}^{\beta} = \Sigma$ and $\Gamma_f^{\beta} = \Gamma$.
As before, all possible projection vertices for such a linear set are equivalent to Γ_f under the action of $\mathrm{P\Gamma L}(7,q^8)$. Since β preserves collinearity of points and the property for a pair of points to harmonically separate another pair, in order to show Point *2*, it is enough to prove that $\boldsymbol{P}^{\boldsymbol{\sigma}}, \boldsymbol{P}^{\boldsymbol{\sigma}^5}, \boldsymbol{Q}, \boldsymbol{Q}^{\boldsymbol{\sigma}^4}$ are collinear and $(\boldsymbol{P}^{\boldsymbol{\sigma}}; \boldsymbol{P}^{\boldsymbol{\sigma}^5}; \boldsymbol{Q}; \boldsymbol{Q}^{\boldsymbol{\sigma}^4}) = -1$. Clearly, $\boldsymbol{\sigma}^4$ fixes the line $\langle \boldsymbol{P}^{\boldsymbol{\sigma}}, \boldsymbol{P}^{\boldsymbol{\sigma}^5}\rangle$ and $\boldsymbol{Q}^{\boldsymbol{\sigma}^4} \in \langle \boldsymbol{P}^{\boldsymbol{\sigma}}, \boldsymbol{P}^{\boldsymbol{\sigma}^5}\rangle$. Now, let us suppose that $\boldsymbol{P} = \langle \mathbf{v}\rangle_{\mathbb{F}_{q^6}}$ with $\mathbf{v}$ a non-zero vector belonging to $\mathbb{F}_{q^8}^8$. If $\boldsymbol{Q} = \langle \mathbf{v}^{\boldsymbol{\sigma}} - \eta \mathbf{v}^{\boldsymbol{\sigma}^5}\rangle_{\mathbb{F}_{q^8}}$, then $\boldsymbol{Q}^{\boldsymbol{\sigma}^4} = \langle \mathbf{v}^{\boldsymbol{\sigma}^5} - \eta^{q^4} \mathbf{v}^{\boldsymbol{\sigma}}\rangle_{\mathbb{F}_{q^8}}$ with $\eta \in \mathbb{F}_{q^8}^*$. Since we have assumed that either $q \leq 11$ or $q \geq 1039891$, by (Timpanella & Zini 2023, Theorem 1.1) we have $\mathrm{N}_{q^8/q^4}(\eta) = -1$. Then

$$(\boldsymbol{P}^{\boldsymbol{\sigma}}; \boldsymbol{P}^{\boldsymbol{\sigma}^5}; \boldsymbol{Q}; \boldsymbol{Q}^{\boldsymbol{\sigma}^4}) = \frac{\begin{vmatrix} 1 & 1 \\ 0 & -\eta \end{vmatrix} \begin{vmatrix} 0 & -\eta^{q^4} \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -\eta^{q^4} \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 0 & 1 \\ 1 & -\eta \end{vmatrix}} = \eta^{q^4+1} = -1.$$

and hence Point *2*.
Now suppose Point *1* and *2* hold. By Theorem 2.3.13, since L is $(1,6)_q$-evasive, $L \cong L_f$ with $f(x) = \eta x^{q^s} + x^{q^{5s}}$, $s \in \{1,3,5,7\}$ and $\eta \in \mathbb{F}_{q^8}^*$. Moreover, the point $P = \langle v\rangle_{\mathbb{F}_{q^8}}$ is an imaginary point (wrt Σ). Since $Q = \langle v^{\sigma} - \eta v^{\sigma^5}\rangle_{\mathbb{F}_{q^8}}$ and $-1 = (P^{\sigma}; P^{\sigma^5}; Q; Q^{\sigma^4}) = \mathrm{N}_{q^8/q^4}(\eta)$, again by (Timpanella & Zini 2023, Theorem 1.1), L_f, and hence L, is equivalent to a scattered linear set of type $L^{3,8}_{r,\eta}$.

2.4.2 A characterization of $L_\delta^{4,6}$

Let $L_\delta^{4,6}$ be the linear set defined as in (2.4) for q odd and $\delta^2+\delta=1$. In Csajbók, Marino & Zullo (2018), it is shown that for $q \equiv 0, \pm 1 \pmod 5$, this is scattered. The remaining cases for q odd were treated later in Marino et al. (2020).

In the following, exploiting the same arguments used by Csajbók, Marino and Zullo in (Csajbók, Marino & Zullo 2018, Thereom 5.7)), we firstly determine conditions under which the maximum scattered linear set $L_\delta^{4,6}$ is simple.

Proposition 2.4.3 *The linear set $L_\delta^{4,6}$, with $q=p^e$ odd and $\delta^2+\delta=1$, is simple if and only if either $q \equiv 0 \pmod 5$ or $p \equiv \pm 2 \pmod 5$ for some prime p.*

Proof. To this aim it is enough to observe that the missing case in (Csajbók, Marino & Zullo 2018, Thereom 5.7)) is when $q=p^{2h+1}$ and $p \equiv \pm 2 \pmod 5$. In this case, $\delta \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$. So, δ and δ^p are two distinct roots of $X^2+X-1=0$ and $X^q+X+1=0$ and we get that U_f and $U_{\hat{f}}$ with $f(x)=x^q+x^{q^3}+\delta x^{q^5}$ are $\Gamma\mathrm{L}(2,q^6)$-equivalent.

Hence, we can prove the following results.

Theorem 2.4.4 *Let L be a $(1,4)_q$-evasive linear set of rank 6 contained in a line Λ of $\mathrm{PG}(5,q^6)$ and let Σ be a canonical subgeometry of $\mathrm{PG}(5,q^6)$, q odd. Then L is equivalent to a maximum scattered linear set of type $L_\delta^{4,6}$ if and only if for each solid Γ of $\mathrm{PG}(5,q^6)$ with $\Gamma\cap\Sigma=\Gamma\cap\Lambda=\emptyset$ such that $L=\mathrm{p}_{\Gamma,\Lambda}(\Sigma)$, the following hold:*

1. *there exist a generator σ of the subgroup of $\mathrm{P\Gamma L}(6,q^6)$ fixing Σ pointwise and a point $P=\langle v\rangle_{\mathbb{F}_{q^6}}$, v a non-zero vector of $\mathbb{F}_{q^6}^6$, such that*
$$\Gamma = \langle P^{\sigma^2}, P^{\sigma^4}, Q, R\rangle \tag{2.20}$$
with $Q \in \langle P^\sigma, P^{\sigma^5}\rangle$ and $R \in \langle P^{\sigma^3}, P^{\sigma^5}\rangle$;
2. *the point $C=\langle v^\sigma - v^{\sigma^3}\rangle_{\mathbb{F}_{q^6}}$ belongs to Γ;*
3. *the points $P^\sigma, P^{\sigma^5}, C^{\sigma^4}$ and Q are collinear and the cross-ratio $(P^\sigma; P^{\sigma^5}; C^{\sigma^4}; Q)$ is in $\mathbb{F}_{q^2}$;*
4. *the points C, P^{σ^5} and $\langle Q, Q^{\sigma^2}\rangle \cap \langle R, R^{\sigma^2}\rangle$ are collinear.*

Moreover, the vertices Γ *such that* $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$ *belong to the same orbit under the action of* $\mathrm{Aut}(\Sigma)$ *if and only if* $q \equiv 0 \pmod 5$ *or* $p \equiv \pm 2 \pmod 5$ *for some prime* p.

Proof. Let us suppose that L is equivalent to a maximum scattered linear set of type $L^{4,6}_{\delta}$, $\delta^2+\delta = 1$. Since $\Lambda_f = \Lambda_{\hat{f}}$ where $f(x) = x^q + x^{q^3} + \delta x^{q^5}$, by Proposition 2.3.12, there exists a collineation β such that $\boldsymbol{\Sigma}^{\beta} = \Sigma$ and either $\Gamma^{\beta}_f = \Gamma$ or $\Gamma^{\beta}_{\hat{f}} = \Gamma$, an imaginary point $\boldsymbol{P} = \langle \mathbf{v}\rangle_{\mathbb{F}_q^6}$ such that

$$\Gamma_f = \langle \boldsymbol{P}^{\boldsymbol{\sigma}^2}, \boldsymbol{P}^{\boldsymbol{\sigma}^4}, \boldsymbol{Q}_1, \boldsymbol{R}_1\rangle$$

where $\boldsymbol{Q}_1 = \langle \mathbf{v}^{\boldsymbol{\sigma}^5} - \delta\mathbf{v}^{\boldsymbol{\sigma}}\rangle_{\mathbb{F}_{q^6}}$ and $\boldsymbol{R}_1 = \langle \mathbf{v}^{\boldsymbol{\sigma}^5} - \delta\mathbf{v}^{\boldsymbol{\sigma}^3}\rangle_{\mathbb{F}_{q^6}}$ and

$$\Gamma_{\hat{f}} = \langle \boldsymbol{P}^{\boldsymbol{\sigma}^2}, \boldsymbol{P}^{\boldsymbol{\sigma}^4}, \boldsymbol{Q}_2, \boldsymbol{R}_2\rangle$$

where $\boldsymbol{Q}_2 = \langle \delta^q\mathbf{v}^{\boldsymbol{\sigma}^5} - \mathbf{v}^{\boldsymbol{\sigma}}\rangle_{\mathbb{F}_{q^6}}$ and $\boldsymbol{R}_2 = \langle \mathbf{v}^{\boldsymbol{\sigma}^5} - \mathbf{v}^{\boldsymbol{\sigma}^3}\rangle_{\mathbb{F}_{q^6}}$. Let us consider the point $\boldsymbol{C}_1 = \langle \mathbf{v}^{\boldsymbol{\sigma}} - \mathbf{v}^{\boldsymbol{\sigma}^3}\rangle_{\mathbb{F}_{q^6}}$. Clearly $\boldsymbol{C}_1 \in \Gamma_f$ and it is easy to see that the point $\boldsymbol{P}^{\boldsymbol{\sigma}}, \boldsymbol{P}^{\boldsymbol{\sigma}^5}, \boldsymbol{C}_1^{\boldsymbol{\sigma}^4}$ and $\boldsymbol{Q}_1$ are collinear. The cross-ratio $(\boldsymbol{P}^{\boldsymbol{\sigma}}; \boldsymbol{P}^{\boldsymbol{\sigma}^5}; \boldsymbol{C}_1^{\boldsymbol{\sigma}^4}; \boldsymbol{Q}_1) = \delta \in \mathbb{F}_{q^2}$ and since $\delta^2+\delta = 1$, elementary argument also shows that point 4 holds true for the points $\boldsymbol{C}_1, \boldsymbol{P}^{\boldsymbol{\sigma}^5}$ and $\langle \boldsymbol{Q}_1, \boldsymbol{Q}_1^{\boldsymbol{\sigma}^2}\rangle \cap \langle \boldsymbol{R}_1, \boldsymbol{R}_1^{\boldsymbol{\sigma}^2}\rangle$ are collinear. Then, if $\Gamma^{\beta}_f = \Gamma$, putting $\sigma = \boldsymbol{\sigma}^{\beta}$, $\boldsymbol{P}^{\beta} = P$, $\boldsymbol{Q}_1^{\beta} = Q, \boldsymbol{R}_1^{\beta} = R$ and $\boldsymbol{C}_1^{\beta} = C$, Point (*1*-*4*) hold true for Γ as well. If $\Gamma^{\beta}_{\hat{f}} = \Gamma$, let us consider the point $\boldsymbol{S} = \langle \delta^q\mathbf{v}^{\boldsymbol{\sigma}^3} - \mathbf{v}^{\boldsymbol{\sigma}}\rangle_{\mathbb{F}_{q^6}} \in \Gamma_{\hat{f}} \cap \langle \boldsymbol{P}^{\boldsymbol{\sigma}}, \boldsymbol{P}^{\boldsymbol{\sigma}^3}\rangle_{\mathbb{F}_{q^6}}$. Clearly, the solid $\Gamma_{\hat{f}} = \langle \boldsymbol{P}^{\boldsymbol{\sigma}^2}, \boldsymbol{P}^{\boldsymbol{\sigma}^4}, \boldsymbol{Q}_2, \boldsymbol{S}\rangle$. Then, it is easy to see that the point $\boldsymbol{P}^{\boldsymbol{\sigma}}, \boldsymbol{P}^{\boldsymbol{\sigma}^5}, \boldsymbol{R}_2^{\boldsymbol{\sigma}^2}$ and $\boldsymbol{Q}_2$ are collinear and their cross-ratio is equal to δ^{-q} which belongs to $\mathbb{F}_{q^2}$. Since $\delta^2+\delta = 1$, elementary argument shows that the points $\boldsymbol{R}_2, \boldsymbol{P}^{\boldsymbol{\sigma}}$ and $\langle \boldsymbol{Q}_2, \boldsymbol{Q}_2^{\boldsymbol{\sigma}^4}\rangle \cap \langle \boldsymbol{S}, \boldsymbol{S}^{\boldsymbol{\sigma}^4}\rangle$ are collinear. Then, if $\Gamma^{\beta}_{\hat{f}} = \Gamma$, putting $\sigma = \boldsymbol{\sigma}^{5\beta}$, $\boldsymbol{P}^{\beta} = P$, $\boldsymbol{Q}_2^{\beta} = Q, \boldsymbol{S}^{\beta} = R$ and $\boldsymbol{R}_2^{\beta} = C$, Point (*1*-*4*) hold for Γ.

Now we suppose that Point (*1*-*4*) hold true then by Theorem 2.3.13, $L \cong L_f$ with $f(x) = x^{q^s} + \gamma x^{q^{3s}} + \delta x^{q^{5s}} \in \mathbb{F}_{q^6}[x]$ with $s \in \{1,5\}$ and $P = \langle v\rangle_{\mathbb{F}_{q^6}}$ is an imaginary point. Then $Q = \langle v^{\sigma^5} - \delta v^{\sigma}\rangle_{\mathbb{F}_{q^6}}$ and $R = \langle \gamma v^{\sigma^5} - \delta v^{\sigma^3}\rangle_{\mathbb{F}_{q^6}}$. Since $C = \langle v^{\sigma} - v^{\sigma^3}\rangle_{\mathbb{F}_{q^6}} \in \Gamma$, then $\gamma = 1$. By Point *3*, it is easy to see that the cross-ratio $(P^{\sigma}; P^{\sigma^5}; C^{\sigma^4}; Q) = \delta$, and hence $\delta \in \mathbb{F}_{q^2}$. Moreover, by Point *4*, the lines $\langle Q, Q^{\sigma^2}\rangle$ and $\langle R, R^{\sigma^2}\rangle$ meet at the point $\langle v^{\sigma} - (\delta^2+\delta)v^{\sigma^3} + v^{\sigma^5}\rangle_{\mathbb{F}_{q^6}}$ and the collinearity of this point, C and P^{σ^5} implies

$$0 = \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & -(\delta^2+\delta) & 1 \end{vmatrix} = -\delta^2 - \delta + 1, \tag{2.21}$$

obtaining $L \cong L_\delta^{4,6}$.

The family of linear sets of type $L_\delta^{4,6}$ was investigated for also for q even in Bartoli et al. (2024). The conditions assuring the property of being scattered in this case are rather demanding, and it seems quite difficult to translate them in geometric terms, see (Bartoli et al. 2024, Theorem 2.9). However, by using arguments similar to those used in the theorem above, we can state the following result.

Theorem 2.4.5 *Let L be a $(1,4)_q$-evasive linear set of rank 6 contained in a line Λ of $\mathrm{PG}(5,q^6)$ and let Σ be a canonical subgeometry of $\mathrm{PG}(5,q^6)$, q even. If L is equivalent to a maximum scattered linear set of type $L_\delta^{4,6}$, then for each solid Γ of $\mathrm{PG}(5,q^6)$ with $\Gamma \cap \Sigma = \Gamma \cap \Lambda = \emptyset$ such that $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$, the following holds:*

1. *there exist a generator σ of the subgroup of $\mathrm{P\Gamma L}(6,q^6)$ fixing Σ pointwise and a point $P = \langle v \rangle_{\mathbb{F}_{q^6}}$, v a nonzero vector of $\mathbb{F}_{q^6}^6$, such that the solid*

$$\Gamma = \langle P^{\sigma^2}, P^{\sigma^4}, Q, R \rangle \tag{2.22}$$

   with $Q \in \langle P^\sigma, P^{\sigma^5} \rangle$ and $R \in \langle P^{\sigma^3}, P^{\sigma^5} \rangle$;

2. *the point $C = \langle v^\sigma - v^{\sigma^3} \rangle_{\mathbb{F}_{q^6}}$ belongs to Γ;*

3. *the point $P^\sigma, P^{\sigma^5}, C^{\sigma^4}$ and Q are collinear and the cross-ratio $(P^\sigma; P^{\sigma^5}; C^{\sigma^4}; Q)$ does not belong to any subfield of $\mathbb{F}_{q^6}$.*

2.4.3 A characterization of $L_{s,m,h}^{5,t}$

This section is devoted to characterizing via geometrical properties of their projection vertex the scattered linear set of type $L_{s,m,h}^{5,t}$ as in (2.5) with

$$\psi_{t,s,m,h}(x) = x^{q^{s(t-1)}} + h^{1-q^{s(2t-1)}} x^{q^{s(2t-1)}} + m\left(x^{q^s} - h^{1-q^{s(t+1)}} x^{q^{s(t+1)}}\right),$$

where $t \geq 3$ and $(m,h) \in \mathbb{F}_{q^t}^* \times \mathbb{F}_{q^{2t}}^*$. We will proceed by splitting our discussion into two cases according to suitable conditions for the parameters m and h are satisfied, precisely:

(*i*) $m = 1$ and $h \in \mathbb{F}_{q^{2t}}$ with $\mathrm{N}_{q^{2t}/q^t}(h) = -1$

(*ii*) m neither a $(q-1)$-th power nor a $(q+1)$-th power of an element belonging to $\ker \mathrm{Tr}_{q^{2t}/q^t}$ and $q \equiv 1 \pmod 4$ if t is odd.

Actually, for $q \leq 5$ odd and $t \in \{3, 4, 5, 6\}$, `Magma` computation suggests a wider choice of parameters for which the linear set $L^{5,t}_{s,m,h}$ is scattered (see Giannoni et al. (2024)).

