Scuola Superiore Meridionale
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Doctoral Thesis
in Cosmology, Space Science & Space Technology

Analytical Approaches for the Relativistic Two-Body Scattering

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A Nonna T e Nonno Renzo

Abstract

Detection of scattering of binary black holes is incredibly hard, hence the main focus of the community has been for several years the characterization of gravitational wave signals from binaries in circular or eccentric motion. However by leveraging the relation that is known under the name of bound-to-unbound mapping, it should be *in principle* possible to get information about bound observables from the unbound ones at very high order in perturbation theory, potentially informing numerical simulations and/or waveform generators. This is (one of) the main motivations behind the interest in studying the scattering of binary black holes.

Scattering can be investigated analytically through the post-Minkwoskian (PM) approximation: a weak field approximation that computes corrections around the flat, Minkowski, spacetime. This is the natural ground for modeling an hyperbolic encounter that, in this approximation, could be seen as deviation from the straight line motion in presence of a black hole. An alternative (and independent) expansion, the post-Newtonian (PN) approximation, has been used historically to investigate bound orbits. It can be naively defined by restricting the attention to binary systems, where the relative velocity is small relative to the speed of light. Because of the virial theorem in bound systems, a small relative velocity is immediately translated in a large relative distance approximation. This means that, for bound systems, the PM and PN approximations are degenerate while, in a scattering scenario, the two expansions can be performed independently.

Together with these approaches, a third perturbative scheme can be used to obtain analytical results: Self Force. This approximation contains information that are exact in both PN and PM sense, and it has been used in the past to get analytical expressions up to incredibly high orders in PN for bound orbits.

In this Thesis, it will be discussed how to obtain analytically scattering observables by using Self Force techniques, generalizing the usual procedure to the case of a generic, equatorial orbit. These methodologies will be applied to study the scattering of a scalar charge off a Schwarzschild black hole. This methodology will also be applied to compute the fluxes and the waveform for a gravitational perturbation.

A parallel project, focused on scattering, will also be discussed at the end of this Thesis: it will be presented how to compute the Quasi-Keplerian parameters for parametrizing the hyperbolic orbit in Scalar Tensor theories.

Acknowledgements

It is fair to say that this section was not an easy one. Many people deserve a place in these acknowledgments; I am sure I will forget some of them, so, before starting, I apologize to anyone I have not included below.

First of all, I want to thank the person who made all of this possible: Prof. Salvatore Capozziello, with whom I began my journey at the Scuola Superiore Meridionale and who later allowed me to change direction so that I could find my own research line. I would also like to thank Prof. Lorenzo Fatibene and Prof. Matteo Luca Ruggiero, who gave me the possibility to spend part of the past year at the Department of Mathematics in Turin when I wasn't traveling around for research visits.

I am indebted to Donato Bini, who introduced me to my present research topic, encouraged me to spend an extended period at UCD, and helped me find the best postdoc position I could have hoped for. Even if you say you only knocked at doors and I was the one who opened them, it is also true that, without you, I would not have known where to knock (and perhaps not how to open the door either). Andrea Geralico was another essential part of this journey: you were always available and, even when I made the silliest mistake in my calculations, you helped me fix it. Without your help I would still be stuck on an error in the energy loss—thank you. This path would also have been impossible without Chris Kavanagh, who welcomed me to UCD for six months without hesitation, helped me finish my work, and taught me many new things. Every time I brought you a weird idea or puzzling result, you helped me and showed how to make it concrete. You also made me appreciate distribution theory and complex analysis—no small feat.

These acknowledgments would be incomplete without thanking everyone at UCD: professors, PhD students, and postdocs. Beyond the science, the most valuable lesson I learned there is that students learn by osmosis—simply being present and listening to conversations about self-force and black holes was far more valuable than trying to learn everything on my own. Another lesson I learned is: do not try to imitate an Irish person in a pub—though I am still not sure which lesson is more valuable. The Southampton group also deserves mention, in particular Adam and David, who taught me different aspects of self-force and PN theory.

Now that the academic section is done, I can thank the people who were actually with me throughout this journey. First, the “Dottorandi Bravi”: Emanuela, Alessio, Manuel, Toberone and Nicola, who were with me from day -1, from the moment I sent that first email to start getting to know each other. I could not have imagined that that message would create such friendships, and I am glad to have shared this experience with you all. Jakob, you also deserve a place here: one of the things I miss most about Dublin is waiting for you at the office on Friday morning to exchange looks and discuss the latest One Piece chapter. Thanks for your code$-$it made my life much easier$-$but more importantly, thank you for the physics discussions. I must also thank Riccardo. You are the mind for sure; I can try to be the arm, or alternatively the one who challenges all your ideas. I

remain amazed that you always manage to convince me that something is possible. I cannot wait for that fish dinner you promised.

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Per concludere: anche se niente di ciò che ho fatto sarebbe stato possibile senza le persone elencate sopra, è indubbio che senza mamma e papà non potrei essere qui. Sono stati loro a far germogliare il mio interesse per la scienza e sono sempre stati i primi a farmelo coltivare, anche se significava svegliarsi presto per andare a qualche fiera della scienza o a un museo, anche quando fuori diluviava e — guarda un po' — la fiera si svolgeva dall'altra parte della città. Forse all'inizio non era vostra intenzione portarmi fino a qui, non so quanto aveste immaginato tutto questo, però non sarei quello che sono senza quel semino di curiosità che mi avete dato. Non esiste un modo per ringraziarvi pienamente; di questo ne sono certo.

Infine, questa tesi è dedicata a Nonna T e Nonno Renzo, perché mi hanno visto iniziare questo percorso ma non mi hanno visto finirlo. Ci sono tante cose che avrei sperato di farvi ancora vedere, ma purtroppo così va la vita. Mi mancate.

Publication List

This thesis is based on the following publications:

[157] D. Usseglio, D. Trestini, and A. Chowdhuri, *Quasi-Keplerian parametrization for compact binaries on hyperbolic orbits in scalar-tensor theories at second post-Newtonian order*, Phys. Rev. D **112**, 044048 (2025), arXiv:2504.13829 [gr-qc] .

[100] D. Bini, A. Geralico, C. Kavanagh, A. Pound, and D. Usseglio, *Post-Minkowskian self-force in the low-velocity limit: Scalar field scattering*, Phys. Rev. D **110**, 064050 (2024), arXiv:2406.15878 [gr-qc]

In addition to these works, original material that it is planned to appear in subsequent publications will be presented in this thesis.

Additional publications which are not included in this thesis are listed below:

[3] S. Capozziello, N. Menadeo, and D. Usseglio, *Analytical self-force on scalar particle in eccentric orbits around a Schwarzschild black hole* (2025), arXiv:2506.01693 [gr-qc] .

[4] V. De Falco, E. Battista, D. Usseglio, and S. Capozziello, *Radiative losses and radiation-reaction effects at the first post-Newtonian order in Einstein–Cartan theory*, Eur. Phys. J. C **84**, 137 (2024), arXiv:2401.13374 [gr-qc] .

[5] F. Chiaffredo, L. Fatibene, M. Ferraris, E. Ricossa, and D. Usseglio, *A variational framework for higher order perturbations*, Int. J. Geom. Meth. Mod. Phys. **21**, 2440007 (2024), arXiv:2310.12907 [math-ph] .

[6] E. Battista, V. De Falco, and D. Usseglio, *First post-Newtonian N-body problem in Einstein–Cartan theory with the Weyssenhoff fluid: Lagrangian and first integrals*, Eur. Phys. J. C **83**, 112 (2023), arXiv:2301.08954 [gr-qc]

Contents

1 Introduction

1.1 Binary Systems as Sources of Gravitational Waves

The first direct detection of gravitational waves (GWs) in 2015 [1] represented a pivotal milestone in astrophysics, highlighting the importance of accurately modeling their astrophysical sources: binary systems. The most significant sources observed so far are *binary black hole mergers*. However, detections have also been made from systems involving a neutron star–black hole binary, as well as from binary neutron star mergers [2–4].

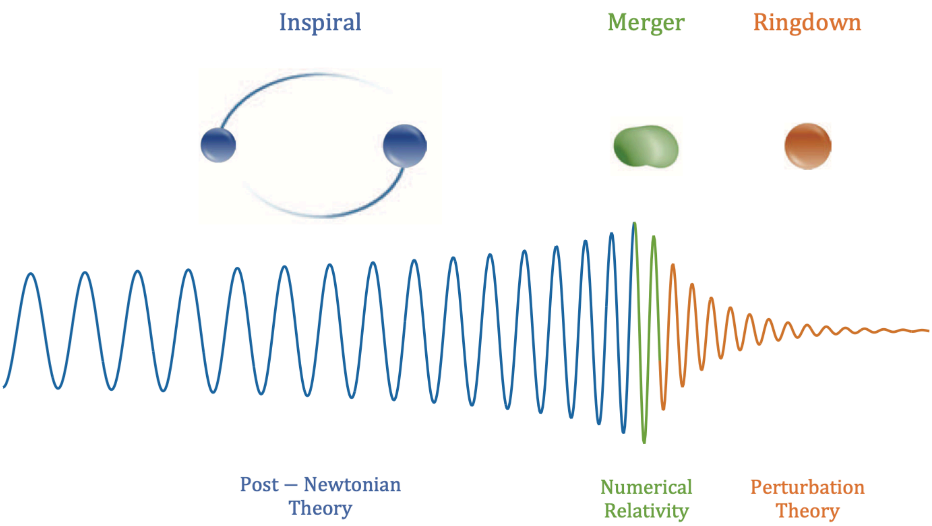

Figure 1.1: Image taken from [5]. This figure illustrates the three phases that characterize the orbital evolution of a binary system, together with the techniques most commonly used to model them.

As GWs propagate outward and reach our detectors, they carry away energy and angular momentum, which induces a back-reaction on the binary system. This reaction alters the orbital dynamics and drives the inspiral process. This sequence is illustrated in Fig. 1.1. The initial and longest portion of the signal corresponds to the inspiral phase, during which the orbit decays gradually over thousands of cycles. As the two objects approach, their velocities increase, leading to stronger gravitational radiation. This enhanced radiation further accelerates the *inspiral*, creating a feedback mechanism that results in a steadily growing waveform amplitude. Eventually, the two bodies *merge*, producing a single, highly perturbed remnant. This remnant emits a final burst of gravitational radiation as it settles into equilibrium. The corresponding waveform segment is known as the *ringdown*, and is dominated by the excitation of quasi-normal modes, whose spectrum depends solely on the mass and spin of the final object [6–9].

A particularly interesting class of binaries is that of Extreme-Mass-Ratio Inspirals (EMRIs), where the mass ratio between the two objects is very small, $m/M \ll 1$. Despite the weak gravitational influence of the smaller companion, EMRIs generate highly intricate waveforms even during the inspiral phase, due to the complex orbital dynamics near the massive body. EMRIs are among the key scientific targets of the future LISA mission [10, 11]. An example of a waveform produced by this class of systems is shown in Fig. 1.2. In these particular snapshots the following orbital parameters have been chosen: $\{M, m, a, p_0, e_0, d_L\} = \{10^5 M_\odot, 30M_\odot, -0.998, 28.3, 0.85, 1\text{Gpc}\}$, where m_1 and m_2 are respectively the mass of the large and small black hole, a is the Kerr parameter for the angular momentum of the large black hole and (p_0, e_0) are the semilatus rectum and the eccentricity at $t = 0$, where the p_0 is adimensional and defined with respect to the larger mass M. The parameter d_L represents the luminosity distance between the system and the detector which detects the waveform. It is easy to see that the signal is strongly variable during time. This is shown in Fig. 1.3 where the geodesic that produces the signal in the early phases is shown in red. Because of the large eccentricity, the small body spend a lot of times away from the central black hole, which explains the separation between peaks in the left panel of Fig. 1.2. During the evolution, the system emits GWs, leading to a gradual change in the orbital parameters. The orbits tend to circularize, and the waveform acquires a structure that is markedly different from the initial one.[1]

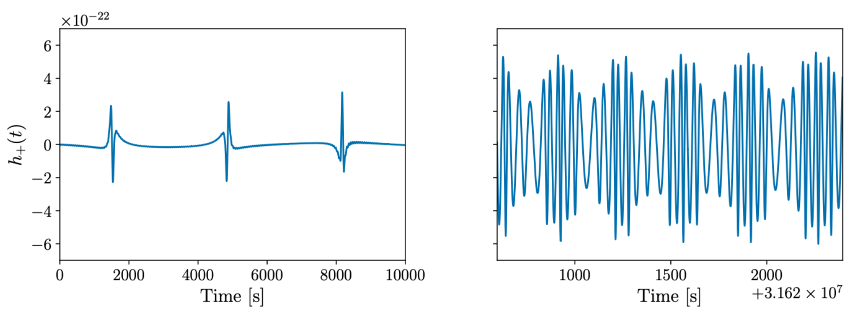

Figure 1.2: Image taken from [12]. Waveform for a retrograde eccentric EMRI into a spinning MBH in the time domain: early and late times in left and right panels, respectively. The orbital parameters are $\{M, m, a, p_0, e_0, d_L\} = \{10^5 M_\odot, 30M_\odot, -0.998, 28.3, 0.85, 1\text{Gpc}\}$.

Because of the intrinsic complexity in the signals, to enable the detection and analysis of GW signals, it is crucial to develop extensive template catalogs that densely cover the parameter space of possible sources. In this endeavor, analytical techniques play a central role. Not only do they provide independent results for validating numerical simulations, but they also facilitate the matching between waveform segments computed in different regimes — such as inspiral, merger, and ringdown — using different methods.

[1]Notably, EMRIs do not always circularize: the eccentricity reduces but, in some specific orbital configurations, the eccentricity can have a sudden increase.

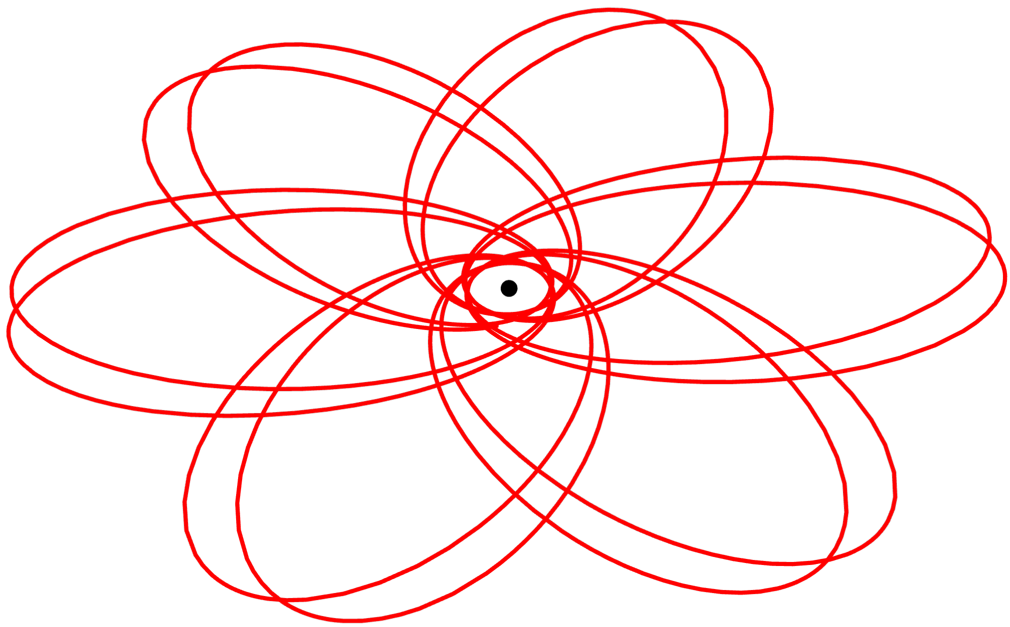

Figure 1.3: Image produced with the Black Hole Perturbation Toolkit [13]. Geodesic with the orbital parameters used in Fig. 1.2.

1.2 Analytical techniques: what, how, why?

Analytical methods have proven extremely powerful in modeling gravitational waveforms across the inspiral–merger–ringdown (IMR) phases. A range of perturbative techniques has been developed to describe binary dynamics in specific physical regimes that could be identified by dissecting Fig. 1.1.

During the early inspiral, the bodies are widely separated. From the virial theorem, large separation implies that the relative velocity is small, enabling a post-Newtonian (PN) expansion [14], organized in powers of $v/c \ll 1$. This approximation yields an accurate analytical description of the system's dynamics in the weak-field, slow-motion regime. However, as the orbit shrinks and the system evolves over many cycles, relativistic effects become more significant, reducing the PN approximation's accuracy [15–17]. In this late stage of the evolution, Numerical Relativity (NR) becomes essential to accurately model the dynamics and gravitational wave emission [18–20], especially near the merger, where nonlinear strong field effects dominate.

To construct a complete waveform, one must match the PN description of the early inspiral with the NR results near and after the merger. This can be achieved with different models, for instance the Effective-One-Body (EOB) formalism [21–23] or the Phenom family of models [24]. In particular, EOB provides a nice map between a two-body problem and the motion of a single effective particle in a deformed background. This framework bridges the gap between analytical and numerical models, extending analytical methods up to merger and ringdown. On the other hand, hybrid waveforms, such as the one produced with phenomenogical models, can be constructed by directly matching PN and NR results. This list is not intended to be exhaustive, a complete description of the various models that are used for generating a complete waveform can be found in [10], with a strong focus on the LISA mission, or in [25] where some discussions on the open challenges in waveform modeling are presented, in relation with a future ground-based detector, the Einstein Telescope.

Beyond PN and EOB, other perturbative techniques are particularly useful in different physical regimes. Among these, two stand out: the Gravitational Self-Force (GSF) formalism [26–28] and the post-Minkowskian (PM) expansion [29–32].

The GSF approach does not assume small velocities or large separations, but instead requires a hierarchy between the masses of the binary components: a primary black hole of mass M, and a secondary compact object of mass $m \ll M$. This makes it ideally suited for modeling EMRIs. Since the expansion parameter is the mass ratio itself, GSF techniques contain exact information about the orbital dynamics with respect to distance and velocity. This feature has enabled the calculation of several gauge-invariant observables to very high PN orders, for both circular, eccentric and recently also spherical orbits, in Schwarzschild [33–37] and Kerr spacetimes [38–42].

In contrast, the PM expansion assumes weak gravitational fields, i.e., an expansion in powers of $G \ll 1$, while making no assumptions on the velocity. As a result, it is well suited for describing high-speed interactions, in particular unbound scattering events. In fact, while the PN expansion is naturally tied to bound systems and slow motion, PM is the natural choice for computing observables in gravitational scattering scenarios. A more detailed discussion on PM and scattering is given in the next Sections.

Finally, it is crucial to highlight that the dynamics of a binary system are governed not only by dissipative effects (e.g., GW emission), but also — and predominantly — by conservative dynamics. The leading-order corrections to Newtonian motion are conservative in nature, as exemplified by the famous calculation of Mercury's perihelion precession by Einstein. The accurate modeling of the conservative sector is essential, as it forms the core input for both analytical and numerical frameworks used to calculate dissipative effects. Therefore, much of the analytical effort is devoted not directly to waveform generation, but to computing *gauge-invariant observables* that serve as reliable inputs for waveform construction.

1.3 Self Force

It is worthwhile to devote a few more words to the concept of Self Force, since it constitutes one of the two main topics of this Thesis. As already mentioned briefly, EMRI systems can be naturally studied within the Self Force formalism, which essentially describes perturbations of the black hole spacetime induced by a small massive body, modeled as a point particle.

The goal of the Self Force approach is to investigate perturbatively the motion of a light body around a massive black hole, using the mass ratio between the two objects as the expansion parameter.

At zeroth order, only geodesic motion is considered: the large black hole determines the background geometry in which the small body moves, with geodesics providing the natural generalization of straight lines in curved spacetime. Deviations from this motion arise at the first subleading order, where the particle's trajectory is influenced not only by the central black hole but also by its own gravitational field. These corrections, which become increasingly intricate at higher orders due to nonlinear interactions of the field with the particle and with itself, encode the imprint of gravitational-wave emission. To model the resulting *radiation-reaction* effects on the orbit of an EMRI, the gravitational field of the small object is treated as a perturbation of the background geometry, obtained by solving a linearized version of Einstein's equations. This perturbation produces an effective acceleration that displaces the particle from its original geodesic, an effect identified as the *gravitational Self Force*. Within this framework, the inspiral of a binary system emitting GWs is ultimately driven by the action of the Self Force.

The point-particle approximation, however, presents a conceptual limitation: for gravitating sources in GR it is not strictly valid, since the field of the small object cannot be exactly represented as that of a point particle. This issue can be fully understood using a rigorous derivation involving matched asymptotic expansions, from which the point-particle result emerges as a particular limit. However, for the purpose of this Thesis, such detail is not relevant and the point-particle approximation will suffice.

At first order in Self-Force, the solution to the linearized Einstein equations can be written in terms of the retarded Green's function, which splits into a singular and a regular piece, usually referred to as the Detweiler–Whiting decomposition [43]. The singular field diverges as $\sim 1/r$, with r measuring the distance to the particle's worldline, while the regular field remains smooth. This decomposition is crucial to extract meaningful information and compute the deviation from geodesic motion due to the Self Force. The divergent part can be obtained by performing a near-particle expansion, as discussed in [44, 45]. Subtracting it from the retarded field yields the regular part of the field.

This subtraction scheme is implemented through the *mode-sum regularization* method. Instead of removing the singular field directly from the retarded solution, the procedure is carried out at the level of the multipolar modes. There are several reasons why working mode by mode is more convenient. First, the differential equations to be solved are already projected onto a basis of spherical (or spheroidal) harmonics. Second, at first order in Self Force, the source is modeled by a point particle described by a Dirac delta distribution localized at the particle's position. This distribution can be decomposed in spherical harmonics, and each (l, m) mode turns out to be finite; the divergence only arises from the infinite sum over all modes. The same property holds for the singular field: its individual harmonic modes are finite. Therefore, subtracting the singular contribution from each mode of the retarded field involves only finite quantities.

Once the subtraction is performed at the mode level, the infinite sum over (l, m) can be carried out consistently, yielding the *regularized field*, which can then be used as the fundamental input to compute Self Force corrections to geodesic motion.

1.4 The Scattering Problem

The discussions in previous Sections have been primarily focused on bound systems. In this Section, the scattering problem will be introduced, since it will be the central theme of this Thesis, with a particular emphasis on scattering in the context of EMRIs. An illustration of the orbital configuration for a scattering event is shown in Fig. 1.4.

Unlike GWs emission from bound systems, hyperbolic encounters lack a neat separation between regimes such as inspiral, merger and ringdown. In a scattering event there is an incoming massive object from the past infinity, characterized by its asymptotic energy E and orbital angular momentum L, that approaches the primary black hole up to the minimum approach distance $r_{\min}$, then escapes

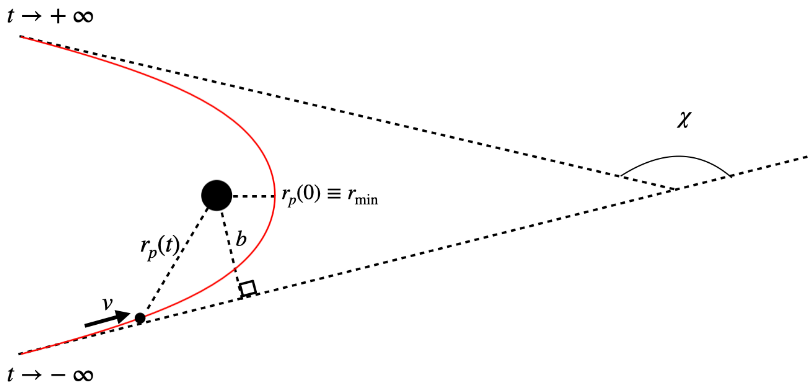

Figure 1.4: A sketch of a hyperbolic encounter. The small particle is approaching from the infinite past at a certain velocity v. The particle is on a geodesic with the radial motion expressed by the quantity $r_p(t)$ parametrized with respect to the coordinate time t. It reaches the impact parameter b and scatters back to infinity. The difference between the incoming and outgoing direction is parametrized by the scattering angle χ.

to future infinity. This is the simplest case of a scattering event, where the "endpoints" are well-defined. It is important to point out that, as for the bound case, the system can be completely modeled by fixing two orbital parameters, as the semilatus rectum and eccentricity, or energy and angular momentum. In addition, in a scattering scenario, also the impact parameter b and the velocity at the infinite past v could be used to uniquely characterize the motion.

The scattering problem has also been investigated for systems that are not EMRIs by NR and the EOB formalism [46–53] and, for EMRIs, by GSF techniques (though mostly limited to scalar charges for now) [54–58].

Interestingly, if sufficient energy and angular momentum are radiated away, the trajectory can transition from scattering to a dynamical capture [59, 60], resulting in a merger and ringdown. This scenario has been applied to explain a particular GWs detection in [61].

From an observational viewpoint, detecting GWs from scattering events is extremely challenging due to their short, non-periodic signals. In Figs. 1.5 and 1.6 the two polarizations of the GW signal are presented for an hyperbolic scattering where $(b, v) = (50M, 0.3c)$. It is clear from both images that the sudden burst of radiation corresponds to nera $t = 0$, i.e. when the incoming particle is close to $r_{\min}$. Even if the event is confined on the equatorial plane $\theta = \pi/2$, the signal is emitted spherically and hence, different sets of angles (θ, ϕ) can be selected in order to plot the waveform. Notably, for some specific choices of the angles, for both h_+ and $h_\times$, the asymptotic values of the waveforms for $t \to \pm\infty$ are different. This is due to the gravitational *linear memory* effects, see [62, 63]. These images have been produced by using the techniques described in Chapter 5 for the $l = 2$ multipole and selecting different sets of θ, ϕ in order to highlight different features of the two polarizations. Besides $G = c = 1$, the expressions plotted are in units of $M\nu$, where $\nu = mM/(m + M)^2$ is the symmetric mass ratio of the binary and m is the mass of the incoming particle.

The non-zero values of the waveform at past and future infinity is a well known issue, related to the choice of the Bondi-Metzner-Sachs (BMS) frame, see [64], and there are various techniques for solving this issue, from both analytical and numerical perspective [65, 66].

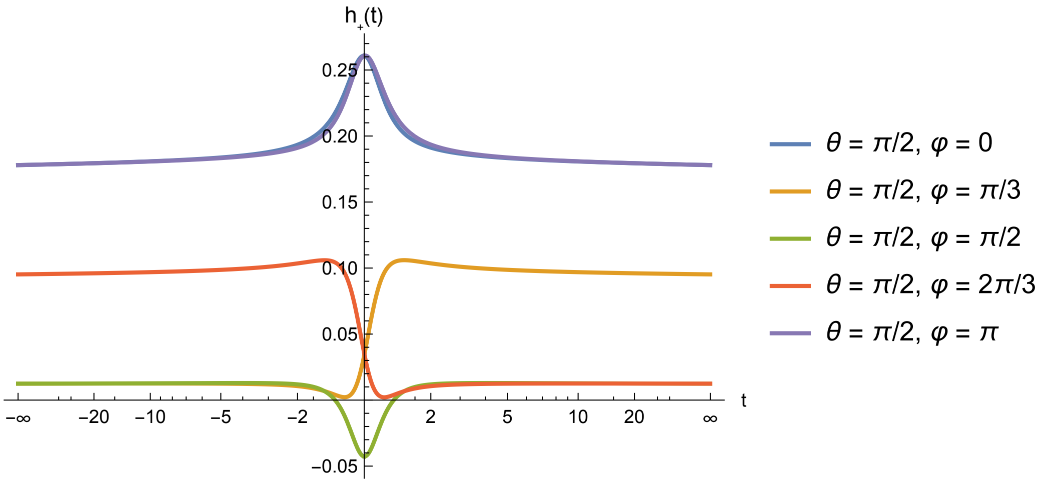

Figure 1.5: The "plus" polarization of a gravitational wave emitted by a scattering event obtained analytically from the techniques of Chap. 5.

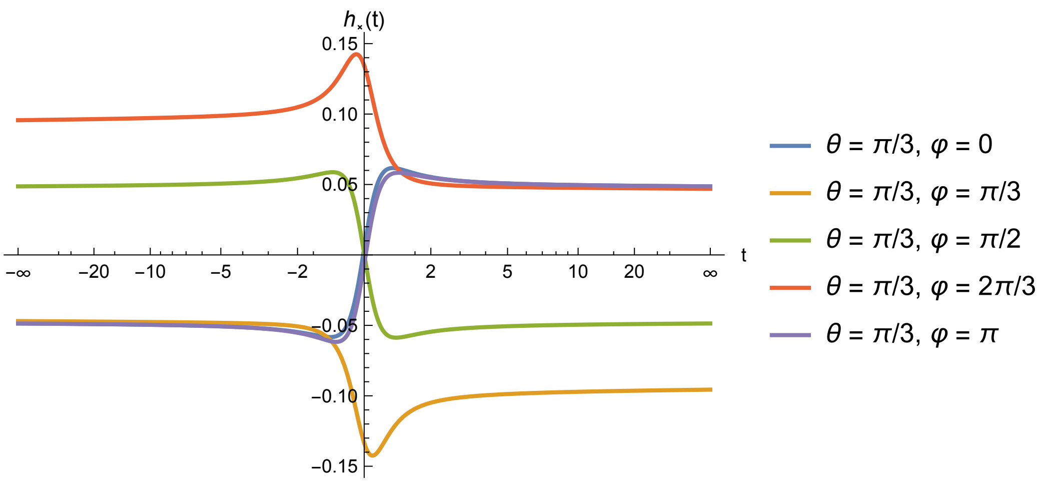

Figure 1.6: The "cross" polarization of a gravitational wave emitted by a scattering event obtained analytically from the techniques of Chap. 5. Notably, when $\theta = \pi/2$, $h_\times$ is identically zero for any value of ϕ.

Despite observational difficulties, scattering processes offer deep theoretical insights. Notably, analytical results for hyperbolic scattering can inform our understanding of bound motion via the boundary-to-bound (B2B) map, see [67–73]. In this mapping, it is possible to relate gauge invariant observables computed in scattering with observables for bound motion, for instance the scattering angle and the periastron advance, but it is also possible to directly compute bound waveforms from scattering amplitudes.

In order to use the B2B map, it is however necessary to compute observables at high order in perturbation theory. In this view, the largest portion of analytical results obtained for hyperbolic encounters have been derived mainly via EOB Hamiltonians [74–80] and using modern effective field theory and scattering amplitude methods from particle physics [31, 32, 81–90].

In particular, by using world-line quantum field theory techniques (WQFT) [91], it was possible to reach an almost complete analytical knowledge of the 5PM order [92, 93] for both the conservative and dissipative sector, with resummed information on the velocity, i.e. to all order in the PN expansion, that was also recently used for checking results obtained in several works, such as [75, 76, 94–96], and also getting new analytical information, see [97].

Despite the large number of results, it should also be pointed out that there are several open problems. The most pressing one is indeed related to the B2B mapping which, at the present time, does not work for a particular sector of the dynamics, called *non-local*. This sector contains information about the whole past history of the gravitational source. At the present time, there is no B2B prescription that works for the full dynamics. However, there are strong suggestions that this prescription should exist, in particular from the recent work [98].

The second problem is that the calculations are unbelievably hard. The 5PM results mentioned above reached the present computational threshold of the community: by using the WQFT approach, it was necessary to compute 363 Feynman diagram for the conservative and 426 in the dissipative sector, in order to reach 5PM accuracy at the first order in the Self Force expansion. This number of diagrams required then 300k core hours of computations for each sector only for *setting up* the problem. This represents a very stringent computational challenge. Together with it, also the presence of new functions that were never observed in High Energy Physics computations, see [93], pose the question if these techniques can be pushed to higher precision.

Some preliminary works in SF are now trying to obtain analytical information at very high PM-PN orders by using a Self Force expansion on top, for instance [64, 99–101]. The discussion about these techniques is postponed and it will be the core of this Thesis.

1.5 Organization of the Thesis

The main goal of this Thesis is to generalize techniques from the GSF program to include sources on hyperbolic orbits. The general formalism presented in Chapter 2 and 3 is valid in the general case of a rotating black hole. However, the applications will be of the successive Chapter will restrict the attention to the non-rotating case, which is also used in the first Chapters to highlight the analytical features of the derived expressions, which appears less convoluted in the non-rotating case and hence, more explicative. The Chapters are organized as follows.

In Chapter 2 it will be provided a short review on Black Hole Perturbation Theory, how to formulate the SF problem via the Teukolsky formalism and how to solve the homogeneous problem.

In Chapter 3 it will be presented how to solve the inhomogeneous problem with a source that is in hyperbolic motion. It order to do so, the formalism adopted in previous analytical SF calculations will be extended in order to incorporate an orbit which is in a generic motion on the equatorial plane.

Finally, in Chapter 4 the formal results obtained in the previous Chapters will be adapted to treat the case of a scalar field that scatters off a Schwarzschild black hole, by providing explicit results and analytical results expressed by combining post-Newtonian and post-Minkowskian expansion. The results presented will be valid up to 5PM and 4.5PN.

In Chapter 5 the focus will discuss how to compute gravitational fluxes and waveforms for an hyperbolic scattering event.

In conclusion, in Chapter 6 an independent computation of the observables for a scattering event in the context of Scalar-Tensor theories will be presented. This will provide another way of investigating a scattering event but in another framework where also deviations from GR will be included. These calculations are a first step in investigating analytically scattering also in the context of modified gravity. In addition, it is notable that Scalar-Tensor theories can also be used to model systems where the binary is surrounded by an environment, such as dust, see [102, 103].

2 Introduction to Black Hole Perturbation Theory and Self Force

The main topic of this Section concerns perturbations of black hole spacetimes; see [28, 104] for more detailed discussions. This Chapter aims to first introduce the general strategy for formulating the differential equation that governs the evolution of perturbations in General Relativity, namely the Teukolsky equation. Subsequently, it presents how to solve the corresponding homogeneous equation analytically, through perturbative expansions that yield closed-form expressions up to a given order in perturbation theory. Unless otherwise stated, this Chapter adopts $G = c = 1$.

2.1 Black Hole Perturbation Theory

The starting point of the perturbative approach consists of considering the full set of Einstein's field equations (EFEs):

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \,, \tag{2.1}$$

where $G_{\mu\nu}$ is the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \,, \tag{2.2}$$

which can be naively interpreted as a function of the second derivatives of the metric tensor $g_{\mu\nu}$, or its inverse $g^{\mu\nu}$, with respect to the spacetime coordinates x^μ. $R_{\mu\nu}$ and R are respectively the Ricci tensor, constructed from the index contraction of the Riemann tensor $R^{\alpha}{}_{\mu\alpha\nu} = R_{\mu\nu}$, and the Ricci scalar $R^{\alpha}{}_{\mu\alpha\nu} g^{\mu\nu} = R$. On the right hand side, $T_{\mu\nu}$ represents the stress-energy tensor of matter.

A solution to Eq. (2.1) can be written as a perturbative expansion in terms of a small parameter ϵ,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu} + O(\epsilon^3) \,, \tag{2.3}$$

where $\bar{g}_{\mu\nu}$ is the background solution. Similarly, the stress-energy tensor can be expanded in terms of the same parameter

$$T_{\mu\nu} = \epsilon T^{(1)}_{\mu\nu} + \epsilon^2 T^{(2)}_{\mu\nu} + O(\epsilon^3) \,, \tag{2.4}$$

under the assumption that the zeroth-order term vanishes. The reason behind this assumption is that the background metric $\bar{g}_{\mu\nu}$ should be the one of a isolated black hole, i.e. no matter distribution around the black hole. Perturbative solutions of EFEs are not always valid and the small parameter ϵ should be properly defined to permit a perturbative expansion of the metric. As an example, ϵ could be related with the Newtonian gravitational potential M/r, then perturbative solutions can be assumed in the weak field limit, when $M/r << 1$. On the opposite region of the parameter space, where strong field effects dominate and $M/r \sim 1$, perturbative solutions cannot capture the full non-linear dynamics. Regarding this Thesis, perturbative corrections to black hole solutions due to a second body are captured by an expansion in powers of the mass ratio, $\epsilon = m/M \ll 1$.

Inserting these expansions into the field equations generates a hierarchy of differential equations to be solved iteratively in powers of ϵ :

$$\begin{aligned} G^{(0)}_{\mu\nu}[\bar{g}] &= 0 && 0^{\text{th}}\text{ order}\,, \\ G^{(1)}_{\mu\nu}[h^{(1)}] &= 8\pi T^{(1)}_{\mu\nu} && 1^{\text{st}}\text{ order}\,, \\ G^{(2)}_{\mu\nu}[h^{(2)}] &= 8\pi T^{(2)}_{\mu\nu} - G^{(2)}_{\mu\nu}[h^{(1)}, h^{(1)}] && 2^{\text{nd}}\text{ order}\,, \\ &\dots \end{aligned}$$

where the zeroth-order equation is solved in many textbooks [105, 106] to give explicit forms of black hole spacetime metrics. The most important from the observational point of view are the Schwarzschild and the Kerr black holes. Coupling Einstein's equations to Maxwell's equations yields generalizations of these metrics, namely the Reissner–Nordström and Kerr–Newman black holes.

The Schwarzschild solution represents a static, non-rotating, spherically symmetric black hole with mass M. It is described by the following line element

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,, \tag{2.5}$$

where $f(r) := 1 - 2M/r$[1]. The Kerr solution generalizes this to a rotating, axisymmetric black hole characterized by its mass M and angular momentum per unit mass a. In Boyer–Lindquist coordinates, the line element reads

$$ds^2 = -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\phi - adt\right]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \tag{2.6}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. Unlike the Schwarzschild case, which features a single event horizon at $r_H = 2M$, the Kerr metric possesses two horizons at $r_\pm := M \pm \sqrt{M^2 - a^2}$, known as the *inner* and *outer* horizon. For $a = 0$, it is easy to see that $r_+ = r_H$ and $r_- = 0$.

At first order in ϵ, the relevant equation becomes

$$G^{(1)}_{\mu\nu}[h^{(1)}] = 8\pi T^{(1)}_{\mu\nu}\,, \tag{2.7}$$

where the left hand side is

$$G^{(1)}_{\mu\nu}[h^{(1)}] = -\frac{1}{2}\left(\bar{g}_\mu{}^\alpha \bar{g}_\nu{}^\beta - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\right)\left(\Box h^{(1)}_{\alpha\beta} + 2\bar{R}_\alpha{}^\gamma{}_\beta{}^\sigma h^{(1)}_{\gamma\sigma} - 2\bar{h}_{\gamma(\alpha}{}^{;\gamma}{}_{\beta)}\right)\,, \tag{2.8}$$

and the Riemann tensor $\bar{R}_\alpha{}^\gamma{}_\beta{}^\sigma$ is computed from the background metric. The d'Alembertian operator is defined as $\Box := \bar{g}^{\mu\nu}\nabla_\mu\nabla_\nu$, where ∇_μ is the covariant derivative compatible with the

[1]The chosen signature is the *mostly plus* one, i.e. $(-,+,+,+)$

background metric, which can be also indicated with semicolons. For compactness, the trace-reversed perturbation has been defined as

$$\bar{h}_{\mu\nu} := h^{(1)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}h^{(1)}_{\alpha\beta} \,. \tag{2.9}$$

As discussed in [28, 107], one can impose the Lorenz gauge condition $\nabla_\alpha \bar{h}^{\alpha\beta} = 0$ via a gauge transformation, simplifying the structure of the equations.

However, Eq. (2.7) is not the equation typically solved in practical applications. In the Schwarzschild case, two master equations—governing odd and even parity modes—were introduced by Regge and Wheeler [108] and Zerilli [109], respectively. Their solutions are directly related to the metric perturbations by a straightforward reconstruction procedure.

For rotating black holes, such a decomposition is not straightforward due to the more complex structure of the Kerr background. Nevertheless, as shown by Teukolsky and Press [110–112], it is possible to derive a master equation within the Newman–Penrose formalism [113]. In this framework, the basic variables are the tetrad components of the curvature tensor, and the so-called Teukolsky equation governs the evolution of specific null tetrad components of the Weyl tensor, namely ψ_0 or ψ_4.

In this respect, the Teukolsky formalism offers a more general framework than the Regge–Wheeler approach. Its main limitation lies in the fact that it does not directly yield the metric perturbation. A more involved reconstruction procedure is thus required [114–120] also in this case. The next Section will be devoted to discuss with more details the Teukolsky equation.

The second-order perturbation equations present significant analytical and computational challenges. This Thesis will not address these aspects, but the interested reader is referred to [28, 121–124] for further discussion on both analytical and numerical approaches.

2.2 Teukolsky formalism

2.2.1 Null tetrads and Geroch-Held-Penrose formalism

Black hole spacetimes are exact solutions of the Einstein field equations (EFEs) that belong to the Petrov type-D class. The Petrov classification is a mathematical framework used to characterize the algebraic symmetries of the Weyl tensor at each spacetime point. In type-D geometries, the Weyl tensor admits two distinct, non-degenerate principal null directions. This property naturally motivates the introduction of a complex null tetrad in which two legs are aligned with these principal directions.

Adopting standard notation

$$l^\alpha := e^\alpha_{(1)}\,, \qquad n^\alpha := e^\alpha_{(2)}\,, \qquad m^\alpha := e^\alpha_{(3)}\,, \qquad \bar{m}^\alpha := e^\alpha_{(4)}\,, \tag{2.10}$$

where l^α and n^α represent the outward and inward null directions, respectively, and satisfy the normalization condition $l^\alpha n_\alpha = -1$. The remaining vectors, m^α and $\bar{m}^\alpha$ are complex conjugates constructed from linear combinations of two orthonormal spacelike unit vectors, i.e. in spherical coordinates $(\hat{\theta}, \hat{\phi})$. Their explicit expressions depend on the coordinate system; again in spherical coordinates, a standard choice is

$$m^\alpha = \frac{1}{\sqrt{2}r}\left(\hat{\theta} + i\hat{\phi}\right)^\alpha \,, \tag{2.11}$$

with $m^\alpha \bar{m}_\alpha = 1$, while all other inner products vanish.

The most commonly employed tetrads in Kerr spacetime are those introduced by Carter [125] and Kinnersley [126]. The primary distinction between these lies in their transformation properties under discrete symmetries, for instance $\{t, \phi\} \to \{-t, -\phi\}$. Nonetheless, the physical results derived from either choice are equivalent and invariant under such transformations.

The null tetrad $\{e^\alpha_{(a)}\} = \{l^\alpha, n^\alpha, m^\alpha, \bar{m}^\alpha\}$ constitutes the basis of the Newman–Penrose formalism [113], which was originally employed by Teukolsky to derive his master equation. However, a more compact and symmetric reformulation is provided by the Geroch–Held–Penrose (GHP) formalism [127]. Despite its more abstract nature, the GHP approach proves particularly efficient for handling perturbations of type-D spacetimes such as Kerr. Additional details can be found in [128–130] but the following discussion (and notation) is mainly based on [28].

The Kerr metric can be concisely expressed in terms of the tetrad vectors as

$$g_{\alpha\beta} = -2l_{(\alpha}n_{\beta)} + 2m_{(\alpha}\bar{m}_{\beta)} \,. \tag{2.12}$$

For vacuum spacetimes, the non-vanishing components of the Riemann tensor coincide with those of the Weyl tensor $C_{\alpha\beta\mu\nu}$, and their contraction with the null tetrad yields five complex scalars:

$$\Psi_0 = C_{lmlm} \,, \qquad \Psi_1 = C_{lnlm} \,, \qquad \Psi_2 = C_{lm\bar{m}n} \,, \qquad \Psi_3 = C_{ln\bar{m}n} \,, \qquad \Psi_4 = C_{n\bar{m}n\bar{m}} \,, \tag{2.13}$$

where a short notation has been used, i.e. $C_{lmlm} = C_{\alpha\beta\mu\nu} l^\alpha m^\beta l^\mu m^\nu$. For type-D backgrounds [2], such as Schwarzschild and Kerr, most of these Weyl scalars vanish:

$$\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0 \,, \tag{2.14}$$

while the only non-zero component is given by

$$\Psi_2 = -M\zeta^{-3} := -M(r + ia\cos\theta)^{-3} \,, \tag{2.15}$$

where ζ also has a coordinate-related expression $\Sigma = \zeta\bar{\zeta}$.

From these expressions, it becomes evident that studying perturbations of the Weyl scalars—rather than directly perturbing the full metric—is a more efficient and insightful strategy in the context of Kerr perturbation theory. In doing this, GHP formalism plays a crucial role.

Given the null tetrad $e^\alpha_{(a)}$, one can first construct the four directional derivatives:

$$D := \nabla_l = l^\alpha \nabla_\alpha \,, \quad \Delta := \nabla_n = n^\alpha \nabla_\alpha \,, \quad \delta := \nabla_m = m^\alpha \nabla_\alpha \,, \quad \bar{\delta} := \nabla_{\bar{m}} = \bar{m}^\alpha \nabla_\alpha \,, \tag{2.16}$$

and by looking at their action on the tetrad vectors, it is possible to define four spin coefficients:

$$\kappa = -l^\mu m^\nu \nabla_\mu l_\nu \,, \quad \sigma = -m^\mu m^\nu \nabla_\mu l_\nu \,, \quad \rho = -\bar{m}^\mu m^\nu \nabla_\mu l_\nu \,, \quad \tau = -n^\mu m^\nu \nabla_\mu l_\nu \,, \tag{2.17}$$

which do not exhaust the full set of spin coefficients that can be defined, which are 12 in total.

To define the remaining 8 spin coefficients, it is necessary to introduce a set of three discrete transformations corresponding to simultaneous exchanges of tetrad vectors:

1. $' : l^\alpha \leftrightarrow n^\alpha$ and $m^\alpha \leftrightarrow \bar{m}^\alpha$;

[2] To be precise, the Goldberg- Sachs theorem applied to the case of Petrov type-D spacetime assures that most of the complex Weyl scalars vanish; see [131].

2. $^{-}: m^\alpha \leftrightarrow \bar{m}^\alpha$;
3. $*: l^\alpha \to m^\alpha\,,\;\; n^\alpha \to -\bar{m}^\alpha\,,\;\; m^\alpha \to -l^\alpha\,,\;\; \bar{m}^\alpha \to -\bar{n}^\alpha\,.$

The second quartet of spin coefficients can then be constructed by simply acting with the "prime" transformation on κ, σ, ρ and τ.

The final four spin coefficients are used to define the GHP derivative operators, which represent a fundamental ingredient in the construction of the Teukolsky equation:

$$Þ := l^\alpha \nabla_\alpha - p\varepsilon - q\bar{\varepsilon}\,, \qquad ð := m^\alpha \nabla_\alpha - p\beta + q\bar{\beta}'\,, \tag{2.18}$$

along with their prime variants $Þ'$ and $ð'$ and where

$$\beta = \frac{1}{2}\left(m^\mu \bar{m}^\nu \nabla_\mu m_\nu - m^\mu n^\nu \nabla_\mu l_\nu\right)\,, \qquad \varepsilon = \frac{1}{2}\left(l^\mu \bar{m}^\nu \nabla_\mu m_\nu - l^\mu n^\nu \nabla_\mu l_\nu\right)\,. \tag{2.19}$$

The two coefficients p and q are integer numbers that identify the so-called GHP type of an object. Every quantity defined within this formalism has a well-defined type, which is related to its spin and boost weights, given by $s = (p-q)/2$ and $b = (p+q)/2$. For our purposes, it is sufficient to look how the prime and bar operations affect these types:

$$' : \{p, q\} \to \{-p, -q\}\,, \qquad ^{-} : \{p, q\} \to \{q, p\}\,, \tag{2.20}$$

while

$$\kappa : \{3, 1\}\,, \qquad \sigma : \{3, -1\}\,, \qquad \rho : \{1, 1\}\,, \qquad \tau : \{1, -1\}\,, \tag{2.21}$$

and their primed variants have types that are straightforward to determine. The standard way to identify the GHP type of a object is by applying a boost and a rotation to it and see how it transforms. For instance, if the combination of these two transformations is parametrized by χ, by applying this transformation to a generic object Ξ:

$$\Xi \to \chi^p \bar{\chi}^q \Xi\,, \tag{2.22}$$

the GHP type of Ξ can be easily identified. χ is usually a Lorentz transformation, composed by a boost and a rotation, which lead to a generic expression of $\chi^2 = re^{i\theta}$, where r contains information about the boost and the exponential about the rotation. It is easy to see that the difference $p - q$ is related to rotation of Ξ, while $p + q$ is related to the boost. Indeed the factor $1/2$ in the definition of s and b comes from the definition of χ.

The last four spin coefficients of Eq. (2.18) have been defined in order to produce a quartet with well defined GHP type which can also be used as derivative operators. The directional derivatives of Eq. (2.16) fails in producing a new object with well defined GHP type, hence they are not suitable for this formalism. These operators are then combined with the spin coefficients defined in Eq. (2.19) and their primed and barred variants, which do not posses a well defined GHP type, in order to produce new derivative operators which transform well under Lorentz transformations. Moreover, when Þ and ð act on a quantity with GHP type p, q, the result transforms as $\{p, q\} \to \{p + r, q + t\}$, where $\{r, t\}$ depends on the operator:

$$Þ : \{1, 1\}\,, \qquad Þ' : \{-1, -1\}\,, \qquad ð : \{1, -1\}\,, \qquad ð' : \{-1, 1\}\,, \tag{2.23}$$

hence one can interpret Þ (ð) and Þ$'$ (ð$'$) as boost (spin) raising and lowering operators. More details can be found in [130].

A final remark concerns the GHP type of the five Weyl scalars:

$$\Psi_0 = \{4,0\}\ , \quad \Psi_1 = \{2,0\}\ , \quad \Psi_2 = \{0,0\}\ , \quad \Psi_3 = \{-2,0\}\ , \quad \Psi_4 = \{-4,0\}\ , \tag{2.24}$$

and in terms of spin weight, this implies that Ψ_2 is a spin-0 scalar perturbation, Ψ_1 and Ψ_3 have $|s| = 1$ (vector perturbations), and Ψ_0 and Ψ_4 correspond to $|s| = 2$ (gravitational perturbations).

2.2.2 Teukolsky equation

It was shown in [132] that by perturbing the Weyl scalars as

$$\Psi_i = \bar{\Psi}_i + \epsilon\psi_i\,, \tag{2.25}$$

where $\bar{\Psi}_i$ are the background Weyl scalars previously discussed, ψ_0 and ψ_4 are two gauge invariant variables and they satisfy decoupled and fully separable second order differential equations. These perturbed scalars are related to the metric perturbation $h^{(1)}_{\mu\nu}$ introduced in the previous section.

The Teukolsky equations are the two master equations governing ψ_0 and ψ_4:

$$\mathcal{O}\psi_0 = 8\pi\mathcal{J}_0 T\,, \qquad \mathcal{O}'\psi_4 = 8\pi\mathcal{J}_4 T\,, \tag{2.26}$$

where $\mathcal{O}$ and $\mathcal{O}'$ are second-order differential operators, expressible in GHP formalism as

$$\mathcal{O} = (\text{Þ} - 2s\rho - \bar{\rho})\left(\text{Þ}' - \rho'\right) - (\eth - 2s\tau - \bar{\tau}')\left(\eth' - \tau'\right) + \frac{1}{2}\left[(6s-2) - 4s^2\right]\Psi_2\,, \tag{2.27}$$

where s is a spin-weight parameter. The right-hand sides of Eq. (2.26) contain the projections of the stress-energy tensor, and $\mathcal{J}_0$ and $\mathcal{J}_4$ are second-order differential operators acting on $T_{\mu\nu}$ to produce the so-called *Teukolsky sources*, which will be discussed in the next Chapter. The operators can be also written in GHP form as

$$\begin{aligned}
(\mathcal{J}_0)_{\mu\nu} &= \frac{1}{2}\left(\eth - \bar{\tau}' - 4\tau\right)\left[(\text{Þ} - 2\bar{\rho})\, l_{(\mu}m_{\nu)} - (\eth - \bar{\tau}')\, l_\mu l_\nu\right] \\
&\quad + \frac{1}{2}\left(\text{Þ} - 4\rho - \bar{\rho}\right)\left[(\eth - 2\bar{\tau}')\, l_{(\mu}m_{\nu)} - (\text{Þ} - \bar{\rho})\, m_\mu m_\nu\right]\,, \qquad (2.28\text{a}) \\
(\mathcal{J}_4)_{\mu\nu} &= \frac{1}{2}\left(\eth' - \bar{\tau} - 4\tau'\right)\left[\left(\text{Þ}' - 2\bar{\rho}'\right) n_{(\mu}\bar{m}_{\nu)} - (\eth' - \bar{\tau})\, n_\mu n_\nu\right] \\
&\quad + \frac{1}{2}\left(\text{Þ}' - 4\rho' - \bar{\rho}'\right)\left[(\eth' - 2\bar{\tau})\, n_{(\mu}\bar{m}_{\nu)} - \left(\text{Þ}' - \bar{\rho}'\right)\bar{m}_\mu\bar{m}_\nu\right]\,, \qquad (2.28\text{b})
\end{aligned}$$

where the spacetime components of these operators have been restored.

In the following, only the equation for the spin $s = -2$ perturbation will be investigated. The reason is that it is possible to relate the homogeneous solutions of Eqs. (2.26) by means of Teukolsky-Starobinsky identities:

$$\begin{aligned}
\text{Þ}^4\zeta^4\psi_4 &= \eth'^4\zeta^4\psi_0 - 3M\mathcal{L}_\xi\bar{\psi}_0\,, \qquad (2.29\text{a}) \\
\text{Þ}'^4\zeta^4\psi_0 &= \eth^4\zeta^4\psi_4 + 3M\mathcal{L}_\xi\bar{\psi}_4\,, \qquad (2.29\text{b})
\end{aligned}$$

where the Lie derivative operator $\mathcal{L}_\xi$ is defined as

$$\mathcal{L}_\xi = -\zeta\left(-\rho'\text{Þ} + \rho\text{Þ}' + \tau'\eth - \tau\eth'\right) - \frac{p}{2}\zeta\Psi_2 - \frac{q}{2}\bar{\zeta}\bar{\Psi}_2\,, \tag{2.30}$$

and ξ is the Killing vector on the t direction of the background spacetime

$$\xi = -\zeta \left(-\rho' l^\alpha + \rho n^\alpha + \tau' m^\alpha - \tau \bar{m}^\alpha\right) . \tag{2.31}$$

At this stage, the GHP formalism gives a well-suited framework for formulating the Teukolsky equations. To devise a practical strategy for solving them, it is necessary to express the operators in terms of the spacetime coordinates. For $s = -2$, one can exploit the identity

$$\mathcal{O}' \psi_4 = \zeta^{-4} \mathcal{O} \zeta^4 \psi_4 \,, \tag{2.32}$$

which leads to the alternative form

$$\mathcal{O}(\zeta^4 \psi_4) = 8\pi \zeta^4 \mathcal{J}_4 T \,, \tag{2.33}$$

which is identifying a single differential operator that acts identically on spin ± 2. The price to pay is that for the spin -2 perturbation, the field is not only the Weyl scalar but there is also the prefactor ζ^4 in the Kinnersley tetrad.

The operator $\mathcal{O}$ can then be written in terms of the spacetime coordinates as

$$\begin{aligned}
\mathcal{O} = &-\left(\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2\theta\right)\partial_t^2 - \frac{4Mar}{\Delta}\partial_t\partial_\phi - \left(\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right)\partial_\phi^2 + \Delta^{-s}\partial_r(\Delta^{s+1}\partial_r) \\
&+ \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + 2s\left(\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right)\partial_\phi + 2s\left(\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right)\partial_t \\
&- s(s\cot^2\theta - 1) \,,
\end{aligned} \tag{2.34}$$

where $s = -2$. This expression makes it evident that solving analytically the equation directly in the time domain is not tractable analytically in general. However, due to its separability, one can pass to the frequency domain and also decompose the field into radial and angular parts:

$$(\zeta^4 \psi_4)_\omega = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} e^{im\phi} {}_sS_{lm}(\theta) R_{lm\omega}(r) \,, \tag{2.35}$$

where $(\zeta^4\psi_4)_\omega$ is the frequency-domain solution, $R_{lm\omega}(r)$ and ${}_sS_{lm}(\theta)$ are respectively the radial and angular function. It is important to stress that the equation is defined only for $l \geq |s|$, for the so-called *radiative* modes[3] hence, the solution is not capturing information from $l = 0, 1$ which are called *non-radiative* modes. However, this definitions is not that simple as just described and, in particular for Kerr perturbations, there few subtleties with the low multipoles perturbations which are described in [133], but in general these contributions can be computed from standard techniques, see [35, 133].

Going back to the time domain, by simply performing an Inverse Fourier Transform[4] (IFT) on Eq. (2.35)

$$\zeta^4 \psi_4 = \frac{1}{2\pi} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \int d\omega \, e^{-i\omega t} e^{im\phi} {}_sS_{lm}(\theta) R_{lm\omega}(r) \,. \tag{2.36}$$

[3] More discussion on radiative modes in Chapter 5.

[4] The notation is: the Fourier Transform is the integral transform that takes a function in the time domain and gives its counterpart in the frequency domain, while the Inverse Fourier Transform does the opposite.

The resulting equations for the radial and the angular part are

$$\left[\Delta^2\frac{d}{dr}\left(\Delta^{-1}\frac{d}{dr}\right)+\frac{K^2+4i(r-M)K}{\Delta}-8i\omega r-{}_{-2}\lambda_{lm}\right]R_{lm\omega}=T_{lm\omega}\,, \tag{2.37}$$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)-a^2\omega^2\sin^2\theta \right.$$
$$\left. -\frac{(m-2\cos\theta)^2}{\sin^2\theta}+4a\omega\cos\theta-2+{}_{-2}\lambda_{lm}+2am\omega\right]{}_sS_{lm}=0\,, \tag{2.38}$$

where $K=(r^2+a^2)\omega-am$ and ${}_{-2}\lambda_{lm}$ are the eigenvalues of the equation for ${}_sS_{lm}={}_sS_{lm}(\theta)$. As a last remark, these equations can be generalized to a generic value of s, providing a framework that could work not only for gravitational perturbations but also for scalar and vector ones (together with also neutrinos). For the sake of completeness, the equations are

$$\left[\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\right)+\frac{K^2-2is(r-M)K}{\Delta}+4is\omega r-{}_s\lambda_{lm}\right]{}_s\Psi_{lm\omega}={}_sT_{lm\omega}\,, \tag{2.39}$$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)-a^2\omega^2\sin^2\theta \right.$$
$$\left. -\frac{(m+s\cos\theta)^2}{\sin^2\theta}-2as\omega\cos\theta+s+{}_s\lambda_{lm}+2am\omega\right]{}_sS_{lm}=0\,, \tag{2.40}$$

where the radial function is now generically called ${}_s\Psi_{lm\omega}$.

2.3 Analytical solution of the homogeneous radial equation

The problem of finding analytical solutions to the homogeneous equation, it is now reduced to solving two decoupled second-order differential equations in the radial and angular variables. The procedure is standard: one first determines solutions to the associated homogeneous equations with appropriate boundary conditions, and then constructs the inhomogeneous solution by convolving the homogeneous solutions with the source term. However, if analytical solutions to these equations are sought, perturbative techniques become essential. The most commonly employed method in this context is the *post-Newtonian* (PN) expansion.

This Section aims to define a systematic procedure for computing analytical expressions for the solutions of the Teukolsky angular and radial equations using a perturbative approach. In this framework, the PN approximation is interpreted as a small-frequency expansion in the parameter ω. A more precise definition of the PN expansion will be provided in later chapters, where it will be clarified how the PN regime is intrinsically connected to the presence of the perturbing source located at a large radial distance from the central black hole.

2.3.1 Angular Equation

The angular equation (2.40) for generic s admits solutions called spin-weighted spheroidal harmonics ${}_sS_{lm}(\theta,\phi;a\omega)$ [134]. These eigenfunctions of Eq. (2.40) have a spectrum of eigenvalues labeled by ${}_s\lambda_{lm}(a\omega)$.

In the PN approximation, it is possible to treat the ${}_sS_{lm}(\theta,\phi;a\omega)$ perturbatively by expanding for $a\omega\to 0$ and get the so-called spin-weighted spherical harmonics ${}_sY_{lm}(\theta,\phi)$, see [38, 111] and the arising structure is the following:

$$
\begin{aligned}
{}_sS_{lm}(a\omega) &= {}_sY_{lm}\\
&+\frac{a\omega s}{\sqrt{1+2l}}\left(\frac{{}_sY_{(-1+l)m}\beta_l}{l^2\sqrt{-1+2l}}-\frac{{}_sY_{(1+l)m}\beta_{1+l}}{(1+l)^2\sqrt{3+2l}}\right)\\
&+\frac{(a\omega)^2}{2}\left\{\frac{2ms\left(l^2-2s^2\right){}_sY_{(-1+l)m}\beta_l}{l^4\sqrt{-1+2l}\sqrt{1+2l}\,(-1+l^2)}-\frac{\left(l-2s^2\right){}_sY_{(-2+l)m}\beta_{-1+l}\beta_l}{(1-2l)^2(-1+l)l^2\sqrt{-3+2l}\sqrt{1+2l}}\right.\\
&-\frac{2ms\left((1+l)^2-2s^2\right){}_sY_{(1+l)m}\beta_{1+l}}{l(1+l)^4(2+l)\sqrt{1+2l}\sqrt{3+2l}}+s^2{}_sY_{lm}\left(\frac{\beta_l^2}{l^4-4l^6}-\frac{\beta_{1+l}^2}{(1+l)^4(3+4l(2+l))}\right)\\
&\left.+\frac{\left(1+l+2s^2\right){}_sY_{(2+l)m}\beta_{1+l}\beta_{2+l}}{(1+l)^2(2+l)\sqrt{1+2l}(3+2l)^2\sqrt{5+2l}}\right\}\\
&+\frac{(a\omega)^3}{6}\left\{\frac{6ms^2\left(l+2l^2-4s^2\right){}_sY_{(-2+l)m}\beta_{-1+l}\beta_l}{(1-2l)^2(2-l)(1-l)l^4(1+l)\sqrt{-3+2l}\sqrt{1+2l}}\right.\\
&+\frac{s\left(1-3l+2s^2\right){}_sY_{(-3+l)m}\beta_{-2+l}\beta_{-1+l}\beta_l}{(1-2l)^2(-2+l)(-1+l)^2l^2\sqrt{-5+2l}(-3+2l)\sqrt{1+2l}}\\
&+6ms^2{}_sY_{lm}\left[-\frac{\left(l^2-m^2\right)\left(l^4-3l^2s^2+2s^4\right)}{l^6-5l^8+4l^{10}}-\frac{\left((1+l)^2-2s^2\right)\beta_{1+l}^2}{l(1+l)^6(2+l)(1+2l)(3+2l)}\right]\\
&+\frac{3{}_sY_{(-1+l)m}\beta_l}{l^6(-1+2l)^{3/2}(1+2l)^{3/2}}\left[\frac{l^3(1+2l)\left(s-3ls+2s^3\right)\beta_{-1+l}^2}{(1-l)^2(3-4(2-l)l)}-3s^3\beta_l^2\right.\\
&\left.+2(1-4l^2)m^2s^3\left(\frac{2s^2}{1-l^2}+\frac{(l+s^2)}{(1+l)^2}-\frac{(l-s^2)}{(1-l)^2}+\frac{l^3\left(1+3l+2l^2+(2+l)s^2\right)\beta_{1+l}^2}{2(1+l)^4(3+4l(2+l))m^2s^2}\right)\right]\\
&+\frac{6ms^2\left(1+l(3+2l)-4s^2\right){}_sY_{(2+l)m}\beta_{1+l}\beta_{2+l}}{l(1+l)^4(2+l)(3+l)\sqrt{1+2l}(3+2l)^2\sqrt{5+2l}}\\
&+\frac{3{}_sY_{(1+l)m}\beta_{1+l}}{(1+l)^6(1+2l)^{3/2}(3+2l)^{3/2}}\left[\frac{(1+l)^3(3+2l)s\left(l+2l^2+s^2-ls^2\right)\beta_l^2}{l^4(-1+2l)}+3s^3\beta_{1+l}^2\right.\\
&\left.+(1+2l)(3+2l)m^2s^3\left(\frac{8\left((1+l)^2-s^2\right)}{l^2(2+l)^2}-\frac{(1+l)^3\left(4+3l+2s^2\right)\beta_{2+l}^2}{(2+l)^2(3+2l)^2(5+2l)m^2s^2}\right)\right]\\
&\left.-\frac{s\left(4+3l+2s^2\right){}_sY_{(3+l)m}\beta_{1+l}\beta_{2+l}\beta_{3+l}}{(1+l)^2(2+l)^2(3+l)\sqrt{1+2l}(3+2l)^2(5+2l)\sqrt{7+2l}}\right\}+O[(a\omega)^4]\,,
\end{aligned}
\tag{2.41}
$$

where the (θ,ϕ) dependence is implicit in the $Y_{lm}(\theta,\phi)$ and the parameter $\beta_l=\sqrt{l^2-s^2}\sqrt{l^2-m^2}$, was introduced.

It is possible to provide expressions at very high order in this expansion by using the `SpinWeightedSpheroidalHarmonics` library of the `BlackHolePerturbationToolkit`, see [13], but the general structure at any order in $a\omega$ is

$$
{}_sS_{lm}(a\omega) = \sum_{i=0}^{N} (a\omega)^i \sum_{j=-i}^{i} A_{i,j}(l,m,s)\, {}_sY_{(l+j)m}\,, \tag{2.42}
$$

which reflects the explicit expression of Eq. (2.41).

Similarly, it is possible to series expand also the eigenvalues ${}_s\lambda_{lm}(a\omega)$ and get

$$
\begin{aligned}
{}_s\lambda_{lm}(a\omega) = {} & l(1+l) - s(1+s) - a\omega\left(2m + \frac{2ms^2}{l(1+l)}\right) \\
& + (a\omega)^2\left[\frac{2\left(-1+l+l^2\right)}{-3+4l(1+l)} + \frac{4s^2}{-3+4l(1+l)} - \frac{6s^4}{l\left(-3+l+4l^2(2+l)\right)}\right. \\
& \left. + m^2\left(\frac{2}{-3+4l(1+l)} - \frac{12s^2}{l\left(-3+l+4l^2(2+l)\right)} + \frac{(6+10l(1+l))s^4}{l^3(1+l)^3(-3+4l(1+l))}\right)\right] \\
& + 4(a\omega)^3 ms^2\left[\frac{-1+3l+3l^2}{l\left(6-5l-18l^2+l^3+12l^4+4l^5\right)}\right. \\
& - \frac{10s^2}{l\left(6-5l-18l^2+l^3+12l^4+4l^5\right)} + \frac{\left(6+7l+7l^2\right)s^4}{l^3(1+l)^3\left(6-11l-7l^2+8l^3+4l^4\right)} \\
& - m^2\left(\frac{5}{l\left(6-5l-18l^2+l^3+12l^4+4l^5\right)}\right. \\
& \left.\left. - \frac{2\left(6+7l+7l^2\right)s^2}{l^3(1+l)^3\left(6-11l-7l^2+8l^3+4l^4\right)} + \frac{(6+l(1+l)(19+9l(1+l)))s^4}{(1-l)l^5(1+l)^5(2+l)(1-2l)(3+2l)}\right)\right] \\
& + \frac{(a\omega)^4}{(3-2l)(5+2l)}\left\{-\frac{6-2l(1+l)(17+l(1+l)(-3+2l)(5+2l))}{(3-4l(1+l))^3}\right. \\
& - \frac{8(15+l(1+l)(-13+12l(1+l)))s^2}{(3-4l(1+l))^3} + \frac{4(90+l(1+l)(-141+220l(1+l)))s^4}{l(1+l)(3-4l(1+l))^3} \\
& + \frac{72(5-28l(1+l))s^6}{l(1+l)(3-4l(1+l))^3} - \frac{18(45-l(1+l)(57+68l(1+l)))s^8}{l^3\left(3-l-4l^2(2+l)\right)^3} \\
& + m^2\left[-\frac{60-4l\left(13+l-24l^2-12l^3\right)}{(3-4l(1+l))^3} + \frac{16(45+l(1+l)(-87+100l(1+l)))s^2}{l(1+l)(3-4l(1+l))^3}\right. \\
& + \frac{(720-8l(1+l)(159+l(1+l)(704+3l(1+l)(-933+476l(1+l)))))s^4}{(1-2l)^3l^3(1+l)^3(3+2l)^3\left(-2+l+l^2\right)} \\
& + \frac{16(1260+l(1+l)(-2226+l(1+l)(-991+1332l(1+l))))s^6}{(1-2l)^3l^3(1+l)^3(3+2l)^3\left(-2+l+l^2\right)} \\
& \left. - \frac{4(2430+l(1+l)(567+l(1+l)(-10089+13l(1+l)(183+220l(1+l)))))s^8}{l^5(1+l)^5\left(-2+l+l^2\right)(3-4l(1+l))^3}\right] \\
& + m^4\left[\frac{66+40l(1+l)}{(3-4l(1+l))^3} + \frac{72(5-28l(1+l))s^2}{l(1+l)(3-4l(1+l))^3}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4(3330+l(1+l)(-5883+l(1+l)(-2693+3276l(1+l))))s^4}{(1-2l)^3l^3(1+l)^3(3+2l)^3\left(-2+l+l^2\right)}\\
&-\frac{8(2430+l(1+l)(567+l(1+l)(-10089+13l(1+l)(183+220l(1+l)))))s^6}{l^5(1+l)^5\left(-2+l+l^2\right)(3-4l(1+l))^3}\\
&+\frac{9100s^8}{l^7(1+l)^7\left(-2+l+l^2\right)\left(3-4l-4l^2\right)^3}\left(1+\frac{67l}{30}-\frac{11l^2}{5}-\frac{9697l^3}{675}-\frac{34141l^4}{2025}\right.\\
&\left.\left.\left.+\frac{836l^5}{675}+\frac{53021l^6}{2025}+\frac{20786l^7}{675}+\frac{929l^8}{50}+\frac{2938l^9}{405}+\frac{2938l^{10}}{2025}\right)\right]\right\}+O[(a\omega)^5]\,.
\end{aligned}
\tag{2.43}
$$

It is useful at this point to provide some properties of the ${}_sS_{lm}(\theta,\phi;a\omega)$ that will be useful in the following. First of all, the azimuthal dependence can be factored out as

$$
{}_sS_{lm}(\theta,\phi;a\omega)=e^{im\phi}{}_sS_{lm}(\theta,0;a\omega)\,, \tag{2.44}
$$

and two useful symmetry properties can be exploited

$$
{}_sS_{lm}(\theta,\phi;a\omega)=(-1)^{l+m}{}_{-s}S_{lm}(\pi-\theta,\phi;a\omega)\,, \tag{2.45a}
$$

$$
{}_sS_{lm}(\theta,\phi;a\omega)=(-1)^{l+s}{}_s\bar{S}_{l-m}(\pi-\theta,\phi;-a\omega)\,, \tag{2.45b}
$$

and by combining them it is possible to write an identity that relates the harmonics characterized by $(s,l,m,a\omega)$ with $(-s,l,-m,-a\omega)$ as follows

$$
{}_sS_{lm}(\theta,\phi;a\omega)=(-1)^{m+s}{}_{-s}\bar{S}_{l-m}(\theta,\phi;-a\omega)\,. \tag{2.46}
$$

These properties are inherited by the eigenvalues as

$$
{}_s\lambda_{lm}(a\omega)={}_{-s}\lambda_{lm}(a\omega)-2s\,, \tag{2.47a}
$$

$$
{}_s\lambda_{l-m}(a\omega)={}_s\lambda_{l-m}(-a\omega)\,, \tag{2.47b}
$$

and their combination gives

$$
{}_s\lambda_{lm}(a\omega)={}_{-s}\lambda_{l-m}(-a\omega)-2s\,. \tag{2.48}
$$

The basis of the spin-weighted spheroidal harmonics is also orthonormal over the entire sphere

$$
\int {}_sS_{lm}(\theta,\phi;a\omega){}_s\bar{S}_{l'm'}(\theta,\phi;a\omega)\,d\Omega=\delta_{l,l'}\delta_{m,m'} \tag{2.49}
$$

where $d\Omega=\sin\theta d\theta d\phi$ is the volume element on the two-sphere. Together with the orthonormality, these functions provide also a complete basis (at least for real values of ω), i.e.

$$
\delta\left(\theta-\theta'\right)\delta(\phi-\phi')=\sin\theta'\sum_{l,m}{}_sS_{lm}(\theta,\phi;a\omega){}_s\bar{S}_{lm}(\theta',\phi';a\omega)\,, \tag{2.50}
$$

and this property will be important in following discussions.

2.3.2 Radial Equation

The radial equation (2.39) includes a source term on the right-hand side, which originates from the decomposition of the Teukolsky source into radial and angular variables. However, the first step involves solving the associated homogeneous equation

$$\left[\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\right)+\frac{K^2-2is(r-M)K}{\Delta}+4is\omega r-{}_s\lambda_{lm}\right]{}_s\Psi_{lm\omega}=0\,, \tag{2.51}$$

by imposing appropriate boundary conditions at the horizon and at infinity. There exist four common boundary conditions that can be imposed:

- *in* : describes waves incoming from past null infinity $\mathscr{I}^-$, which are partially absorbed by the horizon $\mathscr{H}^+$ and partially scattered back to future null infinity $\mathscr{I}^+$. These modes are *ingoing* at the horizon;
- *up* : describes waves emerging from the past horizon $\mathscr{H}^-$, which are partially transmitted to future null infinity $\mathscr{I}^+$ and partially scattered back into the horizon $\mathscr{H}^+$. These modes are purely *outgoing* at infinity;
- *out*: describes waves coming from both $\mathscr{I}^-$ and $\mathscr{H}^-$ and traveling outward to $\mathscr{I}^+$. These modes are purely *outgoing* from the horizon;
- *down*: describes waves coming from $\mathscr{I}^-$ and $\mathscr{H}^-$ and traveling inward toward the future horizon $\mathscr{H}^+$. These modes are purely *ingoing* at infinity.

These boundary conditions can be expressed in terms of asymptotic behaviors as follows:

$${}_sR^{\text{in}}_{lm\omega}(r)\backsim\begin{cases}{}_sR^{\text{in,trans}}_{lm\omega}\Delta^{-s}e^{-ikr_*} & r\to r_+\\ {}_sR^{\text{in,ref}}_{lm\omega}r^{-1-2s}e^{i\omega r_*}+{}_sR^{\text{in,inc}}_{lm\omega}r^{-1}e^{-i\omega r_*} & r\to\infty\end{cases} \tag{2.52a}$$

$${}_sR^{\text{up}}_{lm\omega}(r)\backsim\begin{cases}{}_sR^{\text{up,inc}}_{lm\omega}e^{ikr_*}+{}_sR^{\text{up,ref}}_{lm\omega}\Delta^{-s}e^{-ikr_*} & r\to r_+\\ {}_sR^{\text{up,trans}}_{lm\omega}r^{-1-2s}e^{i\omega r_*} & r\to\infty\end{cases} \tag{2.52b}$$

$${}_sR^{\text{out}}_{lm\omega}(r)\backsim\begin{cases}{}_sR^{\text{out,trans}}_{lm\omega}e^{ikr_*} & r\to r_+\\ {}_sR^{\text{out,inc}}_{lm\omega}r^{-1-2s}e^{i\omega r_*}+{}_sR^{\text{out,ref}}_{lm\omega}r^{-1}e^{-i\omega r_*} & r\to\infty\end{cases} \tag{2.52c}$$

$${}_sR^{\text{down}}_{lm\omega}(r)\backsim\begin{cases}{}_sR^{\text{down,ref}}_{lm\omega}e^{ikr_*}+{}_sR^{\text{down,inc}}_{lm\omega}\Delta^{-s}e^{-ikr_*} & r\to r_+\\ {}_sR^{\text{down,trans}}_{lm\omega}r^{-1}e^{-i\omega r_*} & r\to\infty\end{cases} \tag{2.52d}$$

where

$$k:=\omega-m\Omega_+,\;\Omega_+:=\frac{a}{2Mr_+},\;r_*:=r+\frac{1}{2\kappa_+}\ln\frac{r-r_+}{2M}+\frac{1}{2\kappa_-}\frac{r-r_-}{2M},\;\kappa_\pm:=\frac{r_\pm-r_\mp}{2r_\pm^2+a^2} \tag{2.53}$$

with Ω_+ denoting the angular velocity of the horizon, $\kappa_\pm$ the surface gravity at the outer/inner horizon. Among the available options, the *in* and *up* solutions are those most commonly used in this context, because they corresponds to the retarded Green's function.

The general solution to the radial equation may be expressed in terms of confluent Heun functions [135], which, however, are not well suited for analytical calculations when a post-Newtonian expansion is required. In order to obtain analytically tractable expressions, one may either fix the l-mode, solve the equation exactly, and subsequently expand the solution in the PN limit, or alternatively, keep l generic, expand the radial equation itself, and solve it perturbatively, order by order. Both methods are valid and, in practice, both are necessary to reconstruct the full solution of the Teukolsky equation, Eq. (2.36), which involves a sum over all (l, m)-modes. The strategy is to compute exact solutions for $l \leq l_{\text{max}}$, while for $l > l_{\text{max}}$, a generic-l solution is constructed, which also facilitates taking the large-l limit.