The linear sets of the type $L^{5,t}_{s,m,h}$, $m = 1$

To investigate the geometrical properties of the projection vertex of a linear set of type $L^{5,t}_{s,h} := L^{5,t}_{s,1,h}$ in $\mathrm{PG}(1, q^{2t})$, we first observe that, by Theorem 2.3.9, (Longobardi & Zanella 2021, Proposition 3.1) and (Neri et al. 2022, Proposition 4.17), for any $t \geq 5$ $\psi_{t,h,s}(x) := \psi_{t,s,1,h}(x)$ is $\Gamma\mathrm{L}(2t, q^{2t})$-equivalent to its adjoint. For $t = 3$, using the same argument as done in (Longobardi et al. 2023, Theorem 4.7), the simplicity of $L^{5,t}_{s,h}$ follows. Therefore, we get the following.

Theorem 2.4.6 *The linear set $L^{5,t}_{s,h}$ of $\mathrm{PG}(1, q^{2t})$ is simple, for $t = 3$ or $t \geq 5$.*

Now, as we did for the previous examples, we prove the following result.

Theorem 2.4.7 *Let L be an $(1, 2(t-1))$-evasive linear set of rank $2t$ in a line Λ of $\mathrm{PG}(2t-1, q^{2t})$. Let Σ be a canonical subgeometry of $\mathrm{PG}(2t-1, q^{2t})$, q odd and $t \geq 3$. Then L is equivalent to a scattered linear set of the type $L^{5,t}_{s,h}$ if and only if for each $(2t-3)$ dimensional subspace Γ of $\mathrm{PG}(2t-1, q^{2t})$ such that $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$, the following holds:*

1. *there exist a generator σ of the subgroup of $\mathrm{P\Gamma L}(2t, q^{2t})$ fixing Σ pointwise and a point $P = \langle v \rangle_{\mathbb{F}_{q^{2t}}} \in \mathrm{PG}(2t-1, q^{2t})$ such that*

$$\Gamma = \langle P^{\sigma^i}, Q, R, S : i \notin \{0, 1, t-1, t+1, 2t-1\} \rangle$$

with $Q \in \langle P^{\sigma}, P^{\sigma^{2t-1}} \rangle$, $R \in \langle P^{\sigma^{t-1}}, P^{\sigma^{2t-1}} \rangle$ and $S \in \langle P^{\sigma^{t+1}}, P^{\sigma^{2t-1}} \rangle$,

2. *the point $X = \langle v^{\sigma} - v^{\sigma^{t-1}} \rangle_{\mathbb{F}_{q^{2t}}}$ belongs to Γ,*

3. *there exists $h \in \mathcal{H} = \{z \in \mathbb{F}_{q^{2t}} : \mathrm{N}_{q^{2t}/q^t}(z) = -1\}$ such that*

(a) the points $P^{\sigma}, P^{\sigma^{2t-1}}, D, Q$ *where* $D = \langle v^{\sigma} + v^{\sigma^{2t-1}}\rangle_{\mathbb{F}_{q^{2t}}}$ *are collinear and the cross-ratio* $(P^{\sigma}; P^{\sigma^{2t-1}}; D; Q) = -h^{1-q^{s(2t-1)}}$,

(b) the points $P^{\sigma}, P^{\sigma^{t+1}}, R^{\sigma^2}, Y$, *where* $Y = \langle Q, S\rangle \cap \langle P^{\sigma}, P^{\sigma^{t+1}}\rangle$, *are collinear and the cross-ratio* $(P^{\sigma}; P^{\sigma^{t+1}}; R^{\sigma^2}; Y) = h^{1+q^{2s}}$.

Proof. Let us suppose that L is equivalent to a maximum scattered linear set of type $L^{5,t}_{s,h}$. By Theorem 2.3.11, there exists β such that $\boldsymbol{\Sigma}^{\beta} = \Sigma$ and either $\Gamma^{\beta}_{\psi_{t,h,s}} = \Gamma$ or $\Gamma^{\beta}_{\hat{\psi}_{t,h,s}} = \Gamma$. If $t = 3$ or $t \geq 5$, since $L^{5,t}_{s,h}$ is simple, it is enough to prove the statement supposing that $\Gamma^{\beta}_{\psi_{t,h,s}} = \Gamma$.
Let $\boldsymbol{\rho} = \boldsymbol{\sigma}^s$ and $\boldsymbol{P} = \langle \mathbf{v}\rangle_{\mathbb{F}_{q^{2t}}}$ be an imaginary point of $\mathrm{PG}(2t-1, q^{2t})$ (wrt $\boldsymbol{\Sigma}$), $\boldsymbol{Q} \in \langle \boldsymbol{P}^{\boldsymbol{\rho}}, \boldsymbol{P}^{\boldsymbol{\rho}^{2t-1}}\rangle$, $\boldsymbol{R} \in \langle \boldsymbol{P}^{\boldsymbol{\rho}^{t-1}}, \boldsymbol{P}^{\boldsymbol{\rho}^{2t-1}}\rangle$ and $\boldsymbol{S} \in \langle \boldsymbol{P}^{\boldsymbol{\rho}^{t+1}}, \boldsymbol{P}^{\boldsymbol{\rho}^{2t-1}}\rangle$ and

$$\Gamma_{\psi_{t,h,s}} = \langle \boldsymbol{P}^{\boldsymbol{\rho}^i}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{S} : i \notin \{1, t-1, t+1, 2t-1\}\rangle$$

where

$$\boldsymbol{Q} = \langle \mathbf{v}^{\boldsymbol{\rho}^{2t-1}} - h^{1-q^{s(2t-1)}}\mathbf{v}^{\boldsymbol{\rho}}\rangle_{\mathbb{F}_{q^{2t}}}, \boldsymbol{R} = \langle \mathbf{v}^{\boldsymbol{\rho}^{2t-1}} - h^{1-q^{s(2t-1)}}\mathbf{v}^{\boldsymbol{\rho}^{t-1}}\rangle_{\mathbb{F}_{q^{2t}}}, \text{ and}$$

$$\boldsymbol{S} = \langle h^{1-q^{s(t+1)}}\mathbf{v}^{\boldsymbol{\rho}^{2t-1}} + h^{1-q^{s(2t-1)}}\mathbf{v}^{\boldsymbol{\rho}^{t+1}}\rangle_{\mathbb{F}_{q^{2t}}}.$$

It is straightforward to check that the lines $\langle \boldsymbol{Q}, \boldsymbol{R}\rangle$ and $\langle \boldsymbol{P}^{\boldsymbol{\rho}}, \boldsymbol{P}^{\boldsymbol{\rho}^{t-1}}\rangle$ meet at the point $\boldsymbol{X} = \langle \mathbf{v}^{\boldsymbol{\rho}^{t-1}} - \mathbf{v}^{\boldsymbol{\rho}}\rangle_{\mathbb{F}_{q^{2t}}}$ and this belongs to $\Gamma_{\psi_{t,h,s}}$. Also, the lines $\langle \boldsymbol{Q}, \boldsymbol{S}\rangle$ and $\langle \boldsymbol{P}^{\boldsymbol{\rho}}, \boldsymbol{P}^{\boldsymbol{\rho}^{t+1}}\rangle$ meet at the point $\boldsymbol{Y} = \langle h^{1-q^{s(t+1)}}\mathbf{v}^{\boldsymbol{\rho}} + \mathbf{v}^{\boldsymbol{\rho}^{t+1}}\rangle_{\mathbb{F}_{q^{2t}}}$. Let $\boldsymbol{D} = \langle \mathbf{v}^{\boldsymbol{\rho}} + \mathbf{v}^{\boldsymbol{\rho}^{2t-1}}\rangle_{\mathbb{F}_{q^{2t}}}$, then the points $\boldsymbol{P}^{\boldsymbol{\rho}}, \boldsymbol{P}^{\boldsymbol{\rho}^{2t-1}}, \boldsymbol{D}, \boldsymbol{Q}$ and $\boldsymbol{P}^{\boldsymbol{\rho}}, \boldsymbol{P}^{\boldsymbol{\rho}^{t+1}}, \boldsymbol{R}^{\boldsymbol{\rho}^2}, \boldsymbol{Y}$ are collinear. Moreover,

$$(\boldsymbol{P}^{\boldsymbol{\rho}}; \boldsymbol{P}^{\boldsymbol{\rho}^{2t-1}}; \boldsymbol{D}; \boldsymbol{Q}) = \frac{\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\begin{vmatrix} 0 & 1 \\ 1 & -h^{1-q^{s(2t-1)}} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 0 & -h^{1-q^{s(2t-1)}} \end{vmatrix}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}} = -h^{1-q^{s(2t-1)}}$$

and

$$(\boldsymbol{P}^{\boldsymbol{\rho}}; \boldsymbol{P}^{\boldsymbol{\rho}^{t+1}}; \boldsymbol{R}^{\boldsymbol{\rho}^2}; \boldsymbol{Y}) = \frac{\begin{vmatrix} 1 & 1 \\ 0 & -h^{q^{2s}-q^s} \end{vmatrix}\begin{vmatrix} 0 & h^{1-q^{s(t+1)}} \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & h^{1-q^{s(t+1)}} \\ 0 & 1 \end{vmatrix}\begin{vmatrix} 0 & 1 \\ 1 & -h^{q^{2s}-q^s} \end{vmatrix}} = h^{q^{2s}+1}.$$

Therefore, since β preserve the collinearity and if $(A_1; A_2; A_3; A_4) \in \mathcal{H}$ then $(A_1^\beta; A_2^\beta; A_3^\beta; A_4^\beta)$ belongs to $\mathcal{H}$, the Points *1-3* hold true.
For $t = 4$, we have

$$\Gamma_{\hat{\psi}_{t,h,s}} = \langle \boldsymbol{P}^{\boldsymbol{\rho}^i}, \boldsymbol{Q}, \boldsymbol{R}, \boldsymbol{S} : i \notin \{1,3,5,7\}\rangle$$

where

$$\boldsymbol{Q} = \langle \mathbf{v}^{\boldsymbol{\rho}} - h^{q^s-1}\mathbf{v}^{\boldsymbol{\rho}^7}\rangle_{\mathbb{F}_{q^8}}, \quad \boldsymbol{R} = \langle \mathbf{v}^{\boldsymbol{\rho}^3} + h^{q^{3s}-1}\mathbf{v}^{\boldsymbol{\rho}^7}\rangle_{\mathbb{F}_{q^8}}, \text{ and}$$

$$\boldsymbol{S} = \langle \mathbf{v}^{\boldsymbol{\rho}^7} - \mathbf{v}^{\boldsymbol{\rho}^5}\rangle_{\mathbb{F}_{q^8}}.$$

If $\boldsymbol{\Sigma}^\beta = \Sigma$ and $\Gamma^\beta_{\hat{\psi}_{t,h,s}} = \Gamma$, let $\boldsymbol{D} = \langle \mathbf{v}^{\rho} + \mathbf{v}^{\boldsymbol{\rho}^7}\rangle_{\mathbb{F}_{q^8}}$, $\boldsymbol{T} = \langle h^{q^{3s}-q^s}\mathbf{v}^{\boldsymbol{\rho}} - \mathbf{v}^{\boldsymbol{\rho}^3}\rangle_{\mathbb{F}_{q^8}}$ and $\boldsymbol{Z} = \langle \mathbf{v}^{\boldsymbol{\rho}} - h^{1-q^s}\mathbf{v}^{\boldsymbol{\rho}^5}\rangle_{\mathbb{F}_{q^8}}$. The result follows noting that

$$\Gamma_{\hat{\psi}_{t,h,s}} = \langle \boldsymbol{P}^{\boldsymbol{\rho}^i}, \boldsymbol{Q}, \boldsymbol{Z}, \boldsymbol{T} : i \notin \{1,3,5,7\}\rangle$$

and putting $\sigma = \beta^{-1}\boldsymbol{\rho}^7\beta$, $P = \boldsymbol{P}^\beta$, $Q = \boldsymbol{Q}^\beta$, $R = \boldsymbol{Z}^\beta$, $S = \boldsymbol{T}^\beta$, $X = \boldsymbol{S}^\beta$ and $D = \boldsymbol{D}^\beta$.
Now, let suppose that Points *1 – 3* hold true. By Theorem 2.3.13, there exist a subgeometry Σ, a collineation σ fixing Σ pointwise and an imaginary point (wrt Σ) $P = \langle v\rangle_{\mathbb{F}_{q^{2t}}}$, such that $L \cong L_f$ with $f(x) = x^{q^s} + \alpha x^{q^{s(t-1)}} + \beta x^{q^{s(t+1)}} + \gamma x^{q^{s(2t-1)}} \in \mathbb{F}_{q^n}[x]$, where $Q = \langle v^{\sigma^{2t-1}} - \gamma v^{\sigma}\rangle_{\mathbb{F}_{q^{2t}}}$, $R = \langle \alpha v^{\sigma^{2t-1}} - \gamma v^{\sigma^{t-1}}\rangle_{\mathbb{F}_{q^{2t}}}$ and $S = \langle \beta v^{\sigma^{2t-1}} - \gamma v^{\sigma^{t+1}}\rangle_{\mathbb{F}_{q^{2t}}}$. Again, it is straightforward to see that $X = \langle v^{\sigma^{t-1}} - \alpha v^{\sigma}\rangle_{\mathbb{F}_{q^{2t}}}$ and, hence by *2*, $\alpha = 1$. By Point *3(b)*, saying $D = \langle v^\sigma + v^{\sigma^{2t-1}}\rangle_{\mathbb{F}_{q^{2t}}}$, we get $\gamma = h^{1-q^{s(2t-1)}}$ for some $h \in \mathbb{F}_{q^{2t}}$ with $\mathrm{N}_{q^{2t}/q^t}(h) = -1$. Finally, since $Y = \langle \beta v^\sigma - v^{\sigma^{t+1}}\rangle_{\mathbb{F}_{q^{2t}}}$, *3(c)* implies that $\beta\gamma^{q^{2s}} = h^{1+q^{2s}}$, so $\beta = \frac{h^{1+q^{2s}}}{h^{q^{2s}-q^s}} = h^{1+q^s} = -h^{1-q^{s(t+1)}}$. Therefore, $L \cong L^{5,t}_{s,h}$. By Point $3(a-b)$ of Theorem above, if $h \in \mathbb{F}_{q^t}$, then the cross-ratio $(P^\sigma, P^{\sigma^{2t-1}}, D, Q)$ and $(P^\sigma, P^{\sigma^{t+1}}, R^{\sigma^2}, Y)$ are equal to -1. Thus, setting $L^{5,t}_s := L^{5,t}_{h,s}$ with $h \in \mathbb{F}_{q^t}$, we can state the following.

Corollary 2.4.8 *Let L be an $(1, 2(t-1))$-evasive linear set of rank $2t$ in a line Λ of $\mathrm{PG}(2t-1, q^{2t})$. Let Σ be a canonical subgeometry of $\mathrm{PG}(2t-1, q^{2t})$, q odd. Then L is equivalent to a scattered linear set of the type $L^{5,t}_s$ if and only if for each $(2t-3)$ dimensional subspace Γ of $\mathrm{PG}(2t-1, q^{2t})$ such that $L = \mathrm{p}_{\Gamma,\Lambda}(\Sigma)$, the following holds:*

1. *there exists a generator σ of the subgroup of $\mathrm{P\Gamma L}(2t, q^{2t})$ fixing Σ pointwise and*

a point $P=\langle v\rangle_{\mathbb{F}_{q^{2t}}}\in \mathrm{PG}(2t-1,q^{2t})$ *such that*

$$\Gamma=\langle P^{\sigma^i},Q,R,S : i\notin\{0,1,t-1,t+1,2t-1\}\rangle$$

with $Q\in\langle P^{\sigma},P^{\sigma^{2t-1}}\rangle$, $R\in\langle P^{\sigma^{t-1}},P^{\sigma^{2t-1}}\rangle$ *and* $S\in\langle P^{\sigma^{t+1}},P^{\sigma^{2t-1}}\rangle$,

2. *the point* $X=\langle v^{\sigma}-v^{\sigma^{t-1}}\rangle_{\mathbb{F}_{q^{2t}}}$ *belongs to* Γ.

3. (a) *the pair* $\{P^{\sigma},P^{\sigma^{2t-1}}\}$ *separates* $\{D,Q\}$ *harmonically, where* $D=\langle v^{\sigma}+v^{\sigma^{2t-1}}\rangle_{\mathbb{F}_{q^{2t}}}$.

   (b) *the pair* $\{P^{\sigma},P^{\sigma^{t+1}}\}$ *separates* $\{R^{\sigma^2},Y\}$ *harmonically, where* $Y=\langle Q,S\rangle\cap\langle P^{\sigma},P^{\sigma^{t+1}}\rangle$.

4. *If* t *is odd, then* $q\equiv 1\pmod 4$.

The linear sets of the type $L^{5,t}_{m,s}$, for $m\in\mathbb{F}_{q^t}\setminus\{0,1\}$

In this subsection we will deal with the family of scattered linear sets of $\mathrm{PG}(1,q^{2t})$, $t\geq 3$ and q odd, recently studied in Smaldore et al. (2024) (cf. (2.5)). The elements in this class have the following fashion:

$$L^{5,t}_{m,s}=\{\langle(x,m(x^{q^s}-x^{q^{s(t+1)}})+x^{q^{s(t-1)}}+x^{q^{s(2t-1)}}\rangle_{\mathbb{F}_{q^{2t}}}:x\in\mathbb{F}^*_{q^{2t}}\},$$

where $\gcd(s,2t)=1$. Here, $m\in\mathbb{F}^*_{q^t}$ is neither a $(q-1)$-th power nor a $(q+1)$-th power of any element of the vector space $\ker\mathrm{Tr}_{q^{2t}/q^t}$ and if t is odd, $q\equiv 1\pmod 4$, see (Smaldore et al. 2024, Theorem 2.3). Firstly, by (Smaldore et al. 2024, Theorem 4.5) and Theorem 2.3.9, for this class of scattered linear sets with parameter $m\in\mathbb{F}^*_{q^t}$ as above, we can state the following.

Theorem 2.4.9 *For* $t>4$, *the* $\Gamma\mathrm{L}$*-class of the scattered linear set* $L^{5,t}_{m,s}$ *is two.*

Similarly to what has been before, we are now ready to characterize the projection vertex of a linear set equivalent to an element of the type $L^{5,t}_{m,s}$.