To distinguish between these two approaches, the generic-l solution is referred to as the *PN solution*, while the fixed-l method is based on the *Mano–Suzuki–Takasugi* (MST) formalism. Both computational schemes will be presented in the following, along with some explicit solutions.[5] This terminology could generate confusion: at the end both solutions, PN and MST, will be PN-expanded. The reason for this terminology is that MST formalism provides *exact* solutions that will be expanded as second step. On the other side, the PN solution are obtained perturbatively by solving the PN expanded differential equation order-by-order.

As a final remark, recent years have seen the development of alternative techniques that successfully address the radial Teukolsky equation—even in a non-perturbative regime. However, such methods are not yet fully optimized for computing gravitational observables at high perturbative orders; see [137, 138].

PN solutions

The post-Newtonian (PN) approach provides analytic solutions to the radial Teukolsky equation that are valid for arbitrary values of the multipole number l. A general ansatz for the solution takes the form

$$ {}_s\Psi^{\text{PN}}_{lm\omega}(r) = \sum_{i=0}^{N} {}_s\Psi^{i}_{lm\omega}(r)\eta^i, \tag{2.54} $$

where $\eta = 1/c$ is the expansion parameter controlling the PN series. It is worth mentioning that this is not the only ansatz that could be used, i.e. an alternative and more computationally efficient assumption is also presented in Sec. III.B.2 of [38]. The differential equation (2.39) is also treated perturbatively in η. While dimensional analysis alone does not uniquely determine the rescalings of r and ω, one can consistently restore powers of η by rescaling the radial coordinate and frequency as $r \to r\,\eta^{-2}$ and $\omega \to \omega\,\eta^3$. A more detailed justification for this rescaling will be presented in the next Chapter.

This perturbative procedure yields a hierarchy of differential equations at successive PN orders:

$$ \frac{\left(-l - l^2 + s + s^2\right) {}_s\Psi^0_{lm\omega}(r)}{r^2} + \frac{2(1+s)\,{}_s\Psi'^0_{lm\omega}(r)}{r} + {}_s\Psi''^0_{lm\omega}(r) = 0, \tag{2.55a} $$

$$ \frac{\left(-l - l^2 + s + s^2\right) {}_s\Psi^1_{lm\omega}(r)}{r^2} + \frac{2(1+s)\,{}_s\Psi'^1_{lm\omega}(r)}{r} + {}_s\Psi''^1_{lm\omega}(r) = -\frac{2is\omega\,{}_s\Psi^0_{lm\omega}(r)}{r}, \tag{2.55b} $$

[5] Notably, no public library or code for MST solutions was available until the release of the `SFPN` package [136] of the BHPT. This thesis does not make use of that package, as the results presented here were obtained independently before the code became public. Nevertheless, it has been verified that the homogeneous solutions computed here are in agreement with those produced by `SFPN`.

$$\frac{\left(-l-l^2+s+s^2\right){}_s\Psi^2_{lm\omega}(r)}{r^2}+\frac{2(1+s){}_s\Psi'^2_{lm\omega}(r)}{r}+{}_s\Psi''^2_{lm\omega}(r)=-\frac{2is\omega{}_s\Psi^1_{lm\omega}(r)}{r}$$
$$-\frac{2M(1+s){}_s\Psi'^0_{lm\omega}(r)}{r^2}-\frac{\left(-2lM-2l^2M+2Ms(1+iam+s)+r^3\omega^2\right){}_s\Psi^0_{lm\omega}(r)}{r^3}\,. \tag{2.55c}$$

It is evident that, at each increasing order in η, the right-hand side accumulates contributions involving solutions from previous orders.

To solve each of these equations, suitable boundary conditions are needed. However, the PN-expanded equations do not encode information about the black hole horizons, so the physical boundary conditions given in Eq. (2.52) can no longer be directly imposed. For instance, the general solution to the Newtonian equation (i.e., the equation at order η^0) takes the form

$${}_s\Psi^0_{lm\omega}(r)=C_1\,{}_sR^{0,\text{in}}_{lm\omega}(r)+C_2\,{}_sR^{0,\text{up}}_{lm\omega}(r), \tag{2.56}$$

where the functions ${}_sR^{(0),\text{in}}$ and ${}_sR^{(0),\text{up}}$ refer to the standard ingoing and outgoing solutions, respectively. The constants C_1 and C_2 must be chosen in accordance with the desired boundary behavior:

$$\lim_{r\to 0}{}_s\Psi^0_{lm\omega}(r)={}_sR^{0,\text{in}}_{lm\omega}(r)\ ,\ \lim_{r\to+\infty}{}_s\Psi^0_{lm\omega}(r)={}_sR^{0,\text{up}}_{lm\omega}(r)\,, \tag{2.57}$$

For instance, selecting the in solution corresponds to setting $C_1=1$, $C_2=0$. Clearly, the up solution is selected for $C_1=0$, $C_2=1$.

At leading order, the in solution reads simply

$${}_sR^{0,\text{in}}_{lm\omega}(r)=r^{l-s}\,, \tag{2.58}$$

and plugging this into the equation at the next order, it gives

$${}_sR^{1,\text{in}}_{lm\omega}(r)=-is\omega\frac{r^{1+l-s}}{1+l}, \tag{2.59}$$

which has two notable features: it is purely imaginary, and it is proportional to the spin s. Therefore, this correction vanishes for scalar perturbations ($s=0$).

Interestingly, the spin a of the black hole starts to contribute at η^2 where

$${}_sR^{2,\text{in}}_{lm\omega}(r)=\frac{1}{2}r^{-1+l-s}\left(-2lM+\frac{2(l+iam)Ms}{l}-\frac{r^3\left(1+l+2s^2\right)\omega^2}{(1+l)(3+2l)}\right)\,. \tag{2.60}$$

Rather than repeating the entire procedure for the up-going solution, one can exploit the symmetry of the Teukolsky equation under the transformation $l\to-l-1$, which allows for the generation of the second independent solution from the first.

The ${}_{-2}R^{\text{in}}_{lm\omega}(r)$ up to η^{20} can be schematically written as

$${}_{-2}R^{\text{in}}_{lm\omega}(r)=\sum_{i=0}^{20}\sum_{j=0}^{\lfloor i/2\rfloor}c^{[i,j]}_{lm}(r,\omega)a^j\eta^i+\sum_{i=6}^{20}\sum_{j=0}^{\lfloor i/2\rfloor}c^{[i,j]}_{lm,\log}(r,\omega)a^j\eta^i+\sum_{i=12}^{20}\sum_{j=0}^{\lfloor i/2\rfloor}c^{[i,j]}_{lm,\log^2}(r,\omega)a^j\eta^i$$
$$+\sum_{i=18}^{20}\sum_{j=0}^{\lfloor i/2\rfloor}c^{[i,j]}_{lm,\log^3}(r,\omega)a^j\eta^i\,, \tag{2.61}$$

where all functions $c^{[i,j]}_{lm,\log^k}(r,\omega)$ are polynomial functions of r and ω, times powers of $\log(r/M)$ which are denoted in the subscripts. The coefficients of the polynomials are rational functions of l and m, where there is an explicit dependence on m only in the Kerr case. It has been also introduce the notation $\lfloor\cdot\rfloor$ for the floor function.

At first glance, this solution seems to provide a generic analytic expression for the radial function at arbitrary PN order and arbitrary l. However, two crucial issues arise.

First, as emphasized earlier, these PN-expanded solutions do not automatically satisfy the physical boundary conditions (2.52), which leads to potential inconsistencies when reconstructing physical observables.

Second, several of the PN coefficients $c^{[i,j]}_{lm,\log^k}(r)$ exhibit divergences for specific values of l, especially when the transformation $l \to -l-1$ is applied to obtain the second independent solution. For instance, the coefficient $clm^{[8,0]}(r,\omega)$ is given by:

$$
\begin{aligned}
c^{[8,0]}_{lm}(r) &= \frac{(-3+l)(-2+l)(-1+l)l(1+l)(2+l)M^4}{6\left(3-8l+4l^2\right)r^4} \\
&+ \frac{\left(162+35l+l^2\right)r^8\omega^8}{384(1+l)(2+l)(3+2l)(5+2l)(7+2l)(9+2l)} \\
&- \frac{\left(57288+69086l+27963l^2+4335l^3+165l^4-5l^5\right)Mr^5\omega^6}{240(1+l)^2(2+l)(3+l)(3+2l)(5+2l)(7+2l)} \\
&- \frac{\left(648+2052l+4170l^2+301l^3-6232l^4-5669l^5-1618l^6-140l^7+8l^8\right)M^3\omega^2}{6l(1+l)(-1+2l)(1+2l)^2(3+2l)r} \\
&- \frac{20160M^2r^2\omega^4}{l(1+l)^3(2+l)(-1+2l)(1+2l)^2(3+2l)^3(5+2l)}f_l\,,
\end{aligned}
$$

where f_l is a high-degree polynomial in l

$$
\begin{aligned}
f_l = 1 &+ \frac{9463l}{1920} + \frac{77503l^2}{6720} + \frac{7374637l^3}{483840} + \frac{3500563l^4}{483840} - \frac{226957l^5}{26880} - \frac{1276703l^6}{80640} \\
&- \frac{599359l^7}{53760} - \frac{664211l^8}{161280} - \frac{2323l^9}{3024} - \frac{2873l^{10}}{60480} + \frac{29l^{11}}{10080} + \frac{l^{12}}{10080}\,.
\end{aligned} \tag{2.62}
$$

Although this coefficient is well-defined for all $l \geq 2$, applying the symmetry $l \to -l-1$ introduces divergences. For example, the third term becomes

$$
c^{[8,0]}_{-l-1\,m}(r) = \ldots + \frac{\left(12000-25480l+15998l^2-3625l^3+190l^4+5l^5\right)Mr^5\omega^6}{240(-2+l)(-1+l)l^2(-5+2l)(-3+2l)(-1+2l)} + \ldots \tag{2.63}
$$

with singularities highlighted in red.

As one proceeds to higher PN orders, further divergences appear at larger values of l, eventually compromising the validity and predictive power of the expansion.

For these reasons—(i) the absence of proper boundary conditions and (ii) the emergence of divergences in the homogeneous solutions—it becomes necessary to develop alternative analytic methods that remain well-defined for all relevant values of l.

2.3.3 MST solutions for the radial equation

This Section provides an introduction to the Mano-Suzuki-Takasugi (MST) formalism for solving the homogeneous radial Teukolsky equation through a reformulation of Leaver's method [139]. More details can be found in the original papers [140–142] or in the review [104].

The homogeneous radial equation for generic spin s is recalled here for clarity:

$$\left[\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\right)+\frac{K^2-2is(r-M)K}{\Delta}+4is\omega r-{}_s\lambda_{lm}\right]{}_s\Psi_{lm\omega}=0\,, \tag{2.64}$$

and it is well known that it possesses three singularities: two regular singularities at the inner and outer horizons $r=r_\pm$, and one irregular singularity at $r\to+\infty$. This structure prevents the equation from being cast in the form of a hypergeometric equation. However, near the horizon, the solution can be approximated by a hypergeometric function. This motivates the representation of the in-solution as *an infinite sum of hypergeometric functions*.

Before proceeding, it is useful to introduce several coefficients to simplify the notation:

$$\begin{gathered}z=\omega r,\;\; z_\pm=\omega r_\pm,\;\; \epsilon=2M\omega,\;\; q=\frac{a}{M},\;\; \kappa=\sqrt{1-q^2},\;\; x=\frac{z_+-z}{\epsilon\kappa}\\ \tau=\frac{\epsilon-mq}{\kappa},\;\; \epsilon_\pm=\frac{\epsilon\pm\tau}{2},\;\; \hat{z}=\omega(r-r_-)=\epsilon\kappa(1-x)\,.\end{gathered} \tag{2.65}$$

Horizon solution in terms of Hypergeometric functions

The in-solution can be modeled as an ingoing wave solution as

$${}_s\Psi^{\text{in}}_{lm\omega}=e^{i\epsilon\kappa x}(-x)^{-s-i(\epsilon+\tau)/2}(1-x)^{i(\epsilon-\tau)/2}f_{\text{in}}(x)\,, \tag{2.66}$$

which is a highly non-trivial ansatz and some discussion on it can be found in [143]. However, it enables the radial equation to be rewritten as a hypergeometric equation in the variable x on the left-hand side, with the right-hand side containing a term proportional to ϵ:

$$x(1-x)f''_{\text{in}}+[c-(a+b+1)x]f'_{\text{in}}-abf_{\text{in}}=-\epsilon\{2i\kappa[x(1-x)f'_{\text{in}}-(c+2i\epsilon)xf_{\text{in}}]+[\epsilon-i\kappa(1-2s)]f_{\text{in}}\}\,, \tag{2.67}$$

where

$$a=\frac{1}{2}\left(1-\sqrt{(1+2s)^2+4{}_s\lambda_{lm}}-2i\tau\right)\,, \tag{2.68a}$$

$$b=\frac{1}{2}\left(1+\sqrt{(1+2s)^2+4{}_s\lambda_{lm}}-2i\tau\right)\,, \tag{2.68b}$$

$$c=1-s-i(\epsilon+\tau)\,. \tag{2.68c}$$

In the small-ω limit, also $\epsilon\to0$ and the solution reduces to a simple hypergeometric function around $x=0$:

$$\lim_{\epsilon\to0}f_{\text{in}}(x)={}_2F_1(-l-i\tau,l+1-i\tau,1-s-i\tau;x)\,, \tag{2.69}$$

hence it is arguable that, ϵ can be used as an expansion parameter for obtaining information on the exact solution for a general value of ϵ. A crucial step for constructing such convergent series of hypergeometric functions consists in promoting l to a non-integer value ν called *renormalized* *angular momentum*. It is worth noting that, although originally introduced as a computational trick, ν now has a precise interpretation within the framework of renormalization group techniques applied to the Teukolsky equation (see [98]).

The formal in-solution can then be written as

$$f^{\nu}_{\text{in}}(x) = \sum_{n=-\infty}^{\infty} a_n f^{\nu+n}(x)\,, \tag{2.70}$$

where

$$f^{\nu+n}(x) = {}_2F_1(n+\nu+1-i\tau, -n-\nu-i\tau, 1-s-i\epsilon-i\tau; x)\,, \tag{2.71}$$

and, by substituting this expansion into the radial equation, a three-term recurrence relation for the coefficients a_n is obtained:

$$\alpha^{\nu}_n a_{n+1} + \beta^{\nu}_n a_n + \gamma^{\nu}_n a_{n-1} = 0\,, \tag{2.72}$$

where

$$\alpha^{\nu}_n = \frac{i\epsilon\kappa(n+\nu+1+s+i\epsilon)(n+\nu+1+s-i\epsilon)(n+\nu+1+i\tau)}{(n+\nu+1)(2n+2\nu+3)}\,, \tag{2.73a}$$

$$\beta^{\nu}_n = -{}_s\lambda_{lm} - s(s+1) + (n+\nu)(n+\nu+1) + \epsilon^2 + \epsilon(\epsilon-mq) + \frac{\epsilon(\epsilon-mq)(s^2+\epsilon^2)}{(n+\nu)(n+\nu+1)}\,, \tag{2.73b}$$

$$\gamma^{\nu}_n = -\frac{i\epsilon\kappa(n+\nu-s+i\epsilon)(n+\nu-s-i\epsilon)(n+\nu-i\tau)}{(n+\nu)(2n+2\nu-1)}\,. \tag{2.73c}$$

In deriving these expressions, the recurrence relations of hypergeometric functions have also been used, namely:

$$\begin{aligned} xp_{n+\nu} = &-\frac{(n+\nu+1-s-i\epsilon)(n+\nu+1-i\tau)}{2(n+\nu+1)(2n+2\nu+1)}p_{n+\nu+1} + \frac{1}{2}\left[1+\frac{i\tau(s+i\epsilon)}{(n+\nu)(n+\nu+1)}\right]p_{n+\nu} \\ &-\frac{(n+\nu+s+i\epsilon)(n+\nu+i\tau)}{2(n+\nu)(2n+2\nu+1)}p_{n+\nu-1} \end{aligned} \tag{2.74}$$

$$\begin{aligned} x(1-x)p'_{n+\nu} = &\frac{(n+\nu+i\tau)(n+\nu+1-i\tau)(n+\nu+1-s-i\epsilon)}{2(n+\nu+1)(2n+2\nu+1)}p_{n+\nu+1} \\ &+\frac{s+i\epsilon}{2}\left[1+\frac{i\tau(1-i\tau)}{(n+\nu)(n+\nu+1)}\right]p_{n+\nu} \\ &-\frac{(n+\nu+1-i\tau)(n+\nu+i\tau)(n+\nu+s+i\epsilon)}{2(n+\nu)(2n+2\nu+1)}p_{n+\nu-1}\,. \end{aligned} \tag{2.75}$$

By solving the three-term recurrence relation, one can determine the coefficients a_n of the in-solution of Eq. (2.70). The convergence of the series depends on the asymptotic behavior of these coefficients as $n \to \pm\infty$. Without delving into the technical details of Eq.(2.72), the full in-solution is written as:

$${}_s\Psi^{\text{in}}_{lm\omega} = R^{\nu}_0 + R^{-\nu-1}_0\,, \tag{2.76}$$

where

$$R_0^\nu = e^{i\epsilon\kappa x}(-x)^{-s-(i/2)(\epsilon+\tau)}(1-x)^{(i/2)(\epsilon+\tau)+\nu} \sum_{n=-\infty}^{+\infty} p_n^\nu \frac{\Gamma(1-s-i\epsilon-i\tau)}{\Gamma(n+\nu+1-i\tau)}(1-x)^n$$
$$\times \frac{\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+1-s-i\epsilon)} {}_2F_1\left(-n-\nu-i\tau, -n-\nu-s-i\epsilon, -2n-2\nu; \frac{1}{1-x}\right), \quad (2.77)$$

where p_n^ν being the minimal solutions to Eq. (2.72). For further details on the theory of three-term recurrence relations, see Sec. 4 of [104].

Solution at infinity in terms of Coulomb functions

The solution discussed above is convergent for any finite value of r but diverges as $r \to +\infty$. To properly treat this behavior, a new solution must be defined that is regular at infinity. This solution can then be matched in the overlapping region with the near-horizon solution to construct the full radial solution of the Teukolsky equation with the correct boundary conditions.

The solution at infinity, denoted ${}_s\Psi^\infty_{lm\omega}$ is not directly the up solution ${}_s\Psi^{\rm up}_{lm\omega}$ defined earlier, but it contains both incoming and outgoing wave components.

The ansatz for this solution at infinity is

$${}_s\Psi^\infty_{lm\omega}(\hat{z}) = \hat{z}^{-1-s}\left(1-\frac{\epsilon\kappa}{\hat{z}}\right)^{-s-i(\epsilon+\tau)/2} f(\hat{z})\,, \quad (2.78)$$

which leads to the following differential equation:

$$\begin{aligned}\hat{z}^2 f'' + [\hat{z}^2 + (2\epsilon+2is)\hat{z} - {}_s\lambda_{lm} - s(s+1)]f = \epsilon\{&\kappa\hat{z}(f''+f) + \kappa(s-1+2i\epsilon)f' \\ &- [\kappa - i(\epsilon - mq)](s-1+i\epsilon)\frac{f}{\hat{z}} \\ &- [2\epsilon - mq - \kappa(\epsilon+is)]f\}\,. \end{aligned} \quad (2.79)$$

The left-hand side resembles the Coulomb wave equation:

$$f'' + \left[1 + \frac{2(\epsilon+is)}{\hat{z}} - \frac{{}_s\lambda_{lm} + s(s+1)}{\hat{z}^2}\right] f = 0\,, \quad (2.80)$$

where $\hat{z}^{-2}$ coefficient reduces to $l(l+1)$ in the spherically symmetric case. The right-hand side is proportional to ϵ, which means that by using ϵ as an expansion parameter, as was done for the in-solution, one can write for $\epsilon \to 0$

$$\begin{aligned} f(\hat{z}) &= F_l(-is-\epsilon;\hat{z}) \\ &= e^{-i\hat{z}}2^l\hat{z}^{l+1}\frac{\Gamma(l+1-s+i\epsilon)}{\Gamma(2l+2)}\Phi(l+1-s+i\epsilon, 2l+2; 2i\hat{z})\,, \end{aligned} \quad (2.81)$$

where $\Phi(\cdot,\cdot;\cdot)$ is the regular confluent hypergeometric function, regular at $\hat{z}=0$, which satisfies the Coulomb equation.

As before, by introducing the renormalized angular momentum ν, the above solution can be written in other terms, as an infinite sum over Coulomb wave functions as

$$f_\nu(\hat{z}) = \sum_{n=-\infty}^{\infty} (-i)^n \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} b_n F_{n+\nu}(-is-\epsilon;\hat{z})\,, \quad (2.82)$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the notation for the Pochhammer symbol.

Substituting this expansion into the differential equation yields a recurrence relation for the coefficients b_n, which is identical to Eq. (2.72). This confirms that the same set of minimal solutions p_n^ν can be used for b_n as well.

The solution at infinity can therefore be written as the sum of incoming and outgoing components:

$$ {}_s\Psi^{\infty}_{lm\omega}(\hat{z}) := R^\nu_\infty = R^\nu_+ + R^\nu_- \,, \tag{2.83}$$

where the up-solution is then

$$\begin{aligned} {}_s\Psi^{\text{up}}_{lm\omega} &= R^\nu_- \\ &= 2^\nu e^{-\pi\epsilon} e^{-i\pi(\nu+1+s)} e^{i\hat{z}} \hat{z}^{\nu+i\epsilon_+} (\hat{z}-\epsilon\kappa)^{-s-i\epsilon_+} \sum_{n=-\infty}^{\infty} i^n p_n^\nu \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} (2\hat{z})^n \\ &\qquad \times \Psi(n+\nu+1+s-i\epsilon, 2n+2\nu+2; -2i\hat{z}) \,, \end{aligned} \tag{2.84}$$

where $\Psi(\cdot,\cdot;\cdot)$ is the irregular confluent hypergeometric function.

Matching procedure and PN expansion of the exact solutions

The two solutions in Eqs. (2.76) and (2.84) have been derived independently. However, a matching procedure is required to construct a globally valid ingoing solution that remains convergent over the entire domain $r > r_+$, including also $r \to +\infty$.

Both R^ν_∞ and R^ν_0 are convergent in the region $\epsilon\kappa < \hat{z} < \infty$. It can be shown that they possess the same analytic structure up to a multiplicative constant:

$$ R^\nu_0 = K_\nu R^\nu_\infty \,, \tag{2.85}$$

where K_ν, known as the (tidal) response function, is given by

$$\begin{aligned} K_\nu =& \frac{e^{i\epsilon\kappa}(2\epsilon\kappa)^{s-\nu-r} 2^{-s} i^r \Gamma(1-s-2i\epsilon_+)\Gamma(r+2\nu+2)}{\Gamma(r+\nu+1-s+i\epsilon)\Gamma(r+\nu+1+i\tau)\Gamma(r+\nu+1+s+i\epsilon)} \\ &\times \left(\sum_{n=r}^{\infty} (-1)^n \frac{\Gamma(n+r+2\nu+1)}{(n-r)!} \frac{\Gamma(n+\nu+1+s+i\epsilon)}{\Gamma(n+\nu+1-s-i\epsilon)} \frac{\Gamma(n+\nu+1+i\tau)}{\Gamma(n+\nu+1-i\tau)} p_n^\nu \right) \\ &\times \left(\sum_{n=-\infty}^{r} \frac{(-1)^n}{(r-n)!(r+2\nu+2)_n} \frac{(\nu+1+s-i\epsilon)_n}{(\nu+1-s+i\epsilon)_n} p_n^\nu \right)^{-1} \,, \end{aligned} \tag{2.86}$$

where $r \in \mathbb{Z}$. Although the dependence on r is not manifestly canceled in this expression, it can be shown—both analytically by expanding for small ϵ and numerically—that K_ν is indeed independent of r.

By combining Eqs. (2.76) and (2.85), the full ingoing solution can be written in terms of Coulomb wave functions:

$$ {}_s\Psi^{\text{in}}_{lm\omega} = K_\nu R^\nu_\infty + K_{-\nu-1} R^{-\nu-1}_\infty \,, \tag{2.87}$$

which is regular also at $r = \infty$.

This matching procedure yields two analytic and exact, linearly independent solutions of the radial Teukolsky equation. By adopting the post-Newtonian scalings defined earlier,

$$r = r\eta^{-2},\ \omega = \omega\eta^3\,,$$

these exact solutions can be expanded in powers of η. As an illustrative case, here are reported both expressions for $s=-2$ and $l=2$ up to η^{20} in Schwarzschild:

$$\begin{aligned}
{}_{-2}\Psi^{\rm in}_{2m\omega}(r) ={}& \frac{4r^4\omega^4}{5} + \frac{8ir^5\omega^5}{15}\eta - \frac{2}{105}r^3\omega^4\left(168+11r^3\omega^2\right)\eta^2 + \frac{2i}{35}r^4\omega^5\left(21 - r^3\omega^2 - 28\gamma_E\right)\eta^3 \\
&+ \left[\frac{16r^2\omega^4}{5} + \frac{23r^8\omega^8}{1890} - \frac{8r^5\omega^6}{315}\left(73-42\gamma_E\right)\right]\eta^4 + i\left[\frac{2}{945}r^9\omega^9 - \frac{16}{5}r^3\omega^5\left(3-2\gamma_E\right) - \frac{r^6\omega^7}{315}\left(289-132\gamma_E\right)\right]\eta^5 \\
&- \Bigg\{\frac{13r^{10}\omega^{10}}{41580} - r^7\omega^8\left(\frac{271}{945} - \frac{4\gamma_E}{35}\right) + r^4\omega^6\Bigg[\frac{856}{525}\log(-2i\omega r\eta) - \frac{4\left(17284 - 1605i\pi - 525\pi^2 + 1515\gamma_E - 3150\gamma_E^2\right)}{7875}\Bigg]\Bigg\}\eta^6 \\
&- i\Bigg\{\frac{r^{11}\omega^{11}}{24948} - \frac{16}{5}r^2\omega^5\left(3-2\gamma_E\right) - \frac{r^8\omega^9\left(755-276\gamma_E\right)}{11340} + r^5\omega^7\Bigg[\frac{1712\log(-2i\omega r\eta)}{1575} \\
&- \frac{2\left(25291 - 6420i\pi - 2100\pi^2 + 30960\gamma_E - 12600\gamma_E^2\right)}{23625}\Bigg]\Bigg\}\eta^7 \\
&+ \Bigg\{\frac{59r^{12}\omega^{12}}{12972960} - r^9\omega^{10}\left(\frac{19303}{1559250} - \frac{4\gamma_E}{945}\right) - r^3\omega^6\Bigg[\frac{4\left(21001 - 6420i\pi - 2100\pi^2 + 24960\gamma_E - 12600\gamma_E^2\right)}{7875} \\
&- \frac{3424}{525}\log(-2i\omega r\eta)\Bigg] + r^6\omega^8\Bigg[\frac{4708\log(-2i\omega r\eta)}{11025} - \frac{57341 - 247170i\pi - 80850\pi^2 + 1629810\gamma_E - 485100\gamma_E^2}{1157625}\Bigg]\Bigg\}\eta^8 \\
&+ i\Bigg\{\frac{r^{13}\omega^{13}}{2162160} - \frac{r^{10}\omega^{11}\left(4001-1300\gamma_E\right)}{2079000} + r^7\omega^9\Bigg[\frac{428\log(-2i\omega r\eta)}{3675} + \frac{173177i}{1157625} - \frac{214\pi}{3675} + \frac{2i\pi^2}{105} - \frac{15118i\gamma_E}{33075} + \frac{4i\gamma_E^2}{35}\Bigg] \\
&- r^4\omega^7\Bigg[\frac{428\left(3-4\gamma_E\right)}{525}\log(-2i\omega r\eta) + \frac{214i\pi}{175} + \frac{1486\pi^2}{1575} + \left(\frac{157532}{7875} - \frac{856i\pi}{525} - \frac{8\pi^2}{15}\right)\gamma_E - \frac{452\gamma_E^2}{525} - \frac{16\gamma_E^3}{15} \\
&- \frac{2(93757+8400\zeta(3))}{7875}\Bigg]\Bigg\}\eta^9 - \Bigg\{\frac{83r^{14}\omega^{14}}{1945944000} - r^{11}\omega^{12}\left(\frac{18272}{70945875} - \frac{\gamma_E}{12474}\right) + r^8\omega^{10}\Bigg[\frac{2461\log(-2i\omega r\eta)}{99225} \\
&+ \frac{40593821}{687629250} + \frac{2461i\pi}{198450} + \frac{23\pi^2}{5670} - \frac{21503\gamma_E}{198450} + \frac{23\gamma_E^2}{945}\Bigg] \\
&+ r^2\omega^6\Bigg[\frac{3424}{525}\log(-2i\omega r\eta) - \frac{9764}{7875} + \frac{1712i\pi}{525} + \frac{16\pi^2}{15} - \frac{6656\gamma_E}{525} + \frac{32\gamma_E^2}{5}\Bigg] - r^5\omega^8\Bigg[\frac{1712\log(-2i\omega r\eta)\left(73-42\gamma_E\right)}{33075} \\
&- \frac{65418022}{3472875} + \frac{62488i\pi}{33075} + \frac{1544\pi^2}{1575} + \left(\frac{1333028}{165375} - \frac{1712i\pi}{1575} - \frac{16\pi^2}{45}\right)\gamma_E + \frac{2416\gamma_E^2}{1575} - \frac{32\gamma_E^3}{45} - \frac{64\zeta(3)}{45}\Bigg]\Bigg\}\eta^{10} \\
&- i\Bigg\{\frac{r^{15}\omega^{15}}{277992000} - \frac{r^{12}\omega^{13}\left(137479-41300\gamma_E\right)}{4540536000} + r^9\omega^{11}\Bigg[\frac{428\log(-2i\omega r\eta)}{99225} + \frac{19763927}{1375258500} + \frac{214i\pi}{99225} + \frac{2\pi^2}{2835} \\
&- \frac{111581\gamma_E}{5457375} + \frac{4\gamma_E^2}{945}\Bigg] - r^3\omega^7\Bigg[\frac{3424\left(3-2\gamma_E\right)}{525}\log(-2i\omega r\eta) + \frac{1712i\pi}{175} + \frac{8464\pi^2}{1575} + \frac{3232\gamma_E^2}{525} - \frac{64\gamma_E^3}{15} \\
&+ \left(\frac{322088}{7875} - \frac{3424i\pi}{525} - \frac{32\pi^2}{15}\right)\gamma_E - \frac{8(75739+8400\zeta(3))}{7875}\Bigg] - r^6\omega^9\Bigg[\frac{214\log(-2i\omega r\eta)\left(289-132\gamma_E\right)}{33075} - \frac{26087867}{3472875} \\
&+ \frac{30923i\pi}{33075} + \frac{549\pi^2}{1225} + \left(\frac{759764}{385875} - \frac{4708i\pi}{11025} - \frac{44\pi^2}{315}\right)\gamma_E + \frac{10814\gamma_E^2}{11025} - \frac{88\gamma_E^3}{315} - \frac{176\zeta(3)}{315}\Bigg]\Bigg\}\eta^{11} \\
&+ \Bigg\{\frac{2656r\omega^6}{525} + \frac{37r^{16}\omega^{16}}{132324192000} - r^{13}\omega^{14}\left(\frac{8737}{2750517000} - \frac{\gamma_E}{1081080}\right) + r^{10}\omega^{12}\Bigg[\frac{1391\log(-2ir\omega)}{2182950} + \frac{1391i\pi}{4365900} \\
&+ \frac{104095001}{39332393100} + \frac{13\pi^2}{124740} - \frac{70111\gamma_E}{21829500} + \frac{13\gamma_E^2}{20790}\Bigg] + r^4\omega^8\Bigg[\frac{37191923901098}{994023646875} - \frac{58563796i\pi}{5788125} - \frac{555214\pi^2}{165375} + \frac{428i\pi^3}{1575} \\
&+ \frac{2\pi^4}{25} + \frac{91592\log^2(-2ir\omega)}{55125} - \frac{127328\gamma_E^2}{6125} + \frac{856}{525}i\pi\gamma_E^2 + \frac{8}{15}\pi^2\gamma_E^2 + \frac{872\gamma_E^3}{525} + \frac{8\gamma_E^4}{15} \\
&- \log(-2ir\omega\eta)\left(\frac{117127592}{5788125} - \frac{91592i\pi}{55125} - \frac{856\pi^2}{1575} + \frac{86456\gamma_E}{55125} - \frac{1712\gamma_E^2}{525}\right) - \frac{43228i\pi\gamma_E}{55125} - \frac{2116\pi^2\gamma_E}{1575} \\
&+ \gamma_E\left(\frac{158517988}{5788125} + \frac{64\zeta(3)}{15}\right) - \frac{5104\zeta(3)}{525}\Bigg] + r^7\omega^{10}\Bigg[\frac{214\log(-2ir\omega\eta)\left(-271+108\gamma_E\right)}{99225} + \frac{701543978}{343814625} - \frac{990728\gamma_E}{3472875} \\
&+ \frac{8\gamma_E^3}{105} - \pi\left(\frac{28997i}{99225} - \frac{428i\gamma_E}{3675}\right) - \pi^2\left(\frac{13337}{99225} - \frac{4\gamma_E}{105}\right) - \frac{11266\gamma_E^2}{33075} + \frac{16\zeta(3)}{105}\Bigg]\Bigg\}\eta^{12} \\
&+ i\Bigg\{\frac{128\omega^5}{75r} + \frac{r^{17}\omega^{17}}{49621572000} - r^{14}\omega^{15}\left(\frac{103303}{343264521600} - \frac{83\gamma_E}{972972000}\right) + r^{11}\omega^{13}\Bigg[\frac{107\log(-2ir\omega)}{1309770} + \frac{236096513}{589985896500}
\end{aligned}$$

$$+\frac{107i\pi}{2619540}+\frac{\pi^2}{74844}-\frac{184489\gamma_E}{425675250}+\frac{\gamma_E^2}{12474}\Big]+r^5\omega^9\Big[-\frac{349660015676}{60858590625}-\frac{51447782i\pi}{17364375}-\frac{20194\pi^2}{70875}+\frac{856i\pi^3}{4725}+\frac{4\pi^4}{75}$$

$$+\frac{183184\log^2(-2ir\omega)}{165375}-\frac{253532\gamma_E^2}{23625}+\frac{1712i\pi\gamma_E^2}{1575}+\frac{16}{45}\pi^2\gamma_E^2-\frac{1408\gamma_E^3}{4725}+\frac{16\gamma_E^4}{45}-\frac{6272\zeta(3)}{675}$$

$$-\log(-2ir\omega\eta)\left(\frac{102895564}{17364375}-\frac{183184i\pi}{165375}-\frac{1712\pi^2}{4725}+\frac{294464\gamma_E}{55125}-\frac{3424\gamma_E^2}{1575}\right)-\frac{147232i\pi\gamma_E}{55125}-\frac{7552\pi^2\gamma_E}{4725}$$

$$+\gamma_E\left(\frac{183761552}{5788125}+\frac{128\zeta(3)}{45}\right)\Big]-r^2\omega^7\Big[\frac{3424\,(3-2\gamma_E)}{525}\log(-2ir\omega\eta)+\pi\left(\frac{1712i}{175}-\frac{3424i\gamma_E}{525}\right)+\frac{173608\gamma_E}{7875}$$

$$+\frac{3232\gamma_E^2}{525}-\frac{64\gamma_E^3}{15}+\pi^2\left(\frac{8464}{1575}-\frac{32\gamma_E}{15}\right)+\frac{128(367+75\zeta(3))}{1125}\Big]-r^8\omega^{11}\Big[\frac{107\log(-2ir\omega\eta)\,(755-276\gamma_E)}{595350}$$

$$-\frac{1747351621}{4125775500}+\pi\left(\frac{16157i}{238140}-\frac{2461i\gamma_E}{99225}\right)+\pi^2\left(\frac{36269}{1190700}-\frac{23\gamma_E}{2835}\right)+\frac{12119033\gamma_E}{687629250}+\frac{5527\gamma_E^2}{66150}-\frac{46\gamma_E^3}{2835}-\frac{92\zeta(3)}{2835}\Big]\Big\}\eta^{13}$$

$$+\Big\{\frac{3344\omega^6}{525}-\frac{r^{18}\omega^{18}}{738424512000}+r^{15}\omega^{16}\left(\frac{8345581}{320952327696000}-\frac{\gamma_E}{138996000}\right)-r^{12}\omega^{14}\Big[\frac{6313\log(-2ir\omega)}{681080400}$$

$$+\frac{1583976763}{30679266618000}+\frac{6313i\pi}{1362160800}+\frac{59\pi^2}{38918880}-\frac{49901\gamma_E}{972972000}+\frac{59\gamma_E^2}{6486480}\Big]+r^9\omega^{12}\Big[\frac{107\log(-2ir\omega\eta)\,(19303-6600\gamma_E)}{81860625}$$

$$+\pi^2\left(\frac{182201}{32744250}-\frac{4\gamma_E}{2835}\right)+\pi\left(\frac{2065421i}{163721250}-\frac{428i\gamma_E}{99225}\right)-\frac{526571927459}{7374823706250}-\frac{12071953\gamma_E}{3438146250}$$

$$+\frac{29347\gamma_E^2}{1819125}-\frac{8\gamma_E^3}{2835}-\frac{16\zeta(3)}{2835}\Big]+r^6\omega^{10}\Big[\frac{40117726232293546}{4822008710990625}+\frac{14476871i\pi}{40516875}-\frac{108202\pi^2}{385875}-\frac{2354i\pi^3}{33075}$$

$$-\frac{11\pi^4}{525}-\frac{503756\log^2(-2ir\omega\eta)}{1157625}+\frac{62542\gamma_E^2}{18375}-\frac{4708i\pi\gamma_E^2}{11025}-\frac{44}{315}\pi^2\gamma_E^2+\frac{12212\gamma_E^3}{33075}-\frac{44\gamma_E^4}{315}$$

$$+\frac{553618i\pi\gamma_E}{385875}\log(-2ir\omega)\left(\frac{28953742}{40516875}-\frac{503756i\pi}{1157625}-\frac{4708\pi^2}{33075}+\frac{1107236\gamma_E}{385875}-\frac{9416\gamma_E^2}{11025}\right)+\frac{24938\pi^2\gamma_E}{33075}+\frac{137416\zeta(3)}{33075}$$

$$-\frac{88\gamma_E(19764653+1543500\zeta(3))}{121550625}\Big]-r^3\omega^8\Big[\frac{154901600372}{994023646875}-\frac{90042724i\pi}{5788125}-\frac{333796\pi^2}{165375}+\frac{1712i\pi^3}{1575}+\frac{8\pi^4}{25}$$

$$+\frac{366368\log^2(-2ir\omega\eta)}{55125}-\frac{988936\gamma_E^2}{18375}+\frac{3424}{525}i\pi\gamma_E^2+\frac{32}{15}\pi^2\gamma_E^2+\frac{128\gamma_E^3}{525}+\frac{32\gamma_E^4}{15}-\frac{712192i\pi\gamma_E}{55125}-\frac{13504\pi^2\gamma_E}{1575}$$

$$-\log(-2ir\omega\eta)\left(\frac{180085448}{5788125}-\frac{366368i\pi}{55125}-\frac{3424\pi^2}{1575}+\frac{1424384\gamma_E}{55125}-\frac{6848\gamma_E^2}{525}\right)+\gamma_E\left(\frac{710605192}{5788125}+\frac{256\zeta(3)}{15}\right)$$

$$-\frac{27136\zeta(3)}{525}\Big]\Big\}\eta^{14}$$

$$+i\Big\{\frac{128\omega^5}{75r^2}-\frac{r^{19}\omega^{19}}{11732745024000}+r^{16}\omega^{17}\left(\frac{883787}{427936436928000}-\frac{37\gamma_E}{66162096000}\right)+\frac{16}{525}r\omega^7\,(247-332\gamma_E)$$

$$-r^{13}\omega^{15}\Big[\frac{107\log(-2ir\omega\eta)}{113513400}+\frac{238775461}{40905688824000}+\frac{107i\pi}{227026800}+\frac{\pi^2}{6486480}-\frac{17587\gamma_E}{3250611000}+\frac{\gamma_E^2}{1081080}\Big]$$

$$+r^{10}\omega^{13}\Big[\frac{107\log(-2ir\omega\eta)\,(4001-1300\gamma_E)}{109147500}-\frac{12463848086}{1229137284375}+\pi^2\left(\frac{111841}{130977000}-\frac{13\gamma_E}{62370}\right)$$

$$+\pi\left(\frac{428107i}{218295000}-\frac{1391i\gamma_E}{2182950}\right)-\frac{1347934087\gamma_E}{983309827500}+\frac{56201\gamma_E^2}{21829500}-\frac{13\gamma_E^3}{31185}-\frac{26\zeta(3)}{31185}\Big]$$

$$+r^7\omega^{11}\Big[\frac{16879134647742538}{4822008710990625}-\frac{8358541i\pi}{121550625}-\frac{1621952\pi^2}{10418625}-\frac{214i\pi^3}{11025}-\frac{\pi^4}{175}-\frac{45796\log^2(-2ir\omega)}{385875}+\frac{372622\gamma_E^2}{496125}$$

$$-\frac{428i\pi\gamma_E^2}{3675}-\frac{4}{105}\pi^2\gamma_E^2+\frac{14828\gamma_E^3}{99225}-\frac{4\gamma_E^4}{105}-\frac{1617626i\pi\gamma_E}{3472875}\log(-2ir\omega\eta)\left(\frac{16717082}{121550625}+\frac{45796i\pi}{385875}+\frac{428\pi^2}{11025}\right.$$

$$\left.-\frac{3235252\gamma_E}{3472875}+\frac{856\gamma_E^2}{3675}\right)+\frac{22822\pi^2\gamma_E}{99225}+\frac{122104\zeta(3)}{99225}-\gamma_E\left(\frac{50763069578}{12033511875}+\frac{32\zeta(3)}{105}\right)\Big]+r^4\omega^9\Big[\frac{182870514061952}{994023646875}$$

$$+\frac{159002i\pi^3}{165375}+\frac{1171\pi^4}{2625}+\frac{2195176\gamma_E^3}{165375}-\frac{1712i\pi\gamma_E^3}{1575}-\frac{16}{45}\pi^2\gamma_E^3-\frac{2164\gamma_E^4}{1575}-\frac{16\gamma_E^5}{75}+\frac{45796\log^2(-2ir\omega\eta)\,(3-4\gamma_E)}{55125}$$

$$-\gamma_E^2\left(\frac{26964656}{5788125}+\frac{48364i\pi}{55125}-\frac{4\pi^2}{5}+\frac{64\zeta(3)}{15}\right)-\pi^2\left(\frac{287408297}{17364375}+\frac{32\zeta(3)}{45}\right)+\frac{1154672\zeta(3)}{165375}-\frac{128\zeta(5)}{25}$$

$$+\frac{8i\pi(18827423+1572900\zeta(3))}{5788125}-\gamma_E\left(\frac{126117087624676}{994023646875}-\frac{131553332i\pi}{5788125}-\frac{476144\pi^2}{55125}+\frac{856i\pi^3}{1575}-\frac{4\pi^4}{25}-\frac{23776\zeta(3)}{1575}\right)$$

$$+\log(-2ir\omega)\left(\pi\left(\frac{45796i}{18375}-\frac{183184i\gamma_E}{55125}\right)+\frac{263106664\gamma_E}{5788125}-\frac{96728\gamma_E^2}{55125}-\frac{3424\gamma_E^3}{1575}+\pi^2\left(\frac{318004}{165375}-\frac{1712\gamma_E}{1575}\right)\right.$$

$$\left.-\frac{4(75309692+6291600\zeta(3))}{5788125}\right)\Big]\Big\}\eta^{15}+\Big\{\frac{179r^{20}\omega^{20}}{35479820952576000}-r^{17}\omega^{18}\left(\frac{3938497}{25944619071571200}-\frac{\gamma_E}{24810786000}\right)$$

$$+\frac{\omega^6}{r}\left[\frac{256}{75}\log\left(\frac{r}{2M}\right)+\frac{64}{375}\,(9+20\gamma_E)\right]+r^{14}\omega^{16}\Big[\frac{8881\log(-2ir\omega)}{102162060000}+\frac{184080957847}{312928519503600000}+\frac{8881i\pi}{204324120000}$$

$$
\begin{aligned}
&+\frac{83\pi^2}{5837832000}-\frac{2209573\gamma_E}{4290806520000}+\frac{83\gamma_E^2}{972972000}\Bigg]-r^{11}\omega^{14}\Bigg[\pi\left(\frac{1955104i}{7449316875}-\frac{107i\gamma_E}{1309770}\right)-\frac{\gamma_E^3}{18711}-\frac{2\zeta(3)}{18711}\\
&+\frac{107\log(-2ir\omega\eta)\,(36544-11375\gamma_E)}{7449316875}-\frac{1486060993003}{1193082590700000}+\pi^2\left(\frac{144407}{1277025750}-\frac{\gamma_E}{37422}\right)-\frac{406261381\gamma_E}{1474964741250}\\
&+\frac{74857\gamma_E^2}{212837625}\Bigg]-r^2\omega^8\Bigg[\frac{20045917562848}{994023646875}+\frac{11478988i\pi}{1929375}-\frac{185884\pi^2}{165375}-\frac{1712i\pi^3}{1575}-\frac{8\pi^4}{25}-\frac{366368\log^2(-2ir\omega\eta)}{55125}\\
&+\frac{8\left(240931-44940i\pi-14700\pi^2\right)\gamma_E^2}{55125}-\frac{128\gamma_E^3}{525}-\frac{32\gamma_E^4}{15}+\frac{712192i\pi\gamma_E}{55125}+\frac{13504\pi^2\gamma_E}{1575}+\frac{27136\zeta(3)}{525}\\
&+\log(-2ir\omega)\left(\frac{22957976}{1929375}-\frac{366368i\pi}{55125}-\frac{3424\pi^2}{1575}+\frac{1424384\gamma_E}{55125}-\frac{6848\gamma_E^2}{525}\right)-\gamma_E\left(\frac{46056568}{643125}+\frac{256\zeta(3)}{15}\right)\Bigg]\Bigg]\\
&-r^8\omega^{12}\Bigg[\frac{17508826377753849986}{18617775633134803125}-\frac{12227428i\pi}{288804285}-\frac{189592873\pi^2}{4125775500}-\frac{2461i\pi^3}{595350}-\frac{23\pi^4}{18900}-\frac{263327\log^2(-2ir\omega\eta)}{10418625}\\
&+\frac{6289009\gamma_E^2}{49116375}-\frac{2461i\pi\gamma_E^2}{99225}-\frac{23\pi^2\gamma_E^2}{2835}+\frac{11659\gamma_E^3}{297675}-\frac{23\gamma_E^4}{2835}+\frac{2300821i\pi\gamma_E}{20837250}+\frac{129\pi^2\gamma_E}{2450}-\frac{82382\zeta(3)}{297675}\\
&-\gamma_E\left(\frac{13454204147}{14440214250}+\frac{184\zeta(3)}{2835}\right)-\log(-2ir\omega\eta)\left(\frac{24454856}{288804285}+\frac{263327i\pi}{10418625}+\frac{2461\pi^2}{297675}-\frac{2300821\gamma_E}{10418625}+\frac{4922\gamma_E^2}{99225}\right)\Bigg]\Bigg]\\
&-r^5\omega^{10}\Bigg[\frac{629858304345042344}{535778745665625}+\frac{165208i\pi^3}{165375}+\frac{3172\pi^4}{7875}+\frac{4066472\gamma_E^3}{496125}-\frac{3424i\pi\gamma_E^3}{4725}-\frac{32}{135}\pi^2\gamma_E^3+\frac{16\gamma_E^4}{75}\\
&+\frac{32\gamma_E^5}{225}+\frac{183184\log^2(-2ir\omega\eta)\,(73-42\gamma_E)}{3472875}+\frac{258512i\pi\gamma_E^2}{165375}+\frac{1168}{945}\pi^2\gamma_E^2-\gamma_E^2\left(\frac{127175644}{5788125}+\frac{128\zeta(3)}{45}\right)\\
&+\gamma_E\left(\frac{674879068579004}{20874496584375}-\frac{56585908i\pi}{5788125}-\frac{1273964\pi^2}{496125}+\frac{1712i\pi^3}{4725}+\frac{8\pi^4}{75}-\frac{24704\zeta(3)}{1575}\right)+\frac{6864176\zeta(3)}{496125}\\
&+\pi^2\left(\frac{21497242}{1929375}+\frac{64\zeta(3)}{135}\right)-\frac{2i\pi(1331624071+88082400\zeta(3))}{121550625}\log(-2ir\omega\eta)\left(\frac{113171816\gamma_E}{5788125}+\frac{517024\gamma_E^2}{165375}\right.\\
&\left.+\pi\left(\frac{13372432i}{3472875}-\frac{366368i\gamma_E}{165375}\right)+\pi^2\left(\frac{330416}{165375}-\frac{3424\gamma_E}{4725}\right)-\frac{6848\gamma_E^3}{4725}-\frac{4(1331624071+88082400\zeta(3))}{121550625}\right)\\
&+\frac{256\zeta(5)}{75}\Bigg]\Bigg]\Bigg\}\eta^{16}\\
&+i\Bigg\{\frac{1024\omega^5}{525r^3}+\frac{r^{21}\omega^{21}}{3547982095257600}-\omega^7\left[\frac{256}{75}\log\left(\frac{r}{2M}\right)-\frac{8\left(2551-4180\gamma_E\right)}{2625}\right]-r^{18}\omega^{19}\left(\frac{59275313}{5707816195745664000}\right.\\
&\left.-\frac{\gamma_E}{369212256000}\right)+r^{15}\omega^{17}\Bigg[\frac{107\log(-2ir\omega)}{14594580000}+\frac{16762173833}{3129285195036000000}+\frac{107i\pi}{29189160000}+\frac{\pi^2}{833976000}\\
&-\frac{35845259\gamma_E}{802380819240000}+\frac{\gamma_E^2}{138996000}\Bigg]+r^9\omega^{13}\Bigg[-\frac{550624461251410199}{2864273174328431250}+\frac{1667198677i\pi}{144402142500}+\frac{18573449\pi^2}{1875352500}+\frac{214i\pi^3}{297675}\\
&+\frac{\pi^4}{4725}+\frac{45796\log^2(-2ir\omega\eta)}{10418625}-\frac{8509007\gamma_E^2}{491163750}+\frac{428i\pi\gamma_E^2}{99225}+\frac{4\pi^2\gamma_E^2}{2835}-\frac{129002\gamma_E^3}{16372125}+\frac{4\gamma_E^4}{2835}-\frac{822964\zeta(3)}{16372125}\\
&+\log(-2ir\omega\eta)\left(\frac{1667198677}{72201071250}+\frac{45796i\pi}{10418625}+\frac{428\pi^2}{297675}-\frac{23878334\gamma_E}{573024375}+\frac{856\gamma_E^2}{99225}\right)\\
&+\gamma_E\left(\frac{8564054038481}{51623765943750}-\frac{11939167i\pi}{573024375}-\frac{17629\pi^2}{1819125}+\frac{32\zeta(3)}{2835}\right)\Bigg]-r^{12}\omega^{15}\Bigg[\frac{107\log(-2ir\omega\eta)\,(137479-41300\gamma_E)}{238378140000}\\
&-\frac{321472653223}{2386165181400000}+\pi^2\left(\frac{538697}{40864824000}-\frac{59\gamma_E}{19459440}\right)+\pi\left(\frac{14710253i}{476756280000}-\frac{6313i\gamma_E}{681080400}\right)-\frac{1821062807\gamma_E}{43827523740000}\\
&+\frac{286177\gamma_E^2}{6810804000}-\frac{59\gamma_E^3}{9729720}-\frac{59\zeta(3)}{4864860}\Bigg]+r^6\omega^{11}\Bigg[-\frac{66606632446609622}{1607336236996875}-\frac{19581i\pi^3}{42875}-\frac{19531\pi^4}{110250}\\
&-\frac{8\left(1274311-123585i\pi-40425\pi^2\right)\gamma_E^3}{3472875}-\frac{466\gamma_E^4}{11025}+\frac{88\gamma_E^5}{1575}-\frac{22898\log^2(-2ir\omega\eta)\,(289-132\gamma_E)}{3472875}-\frac{10194488\gamma_E^3}{3472875}\\
&+\frac{9416i\pi\gamma_E^3}{33075}+\frac{88}{945}\pi^2\gamma_E^3+\gamma_E\left(\frac{6546349618375048}{4822008710990625}-\frac{318474496i\pi}{121550625}-\frac{136054\pi^2}{385875}+\frac{4708i\pi^3}{33075}+\frac{22\pi^4}{525}-\frac{8784\zeta(3)}{1225}\right)\\
&+\pi^2\left(\frac{3042132857}{729303750}+\frac{176\zeta(3)}{945}\right)+\frac{31561664\zeta(3)}{3472875}+\frac{2i\pi(546887351+34603800\zeta(3))}{121550625}\\
&-\log(-2ir\omega\eta)\left(\pi\left(\frac{6617522i}{3472875}-\frac{1007512i\gamma_E}{1157625}\right)+\pi^2\left(\frac{39162}{42875}-\frac{9416\gamma_E}{33075}\right)+\frac{636948992\gamma_E}{121550625}+\frac{2314196\gamma_E^2}{1157625}\right.\\
&\left.-\frac{18832\gamma_E^3}{33075}+\frac{4(546887351+34603800\zeta(3))}{121550625}\right)+\frac{704\zeta(5)}{525}\Bigg]+r^3\omega^9\Bigg[-\frac{144359913983072}{331341215625}-\frac{905648i\pi^3}{165375}-\frac{5944\pi^4}{2625}\\
&-\frac{6625264\gamma_E^3}{165375}+\frac{6848i\pi\gamma_E^3}{1575}+\frac{64}{45}\pi^2\gamma_E^3+\frac{3616\gamma_E^4}{1575}+\frac{64\gamma_E^5}{75}-\frac{366368\log^2(-2ir\omega\eta)\,(3-2\gamma_E)}{55125}+\gamma_E\left(\frac{184134697154584}{994023646875}\right.\\
&\left.-\frac{295491368i\pi}{5788125}-\frac{275432\pi^2}{18375}+\frac{3424i\pi^3}{1575}+\frac{16\pi^4}{25}-\frac{135424\zeta(3)}{1575}\right)+\frac{12634912\zeta(3)}{165375}+\pi^2\left(\frac{885094088}{17364375}+\frac{128\zeta(3)}{45}\right)
\end{aligned}
$$

$$+\gamma_E^2\left(\frac{415113824}{5788125}-\frac{345824 i\pi}{55125}-\frac{32\pi^2}{5}+\frac{256\zeta(3)}{15}\right)+\frac{8i\pi(66899909+6291600\zeta(3))}{5788125}+\frac{512\zeta(5)}{25}$$

$$-\log(-2ir\omega\eta)\Bigg(\pi\left(\frac{366368i}{18375}-\frac{732736i\gamma_E}{55125}\right)+\frac{590982736\gamma_E}{5788125}+\frac{691648\gamma_E^2}{55125}-\frac{13696\gamma_E^3}{1575}+\pi^2\left(\frac{1811296}{165375}-\frac{6848\gamma_E}{1575}\right)$$

$$-\frac{16(66899909+6291600\zeta(3))}{5788125}\Bigg)\Bigg]\Bigg\}\eta^{17}$$

$$+\Bigg\{-\frac{73r^{22}\omega^{22}}{4896215291455488000}+\frac{\omega^6}{r^2}\left(\frac{256}{75}\log\left(\frac{r}{2M}\right)-\frac{64\left(107-140\gamma_E\right)}{2625}\right)+r^{19}\omega^{20}\Bigg(\frac{189636277}{285390809787283200000}$$

$$-\frac{\gamma_E}{5866372512000}\Bigg)+r^{16}\omega^{18}\Bigg[-\frac{3959\log(-2ir\omega)}{6947020080000}-\frac{2699646214073}{606455470797976800000}-\frac{3959i\pi}{13894040160000}-\frac{37\pi^2}{396972576000}$$

$$+\frac{3809249\gamma_E}{1069841092320000}-\frac{37\gamma_E^2}{66162096000}\Bigg]+r\omega^8\Bigg[-\frac{568384\log(-2ir\omega\eta)}{55125}-\frac{128}{75}\log\left(\frac{r}{2M}\right)+\frac{501588256}{17364375}-\frac{284192i\pi}{55125}$$

$$-\frac{2656\pi^2}{1575}+\frac{261536\gamma_E}{55125}-\frac{5312\gamma_E^2}{525}\Bigg]+r^{13}\omega^{16}\Bigg[\frac{107\log(-2ir\omega\eta)\left(113581-33075\gamma_E\right)}{1877227852500}-\frac{4607613587497}{353849941284840000}$$

$$+\pi^2\left(\frac{73643}{53635081500}-\frac{\gamma_E}{3243240}\right)+\pi\left(\frac{934859i}{288804285000}-\frac{107i\gamma_E}{113513400}\right)-\frac{11168200357\gamma_E}{2147548663260000}+\frac{19969\gamma_E^2}{4469590125}$$

$$-\frac{\gamma_E^3}{1621620}-\frac{\zeta(3)}{810810}\Bigg]+r^{10}\omega^{14}\Bigg[\frac{543612818795911817333}{170389882594449718200000}-\frac{23118255547i\pi}{10324753188750}-\frac{10045083691\pi^2}{5899858965000}$$

$$-\frac{1391i\pi^3}{13097700}-\frac{13\pi^4}{415800}-\frac{148837\log^2(-2ir\omega\eta)}{229209750}+\frac{133597939\gamma_E^2}{70236416250}-\frac{1391i\pi\gamma_E^2}{2182950}-\frac{13\pi^2\gamma_E^2}{62370}+\frac{4699\gamma_E^3}{3638250}-\frac{13\gamma_E^4}{62370}$$

$$-\log(-2ir\omega\eta)\Bigg(\frac{23118255547}{5162376594375}+\frac{148837i\pi}{229209750}+\frac{1391\pi^2}{6548850}-\frac{7501877\gamma_E}{1146048750}+\frac{1391\gamma_E^2}{1091475}\Bigg)+\frac{41917\zeta(3)}{5457375}+\frac{7501877i\pi\gamma_E}{2292097500}$$

$$+\frac{97931\pi^2\gamma_E}{65488500}+\gamma_E\left(-\frac{639072897347}{25811882971875}-\frac{52\zeta(3)}{31185}\right)\Bigg]+r^7\omega^{12}\Bigg[\frac{751767687171210550429}{744711025325392125 00}+\frac{1427059i\pi^3}{10418625}+\frac{17189\pi^4}{330750}$$

$$+\frac{2542544\gamma_E^3}{3472875}-\frac{856i\pi\gamma_E^3}{11025}-\frac{8}{315}\pi^2\gamma_E^3+\frac{3562\gamma_E^4}{99225}-\frac{8\gamma_E^5}{525}+\frac{22898\log^2(-2ir\omega\eta)\left(271-108\gamma_E\right)}{10418625}$$

$$-\pi^2\left(\frac{78050444429}{72201071250}+\frac{16\zeta(3)}{315}\right)-\gamma_E^2\left(\frac{45250872206}{12033511875}-\frac{1205462i\pi}{3472875}-\frac{542\pi^2}{2835}+\frac{32\zeta(3)}{105}\right)-\frac{1239248\zeta(3)}{385875}$$

$$+\gamma_E\Bigg(\frac{28965245680113278}{14466026132971875}+\frac{167036284i\pi}{364651875}-\frac{389786\pi^2}{10418625}-\frac{428i\pi^3}{11025}-\frac{2\pi^4}{175}+\frac{213392\zeta(3)}{99225}\Bigg)$$

$$-\frac{i\pi(18045134873+1121163120\zeta(3))}{7220107125}+\log(-2ir\omega\eta)\Bigg(-\frac{36090269746}{7220107125}+\pi^2\left(\frac{2854118}{10418625}-\frac{856\gamma_E}{11025}\right)$$

$$+\frac{334072568\gamma_E}{364651875}+\pi\left(\frac{6205358i}{10418625}-\frac{91592i\gamma_E}{385875}\right)+\frac{2410924\gamma_E^2}{3472875}-\frac{1712\gamma_E^3}{11025}-\frac{3424\zeta(3)}{11025}\Bigg)-\frac{64\zeta(5)}{175}\Bigg]$$

$$+r^4\omega^{10}\Bigg[\frac{3023038491925919411 66}{2579774660379984375}+\frac{61288658i\pi^3}{17364375}+\frac{987302\pi^4}{2480625}-\frac{214i\pi^5}{2625}-\frac{122\pi^6}{4725}-\frac{3584}{8025}\log\left(\frac{r}{2M\eta^2}\right)$$

$$-\frac{19600688\log^3(-2ir\omega)}{17364375}+\frac{957632\gamma_E^4}{165375}-\frac{856i\pi\gamma_E^4}{1575}-\frac{8}{45}\pi^2\gamma_E^4-\frac{1208\gamma_E^5}{1575}-\frac{16\gamma_E^6}{225}+\log^2(-2ir\omega\eta)\Bigg(\frac{4661257264}{202584375}$$

$$-\frac{9800344i\pi}{5788125}-\frac{91592\pi^2}{165375}+\frac{9250792\gamma_E}{5788125}-\frac{183184\gamma_E^2}{55125}\Bigg)-\frac{93304i\pi\gamma_E^3}{55125}+\frac{808\pi^2\gamma_E^3}{4725}+\gamma_E^3\left(\frac{218428144}{17364375}-\frac{128\zeta(3)}{45}\right)$$

$$+\pi\left(-\frac{827256559749982i}{15021833990625}+\frac{546128i\zeta(3)}{55125}\right)-\frac{2609414176\zeta(3)}{17364375}-\frac{128\zeta(3)^2}{45}+\pi^2\left(-\frac{258436245855194}{20874496584375}+\frac{4432\zeta(3)}{945}\right)$$

$$+\gamma_E^2\Bigg(-\frac{1084852088883484}{6958165528125}+\frac{19454104i\pi}{826875}+\frac{1654844\pi^2}{165375}-\frac{856i\pi^3}{1575}-\frac{4\pi^4}{25}+\frac{16928\zeta(3)}{1575}\Bigg)$$

$$+\log(-2ir\omega\eta)\Bigg(-\frac{1654513119499964}{15021833990625}-\frac{91592i\pi^3}{165375}-\frac{428\pi^4}{2625}+\frac{38908208\gamma_E^2}{826875}-\frac{186608\gamma_E^3}{55125}-\frac{1712\gamma_E^4}{1575}+\frac{1092256\zeta(3)}{55125}$$

$$+\pi^2\left(\frac{26475532}{3472875}+\frac{452824\gamma_E}{165375}-\frac{1712\gamma_E^2}{1575}\right)-\frac{64\gamma_E(61271871+9175250\zeta(3))}{67528125}+\pi\Bigg(\frac{4661257264i}{202584375}+\frac{9250792i\gamma_E}{5788125}$$

$$-\frac{183184i\gamma_E^2}{55125}\Bigg)\Bigg)+\frac{17728\zeta(5)}{525}+\gamma_E\Bigg(\frac{82874067737971324}{321467247399375}+\frac{226412i\pi^3}{165375}+\frac{638\pi^4}{875}+\frac{798016\zeta(3)}{23625}-\frac{256\zeta(5)}{25}$$

$$-\pi^2\left(\frac{9029366}{354375}+\frac{64\zeta(3)}{45}\right)-\frac{32i\pi(61271871+9175250\zeta(3))}{67528125}\Bigg)\Bigg]\Bigg\}\eta^{18}$$

$$+i\Bigg\{\frac{256\omega^5}{105r^4}-\frac{r^{23}\omega^{23}}{1335331443124224000}+r^{20}\omega^{21}\left(\frac{273527797}{684937943489479680000 0}-\frac{179\gamma_E}{17739910476288000}\right)$$

$$-r^{17}\omega^{19}\Bigg[\frac{107\log(-2ir\omega\eta)}{2605132530000}+\frac{1238927186459}{363873282478786080000 0}+\frac{107i\pi}{5210265060000}+\frac{\pi^2}{148864716000}$$

$$-\frac{85142209\gamma_E}{324307738394640000}+\frac{\gamma_E^2}{24810786000}\Bigg]+\frac{\omega^7}{r}\Bigg[\frac{72704}{7875}\log\left(\frac{r}{2M}\right)-\frac{27392\log(-2ir\omega\eta)}{7875}-\frac{512}{75}\log\left(\frac{r}{2M}\right)\gamma_E$$

$$
\begin{aligned}
&+\frac{29348768}{826875}-\frac{13696i\pi}{7875}+\frac{128\pi^2}{75}-\frac{51584\gamma_E}{7875}-\frac{256\gamma_E^2}{75}\Big]+r^{14}\omega^{17}\Big[\frac{107\log(-2ir\omega\eta)\left(516515-146412\gamma_E\right)}{90106936920000}\\
&-\frac{156861885276293}{138001477101087600000}+\pi^2\left(\frac{3328579}{25744839120000}-\frac{83\gamma_E}{2918916000}\right)+\pi\left(\frac{11053421i}{36042774768000}-\frac{8881i\gamma_E}{102162060000}\right)\\
&-\frac{308397521827\gamma_E}{547624909131300000}+\frac{166961\gamma_E^2}{390073320000}-\frac{83\gamma_E^3}{1459458000}-\frac{83\zeta(3)}{729729000}\Big]\\
&+r^{11}\omega^{15}\Big[\frac{573269068740518371999}{127792411945837288650000}-\frac{10813026053i\pi}{30974259566250}-\frac{4352735141\pi^2}{17699576895000}-\frac{107i\pi^3}{7858620}-\frac{\pi^4}{249480}\\
&-\frac{11449\log^2(-2ir\omega)}{137525850}+\left(\frac{122584639}{737482370625}-\frac{107i\pi}{1309770}-\frac{\pi^2}{37422}\right)\gamma_E^2+\frac{43\gamma_E^3}{238875}-\frac{\gamma_E^4}{37422}+\frac{19740323i\pi\gamma_E}{44695901250}\\
&-\log(-2ir\omega\eta)\Big(\frac{10813026053}{15487129783125}+\frac{11449i\pi}{137525850}+\frac{107\pi^2}{3929310}-\frac{19740323\gamma_E}{22347950625}+\frac{107\gamma_E^2}{654885}\Big)+\frac{254039\pi^2\gamma_E}{1277025750}+\frac{30818\zeta(3)}{30405375}\\
&-\gamma_E\left(\frac{51369193906201}{16106614974450000}+\frac{4\zeta(3)}{18711}\right)\Big]\Big]+r^2\omega^9\Big[\frac{9478648347172}{60858590625}+\frac{905648i\pi^3}{165375}+\frac{5944\pi^4}{2625}+\frac{128}{225}\log\left(\frac{r}{2M}\right)\\
&+\frac{4546544\gamma_E^3}{165375}-\frac{6848i\pi\gamma_E^3}{1575}-\frac{64}{45}\pi^2\gamma_E^3-\frac{3616\gamma_E^4}{1575}-\frac{64\gamma_E^5}{75}-\frac{16792352\zeta(3)}{165375}+\frac{366368\log^2(-2ir\omega\eta)\left(3-2\gamma_E\right)}{55125}\\
&+\frac{345824i\pi\gamma_E^2}{55125}+\frac{32}{5}\pi^2\gamma_E^2-\gamma_E^2\left(\frac{76743088}{1929375}+\frac{256\zeta(3)}{15}\right)-\pi^2\left(\frac{21119584}{643125}+\frac{128\zeta(3)}{45}\right)-\gamma_E\Big(\frac{72451726795744}{994023646875}\\
&-\frac{61426616i\pi}{1929375}-\frac{1439528\pi^2}{165375}+\frac{3424i\pi^3}{1575}+\frac{16\pi^4}{25}-\frac{135424\zeta(3)}{1575}\Big)-\frac{16i\pi(20479147+3145800\zeta(3))}{5788125}\\
&-\frac{512\zeta(5)}{25}+\log(-2ir\omega\eta)\Big(\pi^2\left(\frac{1811296}{165375}-\frac{6848\gamma_E}{1575}\right)+\pi\left(\frac{366368i}{18375}-\frac{732736i\gamma_E}{55125}\right)+\frac{122853232\gamma_E}{1929375}\\
&+\frac{691648\gamma_E^2}{55125}-\frac{13696\gamma_E^3}{1575}-\frac{655332704}{5788125}-\frac{27392\zeta(3)}{1575}\Big)\Big]+r^8\omega^{13}\Big[\frac{419312106551592623657}{223413307597617637500}+\frac{3880783i\pi^3}{125023500}\\
&+\frac{15371\pi^4}{1323000}+\frac{146593637\gamma_E^3}{1031443875}-\frac{4922i\pi\gamma_E^3}{297675}-\frac{46\pi^2\gamma_E^3}{8505}+\frac{6737\gamma_E^4}{595350}-\frac{46\gamma_E^5}{14175}+\frac{11449\log^2(-2ir\omega\eta)\left(755-276\gamma_E\right)}{62511750}\\
&+\log(-2ir\omega\eta)\Big(\pi^2\left(\frac{3880783}{62511750}-\frac{4922\gamma_E}{297675}\right)+\pi\left(\frac{1728799i}{12502350}-\frac{526654i\gamma_E}{10418625}\right)+\frac{154804009\gamma_E}{1444021425}+\frac{591389\gamma_E^2}{3472875}-\frac{9844\gamma_E^3}{297675}\\
&-\frac{114601676821}{108301606875}-\frac{19688\zeta(3)}{297675}\Big)+\frac{591389i\pi\gamma_E^2}{6945750}+\frac{151\pi^2\gamma_E^2}{3402}+\gamma_E^2\left(-\frac{2113364017}{2406702375}-\frac{184\zeta(3)}{2835}\right)\\
&-\pi^2\left(\frac{37542817099}{173282571000}+\frac{92\zeta(3)}{8505}\right)+\gamma_E\Big(\frac{15316852667684410537}{18617775633134803125}+\frac{154804009i\pi}{2888042850}-\frac{30763517\pi^2}{1031443875}-\frac{2461i\pi^3}{297675}-\frac{23\pi^4}{9450}\\
&+\frac{145076\zeta(3)}{297675}\Big)-\frac{826492826\zeta(3)}{1031443875}-\frac{368\zeta(5)}{4725}-\frac{i\pi(114601676821+7162986600\zeta(3))}{216603213750}\Big]\\
&+r^5\omega^{11}\Big[-\frac{17406433582930617182}{157945387370203125}+\frac{2698118i\pi^3}{7441875}-\frac{3984791\pi^4}{7441875}-\frac{428i\pi^5}{7875}-\frac{244\pi^6}{14175}-\frac{7168}{24075}\log\left(\frac{r}{2M}\right)\\
&-\frac{39201376\log^3(-2ir\omega\eta)}{52093125}-\frac{4\left(-527141+44940i\pi+14700\pi^2\right)\gamma_E^4}{496125}-\frac{1088\gamma_E^5}{4725}-\frac{32\gamma_E^6}{675}\\
&+\log^2(-2ir\omega\eta)\Big[\frac{1545784916}{202584375}-\frac{19600688i\pi}{17364375}-\frac{183184\pi^2}{496125}+\frac{31507648\gamma_E}{5788125}-\frac{366368\gamma_E^2}{165375}\Big]+\frac{150656i\pi\gamma_E^3}{496125}+\frac{2752\pi^2\gamma_E^3}{4725}\\
&-\frac{16\gamma_E^3(6854489+2058000\zeta(3))}{17364375}-\pi\left(\frac{38947311286i}{9013100394375}-\frac{95872i\zeta(3)}{10125}\right)-\frac{1199313664\zeta(3)}{17364375}-\frac{256\zeta(3)^2}{135}\\
&+\pi^2\left(\frac{96614424538301}{8946212821875}+\frac{256\zeta(3)}{63}\right)+\gamma_E^2\Big(-\frac{204335097768884}{2982070940625}+\frac{8037748i\pi}{643125}+\frac{2082028\pi^2}{496125}-\frac{1712i\pi^3}{4725}-\frac{8\pi^4}{75}\\
&+\frac{60416\zeta(3)}{4725}\Big)+\log(-2ir\omega\eta)\Big(-\frac{77894622572}{9013100394375}+\frac{1545784916i\pi}{202584375}+\frac{3824956\pi^2}{3472875}-\frac{183184i\pi^3}{496125}-\frac{856\pi^4}{7875}+\frac{301312\gamma_E^3}{496125}\\
&-\frac{3424\gamma_E^4}{4725}-\frac{8\left(-18084933+1602860i\pi+524300\pi^2\right)\gamma_E^2}{5788125}+\frac{31507648i\pi\gamma_E}{5788125}+\frac{1616128\pi^2\gamma_E}{496125}\\
&-\frac{32\gamma_E(1374695417+110103000\zeta(3))}{607753125}+\frac{191744\zeta(3)}{10125}\Big)+\gamma_E\Big(\frac{1133703274198000192}{4822008710990625}+\frac{808064i\pi^3}{496125}+\frac{784\pi^4}{1125}\\
&-\frac{4332896\zeta(3)}{496125}-\frac{8\pi^2(45978197+2058000\zeta(3))}{17364375}-\frac{16i\pi(1374695417+110103000\zeta(3))}{607753125}-\frac{512\zeta(5)}{75}\Big)+\frac{1024\zeta(5)}{35}\Big]\Big]\Big\}\eta^{19}\\
&+\Big\{\frac{263r^{24}\omega^{24}}{734432293718323200000}-r^{21}\omega^{22}\left(\frac{16665333991}{73647952373706302592000}-\frac{\gamma_E}{1773991047628800}\right)\\
&+\frac{\omega^6}{r^3}\left[\frac{2048}{525}\log\left(\frac{r}{2M}\right)+\frac{512\left(-197+140\gamma_E\right)}{18375}\right]+\omega^8\Big[-\frac{715616\log(-2ir\omega\eta)}{55125}+\frac{\left(128\left(883-420\gamma_E\right)\right)}{7875}\log\left(\frac{r}{2M}\right)\\
&+\frac{1650942416}{17364375}-\frac{357808i\pi}{55125}+\frac{16\pi^2}{105}+\frac{28304\gamma_E}{11025}-\frac{6688\gamma_E^2}{525}\Big]+r^{18}\omega^{20}\Big[\frac{107\log(-2ir\omega)}{38767286880000}
\end{aligned}
$$

$$+\frac{46709856850849}{19358058627871419456000000}+\frac{107i\pi}{77534573760000}+\frac{\pi^2}{2215273536000}-\frac{36713113\gamma_E}{20385057841948800000}+\frac{\gamma_E^2}{369212256000}\Big]$$

$$+r^{12}\omega^{16}\Big[-\frac{618951839415702344444921}{1129684921601201631666000000}+\frac{6042101731i\pi}{131482571220000}+\frac{56503372727\pi^2}{1840755997080000}+\frac{6313i\pi^3}{4086482400}$$

$$+\frac{59\pi^4}{129729600}+\frac{675491\log^2(-2ir\omega)}{71513442000}-\left(\frac{117385699}{10956880935000}-\frac{6313i\pi}{681080400}-\frac{59\pi^2}{19459440}\right)\gamma_E^2-\frac{751\gamma_E^3}{34398000}+\frac{59\gamma_E^4}{19459440}$$

$$+\gamma_E\left(\frac{11640391485211}{32213229948900000}-\frac{5339407i\pi}{102162060000}-\frac{475567\pi^2}{20432412000}+\frac{59\zeta(3)}{2432430}\right)$$

$$+\log(-2ir\omega\eta)\left(\frac{6042101731}{65741285610000}+\frac{675491i\pi}{71513442000}+\frac{6313\pi^2}{2043241200}-\frac{5339407\gamma_E}{51081030000}+\frac{6313\gamma_E^2}{340540200}\right)-\frac{200609\zeta(3)}{1702701000}\Big]$$

$$+r^{15}\omega^{18}\Big[\frac{107\log(-2ir\omega\eta)\left(-8345581+2309076\gamma_E\right)}{16849997204040000}+\frac{44348304905603851}{4903192481401642428000000}-\frac{892977167i\pi}{33699994408080000}$$

$$-\frac{53493197\pi^2}{4814284915440000}+\left(\frac{3705731311}{68453113641412500}+\frac{107i\pi}{14594580000}+\frac{\pi^2}{416988000}\right)\gamma_E-\frac{29962613\gamma_E^2}{802380819240000}+\frac{\gamma_E^3}{208494000}$$

$$+\frac{\zeta(3)}{104247000}\Big]+r^9\omega^{14}\Big[-\frac{896989066465779154978 31}{319481029864593221625000}-\frac{19495507i\pi^3}{3438146250}-\frac{76427\pi^4}{36382500}-\Big(\frac{116085371}{5157219375}-\frac{856i\pi}{297675}$$

$$-\frac{8\pi^2}{8505}\Big)\gamma_E^3-\frac{40961\gamma_E^4}{16372125}+\frac{8\gamma_E^5}{14175}+\frac{11449\log^2(-2ir\omega\eta)\left(-19303+6600\gamma_E\right)}{8595365625}+\gamma_E^2\Big(\frac{1404795801709}{8603960990625}-\frac{3140129i\pi}{191008125}$$