Theorem 2.4.10 *Let* L *be an* $(1,2(t-1))$*-evasive linear set of rank* $2t$ *in a line* Λ *of* $\mathrm{PG}(2t-1,q^{2t})$. *Let* Σ *be a canonical subgeometry of* $\mathrm{PG}(2t-1,q^{2t})$. *Then* L *is equivalent to a maximum scattered linear set of the type* $L^{5,t}_{m,s}$ *if and only if for each*

$(2t-3)$ dimensional subspace Γ of $\mathrm{PG}(2t-1,q^{2t})$ such that $L=\mathrm{p}_{\Gamma,\Lambda}(\Sigma)$, the following holds:

1. *there exist a generator σ of the subgroup of $\mathrm{P\Gamma L}(2t,q^{2t})$ fixing Σ pointwise and a point $P=\langle v\rangle_{\mathbb{F}_{q^{2t}}}\in\mathrm{PG}(2t-1,q^{2t})$ such that*

$$\Gamma=\langle P^{\sigma^i},Q,R,S : i\notin\{0,1,t-1,t+1,2t-1\}\rangle$$

   with $Q\in\langle P^{\sigma},P^{\sigma^{2t-1}}\rangle$, $R\in\langle P^{\sigma^{t-1}},P^{\sigma^{2t-1}}\rangle$ and $S\in\langle P^{\sigma^{t+1}},P^{\sigma^{2t-1}}\rangle$,

2. *the point $C=\langle v^{\sigma}+v^{\sigma^{t+1}}\rangle_{\mathbb{F}_{q^{2t}}}\in\Gamma$,*

3. *there exists $\mu\in\mathcal{W}=\{z\in\mathbb{F}_{q^t}:\nexists w\in\ker\mathrm{Tr}_{q^{2t}/q^t}\text{ s.t. } z=w^{q-1}\text{ or } z=w^{q+1}\}$ and*

   (a) *the points $P^{\sigma},P^{\sigma^{2t-1}}$, X, Q, where $X=\langle v^{\sigma}-v^{\sigma^{2t-1}}\rangle_{\mathbb{F}_{q^{2t}}}$, are collinear and the cross ratio $(P^{\sigma};P^{\sigma^{2t-1}};X;Q)=\mu$,*

   (b) *the points $P^{\sigma^{t-1}},P^{\sigma^{2t-1}}$, Y, R, where $Y=\langle v^{\sigma^{t-1}}+v^{\sigma^{2t-1}}\rangle_{\mathbb{F}_{q^{2t}}}$, are collinear and the pair $\{P^{\sigma^{t-1}},P^{\sigma^{2t-1}}\}$ separates the pair $\{Y,R\}$ harmonically.*

Proof. Let us suppose that L is equivalent to $L^{5,t}_{m,s}$. By Theorem 2.3.11, there exists β such that $\mathbf{\Sigma}^{\beta}=\Sigma$, saying $\psi_{t,m,s}:=\psi_{t,m,h,s}$ with $h\in\mathbb{F}_{q^t}$, and either $\Gamma^{\beta}_{\psi_{t,m,s}}=\Gamma$ or $\Gamma^{\beta}_{\hat{\psi}_{t,m,s}}=\Gamma$.

If $\beta\in\mathrm{P\Gamma L}(2t,q^{2t})$ such that $\Gamma^{\beta}_{\psi_{t,m,s}}=\Gamma$, then let $\boldsymbol{\rho}=\boldsymbol{\sigma}^s$ with $\gcd(s,2t)=1$, $\boldsymbol{P}=\langle\mathbf{v}\rangle_{\mathbb{F}_{q^{2t}}}$ be an imaginary point of $\mathrm{PG}(2t-1,q^{2t})$ (wrt $\mathbf{\Sigma}$) and

$$\Gamma_{\psi_{t,m,s}}=\langle\boldsymbol{P}^{\boldsymbol{\rho}^i},\boldsymbol{Q},\boldsymbol{R},\boldsymbol{S} : i\notin\{1,t-1,t+1,2t-1\}\rangle$$

where

$$\boldsymbol{Q}=\langle m\mathbf{v}^{\boldsymbol{\rho}^{2t-1}}-\mathbf{v}^{\boldsymbol{\rho}}\rangle_{\mathbb{F}_{q^{2t}}},\quad \boldsymbol{R}=\langle\mathbf{v}^{\boldsymbol{\rho}^{2t-1}}-\mathbf{v}^{\boldsymbol{\rho}^{t-1}}\rangle_{\mathbb{F}_{q^{2t}}},\ \text{and}$$

$$\boldsymbol{S}=\langle m\mathbf{v}^{\boldsymbol{\rho}^{2t-1}}+\mathbf{v}^{\boldsymbol{\rho}^{t+1}}\rangle_{\mathbb{F}_{q^{2t}}}.$$

Let us consider the point $\boldsymbol{C}=\langle\mathbf{v}^{\boldsymbol{\rho}}+\mathbf{v}^{\boldsymbol{\rho}^{t+1}}\rangle_{\mathbb{F}_{q^{2t}}}$; it belongs to $\Gamma_{\psi_{t,m,s}}$. Moreover, let $\boldsymbol{X}=\langle\mathbf{v}^{\boldsymbol{\rho}}-\mathbf{v}^{\boldsymbol{\rho}^{2t-1}}\rangle_{\mathbb{F}_{q^{2t}}}$. Clearly, $\boldsymbol{X},\boldsymbol{Q}\in\langle\boldsymbol{P}^{\boldsymbol{\rho}};\boldsymbol{P}^{\boldsymbol{\rho}^{2t-1}}\rangle$ and $(\boldsymbol{P}^{\boldsymbol{\rho}};\boldsymbol{P}^{\boldsymbol{\rho}^{2t-1}};\boldsymbol{X};\boldsymbol{Q})=1/m$. Finally, $(\boldsymbol{P}^{\boldsymbol{\rho}^{t-1}};\boldsymbol{P}^{\boldsymbol{\rho}^{2t-1}};\boldsymbol{Y};\boldsymbol{R})=-1$ where $\boldsymbol{Y}=\langle\mathbf{v}^{\boldsymbol{\rho}^{t-1}}+\mathbf{v}^{\boldsymbol{\rho}^{2t-1}}\rangle_{\mathbb{F}_{q^{2t}}}$. By setting $\sigma=\beta^{-1}\boldsymbol{\rho}\beta$, $P=\boldsymbol{P}^{\beta}$, $Q=\boldsymbol{Q}^{\beta}$, $R=\boldsymbol{R}^{\beta}$, $S=\boldsymbol{S}^{\beta}$, $C=\boldsymbol{C}^{\beta}$ $X=\boldsymbol{X}^{\beta}$ and $Y=\boldsymbol{Y}^{\beta}$

and noting that β preserves the collinearity, being harmonically separated and if $(A_1;A_2;A_3;A_4)\in\mathcal{W}$ then $(A_1^\beta;A_2^\beta;A_3^\beta;A_4^\beta)\in\mathcal{W}$, the Point *(1–3)* follow.
If $\Gamma^\beta_{\hat\psi_{t,m,s}}=\Gamma$, by Remark 2.3.10,

$$\Gamma_{\hat\psi_{t,m,s}}=\langle \boldsymbol{P}^{\boldsymbol{\rho}^i},\boldsymbol{Q},\boldsymbol{R},\boldsymbol{S}: i\notin\{0,1,t-1,t+1,2t-1\}\rangle$$

where

$$\boldsymbol{Q}=\langle m^{\boldsymbol{\rho}^{t-1}}\mathbf{v}^{\boldsymbol{\rho}}+\mathbf{v}^{\boldsymbol{\rho}^{t-1}}\rangle_{\mathbb{F}_{q^{2t}}},\quad \boldsymbol{R}=\langle \mathbf{v}^{\boldsymbol{\rho}}-\mathbf{v}^{\boldsymbol{\rho}^{t+1}}\rangle_{\mathbb{F}_{q^{2t}}},\text{ and}$$

$$\boldsymbol{S}=\langle m^{\boldsymbol{\rho}^{2t-1}}\mathbf{v}^{\boldsymbol{\rho}}-\mathbf{v}^{\boldsymbol{\rho}^{2t-1}}\rangle_{\mathbb{F}_{q^{2t}}}$$

Let us consider the point $\boldsymbol{C}=\langle \mathbf{v}^{\boldsymbol{\rho}^{2t-1}}+\mathbf{v}^{\boldsymbol{\rho}^{t-1}}\rangle_{\mathbb{F}_{q^{2t}}}$. Since $m\in\mathbb{F}_{q^t}$, $\boldsymbol{C}\in\Gamma_{\hat\psi_{t,m,s}}$. Moreover, let consider the point $\boldsymbol{X}=\langle \mathbf{v}^{\boldsymbol{\rho}}-\mathbf{v}^{\boldsymbol{\rho}^{2t-1}}\rangle_{\mathbb{F}_{q^{2t}}}$. Similarly to the previous case, $\boldsymbol{X},\boldsymbol{S}\in\langle \boldsymbol{P}^{\boldsymbol{\rho}};\boldsymbol{P}^{\boldsymbol{\rho}^{2t-1}}\rangle$ and $(\boldsymbol{P}^{\boldsymbol{\rho}};\boldsymbol{P}^{\boldsymbol{\rho}^{2t-1}};\boldsymbol{X};\boldsymbol{S})=m^{\boldsymbol{\rho}^{2t-1}}$. If $\boldsymbol{Y}=\langle \mathbf{v}^{\boldsymbol{\rho}^{t+1}}+\mathbf{v}^{\boldsymbol{\rho}}\rangle_{\mathbb{F}_{q^{2t}}}$, the cross-ratio $(\boldsymbol{P}^{\boldsymbol{\rho}^{t-1}};\boldsymbol{P}^{\boldsymbol{\rho}^{2t-1}};\boldsymbol{Y};\boldsymbol{R})=-1$. As we have done before, the result follows by setting $\sigma=\beta^{-1}\boldsymbol{\rho}^{2t-1}\beta$, $P=\boldsymbol{P}^\beta$, $Q=\boldsymbol{S}^\beta$, $R=\boldsymbol{R}^\beta$, $S=\boldsymbol{Q}^\beta$, $C=\boldsymbol{C}^\beta$, $X=\boldsymbol{X}^\beta$ and $Y=\boldsymbol{Y}^\beta$. Now, let us suppose that Point *(1–3)* hold true. Then, by Theorem 2.3.13, $L\cong L_f$ with $f(x)=\alpha x^{q^s}+\beta x^{q^{s(t-1)}}+\gamma x^{q^{s(t+1)}}+x^{q^{s(2t-1)}}\in\mathbb{F}_{q^{2t}}[x]$ for some $\alpha,\beta,\gamma\in\mathbb{F}_{q^{2t}}$ and some integer $1\le s\le 2t-1$ such that $\gcd(s,2t)=1$. Furthermore there exist $P=\langle v\rangle_{\mathbb{F}_{q^{2t}}}$, an imaginary point with respect to the subgeometry Σ fixed pointwise by a collineation $\sigma\in\mathrm{P\Gamma L}(2t,q^{2t})$. Thus, let consider the points $Q=\langle \alpha v^{\sigma^{2t-1}}-v^\sigma\rangle_{\mathbb{F}_{q^{2t}}}$, $R=\langle \beta v^{\sigma^{2t-1}}-v^{\sigma^{t-1}}\rangle_{\mathbb{F}_{q^{2t}}}$ and $S=\langle \gamma v^{\sigma^{2t-1}}-v^{\sigma^{t+1}}\rangle_{\mathbb{F}_{q^{2t}}}$. Since $C=\langle v^\sigma+v^{\sigma^{t+1}}\rangle_{\mathbb{F}_{q^{2t}}}\in\Gamma$, then $\alpha=-\gamma$. By Point *3(a)*, the cross-ratio $(P^\sigma;P^{\sigma^{2t-1}};X;Q)=1/\alpha\in\mathcal{W}$. By Point *3(b)*, we get $\beta=1$. Since, if $1/\alpha\in\mathcal{W}$, then $\alpha\in\mathcal{W}$, we get $f(x)=\psi_{t,\alpha,s}(x)$ and hence $L\cong L^{5,t}_{\alpha,s}$.

2.5 An equivalence issue

Motivated by an open problem proposed in the last section of Neri et al. (2022), in this section, we will focus on linear MRD codes with minimum distance $d=n-1$ that are $\mathbb{F}_{q^n}$-subspaces of $\tilde{\mathcal{L}}_{n,q}[x]$. Precisely, in (Neri et al. 2022, Theorem 3.8), the authors introduce a new class of such MRD codes extending two constructions exhibited in Longobardi & Zanella (2021) and Longobardi et al. (2023). This family is denoted by the symbol $C^{5,t}_{1,h,s}$, given by

$$C^{5,t}_{1,h,s} = \langle x, \psi_{h,t,s}(x)\rangle_{\mathbb{F}_{q^{2t}}}, \tag{2.23}$$

where $\psi_{h,t,s}(x) = x^{q^s} + x^{q^{s(t-1)}} + h^{1+q^s} x^{q^{s(t+1)}} + h^{1-q^{s(2t-1)}} x^{q^{s(2t-1)}} \in \tilde{\mathcal{L}}_{2t,q}[x]$, $t \in \mathbb{N}$, $t \geq 3$, $s \in \mathbb{Z}$ with $(2t, s) = 1$ and $h \in \mathbb{F}_{q^{2t}}$ such that $\mathrm{N}_{q^{2t}/q^t}(h) = h^{q^t+1} = -1$. Also, in Neri et al. (2022), it was proven that if $t > 4$, then $C^{5,t}_{1,h,s}$ is not equivalent to any other example known in the literature so far. Moreover, the following result was proven concerning the equivalence of two elements in $C^{5,t}_{1,h,s}$.

Theorem 2.5.1 *(Neri et al. 2022, Theorem 4.6) Let $t \geq 5$ and consider $C^{5,t}_{1,h,s}$ and $C^{5,t}_{1,k,\ell}$ such that $(s, n) = 1 = (\ell, n)$. Then the codes $C^{5,t}_{1,h,s}$ and $C^{5,t}_{1,k,\ell}$ are equivalent if and only if one of the following collection of conditions are satisfied:*

I. $s \equiv \ell \pmod n$, *and there exists* $\rho \in \mathrm{Aut}(\mathbb{F}_{q^n})$ *such that*

$$\rho(h) = \begin{cases} \pm k, & \text{if } t \not\equiv 2 \pmod 4 \\ \lambda k, \ \text{where } \lambda^{q^2+1} = 1 & \text{if } t \equiv 2 \pmod 4 \end{cases}$$

II. $s \equiv -\ell \pmod n$, *and there exists* $\rho \in \mathrm{Aut}(\mathbb{F}_{q^n})$ *such that*

$$\rho(h) = \begin{cases} \pm k^{-1}, & \text{if } t \not\equiv 2 \pmod{)}4 \\ \lambda k^{-1}, \ \text{where } \lambda^{q^2+1} = 1 & \text{if } t \equiv 2 \pmod{)}4 \end{cases}$$

III. if $s \equiv (t-1)\ell \pmod n$, t *even, and there exists* $\rho \in \mathrm{Aut}(\mathbb{F}_{q^n})$ *such that*

$$\rho(h) = \begin{cases} \pm k, & \text{if } t \not\equiv 2 \pmod 4 \\ \lambda k, \ \text{where } \lambda^{q^2+1} = 1 & \text{if } t \equiv 2 \pmod 4 \end{cases}$$

IV. $s \equiv (t+1)\ell \pmod n$, t *even, and there exists* $\rho \in \mathrm{Aut}(\mathbb{F}_{q^n})$ *such that*

$$\rho(h) = \begin{cases} \pm k^{-1}, & \text{if } t \not\equiv 2 \pmod 4 \\ \lambda k^{-1}, \ \text{where } \lambda^{q^2+1} = 1 & \text{if } t \equiv 2 \pmod 4 \end{cases}.$$

The necessity in the statement of this theorem still holds if $t = 3, 4$. However, as underlined in the initial part of (Neri et al. 2022, Section 4), in such cases its sufficiency part, and more generally the equivalence issue for the codes $C^{5,t}_{1,h,s}$ requires more demanding computations and is left as open problems (see (Neri et al. 2022, Section 5)). In this remaining chapter, we will face with these problems, extending Theorem 2.5.1. Our main achievements, Theorems 2.5.5 and 2.5.6, are obtained by heavily exploiting some concepts and results stated in Longobardi & Zanella (2024). For this reason, we dedicated the next part, to recall basic definitions and salient aspects of these tools. As a direct consequence of the results stated in Section 3, the same arguments as those used in Longobardi et al. (2023) and in Neri et al. (2022), lead to the determination of the number of inequivalent codes of type $C^{5,t}_{1,h,s}$ for $t = 3, 4$. Finally, we give direct proof of the fact that the maximum scattered linear set of $\mathrm{PG}(1, q^8)$ associated with $C^{5,4}_{1,h,s}$ is not equivalent to any family known so far.

To extend Theorem 2.5.1 stated above by removing the hypothesis $t \geq 5$, as a first step we recap and adapt some concepts and tools which were introduced in (Longobardi & Zanella 2024, Section 4), and that will be crucial to perform subsequent computations.

Following Longobardi & Zanella (2024), we say that a q-polynomial $F(x) = \sum_{i=0}^{n-1} c_i x^{q^i}$ is in *standard form*, if the greatest common divisor m_F of the set of integers

$$\{(i - j) \pmod{n} : c_i c_j \neq 0 \text{ with } i \neq j\} \cup \{n\},$$

is strictly larger than 1. If this is the case, then $F(x)$ has the following fashion:

$$F(x) = \sum_{j=0}^{n/m-1} c_j x^{q^{s+jm}}, \tag{2.24}$$

where $m = m_F$ and $0 \leq s < m_F$. We stress here that when $F(x)$ is a scattered polynomial, then s must be coprime with m_F, otherwise $F(x)$ would be $\mathbb{F}_{q^r}$-linear for $r = (s, m_F) > 1$, contradicting its scatteredness.

Let $\mathcal{C} \leq \tilde{\mathcal{L}}_{n,q}[x]$ be a linear rank-metric code. Then, its *left idealizer* and *right idealizer* are defined as the sets

$$I_L(\mathcal{C}) = \{\varphi(x) \in \tilde{\mathcal{L}}_{n,q}[x] : \varphi \circ f \in \mathcal{C} \ \forall f \in \mathcal{C}\},$$

and

$$I_R(\mathcal{C}) = \{\varphi(x) \in \tilde{\mathcal{L}}_{n,q}[x] : f \circ \varphi \in \mathcal{C}\ \forall f \in \mathcal{C}\},$$

respectively. By (Lunardon et al. 2017, Corollary 5.6), if $\mathcal{C}$ is a linear MRD code, $I_L(\mathcal{C})$ and $I_R(\mathcal{C})$ are subalgebras of $\tilde{\mathcal{L}}_{n,q}[x]$; more precisely, they are isomorphic to subfields of $\mathbb{F}_{q^n}$. In particular, if $\mathcal{C}$ has left idealizer containing a subfield isomorphic to $\mathbb{F}_{q^n}$, then it is called $\mathbb{F}_{q^n}$-*linear*. Note that any MRD code $\mathcal{C}_f$ associated with a scattered polynomial $f(x)$ is $\mathbb{F}_{q^n}$-linear. The idealizers are useful tools for studying the equivalence issue in this context; indeed MRD codes with non-isomorphic left (resp. right) idealizers cannot be equivalent, see (Lunardon et al. 2017, Section 4).
Now, let $f(x)$ be a scattered q-polynomial and consider G_f the stabilizer in $\mathrm{GL}(2, q^n)$ of the scattered subspace U_f. By (Longobardi et al. 2023, Lemma 4.1), (see also (Longobardi & Zanella 2024, Proposition 3.1)) $G_f^\circ = G_f \cup \{O\}$, where O is the null matrix of order 2, is a subfield of matrices isomorphic to $I_R(\mathcal{C}_f)$ as field. Then, one gets the following.