$$-\frac{19303\pi^2}{2338875}+\frac{32\zeta(3)}{2835}\Big)+\gamma_E\Big(-\frac{1886886067977074989 9}{930888781656740156 25}-\frac{946008589i\pi}{361005356250}+\frac{87334897\pi^2}{10314438750}+\frac{428i\pi^3}{297675}+\frac{2\pi^4}{4725}$$

$$-\frac{1457608\zeta(3)}{16372125}\Big)+\pi\left(\frac{351906993365 42i}{387178244578125}+\frac{1712i\zeta(3)}{297675}\right)+\pi^2\left(\frac{2192176229317}{61948519132500}+\frac{16\zeta(3)}{8505}\right)+\frac{808801442\zeta(3)}{5157219375}$$

$$+\log(-2ir\omega)\Big(\frac{70381398673084}{387178244578125}-\frac{221000047i\pi}{8595365625}-\frac{19495507\pi^2}{1719073125}-\frac{6280258\gamma_E^2}{191008125}+\frac{1712\gamma_E^3}{297675}-\Big(\frac{946008589}{180502678125}$$

$$-\frac{91592i\pi}{10418625}-\frac{856\pi^2}{297675}\Big)\gamma_E+\frac{3424\zeta(3)}{297675}\Big)+\frac{64\zeta(5)}{4725}\Big]+r^6\omega^{12}\Big[\frac{15362825581984471680069 9287}{19522599528905154556875 00}+\frac{31284782i\pi^3}{121550625}$$

$$+\frac{38195033\pi^4}{104186250}+\frac{1177i\pi^5}{55125}+\frac{2816}{24075}\log\left(\frac{r}{2M}\right)+\frac{107803784\log^3(-2ir\omega\eta)}{364651875}-\left(\frac{638954}{385875}-\frac{4708i\pi}{33075}-\frac{44\pi^2}{945}\right)\gamma_E^4$$

$$+\frac{671\pi^6}{99225}+\frac{1324\gamma_E^5}{33075}+\frac{88\gamma_E^6}{4725}-\log^2(-2ir\omega\eta)\Big(\frac{17275308146}{12762815625}-\frac{53901892i\pi}{121550625}-\frac{503756\pi^2}{3472875}+\frac{118474252\gamma_E}{40516875}$$

$$-\frac{1007512\gamma_E^2}{1157625}\Big)-\frac{2613368\gamma_E^3}{3472875}+\frac{9416\gamma_E^4}{33075}-\frac{29407024\zeta(3)}{3472875}+\log(-2ir\omega\eta)\Big(\frac{555375413403322}{37855021656375}$$

$$-\frac{17275308146i\pi}{12762815625}+\frac{5352272\pi^2}{14586075}+\frac{503756i\pi^3}{3472875}+\frac{2354\pi^4}{55125}+\left(-\frac{992371748}{121550625}+\frac{1007512i\pi}{1157625}+\frac{9416\pi^2}{33075}\right)\gamma_E^2$$

$$+\gamma_E\left(\frac{424834758548}{12762815625}-\frac{118474252i\pi}{40516875}-\frac{5336732\pi^2}{3472875}+\frac{75328\zeta(3)}{33075}\right)\Big)+\pi^2\left(-\frac{2244279836904 23603}{2893205226594 3750}-\frac{11672\zeta(3)}{6615}\right)$$

$$+\gamma_E^2\left(\frac{868012466035399 06}{4822008710990625}-\frac{496185874i\pi}{121550625}-\frac{3892852\pi^2}{3472875}+\frac{4708i\pi^3}{33075}+\frac{22\pi^4}{525}-\frac{199504\zeta(3)}{33075}\right)+\frac{2358943736\zeta(3)}{121550625}$$

$$+\frac{704\zeta(3)^2}{945}-\frac{i\pi(277687706701661+160271177391864\zeta(3))}{37855021656375}+\gamma_E^3\left(\frac{7610116}{1500625}-\frac{1306684i\pi}{3472875}-\frac{10348\pi^2}{33075}+\frac{704\zeta(3)}{945}\right)$$

$$-\frac{46688\zeta(5)}{3675}+\gamma_E\Big(-\frac{987806491912669872 2}{1012621829308031 25}-\frac{2668366i\pi^3}{3472875}-\frac{17177\pi^4}{55125}+\pi^2\left(\frac{3175939657}{364651875}+\frac{352\zeta(3)}{945}\right)$$

$$+\frac{1248752\zeta(3)}{128625}+\frac{2i\pi(106208689637+7266798000\zeta(3))}{12762815625}+\frac{1408\zeta(5)}{525}\Big)\Big]+r^3\omega^{10}\Big[\frac{4436792718649306921 96}{2579774660379984375}$$

$$-\frac{43239212i\pi^3}{17364375}+\frac{6969322\pi^4}{2480625}+\frac{856i\pi^5}{2625}+\frac{488\pi^6}{4725}+\frac{46432}{24075}\log\left(\frac{r}{2M}\right)+\frac{78402752\log^3(-2ir\omega\eta)}{17364375}$$

$$+\frac{8\left(-411511+44940i\pi+14700\pi^2\right)\gamma_E^4}{165375}+\frac{2816\gamma_E^5}{1575}+\frac{64\gamma_E^6}{225}+\gamma_E^3\left(\frac{86835824}{17364375}+\frac{13696i\pi}{55125}-\frac{13312\pi^2}{4725}+\frac{512\zeta(3)}{45}\right)$$

$$+\log^2(-2ir\omega\eta)\left(-\frac{8357873576}{202584375}+\frac{39201376i\pi}{5788125}+\frac{366368\pi^2}{165375}-\frac{152409088\gamma_E}{5788125}+\frac{732736\gamma_E^2}{55125}\right)$$

$$+\log(-2ir\omega\eta)\Big(\frac{965094049382048}{15021833990625}-\frac{8357873576i\pi}{202584375}-\frac{5027192\pi^2}{694575}+\frac{366368i\pi^3}{165375}+\frac{1712\pi^4}{2625}$$

$$+\frac{16\left(-6637427+686940i\pi+224700\pi^2\right)\gamma_E^2}{826875}+\frac{27392\gamma_E^3}{55125}+\frac{6848\gamma_E^4}{1575}+\gamma_E\Big(\frac{58212150928}{202584375}-\frac{152409088i\pi}{5788125}$$

$$-\frac{2889856\pi^2}{165375}+\frac{54784\zeta(3)}{1575}\Big)-\frac{5807104\zeta(3)}{55125}\Big)+\frac{5436555424\zeta(3)}{17364375}+\frac{512\zeta(3)^2}{45}-\frac{17408\zeta(5)}{105}$$

$$
\begin{aligned}
&+\gamma_E^2\left(\frac{2288649254044096}{6958165528125}-\frac{53099416i\pi}{826875}-\frac{3923816\pi^2}{165375}+\frac{3424i\pi^3}{1575}+\frac{16\pi^4}{25}-\frac{108032\zeta(3)}{1575}\right)\\
&+\pi^2\left(-\frac{639787208115814}{20874496584375}-\frac{4352\zeta(3)}{189}\right)+\pi\left(\frac{482547024691024i}{15021833990625}-\frac{2903552i\zeta(3)}{55125}\right)\\
&+\gamma_E\Big(-\frac{1297314822179885408}{1607336236996875}-\frac{1444928i\pi^3}{165375}-\frac{3392\pi^4}{875}+\frac{1121216\zeta(3)}{23625}+\pi^2\left(\frac{234929768}{2480625}+\frac{256\zeta(3)}{45}\right)\\
&+\frac{8i\pi(3638259433+440412000\zeta(3))}{202584375}+\frac{1024\zeta(5)}{25}\Big)\Big]\Big\}\eta^{20}\,,
\end{aligned}
\tag{2.88}
$$

$$
\begin{aligned}
{}_{-2}\Psi^{\rm up}_{2m\omega}(r) = &-\frac{3i}{r\omega}-3\eta-\eta^2\left(\frac{3i}{r^2\omega}-\frac{3ir\omega}{2}\right)+\eta^3\left(\frac{r^2\omega^2}{2}+\frac{2+6i\pi-6\gamma_E}{r}\right)\\
&-\eta^4\left\{\frac{24i}{7r^3\omega}+\frac{1}{8}ir^3\omega^3+\omega\left(\frac{13i}{2}-6\pi-6i\gamma_E\right)\right\}+\eta^5\left\{\frac{31r^4\omega^4}{40}-\frac{1}{r^2}\left(\frac{12}{7}-6i\pi+6\gamma_E\right)-3r\omega^2\left(2+i\pi-\gamma_E\right)\right\}\\
&+\eta^6\Big\{\frac{43}{80}ir^5\omega^5-\frac{30i}{7r^4\omega}+r^2\omega^3\left(\frac{31i}{8}-\pi-i\gamma_E\right)+\frac{\omega}{r}\Big(\frac{1265i}{49}-\frac{33\pi}{35}-7i\pi^2-\frac{214}{35}i\log(-2ir\omega\eta)\\
&+\frac{6}{35}(-59i+70\pi)\gamma_E+6i\gamma_E^2\Big)\Big\}+\eta^7\Big\{-\frac{117}{560}r^6\omega^6+\frac{r^3\omega^4}{20}\left(-7+5i\pi-5\gamma_E\right)+\frac{48i\left(i+\pi+i\gamma_E\right)}{7r^3}\\
&+\omega^2\left[\frac{6976}{245}+\frac{348i\pi}{35}-7\pi^2-\frac{214}{35}\log(-2ir\omega\eta)-\left(\frac{669}{35}+12i\pi\right)\gamma_E+6\gamma_E^2\right]\Big\}\\
&+\eta^8\Big\{-\frac{40i}{7r^5\omega}-\frac{769ir^7\omega^7}{13440}+r^4\omega^5\left[\frac{8}{5}i\log(-2ir\omega\eta)+\frac{1}{400}\left(-313i-620\pi+20i\gamma_E\right)\right]\\
&+r\omega^3\left[-\frac{28943i}{1470}+\frac{733\pi}{70}+\frac{7i\pi^2}{2}+\frac{107}{35}i\log(-2ir\omega\eta)+\left(\frac{527i}{35}-6\pi\right)\gamma_E-3i\gamma_E^2\right]\\
&+\frac{\omega}{r^2}\left[\frac{26899i}{735}+\frac{227\pi}{35}-7i\pi^2-\frac{214}{35}i\log(-2ir\omega\eta)+\left(-\frac{94i}{35}+12\pi\right)\gamma_E+6i\gamma_E^2\right]\Big\}\\
&+\eta^9\Big\{\frac{1471r^8\omega^8}{120960}+\frac{5i\left(65i+36\pi+36i\gamma_E\right)}{21r^4}+r^5\omega^6\left[\frac{797}{3150}-\frac{43i\pi}{40}-\frac{16}{15}\log(-2ir\omega\eta)+\frac{\gamma_E}{120}\right]\\
&-r^2\omega^4\left[\frac{52673}{4410}+\frac{3041i\pi}{420}-\frac{7\pi^2}{6}-\frac{107}{105}\log(-2ir\omega\eta)-\left(\frac{3683}{420}+2i\pi\right)\gamma_E+\gamma_E^2\right]+\frac{\omega^2}{r}\Big[\frac{907}{2205}-\frac{36452i\pi}{735}-\frac{122\pi^2}{35}\\
&+6i\pi^3-\frac{428}{105}i\log(-2ir\omega\eta)\left(i-3\pi-3i\gamma_E\right)+\left(\frac{40946}{735}+\frac{494i\pi}{35}-14\pi^2\right)\gamma_E-\left(\frac{568}{35}+12i\pi\right)\gamma_E^2+4\gamma_E^3+8\zeta(3)\Big]\Big\}\\
&+\eta^{10}\Big\{-\frac{8i}{r^6\omega}+\frac{2561ir^9\omega^9}{1209600}-r^6\omega^7\left[\frac{44i}{105}\log(-2ir\omega\eta)-\frac{133207i+589680\pi-1680i\gamma_E}{1411200}\right]\\
&+r^3\omega^5\left[\frac{297511i}{88200}+\frac{139\pi}{168}-\frac{7i\pi^2}{24}-\frac{559}{84}i\log(-2ir\omega\eta)+\left(-\frac{2501i}{420}+\frac{\pi}{2}\right)\gamma_E+\frac{i\gamma_E^2}{4}\right]\\
&+\frac{\omega}{r^3}\left[-\frac{1712i}{245}\log(-2ir\omega\eta)+\frac{4}{735}\left(9703i+3162\pi-1470i\pi^2+12(103i+210\pi)\gamma_E+1260i\gamma_E^2\right)\right]\\
&+\omega^3\Big[\frac{323789i}{8820}-\frac{14795\pi}{294}-\frac{491i\pi^2}{70}-\frac{107}{105}\log(-2ir\omega\eta)\left(13i-12\pi-12i\gamma_E\right)-\left(\frac{51593i}{735}-\frac{1124\pi}{35}-14i\pi^2\right)\gamma_E\\
&+6\pi^3+\left(\frac{883i}{35}-12\pi\right)\gamma_E^2-4i\gamma_E^3-8i\zeta(3)\Big]\Big\}\\
&+\eta^{11}\Big\{-\frac{4159r^{10}\omega^{10}}{13305600}+r^7\omega^8\left[\frac{4}{35}\log(-2ir\omega\eta)+\frac{i\left(46859i+242235\pi+315i\gamma_E\right)}{2116800}\right]+\frac{8i\left(27i+10\pi+10i\gamma_E\right)}{7r^5}\\
&+r^4\omega^6\left[\frac{1068997}{110250}+\frac{989i\pi}{420}+\frac{281\pi^2}{120}+\left(-\frac{16867}{4200}-\frac{i\pi}{10}\right)\gamma_E+\frac{33\gamma_E^2}{20}+\log(-2ir\omega\eta)\left(-\frac{5147}{2100}-\frac{16i\pi}{5}+\frac{16\gamma_E}{5}\right)\right]\\
&+r\omega^4\Big[\frac{59317}{1470}+\frac{24449i\pi}{735}-\frac{1042\pi^2}{105}-3i\pi^3+\frac{214}{35}i\log(-2ir\omega\eta)\left(2i-\pi-i\gamma_E\right)-\left(\frac{37931}{735}+\frac{947i\pi}{35}-7\pi^2\right)\gamma_E\\
&+\left(\frac{634}{35}+6i\pi\right)\gamma_E^2-2\gamma_E^3-4\zeta(3)\Big]+\frac{\omega^2}{r^2}\Big[\frac{8097}{490}-\frac{55082i\pi}{735}-\frac{1276\pi^2}{105}+6i\pi^3+\frac{428}{245}i\log(-2ir\omega\eta)\left(2i+7\pi+7i\gamma_E\right)\\
&+\left(\frac{10246}{147}-\frac{26i\pi}{35}-14\pi^2\right)\gamma_E-\left(\frac{44}{5}+12i\pi\right)\gamma_E^2+4\gamma_E^3+8\zeta(3)\Big]\Big\}\\
&+\eta^{12}\Big\{-\frac{128i}{11r^7\omega}-\frac{6401ir^{11}\omega^{11}}{159667200}+r^8\omega^9\left[\frac{23}{945}i\log(-2ir\omega\eta)-\frac{353041i}{76204800}-\frac{1471\pi}{60480}+\frac{i\gamma_E}{60480}\right]\\
&+\frac{\omega}{r^4}\left[\frac{25681i}{315}+\frac{5192\pi}{147}-10i\pi^2-\frac{428}{49}i\log(-2ir\omega\eta)+\frac{2}{147}(1633i+1260\pi)\gamma_E+\frac{60i\gamma_E^2}{7}\right]\\
&+r^5\omega^7\Big[\log(-2ir\omega)\left(-\frac{6659i}{1800}+\frac{32\pi}{15}+\frac{32i\gamma_E}{15}\right)+\frac{12685741i}{1323000}-\frac{5311\pi}{5040}+\frac{1159i\pi^2}{720}+\frac{127i\gamma_E^2}{120}
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{17663i}{4200}+\frac{\pi}{60}\right)\gamma_E\Big]+r^2\omega^5\Big[-\frac{6193993i}{264600}+\frac{351727\pi}{17640}+\frac{19361i\pi^2}{2520}-\pi^3-\left(\frac{4111i}{420}-2\pi\right)\gamma_E^2\\
&+\frac{1}{420}\log(-2ir\omega\eta)\left(6005i-856\pi-856i\gamma_E\right)+\left(\frac{336797i}{8820}-\frac{3469\pi}{210}-\frac{7i\pi^2}{3}\right)\gamma_E+\frac{2i\gamma_E^3}{3}+\frac{4i\zeta(3)}{3}\Big]\\
&+\frac{\omega^3}{r}\Big[-\frac{28356226421i}{1484408646}+\frac{1174063i\pi^2}{22050}+\frac{757\pi^3}{105}-\frac{121i\pi^4}{30}-\frac{22898i\log^2(-2ir\omega\eta)}{3675}-\left(\frac{242608i}{3675}-\frac{922\pi}{35}-14i\pi^2\right)\gamma_E^2\\
&+\left(\frac{1564i}{105}-8\pi\right)\gamma_E^3-2i\gamma_E^4+\log(-2ir\eta\omega)\left(\frac{2416112i}{55125}-\frac{2354\pi}{1225}-\frac{214i\pi^2}{15}+\left(-\frac{25252i}{1225}+\frac{856\pi}{35}\right)\gamma_E+\frac{428i\gamma_E^2}{35}\right)\\
&-\pi\left(\frac{59686}{2625}+16\zeta(3)\right)+\gamma_E\left(\frac{263418i}{6125}-\frac{371582\pi}{3675}-\frac{766i\pi^2}{105}+12\pi^3-16i\zeta(3)\right)-\frac{2008i\zeta(3)}{105}\Big]\Big\}\\
&+\eta^{13}\Big\{\frac{9439r^{12}\omega^{12}}{2075673600}+r^9\omega^{10}\Big[-\frac{4}{945}\log(-2ir\omega\eta)+\frac{802517}{1047816000}-\frac{2561i\pi}{604800}+\frac{\gamma_E}{604800}\Big]-\frac{1}{r^6}\left(\frac{648}{11}-16i\pi+16\gamma_E\right)\\
&+r^6\omega^8\Big[\log(-2ir\omega\eta)\left(\frac{161947}{88200}+\frac{88i\pi}{105}-\frac{88\gamma_E}{105}\right)-\frac{6187033}{1372000}-\frac{56687i\pi}{141120}-\frac{3161\pi^2}{5040}+\left(\frac{476261}{235200}+\frac{i\pi}{420}\right)\gamma_E-\frac{353\gamma_E^2}{840}\Big]\\
&+\omega^4\Big[-\frac{20780545760521}{742204323000}+\frac{1041133\pi^2}{22050}+\frac{188i\pi^3}{105}-\frac{121\pi^4}{30}-\frac{22898\log^2(-2ir\omega\eta)}{3675}+\left(\frac{2194}{105}+8i\pi\right)\gamma_E^3\\
&-\left(\frac{329548}{3675}+\frac{1552i\pi}{35}-14\pi^2\right)\gamma_E^2-2\gamma_E^4+\log(-2ir\omega\eta)\Big(\frac{387806}{7875}+\frac{24824i\pi}{1225}-\frac{214\pi^2}{15}+\left(-\frac{47722}{1225}-\frac{856i\pi}{35}\right)\gamma_E\\
&+\frac{428\gamma_E^2}{35}\Big)+\gamma_E\left(\frac{4508003}{36750}+\frac{444347i\pi}{3675}-\frac{2971\pi^2}{105}-12i\pi^3-16\zeta(3)\right)-\pi\left(\frac{597787i}{12250}-16i\zeta(3)\right)\\
&-\frac{748\zeta(3)}{105}\Big]+r^3\omega^6\Big[-\frac{45942881}{1323000}-\frac{15662i\pi}{2205}-\frac{829\pi^2}{252}+\frac{i\pi^3}{4}+\log(-2ir\omega\eta)\left(\frac{8753}{350}+\frac{559i\pi}{42}-\frac{559\gamma_E}{42}\right)\\
&+\left(\frac{1400389}{44100}+\frac{979i\pi}{84}-\frac{7\pi^2}{12}\right)\gamma_E+\left(-\frac{1324}{105}-\frac{i\pi}{2}\right)\gamma_E^2+\frac{\gamma_E^3}{6}+\frac{\zeta(3)}{3}\Big]+\frac{\omega^2}{r^3}\Big[\frac{3424}{245}i\log(-2ir\omega\eta)\left(i+\pi+i\gamma_E\right)\\
&+\frac{3719}{63}-\frac{16552i\pi}{147}-\frac{18608\pi^2}{735}+\frac{48i\pi^3}{7}+\left(\frac{67352}{735}-\frac{5008i\pi}{245}-16\pi^2\right)\gamma_E-\left(\frac{64}{245}+\frac{96i\pi}{7}\right)\gamma_E^2+\frac{32\gamma_E^3}{7}+\frac{64\zeta(3)}{7}\Big]\Big\}\\
&+\eta^{14}\Big\{-\frac{192i}{11r^8\omega}+\frac{13441ir^{13}\omega^{13}}{29059430400}+r^{10}\omega^{11}\left(-\frac{13i\log(-2ir\omega\eta)}{20790}+\frac{20689693i}{184415616000}+\frac{4159\pi}{6652800}-\frac{i\gamma_E}{6652800}\right)\\
&+\frac{\omega}{r^5}\Big[-\frac{1712i}{147}\log(-2ir\omega\eta)+\frac{881338i}{6615}+\frac{9928\pi}{147}-\frac{40i\pi^2}{3}+\left(\frac{7360i}{147}+\frac{160\pi}{7}\right)\gamma_E+\frac{80i\gamma_E^2}{7}\Big]\\
&+r^7\omega^9\Big[\log(-2ir\omega\eta)\left(\frac{1213759i}{2116800}-\frac{8\pi}{35}-\frac{8i\gamma_E}{35}\right)-\frac{1870677493i}{1333584000}+\frac{86857\pi}{846720}-\frac{6919i\pi^2}{40320}+\left(\frac{1307477i}{2116800}+\frac{\pi}{3360}\right)\gamma_E\\
&-\frac{767i\gamma_E^2}{6720}\Big]+r^4\omega^7\Big[-\frac{2035391357i}{123480000}-\frac{16400641\pi}{882000}-\frac{65227i\pi^2}{25200}-\frac{157\pi^3}{60}+\left(-\frac{518123i}{88200}+\frac{838\pi}{105}-\frac{19i\pi^2}{20}\right)\gamma_E\\
&+\left(\frac{27161i}{4200}-\frac{33\pi}{10}\right)\gamma_E^2-\frac{13i\gamma_E^3}{6}+\log(-2ir\omega\eta)\left(\frac{851623i}{63000}+\frac{229\pi}{70}+\frac{56i\pi^2}{15}+\left(\frac{5147i}{1050}-\frac{32\pi}{5}\right)\gamma_E-\frac{16i\gamma_E^2}{5}\right)\\
&+\frac{127i\zeta(3)}{15}\Big]+\frac{\omega^3}{r^2}\Big[-\frac{3247112871637i}{30925180125}+\frac{1951303i\pi^2}{22050}+\frac{1537\pi^3}{105}-\frac{121i\pi^4}{30}-\frac{22898i\log^2(-2ir\omega\eta)}{3675}\\
&-\left(\frac{88736i}{1225}-\frac{402\pi}{35}-14i\pi^2\right)\gamma_E^2+\left(\frac{348i}{35}-8\pi\right)\gamma_E^3-2i\gamma_E^4+\pi\left(-\frac{3635501}{55125}-16\zeta(3)\right)\\
&+\log(-2ir\eta\omega)\left(\frac{3627352i}{55125}+\frac{48578\pi}{3675}-\frac{214i\pi^2}{15}+\left(-\frac{20116i}{3675}+\frac{856\pi}{35}\right)\gamma_E+\frac{428i\gamma_E^2}{35}\right)\\
&+\gamma_E\left(\frac{1805527i}{55125}-\frac{167414\pi}{1225}+\frac{1054i\pi^2}{105}+12\pi^3-16i\zeta(3)\right)-\frac{1016i\zeta(3)}{35}\Big]+r\omega^5\Big[\frac{115280572237489i}{371102161500}\\
&-\frac{1272623i\pi^2}{44100}+\frac{1343\pi^3}{210}+\frac{121i\pi^4}{60}+\frac{11449i\log^2(-2ir\omega\eta)}{3675}+\left(\frac{246044i}{3675}-\frac{1161\pi}{35}-7i\pi^2\right)\gamma_E^2\\
&+\left(-\frac{494i}{35}+4\pi\right)\gamma_E^3+i\gamma_E^4+\log(-2ir\eta\omega)\left(-\frac{656632i}{18375}+\frac{78431\pi}{3675}+\frac{107i\pi^2}{15}+\left(\frac{112778i}{3675}-\frac{428\pi}{35}\right)\gamma_E-\frac{214i\gamma_E^2}{35}\right)\\
&+\gamma_E\left(-\frac{305651i}{2625}+\frac{322921\pi}{3675}+\frac{2833i\pi^2}{105}-6\pi^3+8i\zeta(3)\right)-\frac{132i\zeta(3)}{35}+\pi\left(-\frac{1154609}{18375}+8\zeta(3)\right)\Big]\Big\}\\
&+\eta^{15}\Big\{-\frac{6197r^{14}\omega^{14}}{145297152000}+r^{11}\omega^{12}\left[\frac{\log(-2ir\omega)}{12474}-\frac{50587267}{3596104512000}+\frac{6401i\pi}{79833600}-\frac{\gamma_E}{79833600}\right]\\
&+\frac{64i\left(19i+4\pi+4i\gamma_E\right)}{11r^7}+r^8\omega^{10}\Big[\log(-2ir\omega)\left(-\frac{845707}{6350400}-\frac{46i\pi}{945}+\frac{46\gamma_E}{945}\right)+\frac{601386557}{1833678000}+\frac{51577i\pi}{2381400}+\frac{13241\pi^2}{362880}
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{5427283}{38102400}+\frac{i\pi}{30240}\right)\gamma_E+\frac{491\gamma_E^2}{20160}\bigg]+r^5\omega^8\bigg[-\frac{2764249777}{277830000}-\frac{12515183i\pi}{661500}+\frac{713\pi^2}{1575}-\frac{643i\pi^3}{360}\\
&+\left(\frac{2538757}{132300}+\frac{3031i\pi}{360}+\frac{263\pi^2}{360}\right)\gamma_E+\left(-\frac{49801}{6300}-\frac{127i\pi}{60}\right)\gamma_E^2+\frac{17\gamma_E^3}{12}\\
&+\log(-2ir\omega)\left(\frac{2011}{165375}+\frac{2651i\pi}{420}-\frac{112\pi^2}{45}+\left(-\frac{6659}{900}-\frac{64i\pi}{15}\right)\gamma_E+\frac{32\gamma_E^2}{15}\right)-\frac{57\zeta(3)}{10}\bigg]\\
&+\frac{\omega^2}{r^4}\bigg[\frac{12529079}{87318}-\frac{394309i\pi}{2205}-\frac{21061\pi^2}{441}+\frac{60i\pi^3}{7}+\frac{214}{441}i\log(-2ir\omega)\left(65i+36\pi+36i\gamma_E\right)\\
&+\left(\frac{289984}{2205}-\frac{7816i\pi}{147}-20\pi^2\right)\gamma_E+\frac{2}{147}(991-1260i\pi)\gamma_E^2+\frac{40\gamma_E^3}{7}+\frac{80\zeta(3)}{7}\bigg]+r^2\omega^6\bigg[\frac{120624950063969}{4453225938000}\\
&-\frac{1977383\pi^2}{132300}-\frac{7411i\pi^3}{1260}+\frac{121\pi^4}{180}+\frac{11449\log^2(-2ir\omega)}{11025}+\left(\frac{1180153}{22050}+\frac{1299i\pi}{70}-\frac{7\pi^2}{3}\right)\gamma_E^2+\left(-\frac{1513}{210}-\frac{4i\pi}{3}\right)\gamma_E^3\\
&+\frac{\gamma_E^4}{3}+\log(-2ir\omega)\left(-\frac{297678}{6125}-\frac{607627i\pi}{22050}+\frac{107\pi^2}{45}+\left(\frac{676321}{22050}+\frac{428i\pi}{105}\right)\gamma_E-\frac{214\gamma_E^2}{105}\right)\\
&+\pi\left(\frac{23402873i}{661500}-\frac{8i\zeta(3)}{3}\right)-\frac{219\zeta(3)}{35}+\gamma_E\left(-\frac{9017027}{94500}-\frac{2973889i\pi}{44100}+\frac{22357\pi^2}{1260}+2i\pi^3+\frac{8\zeta(3)}{3}\right)\bigg]\\
&+\frac{\omega^4}{r}\bigg[-\frac{7242471875251}{92775540375}-\frac{475411i\pi^3}{11025}-\frac{12029\pi^4}{1575}+\frac{11i\pi^5}{5}+\frac{45796i\log^2(-2ir\omega)\left(-i+3\pi+3i\gamma_E\right)}{11025}\\
&+\left(-\frac{202256}{3675}-\frac{180i\pi}{7}+\frac{28\pi^2}{3}\right)\gamma_E^3+\left(\frac{332}{35}+4i\pi\right)\gamma_E^4-\frac{4\gamma_E^5}{5}-\frac{284672\zeta(3)}{3675}\\
&+\gamma_E\left(-\frac{974435363093}{30925180125}+\frac{1095739\pi^2}{11025}-\frac{46i\pi^3}{21}-\frac{121\pi^4}{15}+\pi\left(-\frac{2096432i}{55125}+32i\zeta(3)\right)-\frac{768\zeta(3)}{35}\right)\\
&+\pi\left(\frac{1285046469766i}{30925180125}+\frac{232i\zeta(3)}{5}\right)+\gamma_E^2\left(\frac{5015854}{55125}+\frac{28296i\pi}{245}-\frac{2264\pi^2}{105}-12i\pi^3-16\zeta(3)\right)\\
&+\pi^2\left(\frac{475948}{7875}+\frac{56\zeta(3)}{3}\right)+\log(-2ir\omega)\bigg(\frac{1107346}{165375}-\frac{4603244i\pi}{55125}-\frac{26108\pi^2}{3675}+\frac{428i\pi^3}{35}\\
&+\left(\frac{5290184}{55125}+\frac{105716i\pi}{3675}-\frac{428\pi^2}{15}\right)\gamma_E-\left(\frac{121552}{3675}+\frac{856i\pi}{35}\right)\gamma_E^2+\frac{856\gamma_E^3}{105}+\frac{1712\zeta(3)}{105}\bigg)-\frac{96\zeta(5)}{5}\bigg]\bigg\}\\
&+\eta^{16}\bigg\{-\frac{3840i}{143r^9\omega}-\frac{8363ir^{15}\omega^{15}}{2324754432000}+r^{12}\omega^{13}\left[\frac{59i\log(-2ir\omega\eta)}{6486480}+\frac{-424433451i-242959860\pi+25740i\gamma_E}{26713919232000}\right]\\
&+\frac{\omega}{r^6}\left[\frac{82683362i}{363825}+\frac{145496\pi}{1155}-\frac{56i\pi^2}{3}-\frac{1712}{105}i\log(-2ir\omega\eta)+\left(\frac{117248i}{1155}+32\pi\right)\gamma_E+16i\gamma_E^2\right]\\
&+r^9\omega^{11}\bigg[\log(-2ir\omega\eta)\left(-\frac{17294311i}{698544000}+\frac{8\pi}{945}+\frac{8i\gamma_E}{945}\right)+\frac{994533333937i}{1613636640000}-\frac{15463027\pi}{4191264000}+\frac{23047i\pi^2}{3628800}\\
&-\left(\frac{55093001i}{2095632000}+\frac{\pi}{302400}\right)\gamma_E+\frac{853i\gamma_E^2}{201600}\bigg]+r^6\omega^9\bigg[-\frac{421712739967i}{46675440000}+\frac{3854653\pi}{432000}+\frac{48011i\pi^2}{1411200}\\
&+\frac{251\pi^3}{360}+\left(\frac{6027023i}{548800}-\frac{285671\pi}{70560}+\frac{697i\pi^2}{2520}\right)\gamma_E+\frac{(-2724359i+593040\pi)\gamma_E^2}{705600}+\frac{47i\gamma_E^3}{84}-\frac{67i\zeta(3)}{30}\\
&+\log(-2ir\omega\eta)\left(\frac{769569i}{392000}-\frac{1363\pi}{420}-\frac{44i\pi^2}{45}+\frac{(-161947i+73920\pi)\gamma_E}{44100}+\frac{88i\gamma_E^2}{105}\right)\bigg]\\
&+r^3\omega^7\bigg[-\frac{19837297960996i}{1391633105625}+\frac{96791i\pi^2}{529200}+\frac{13693\pi^3}{2520}-\frac{121i\pi^4}{720}-\frac{11449i\log^2(-2ir\omega\eta)}{44100}\\
&-\left(\frac{628679i}{11025}-\frac{699\pi}{28}-\frac{7i\pi^2}{12}\right)\gamma_E^2+\left(\frac{899i}{70}-\frac{\pi}{3}\right)\gamma_E^3-\frac{i\gamma_E^4}{12}+\log(-2ir\omega\eta)\bigg(\frac{52259i}{11025}-\frac{1906691\pi}{44100}-\frac{559i\pi^2}{36}\\
&+\left(-\frac{1114327i}{22050}+\frac{559\pi}{21}\right)\gamma_E+\frac{559i\gamma_E^2}{42}\bigg)+\pi\left(\frac{8758027}{132300}-\frac{2\zeta(3)}{3}\right)+\gamma_E\bigg(\frac{7011203i}{94500}-\frac{2533171\pi}{44100}\\
&-\frac{2255i\pi^2}{252}+\frac{\pi^3}{2}-\frac{2i\zeta(3)}{3}\bigg)-\frac{2893i\zeta(3)}{105}\bigg]+\frac{\omega^3}{r^3}\bigg[-\frac{4738606360087823i}{21431149826625}+\frac{10809632i\pi^2}{77175}+\frac{19496\pi^3}{735}-\frac{484i\pi^4}{105}\\
&-\frac{183184i\log^2(-2ir\omega\eta)}{25725}+\frac{8i\left(-272623+20790i\pi+51450\pi^2\right)\gamma_E^2}{25725}-\frac{32}{245}(-37i+70\pi)\gamma_E^3-\frac{16i\gamma_E^4}{7}\\
&-\pi\left(\frac{450693326}{2701125}+\frac{128\zeta(3)}{7}\right)+\log(-2ir\omega\eta)\bigg(\frac{87859384i}{900375}+\frac{902224\pi}{25725}-\frac{1712i\pi^2}{105}+\left(\frac{352672i}{25725}+\frac{6848\pi}{245}\right)\gamma_E\\
&+\frac{3424i\gamma_E^2}{245}\bigg)+\gamma_E\left(-\frac{55326098i}{2701125}-\frac{4890976\pi}{25725}+\frac{25232i\pi^2}{735}+\frac{96\pi^3}{7}-\frac{128i\zeta(3)}{7}\right)-\frac{11328i\zeta(3)}{245}\bigg]
\end{aligned}
$$

$$+\,\omega^5\Big[\frac{24087996816841i}{742204323000}-\frac{697247\pi^3}{22050}+\frac{10001i\pi^4}{6300}+\frac{11\pi^5}{5}+\frac{11449\log^2(-2ir\omega\eta)\left(-13i+12\pi+12i\gamma_E\right)}{11025}$$

$$+\left(\frac{282686i}{3675}-\frac{264\pi}{7}-\frac{28i\pi^2}{3}\right)\gamma_E^3+\left(-\frac{437i}{35}+4\pi\right)\gamma_E^4+\frac{4i\gamma_E^5}{5}+\gamma_E^2\Big(-\frac{20441663i}{110250}+\frac{37641\pi}{245}+\frac{4469i\pi^2}{105}$$

$$-12\pi^3+16i\zeta(3)\Big)+\pi^2\left(\frac{916354i}{55125}-\frac{56i\zeta(3)}{3}\right)+\log(-2ir\omega\eta)\Big(\frac{18450151i}{330750}-\frac{4685099\pi}{55125}-\frac{52537i\pi^2}{3675}$$

$$+\frac{428\pi^3}{35}+\left(-\frac{6917654i}{55125}+\frac{240536\pi}{3675}+\frac{428i\pi^2}{15}\right)\gamma_E+\left(\frac{188962i}{3675}-\frac{856\pi}{35}\right)\gamma_E^2-\frac{856i\gamma_E^3}{105}-\frac{1712i\zeta(3)}{105}\Big)$$

$$+\frac{175892i\zeta(3)}{3675}+\frac{96i\zeta(5)}{5}+\pi\left(\frac{347666419979}{12370072050}+\frac{112\zeta(3)}{5}\right)+\gamma_E\Big(\frac{1536357484549i}{13744524500}-\frac{1198744i\pi^2}{11025}+\frac{332\pi^3}{21}+\frac{121i\pi^4}{15}$$

$$-\frac{72i\zeta(3)}{35}-\pi\left(\frac{10065182}{55125}-32\zeta(3)\right)\Big)\Big]\Big\}$$

$$+\,\eta^{17}\Big\{\frac{33151r^{16}\omega^{16}}{118562476032000}+r^{13}\omega^{14}\left(-\frac{\log(-2ir\omega\eta)}{1081080}+\frac{6532153}{40905688824000}-\frac{13441i\pi}{14529715200}+\frac{\gamma_E}{14529715200}\right)$$

$$+\frac{96i\left(307i+52\pi+52i\gamma_E\right)}{143r^8}+r^{10}\omega^{12}\Big[\log(-2ir\omega\eta)\left(\frac{2688779}{698544000}+\frac{13i\pi}{10395}-\frac{13\gamma_E}{10395}\right)-\frac{609930185327}{62931828960000}$$

$$-\frac{50060551i\pi}{92207808000}-\frac{3403\pi^2}{3628800}+\left(\frac{375608521}{92207808000}+\frac{i\pi}{3326400}\right)\gamma_E-\frac{1387\gamma_E^2}{2217600}\Big]+r^7\omega^{10}\Big[\frac{5612950698343}{1540289520000}+\frac{132545411i\pi}{47628000}$$

$$+\frac{145199\pi^2}{6350400}+\frac{61i\pi^3}{320}+\left(-\frac{497996111}{133358400}-\frac{104611i\pi}{84672}-\frac{1543\pi^2}{20160}\right)\gamma_E+\frac{(420206+80535i\pi)\gamma_E^2}{352800}-\frac{307\gamma_E^3}{2016}$$