Theorem 2.5.2 *Let $\mathcal{C}_F = \langle x, F(x)\rangle_{\mathbb{F}_{q^n}} \subset \tilde{\mathcal{L}}_{n,q}[x]$ be a linear 2-dimensional MRD code with minimum distance $n-1$. The following statements are equivalent:*

(i) $|G_F^\circ| = q^m$, $m > 1$, and all elements of G_F° are diagonal.

(ii) the q-polynomial $F(x)$ is in standard form.

(iii) the right idealizer

$$I_R(\mathcal{C}_F) = \{\alpha x \,:\, \alpha \in \mathbb{F}_{q^m}\}.$$

Moreover, if the above conditions (i), (ii) and (iii) hold, then $m = m_F$ and

$$G_F^\circ = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{q^s} \end{pmatrix} : \alpha \in \mathbb{F}_{q^m} \right\}, \tag{2.25}$$

with $(s, m) = 1$.

Proof. By (Longobardi & Zanella 2024, Theorem 4.3.), (i) is equivalent to (ii).

$(ii) \Rightarrow (iii)$. Since $F(x)$ is as in Formula (2.24), the set

$$\{\alpha x : \alpha \in \mathbb{F}_{q^m}\}$$

is contained in $I_R(\mathcal{C}_F)$. Taking into account $|G_F^\circ| = |I_R(\mathcal{C}_F)| = q^m$, (iii) follows.

$(iii) \Rightarrow (i)$. Let $\omega \in \mathbb{F}_{q^m}$ such that $\mathbb{F}_q(\omega) = \mathbb{F}_{q^m}$. Since $\omega x \in I_R(\mathcal{C}_F)$, then there exists $d \in \mathbb{F}_{q^n}$ such that

$$F(\omega x) = dF(x). \tag{2.26}$$

This implies that if $F(x) = \sum_{i=1}^{n-1} c_i x^{q^i}$, then

$$c_i \omega^{q^i} = dc_i$$

for any $i = 1, \ldots, n-1$. So, if $c_i \neq 0$, $d \in \mathbb{F}_{q^m}$ and $d = \omega^h$ for some $h \in \mathbb{N}$. By (2.26), the matrix

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega^h \end{pmatrix}$$

belongs to G_F, has order $q^m - 1$ and, since $|G_F| = |I_R(\mathcal{C}_F) \setminus \{0\}| = q^m - 1$, it generates G_F. This implies (i).

As a direct consequence of (Longobardi & Zanella 2024, Corollary 4.7), one has that if the MRD code $\mathcal{C}_f = \langle x, f(x)\rangle_{\mathbb{F}_{q^n}}$ has right idealizer $I_R(\mathcal{C}_f)$ not isomorphic to $\mathbb{F}_q$ then, $\mathcal{C}_f$ is equivalent to an MRD code $\mathcal{C}_F$, where F is a q-polynomial shaped as in (2.24). In addition, as a consequence of what is stated in (Longobardi & Zanella 2024, Section 4), we can prove the following result concerning the equivalence issue for codes in the form $\mathcal{C}_F = \langle x, F(x)\rangle_{\mathbb{F}_{q^n}}$ where $F(x)$ is a q-polynomial in standard form. Precisely,

Theorem 2.5.3 *Let $\mathcal{C}_{F_i}$, $i = 1, 2$ be two 2-dimensional MRD codes where F_i, $i = 1, 2$, are scattered polynomials having the form described in (2.24). Then, they are equivalent if and only if there exist $a, b, c, d \in \mathbb{F}_{q^n}^*$ such that*

$$dF_2(x) = F_1^\rho(ax) \quad \text{or} \quad F_1^\rho(bF_2(x)) = cx \tag{2.27}$$

for some $\rho \in \mathrm{Aut}(\mathbb{F}_{q^n})$. In particular,

(i) $\mathcal{C}_{F_2} = \mathcal{C}_{F_1}^\rho \circ ax$ *or,*

(ii) $\mathcal{C}_{F_2} = \mathcal{C}_{F_1}^\rho \circ bF_2(x)$.

Proof. By (Sheekey 2016, Theorem 8), $\mathcal{C}_{F_1}$ and $\mathcal{C}_{F_2}$ are equivalent codes if and only if

$F_1(x)$ and $F_2(x)$ are ΓL-equivalent, or equivalently, $F_1^\rho(x)$ and $F_2(x)$ are GL-equivalent for some $\rho \in \mathrm{Aut}(\mathbb{F}_{q^n})$. Then, there exists a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, q^n)$ such that for any $y \in \mathbb{F}_{q^n}$, there is $z \in \mathbb{F}_{q^n}$ such that

$$A \begin{pmatrix} y \\ F_2(y) \end{pmatrix} = \begin{pmatrix} z \\ F_1^\rho(z) \end{pmatrix}. \tag{2.28}$$

Let $G_{F_1^\rho}$ and G_{F_2} be the stabilizer groups in $\mathrm{GL}(2, q^n)$ of the subspaces $U_{F_1^\rho}$ and U_{F_2}, respectively. It is straightforward to see that

$$A^{-1} G_{F_1^\rho} A = G_{F_2}. \tag{2.29}$$

By Theorem 2.5.2, $G_{F_1^\rho}$ and G_{F_2} are groups made up of diagonal matrices. Therefore, by Formula (2.29) one gets that $a = d = 0$ or $b = c = 0$.
If $a = d = 0$, $b, c \in \mathbb{F}_{q^n}^*$ and $cx = F_1^\rho(bF_2(x))$. So, one gets the second formula in (2.27).
If $b = c = 0$, $a, d \in \mathbb{F}_{q^n}^*$ and, by (2.28), $dF_2(x) = F_1^\rho(ax)$.
Now, if $dF_2(x) = F_1^\rho(ax)$,

$$\begin{aligned} \mathcal{C}_{F_2} &= \langle x, F_2(x)\rangle_{\mathbb{F}_{q^n}} = \langle x, dF_2(x)\rangle_{\mathbb{F}_{q^n}} = \langle x, F_1^\rho(ax)\rangle_{\mathbb{F}_{q^n}} \\ &= \langle x, F_1^\rho(x)\rangle_{\mathbb{F}_{q^n}} \circ ax = \mathcal{C}_{F_1}^\rho \circ ax. \end{aligned} \tag{2.30}$$

If $F_1^\rho(bF_2(x)) = cx$, then

$$\begin{aligned} \mathcal{C}_{F_2} &= \langle x, F_2(x)\rangle_{\mathbb{F}_{q^n}} = \langle cx, F_2(x)\rangle_{\mathbb{F}_{q^n}} = \langle F_1^\rho(bF_2(x)), F_2(x)\rangle_{\mathbb{F}_{q^n}} \\ &= \langle x, F_1^\rho(x)\rangle_{\mathbb{F}_{q^n}} \circ bF_2(x) = \mathcal{C}_{F_1}^\rho \circ bF_2(x). \end{aligned} \tag{2.31}$$

Also, as consequence of Longobardi & Zanella (2024) and the Theorem above, we have the following

Corollary 2.5.4 *Let $\mathcal{C}_{f_i}$ be MRD codes such that $I_R(\mathcal{C}_{f_i})$, are not isomorphic to $\mathbb{F}_q$, $i = 1, 2$. Then, $\mathcal{C}_{f_1}$ and $\mathcal{C}_{f_2}$ are equivalent if and only if there exist two polynomials F_i in standard forms, such that $\mathcal{C}_{f_i}$ is equivalent to $\mathcal{C}_{F_i}$, $i = 1, 2$, and (2.27) holds true.*

By (Longobardi & Zanella 2024, Proposition 3.4, Remark 3.5 and Remark 4.4), one can see that for each $t \geq 3$ the right idealizer of $C^{5,t}_{1,h,s}$ is isomorphic to $\mathbb{F}_{q^2}$. In addition, any code

$$C^{5,t}_{1,h,s} = \langle x, \psi_{h,t,s}(x) \rangle_{\mathbb{F}_{q^{2t}}}$$

with $t \geq 4$ and even, is already expressed through a polynomial in standard form. On the other hand, by (Longobardi & Zanella 2024, Example 4.11), one gets that the code $\mathcal{C}_H$ where

$$H(x) := H_{h,s}(x) = (1-h^{1+q^{2s}})x^{q^s} + (h+h^2)x^{q^{3s}} + h^{1+q^{2s}}(h+h^{q^s})x^{q^{5s}} \in \tilde{\mathcal{L}}_{6,q}[x], \quad (2.32)$$

is MRD, equivalent to $C^{5,3}_{1,h,s}$ and $H(x)$ is in standard form.

2.5.1 On the equivalence issue for $C^{5,t}_{1,h,s}$, with $t = 3, 4$

In the following section, we assume $n = 2t$ and restrict ourselves to the case $t = 3, 4$ and start by studying the equivalence between two different elements in $C^{5,t}_{1,h,s}$. Firstly, we prove the following main result

Theorem 2.5.5 *The MRD codes $C^{5,3}_{1,h,s}$ and $C^{5,3}_{1,k,\ell}$, are equivalent if and only if*

1. $\ell \equiv s \pmod 6$ *and* $h^\rho = \pm k$,
2. $\ell \equiv -s \pmod 6$ *and* $h^\rho = \pm k^{-1}$,

where $\rho \in \mathrm{Aut}(\mathbb{F}_{q^6})$.

Proof. It is not difficult to show that the necessity condition in the statement of this theorem holds. Regarding the sufficiency, we first note that since $(s, 6) = (\ell, 6) = 1$, there exists an integer $-1 \leq r \leq 4$ such that $\ell \equiv sr \pmod 6$. Moreover, it is straightforward to see that r must be coprime with 6. Then $r \notin \{2, 3, 4\}$ and hence if $r \neq \pm 1$, then $C^{5,t}_{1,h,s}$ and $C^{5,t}_{1,k,\ell}$ can not be equivalent. Assume that $C^{5,3}_{1,h,s} = \langle x, \psi_{h,3,s}(x) \rangle_{\mathbb{F}_{q^6}}$ and $C^{5,3}_{1,k,\ell} = \langle x, \psi_{k,3,\ell}(x) \rangle_{\mathbb{F}_{q^6}}$, are equivalent, this may happen only if $r \in \{-1, 1\}$, i.e. $\ell \equiv \pm s \pmod 6$. In addition, by Corollary 2.5.4, $C^{5,3}_{1,h,s}$ and $C^{5,3}_{1,k,\ell}$ are equivalent if and only if Formula (2.27) holds true for any pair of standard forms of the q-polynomials $\psi_{h,3,s}$ and $\psi_{k,3,\ell}$, respectively. As we indicated in the last part of the previous section, in Longobardi & Zanella (2024), it was shown that $\psi_{h,3,s}(x)$ is

ΓL-equivalent to the following polynomial in standard form:

$$H_{h,s}(x) = (1 - h^{1+q^{2s}})x^{q^s} + (h + h^2)x^{q^{3s}} + h^{1+q^{2s}}(h + h^{q^s})x^{q^{5s}}.$$

Following Corollary 2.5.4, to investigate this equivalence issue we may consider the following two equations

$$dH_{k,\ell}(x) = H^{\rho}_{h,s}(ax), \quad \text{and} \quad H^{\rho}_{h,s}(bH_{k,\ell}(x)) = cx \tag{2.33}$$

for $a, b, c, d \in \mathbb{F}^*_{q^6}$ and $\rho \in \mathrm{Aut}(\mathbb{F}_{q^6})$. Since the automorphism ρ acts on h, without loss of generality, we may suppose that it is the identity, see (Neri et al. 2022, Remark 4.5).

Case 1. Suppose $s \equiv \ell \pmod 6$. If $dH_{k,\ell}(x) = H_{h,s}(ax)$, by expanding this equation we get the following conditions

$$\begin{cases} d(1 - k^{1+q^{2s}}) = a^{q^s}(1 - h^{1+q^{2s}}) \\ d(k + k^2) = a^{q^{3s}}(h + h^2) \\ dk^{1+q^{2s}}(k + k^{q^s}) = a^{q^{5s}}h^{1+q^{2s}}(h + h^{q^s}). \end{cases} \tag{2.34}$$

We focus on getting a relation between h and k and eventually the values of a and d. To do this we start by considering the ratio between the second and first equation. Doing so we get

$$a^{q^{3s}-q^s} = \frac{k(k+1)(1 - h^{1+q^{2s}})}{h(h+1)(1 - k^{1+q^{2s}})}. \tag{2.35}$$

Similarly computing the ratio between the third and second equation and between the first and third in (2.34) we get

$$\begin{aligned} a^{q^{5s}-q^{3s}} &= \frac{k^{q^{2s}}(h+1)(k + k^{q^s})}{h^{q^{2s}}(k+1)(h + h^{q^s})} \\ a^{q^s-q^{5s}} &= \frac{(1 - k^{1+q^{2s}})(h + h^{q^s})h^{1+q^{2s}}}{(1 - h^{1+q^{2s}})(k + k^{q^s})k^{1+q^{2s}}}. \end{aligned} \tag{2.36}$$

By rising Equation (2.35) and the two Equations in (2.36) to the q^{2s}-power, and taking into account that $h^{q^{3s}+1} = k^{q^{3s}+1} = -1$, we get the following system of conditions on

h and k:

$$\begin{cases} \left(\frac{k}{h}\right)^{q^{5s}} = \left[\frac{(k^{q^{4s}}+k^{q^{5s}})(h+1)}{(h^{q^{4s}}+h^{q^{5s}})(k+1)}\right]^{q^{2s}+1} \\ \left(\frac{k}{h}\right)^{q^{2s}+q^{4s}} = \frac{(h+h^{q^s})(k^{q^{2s}}+1)}{(k+k^{q^s})(h^{q^{2s}}+1)} \\ \left(\frac{k}{h}\right)^{q^{4s}} = \frac{(h+1)(k^{q^{5s}}+k^{q^{4s}})}{(k+1)(h^{q^{5s}}+h^{q^{4s}})}. \end{cases} \tag{2.37}$$

Putting $\xi = \frac{(k^{q^{4s}}+k^{q^{5s}})(h+1)}{(h^{q^{4s}}+h^{q^{5s}})(k+1)}$, then the system above reads

$$\begin{cases} \left(\frac{k}{h}\right)^{q^{5s}} = \xi^{q^{2s}+1} \\ \left(\frac{k}{h}\right)^{q^{2s}+q^{4s}} = \frac{1}{\xi^{q^{2s}}} \\ \left(\frac{k}{h}\right)^{q^{4s}} = \xi. \end{cases}$$

Since $\xi^{1+q^{3s}} = 1$, then

$$\begin{cases} \left(\frac{k}{h}\right)^{q^{5s}} = \xi^{q^{2s}+1} \\ \left(\frac{k}{h}\right)^{q^{2s}+q^{4s}} = \xi^{q^{5s}} \\ \left(\frac{k}{h}\right)^{q^{4s}} = \xi. \end{cases} \tag{2.38}$$

Now, raising the third equation of (2.38) the q^s-power and taking into account the first one, we get that $\xi^{q^{2s}-q^s+1} = 1$, and so the system becomes

$$\begin{cases} \left(\frac{k}{h}\right)^{q^{5s}} = \xi^{q^s} \\ \left(\frac{k}{h}\right)^{q^{2s}+q^{4s}} = \xi^{q^{5s}} \\ \left(\frac{k}{h}\right)^{q^{4s}} = \xi. \end{cases} \tag{2.39}$$

From the first equation of above System (2.39) and taking into account the expression of ξ, we have

$$\xi = \frac{(\xi h^{q^{4s}} + \xi^{q^s} h^{q^{5s}})(h+1)}{(h^{q^{4s}} + h^{q^{5s}})(\xi^{q^{2s}} h + 1)}.$$

From the latter formula we obtain

$$(\xi^{q^s} - \xi) h^{q^{4s}+1} = (\xi^{q^s} - \xi) h^{q^{5s}}.$$

Now if $\xi \notin \mathbb{F}_q$ (that with condition $\xi^{q^{2s}-q^s+1} = 1$ implies $\xi \neq 1$), then $h^{q^{4s}+1} = h^{q^{5s}}$, a contradiction, (Bartoli et al. 2020, Lemma 2.3 (3)). Hence $\xi = 1$, $h = k$ and, by system

(2.34), $a \in \mathbb{F}_{q^2}$ and $d = a^q$. If otherwise $H_{h,s}(bH_{k,s}(x)) = cx$, then by expanding this formula we get

$$
\begin{aligned}
&\left(k^{q^s+q^{3s}}(1-h^{1+q^{2s}})(k^{q^s}+k^{q^{2s}})b^{q^s} + (h+h^2)(k^{q^{3s}}+k^{2q^{3s}})b^{q^{3s}} + h^{1+q^{2s}}(h+h^{q^s})(1-k^{q^{5s}+q^s})b^{q^{5s}}\right)x\\
&+\left((1-h^{1+q^{2s}})(k^{q^s}+k^{2q^s})b^{q^s} + (h+h^2)(1-k^{q^{3s}+q^{5s}})b^{q^{3s}} + k^{q^s+q^{5s}}h^{1+q^{2s}}(h+h^{q^s})(k^{q^{5s}}+k)b^{q^{5s}}\right)x^{q^{4s}}\\
&+\Big((1-h^{1+q^{2s}})(1-k^{q^s+q^{3s}})b^{q^s} + k^{q^{3s}+q^{5s}}(h+h^2)(k^{q^{3s}}+k^{q^{4s}})b^{q^{3s}}+\\
&+h^{1+q^{2s}}(h+h^{q^s})(k^{q^{5s}}+k^{2q^{5s}})b^{q^{5s}}\Big)x^{q^{2s}}\\
&=cx.
\end{aligned}
$$