$$+\log(-2ir\omega\eta)\left(-\frac{2106473}{2268000}-\frac{4451i\pi}{4320}+\frac{4\pi^2}{15}+\left(\frac{1213759}{1058400}+\frac{16i\pi}{35}\right)\gamma_E-\frac{8\gamma_E^2}{35}\right)+\frac{439\zeta(3)}{720}\Big]$$

$$+\,r^4\omega^8\Big[\frac{95299059570507089}{890645187600000}+\frac{13726289\pi^2}{529200}+\frac{14599i\pi^3}{6300}+\frac{2573\pi^4}{1200}+\frac{354919\log^2(-2ir\omega\eta)}{220500}$$

$$+\left(\frac{510073}{55125}-\frac{2363i\pi}{210}+\frac{167\pi^2}{60}\right)\gamma_E^2+\left(\frac{2497}{420}+\frac{13i\pi}{3}\right)\gamma_E^3-\frac{97\gamma_E^4}{60}+\log(-2ir\omega\eta)\Big(-\frac{1061009749}{23152500}-\frac{651209i\pi}{22050}$$

$$-\frac{3097\pi^2}{1260}-\frac{16i\pi^3}{5}+\left(\frac{2223733}{73500}-\frac{229i\pi}{35}+\frac{112\pi^2}{15}\right)\gamma_E-\frac{2701\zeta(3)}{350}+\left(\frac{5147}{1050}+\frac{32i\pi}{5}\right)\gamma_E^2-\frac{32\gamma_E^3}{15}-\frac{64\zeta(3)}{15}\Big)$$

$$+\pi\left(\frac{8146505827i}{185220000}-\frac{254i\zeta(3)}{15}\right)+\gamma_E\left(-\frac{14594252063}{185220000}+\frac{1125487i\pi}{147000}-\frac{96197\pi^2}{12600}+\frac{61i\pi^3}{30}+\frac{38\zeta(3)}{3}\right)\Big]$$

$$+\frac{\omega^2}{r^5}\Big[\frac{1712}{735}i\log(-2ir\omega\eta)\left(27i+10\pi+10i\gamma_E\right)+\frac{37370264}{121275}-\frac{1970684i\pi}{6615}-\frac{38600\pi^2}{441}+\frac{80i\pi^3}{7}$$

$$+\left(\frac{38476}{189}-\frac{16432i\pi}{147}-\frac{80\pi^2}{3}\right)\gamma_E+\left(\frac{5648}{147}-\frac{160i\pi}{7}\right)\gamma_E^2+\frac{160\gamma_E^3}{21}+\frac{320\zeta(3)}{21}\Big]$$

$$+\frac{\omega^4}{r^2}\Big[-\frac{2015451214328591}{28574866435500}-\frac{866141i\pi^3}{11025}-\frac{2842\pi^4}{225}+\frac{11i\pi^5}{5}+\left(\frac{736}{105}+4i\pi\right)\gamma_E^4-\frac{4\gamma_E^5}{5}$$

$$+\frac{45796i\log^2(-2ir\omega\eta)\left(2i+7\pi+7i\gamma_E\right)}{25725}+\left(-\frac{598328}{11025}-\frac{332i\pi}{21}+\frac{28\pi^2}{3}\right)\gamma_E^3+\gamma_E\Big(-\frac{53409355631}{294525525}+\frac{186471\pi^2}{1225}$$

$$-\frac{358i\pi^3}{21}-\frac{121\pi^4}{15}+\pi\left(-\frac{1259794i}{385875}+32i\zeta(3)\right)-\frac{4384\zeta(3)}{105}\Big)+\gamma_E^2\Big(\frac{36656273}{385875}+\frac{99892i\pi}{735}-\frac{148\pi^2}{35}-12i\pi^3$$

$$-16\zeta(3)\Big)+\pi\left(\frac{4624898296189i}{20616786750}+\frac{6952i\zeta(3)}{105}\right)-\frac{1504816\zeta(3)}{11025}+\pi^2\left(\frac{290340317}{2315250}+\frac{56\zeta(3)}{3}\right)$$

$$+\log(-2ir\omega\eta)\Big(\frac{77407931}{2701125}-\frac{52156808i\pi}{385875}-\frac{273064\pi^2}{11025}+\frac{428i\pi^3}{35}+\frac{856\gamma_E^3}{105}+\frac{1712\zeta(3)}{105}$$

$$+\left(\frac{48035168}{385875}-\frac{5564i\pi}{3675}-\frac{428\pi^2}{15}\right)\gamma_E+\left(-\frac{9416}{525}-\frac{856i\pi}{35}\right)\gamma_E^2\Big)-\frac{96\zeta(5)}{5}\Big]$$

$$+\,r\omega^6\Big[\frac{22408763301923}{742204323000}+\frac{362501i\pi^3}{22050}-\frac{4573\pi^4}{1575}-\frac{11i\pi^5}{10}-\frac{22898i\log^2(-2ir\omega\eta)\left(-2i+\pi+i\gamma_E\right)}{3675}$$

$$+\left(\frac{627764}{11025}+\frac{550i\pi}{21}-\frac{14\pi^2}{3}\right)\gamma_E^3+\left(-\frac{848}{105}-2i\pi\right)\gamma_E^4+\frac{2\gamma_E^5}{5}+\log(-2ir\omega\eta)\Big(\frac{3564809}{55125}+\frac{361428i\pi}{6125}-\frac{222988\pi^2}{11025}$$

$$-\frac{214i\pi^3}{35}+\left(-\frac{84344}{875}-\frac{202658i\pi}{3675}+\frac{214\pi^2}{15}\right)\gamma_E+\left(\frac{135676}{3675}+\frac{428i\pi}{35}\right)\gamma_E^2-\frac{428\gamma_E^3}{105}-\frac{856\zeta(3)}{105}\Big)$$

$$+\gamma_E\left(\frac{23527225720343}{185551080750}-\frac{1718599\pi^2}{22050}-\frac{397i\pi^3}{21}+\frac{121\pi^4}{30}+\pi\left(\frac{484786i}{2625}-16i\zeta(3)\right)-\frac{1648\zeta(3)}{105}\right)$$

$$+ \pi^2\left(\frac{4783949}{110250} - \frac{28\zeta(3)}{3}\right) - \pi\left(\frac{2764236498031i}{92775540375} - \frac{52i\zeta(3)}{15}\right) + \frac{176968\zeta(3)}{11025} - \gamma_E^2\Big(\frac{432167}{2625} + \frac{16970i\pi}{147} - \frac{1194\pi^2}{35}$$

$$- 6i\pi^3 - 8\zeta(3)\Big) + \frac{48\zeta(5)}{5}\Big]\Big\}$$

$$+ \eta^{18}\Big\{ -\frac{3840i}{91 r^{10}\omega} + \frac{43009 i r^{17}\omega^{17}}{2134124568576000} + r^{14}\omega^{15}\Big(-\frac{83i\log(-2ir\omega\eta)}{972972000} + \frac{4612625803i}{314155690168320000} + \frac{6197\pi}{72648576000}$$

$$- \frac{i\gamma_E}{217945728000}\Big) + \frac{\omega}{r^7}\Big[\frac{20813201936i}{52026975} + \frac{269056\pi}{1155} - \frac{896i\pi^2}{33} - \frac{27392i\log(-2ir\omega\eta)}{1155} + \frac{128(1781i+420\pi)\gamma_E}{1155}$$

$$+ \frac{256i\gamma_E^2}{11}\Big] + r^{11}\omega^{13}\Big[\log(-2ir\omega\eta)\Big(\frac{8018599i}{15567552000} - \frac{\pi}{6237} - \frac{i\gamma_E}{6237}\Big) - \frac{12880904243209i}{9817365317760000} + \frac{496174171\pi}{7192209024000}$$

$$- \frac{5237i\pi^2}{43545600} + \Big(\frac{1953470903i}{3596104512000} + \frac{\pi}{39916800}\Big)\gamma_E - \frac{79i\gamma_E^2}{985600}\Big] + r^8\omega^{11}\Big[\frac{601493494129069i}{609954649920000} - \frac{233949307\pi}{359251200}$$

$$+ \frac{402181i\pi^2}{45722880} - \frac{1051\pi^3}{25920} + \Big(-\frac{40449720281i}{44008272000} + \frac{2713481\pi}{9525600} - \frac{979i\pi^2}{60480}\Big) + \gamma_E + \Big(\frac{420061i}{1524096} - \frac{491\pi}{10080}\Big)\gamma_E^2 - \frac{589i\gamma_E^3}{18144}$$

$$+ \log(-2ir\omega\eta)\Big(-\frac{330947587i}{1257379200} + \frac{21913\pi}{90720} + \frac{23i\pi^2}{405} + \Big(\frac{845707i}{3175200} - \frac{92\pi}{945}\Big)\gamma_E - \frac{46i\gamma_E^2}{945}\Big) + \frac{841i\zeta(3)}{6480}\Big]$$

$$+ \frac{\omega^3}{r^4}\Big[-\frac{54751137055689161i}{128586898959750} + \frac{10652942i\pi^2}{46305} + \frac{6904\pi^3}{147} - \frac{121i\pi^4}{21} - \frac{45796i\log^2(-2ir\omega)}{5145} - \frac{20i\gamma_E^4}{7}$$

$$- \Big(\frac{186714i}{1715} + \frac{5248\pi}{147} - 20i\pi^2\Big)\gamma_E^2 - \frac{4}{441}(349i+1260\pi)\gamma_E^3 + \log(-2ir\omega\eta)\Big(\frac{49789634i}{324135} + \frac{1111088\pi}{15435} - \frac{428i\pi^2}{21}$$

$$+ \Big(\frac{698924i}{15435} + \frac{1712\pi}{49}\Big)\gamma_E + \frac{856i\gamma_E^2}{49}\Big) + \pi\Big(-\frac{3891153316}{10696455} - \frac{160\zeta(3)}{7}\Big) + \gamma_E\Big(-\frac{1426566433i}{10696455} - \frac{1469746\pi}{5145}$$

$$+ \frac{33134i\pi^2}{441} + \frac{120\pi^3}{7} - \frac{160i\zeta(3)}{7}\Big) - \frac{33608i\zeta(3)}{441}\Big] + r^5\omega^9\Big[\frac{3807449161258709i}{83497986337500} + \frac{2811043i\pi^2}{127008} + \frac{2687\pi^3}{8400} + \frac{31481i\pi^4}{21600}$$

$$+ \frac{492307i\log^2(-2ir\omega\eta)}{441000} + \Big(-\frac{7975579i}{441000} + \frac{37123\pi}{2520} + \frac{211i\pi^2}{120}\Big)\gamma_E^2 + \Big(\frac{29243i}{3780} - \frac{17\pi}{6}\Big)\gamma_E^3 - \frac{383i\gamma_E^4}{360}$$

$$+ \log(-2ir\omega\eta)\Big(-\frac{1041057547i}{46305000} + \frac{330361\pi}{88200} - \frac{48823i\pi^2}{7560} + \frac{32\pi^3}{15} + \Big(\frac{1460833i}{661500} + \frac{2651\pi}{210} + \frac{224i\pi^2}{45}\Big)\gamma_E$$

$$+ \Big(\frac{6659i}{900} - \frac{64\pi}{15}\Big)\gamma_E^2 - \frac{64i\gamma_E^3}{45} - \frac{128i\zeta(3)}{45}\Big) + \gamma_E\Big(-\frac{44865358i}{17364375} - \frac{45105317\pi}{1323000} - \frac{278339i\pi^2}{37800} - \frac{259\pi^3}{180} + \frac{77i\zeta(3)}{9}\Big)$$

$$- \frac{133463i\zeta(3)}{9450} + \pi\Big(\frac{2595065923}{277830000} + \frac{57\zeta(3)}{5}\Big)\Big] + r^2\omega^7\Big[-\frac{357960470920729i}{22266129690000} + \frac{1314869\pi^3}{264600} + \frac{263743i\pi^4}{75600} - \frac{11\pi^5}{30}$$

$$- \frac{11449\log^2(-2ir\omega\eta)\,(-31i+8\pi+8i\gamma_E)}{44100} + \Big(-\frac{3082423i}{66150} + \frac{865\pi}{63} + \frac{14i\pi^2}{9}\Big)\gamma_E^3 + \Big(\frac{4967i}{1260} - \frac{2\pi}{3}\Big)\gamma_E^4 - \frac{2i\gamma_E^5}{15}$$

$$+ \gamma_E^2\Big(\frac{10026451i}{6750} - \frac{856147\pi}{8820} - \frac{2817i\pi^2}{140} + 2\pi^3 - \frac{8i\zeta(3)}{3}\Big) + \pi^2\Big(-\frac{107401051i}{7938000} + \frac{28i\zeta(3)}{9}\Big)$$

$$+ \pi\Big(-\frac{26714891373581}{742204323000} + \frac{502\zeta(3)}{45}\Big) + \gamma_E\Big(-\frac{79876446817543i}{742204323000} + \frac{8002073i\pi^2}{132300} - \frac{1739\pi^3}{126} - \frac{121i\pi^4}{90}$$

$$+ \pi\Big(\frac{14495167}{94500} - \frac{16\zeta(3)}{3}\Big) + \frac{4798i\zeta(3)}{315}\Big) + \frac{700727i\zeta(3)}{33075} + \log(-2ir\omega\eta)\Big(-\frac{10606433i}{198450} + \frac{18220141\pi}{220500} + \frac{4047307i\pi^2}{132300}$$

$$- \frac{214\pi^3}{105} + \Big(\frac{1784429i}{15750} - \frac{653423\pi}{11025} - \frac{214i\pi^2}{45}\Big)\gamma_E + \Big(-\frac{722117i}{22050} + \frac{428\pi}{105}\Big)\gamma_E^2 + \frac{428i\gamma_E^3}{315} + \frac{856i\zeta(3)}{315}\Big) - \frac{16i\zeta(5)}{5}\Big]$$

$$+ \frac{\omega^5}{r}\Big[-\frac{386525631217745999659i}{6356564763176281500} + \frac{1554809i\pi^4}{55125} + \frac{17743\pi^5}{3150} - \frac{127i\pi^6}{126} - \frac{4900172i\log^3(-2ir\omega\eta)}{1157625}$$

$$+ \Big(\frac{55294i}{1575} - \frac{254\pi}{15} - \frac{14i\pi^2}{3}\Big)\gamma_E^4 + \frac{4}{105}(-121i+42\pi)\gamma_E^5 + \frac{4i\gamma_E^6}{15} + \pi^3\Big(-\frac{183422969}{2315250} - 16\zeta(3)\Big)$$

$$+ \log^2(-2ir\omega\eta)\Big(\frac{1446888026i}{40516875} - \frac{251878\pi}{128625} - \frac{22898i\pi^2}{1575} + \Big(-\frac{2701964i}{128625} + \frac{91592\pi}{3675}\Big)\gamma_E + \frac{45796i\gamma_E^2}{3675}\Big)$$

$$+ \pi^2\Big(-\frac{20676541268534i}{463877701875} - \frac{6968i\zeta(3)}{105}\Big) + \gamma_E^3\Big(-\frac{115359136i}{1157625} + \frac{1046188\pi}{11025} + \frac{836i\pi^2}{35} - 8\pi^3 + \frac{32i\zeta(3)}{3}\Big)$$

$$+ \frac{3325328i\zeta(3)}{33075} + \frac{32i\zeta(3)^2}{3} + \gamma_E^2\Big(\frac{9358613463113i}{154625900625} - \frac{56081i\pi^2}{525} + \frac{1054\pi^3}{105} + \frac{121i\pi^4}{15} + \frac{592i\zeta(3)}{105}$$

$$- \pi\Big(\frac{47653366}{385875} - 32\zeta(3)\Big)\Big) + \log(-2ir\omega\eta)\Big(\frac{218308363129 09i}{5007277996875} + \frac{101891479i\pi^2}{1157625} + \frac{161998\pi^3}{11025} - \frac{12947i\pi^4}{1575}$$

$$+\bigg(-\frac{9027436i}{77175}+\frac{197308\pi}{3675}+\frac{428i\pi^2}{15}\bigg)\gamma_E^2-\frac{856(-391i+210\pi)\gamma_E^3}{11025}-\frac{428i\gamma_E^4}{105}-\frac{429712i\zeta(3)}{11025}$$

$$-\frac{2\pi(110527087+73402000\zeta(3))}{4501875}+\gamma_E\bigg(\frac{783725504i}{13505625}-\frac{65956684\pi}{385875}-\frac{163924i\pi^2}{11025}+\frac{856\pi^3}{35}-\frac{3424i\zeta(3)}{105}\bigg)\bigg)$$

$$+\gamma_E\bigg(\frac{859864496446303 13i}{535778745665625}-\frac{788824\pi^3}{11025}+\frac{11111i\pi^4}{1575}+\frac{22\pi^5}{5}+\pi^2\bigg(-\frac{38037233i}{1157625}-\frac{112i\zeta(3)}{3}\bigg)+\frac{255664i\zeta(3)}{2205}$$

$$+\pi\bigg(\frac{5257916983282}{154625900625}+\frac{1264\zeta(3)}{21}\bigg)+\frac{192i\zeta(5)}{5}\bigg)+\frac{2976i\zeta(5)}{35}+\pi\bigg(\frac{164067896615776837}{1071557491331250}+\frac{1922888\zeta(3)}{11025}+\frac{192\zeta(5)}{5}\bigg)\bigg]\bigg\}$$

$$+\eta^{19}\bigg\{-\frac{54911r^{18}\omega^{18}}{40548366802944000}+r^{15}\omega^{16}\bigg(\frac{\log(-2ir\omega\eta)}{138996000}-\frac{3284693057}{267032336643072 0000}+\frac{8363i\pi}{1162377216000}-\frac{\gamma_E}{3487131648000}\bigg)$$

$$+\frac{7680i\,(50i+7\pi+7i\gamma_E)}{1001r^9}+r^{12}\omega^{14}\bigg[\log(-2ir\omega\eta)\bigg(-\frac{507623}{8382528000}-\frac{59i\pi}{3243240}+\frac{59\gamma_E}{3243240}\bigg)+\frac{54684598769}{350620189920000}$$

$$+\frac{365156287i\pi}{46749358656000}+\frac{7723\pi^2}{566092800}+\bigg(-\frac{5959061099}{93498717312000}-\frac{i\pi}{518918400}\bigg)\gamma_E+\frac{1049\gamma_E^2}{115315200}\bigg]$$

$$+r^9\omega^{12}\bigg[\log(-2ir\omega\eta)\bigg(\frac{436420913}{7858620000}+\frac{322199i\pi}{7128000}-\frac{4\pi^2}{405}-\bigg(\frac{17294311}{349272000}+\frac{16i\pi}{945}\bigg)\gamma_E+\frac{8\gamma_E^2}{945}\bigg)-\frac{4061229852596531}{19823526122400000}$$

$$-\frac{6050426057i\pi}{49397040000}-\frac{614501\pi^2}{285768000}-\frac{1829i\pi^3}{259200}+\bigg(\frac{86555528603}{484090992000}+\frac{110189533i\pi}{2095632000}+\frac{1709\pi^2}{604800}\bigg)\gamma_E$$

$$-\bigg(\frac{53487967}{1047816000}+\frac{853i\pi}{100800}\bigg)\gamma_E^2+\frac{5119\gamma_E^3}{907200}-\frac{1463\zeta(3)}{64800}\bigg]$$

$$+\frac{\omega^2}{r^6}\bigg[\frac{1209979788}{1926925}-\frac{187207564i\pi}{363825}-\frac{551608\pi^2}{3465}+16i\pi^3+\frac{1712i\log(-2ir\omega\eta)\,(81i+22\pi+22i\gamma_E)}{1155}$$

$$+\bigg(\frac{121685044}{363825}-\frac{253328i\pi}{1155}-\frac{112\pi^2}{3}\bigg)\gamma_E+\bigg(\frac{98416}{1155}-32i\pi\bigg)\gamma_E^2+\frac{32\gamma_E^3}{3}+\frac{64\zeta(3)}{3}\bigg]$$

$$+r^6\omega^{10}\bigg[-\frac{10457784872905793317}{1234434230013600000}-\frac{647203\pi^2}{66150}+\frac{279541i\pi^3}{705600}-\frac{86119\pi^4}{151200}-\frac{148837\log^2(-2ir\omega\eta)}{343000}$$

$$+\bigg(\frac{2452243}{196000}+\frac{102931i\pi}{14112}-\frac{589\pi^2}{840}\bigg)\gamma_E^2+\bigg(-\frac{803987}{211680}-\frac{47i\pi}{42}\bigg)\gamma_E^3+\frac{151\gamma_E^4}{360}+\gamma_E\bigg(-\frac{283187154931}{23337720000}$$

$$-\frac{491472143i\pi}{24696000}+\frac{7405193\pi^2}{2116800}-\frac{701i\pi^3}{1260}-\frac{211\zeta(3)}{63}\bigg)+\pi\bigg(\frac{307326388549i}{23337720000}+\frac{67i\zeta(3)}{15}\bigg)+\frac{3753521\zeta(3)}{529200}$$

$$+\log(-2ir\omega\eta)\bigg(\frac{11543798753}{1944810000}-\frac{3045409i\pi}{1481760}+\frac{181529\pi^2}{52920}+\frac{88i\pi^3}{105}+\bigg(\frac{4196287}{1372000}+\frac{1363i\pi}{210}-\frac{88\pi^2}{45}\bigg)\gamma_E$$

$$+\bigg(-\frac{161947}{44100}-\frac{176i\pi}{105}\bigg)\gamma_E^2+\frac{176\gamma_E^3}{315}+\frac{352\zeta(3)}{315}\bigg)\bigg]+\frac{\omega^4}{r^3}\bigg[-\frac{758863121797169}{5143475958390}-\frac{10036288i\pi^3}{77175}-\frac{231752\pi^4}{11025}+\frac{88i\pi^5}{35}$$

$$+\frac{366368i\log^2(-2ir\omega)\,(i+\pi+i\gamma_E)}{25725}-\bigg(\frac{4375664}{77175}+\frac{736i\pi}{147}-\frac{32\pi^2}{3}\bigg)\gamma_E^3+\frac{32}{735}(109+105i\pi)\gamma_E^4-\frac{32\gamma_E^5}{35}$$

$$+\gamma_E\bigg(-\frac{7329189186654088}{21431149826625}+\frac{17637152\pi^2}{77175}-\frac{5744i\pi^3}{147}-\frac{968\pi^4}{105}+\pi\bigg(\frac{335761708i}{2701125}+\frac{256i\zeta(3)}{7}\bigg)-\frac{54272\zeta(3)}{735}\bigg)$$

$$+\gamma_E^2\bigg(\frac{169783414}{2701125}+\frac{17776i\pi}{105}+\frac{4416\pi^2}{245}-\frac{96i\pi^3}{7}-\frac{128\zeta(3)}{7}\bigg)+\pi\bigg(\frac{422048979477457i}{857245993065}+\frac{10688i\zeta(3)}{105}\bigg)$$

$$-\frac{17379808\zeta(3)}{77175}+\pi^2\bigg(\frac{2141626003}{8103375}+\frac{64\zeta(3)}{3}\bigg)+\log(-2ir\omega\eta)\bigg(\frac{812193482}{8103375}-\frac{62847216i\pi}{300125}$$

$$-\frac{3982112\pi^2}{77175}+\frac{3424i\pi^3}{245}+\bigg(\frac{50024336}{300125}-\frac{1071712i\pi}{25725}-\frac{3424\pi^2}{105}\bigg)\gamma_E-\bigg(\frac{13696}{25725}+\frac{6848i\pi}{245}\bigg)\gamma_E^2$$

$$+\frac{6848\gamma_E^3}{735}+\frac{13696\zeta(3)}{735}\bigg)-\frac{768\zeta(5)}{35}\bigg]+r^3\omega^8\bigg[-\frac{2592107555927213}{12370072050000}+\frac{1842457i\pi^3}{264600}-\frac{200707\pi^4}{37800}$$

$$+\frac{11i\pi^5}{120}+\frac{11449i\log^2(-2ir\omega)\,(7i+5\pi+5i\gamma_E)}{110250}-\bigg(\frac{1820246}{33075}+\frac{3215i\pi}{126}-\frac{7\pi^2}{18}\bigg)\gamma_E^3+\bigg(\frac{5443}{630}+\frac{i\pi}{6}\bigg)\gamma_E^4-\frac{\gamma_E^5}{30}$$

$$+\gamma_E^2\bigg(\frac{51733103}{661500}+\frac{223138i\pi}{2205}-\frac{514\pi^2}{21}-\frac{i\pi^3}{2}-\frac{2\zeta(3)}{3}\bigg)+\pi\bigg(-\frac{27427810151881i}{3711021615000}+\frac{499i\zeta(3)}{9}\bigg)$$

$$+\pi^2\bigg(-\frac{145108711}{1984500}+\frac{7\zeta(3)}{9}\bigg)+\frac{2976776\zeta(3)}{33075}+\log(-2ir\omega\eta)\bigg(\frac{7862225003}{69457500}+\frac{1764533i\pi}{110250}+\frac{2951429\pi^2}{66150}+\frac{559i\pi^3}{42}$$

$$+\frac{884894+9647945i\pi-3423875\pi^2}{110250}\gamma_E-\bigg(\frac{562888}{11025}+\frac{559i\pi}{21}\bigg)\gamma_E^2+\frac{559\gamma_E^3}{63}+\frac{1118\zeta(3)}{63}\bigg)+\gamma_E\bigg(\frac{157134638242487}{1855510807500}$$

$$
\begin{aligned}
&+\frac{11902507\pi^2}{264600}+\frac{3077i\pi^3}{1260}-\frac{121\pi^4}{360}-\frac{11768\zeta(3)}{315}+\frac{4i\pi(-4812067+55125\zeta(3))}{165375}\Bigg)-\frac{4\zeta(5)}{5}\Bigg]\\
&+\omega^6\Bigg[-\frac{52691553736189625774911}{635656476317628150000}+\frac{1795753\pi^4}{110250}-\frac{3674i\pi^5}{1575}-\frac{127\pi^6}{126}-\frac{4900172\log^3(-2ir\omega\eta)}{1157625}\\
&+\left(\frac{77344}{1575}+\frac{344i\pi}{15}-\frac{14\pi^2}{3}\right)\gamma_E^4-\left(\frac{122}{21}+\frac{8i\pi}{5}\right)\gamma_E^5+\frac{4\gamma_E^6}{15}-\frac{468862\zeta(3)}{33075}+\frac{32\zeta(3)^2}{3}\\
&+\log^2(-2ir\omega\eta)\left(\frac{1670486996}{40516875}+\frac{2656168i\pi}{128625}-\frac{22898\pi^2}{1575}-\left(\frac{5106254}{128625}+\frac{91592i\pi}{3675}\right)\gamma_E+\frac{45796\gamma_E^2}{3675}\right)\\
&+\pi^2\left(\frac{7980651295447}{412335735000}-\frac{4028\zeta(3)}{105}\right)+\gamma_E^2\Bigg(\frac{43046157166049}{206167867500}-\frac{72221\pi^2}{525}-\frac{2944i\pi^3}{105}+\frac{121\pi^4}{15}\\
&+\pi\left(\frac{111220471i}{385875}-32i\zeta(3)\right)-\frac{1928\zeta(3)}{105}\Bigg)+\pi^3\left(\frac{38122289i}{2315250}+16i\zeta(3)\right)+\gamma_E^3\Bigg(-\frac{206833981}{1157625}-\frac{1461358i\pi}{11025}+\frac{1326\pi^2}{35}\\
&+8i\pi^3+\frac{32\zeta(3)}{3}\Bigg)+\log(-2ir\omega\eta)\Bigg(-\frac{119938449492599}{20029111987500}+\frac{87667969\pi^2}{1157625}+\frac{40232i\pi^3}{11025}-\frac{12947\pi^4}{1575}\\
&-\frac{4\left(3187117+1743672i\pi-550515\pi^2\right)\gamma_E^2}{77175}+\frac{428(1097+420i\pi)\gamma_E^3}{11025}-\frac{428\gamma_E^4}{105}+\gamma_E\Bigg(\frac{873473443}{4501875}+\frac{81528394i\pi}{385875}\\
&-\frac{635794\pi^2}{11025}-\frac{856i\pi^3}{35}-\frac{3424\zeta(3)}{105}\Bigg)-\frac{160072\zeta(3)}{11025}+\frac{i\pi(-949933333+440412000\zeta(3))}{13505625}\Bigg)\\
&+\pi\left(-\frac{287460560462061643i}{4286229965325000}-\frac{1135388i\zeta(3)}{11025}-\frac{192i\zeta(5)}{5}\right)+\frac{1968\zeta(5)}{35}\\
&+\gamma_E\Bigg(\frac{63137383760774341}{1071557491331250}+\frac{737479i\pi^3}{11025}-\frac{15893\pi^4}{3150}-\frac{22i\pi^5}{5}+\pi^2\left(\frac{126154837}{1157625}-\frac{112\zeta(3)}{3}\right)\\
&+\pi\left(-\frac{6522482409664i}{51541966875}-\frac{256i\zeta(3)}{21}\right)+\frac{179056\zeta(3)}{2205}+\frac{192\zeta(5)}{5}\Bigg)\Bigg]\Bigg\}\\
&+\eta^{20}\Bigg\{-\frac{6144i}{91r^{11}\omega}-\frac{5317ir^{19}\omega^{19}}{62382102773760000}+r^{16}\omega^{17}\Bigg[\frac{37i\log(-2ir\omega\eta)}{66162096000}-\frac{17327118227i}{181581988917288960000}-\frac{33151\pi}{59281238016000}\\
&+\frac{i\gamma_E}{59281238016000}\Bigg]+\frac{\omega}{r^8}\Bigg[-\frac{13696}{385}i\log(-2ir\omega\eta)+\frac{1388647032i}{1926925}+\frac{2152064\pi}{5005}-\frac{448i\pi^2}{11}+\left(\frac{1884992i}{5005}+\frac{768\pi}{11}\right)\gamma_E\\
&+\frac{384i\gamma_E^2}{11}\Bigg]+r^{13}\omega^{15}\Bigg[\log(-2ir\omega\eta)\left(-\frac{5815283i}{915372057600}+\frac{\pi}{540540}+\frac{i\gamma_E}{540540}\right)+\frac{1948100188157i}{117808383813120000}\\
&-\frac{207008003\pi}{261796408473600}+\frac{1571i\pi^2}{1132185600}-\left(\frac{4366956241i}{654491021184000}+\frac{\pi}{7264857600}\right)\gamma_E+\frac{13439i\gamma_E^2}{14529715200}\Bigg]\\
&+\frac{\omega^3}{r^5}\Bigg[-\frac{2033986379193058921i}{2507444529715125}+\frac{54421846i\pi^2}{138915}+\frac{36632\pi^3}{441}-\frac{484i\pi^4}{63}-\frac{183184i\log^2(-2ir\omega\eta)}{15435}\\
&+\left(-\frac{1412812i}{9261}-\frac{4336\pi}{49}+\frac{80i\pi^2}{3}\right)\gamma_E^2-\frac{64}{147}(41i+35\pi)\gamma_E^3-\frac{80i\gamma_E^4}{21}-\pi\left(\frac{39774375118}{53482275}+\frac{640\zeta(3)}{21}\right)\\
&+\log(-2ir\omega\eta)\left(\frac{247774628i}{972405}+\frac{2124592\pi}{15435}-\frac{1712i\pi^2}{63}+\left(\frac{315008i}{3087}+\frac{6848\pi}{147}\right)\gamma_E+\frac{3424i\gamma_E^2}{147}\right)\\
&+\gamma_E\left(-\frac{19332968308i}{53482275}-\frac{4243160\pi}{9261}+\frac{65216i\pi^2}{441}+\frac{160\pi^3}{7}-\frac{640i\zeta(3)}{21}\right)-\frac{18944i\zeta(3)}{147}\Bigg]\\
&+r^7\omega^{11}\Bigg[\frac{22638886347596064181i}{12220898877134640000}-\frac{673971043i\pi^2}{228614400}-\frac{2071729\pi^3}{12700800}-\frac{188281i\pi^4}{1209600}-\frac{8804281i\log^2(-2ir\omega\eta)}{74088000}\\
&+\left(\frac{13526987i}{2976750}-\frac{319751\pi}{141120}-\frac{3833i\pi^2}{20160}\right)\gamma_E^2-\frac{(746999i-193410\pi)\gamma_E^3}{635040}+\frac{329i\gamma_E^4}{2880}-\pi\left(\frac{4447527101951}{770144760000}+\frac{439\zeta(3)}{360}\right)\\
&+\log(-2ir\omega\eta)\Bigg(\frac{44134399819i}{48134047500}+\frac{6288811\pi}{4939200}+\frac{1403429i\pi^2}{1270080}-\frac{8\pi^3}{35}+\left(\frac{180021511i}{111132000}-\frac{4451\pi}{2160}-\frac{8i\pi^2}{15}\right)\gamma_E\\
&-\left(\frac{1213759i}{1058400}-\frac{16\pi}{35}\right)\gamma_E^2+\frac{16i\gamma_E^3}{105}+\frac{32i\zeta(3)}{105}\Bigg)+\frac{3547559i\zeta(3)}{1587600}+\gamma_E\Bigg(-\frac{545200033471i}{85571640000}+\frac{4560260993\pi}{666792000}\\
&+\frac{2242249i\pi^2}{2116800}+\frac{171\pi^3}{1120}-\frac{461i\zeta(3)}{504}\Bigg)\Bigg]+r^{10}\omega^{13}\Bigg[\log(-2ir\omega\eta)\Bigg(\frac{24258363523i}{2568646080000}-\frac{16777\pi}{2376000}-\frac{13i\pi^2}{8910}\\
&+\frac{(-2688779i+873600\pi)\gamma_E}{349272000}+\frac{13i\gamma_E^2}{10395}\Bigg)-\frac{393417734660974453i}{1133905694201280000}+\frac{692951731379\pi}{35961045120000}-\frac{212559509i\pi^2}{553246848000}+\frac{2971\pi^3}{2851200}\\
&+\left(\frac{241892036929i}{8390910528000}-\frac{375601459\pi}{46103904000}+\frac{2771i\pi^2}{6652800}\right)\gamma_E-\left(\frac{730527349i}{92207808000}-\frac{1387\pi}{1108800}\right)\gamma_E^2+\frac{8321i\gamma_E^3}{9979200}-\frac{2377i\zeta(3)}{712800}\Bigg]\\
&+r\omega^7\Bigg[\frac{26663346157427934243349i}{317828238158814075000}-\frac{662449i\pi^4}{110250}+\frac{5357\pi^5}{6300}+\frac{127i\pi^6}{252}+\frac{2450086i\log^3(-2ir\omega\eta)}{1157625}
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\frac{56167i}{1575} - \frac{227\pi}{15} - \frac{7i\pi^2}{3}\right)\gamma_E^4 + \left(\frac{382i}{105} - \frac{4\pi}{5}\right)\gamma_E^5 - \frac{2i\gamma_E^6}{15} + \pi^2\left(-\frac{30232972364519i}{1391633105625} + \frac{652i\zeta(3)}{315}\right) \\
&- \log^2(-2ir\omega\eta)\left(\frac{1294062173i}{40516875} - \frac{8392117\pi}{385875} - \frac{11449i\pi^2}{1575} - \left(\frac{12067246i}{385875} - \frac{45796\pi}{3675}\right)\gamma_E + \frac{22898i\gamma_E^2}{3675}\right) \\
&+ \gamma_E^3\left(\frac{176320048i}{1157625} - \frac{1096954\pi}{11025} - \frac{8662i\pi^2}{315} + 4\pi^3 - \frac{16\zeta(3)}{3}\right) + \gamma_E^2\Big(-\frac{207263406653339i}{927755403750} + \frac{332123i\pi^2}{3150} \\
&- \frac{2627\pi^3}{105} - \frac{121i\pi^4}{30} + \pi\left(\frac{102425623}{385875} - 16\zeta(3)\right) + \frac{2504i\zeta(3)}{105}\Big) + \frac{291992i\zeta(3)}{6615} - \frac{16i\zeta(3)^2}{3} - \pi^3\left(\frac{66841171}{4630500} - 8\zeta(3)\right) \\
&+ \log(-2ir\omega\eta)\Big(\frac{319701054873731i}{10014555993750} - \frac{37479133i\pi^2}{771750} + \frac{143701\pi^3}{11025} + \frac{12947i\pi^4}{3150} \\
&+ \frac{2i\left(4926295 + 2608767i\pi - 550515\pi^2\right)\gamma_E^2}{77175} + \frac{428(-247i + 70\pi)\gamma_E^3}{3675} + \frac{214i\gamma_E^4}{105} - \frac{9416i\zeta(3)}{1225} \\
&+ \pi\left(-\frac{3946207057}{40516875} + \frac{1712\zeta(3)}{105}\right) + \gamma_E\Big(-\frac{7828393576i}{40516875} + \frac{62324162\pi}{385875} + \frac{606262i\pi^2}{11025} - \frac{428\pi^3}{35} \\
&+ \frac{1712i\zeta(3)}{105}\Big)\Big) + \pi\left(-\frac{75939597986335271}{1071557491331250} - \frac{311564\zeta(3)}{11025} - \frac{96\zeta(5)}{5}\right) \\
&+ \gamma_E\Big(-\frac{60994582325626891i}{2143114982662500} + \frac{168734\pi^3}{3675} + \frac{3471i\pi^4}{350} - \frac{11\pi^5}{5} + \pi^2\left(-\frac{104454419i}{771750} + \frac{56i\zeta(3)}{3}\right) \\
&- \frac{87736i\zeta(3)}{2205} + \pi\left(-\frac{72822489575903}{463877701875} + \frac{488\zeta(3)}{21}\right) - \frac{96i\zeta(5)}{5}\Big) - \frac{368i\zeta(5)}{35}\Big] \\
&+ r^4\omega^9\Big[\frac{233282598583884433307i}{280553234094000000} - \frac{74566553\pi^3}{2646000} - \frac{219251i\pi^4}{108000} - \frac{2431\pi^5}{1800} + \frac{183184i\log^3(-2ir\omega\eta)}{165375} \\
&- \frac{11449\log^2(-2ir\omega)\left(313i + 620\pi - 20i\gamma_E\right)}{2205000} - \frac{i\left(715259 - 427000i\pi + 191590\pi^2\right)\gamma_E^3}{44100} \\
&- \left(\frac{47749i}{12600} - \frac{97\pi}{30}\right)\gamma_E^4 + \frac{43i\gamma_E^5}{50} + \gamma_E^2\left(\frac{22781312947i}{185220000} - \frac{2213479\pi}{88200} + \frac{42389i\pi^2}{4200} - \frac{7\pi^3}{6} - \frac{42i\zeta(3)}{5}\right) \\
&+ \gamma_E\left(-\frac{20207048856809351i}{89064518760000} - \frac{10306871i\pi^2}{661500} + \frac{9028\pi^3}{1575} - \frac{3847i\pi^4}{1800} + \pi\left(\frac{15141862741}{92610000} - \frac{76\zeta(3)}{3}\right) + \frac{2803i\zeta(3)}{315}\right) \\
&+ \pi^2\left(-\frac{50702569247i}{1111320000} + \frac{953i\zeta(3)}{45}\right) + \frac{1090211i\zeta(3)}{12250} + \pi\left(-\frac{10913554954125109}{55665324225000} + \frac{2144\zeta(3)}{315}\right) \\
&+ \log(-2ir\omega\eta)\Big(-\frac{2868092356769833i}{222661296900000} + \frac{16005901i\pi^2}{441000} + \frac{3457\pi^3}{3150} + \frac{484i\pi^4}{225} + \frac{(-5147i + 6720\pi)\gamma_E^3}{1575} + \frac{16i\gamma_E^4}{15} \\
&- \left(\frac{6648301i}{220500} + \frac{229\pi}{35} + \frac{112i\pi^2}{15}\right)\gamma_E^2 + \gamma_E\left(\frac{2046765221i}{23152500} - \frac{515852\pi}{7875} + \frac{3097i\pi^2}{630} - \frac{32\pi^3}{5} + \frac{128i\zeta(3)}{15}\right) \\
&- \frac{10294i\zeta(3)}{1575} + \pi\left(\frac{23793731}{315000} + \frac{128\zeta(3)}{15}\right)\Big) - \frac{764i\zeta(5)}{25}\Big] + \frac{\omega^5}{r^2}\Big[-\frac{151778286109871900201i}{79457059539703518750} + \frac{9175757i\pi^4}{165375} \\
&+ \frac{26323\pi^5}{3150} - \frac{127i\pi^6}{126} - \frac{4900172i\log^3(-2ir\omega)}{1157625} - \frac{2i\left(-177509 - 66045i\pi + 25725\pi^2\right)\gamma_E^4}{11025} + \left(-\frac{76i}{21} + \frac{8\pi}{5}\right)\gamma_E^5 \\
&+ \frac{4i\gamma_E^6}{15} - \pi^3\left(\frac{348323659}{2315250} + 16\zeta(3)\right) + \log^2(-2ir\omega\eta)\Big(\frac{784702262i}{13505625} + \frac{5197846\pi}{385875} - \frac{22898i\pi^2}{1575} \\
&+ \left(-\frac{2152412i}{385875} + \frac{91592\pi}{3675}\right)\gamma_E + \frac{45796i\gamma_E^2}{3675}\Big) + \gamma_E^3\left(-\frac{123500126i}{1157625} + \frac{1084948\pi}{11025} + \frac{3884i\pi^2}{315} - 8\pi^3 + \frac{32i\zeta(3)}{3}\right) \\
&+ \pi^2\left(-\frac{27696839199877i}{103083933750} - \frac{28184i\zeta(3)}{315}\right) + \frac{60718564i\zeta(3)}{231525} + \frac{32i\zeta(3)^2}{3} \\
&+ \gamma_E^2\left(\frac{32592750893818i}{154625900625} - \frac{1565461i\pi^2}{11025} - \frac{506\pi^3}{105} + \frac{121i\pi^4}{15} + \frac{2672i\zeta(3)}{105} + \pi\left(-\frac{48218756}{385875} + 32\zeta(3)\right)\right) \\
&+ \log(-2ir\omega\eta)\Big(-\frac{696660635226086i}{5007277996875} + \frac{185056159i\pi^2}{1157625} + \frac{328918\pi^3}{11025} - \frac{12947i\pi^4}{1575} - \frac{856(-87i + 70\pi)\gamma_E^3}{3675} \\
&+ \frac{4\left(-2509379i + 451647\pi + 550515i\pi^2\right)\gamma_E^2}{77175} - \frac{428i\gamma_E^4}{105} + \gamma_E\Big(\frac{795325214i}{13505625} - \frac{93917924\pi}{385875} + \frac{225556i\pi^2}{11025} \\
&+ \frac{856\pi^3}{35} - \frac{3424i\zeta(3)}{105}\Big) - \frac{217424i\zeta(3)}{3675} - \frac{4\pi(389695393 + 110103000\zeta(3))}{13505625}\Big) + \gamma_E\Big(\frac{2073465136261921i}{1071557491331250} - \frac{467788\pi^3}{3675} \\
&+ \frac{8947i\pi^4}{525} + \frac{22\pi^5}{5} + \frac{471472i\zeta(3)}{2205} + \pi\left(\frac{51526984225007}{154625900625} + \frac{2096\zeta(3)}{21}\right) - \frac{2i\pi^2(7520297 + 3087000\zeta(3))}{165375} + \frac{192i\zeta(5)}{5}\Big) \\
&+ \frac{544i\zeta(5)}{5} + \pi\left(\frac{224804224826795527}{1071557491331250} + \frac{3335768\zeta(3)}{11025} + \frac{192\zeta(5)}{5}\right)\Big]\Big\},
\end{aligned}
$$

where M has been fixed to 1 but it has been restored to ensure the right dimensionality in the argument of the logarithms.