Now, comparing the coefficients of $x, x^{q^{2s}}$ and $x^{q^{4s}}$ on the left and right side, and taking into account that $h^{q^{3s}+1} = k^{q^{3s}+1} = -1$, we obtain the following linear system in the unknowns $b^{q^s}, b^{q^{3s}}$ and $b^{q^{5s}}$:

$$
A_{h,k,s}\begin{pmatrix} b^{q^s} \\ b^{q^{3s}} \\ b^{q^{5s}} \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix} \tag{2.40}
$$

where

$$
A_{h,k,s} = \begin{pmatrix} A_{h_1,k_1,s_1} & A_{h_2,k_2,s_2} & A_{h_3,k_3,s_3} \end{pmatrix} \tag{2.41}
$$

$$
\begin{aligned}
A_{h_1,k_1,s_1} &= \begin{pmatrix} k^{q^s+q^{3s}}(1-h^{1+q^{2s}})(k^{q^s}+k^{q^{2s}}) & (h+h^2)(k^{q^{3s}}+k^{2q^{3s}}) & h^{1+q^{2s}}(h+h^{q^s})(1-k^{q^{5s}+q^s}) \end{pmatrix}^T,\\
A_{h_2,k_2,s_2} &= \begin{pmatrix} (1-h^{1+q^{2s}})(k^{q^s}+k^{2q^s}) & (h+h^2)(1-k^{q^{3s}+q^{5s}}) & k^{q^{5s}+q^s}h^{1+q^{2s}}(h+h^{q^s})(k^{q^{5s}}+k) \end{pmatrix}^T,\\
A_{h_3,k_3,s_3} &= \begin{pmatrix} (1-h^{1+q^{2s}})(1-k^{q^s+q^{3s}}) & k^{q^{3s}+q^{5s}}(h+h^2)(k^{q^{3s}}+k^{q^{4s}}) & h^{1+q^{2s}}(h+h^{q^s})(k^{q^{5s}}+k^{2q^{5s}}) \end{pmatrix}^T.
\end{aligned}
$$

The determinant of $A_{h,k,s}$ is

$$
h^{1+q^{2s}}(h^{1+q^{2s}}-1)(h+h^2)(h+h^{q^s})(k^{q^s+q^{3s}+q^{5s}}+1)^3. \tag{2.42}
$$

First, we see that this expression cannot be equal to zero. By way of contradiction, let us assume that this is the case. We can easily see that none of the terms containing h can be equal to zero because this is in contradiction with either the fact that h is non-zero or that $h^{q^{3s}+1}=-1$. The term with k needs more attention. But note that if it is zero, then

$$k^{q^s+q^{3s}+q^{5s}}=-1.$$

Rising the above equation to the q^{3s} power and then taking ratio with $k^{1+q^{3s}}=-1$ we get

$$k^{q^{2s}-q^s+1}=1$$

which finally leads to a contradiction as $-1=k^{1+q^{3s}}=(k^{q^{2s}-q^s+1})^{q^s+1}=1$. Hence, the determinant of $A_{h,k,s}$ is different from zero.
As by the assumption $c\neq 0$ the unique solution of the System (2.40) is $\mathbf{b}=\left(b^{q^s},b^{q^{3s}},b^{q^{5s}}\right)\in\mathbb{F}_{q^6}^3$, with

$$\begin{aligned}
b^{q^s} &= \frac{ck^{q^{5s}}(1-k^{q^{3s}+q^{5s}})}{(h^{1+q^{2s}}-1)(k^{q^s+q^{3s}+q^{5s}}+1)^2}\\
b^{q^{3s}} &= \frac{-ck^{1+q^s+q^{5s}}(k^{q^{3s}}+1)}{(h+h^2)(k^{q^s+q^{3s}+q^{5s}}+1)^2}\\
b^{q^{5s}} &= \frac{c(1-k^{q^s+q^{3s}})}{h^{1+q^{2s}}(h+h^{q^s})(k^{q^s+q^{3s}+q^{5s}}+1)^2}.
\end{aligned}\tag{2.43}$$

Also in this case, we focus on getting a relation between h and k and eventually the values of b and c satisfying the second equation expressed in (2.33). In order to do this we start by computing the ratio between the second and first equation in (2.43). From this we get

$$b^{q^{3s}-q^s}=-\frac{k^{1+q^s}(k^{q^{3s}}+1)(h^{1+q^{2s}}-1)}{h(h+1)(1-k^{q^{3s}+q^{5s}})}.\tag{2.44}$$

Similarly, taking the ratio of the third and second, and of the first and the third equation in (2.43) we get

$$b^{q^{5s}-q^{3s}}=-\frac{(1-k^{q^s+q^{3s}})(h+1)}{h^{q^{2s}}(h+h^{q^s})k^{1+q^s+q^{5s}}(k^{q^{3s}}+1)},\tag{2.45}$$

$$b^{q^s-q^{5s}}=\frac{h^{1+q^{2s}}(h+h^{q^s})k^{q^{5s}}(1-k^{q^{3s}+q^{5s}})}{(1-k^{q^s+q^{3s}})(h^{1+q^{2s}}-1)},\tag{2.46}$$

respectively. Since the q^{2s}-power of Formulas (2.44),(2.45) and (2.46) returns Formulas

(2.45), (2.46) and (2.44), respectively, we get

$$\begin{cases} k^{q^s}h^{q^{2s}} = \left[\frac{(1-k^{q^s+q^{5s}})(h+1)}{(h^{q^{4s}}+h^{q^{5s}})(k^{q^{3s}}+1)}\right]^{q^{2s}+1} \\ k^{q^{3s}}h^{q^{4s}} = \frac{(k^{q^{3s}}+1)(h^{q^{4s}}+h^{q^{5s}})}{(h+1)(1-k^{q^{5s}+q^s})} \\ k^{q^s+q^{3s}}h^{q^{2s}+q^{4s}} = \frac{(1-k^{q^s+q^{3s}})(h^{q^{2s}}+1)}{(k^{q^{5s}}+1)(h+h^{q^s})}. \end{cases} \tag{2.47}$$

Putting $\xi = \frac{(1-k^{q^s+q^{5s}})(h+1)}{(h^{q^{4s}}+h^{q^{5s}})(k^{q^{3s}}+1)}$, conditions above reads

$$\begin{cases} k^{q^s}h^{q^{2s}} = \xi^{q^{2s}+1} \\ k^{q^{3s}}h^{q^{4s}} = 1/\xi \\ k^{q^s+q^{3s}}h^{q^{2s}+q^{4s}} = \xi^{q^{2s}}. \end{cases} \tag{2.48}$$

By the first and the second equation, one gets that $\xi^{q^{4s}+q^{2s}+1} = 1$ and since $h^{1+q^{3s}} = k^{1+q^{3s}} = -1$, $\xi^{q^{3s}+1} = 1$ and so $\xi^{q^{2s}-q^s+1} = 1$. Therefore System (2.48) becomes

$$\begin{cases} k^{q^s}h^{q^{2s}} = \xi^{q^s} \\ k^{q^{3s}}h^{q^{4s}} = \xi^{q^{3s}} \\ k^{q^s+q^{3s}}h^{q^{2s}+q^{4s}} = \xi^{q^{2s}}. \end{cases} \tag{2.49}$$

From the first equation in System (2.49), and taking into account the expressions of ξ, we have

$$\begin{aligned} &\xi = \frac{(h^{q^{2s}+1}-\xi)(h+1)h^{q^{4s}}}{h^{q^{2s}+1}(h^{q^{4s}}+h^{q^{5s}})(\xi^{q^{3s}}+h^{q^{4s}})} \\ &\xi(h^{1+q^{2s}+q^{4s}}+1)h^{q^{4s}} = h(h^{1+q^{2s}+q^{4s}}+1) \\ &\xi = h^{1-q^{4s}}, \end{aligned} \tag{2.50}$$

obtaining $h = -k$. Then, taking this into account in (2.43), we get

$$\begin{aligned} b^{q^s} &= \frac{ck^{q^{5s}}(1-k^{q^{3s}+q^{5s}})}{(k^{1+q^{2s}}-1)(k^{q^s+q^{3s}+q^{5s}}+1)^2} \\ b^{q^{3s}} &= -\frac{ck^{q^s+q^{5s}}(k^{q^{3s}}+1)}{(k-1)(k^{q^s+q^{3s}+q^{5s}}+1)^2} \end{aligned} \qquad b^{q^{5s}} = -\frac{c(1-k^{q^s+q^{3s}})}{k^{1+q^{2s}}(k+k^{q^s})(k^{q^s+q^{3s}+q^{5s}}+1)^2}. \tag{2.51}$$

Finally, by raising one of these three equations to the q^{2s}-power and comparing it with the other ones, we obtain

$$c = \lambda k^{q^{2s}+1},$$

for some $\lambda \in \mathbb{F}_{q^2}$, and so $b = \frac{\lambda^q k^{q^{4s}}}{(k^{1+q^{2s}+q^{4s}}+1)^2}$.

Case 2. Now, we assume $s \equiv -\ell \pmod 6$. In the following we will show that elements $a, d \in \mathbb{F}_{q^6}$ exist such that

$$dH_{k,-s}(x) = H_{k^{-1},s}(ax), \tag{2.52}$$

holds true. Indeed, one easily sees that this is realized by putting $a = -k^{-1}$ and $d = k^{-2}$. This implies that $\mathcal{C}_{k,3,-s}$ is equivalent to $\mathcal{C}_{k^{-1},3,s}$.

Now, using the transitivity of the equivalence relation we have that $C^{5,3}_{1,h,s}$ is equivalent to $C^{5,3}_{1,k,-s}$ if and only if $C^{5,3}_{1,h,s}$ is equivalent to $C^{5,3}_{1,k^{-1},s}$.

So, using the same arguments as in Case 1, we have that $C^{5,3}_{1,k,-s}$ is equivalent to $C^{5,3}_{1,h,s}$ if and only if $h^\rho = \pm k^{-1}$.

Regarding the case $t = 4$, we prove the following result.

Theorem 2.5.6 *The MRD codes $C^{5,4}_{1,h,s}$ and $C^{5,4}_{1,k,\ell}$, are equivalent if and only if*

1. $\ell \equiv s \pmod 8$, *then* $h^\rho = \pm k$,
2. $\ell \equiv -s \pmod 8$, *then* $h^\rho = \pm k^{-1}$,
3. *if* $\ell \equiv 3s \pmod 8$, *then* $h^\rho = \pm k$,
4. *if* $\ell \equiv 5s \pmod 8$, *then* $h^\rho = \pm k^{-1}$,

where $\rho \in \mathrm{Aut}(\mathbb{F}_{q^8})$.

Proof. As before the necessity condition in the statement, is easy to prove. Concerning the sufficiency, using the same notation and arguments as in the proof of the previous Theorem 2.5.5 we get $r \in \{-1, 1, 3, 5\}$. Hence, $C^{5,4}_{1,h,s} = \langle x, \psi_{h,4,s}(x)\rangle_{\mathbb{F}_{q^8}}$ and $C^{5,4}_{1,k,\ell} = \langle x, \psi_{k,4,\ell}(x)\rangle_{\mathbb{F}_{q^8}}$, are equivalent if and only if Formula (2.27) holds true for any pair of standard forms of the q-polynomials $\psi_{h,4,s}$ and $\psi_{k,4,\ell}$, respectively. As mentioned before, the polynomial $\psi_{h,t,s}$ with $t \geq 4$ and even, is already expressed in standard form. Therefore, by Corollary 2.5.4, to investigate the equivalence issue between two elements in this class, we only need to consider the two following equations:

$$d\psi_{k,4,\ell}(x) = \psi^{\rho}_{h,4,s}(ax) \quad \text{and} \quad \psi_{h,4,s}\left(b\psi^{\rho}_{k,4,\ell}(x)\right) = cx \tag{2.53}$$

for $a, b, c, d \in \mathbb{F}^*_{q^8}$ and $\rho \in \mathrm{Aut}(\mathbb{F}_{q^8})$. Also here, without loss of generality, again by (Neri et al. 2022, Remark 4.5), we will suppose ρ to be the identity. Because of what is asserted in the first part of the proof, we can proceed by dividing the remaining part of the argument into four cases.

Case 1. We assume $\ell \equiv s \pmod 8$. Now, if $d\psi_{k,4,s}(x) = \psi_{h,4,s}(ax)$, then by comparing the coefficients, one easily gets $d = a^{q^s}$ with $a \in \mathbb{F}_{q^2}$ and $h = \pm k$.

If otherwise $\psi_{h,4,s}(b\psi_{k,4,\ell}(x)) = cx$, by expanding left side of this equation, one get

$$\begin{aligned}
&\left(b^{q^s} + b^{q^{3s}}k^{q^{3s}-q^{2s}} + b^{q^{5s}}h^{1-q^{5s}}k^{q^{5s}-q^{2s}} + b^{q^{7s}}h^{1-q^{7s}}\right)x^{q^{2s}}\\
+&\left(b^{q^s} + b^{q^{3s}} - b^{q^{5s}}k^{q^{5s}-q^{4s}}h^{1-q^{5s}} - b^{q^{7s}}h^{1-q^{7s}}k^{q^{7s}-q^{4s}}\right)x^{q^{4s}}\\
+&\left(-k^{q^s-q^{6s}}b^{q^s} + b^{q^{3s}} - h^{1-q^{5s}}b^{q^{5s}} + h^{1-q^{7s}}k^{q^{7s}-q^{6s}}b^{q^{7s}}\right)x^{q^{6s}}\\
+&\left(b^{q^s}k^{q^s-1} - b^{q^{3s}}k^{q^{3s}-1} - h^{1-q^{5s}}b^{q^{5s}} + h^{1-q^{7s}}b^{q^{7s}}\right)x = cx.
\end{aligned}$$

By equating the coefficients of the terms $x, x^{q^{2s}}, x^{q^{4s}}$, and $x^{q^{6s}}$ on the left and right side, and taking into account that $h^{q^{4s}+1} = k^{q^{4\ell}+1} = -1$, we obtain the following linear system in $b^{q^s}, b^{q^{3s}}, b^{q^{5s}}$, and $b^{q^{7s}}$:

$$\begin{pmatrix} 1 & k^{q^{3s}-q^{2s}} & h^{1-q^{5s}}k^{q^{5s}-q^{2s}} & h^{1-q^{7s}} \\ 1 & 1 & -k^{q^{5s}-q^{4s}}h^{1-q^{5s}} & -h^{1-q^{7s}}k^{q^{7s}-q^{4s}} \\ -k^{q^s-q^{6s}} & 1 & -h^{1-q^{5s}} & h^{1-q^{7s}}k^{q^{7s}-q^{6s}} \\ k^{q^s-1} & -k^{q^{3s}-1} & -h^{1-q^{5s}} & h^{1-q^{7s}} \end{pmatrix}\begin{pmatrix} b^{q^s} \\ b^{q^{3s}} \\ b^{q^{5s}} \\ b^{q^{7s}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix}. \tag{2.54}$$

Let us denote by $A_{h,k,s} \in \mathbb{F}^{4\times 4}_{q^8}$ the matrix associated with it. First, we show that its determinant is not zero. In fact, by recalling that $k^{1+q^{4s}} = -1$ and that $(s, 8) = 1$, we get

$$\det(A_{h,k,s}) = -\mathrm{Tr}_{q^8/q}(k^{q^s+1} + k^{q^{3s}+1})h^{2-q^{5s}-q^{7s}}.$$

If $\det(A_{h,k,s}) = 0$, since

$$\mathrm{Tr}_{q^8/q}(k^{q^s+1} + k^{q^{3s}+1}) = \mathrm{Tr}_{q^8/q^2}(k)^{q^s+1},$$

we would get that

$$k + k^{q^{2s}} + k^{q^{4s}} + k^{q^{6s}} = 0$$

or equivalently that

$$k + k^{q^{2s}} = \frac{1}{k} + \frac{1}{k^{q^{2s}}}.$$

Then

$$k^{q^{2s}+1} = 1,$$

which, by (Longobardi et al. 2023, Proposition 3.2), can not be the case. Since $c \neq 0$, then the unique solution of System (2.54) is $\mathbf{b} = \left(b^{q^s}, b^{q^{3s}}, b^{q^{5s}}, b^{q^{7s}}\right) \in \mathbb{F}_{q^8}^4$, with

$$\begin{aligned}
b^{q^s} &= \frac{c}{T}(k^{q^{5s}+q^{7s}} - 1)(k^{q^{6s}-q^{4s}} + 1) \\
b^{q^{3s}} &= -\frac{c}{T}(k^{q^{5s}+q^{7s}} - 1)(k^{q^{2s}-q^{4s}} + 1) \\
b^{q^{5s}} &= -\frac{c}{h^{1-q^{5s}}T}(k^{q^s} + k^{q^{7s}})(k^{q^{2s}} + k) \\
b^{q^{7s}} &= \frac{c}{h^{1-q^{7s}}T}(k^{q^{3s}} + k^{q^{5s}})(k^{q^{6s}} + k),
\end{aligned} \tag{2.55}$$

where $T = \mathrm{Tr}_{q^8/q}(k^{1+q^s} + k^{1+q^{3s}})$. By computing the ratio between the second and first component of $\mathbf{b}$ we have

$$(b^{q^s})^{q^{2s}-1} = -\frac{k^{q^{2s}-q^{4s}} + 1}{k^{q^{6s}-q^{4s}} + 1} = -\frac{k^{q^{2s}} + k^{q^{4s}}}{k^{q^{4s}} + k^{q^{6s}}} = -\frac{1}{(k^{q^{2s}} + k^{q^{4s}})^{q^{2s}-1}}. \tag{2.56}$$

In the same way, computing the ratio between the fourth and third equation in (2.55) we have

$$\begin{aligned}
(b^{q^{5s}})^{q^{2s}-1} &= -h^{q^{7s}-q^{5s}} \frac{(k^{q^{6s}} + k)(k^{q^{3s}} + k^{q^{5s}})}{(k^{q^{2s}} + k)(k^{q^s} + k^{q^{7s}})} \\
&= -\left(\frac{h^{q^{5s}} k^{q^s}}{k^{q^{6s}} + k}\right)^{q^{2s}-1} \frac{(k^{q^{5s}-q^{3s}} + 1)}{(k^{q^{7s}-q^s} + 1)} = -\left(\frac{h^{q^{5s}} k^{q^s}}{k^{q^{6s}} + k}\right)^{q^{2s}-1}.
\end{aligned} \tag{2.57}$$

Let ω be an element in $\mathbb{F}_{q^s}$ such that $\omega^{q^{2s}-1} = -1$; then, from the last two Expressions

(2.56) and (2.57) we get

$$b^{q^s} = \frac{\lambda\omega}{k^{q^{2s}} + k^{q^{4s}}} \quad \text{and} \quad b^{q^{5s}} = \frac{h^{q^{5s}} k^{q^s} \mu\omega}{k^{q^{6s}} + k}, \tag{2.58}$$

respectively, where $\lambda, \mu \in \mathbb{F}_{q^2}$. Raising the two sides of the first equation in (2.58) to the q^{4s} power, and taking into account the second equation, we have that $h^{q^{5s}} k^{q^s}$ must also belong to $\mathbb{F}_{q^2}$. In other words, we get

$$h^{q^{5s}} = \nu k^{-q^s}, \tag{2.59}$$

for some $\nu \in \mathbb{F}_{q^2}^*$.