3 Black Hole Perturbations sourced by Point Particles

In the previous Chapter, perturbations of the background black hole metric were discussed without specifying a particular source term; only the homogeneous solutions to the Teukolsky equation were considered.

This Chapter focuses on a specific class of source terms: point particles. The background metric is associated with a central black hole of mass M and angular momentum per unit mass a, while a point particle of mass $m \ll M$ induces perturbations of the spacetime. The parameter ϵ introduced in Eq. (2.3) can thus be explicitly related to the mass ratio $\mu = m/M$.

In solving the Teukolsky equation with a point-like source, the homogeneous solutions must be convolved with appropriate source terms. These have been formally defined in Eqs. (2.28), but require explicit computation based on the particle's motion.

The first Section briefly reviews the geodesic motion of point particles in black hole spacetimes. This discussion is not intended to be exhaustive, as the topic is vast and has been treated in detail in several reviews (see, e.g., [26, 27]).

Following this general overview, the Teukolsky source will be explicitly constructed for a particle in equatorial orbit, either bound or unbound. The resulting expressions will depend directly on the components of the particle's four-velocity.

Once the Teukolsky source is obtained, a more precise definition of the post-Newtonian expansion will be introduced, clarifying the connection between the low-frequency expansion used for the homogeneous solutions and the low-velocity approximation inherent in the PN framework. The general procedure for constructing the solution of the Teukolsky equation for a generic equatorial orbit will then be presented.

3.1 Point particles on geodesic motion

It is well known that point particles are not well defined in General Relativity: a mass distribution cannot reach an arbitrarily high density without collapsing into a black hole, which is not a point-like object. This inconsistency lies at the core of the difficulties encountered when solving the EFEs at second order in ϵ[1]: in general, the EFEs do not admit a well-defined solution if the source is modeled as a point particle, see [144, 145]. As a consequence, this issue also manifests itself within the perturbative framework.

[1] Note that this ϵ is not the ϵ used in the MST formalism described in the previous Chapter.

Nevertheless, it has been proven that the point-particle approximation remains valid at first order, by modeling the source as a finite-size object and taking the point-particle limit via a suitable limiting procedure, see [27].

At first order, the equation of motion for the point particle can be written perturbatively as

$$\frac{D^2 z^\mu}{d\tau^2} := a^\mu = f^\mu_{(0)} + \epsilon f^\mu_{(1)} \tag{3.1}$$

where τ is the proper time as measured in the background black hole's spacetime, while $z^\mu(\tau)$ is the perturbed worldline. The operator

$$\frac{D^2}{d\tau^2} = \frac{dz^\mu}{d\tau}\nabla_\mu \frac{d}{d\tau}\,, \tag{3.2}$$

applied to the perturbed geodesic defines the covariant acceleration a^μ, with respect to the background metric $g_{\mu\nu}$. In general, $f^\mu_{(0)} \neq 0$; however, it can be neglected under the assumption that the particle follows a geodesic in the background spacetime, i.e. no external forces are present. The term $f^\mu_{(1)}$ provides the first-order correction to the acceleration and can be computed once the first-order metric perturbations are known. This term gives rise to a self-acceleration: it corresponds to the force experienced by the particle due to the perturbation it induces on the background geometry, which in turn affects its own motion. For this reason, $f^\mu_{(1)}$ is referred to as the *self-force*, and it is the quantity that must be computed from the solution of the Teukolsky equation.

The computation of first-order corrections requires knowledge of the unperturbed geodesic motion. The most general equations for timelike geodesics of a test non-spinning particle in Kerr spacetime, including inclined motion, read

$$\Sigma\frac{dt}{d\tau} = -\left(a\hat{\mathcal{E}} - \frac{\hat{l}_z}{\sin^2\theta}\right) a\sin^2\theta + \frac{r^2+a^2}{\Delta}\left(\hat{\mathcal{E}}(r^2+a^2) - a\hat{l}_z\right), \tag{3.3a}$$

$$\Sigma\frac{dr}{d\tau} = \epsilon_r\left\{\left[\hat{\mathcal{E}}(r^2+a^2) - a\hat{l}_z\right]^2 - \Delta\left[(\hat{\mathcal{E}}a - \hat{l}_z)^2 + r^2 + \hat{C}\right]\right\}^{1/2}, \tag{3.3b}$$

$$\Sigma\frac{d\theta}{d\tau} = \epsilon_r\left\{\hat{C} - \cos^2\theta\left[a^2(1-\hat{\mathcal{E}}^2) + \frac{\hat{l}_z^2}{\sin^2\theta}\right]\right\}^{1/2}, \tag{3.3c}$$

$$\Sigma\frac{d\phi}{d\tau} = -\left(a\hat{\mathcal{E}} - \frac{\hat{l}_z}{\sin^2\theta}\right) + \frac{a}{\Delta}\left[\hat{\mathcal{E}}(r^2+a^2) - a\hat{l}_z\right], \tag{3.3d}$$

where $\hat{\mathcal{E}}$, $\hat{l}_z$, and $\hat{C}$ denote the specific energy, the z-component of the specific angular momentum, and the normalized Carter constant, respectively. The hat indicates that these quantities are normalized with respect to the particle mass m, i.e., $\mathcal{E} = m\hat{\mathcal{E}}$, $l_z = m\hat{l}_z$, and $C = m^2\hat{C}$. The sign $\epsilon_r = \pm 1$ specifies the direction of the radial motion.

This set of equations can be simplified by considering equatorial orbits, i.e. $\theta = \pi/2$. In this case, the equations reduce to:

$$\frac{dt}{d\tau} = -\frac{a^2}{r^2}E + \frac{a}{r^2}J + \frac{(r^2+a^2)^2}{r^2\Delta}E - \frac{(r^2+a^2)a}{r^2\Delta}J\,, \tag{3.4a}$$

$$\frac{dr}{d\tau} = \frac{\epsilon_r}{r^2}\left\{\left[E(r^2+a^2) - aJ\right]^2 - \Delta\left[(Ea-J)^2 + r^2\right]\right\}^{1/2}, \tag{3.4b}$$

$$\frac{d\theta}{d\tau} = 0\,, \tag{3.4c}$$
$$\frac{d\phi}{d\tau} = -\frac{aE-J}{r^2} + \frac{a}{r^2\Delta}\left[E(r^2+a^2) - aJ\right]\,, \tag{3.4d}$$

where $E = -u_t$ and $J = u_\varphi$ are the conserved specific energy and orbital angular momentum.

Using the Darwin parameterization, it is possible to write an explicit solution for the radial coordinate as a function of the relativistic anomaly ξ:

$$r_p(\xi) = \frac{pM}{1+e\cos\xi},\,, \tag{3.5}$$

where e is the eccentricity and p the semi-latus rectum. These parameters also define the energy and angular momentum implicitly via

$$E^2 = \frac{(1-e^2)^2}{p^3}\hat{J}^2 + 1 - \frac{1-e^2}{p}, \qquad E = -\frac{p-3-e^2}{2qp}\hat{J} - \frac{q^2-p}{2q\hat{J}}, \tag{3.6}$$

with $q = a/M$ and $\hat{J} = J/M - qE$, see [41, 146].

At this stage of the discussion, the orbit remains generic, and no assumptions are made about whether it is bound or unbound. The next Subsection present explicit expressions for the cases of hyperbolic orbits in Schwarzschild.

3.1.1 Hyperbolic orbits

The discussion will follow the one proposed in [100].

By recalling the Schwarzschild metric in standard spherical coordinates

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{3.7}$$

where $f = 1 - 2M/r$. A generic timelike geodesic orbit on the equatorial plane ($\theta = \pi/2$) is described by the following equations

$$\dot{t}_p = \frac{E}{f(r_p)}, \qquad \dot{\phi}_p = \frac{J}{r_p^2}, \qquad \dot{r}_p = \epsilon_r\sqrt{E^2 - V(r_p;J)}\,, \tag{3.8}$$

where the overdot indicates $d/d\tau$, that can be straightforwardly obtained from Eqs. (3.4) sending $a \to 0$. Besides $V(r_p;J)$ is the radial potential in the radial equation, defined as

$$V(r;J) = f(r)\left(1 + \frac{J^2}{r^2}\right). \tag{3.9}$$

Hyperbolic orbits are characterized by $E > 1$ and $J > J_{\rm crit}(E)$, with

$$J_{\rm crit}(E) = \frac{M}{vE}\sqrt{\frac{27E^4 + 9\beta E^3 - 36E^2 - 8\beta E + 8}{2}}, \tag{3.10}$$

where $\beta := \sqrt{9E^2 - 8}$; see [55, 57]. Alternatively, rather than using E and J, one can characterize hyperbolic orbits using the impact parameter b and initial velocity at infinity v. These are related to E and J by

$$E = \frac{1}{\sqrt{1-v^2}}, \qquad J = bvE, \tag{3.11}$$

and they range over $0 < v < 1$ and $b > b_{\rm crit}$, where $b_{\rm crit} = J_{\rm crit}/(vE)$. [2]

The solution to the radial equation is also often given in terms of the relativistic anomaly ξ:

$$r_p(\xi) = \frac{Mp}{1 + e\cos\xi}, \tag{3.12}$$

where p is the semi-latus rectum and e the eccentricity. The sign difference between the ingoing and outgoing directions is then encoded in ξ, which takes values in the interval $(-\xi_\infty, \xi_\infty)$, where $\xi_\infty = \arccos(-1/e)$. The parameters p and e are related to E and L by the usual relations

$$E^2 = \frac{(p-2)^2 - 4e^2}{p(p-3-e^2)}, \qquad J^2 = \frac{p^2 M^2}{p-3-e^2}, \tag{3.13}$$

and for hyperbolic orbits they range over $e > 1$ and $p > 6 + 2e$. The complete treatment of hyperbolic orbits in terms of all these different orbital parameters is discussed in Ref. [55].

In the Schwarzschild case, there are several ways to solve the geodesic equation exactly. For instance by using the inverse radial distance $u = M/r$ and combining the equation for ϕ_p and r_p, one finds

$$\left(\frac{du}{d\phi}\right)^2 = 2(u-u_1)(u-u_2)(u-u_3), \tag{3.14}$$

where u_1, u_2 and u_3 are the ordered roots of the equation

$$2u^3 - u^2 + \frac{2M^2 u}{J^2} + \frac{M^2(E^2-1)}{J^2} = 0, \tag{3.15}$$

and the subscript p's has been omitted to avoid confusion with the subscripts 1,2,3. For scattering orbits one has $u_1 < 0 < u \leq u_2 < u_3$, with u_2 corresponding to the closest-approach distance $r_{\rm min} = Mu_2$. The condition $u_2 = u_3$ gives the critical value of the angular momentum corresponding to capture by the black hole for fixed energy value.

In order to solve the geodesic equation, a boundary condition is required and a convenient choice is

$$\phi(u_2) = 0 \tag{3.16}$$

(i.e., vanishing ϕ at periastron), one can express the solution to Eq. (3.14) in terms of elliptic integrals as follows:

$$\phi(u) = \frac{\sqrt{2}}{\sqrt{u_3 - u_1}}\left[K(n) - F(\psi, n)\right], \tag{3.17}$$

with

$$n = \sqrt{\frac{u_2 - u_1}{u_3 - u_1}}, \qquad \psi = \sqrt{\frac{u - u_1}{u_2 - u_1}}, \tag{3.18}$$

where $F(\varphi, k)$ and $K(k)$ are the incomplete and complete elliptic integrals of the first kind, respectively, defined by

$$F(\varphi, k) = \int_0^\varphi \frac{dz}{\sqrt{1 - k^2\sin^2 z}}, \qquad K(k) = F(\pi/2, k). \tag{3.19}$$

[2] In the literature v can be defined also as v_∞, since it is the relative velocity of the small body as $r_p \to \infty$. The subscript will be dropped for simplicity.

The total change of ϕ during a hyperbolic encounter is called *scattering angle* and can be formally defined as:

$$\chi = 2\int_0^{u_2} du \frac{d\phi_p}{du} - \pi \,, \tag{3.20}$$

where the subscript p has been restored for clarity.

Similarly with the eccentric case in Kerr, where complete analytical solutions are available, for practical Self Force applications, it is more convenient to compute perturbative solutions. In hyperbolic orbits there is no virial theorem that relates the orbital radius with the frequencies, hence large distances do not automatically means low velocity. This means that, for these orbits, there are two, orthogonal expansions that can be performed: large radial distance and low velocity. These two expansions are easily related respectively with the PM and PN expansion.

In the following, two methods will be proposed for computing analytically perturbative expressions for the geodesics: one chooses u as the independent variable and a rescaled version of (E, J) as orbital parameters; in the second, one uses a rescaled coordinate time as the independent variable and uses (b, v) as orbital parameters. It is always possible to go from one parameterization to the other and, indeed they produce the same result.

(E, J) *parameterization*

Using u as the parameter along the geodesic, Eq. (3.8) can be rewritten as

$$\begin{aligned} \frac{dt_p}{du} &= -\epsilon_r \frac{M\sqrt{1+2\bar{E}}}{ju^2(1-2u)}\left(2u^3 - u^2 + \frac{2u}{j^2} + \frac{2\bar{E}}{j^2}\right)^{-1/2} \,, \\ \frac{d\phi_p}{du} &= -\epsilon_r \left(2u^3 - u^2 + \frac{2u}{j^2} + \frac{2\bar{E}}{j^2}\right)^{-1/2} \,, \end{aligned} \tag{3.21}$$

where rescaled (dimensionless) energy and angular momentum parameters $\bar{E}$ and j have been defined by

$$E = \sqrt{1+2\bar{E}}, \qquad J = Mj \,. \tag{3.22}$$

From these definitions and from Eq. (3.11), $\bar{E} \sim v^2$ and $j \sim \frac{1}{v}$. In this parameterization, it is possible to produce a low-velocity expansion, i.e. a PN expansion. On the other hand, u is constrained as $u < 1/b_{\text{crit}} \sim v^2$. To track these scalings, one can introduce the standard PN small parameter $\eta = 1/c$ (essentially restoring powers of c and using it to track small velocities with respect to c) and make the replacements

$$u \to u\eta^2, \quad j \to j/\eta, \quad \bar{E} \to \bar{E}\eta^2 \,, \tag{3.23}$$

interestingly, these replacements are identical to the ones that could be done for bound orbits.

Solving Eq. (3.21) for small η, one can obtain the expressions for the coordinate time and ϕ_p in terms of u as

$$\begin{aligned} t_p(u) = \epsilon_r M \Bigg\{ &\frac{\alpha j}{u}\mathcal{S}(\alpha, j; u) - \alpha^3 j^3 \mathcal{L}(\alpha, j; u) \\ &+ \eta^2 \left[\frac{1+\alpha^2+2\alpha^2 j^2(\alpha^2+2)u + j^4\alpha^2(\alpha^2-1)u^2}{2u\alpha j(\alpha^2+1)\mathcal{S}(\alpha, j; u)} + \frac{3}{2}\alpha j \mathcal{L}(\alpha, j; u)\right] + O(\eta^4)\Bigg\} \,, \end{aligned} \tag{3.24a}$$

$$\phi_p(u) = \epsilon_r \left\{ B(\alpha, j; u) + \frac{\eta^2}{j^2}\left[3B(\alpha, j; u) + \frac{2 + 3\alpha^2 + u\alpha^2(6\alpha^2 + 5)j^2 - u^2\alpha^2(\alpha^2 + 1)j^4}{\alpha(\alpha^2 + 1)\mathcal{S}(\alpha, j; u)}\right] + O(\eta^4)\right\}, \tag{3.24b}$$

where initial conditions have been chosen as $t_p(u_2) = 0 = \phi_p(u_2)$ at the closest approach, with

$$u_2(\bar{E}, j) = \left(1+\sqrt{1 + 2\bar{E}j^2}\right)/j^2 \equiv \left(1+\sqrt{1 + 1/\alpha^2}\right)/j^2 \tag{3.25}$$

at the lowest order, and

$$\begin{aligned}
\mathcal{S}(\alpha, j; u) &= \sqrt{1 + 2uj^2\alpha^2 - u^2j^4\alpha^2}\,, \\
\mathcal{L}(\alpha, j; u) &= \ln\left(\frac{1 + uj^2\alpha^2 + \mathcal{S}(\alpha, j; u)}{j^2u\alpha\sqrt{\alpha^2 + 1}}\right), \\
B(\alpha, j; u) &= \frac{\pi}{2} + \arctan\left(\frac{(1 - uj^2)\alpha}{\mathcal{S}(\alpha, j; u)}\right).
\end{aligned} \tag{3.26}$$

Note that in the above expressions αj and uj^2 are Newtonian quantities, with

$$\alpha \equiv 1/\sqrt{2\bar{E}j^2} = 1/(p_\infty j), \tag{3.27}$$

where $p_\infty = \sqrt{2\bar{E}}$.

The first few terms of the PN expansion of the geodesic scattering angle (3.20) are given by

$$\frac{\chi}{2} = B(\alpha) - \frac{\pi}{2} + \frac{\eta^2}{j^2}\left[3B(\alpha) + \frac{2 + 3\alpha^2}{\alpha(\alpha^2 + 1)}\right] + O(\eta^4)\,, \tag{3.28}$$

where $B(\alpha) = B(\alpha, j; 0) = \frac{\pi}{2} + \arctan(\alpha)$. Taking the series expansion for large values of the angular momentum parameter j gives the PM expansion at each PN order,

$$\chi = \frac{2}{j}\left(\frac{1}{p_\infty} + 2p_\infty\eta^2\right) + \frac{3\pi\eta^2}{j^2}\left(1 + \frac{5}{4}p_\infty^2\eta^2\right) + O\left(\frac{1}{j^3}\right). \tag{3.29}$$

(b, v) *parameterization*

By recalling Eq. (3.12) and Eq. (3.8) in terms of the relativistic anomaly, keeping (p, e) as parameters the geodesic equations are:

$$\frac{dt_p}{d\xi} = \frac{Mp^2}{(p - 2 - 2e\cos\xi)(1 + e\cos\xi)^2}\sqrt{\frac{(p-2)^2 - 4e^2}{p - 6 - 2e\cos\xi}}\,, \tag{3.30}$$

$$\frac{d\phi_p}{d\xi} = \sqrt{\frac{p}{p - 6 - 2e\cos\xi}}\,, \tag{3.31}$$

and by using the relations of Eqs. (3.13), one can rewrite everything in terms of the parameters (b, v) first inverting these relations and then combining with Eq. (3.11).

Expanding Eq. (3.30) for large b, one gets

$$\begin{aligned}\xi(\bar{t}) =& \arctan\bar{t} + \frac{M}{bv^2}\left[\frac{\bar{t}}{\sqrt{1+\bar{t}^2}} + \frac{\left(1-3v^2\right)\operatorname{arcsinh}\bar{t}}{1+\bar{t}^2}\right] \\ &+ \frac{M^2}{(1+\bar{t}^2)b^2v^4}\Bigg[2\bar{t}v^2 + \frac{\left(2-9v^2+9v^4\right)\operatorname{arcsinh}\bar{t}}{\sqrt{1+\bar{t}^2}} - \frac{\bar{t}\left(1-3v^2\right)^2\operatorname{arcsinh}^2\bar{t}}{1+\bar{t}^2} \\ &\qquad - \frac{15}{2}v^4\arctan\bar{t}\Bigg] + O(b^{-3}),\end{aligned} \tag{3.32}$$

where a rescaled coordinate time has been introduced as $\bar{t} = vt/b$.

By substituting this into Eqs. (3.12) and (3.31), one finds

$$\begin{aligned}r_p(\bar{t}) =& b\sqrt{1+\bar{t}^2} - \frac{M}{v^2} + \frac{M\left(\bar{t}-3\bar{t}v^2\right)\operatorname{arcsinh}\bar{t}}{\sqrt{1+\bar{t}^2}v^2} \\ &+ \frac{M^2}{b}\left[\frac{1-4v^2}{2\sqrt{1+\bar{t}^2}v^4} + \frac{\bar{t}\left(1-3v^2\right)^2\operatorname{arcsinh}\bar{t}}{\left(1+\bar{t}^2\right)v^4} + \frac{\left(1-3v^2\right)^2\operatorname{arcsinh}^2\bar{t}}{2\left(1+\bar{t}^2\right)^{3/2}v^4} - \frac{15\bar{t}\arctan\bar{t}}{2\sqrt{1+\bar{t}^2}}\right] \\ &+ \frac{M^3}{b^2}\Bigg[-\frac{8v^4+\bar{t}^2\left(1+7v^2\left(-1+5\left(v^2+v^4\right)\right)\right)}{2\left(1+\bar{t}^2\right)v^6} + \frac{\left(-2+\bar{t}^2\right)\left(-1+3v^2\right)^3\operatorname{arcsinh}^2\bar{t}}{2\left(1+\bar{t}^2\right)^2v^6} \\ &\quad + \frac{\bar{t}\left(-1+3v^2\right)^3\operatorname{arcsinh}^3\bar{t}}{2\left(1+\bar{t}^2\right)^{5/2}v^6} + \frac{15\bar{t}\left(-1+3v^2\right)\arctan\bar{t}}{2\left(1+\bar{t}^2\right)v^2} \\ &\quad + \operatorname{arcsinh}\bar{t}\left(\frac{\bar{t}\left(1-11v^2+27v^4-9v^6\right)}{2\left(1+\bar{t}^2\right)^{3/2}v^6} + \frac{15\left(-1+3v^2\right)\arctan\bar{t}}{2\left(1+\bar{t}^2\right)^{3/2}v^2}\right)\Bigg] + O(b^{-3}), \\ \phi_p(\bar{t}) =& \arctan\bar{t} + \frac{M}{bv^2\sqrt{1+\bar{t}^2}}\left[\bar{t}\left(1+v^2\right) + \frac{\left(1-3v^2\right)\operatorname{arcsinh}\bar{t}}{\sqrt{1+\bar{t}^2}}\right] \\ &+ \frac{M^2}{4b^2v^4\left(1+\bar{t}^2\right)}\Bigg\{3\bar{t}v^2\left(4+v^2\right) + 3v^2\left[4-9v^2+\bar{t}^2\left(4+v^2\right)\right] \\ &\qquad + \frac{8\left(1-4v^2+3v^4\right)\operatorname{arcsinh}\bar{t}}{\sqrt{1+\bar{t}^2}} - \frac{4\bar{t}\left(1-3v^2\right)^2\operatorname{arcsinh}^2\bar{t}}{1+\bar{t}^2}\Bigg\} + O(b^{-3}).\end{aligned} \tag{3.33}$$

Finally, using the result for ϕ_p, the large b expansion for the geodesic scattering angle (3.20) reads

$$\chi = \frac{2M}{bv^2}(1+v^2) + \frac{3M^2\pi}{4b^2v^4}(4+v^2) - \frac{2M^3}{3b^3v^6}\left[1-5v^2\left(3+9v^2+v^4\right)\right] + O(b^{-4}), \tag{3.34}$$

which can be shown to agree with Eq. (3.29).

3.2 Teukolsky source

The two master equations in Eqs. (2.26) are written in terms of the same differential operator $\mathcal{O}$:

$$\mathcal{O}\psi_0 = 8\pi \mathcal{J}_0 T\,, \qquad \mathcal{O}(\zeta^4\psi_4) = 8\pi\zeta^4 \mathcal{J}_4 T\,, \tag{3.35}$$

where, on the right-hand side, the two operators $\mathcal{J}_{0/4}$ act on the stress-energy tensor of a point particle of mass m, defined as:

$$T^{\mu\nu} = \frac{m}{\Sigma \sin\theta}\frac{d\tau}{dt}\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau}\delta(r-r(t))\delta(\theta-\theta(t))\delta(\phi-\phi(t))\,, \tag{3.36}$$

where z^μ denotes the geodesic trajectory followed by the massive particle and $\tau = \tau(t)$ represents the proper time along this trajectory.

The components of the differential operators acting on the source have been defined in Eqs. (2.28), and the source takes the form:

$$\begin{aligned}
\mathcal{J}_0 T &= \frac{1}{2}\left(\eth - \bar{\tau}' - 4\tau\right)\left[\left(\text{Þ} - 2\bar{\rho}\right)T_{(lm)} - \left(\eth - \bar{\tau}'\right)T_{ll}\right] \\
&+ \frac{1}{2}\left(\text{Þ} - 4\rho - \bar{\rho}\right)\left[\left(\eth - 2\bar{\tau}'\right)T_{(lm)} - \left(\text{Þ} - \bar{\rho}\right)T_{mm}\right]\,, \qquad (3.37\text{a}) \\
\mathcal{J}_4 T &= \frac{1}{2}\left(\eth' - \bar{\tau} - 4\tau'\right)\left[\left(\text{Þ}' - 2\bar{\rho}'\right)T_{(n\bar{m})} - \left(\eth' - \bar{\tau}\right)T_{nn}\right] \\
&+ \frac{1}{2}\left(\text{Þ}' - 4\rho' - \bar{\rho}'\right)\left[\left(\eth' - 2\bar{\tau}\right)T_{(n\bar{m})} - \left(\text{Þ}' - \bar{\rho}'\right)T_{\bar{m}\bar{m}}\right]\,, \qquad (3.37\text{b})
\end{aligned}$$

where the subscripts indicate contraction with the corresponding tetrad legs, e.g. $T_{ll} = T^{\mu\nu}l_\mu l_\nu$.

For brevity, the generic spin-s Teukolsky source is denoted as ${}_sT$, to emphasize the generality of the discussion. The structure of the Teukolsky source is then:

$$\begin{aligned}
{}_sT &= {}_sA(r,r(t),\theta,\theta(t),\phi,\phi(t))\,\delta(r-r(t)) + {}_sB(r,r(t),\theta,\theta(t),\phi,\phi(t))\,\delta'(r-r(t)) \\
&+ {}_sC(r,r(t),\theta,\theta(t),\phi,\phi(t))\,\delta''(r-r(t)), \qquad (3.38)
\end{aligned}$$

where the dependence on $(\theta,\theta(t),\phi,\phi(t))$ differs across the functions ${}_sA$, ${}_sB$, and ${}_sC$. In particular, each of them contains products of Dirac delta functions in the angular variables, such as $\delta\left(\theta-\theta(t)\right)\delta(\phi-\phi(t))$, along with their first and second derivatives.

Using properties of the Dirac delta function:

$$\begin{aligned}
f(x)\delta(x-x') &= f(x')\delta(x-x')\,, \qquad (3.39\text{a}) \\
f(x)\delta'(x-x') &= f(x')\delta'\left(x-x'\right) - f'(x')\delta(x-x')\,, \qquad (3.39\text{b}) \\
f(x)\delta''(x-x') &= -2f'\left(x'\right)\delta'\left(x-x'\right) + f\left(x'\right)\delta''\left(x-x'\right) + f''\left(x'\right)\delta\left(x-x'\right)\,, \qquad (3.39\text{c})
\end{aligned}$$

it becomes possible to eliminate the r-dependence from the coefficients in front of the radial delta functions, and absorb all θ-dependence into the angular delta functions.

This allows the angular delta functions to be expanded in the basis of spin-weighted spheroidal harmonics:

$$\delta\left(\theta-\theta(t)\right)\delta(\phi-\phi(t)) = \sin\theta(t)\sum_{l,m} {}_sS_{lm}(\theta,\phi;a\omega)\,{}_sS^*_{lm}(\theta(t),\phi(t);a\omega), \tag{3.40}$$

using the completeness relation introduced in the previous Chapter.

The derivatives of the delta functions with respect to angles can be expressed similarly:

$$\begin{aligned}
\partial_\theta \delta\left(\theta-\theta(t)\right)\delta(\phi-\phi(t)) &= -\partial_{\theta(t)}\delta\left(\theta-\theta(t)\right)\delta(\phi-\phi(t)) \\
&= -\sin\theta(t)\sum_{l,m} {}_sS_{lm}(\theta,\phi;a\omega)\partial_{\theta(t)}({}_sS^*_{lm}(\theta(t),\phi(t);a\omega)) \\
&\quad - \cos\theta(t)\sum_{l,m} {}_sS_{lm}(\theta,\phi;a\omega){}_sS^*_{lm}(\theta(t),\phi(t);a\omega), \qquad (3.41\text{a}) \\
\partial^2_\theta \delta\left(\theta-\theta(t)\right)\delta(\phi-\phi(t)) &= \partial^2_{\theta(t)}\delta\left(\theta-\theta(t)\right)\delta(\phi-\phi(t)) \\
&= \cos\theta(t)\sum_{l,m} {}_sS_{lm}(\theta,\phi;a\omega)\partial_{\theta(t)}({}_sS^*_{lm}(\theta(t),\phi(t);a\omega)) \\
&\quad + \sin\theta(t)\sum_{l,m} {}_sS_{lm}(\theta,\phi;a\omega)\partial^2_{\theta(t)}({}_sS^*_{lm}(\theta(t),\phi(t);a\omega)) \\
&\quad - \sin\theta(t)\sum_{l,m} {}_sS_{lm}(\theta,\phi;a\omega){}_sS^*_{lm}(\theta(t),\phi(t);a\omega) \\
&\quad + \cos\theta(t)\sum_{l,m} {}_sS_{lm}(\theta,\phi;a\omega)\partial_{\theta(t)}({}_sS^*_{lm}(\theta(t),\phi(t);a\omega)), \qquad (3.41\text{b}) \\
\partial_\phi \delta\left(\theta-\theta(t)\right)\delta(\phi-\phi(t)) &= im\,\sin\theta(t)\sum_{l,m} {}_sS_{lm}(\theta,\phi;a\omega){}_sS^*_{lm}(\theta(t),\phi(t);a\omega), \qquad (3.41\text{c}) \\
\partial^2_\phi \delta\left(\theta-\theta(t)\right)\delta(\phi-\phi(t)) &= (im)^2\,\sin\theta(t)\sum_{l,m} {}_sS_{lm}(\theta,\phi;a\omega){}_sS^*_{lm}(\theta(t),\phi(t);a\omega)\,. \qquad (3.41\text{d})
\end{aligned}$$

These expressions represent the most general form of the angular dependence of the Dirac delta functions and their derivatives in terms of spin-weighted spheroidal harmonics. It is also useful to rewrite the second derivative of ${}_sS_{lm}(\theta,\phi;a\omega)$ with respect to θ using Eq. (2.38).

This yields the most general representation of the Teukolsky source generated by a point mass on a generic trajectory, including motion outside the equatorial plane. However, from this point on, the discussion restricts to $\theta(t)=\pi/2$, corresponding to motion confined to the equatorial plane.

Before addressing the inhomogeneous solution of the radial Teukolsky equation, it is helpful to examine the structure of the source terms. The Fourier coefficients of the Teukolsky source are defined as:

$$ {}_sT_{lm\omega}(r) = \int_{-\infty}^{\infty} dt e^{i\omega t}{}_sT_{\ell m}(t,r)\,, \qquad (3.42)$$

where ${}_sT_{\ell m}(t,r)$ denotes the time-domain, mode-decomposed source, which takes the form:

$$ {}_sT_{lm}(t,r) = {}_sT^{(0)}_{lm}(t)\delta(r-r(t)) + {}_sT^{(1)}_{lm}(t)\delta'(r-r(t)) + {}_sT^{(2)}_{lm}(t)\delta''(r-r(t))\,, \qquad (3.43)$$

with the coefficients ${}_sT^{(i)}_{lm}(t)$ encoding information about the particle's geodesic motion. Furthermore, it will be essential for further discussions to remember that each coefficient has the following property:

$$ {}_sT^{(i)}_{lm}(t) = {}_sT^{(i)}_{lm}(t;a\omega)e^{-im\phi(t)}\,, \qquad (3.44)$$

where the dependence of $a\omega$ has been made explicit to remind that inside these coefficients are contained ${}_sS^*_{lm}(\theta(t),\phi(t);a\omega)$ and its derivatives with respect to $\theta(t)$, which should be then fixed to $\pi/2$ in the equatorial plane. Besides the dependence from the angular motion has been factored out by using properties of the harmonics and, from the previous discussion on geodesic solutions, it is clear that the source exhibits periodic behavior for circular and eccentric motion, while for unbound trajectories the presence of inverse trigonometric or hyperbolic functions indicates non-periodic behavior.

This property of the source will be central for the construction of the inhomogeneous solutions.

3.3 Inhomogeneous solutions of the radial equation

The general expression for the spin-s inhomogeneous solution of Teukolsky equation is

$$ {}_s\psi_{\ell m}(x^\mu) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\, {}_s\psi_{\ell m\omega}(r)\, {}_sS_{lm}(\theta,\phi;a\omega)\,, \tag{3.45} $$

where ${}_s\psi_{\ell m\omega}(r)$ can be written in the basis of the homogeneous solutions *in* and *up* of Eqs. (2.76)-(2.84) as

$$ {}_s\psi_{\ell m\omega}(r) = {}_sC^{\rm in}_{lm\omega}(r)\, {}_sR^{\rm in}_{lm\omega}(r) + {}_sC^{\rm up}_{lm\omega}(r)\, {}_sR^{\rm up}_{lm\omega}(r)\,, \tag{3.46} $$

where the two coefficients ${}_sC^{\rm in/up}_{lm\omega}(r)$ are defined through variation of parameters as

$$ {}_sC^{\rm in}_{lm\omega}(r) = \int_r^\infty dr'\, \frac{{}_sR^{\rm up}_{lm\omega}(r')}{W(r')\Delta}\, {}_sT_{lm\omega}(r')\,, \tag{3.47a} $$

$$ {}_sC^{\rm up}_{lm\omega}(r) = \int_{r_+}^r dr'\, \frac{{}_sR^{\rm in}_{lm\omega}(r')}{W(r')\Delta}\, {}_sT_{lm\omega}(r')\,, \tag{3.47b} $$

where $W(r')$ is the Wronskian defined from the two homogeneous solutions as

$$ W(r) = {}_sR^{\rm in}_{lm\omega}(r)\partial_r\left[{}_sR^{\rm up}_{lm\omega}(r)\right] - {}_sR^{\rm up}_{lm\omega}(r)\partial_r\left[{}_sR^{\rm in}_{lm\omega}(r)\right]\,. \tag{3.48} $$

Another, completely equivalent formulation for constructing the inhomogeneous solution is from the retarded Green functions

$$ {}_sG_{lm\omega}(r,r') = \frac{1}{\Delta^{s+1}W_{lm\omega}}\left\{ {}_sR^{\rm in}_{lm\omega}(r)\, {}_sR^{\rm up}_{lm\omega}(r')H(r'-r) + {}_sR^{\rm in}_{lm\omega}(r')\, {}_sR^{\rm up}_{lm\omega}(r)H(r-r')\right\}\,, \tag{3.49} $$

where $H(\cdot)$ is the Heaviside theta function and, the *invariant Wronskian* has been defined as $W_{lm\omega} := \Delta^{s+1}W(r) = \text{const}$. Hence Eq. (3.45) can be rewritten as

$$ {}_s\psi_{\ell m}(x^\mu) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t}\int dr'\Delta^s\, {}_sG_{\ell m\omega}(r,r')\, {}_sT_{lm\omega}(r')S_{lm}(\theta,\phi;a\omega)\,. \tag{3.50} $$

The radial integral is straightforward in both cases because of the delta functions contained in the source term, besides the Heaviside theta functions in the Green function can be used to generate the coefficients of Eqs. (3.47).