Now, taking into account Equations (2.58), recalling that $\omega^{q^{2s}} = -\omega$ and that $\lambda \in \mathbb{F}_{q^2}$, the first equation of the Linear System (2.54) becomes

$$\frac{1}{k^{q^{2s}} + k^{q^{4s}}} - \frac{k^{q^{3s}-q^{2s}}}{k^{q^{4s}} + k^{q^{6s}}} + \frac{h^{1-q^{5s}} k^{q^{5s}-q^{2s}}}{k^{q^{6s}} + k} - \frac{h^{1-q^{7s}}}{k + k^{q^{2s}}} = 0. \tag{2.60}$$

Plugging (2.59), latter equation reads

$$\frac{1}{k^{q^{2s}} + k^{q^{4s}}} - \frac{k^{q^{3s}-q^{2s}}}{k^{q^{4s}} + k^{q^{6s}}} = \nu^{q^s-1} \left(\frac{k^{-q^{4s}-q^{2s}}}{k^{q^{6s}} + k} + \frac{k^{q^{3s}-q^{4s}}}{k + k^{q^{2s}}} \right), \tag{2.61}$$

whence

$$\frac{1}{k^{q^{2s}} + k^{q^{4s}}} - \frac{k^{q^{3s}}}{k^{q^{4s}+q^{2s}} - 1} = -\nu^{q^s-1} \left(\frac{1}{k^{q^{2s}} + k^{q^{4s}}} - \frac{k^{q^{3s}}}{k^{q^{4s}+q^{2s}} - 1} \right). \tag{2.62}$$

We observe here that the term on the left side of Equation (2.62), can not be equal to zero. Indeed, if it was

$$\frac{1}{k^{q^{2s}} + k^{q^{4s}}} - \frac{k^{q^{3s}}}{k^{q^{4s}+q^{2s}} - 1} = 0;$$

then, raising this equation to the $(q^{4s}+1)$-th power, and taking into account that $k^{q^{4s}+1} = -1$, we would get

$$(k^{q^{2s}} - k^{q^{6s}})(k - k^{q^{4s}}) = 0, \tag{2.63}$$

and so $k \in \mathbb{F}_{q^4}$. In such a case, the condition $k^{1+q^{4s}} = -1$ implies that $k \in \mathbb{F}_{q^2}$. Then, by (2.56), $b^{q^s} = \lambda\omega$ where $\omega^{q^{2s}-1} = -1$. Using the expression of $b^{q^{5s}}$ found in (2.57), we get that $h \in \mathbb{F}_{q^2}$ as well. Then, the first and third equations of Linear System (2.54), can be rewritten in the following form

$$\begin{cases} (1 - k^{q^s-1})(1 - h^{1-q^s}) = 0 \\ (1 + k^{q^s-1})(1 + h^{1-q^s}) = 0. \end{cases} \tag{2.64}$$

These two equations are verified together only when either $k \in \mathbb{F}_q$ and $h \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ with $h^{q^s-1} = -1$ or $h \in \mathbb{F}_q$ and $k \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ with $k^{q^s-1} = -1$. However, both when $q \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$, it is straightforward to see that these cases cannot occur. Then, the expression in (2.63) is not zero and hence, $\nu^{q^s-1} = -1$. On the other hand, since $\nu \in \mathbb{F}_{q^2}$, we also have $k^{1+q^{4s}} = \frac{\nu^2}{h^{1+q^{4s}}}$, whence $\nu^2 = 1$, a contradiction. **Case 2.** We assume $\ell \equiv -s \pmod 8$. Note that, by putting b and c as follows

$$b = \frac{k^{q^{4s}}}{k^{q^{7s}} + k^{q^{5s}}}, \qquad c = k^{q^{7s}-1} + k^{q^{5s}-1},$$

one gets $\psi_{k^{-1},4,s}(b\psi_{k,4,-s}(x)) = cx$. Hence, we have shown that $C^{5,4}_{1,k^{-1},s}$ is equivalent to $C^{5,4}_{1,k,-s}$. Now, using the transitivity of the equivalence we have that $C^{5,4}_{1,h,s}$ is equivalent to $C^{5,4}_{1,k,-s}$ if and only if $C^{5,4}_{1,h,s}$ is equivalent to $C^{5,4}_{1,k^{-1},s}$. By Case 1, we have that $C^{5,4}_{1,h,s}$ is equivalent to $C^{5,4}_{1,k,-s}$ if and only if $h^\rho = \pm k^{-1}$. **Case 3.** We assume now $\ell \equiv 3s \pmod 8$. If $d\psi_{k,4,3s}(x) = \psi_{h,4,s}(ax)$, by expanding and comparing the coefficients, we get the following set of conditions

$$\begin{cases} d = a^{q^s} \\ d = a^{q^{3s}} \\ dk^{1-q^{5s}} = -a^{q^{5s}} h^{1-q^{5s}} \\ -dk^{1-q^{7s}} = a^{q^{7s}} h^{1-q^{7s}}. \end{cases} \tag{2.65}$$

By the first and the second, $d = a^{q^s}$ with $a \in \mathbb{F}_{q^2}$ and plugging this into the third equation, we obtain

$$\left(\frac{h}{k}\right)^{q^{5s}-1} = -1. \tag{2.66}$$

Then, $h = \omega k$, where $\omega \in \mathbb{F}_{q^2}$ such that $\omega^{q^s-1} = -1$. Since on the other hand, we also have $\omega^2 = (h/k)^{1+q^{4s}} = 1$, this leads to a contradiction; hence, this case cannot occur.

Assume that $\psi_{h,4,s}(b\psi_{k,4,3s}(x)) = cx$. Expanding this equation we get

$$\begin{aligned} &\left(b^{q^s} - k^{q^{3s}-q^{2s}}b^{q^{3s}} - b^{q^{5s}}h^{1-q^{5s}}k^{q^{5s}-q^{2s}} + b^{q^{7s}}h^{1-q^{7s}}\right)x^{q^{2s}} \\ +&\left(b^{q^s} + b^{q^{3s}} + b^{q^{5s}}h^{1-q^{5s}}k^{q^{5s}-q^{4s}} + b^{q^{7s}}h^{1-q^{7s}}k^{q^{7s}-q^{4s}}\right)x^{q^{4s}} \\ +&\left(b^{q^s}k^{q^s-q^{6s}} + b^{q^{3s}} - b^{q^{5s}}h^{1-q^{5s}} - b^{q^{7s}}k^{q^{7s}-q^{6s}}h^{1-q^{7s}}\right)x^{q^{6s}} \\ +&\left(-b^{q^s}k^{q^s-1} + b^{q^{3s}}k^{q^{3s}-1} - b^{q^{5s}}h^{1-q^{5s}} + b^{q^{7s}}h^{1-q^{7s}}\right)x = cx. \end{aligned}$$

Now, by comparing the coefficients of $x, x^{q^{2s}}, x^{q^{4s}}$, and $x^{q^{6s}}$ on the left and right side, the equation above gives the following linear system in $b^{q^s}, b^{q^{3s}}, b^{q^{5s}}, b^{q^{7s}}$:

$$\begin{pmatrix} 1 & -k^{q^{3s}-q^{2s}} & -h^{1-q^{5s}}k^{q^{5s}-q^{2s}} & h^{1-q^{7s}} \\ 1 & 1 & k^{q^{5s}-q^{4s}}h^{1-q^{5s}} & h^{1-q^{7s}}k^{q^{7s}-q^{4s}} \\ k^{q^s-q^{6s}} & 1 & -h^{1-q^{5s}} & -h^{1-q^{7s}}k^{q^{7s}-q^{6s}} \\ -k^{q^s-1} & k^{q^{3s}-1} & -h^{1-q^{5s}} & h^{1-q^{7s}} \end{pmatrix} \begin{pmatrix} b^{q^s} \\ b^{q^{3s}} \\ b^{q^{5s}} \\ b^{q^{7s}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix}. \tag{2.67}$$

The determinant of the matrix $A_{h,k,3s}$ associated with linear system (2.67) is equal to

$$\mathrm{Tr}_{q^8/q}(k^{q^s+1} + k^{q^{3s}+1})h^{2-q^{5s}-q^{7s}}$$

and hence it is different from zero. Since $c \neq 0$ then, the unique solution of System (2.67) has components

$$\begin{aligned} b^{q^s} &= -\frac{c}{T}(k^{q^{5s}+q^{7s}} - 1)(k^{q^{6s}-q^{4s}} + 1) \\ b^{q^{3s}} &= \frac{c}{T}(k^{q^{5s}+q^{7s}} - 1)(k^{q^{2s}-q^{4s}} + 1) \\ b^{q^{5s}} &= -\frac{c}{h^{1-q^{5s}}T}(k^{q^s} + k^{q^{7s}})(k^{q^{2s}} + k) \\ b^{q^{7s}} &= \frac{c}{h^{1-q^{7s}}T}(k^{q^{3s}} + k^{q^{5s}})(k^{q^{6s}} + k), \end{aligned} \tag{2.68}$$

which is the same solution obtained for System (2.54) in the case where $\ell \equiv s \pmod 8$. Then arguing as in that case, we get $h^{q^{5s}} = \nu k^{-q^s}$ and $b^{q^s} = \frac{\lambda\omega}{k^{q^{2s}}+k^{q^{4s}}}$ for some $\nu, \lambda \in \mathbb{F}_{q^2}^*$ and $\omega \in \mathbb{F}_{q^8}$ such that $\omega^{q^{2s}-1} = -1$. Since $1 = (h^{q^{5s}}k^{q^s})^{1+q^{4s}} = \nu^2$, then

$\nu = \pm 1$ and so $h = \pm k$,

$$b = -\frac{\lambda^{q^s}\omega^{q^s}}{k^{q^{3s}} + k^{q^s}} \text{ and } c = \lambda\omega(k^{q^s} + k^{q^{3s}}).$$

$$\begin{aligned}
c &= -\lambda\omega\left(\frac{k^{q^s-1}}{k^{q^{2s}}+k^{q^{4s}}} + \frac{k^{q^{3s}-1}}{k^{q^{4s}}+k^{q^{6s}}} + \frac{k^{1-q^{5s}}}{k+k^{q^{6s}}} + \frac{k^{1-q^{7s}}}{k^{q^{2s}}+k}\right)\\
&= -\lambda\omega\left(\frac{k^{q^s-1}}{k^{q^{2s}}+k^{q^{4s}}} + \frac{k^{q^{3s}-1}}{k^{q^{4s}}+k^{q^{6s}}} + \frac{k^{q^s-q^{4s}}}{k+k^{q^{6s}}} + \frac{k^{q^{3s}-q^{4s}}}{k^{q^{2s}}+k}\right)\\
&= -\lambda\omega\left(\frac{k^{q^s}}{k^{q^{2s}+1}-1} + \frac{k^{q^{3s}}}{k^{q^{6s}+1}-1} + \frac{k^{q^s}}{k^{q^{6s}+q^{4s}}-1} + \frac{k^{q^{3s}}}{k^{q^{2s}+q^{4s}}-1}\right)\\
&= -\lambda\omega\left[k^{q^s}\left(\frac{1}{k^{q^{2s}+1}-1} + \frac{1}{k^{q^{6s}+q^{4s}}-1}\right) + k^{q^{3s}}\left(\frac{1}{k^{q^{6s}+1}-1} + \frac{1}{k^{q^{2s}+q^{4s}}-1}\right)\right]\\
&= \lambda\omega(k^{q^s} + k^{q^{3s}}).
\end{aligned} \tag{2.69}$$

Case 4. Finally, we assume $\ell \equiv 5s \pmod 8$. Note that $d\psi_{k,4,5s}(x) = \psi_{k^{-1},4,s}(ax)$ holds true for $a = -\omega^{q^s}k^{-1}$ and $d = -\frac{\omega}{k}$ where $\omega^{q^{2s}} = -\omega$. So, we have shown that $C^{5,4}_{1,k,5s}$ is equivalent to $C^{5,4}_{1,k^{-1},s}$. Now, using the transitivity of the equivalence we have that $C^{5,4}_{1,k,5s}$ is equivalent to $C^{5,4}_{1,h,s}$ if and only if $C^{5,4}_{1,h,s}$ is equivalent to $C^{5,4}_{1,k^{-1},s}$. By Case 1, we have that $C^{5,4}_{1,h,s}$ is equivalent to $C^{5,4}_{1,k,5s}$ if and only if $h^\rho = \pm k^{-1}$. This concludes the proof.

We conclude this section pointing out that as a consequence of Theorems 2.5.5 and 2.5.6 above, by using same argument as developed in (Neri et al. 2022, Theorem 4.10), it is easy to determine the exact number of the equivalence classes of codes in the family $C^{5,t}_{1,h,s}$ (or equivalently the number of inequivalent codes in it), also for $t = 3, 4$. As in (Neri et al. 2022, Theorem 4.12), a lower bound for this number in the case $t = 3, 4$, turns out to be $\left\lfloor \frac{\varphi(t)(q^t+1)}{8rt} \right\rfloor$, where $q = p^r$ and φ is the Euler's totient function.

2.5.2 Equivalence issue of linear sets

In this section, we will denote by $L^{1,n}_s$, $L^{2,n}_{s,\eta}$ and $L^{3,n}_{s,\delta}$ the linear sets associated with the codes $C^{1,n}_s$, $C^{2,n}_{s,\eta}$ and $C^{3,n}_{s,\delta}$, respectively. Also, we will denote by $L^{5,t}_{1,h,s}$ the linear set of $\mathrm{PG}(1, q^{2t})$ associated with the code $C^{5,t}_{1,h,s}$, in the following we will deal with it in the case where $t \in \{3, 4\}$.

Regarding the equivalence issue for $L^{5,3}_{1,h,1}$, this was completely solved in Bartoli et al.

(2020). As a consequence of Theorem 2.5.5, we may state the following result.

Theorem 2.5.7 *If $h \in \mathbb{F}_{q^2}$, the linear set $L^{5,3}_{1,h,s} \subset \mathrm{PG}(1,q^6)$ is $\mathrm{P\Gamma L}$-equivalent to some*

$$L^{4,6}_{\zeta} = \{\langle (x, x^q + x^{q^3} + \zeta x^{q^5})\rangle_{\mathbb{F}_{q^6}} : x \in \mathbb{F}^*_{q^6}\}$$

where $\xi \in \mathbb{F}_{q^6}$ such that $\xi^2 + \xi = 1$ if and only if $h \in \mathbb{F}_q$ and q is a power of 5. If $h \notin \mathbb{F}_{q^2}$, then $L^{5,3}_{1,h,s}$ is not $\mathrm{P\Gamma L}$-equivalent to $L^{1,n}_s$, $L^{2,6}_{s,\eta}$ and $L^{3,6}_{s,\delta}$ in $\mathrm{PG}(1,q^6)$.

Now we investigate the same issue for $L^{5,4}_{1,h,s}$. In Longobardi & Zanella (2021), by using projection techniques, the authors solved the problem for the linear set associated with the code $\mathcal{C}_t$, showing that this is not equivalent to any other linear sets previously constructed.

Following the same approach, we will show the same result for any value of the parameter h. We will work in the following framework. Let $x_0, x_1, \dots, x_7$ be the homogeneous coordinates of $\mathrm{PG}(7,q^8)$ and consider

$$\Sigma = \{\langle (x, x^q, \dots, x^{q^7})\rangle_{\mathbb{F}_{q^8}} : x \in \mathbb{F}^*_{q^8}\}$$

to be a fixed (canonical) subgeometry of $\mathrm{PG}(7,q^8)$. Let

$$\hat{\sigma} : \langle (x_0, x_1, \dots, x_7)\rangle_{\mathbb{F}_{q^8}} \in \mathrm{PG}(7,q^8) \longrightarrow \langle (x_7^q, x_0^q, x_1^q, \dots, x_6^q)\rangle_{\mathbb{F}_{q^8}} \in \mathrm{PG}(7,q^8)$$

be a collineation of $\mathrm{PG}(7,q^8)$. It is straightforward to see that $\hat{\sigma}^m$ fixes Σ pointwise for any $m \in \{0,\dots,7\}$, and $\hat{\sigma}^u$ is a generator of $\mathrm{Fix}(\Sigma)$ for any u such that $(u,8)=1$.

As proven in (Lunardon & Polverino 2004, Theorems 1 and 2) each linear set L of $\mathrm{PG}(1,q^8)$ can be obtained by projecting Σ from a 5-dimensional projective subspace Γ of $\mathrm{PG}(7,q^8)$ onto a line of $\mathrm{PG}(7,q^8)$ which is disjoint from Γ. By (Zanella & Zullo 2020, Theorem 1.1 and 3.5), as done in (Bartoli et al. 2020, Result 3.4), we can assert that if L is scattered and the least positive integer γ such

$$\dim(\Gamma \cap \Gamma^{\hat{\sigma}} \cap \dots \cap \Gamma^{\hat{\sigma}^{\gamma}}) > 5 - 2\gamma,$$

does not belong to $\{1,2\}$, then L is neither equivalent to $L^{1,n}_s$ nor to $L^{2,n}_{s,\eta}$ (for more details see also Longobardi & Zanella (2021)). With this in mind, we prove the following

Theorem 2.5.8 *The linear set $L^{5,4}_{1,h,s}$ is neither equivalent to $L^{1,4}_{s}$ nor to $L^{2,4}_{s,\eta}$.*

Proof. Let $h^{1+q^{4s}} = -1$ and $(s,4) = 1$. The linear set $L^{5,4}_{1,h,s}$ can be seen as the projection of the canonical subgeometry Σ from the

$$\Gamma_s : \begin{cases} x_0 = 0 \\ x_s + x_{3s} - h^{1-q^{5s}} x_{5s} + h^{1-q^{7s}} x_{7s} = 0 \end{cases} \tag{2.70}$$

onto the line $\mathrm{PG}(1,q^8) \subset \mathrm{PG}(7,q^8)$ with equations $x_{2s} = x_{3s} = \cdots = x_{7s} = 0$, where indices in system above are taken modulo 8. Let

$$\Gamma_s^{\hat{\sigma}^u} : \begin{cases} x_u = 0 \\ x_{s+u} + x_{3s+u} - h^{q^u - q^{5s+u}} x_{5s+u} + h^{q^u - q^{7s+u}} x_{7s+u} = 0, \end{cases} \tag{2.71}$$

where $u \in \{s, 3s, 5s, 7s\} \pmod 8$. Then

$$\Gamma_s \cap \Gamma_s^{\hat{\sigma}^u} : \begin{cases} x_0 = 0 \\ x_u = 0 \\ x_s + x_{3s} - h^{1-q^{5s}} x_{5s} + h^{1-q^{7s}} x_{7s} = 0 \\ x_{s+u} + x_{3s+u} - h^{q^u - q^{5s+u}} x_{5s+u} + h^{q^u - q^{7s+u}} x_{7s+u} = 0. \end{cases} \tag{2.72}$$

It is straightforward to see that, for any choice of s and u, none of the unknowns in the third equation appears among those of the fourth one; hence, the four equations in System (2.72) are independent. As a consequence, $\dim(\Gamma_s \cap \Gamma_s^{\hat{\sigma}^u}) = 3$. Now consider,

$$\Gamma_s \cap \Gamma_s^{\hat{\sigma}^u} \cap \Gamma_s^{\hat{\sigma}^{2u}} : \begin{cases} x_0 = 0 \\ x_u = 0 \\ x_{2u} = 0 \\ x_s + x_{3s} - h^{1-q^{5s}} x_{5s} + h^{1-q^{7s}} x_{7s} = 0 \\ x_{s+u} + x_{3s+u} - h^{q^u - q^{5s+u}} x_{5s+u} + h^{q^u - q^{7s+u}} x_{7s+u} = 0 \\ x_{s+2u} + x_{3s+2u} - h^{q^{2u} - q^{5s+2u}} x_{5s+2u} + h^{q^{2u} - q^{7s+2u}} x_{7s+2u} = 0. \end{cases} \tag{2.73}$$

As before, we note that for any $u \in \{s, 3s, 5s, 7s\}$:

$$\{s + 2u, 3s + 2u, 5s + 2u, 7s + 2u\} = \{s, 3s, 5s, 7s\}$$

and

$$\{s + u, 3s + u, 5s + u, 7s + u\} = \{0, 2s, 4s, 6s\},$$

where the integers are taken modulo 8. So the fourth and sixth equations in System (2.73) have the same unknowns; in addition, none of the unknowns of the fifth equation is among these. So, suppose that $u \equiv s \pmod 8$, then (2.73) becomes

$$\Gamma_s \cap \Gamma_s^{\hat{\sigma}^u} \cap \Gamma_s^{\hat{\sigma}^{2u}} : \begin{cases} x_0 = 0 \\ x_s = 0 \\ x_{2s} = 0 \\ x_{3s} - h^{1-q^{5s}} x_{5s} + h^{1-q^{7s}} x_{7s} = 0 \\ x_{4s} - h^{q^s - q^{6s}} x_{6s} = 0 \\ x_{3s} + x_{5s} - h^{q^{2s}-q^{7s}} x_{7s} = 0. \end{cases} \tag{2.74}$$

By the fourth and sixth equations, one gets $(1 + h^{1-q^{5s}})x_{5s} = h^{1-q^{7s}}(1 + h^{q^{2s}-1})x_{7s}$ and since $h^{q^{2s}-1} \neq -1$ (see (Longobardi et al. 2023, Proposition 3.2)), all equations in (2.74) are independent. The same argument can be applied to all the other congruences leading to $\dim(\Gamma_s \cap \Gamma_s^{\hat{\sigma}^u} \cap \Gamma_s^{\hat{\sigma}^{2u}}) = 1$ for any $u \in \{s, 3s, 5s, 7s\} \pmod 8$. Hence, we have that $\gamma \notin \{1, 2\}$; therefore, $L^{5,4}_{1,h,s}$ is neither equivalent to $L^{1,4}_s$ nor to $L^{2,4}_{s,\eta}$.

As a consequence of this achievement, we may derive the following result about the equivalence of the codes $C^{5,t}_{1,h,s}$ with ones belonging to other families.

Theorem 2.5.9 *Let $n = 2t$, $t = 3, 4$, $h, k, \eta \in \mathbb{F}_{q^n}$ satisfying $\mathrm{N}_{q^n/q^t}(h) = \mathrm{N}_{q^n/q^t}(k) = -1$ and $\mathrm{N}_{q^n/q}(\eta) \neq 1$. Let $s \in \mathbb{N}$ such that $(n, s) = 1$.*

(a) $C^{2,n}_{s,\eta}$ and $C^{5,t}_{1,h,s}$ are not equivalent.

(b) Assume $\delta \in \mathbb{F}_{q^{2t}}$ such that $\mathrm{N}_{q^n/q^{n/2}}(\delta) \notin \{0, 1\}$. Then, the codes $C^{3,6}_{s,\delta}$ and $C^{5,t}_{1,h,s}$ are not equivalent.

(c) Assume $\zeta \in \mathbb{F}_{q^6}$ such that $\zeta^2 + \zeta = 1$. Then, the codes $C^{4,6}_{\zeta}$ and $C^{5,3}_{1,h,s}$ are not equivalent except $h \in \mathbb{F}_q$ and q a power of 5.

Proof. (a) First assume $\eta = 0$, then $C^{2,n}_{s,0} = C^{1,n}_{s}$. This has left and right idealizers both isomorphic to $\mathbb{F}_{q^n}$. Since $I_R(C^{5,t}_{1,h,s})$ is isomorphic to $\mathbb{F}_{q^2}$ for both $t = 3, 4$, then $C^{5,t}_{1,h,s}$ can not be equivalent to $C^{1,t}_{s}$. On the other hand, if $\eta \neq 0$ and $t = 3$, by (Bartoli et al. 2020, Result 3.4), we get that $C^{5,t}_{1,h,s}$ is not equivalent to $C^{2,n}_{s,\eta}$. For $t = 4$ the result directly follows from Theorem 2.5.8.
(b) The statement is a direct consequence of the fact that the right idealizer of $C^{3,n}_{s,\delta}$ is isomorphic to $\mathbb{F}_{q^t}$, Csajbók, Marino, Polverino & Zanella (2018).
(c) The statement is a direct consequence of Theorem 2.5.5 and (Bartoli et al. 2020, Proposition 3.10).

In Csajbók, Marino, Polverino & Zanella (2018), the scattered subspace

$$U_{n,s,\delta} = \{(x, \delta x^{q^s} + x^{q^{s+n/2}}) \colon x \in \mathbb{F}_{q^n}\} \tag{2.75}$$

associated with the code $C^{3,n}_{s,\delta}$ was studied. As a direct consequence of (Csajbók, Marino & Zullo 2018, Propositions 4.1 and 4.2), we can say that the scattered linear set associated with $U_{8,s,\delta}$, is equivalent to $C^{5,4}_{1,h,s}$ if and only if the underlying subspaces are ΓL-equivalent and, hence, if and only if the MRD codes $C^{5,4}_{1,h,s}$ and $C^{3,8}_{s,\delta}$ are equivalent. However, as shown in Theorem 2.5.9(b), this is not the case. Therefore, we can state the following.

Theorem 2.5.10 *The linear set $L^{5,4}_{1,h,s}$ is not* PΓL*-equivalent to $L^{1,n}_{s}$ and $L^{2,n}_{s,\eta}$ and $L^{3,8}_{s,\delta}$ of* PG$(1, q^8)$.

2.6 Conclusion

In this chapter, we start by giving a geometric description of the vertex of an evasive linear set over a projective space PG$(r-1, q^n)$. We have seen that the vertex and axis of the projection can be seen as generated by the *imaginary* points and its conjugate. We gave a purely geometric description of the vertex of the maximum scattered linear set as initiated by Csajbók & Zanella (2016*c*) and Zanella & Zullo (2020) for pseudoregulas and LP-type linear set respectively. We saw their approach using intersection numbers cannot be used further. We then used the very recent result by Pepe (2024) and the fundamental result we developed for obtaining the vertex for evasive subspace to give a purely geometric description of vertex of known maximum scattered linear sets of rank n over the projective line $\mathrm{PG}(1, q^n)$. More interestingly, we see that if we can fix distances between some of imaginary and its conjugate points (which we attain by the notion of crossratios) we can get the projection of a subgeometry from vertex to the axis as maximum scattered linear sets.

Further we generalised the result by (Neri et al. 2022, Theorem 4.6) (see Theorem 2.5.1) for $t = 3, 4$. We made use of scattered polynomials introduced by Longobardi & Zanella (2021) to notice that in place of using an invertible matrix to obtain the inequivalence, we can reduce it to two consecutive cases one obtained through the diagonal matrix and one through the anti-diagonal matrix (see Corollary 2.5.4). This preciously gave us more equations to investigate the inequivalence issue. We finally show that the linear set obtained via the class for $t = 4$ is really new, i.e, not equivalent to the known linear sets.

CHAPTER 3

Affine vector space and quadrics

A vector space partition $\mathcal{P}$ of the projective space $\mathrm{PG}(n-1,q)$ is a set of subspaces in $\mathrm{PG}(n-1,q)$ which partitions the set of points. They relate to finite projective planes, design theory, and error-correcting codes. It is natural to consider a vector space partition $\mathcal{P}$ of the affine space $\mathrm{AG}(n,q)$ as a set of subspaces in $\mathrm{AG}(n,q)$ that partitions the set of point of $\mathrm{AG}(n,q)$. We call this partition affine vector space partition (or abbreviated as *avsp*). In this chapter, after introducing these affine vector space partition, we define avsps of different kind particularly hyperbolic, elliptic, and parabolic types. We see that avsps have a direct link with the spreads of quadrics. More preciously, we will see that a hyperbolic, parabolic, or elliptic affine of $\mathrm{PG}(n,q)$ is equivalent to a spread of $\mathcal{Q}^+(n+1,q)$, $\mathcal{Q}(n+1,q)$ or $\mathcal{Q}^-(n+1,q)$, respectively. We finally see how avsp of our kind gives us properties like irreducibility and tightness.

3.1 Our setting and construction

Definition 3.1.1 Let H be a hyperplane of $\mathrm{PG}(n+1,q)$. An *affine vector space partition* of $\mathrm{PG}(n+1,q)$ is a set $\mathcal{P}$ of subspaces of $\mathrm{PG}(n+1,q)$ whose members are not contained in H and such that every point of $\mathrm{PG}(n+1,q)\setminus H$ is contained in exactly one element of $\mathcal{P}$.

The *type* of an avsp $\mathcal{P}$ of $\mathrm{PG}(n+1,q)$ is given by $(n+1)^{m_{n+1}}\dots 2^{m_2}1^{m_1}$, where $m_i>0$ denotes the number of $(i-1)$-spaces of $\mathcal{P}$ for $1\le i\le n+1$. It is now natural to call an avsp of type i^{m_i} as *affine spread*. From here, we observe that if $\mathcal{P}$ is of type $(n+1)^{m_{n+1}} \dots 2^{m_2} 1^{m_1}$ then by just counting points outside of H we have the following condition

$$\sum_{i=1}^{n-1} m_i \cdot q^i = q^{n-1}.$$

This is called the *packing condition*. It is easy to see that it is always possible to consider an avsp satisfying this condition.

Theorem 3.1.2 *Bamberg et al. (2023) For each type $(n+1)^{m_{n+1}} \dots 2^{m_2} 1^{m_1}$ that satisfies the packing condition there exists an avsp $\mathcal{P}$ of* $\mathrm{PG}(n-1, q)$ *attaining that type.*

So we see avsps always exist for a particular packing condition. Further, introducing the condition of tightness and irreducibility makes the problem more significant, challenging and has applications in logical formulas of full disjunctive normal form (DNF). Now, we introduce them formally.

Definition 3.1.3 Let $\mathcal{P}$ be an avsp of $\mathrm{PG}(n+1, q)$. Then $\mathcal{P}$ is said to be *reducible* if there exists a proper subspace U of $\mathrm{PG}(n+1, q)$ not in $\mathcal{P}$ such that the members of $\mathcal{P}$ contained in U form an avsp of U. If $\mathcal{P}$ is not reducible, then it is said to be *irreducible*.

Definition 3.1.4 Let $\mathcal{P}$ be an avsp of $\mathrm{PG}(n+1, q)$. Then $\mathcal{P}$ is said to be *tight* if no point of $\mathrm{PG}(n+1, q)$ belongs to each of the members of $\mathcal{P}$.

In (Bamberg et al. 2023, Section 5.2) the authors studied a particular avsp $\{S_1, \dots, S_{q^3}\}$ of $\mathrm{PG}(6, q)$, q even, of type 4^{q^3}, such that the set $\{S_i \cap H \mid i = 1, \dots, q^3\}$ consists of the q^3 planes of a Klein quadric $\mathcal{Q}^+(5, q)$ that are disjoint from a fixed plane of $\mathcal{Q}^+(5, q)$. Moreover, they conjecture that a similar construction can be realized for all prime powers (Bamberg et al. 2023, Conjecture 2). Motivated by their example, we introduce the definition of *hyperbolic, parabolic, or elliptic avsp*.

Definition 3.1.5 Let $\mathcal{P} = \left\{S_1, S_2, \dots, S_{q^{\frac{n+e+1}{2}}}\right\}$ be an avsp of $\mathrm{PG}(n+1, q)$ of type $\left(\frac{n-e+3}{2}\right)^{q^{\frac{n+e+1}{2}}}$. Then $\mathcal{P}$ is said *hyperbolic, parabolic or elliptic* if the set

$$\mathcal{G} = \left\{\Pi_i = S_i \cap H \mid i = 1, \dots, q^{\frac{n+e+1}{2}}\right\}$$

consists of $q^{\frac{n+e+1}{2}}$ generators of a quadric $\mathcal{Q}_{n,e} \subset H$, where $e = 0, 1, 2$, respectively, such that

(1) elements of $\mathcal{G}$ are disjoint from a fixed generator Π of $\mathcal{Q}_{n,e}$;

(2) distinct members of $\mathcal{G}$ (or of $\mathcal{P}$) meet at most in one point;

(3) if $|\Pi_i \cap \Pi_j| = 1$, then $\langle S_i, S_j \rangle \cap \mathcal{Q}_{n,e} = \mathcal{Q}^+(n-e, q)$.

Remark 3.1.6 Let n be odd, let $\mathcal{P} = \left\{S_1, S_2, \dots, S_{q^{\frac{n+1}{2}}}\right\}$ be a hyperbolic avsp of $\mathrm{PG}(n+1, q)$ and let $\Pi_i = S_i \cap H$, $i = 1, \dots, q^{\frac{n+1}{2}}$. Since members of $\mathcal{P}$ are $\left(\frac{n+1}{2}\right)$-spaces, then $\langle S_i, S_j \rangle = \mathrm{PG}(n+1, q)$ and $|S_i \cap S_j| = 1$, if $i \neq j$. Hence $|\Pi_i \cap \Pi_j| = 1$, if $i \neq j$. Therefore, from the definition of a hyperbolic avsp the point *(2)* is equivalent to the requirement that distinct members of $\mathcal{G}$ pairwise intersect at one point, whereas the point *(3)* is redundant.

If $n \equiv -1 \pmod 4$, then members of $\mathcal{G}$ and Π belong to the same system of generators and hence $|\Pi_i \cap \Pi_j| = 0$, if $i \neq j$. Therefore hyperbolic avsp of $\mathrm{PG}(n+1, q)$, $n \equiv -1 \pmod 4$, do not exist.

Remark 3.1.7 If $n = e + 1$, then there are not q^{e+1} many generators of $\mathcal{Q}_{n,e}$ disjoint from Π. Hence there is no hyperbolic, parabolic, or elliptic avsp for $(n, e) \in \{(1, 0), (2, 1), (3, 2)\}$. If $n = e + 3$, then requirement *(2)* is redundant.

Remark 3.1.8 A hyperbolic affine spread $\mathcal{P}$ of $\mathrm{PG}(6, q)$ is a set of q^3 solids such that $S_i \cap H$, $i = 1, \dots q^3$, are planes of a Klein quadric $\mathcal{Q}^+(5, q)$ that are disjoint from a fixed plane of $\mathcal{Q}^+(5, q)$. Therefore the affine spreads studied in Bamberg et al. (2023) are equivalent to the hyperbolic affine spread of $\mathrm{PG}(6, q)$ consisting of solids and the existence of a hyperbolic avsp of $\mathrm{PG}(6, q)$ is equivalent to the "extension problem" formulated in (Bamberg et al. 2023, Conjecture 2) which states that The extension problem for $\mathcal{P}$ admits a solution for all prime powers q.

In the following we will need the next result, which in the hyperbolic case has been proved in (Dempwolff & Kantor 2015, Example 7.6).

Lemma 3.1.9 *Let* $\left\{\Sigma_1, \dots, \Sigma_{q^{\frac{n+e+1}{2}}+1}\right\}$ *be a spread of a quadric* $\mathcal{Q}_{n+2,e}$ *of* $\mathrm{PG}(n+2, q)$. *Fix a point* $P \in \Sigma_{q^{\frac{n+e+1}{2}}+1}$ *and an* n*-space* $H \subset P^\perp$ *such that* $P \notin H$. *Set*

$\mathcal{Q}_{n,e} = H \cap \mathcal{Q}_{n+2,e}$ and

$$\Pi_i = \langle P, \Sigma_i \rangle \cap H, \quad i = 1, \ldots, q^{\frac{n+e+1}{2}}.$$

The following hold.

i) If $e \in \{1, 2\}$, i.e., $\mathcal{Q}_{n+2,e}$ is parabolic or elliptic, then Π_i and Π_j, $i \neq j$, intersect in at most one point.

ii) If $e = 0$, i.e., $\mathcal{Q}_{n+2,0}$ is hyperbolic and $n \equiv 1 \pmod 4$, then Π_i and Π_j, $i \neq j$, have exactly one point in common.

iii) Each point of $\mathcal{Q}_{n,e} \setminus \Sigma_{q^{\frac{n+e+1}{2}}+1}$ lies in precisely q members of $\left\{\Pi_i \mid i = 1, \ldots, q^{\frac{n+e+1}{2}}\right\}$.

Proof. Since $|(\Sigma_i \cap P^\perp) \cap (\Sigma_j \cap P^\perp)| = 0$, it follows that $\Pi_i \cap \Pi_j$ is at most one point. Assume that $e = 0$, i.e., $\mathcal{Q}_{n+2,0}$ is hyperbolic and $n \equiv 1 \pmod 4$. Consider the $\left(\frac{n+1}{2}\right)$-space $T_i = \langle P, \Sigma_i \cap P^\perp \rangle$. Then T_i is a generator of $\mathcal{Q}_{n+2,0}$ intersecting Σ_i in an $\left(\frac{n-1}{2}\right)$-space. Therefore T_i and Σ_j, $i \neq j$, are in different systems of generators and hence they have at least one point in common. It follows that Π_i and Π_j, $i \neq j$, have at least one point in common.

Finally if R is a point of $\mathcal{Q}_{n,e} \setminus \Sigma_{q^{\frac{n+e+1}{2}}+1}$, since $P \in \Sigma_{q^{\frac{n+e+1}{2}}+1}$ and $R \notin \Sigma_{q^{\frac{n+e+1}{2}}+1}$, there are exactly q distinct members of $\left\{\Sigma_1, \ldots, \Sigma_{q^{\frac{n+e+1}{2}}+1}\right\}$ intersecting the line $\langle P, R \rangle$ in one point distinct from P. Hence, through a point of $\mathcal{Q}_{n,e} \setminus \Sigma_{q^{\frac{n+e+1}{2}}+1}$, there pass precisely q members of $\left\{\Pi_i \mid i = 1, \ldots, q^{\frac{n+e+1}{2}}\right\}$.

In what follows we observe that a hyperbolic, elliptic, or parabolic avsp of $\mathrm{PG}(n+1, q)$ can be obtained starting from a spread of a hyperbolic, elliptic, or parabolic quadric of $\mathrm{PG}(n+2, q)$.

Construction 3.1.10 *Let $\left\{\Sigma_1, \ldots, \Sigma_{q^{\frac{n+e+1}{2}}+1}\right\}$ be a spread of a quadric $\mathcal{Q}_{n+2,e}$ of $\mathrm{PG}(n+2, q)$. Fix a point $P \in \Sigma_{q^{\frac{n+e+1}{2}}+1}$ and a hyperplane $\mathcal{U} \cong \mathrm{PG}(n+1, q)$ not containing the point P. Then $H = \mathcal{U} \cap P^\perp$ is an n-space of $\mathrm{PG}(n+2, q)$ such that $\mathcal{Q}_{n,e} = H \cap \mathcal{Q}_{n+2,e}$. Let*

$$S_i = \langle P, \Sigma_i \rangle \cap \mathcal{U}, \quad i = 1, \ldots, q^{\frac{n+e+1}{2}},$$

and let $\mathcal{P} = \left\{S_1, \ldots, S_{q^{\frac{n+e+1}{2}}}\right\}$*. Then* $\mathcal{P}$ *consists of* $q^{\frac{n+e+1}{2}}$ $\left(\frac{n-e+1}{2}\right)$*-spaces of* $\mathcal{U}$*, none of them is contained in* H*.*

Proposition 3.1.11 *The set* $\mathcal{P}$ *is a hyperbolic, parabolic or elliptic avsp of* $\mathrm{PG}(n+1, q)$ *according as* $\mathcal{Q}_{n+2,e}$ *is hyperbolic, parabolic or elliptic, respectively.*

Proof. To prove that $\mathcal{P}$ is an avsp of $\mathrm{PG}(n+1,q)$ it is enough to show that for every point of $\mathcal{U} \setminus H$, there exists a member of $\mathcal{P}$ containing it. Let R be a point of $\mathcal{U} \setminus H$ and let ℓ be the line joining P and R. Since ℓ passes through P and is not contained in $P^{\perp}$, it is secant to $\mathcal{Q}_{n+2,e}$. Therefore there is a point, say R', distinct from P, such that $R' \in \mathcal{Q}_{n+2,e}$. Moreover $R' \in \Sigma_j$, for some $j \in \left\{1, \ldots, q^{\frac{n+e+1}{2}}\right\}$ and hence $R \in S_j$, by construction.

Set $\mathcal{G} = \left\{\Pi_i = S_i \cap H \mid i = 1, \ldots, q^{\frac{n+e+1}{2}}\right\}$. Then $\mathcal{G}$ consists of generators of $\mathcal{Q}_{n,e}$ disjoint from $\Sigma_{q^{\frac{n+e+1}{2}}+1} \cap H \subset \mathcal{Q}_{n,e}$. By Lemma 3.1.9, $|\Pi_i \cap \Pi_j|$, $i \neq j$, equals one, if $\mathcal{Q}_{n+2,e}$ is hyperbolic, or is at most one, otherwise.

Let $S_i, S_j \in \mathcal{P}$, with $|S_i \cap S_j| = 1$. Then $\langle S_i, S_j, P\rangle = \langle \Sigma_i, \Sigma_j\rangle$ is an $(n+2-e)$-space of $\mathrm{PG}(n+2,q)$ containing two disjoint generators of $\mathcal{Q}_{n+2,e}$. Hence $\langle S_i, S_j, P\rangle \cap \mathcal{Q}_{n+2,e} = \mathcal{Q}_{n+2-e,0} \simeq \mathcal{Q}^+(n+2-e,q)$. Such a quadric $\mathcal{Q}_{n+2-e,0}$ meets $P^{\perp}$ in a cone having as vertex the point P and as base a $\mathcal{Q}_{n-e,0} \simeq \mathcal{Q}^+(n-e,q)$. It follows that $\langle S_i, S_j\rangle \cap \mathcal{Q}_{n,e} = \mathcal{Q}^+(n-e,q)$, as required.

We have seen that a spread of a hyperbolic, elliptic, or parabolic quadric of $\mathrm{PG}(n+2,q)$ gives rise to a hyperbolic, elliptic, or parabolic avsp of $\mathrm{PG}(n+1,q)$, respectively. The converse also holds, as shown below.

3.2 AVSP iff Spread of quadrics?

Theorem 3.2.1 *If* $\mathcal{P}$ *is a hyperbolic, parabolic, or elliptic avsp of* $\mathrm{PG}(n+1,q)$*, then there is a spread of* $\mathcal{Q}_{n+2,e}$*, where* $\mathcal{Q}_{n+2,e}$ *is a hyperbolic, parabolic or elliptic quadric, respectively.*

Proof. Let $\mathcal{P} = \left\{S_1, \ldots, S_{q^{\frac{n+e+1}{2}}}\right\}$ be a hyperbolic, parabolic or elliptic avsp of $\mathrm{PG}(n+1,q)$. Then there exists a hyperplane H of $\mathrm{PG}(n+1,q)$, a quadric $\mathcal{Q}_{n,e}$ of H, that is hyperbolic, parabolic or elliptic, respectively, and a fixed generator Π of $\mathcal{Q}_{n,e}$, such that $\Pi_i = S_i \cap H$ is a generator of $\mathcal{Q}_{n,e}$ disjoint from Π. Embed $\mathrm{PG}(n+1,q)$ as a hyperplane section, say $\mathcal{U}$, of $\mathrm{PG}(n+2,q)$ and fix a quadric $\mathcal{Q}_{n+2,e}$ of $\mathrm{PG}(n+2,q)$ in such a way that $H \cap \mathcal{Q}_{n+2,e} = \mathcal{Q}_{n,e}$ and $H^{\perp}$ is a line secant to $\mathcal{Q}_{n+2,e}$. Here $\perp$ is the polarity of $\mathrm{PG}(n+2,q)$ associated with $\mathcal{Q}_{n+2,e}$. Let P be one of the two points of $\mathcal{Q}_{n+2,e}$ on $H^{\perp}$. For $i = 1, \dots, q^{\frac{n+e+1}{2}}$, consider the following $\left(\frac{n-e+3}{2}\right)$-space

$$\mathcal{F}_i = \langle P, S_i \rangle.$$

Hence $\mathcal{F}_i$ meets $P^{\perp}$ in the $\left(\frac{n-e+1}{2}\right)$-space spanned by P and Π_i, that is a generator of $\mathcal{Q}_{n+2,e}$. We claim that $\mathcal{F}_i$ contains a further generator of $\mathcal{Q}_{n+2,e}$ besides $\langle P, \Pi_i \rangle$. Indeed, if $\langle P, \Pi_i \rangle$ were the unique generator of $\mathcal{Q}_{n+2,e}$ contained in $\mathcal{F}_i$, then $\mathcal{F}_i \subset \langle P, \Pi_i \rangle^{\perp} = P^{\perp} \cap \Pi_i^{\perp} \subset P^{\perp}$, contradicting the fact that $\mathcal{F}_i \cap P^{\perp}$ is an $\left(\frac{n-e+1}{2}\right)$-space. Therefore $\mathcal{F}_i$ has to contain a further generator of $\mathcal{Q}_{n+2,e}$, say Σ_i. Denote by $\Sigma_{q^{\frac{n+e+1}{2}}+1}$ the generator of $\mathcal{Q}_{n+2,e}$ spanned by P and Π. We claim that

$$\left\{ \Sigma_1, \dots, \Sigma_{q^{\frac{n+e+1}{2}}+1} \right\}$$

is a spread of $\mathcal{Q}_{n+2,e}$. Assume by contradiction that there is a point $Q' \in \Sigma_i \cap \Sigma_{q^{\frac{n+e+1}{2}}+1}$, for some $i \in \left\{1, \dots, q^{\frac{n+e+1}{2}}\right\}$. Let $Q = \langle P, Q' \rangle \cap H$, then $Q \in \Pi_i \cap \Pi$, a contradiction. Assume by contradiction that $|\Sigma_i \cap \Sigma_j| > 0$, for some $1 \le i < j \le q^{\frac{n+e+1}{2}}$. Then necessarily $\Sigma_i \cap \Sigma_j$ is a point, otherwise $|S_i \cap S_j| > 1$. Moreover, we infer that such a point, say $R' = \Sigma_i \cap \Sigma_j$, must be in $P^{\perp}$. Indeed, if $R' \notin P^{\perp}$, then the point $R = \langle P, R' \rangle \cap \mathcal{U}$ belongs to both $S_i \setminus H$ and $S_j \setminus H$, contradicting the fact that $S_1, \dots, S_{q^{\frac{n+e+1}{2}}}$ is an avsp. In particular, $|S_i \cap S_j| = 1$.

If $e = 0$, by Remark 3.1.6, we may assume that $n \equiv 1 \pmod 4$. Hence $n+2 \equiv -1 \pmod 4$ and Σ_k and $\Sigma_{q^{\frac{n+e+1}{2}}+1}$ belong to the same system of generators of $\mathcal{Q}_{n+2,0}$, for $k = 1, \dots, q^{\frac{n+e+1}{2}}$. It follows that if $|\Sigma_i \cap \Sigma_j| > 0$, then they have at least a line in common, a contradiction.

If $e \in \{1, 2\}$, observe that $\langle \mathcal{F}_i, \mathcal{F}_j \rangle$ is a $\mathrm{PG}(n+2-e, q)$ containing the cone having as vertex the point P and as base the hyperbolic quadric $\mathcal{Q}^{+}(n-e, q) = \langle S_i, S_j \rangle \cap \mathcal{Q}_{n,e}$. Furthermore, two more generators of $\mathcal{Q}_{n+2,e}$, namely Σ_i, Σ_j, are contained in $\langle \mathcal{F}_i, \mathcal{F}_j \rangle$ and do not pass through P. Therefore necessarily we have that $\langle \mathcal{F}_i, \mathcal{F}_j \rangle \cap \mathcal{Q}_{n+2,e} = \mathcal{Q}_{n+2-e,0}$. Let us denote by $\mathcal{Q}^{+}(n+2-e, q)$ the hyperbolic quadric $\langle \mathcal{F}_i, \mathcal{F}_j \rangle \cap \mathcal{Q}_{n+2,e}$, so that generators of $\mathcal{Q}^{+}(n+2-e, q)$ are $\left(\frac{n-e+1}{2}\right)$-spaces. Since $\Sigma_i \cap \langle P, \Pi_i \rangle$ and

$\Sigma_j \cap \langle P, \Pi_j \rangle$ are $\left(\frac{n-e-1}{2}\right)$-spaces, we have that Σ_i and $\langle P, \Pi_i \rangle$ lie in distinct systems of generators of $\mathcal{Q}^+(n+2-e, q)$. Similarly for Σ_j and $\langle P, \Pi_j \rangle$. Two possibilities arise: either $n+2-e \equiv -1 \pmod 4$ or $n+2-e \equiv 1 \pmod 4$. Since $\langle P, \Pi_i \rangle \cap \langle P, \Pi_j \rangle$ is a line, if the former case occurs, then $\langle P, \Pi_i \rangle$, $\langle P, \Pi_j \rangle$ belong to the same system of generators of $\mathcal{Q}^+(n+2-e, q)$. Hence Σ_i, Σ_j belong to the same system of generators of $\mathcal{Q}^+(n+2-e, q)$ and if $|\Sigma_i \cap \Sigma_j| > 0$, then they have at least a line in common, a contradiction. In the latter case, $\langle P, \Pi_i \rangle$, $\langle P, \Pi_j \rangle$ are in different systems of generators of $\mathcal{Q}^+(n+2-e, q)$. Therefore Σ_i, $\langle P, \Pi_j \rangle$ belong to the same system of generators of $\mathcal{Q}^+(n+2-e, q)$. Similarly for Σ_j, $\langle P, \Pi_i \rangle$. It follows that Σ_i, Σ_j belong to different systems of generators of $\mathcal{Q}^+(n+2-e, q)$ and again if $|\Sigma_i \cap \Sigma_j| > 0$, then they have at least a line in common, a contradiction.

Following the proof of Theorem 3.2.1, it can be observed that for any hyperbolic, parabolic, or elliptic affine spread $\mathcal{P}$, it is possible to construct a spread $\mathcal{S}$ of a hyperbolic, parabolic, or elliptic quadric, respectively, such that if we apply Construction 3.1.10 starting from $\mathcal{S}$, we obtain $\mathcal{P}$. As pointed out in (Bamberg et al. 2023, Section 5.2, Theorem 2), tightness and irreducibility follow immediately for a hyperbolic avsp of $\mathrm{PG}(5, q)$. We will show that the same occurs in higher dimensions and for parabolic, elliptic, or hyperbolic avsp.

Lemma 3.2.2 *Let $e \in \{1, 2\}$, let $\mathcal{S} = \left\{\Sigma_1, \ldots, \Sigma_{q^{\frac{n+e+1}{2}}+1}\right\}$ be a spread of $\mathcal{Q}_{n+2,e}$. Then at most $q+1$ members of $\mathcal{S}$ are contained in a quadric $\mathcal{Q}_{n+1,e-1} \subset \mathcal{Q}_{n+2,e}$.*

Proof. If $e = 1$, then exactly 2 members of $\mathcal{S}$ are contained in a hyperbolic hyperplane section $\mathcal{Q}_{n+1,0} \simeq \mathcal{Q}^+(n+1, q)$ of $\mathcal{Q}_{n+2,1} \simeq \mathcal{Q}(n+2, q)$, whereas, if $e = 2$, then there are $q+1$ elements of $\mathcal{S}$ contained in a parabolic hyperplane section $\mathcal{Q}_{n+1,1} \simeq \mathcal{Q}(n+1, q)$ of $\mathcal{Q}_{n+2,2} \simeq \mathcal{Q}^-(n+2, q)$.

3.3 Tightness and Irreducibility

Proposition 3.3.1 *Let $\mathcal{P}$ be a hyperbolic, parabolic, or elliptic avsp of* $\mathrm{PG}(n+1, q)$, *then $\mathcal{P}$ is tight and irreducible.*

Proof. Let $\mathcal{P} = \left\{S_1, S_2, \ldots, S_{q^{\frac{n+e+1}{2}}}\right\}$ be a hyperbolic, parabolic or elliptic avsp of $\mathrm{PG}(n+1, q)$. If $e = 0$, to prove that $\mathcal{P}$ is irreducible, it is enough to observe that, if

$i \neq j$, then the span of S_i and S_j is the whole $\mathrm{PG}(n+1,q)$. Similarly, if $e=1$ and $|S_i \cap S_j| = 0$, then $\langle S_i, S_j\rangle = \mathrm{PG}(n+1,q)$. If either $e=1$ and $|S_i \cap S_j| = 1$ or $e=2$, we claim that the elements of $\mathcal{P}$ contained in $\langle S_i, S_j\rangle$, where $i \neq j$, do not cover all the points of $\langle S_i, S_j\rangle \setminus H$. By Theorem 3.2.1, there is a quadric $\mathcal{Q}_{n+2,e}$ with a spread $\mathcal{S} = \left\{\Sigma_1, \ldots, \Sigma_{q^{\frac{n+e+1}{2}}+1}\right\}$ such that $\mathcal{P}$ can be obtained via Construction 3.1.10. Then the number of elements of $\mathcal{P}$ contained in $\langle S_i, S_j\rangle$ equals the number of elements of $\mathcal{S}$ contained in $\langle P, \Sigma_i, \Sigma_j\rangle$. Since $\langle P, \Sigma_i, \Sigma_j\rangle \cap \mathcal{Q}_{n+2,e}$ is either a hyperbolic quadric $\mathcal{Q}^+(n+2-e,q)$, if $|S_i \cap S_j| = 1$, or a parabolic quadric $\mathcal{Q}(n+1,q)$, if $|S_i \cap S_j| = 0$ and $e=2$, such a number cannot exceed $q+1$ by Lemma 3.2.2. It follows that $\mathcal{P}$ is irreducible.

By Lemma 3.1.9 *(iii)*, through a point of H there pass precisely q members $\mathcal{P}$. Hence tightness follows.

3.4 Conclusion

We have seen that a hyperbolic, parabolic, or elliptic affine spread of $\mathrm{PG}(n,q)$ is equivalent to a spread of $\mathcal{Q}^+(n+1,q)$, $\mathcal{Q}(n+1,q)$ or $\mathcal{Q}^-(n+1,q)$, respectively. Furthermore, such an affine spread is tight and irreducible. Based on Proposition 3.1.11 and Theorem 3.2.1, the extension problem formulated in (Bamberg et al. 2023, Conjecture 2) is equivalent to that of the existence of a spread of the triality quadric $\mathcal{Q}^+(7,q)$, which in turn is equivalent to that of the existence of an ovoid of $\mathcal{Q}^+(7,q)$. We remark that the existence of a spread of $\mathcal{Q}^+(7,q)$ has been established in the cases when q is even or when q is odd, with q a prime or $q \equiv 0$ or $2 \pmod 3$. Therefore, a positive answer to the extension problem follows in the cases when q is even or when q is odd, with q a prime or $q \equiv 0$ or $2 \pmod 3$.

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