PN expansion revisited

In the previous Chapter, the homogeneous solutions were computed by asserting that, through an appropriate rescaling of the PN-weight $\eta = 1/c$ of the variables (r, ω), it was possible to obtain PN-expanded homogeneous solutions. However, this scaling was only justified through dimensional analysis.

The idea is that the homogeneous solution is solved by identifying two "zones": the *near-zone* next to the horizon and the *far-zone* when $GM/c^2 r \ll 1$, where the in and up solutions have been respectively constructed. The far-zone is also called the weak field zone or PM zone. Upon introducing a particle, it becomes useful to identify an additional zone located, in principle, anywhere within $r \in (2M, +\infty)$.

However, in order to perform analytical calculations, it is necessary to restrict the analysis to particles situated in the so-called *PN-zone*, where their orbital velocity v is small, i.e., $v/c \ll 1$. It is important to emphasize that the PN-zone is a region where both the in and up solutions are valid.

For a particle in a circular orbit around a black hole with $M \gg m$ in the PN-zone, v is related to the orbital radius by $v^2 = M/r_0$, according to the virial theorem. This relation implies that, if v scales as η, then the PN scaling can be associated either with $M \to M\eta^2$ or with $r_0 \to r_0\eta^{-2}$. Moreover, for circular orbits, $r \sim r_0$ and $\omega \sim m\Omega_\phi$, where $\Omega_\phi = (M/r_0^3)^{1/2}$, indicating that if M rescales with η^2, then $\omega \to \omega\eta$, while if r_0 carries the PN-weight, then $\omega \to \omega\eta^3$. This justifies the scaling adopted in the previous Chapter.

For quasi-circular orbits, where an expansion in small eccentricity is valid, the same reasoning applies. However, for hyperbolic encounters, this approach cannot be directly used for the PM zone that now also contains the particle. This issue affects *only* hyperbolic motion, where the radial trajectory is bounded only from below, i.e., $r(t) \in [b, +\infty)$, with b being the impact parameter or distance of closest approach. In contrast, for bound motion, the radial coordinate remains confined between the periastron and apastron.

A possible strategy to overcome this difficulty in scattering scenarios is to adopt a PM expansion by identifying the impact parameter b as the PM-weight. The idea is that, in the weak-field approximation, $GM/b \ll 1$, and thus a small-G expansion corresponds to a large-b expansion. Accordingly, the adopted rescalings for both worldline coordinates and field points are

$$\bar{t} = \frac{vt}{b}\,, \qquad \bar{r} = \frac{r}{b}\,, \qquad \bar{r}_p(\bar{t}) = \frac{r_p(t(\bar{t}))}{b}\,, \tag{3.51}$$

Along with the time rescaling, it is natural to define a rescaled Fourier frequency $\bar{\omega} = b\omega/v$ that keeps both ωt and ωr of order $O(1)$.

It is important to stress that this scaling is valid throughout the region $[b, +\infty)$. Since for unbound orbits there is no virial theorem for connecting the orbital velocity and the distance between the black hole and the particle, the large-b expansion does not impose a constraint on the particle's velocity. However, subsequently imposing the condition $v/c \ll 1$ simplifies the computation considerably and effectively localizes the distance of closest approach within the PN zone. This enables the combination of the PM and PN expansions, yielding homogeneous solutions expanded in both large-b and small-v regimes. Nevertheless, the double PM–PN expansion will be a crucial ingredient for completing the calculation in scattering.

Interestingly, it is clear that, when the virial theorem is added and hence the particle is again on a bound trajectory, then the two expansions collapses and large-b implies small-v. Notably the distance of minimum approach coincide with the periastron for eccentric motion or with the orbital radius for circular orbits. Hence, the proposed rescalings can provide a unified framework for treating all possible equatorial orbits in both Schwarzschild and Kerr backgrounds.

3.3.1 Source against homogeneous solutions

By substituting the expression of the frequency domain source term in Eq. (3.50) one gets

$$ {}_s\psi_{\ell m}(x^\mu) = \int \frac{d\omega}{2\pi}\, e^{-i\omega t} \int dr' \Delta^s\, {}_sG_{\ell m\omega}(r,r') \int_{-\infty}^{\infty} dt' e^{i\omega t'}\, {}_sT_{\ell m}(t',r')\, {}_sS_{lm}(\theta,\phi;a\omega)\,, \tag{3.52} $$

which provides the most generic expression for the inhomogeneous solution of Teukolsky equation for generic orbit for each (l,m)-mode. The triple integral must be handed very carefully depending on: i) the type of orbit and ii) if the background is Kerr or Schwarzshild. In the following, a general procedure will be presented for treating *at the same time* both bound and unbound orbits on the equator. On the other side, if the primary is spinning, the structure of the ${}_sS_{lm}(\theta,\phi;a\omega)$ must be taken into account. However, by leveraging the expansion presented in Eq. (2.41) it is possible to remove the ω-dependence from the ${}_sS_{lm}(\theta,\phi;a\omega)$ and produce polynomials in $a\omega$ where the coefficients are simple ${}_sY_{lm}(\theta,\phi)$. This does not affect only the spin-weighted spheroidal harmonics but also the time domain source, which has the property depicted in Eq. (3.44). This means that *there are no conceptual difficulties in generalizing the following computation to perturbations of a Kerr black hole* at this level. For this reason, the following discussion will be restricted to the Schwarzschild case without loss of generality, furthermore, by renaming

$$ {}_s\tilde{G}_{\ell m\omega}(r,r') = \Delta^s\, {}_sG_{\ell m\omega}(r,r')\, {}_sS_{lm}(\theta,\phi)\,, \tag{3.53} $$

it will be possible to lighten the notation.

Preliminaries

In order to devise a strategy for handling a generic orbit, it is first necessary to examine the structure of the two fundamental ingredients appearing in Eq. (3.52), namely the Green function and the source term. By employing the rescaled variables and the expanded form of the homogeneous solutions in Schwarzschild spacetime, the ω-dependence can be factored out as

$$ {}_s\tilde{G}_{\ell m\omega}(r,r') = \frac{1}{b} \sum_{j,k=0} v^j \bar{\omega}^j \log^k(-i\bar{\omega})\, {}_s\mathcal{G}^{(j,k)}_{\ell m}(\bar{r},\bar{r}')\,, \tag{3.54} $$

where the coefficients ${}_s\mathcal{G}^{(j,k)}_{\ell m}(\bar{r},\bar{r}')$ are dimensionless. It is convenient to split also the Green function as

$$ \begin{aligned} {}_s\tilde{G}_{\ell m\omega}(r,r') &= {}_s\tilde{G}^{\rm L}_{\ell m\omega}(r,r') + {}_s\tilde{G}^{\rm NL}_{\ell m\omega}(\bar{r},\bar{r}') \\ &= \frac{1}{b}\sum_j v^j \bar{\omega}^j_s \mathcal{G}^{\rm L}_{\ell m;j}(\bar{r},\bar{r}') + \frac{1}{b}\sum_{j=0}\sum_{k=1} v^j\bar{\omega}^j \log^k(-i\bar{\omega})\, {}_s\mathcal{G}^{\rm NL}_{\ell m;jk}(\bar{r},\bar{r}')\,, \end{aligned} \tag{3.55} $$

where the superscript "(N)L" denotes (*non-*)*local*. This terminology follows the standard PN literature; however, its meaning is not immediately apparent at this stage. The distinction becomes clear only after computing the convolution of the source with the Green function. It is therefore convenient to treat the two sectors separately.

Local terms

From the local sector of the Green function, one can compute the local solution defined as

$$
\begin{aligned}
{}_s\psi^{\rm L}_{\ell m}(x^\mu) &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{-i\omega t}\int dr'\,{}_s\tilde{G}^{\rm L}_{\ell m\omega}(r,r')\int_{-\infty}^{\infty}dt'\,e^{i\omega t'}\,{}_sT_{\ell m}(t',r') \\
&= \sum_j\frac{\upsilon^j}{2\pi}\int_{-\infty}^{\infty}d\bar\omega\,e^{-i\bar\omega\bar t}\int d\bar r'\,\bar\omega^j\,{}_s\mathcal{G}^{\rm L}_{\ell m;j}(\bar r,\bar r')\int_{-\infty}^{\infty}d\bar t'\,e^{i\bar\omega\bar t'}\,{}_sT_{\ell m}(\bar t',\bar r')\,,
\end{aligned}
\tag{3.56}
$$

where the source term has the structure depicted in Eqs. (3.43) and (3.44) that can be used to perform trivially the radial integral as follows

$$
\begin{aligned}
{}_s\psi^{\rm L}_{\ell m}(x^\mu) &= \sum_j\frac{\upsilon^j}{2\pi b}\int_{-\infty}^{\infty}d\bar\omega\,\bar\omega^j\,e^{-i\bar\omega\bar t}\int_{-\infty}^{\infty}d\bar t'\,e^{i\bar\omega\bar t'}\int d\bar r'\,{}_s\mathcal{G}^{\rm L}_{\ell m;j}(\bar r,\bar r')\Big\{{}_sT^{(0)}_{lm}(\bar t')\delta(\bar r'-\bar r(\bar t')) \\
&\qquad + \frac{1}{b}\,{}_sT^{(1)}_{lm}(\bar t')\delta'(\bar r'-\bar r(\bar t')) + \frac{1}{b^2}\,{}_sT^{(2)}_{lm}(\bar t')\delta''(\bar r'-\bar r(\bar t'))\Big\} \\
&= \sum_j\frac{\upsilon^j}{2\pi b}\int_{-\infty}^{\infty}d\bar\omega\,\bar\omega^j e^{-i\bar\omega\bar t}\int_{-\infty}^{\infty}d\bar t'\,e^{i\bar\omega\bar t'}\,{}_s\hat{\mathcal{G}}^{\rm L}_{\ell m;j}(\bar r,\bar r_p(\bar t'))\,,
\end{aligned}
\tag{3.57}
$$

where

$$
\begin{aligned}
{}_s\hat{\mathcal{G}}^{\rm L}_{\ell m;j}(\bar r,\bar r_p(\bar t')) &= {}_s\mathcal{G}^{\rm L}_{\ell m;j}(\bar r,\bar r_p(\bar t'))\,{}_sT^{(0)}_{lm}(\bar t') - \frac{1}{b}\frac{\partial}{\partial\bar r'}\left[{}_s\mathcal{G}^{\rm L}_{\ell m;j}(\bar r,\bar r')\,{}_sT^{(1)}_{lm}(\bar t')\right]\Bigg|_{\bar r'=\bar r_p(\bar t')} \\
&\quad + \frac{1}{b^2}\frac{\partial^2}{\partial\bar r'^2}\left[{}_s\mathcal{G}^{\rm L}_{\ell m;j}(\bar r,\bar r')\,{}_sT^{(2)}_{lm}(\bar t')\right]\Bigg|_{\bar r'=\bar r_p(\bar t')}\,,
\end{aligned}
\tag{3.58}
$$

and the subscript $_p$ to $\bar r_p(\bar t')$ has been added to distinguish between $\bar r$, i.e. field points, and $\bar r_p(\bar t')$, i.e. points on the trajectory or *particle points*.

The remaining two integrals are now trivial and by using the integral representations of derivatives of Dirac delta functions one gets

$$
\begin{aligned}
{}_s\psi^{\rm L}_{\ell m}(x^\mu) &= \sum_j\frac{\upsilon^j}{2\pi b}\int_{-\infty}^{\infty}d\bar t'\,{}_s\hat{\mathcal{G}}^{\rm L}_{\ell m;j}(\bar r,\bar r_p(\bar t'))\int_{-\infty}^{\infty}d\bar\omega\,e^{i\bar\omega(\bar t'-\bar t)}\bar\omega^j \\
&= \sum_j\frac{\upsilon^j}{2\pi b}\int_{-\infty}^{\infty}d\bar t'\,{}_s\hat{\mathcal{G}}^{\rm L}_{\ell m;j}(\bar r,\bar r_p(\bar t'))\frac{2\pi}{i^j}\delta^{(j)}(\bar t'-\bar t) \\
&= \sum_j\frac{i^j\upsilon^j}{b}\frac{\partial^{(j)}}{\partial\bar t'^{(j)}}\left[{}_s\hat{\mathcal{G}}^{\rm L}_{\ell m;j}(\bar r,\bar r_p(\bar t'))\right]\Bigg|_{\bar t'=\bar t}\,,
\end{aligned}
\tag{3.59}
$$

where all these derivatives acts on both the Green function and on the source. It is convenient to restore the spherical harmonics contained inside the Green function and write the final expression for the local field as

$$
{}_s\psi^{\rm L}_{\ell m}(x^\mu) = \sum_j\frac{i^j\upsilon^j}{b}\frac{\partial^{(j)}}{\partial\bar t'^{(j)}}\left[{}_s\hat{\mathcal{G}}^{\rm L}_{\ell m;j}(\bar r,\bar r_p(\bar t'))e^{-im\phi_p(t)}\right]\Bigg|_{\bar t'=\bar t}\,{}_sY_{lm}(\theta,\phi)\,.
\tag{3.60}
$$

Non-local terms

The non-local field is

$$
{}_s\psi^{\rm NL}_{\ell m}(x^\mu) = \sum_{j,k=1} \frac{\upsilon^j}{2\pi} \int_{-\infty}^{\infty} d\bar\omega\, e^{-i\bar\omega \bar t} \int d\bar r' \log^k(-i\bar\omega)\bar\omega^j\, {}_s\mathcal{G}^{\rm NL}_{\ell m;jk}(\bar r,\bar r') \int_{-\infty}^{\infty} d\bar t' e^{i\bar\omega \bar t'}\, {}_sT_{\ell m}(\bar t',\bar r')\,, \tag{3.61}
$$

where both j and k starts from 1. The non-local sector is considerably more delicate, and a suitable regularization procedure is required to extract its finite part. This is because, although the limit $\bar\omega \to 0$ is well-defined, the integrands diverge when $\bar\omega \to \pm\infty$. The origin of this issue lies in the fact that the homogeneous solutions are constructed as a small-frequency expansion, whereas the direct and inverse Fourier transforms are defined over the entire frequency domain. This necessitates the introduction of a *regularized* non-local field:

$$
\begin{aligned}
{}_s\psi^{\rm NL,\epsilon}_{\ell m}(x^\mu) = \sum_{j,k=1} \frac{\upsilon^j}{2\pi} \int_{-\infty}^{\infty} d\bar\omega\, e^{-\epsilon|\bar\omega|}\, e^{-i\bar\omega \bar t} \int d\bar r' \log^k(-i\bar\omega)\bar\omega^j\, {}_s\mathcal{G}^{\rm NL}_{\ell m;jk}(\bar r,\bar r') \\
\times \int_{-\infty}^{\infty} d\bar t' e^{i\bar\omega \bar t'}\, {}_sT_{\ell m}(\bar t',\bar r')\,,
\end{aligned} \tag{3.62}
$$

which leads to the non-local field by performing the limit for small positive ϵ as

$$
{}_s\psi^{\rm NL}_{\ell m}(x^\mu) = \lim_{\epsilon\to 0^+} {}_s\psi^{\rm NL,\epsilon}_{\ell m}(x^\mu)\,. \tag{3.63}
$$

This definition provides a robust consistency check of the calculation: once ${}_s\psi^{\rm NL,\epsilon}\ell m(x^\mu)$ is obtained, the limit yields a finite result that is independent of ϵ. This feature has been examined in [100], where it has been shown that standard Hadamard regularization reproduces the correct finite part, but the final result depends explicitly on the scale ϵ. It is worth noting that this is not the only way extract the finite part of this integral, but the following procedure has proven quite advantageous because it provides quite simple results that could be checked immediately via similar computations in distribution theory. Looking at the distributional behaviour of these integrals provides a strong analytical benchmark for this calculation and, in App. B it is presented a calculation with this formalism.

As for the local sector, the radial integral can be straightforwardly performed:

$$
{}_s\psi^{\rm NL,\epsilon}_{\ell m}(x^\mu) = \sum_{j,k=1} \frac{\upsilon^j}{2\pi b} \int_{-\infty}^{\infty} d\bar\omega\, e^{-\epsilon|\bar\omega|} \log^k(-i\bar\omega)\bar\omega^j e^{-i\bar\omega \bar t} \int_{-\infty}^{\infty} d\bar t' e^{i\bar\omega \bar t'}\, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;jk}(\bar r,\bar r_p(\bar t'))\,, \tag{3.64}
$$

where ${}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;jk}(\bar r,\bar r_p(\bar t'))$ has been introduced accordingly with Eq. (3.58). Since the integrands are well-behaving, the order of integration can be interchanged, giving

$$
{}_s\psi^{\rm NL,\epsilon}_{\ell m}(x^\mu) = \sum_{j,k=1} \frac{\upsilon^j}{2\pi b} \int_{-\infty}^{\infty} d\bar t'\, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;jk}(\bar r,\bar r_p(\bar t')) \int_{-\infty}^{\infty} d\bar\omega\, e^{-\epsilon|\bar\omega|} \log^k(-i\bar\omega)\bar\omega^j e^{i\bar\omega(\bar t'-\bar t)}\,. \tag{3.65}
$$

At this stage the ω-integral can be rewritten by using the identity

$$
\lim_{x\to 0^+} \frac{\partial^k(-i\omega)^x}{\partial x^k} := \log^k(-i\omega)\,, \tag{3.66}
$$

which could be inserted in the integral and, allowing the derivative with respect to x to be taken outside:

$$ {}_s\psi_{\ell m}^{\mathrm{NL},\epsilon}(x^\mu) = \lim_{x\to 0^+} \sum_{j,k=1} \frac{v^j}{2\pi b} \int_{-\infty}^{\infty} d\bar{t}'\, {}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;jk}(\bar{r},\bar{r}_p(\bar{t}'))\, \partial_x^{(k)} \int_{-\infty}^{\infty} d\bar{\omega}\, e^{-\epsilon|\bar{\omega}|}(-i\omega)^x \bar{\omega}^j e^{i\bar{\omega}(\bar{t}'-\bar{t})}\,, \tag{3.67} $$

where the frequency integral now depends on the three parameters (ϵ, j, x), with

$$ (\epsilon, x) \in \mathbb{R}^+\,, \quad j \in \mathbb{N}\,, \tag{3.68} $$

and it becomes a function of the time interval $\bar{t}' - \bar{t}$, which can be immediately computed as

$$ \begin{aligned} &I_{\mathrm{NL}}(\epsilon, j, x; \bar{t}' - \bar{t}) = \int_{-\infty}^{\infty} d\bar{\omega}\, e^{i\bar{\omega}(\bar{t}'-\bar{t})} e^{-\epsilon|\bar{\omega}|} \omega^j (-i\bar{\omega})^x \\ &= \frac{e^{-\frac{1}{2}i\pi x}\,\Gamma(1+j+x)}{\left((\bar{t}'-\bar{t})^2+\epsilon^2\right)^{1+j+x}} \left(e^{i\pi(j+x)}\left(\epsilon - i\left(\bar{t}'-\bar{t}\right)\right)^{1+j+x} + \left(\epsilon + i\left(\bar{t}'-\bar{t}\right)\right)^{1+j+x} \right), \end{aligned} \tag{3.69} $$

which can then be differentiated k times with respect to x:

$$ \begin{aligned} &\partial_x^{(k)} I_{\mathrm{NL}}(\epsilon, j, x; s) = \sum_{a,b=0}^{k} \frac{k!\, i^x (-1)^a \left(s^2+\epsilon^2\right)^{-1-j-x}}{a!\,b!\,(k-a-b)!} \frac{\partial^{(-a-b+k)}\Gamma(1+j+x)}{\partial x^{(-a-b+k)}} \log^a\left(s^2+\epsilon^2\right) \\ &\times \left\{ (-1)^j(-is+\epsilon)^{1+j+x}\left(\frac{i\pi}{2}+\log(-is+\epsilon)\right)^b + (-1)^x(is+\epsilon)^{1+j+x}\left(-\frac{i\pi}{2}+\log(is+\epsilon)\right)^b \right\}, \end{aligned} \tag{3.70} $$

where $k-a-b \geq 0$ and $s = \bar{t}' - \bar{t}$.

This result must be now injected in the full expression of the non-local field and integrated against ${}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;jk}(\bar{r},\bar{r}_p(\bar{t}'))$ in t'. In general, this computation is quite involved: a closed expression for arbitrary j can be obtained by induction, but it is considerably more difficult for general k. The derivation for $k=j=1$ is presented here, while Appendix A reports expressions up to $k=10$.

In this specific case, the non-local integral reads

$$ \begin{aligned} &I_{\mathrm{NL}}(\epsilon, 1, x; \bar{t}' - \bar{t}) = \int d\bar{\omega} e^{i\bar{\omega}(\bar{t}'-\bar{t})} e^{-\epsilon|\bar{\omega}|} \bar{\omega}(-i\bar{\omega})^x \\ &= \frac{\Gamma(2+x)}{((\bar{t}'-\bar{t})^2+\epsilon^2)^{2+x}} \left(-\, i^x(-i(\bar{t}'-\bar{t})+\epsilon)^{2+x} + (-i)^x(i(\bar{t}'-\bar{t})+\epsilon)^{2+x} \right), \end{aligned} \tag{3.71} $$

and, since both j and k are fixed, one can take the derivative with respect to x and then perform the limit $x \to 0^+$, obtaining

$$ \begin{aligned} &\lim_{x\to 0^+} \partial_x\, I_{\mathrm{NL}}(\epsilon, 1, x; \bar{t}' - \bar{t}) = \\ &= \frac{i\left(4(1-\gamma_E)\left(\bar{t}'-\bar{t}\right)\epsilon + \pi\left(\left(\bar{t}'-\bar{t}\right)^2-\epsilon^2\right)\right)}{\left(\left(\bar{t}'-\bar{t}\right)^2+\epsilon^2\right)^2} + \frac{\log\left(-i\left(\bar{t}'-\bar{t}\right)+\epsilon\right)}{\left(\bar{t}'-\bar{t}+i\epsilon\right)^2} - \frac{\log\left(i\left(\bar{t}'-\bar{t}\right)+\epsilon\right)}{\left(\bar{t}'-\bar{t}-i\epsilon\right)^2} \\ &= f_1^\epsilon(\bar{t}'-\bar{t}) + f_2^\epsilon(\bar{t}'-\bar{t}) + f_3^\epsilon(\bar{t}'-\bar{t})\,, \end{aligned} \tag{3.72} $$

so that the integral to be evaluated becomes

$$
\begin{aligned}
{}_s\psi^{\mathrm{NL},\epsilon}_{\ell m}(k=j=1;x^\mu) &= \frac{v}{2\pi b}\int_{-\infty}^{\infty} d\bar{t}'\; {}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\left[f_1^\epsilon(\bar{t}'-\bar{t})+f_2^\epsilon(\bar{t}'-\bar{t})+f_3^\epsilon(\bar{t}'-\bar{t})\right] \\
&= I_1^\epsilon + I_2^\epsilon + I_3^\epsilon\,, \qquad (3.73)
\end{aligned}
$$

and it is convenient to evaluate I_1^ϵ first, followed by I_2^ϵ and I_3^ϵ together.

The first integral reads

$$
\begin{aligned}
I_1^\epsilon = i\frac{v}{b}\Bigg\{ &\int d\bar{t}'\,\frac{2(1-\gamma_E)\epsilon(\bar{t}'-\bar{t})}{\pi((\bar{t}'-\bar{t})^2+\epsilon^2)^2}\,{}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}')) \\
&+\int d\bar{t}'\,\frac{((\bar{t}'-\bar{t})^2-\epsilon^2)}{2((\bar{t}'-\bar{t})^2+\epsilon^2)^2}\,{}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\Bigg\}\,, \qquad (3.74)
\end{aligned}
$$

where the first step consists in integrating by parts using

$$
\partial_{\bar{t}'}\frac{1}{(\bar{t}'-\bar{t})^2+\epsilon^2} = -2\frac{\bar{t}'-\bar{t}}{((\bar{t}'-\bar{t})^2+\epsilon^2)^2}\,. \qquad (3.75)
$$

The integrals can thus be rewritten as

$$
\begin{aligned}
I_1^\epsilon = -i\frac{v}{b}\Bigg\{ &\int d\bar{t}'\,\frac{(1-\gamma_E)\epsilon}{\pi}\,{}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\partial_{\bar{t}'}\frac{1}{(\bar{t}'-\bar{t})^2+\epsilon^2} \\
&+\int d\bar{t}'\,\frac{(\bar{t}'-\bar{t})^2-\epsilon^2}{4(\bar{t}'-\bar{t})}\,{}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\partial_{\bar{t}'}\frac{1}{(\bar{t}'-\bar{t})^2+\epsilon^2}\Bigg\}\,. \qquad (3.76)
\end{aligned}
$$

Integrating by parts gives

$$
\begin{aligned}
I_1^\epsilon = -i\frac{v}{b}\Bigg\{ &\frac{(1-\gamma_E)\epsilon}{\pi}\,{}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\frac{1}{(\bar{t}'-\bar{t})^2+\epsilon^2}\Bigg|_{-\infty}^{+\infty} \\
&-\int d\bar{t}'\,\frac{(1-\gamma_E)\epsilon}{\pi}\,\frac{\partial_{\bar{t}'}\,{}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))}{(\bar{t}'-\bar{t})^2+\epsilon^2} \\
&+\frac{((\bar{t}'-\bar{t})^2-\epsilon^2)}{4(\bar{t}'-\bar{t})}\,{}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\frac{1}{(\bar{t}'-\bar{t})^2+\epsilon^2}\Bigg|_{-\infty}^{+\infty} \\
&-\int d\bar{t}'\partial_{\bar{t}'}\left[\frac{((\bar{t}'-\bar{t})^2-\epsilon^2)}{4(\bar{t}'-\bar{t})}\,{}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\right]\frac{1}{(\bar{t}'-\bar{t})^2+\epsilon^2}\Bigg\}\,, \qquad (3.77)
\end{aligned}
$$

where the boundary terms vanishes for any ϵ since the denominator dominates for large t', leading to zero contribution. No assumptions are made yet about the convergence of the Green function coefficients for large t', which might be critical when $\bar{t}'\to\pm\infty$ as $\bar{r}_p(\bar{t}')\to+\infty$. However, the Green function is well behaved at infinity, vanishing smoothly as $\bar{t}'\to\pm\infty$.

The remaining integrals are then

$$
I_1^\epsilon = i\frac{v}{b}\Bigg\{\int d\bar{t}'\,\frac{(1-\gamma_E)\epsilon}{\pi}\,\frac{\partial_{\bar{t}'}\,{}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))}{(\bar{t}'-\bar{t})^2+\epsilon^2}
$$

$$+ \int d\bar{t}' \partial_{\bar{t}'} \left[\frac{((\bar{t}' - \bar{t})^2 - \epsilon^2)}{4(\bar{t}' - \bar{t})} \, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}')) \right] \frac{1}{(\bar{t}' - \bar{t})^2 + \epsilon^2} \Bigg\}$$

$$= i\frac{v}{b} \Bigg\{ \int dx \frac{(1-\gamma_E)}{\pi} \frac{\partial_{\bar{t}'} \, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}'))|_{\bar{t}'=\epsilon x + \bar{t}}}{x^2+1}$$

$$+ \int d\bar{t}' \frac{{}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}'))}{2(\bar{t}' - \bar{t})^2 + \epsilon^2}$$

$$- \int d\bar{t}' \frac{(\bar{t}' - \bar{t})^2 - \epsilon^2}{4(\bar{t}' - \bar{t})^2} \frac{{}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}'))}{(t' - t)^2 + \epsilon^2}$$

$$+ \int d\bar{t}' \frac{(\bar{t}' - \bar{t})^2 - \epsilon^2}{4(\bar{t}' - \bar{t})} \frac{\partial_{\bar{t}'} \, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}'))}{(\bar{t}' - \bar{t})^2 + \epsilon^2} \Bigg\}, \tag{3.78}$$

where the first integral follows from the rescaling $\bar{t}' = \epsilon x + \bar{t}$. Taking the limit $\epsilon \to 0^+$ yields

$$I_1 = i\frac{v}{b} \Bigg\{ \frac{1-\gamma_E}{\pi} \partial_{\bar{t}} \, {}_s\tilde{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t})) \int dx \, \frac{1}{x^2+1}$$

$$+ \int d\bar{t}' \frac{{}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}'))}{2(\bar{t}' - \bar{t})^2} - \int d\bar{t}' \frac{{}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}'))}{4(t' - t)^2}$$

$$+ \int d\bar{t}' \frac{\partial_{\bar{t}'} \, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}'))}{4(\bar{t}' - \bar{t})} \Bigg\}, \tag{3.79}$$

where the first integral can be trivially computed, but the others are clearly ill-defined when $\bar{t}' = \bar{t}$, hence they should be intended as the *Principal Value* of the ill-defined integral, which is defined as

$$\text{p.v.} \int d\bar{t}' \frac{{}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}'))}{2(\bar{t}' - \bar{t})^2} :=$$

$$\lim_{\alpha \to 0^+} \left\{ \int_{-\infty}^{\bar{t}-\alpha} d\bar{t}' \frac{{}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}'))}{2(\bar{t}' - \bar{t})^2} + \int_{\bar{t}+\alpha}^{+\infty} d\bar{t}' \frac{{}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}'))}{2(\bar{t}' - \bar{t})^2} \right\}. \tag{3.80}$$

This gives

$$I_1 = i\frac{v}{b} \Bigg\{ (1-\gamma_E) \partial_{\bar{t}} \, {}_s\tilde{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}))$$

$$+ \text{p.v.} \int d\bar{t}' \frac{{}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}'))}{4(\bar{t}' - \bar{t})^2} + \text{p.v.} \int d\bar{t}' \frac{\partial_{\bar{t}'} \, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}'))}{4(\bar{t}' - \bar{t})} \Bigg\}$$

$$= i\frac{v}{b} \Bigg\{ (1-\gamma_E) \partial_{\bar{t}} \, {}_s\tilde{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t})) + \text{p.v.} \int dt' \frac{\partial_{\bar{t}'} \, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}'))}{2(\bar{t}' - \bar{t})} \Bigg\}, \tag{3.81}$$

where the two p.v. integrals have been combined into a single one involving only the derivative of the Green function coefficient. Formally, one could revise this procedure in order to avoid the appearance of ill-defined integrals. It is sufficient to split the integrals in Eq. (3.78) into $(-\infty, \bar{t}-\alpha)$ and $(\bar{t}+\alpha, +\infty)$, with the limit $\alpha \to 0^+$ taken after integration. This ensures well-defined expressions, allows the ϵ-limit to be performed inside the integrals, and automatically produces p.v. terms without divergences.

By using similar techniques, one can immediately handle the two remaining integrals together, namely $I_2^\epsilon + I_3^\epsilon$. It is more convenient to treat them simultaneously because they share the same functional form and, as will be shown, it is straightforward to combine them into a compact expression.

Hence, the integral that needs to be evaluated is

$$I_2^\epsilon + I_3^\epsilon = \frac{v}{2\pi b}\left\{ \int d\bar{t}' \left[\frac{\log\left(\epsilon - i\left(\bar{t}'-\bar{t}\right)\right)}{\left(\bar{t}'-\bar{t}+i\epsilon\right)^2} - \frac{\log\left(\epsilon + i\left(\bar{t}'-\bar{t}\right)\right)}{\left(\bar{t}'-\bar{t}-i\epsilon\right)^2} \right] {}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}')) \right\}, \quad (3.82)$$

where the divergence at $\bar{t}' = \bar{t}$ cannot be cured by simply evaluating the principal value of the integral. This leads to the identification of three domains (rather than the two automatically identified through the principal value), i.e.

$$\begin{aligned} I_2^\epsilon + I_3^\epsilon = \frac{v}{2\pi b} \lim_{\alpha\to 0^+} \Bigg\{ & \int_{-\infty}^{\bar{t}-\alpha} d\bar{t}' \left[\frac{\log\left(\epsilon - i\left(\bar{t}'-\bar{t}\right)\right)}{\left(\bar{t}'-\bar{t}+i\epsilon\right)^2} - \frac{\log\left(\epsilon + i\left(\bar{t}'-\bar{t}\right)\right)}{\left(\bar{t}'-\bar{t}-i\epsilon\right)^2} \right] {}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}')) \\ & + \int_{t+\alpha}^{\infty} d\bar{t}' \left[\frac{\log\left(\epsilon - i\left(\bar{t}'-\bar{t}\right)\right)}{\left(\bar{t}'-\bar{t}+i\epsilon\right)^2} - \frac{\log\left(\epsilon + i\left(\bar{t}'-\bar{t}\right)\right)}{\left(\bar{t}'-\bar{t}-i\epsilon\right)^2} \right] {}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}')) \\ & + \int_{\bar{t}-\alpha}^{\bar{t}+\alpha} d\bar{t}' \left[\frac{\log\left(\epsilon - i\left(\bar{t}'-\bar{t}\right)\right)}{\left(\bar{t}'-\bar{t}+i\epsilon\right)^2} - \frac{\log\left(\epsilon + i\left(\bar{t}'-\bar{t}\right)\right)}{\left(\bar{t}'-\bar{t}-i\epsilon\right)^2} \right] {}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}')) \Bigg\}, \end{aligned} \quad (3.83)$$

where the limit for $\alpha \to 0^+$ must be taken after the integration. By recalling that also a limit in ϵ is required for computing the finite part, one can straightforwardly perform it for the first two integrals and keep ϵ only on the last one, where the integral domain contains the singularity. Hence, one can write immediately

$$\begin{aligned} I_2^\epsilon + I_3^\epsilon = \frac{v}{2\pi b} \lim_{\alpha\to 0^+} \Bigg\{ & \int_{-\infty}^{\bar{t}-\alpha} d\bar{t}' \frac{i\pi}{\bar{t}'-\bar{t}} \partial_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}')) - \int_{\bar{t}+\alpha}^{\infty} d\bar{t}' \frac{i\pi}{\bar{t}'-\bar{t}} \partial_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}')) \\ & + \lim_{\epsilon\to 0^+} \int_{\bar{t}-\alpha}^{\bar{t}+\alpha} d\bar{t}' \left[\frac{\log\left(\epsilon - i\left(\bar{t}'-\bar{t}\right)\right)}{\left(\bar{t}'-\bar{t}+i\epsilon\right)^2} - \frac{\log\left(\epsilon + i\left(\bar{t}'-\bar{t}\right)\right)}{\left(\bar{t}'-\bar{t}-i\epsilon\right)^2} \right] {}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}')) \Bigg\}, \end{aligned} \quad (3.84)$$

where the first two lines are obtained by using the known sign of $t' - t$ in the different integration domains and applying integration by parts to shift the derivative from the denominator to the Green function coefficients.

By splitting the last term into two integrals and integrating by parts, one obtains

$$\begin{aligned} I_2^\epsilon + I_3^\epsilon = \frac{v}{2\pi b} \lim_{\alpha\to 0^+} \Bigg\{ & \int_{-\infty}^{\bar{t}-\alpha} d\bar{t}' \frac{i\pi}{\bar{t}'-\bar{t}} \partial_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}')) \\ & - \int_{\bar{t}+\alpha}^{\infty} d\bar{t}' \frac{i\pi}{\bar{t}'-\bar{t}} \partial_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\mathrm{NL}}_{\ell m;11}(\bar{r}, \bar{r}_p(\bar{t}')) \end{aligned}$$

$$
\begin{aligned}
&- \lim_{\epsilon\to 0^+} \log\left(\epsilon - i\left(\bar{t}' - \bar{t}\right)\right)\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\frac{1}{(\bar{t}'-\bar{t}+i\epsilon)}\Bigg|_{\bar{t}-\alpha}^{\bar{t}+\alpha}\\
&+ \lim_{\epsilon\to 0^+}\int_{\bar{t}-\alpha}^{\bar{t}+\alpha} d\bar{t}'\frac{1}{(\bar{t}'-\bar{t}+i\epsilon)}\partial_{\bar{t}'}\left[\log\left(\epsilon - i\left(\bar{t}'-\bar{t}\right)\right)\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\right]\\
&+ \lim_{\epsilon\to 0^+} \log\left(\epsilon + i\left(\bar{t}' - \bar{t}\right)\right)\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\frac{1}{(\bar{t}'-\bar{t}-i\epsilon)}\Bigg|_{\bar{t}-\alpha}^{\bar{t}+\alpha}\\
&- \lim_{\epsilon\to 0^+}\int_{\bar{t}-\alpha}^{\bar{t}+\alpha} d\bar{t}'\frac{1}{(\bar{t}'-\bar{t}-i\epsilon)}\partial_{\bar{t}'}\left[\log\left(\epsilon + i\left(\bar{t}'-\bar{t}\right)\right)\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\right]\Bigg\}\,,
\end{aligned}
\tag{3.85}
$$

where the boundary terms give a rather simple answer

$$
\begin{aligned}
&- \lim_{\epsilon\to 0^+}\Bigg[\log\left(\epsilon - i\left(\bar{t}' - \bar{t}\right)\right)\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\frac{1}{(\bar{t}'-\bar{t}+i\epsilon)}\Bigg|_{\bar{t}-\alpha}^{\bar{t}+\alpha}\\
&\quad - \log\left(\epsilon + i\left(\bar{t}' - \bar{t}\right)\right)\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\frac{1}{(\bar{t}'-\bar{t}-i\epsilon)}\Bigg|_{\bar{t}-\alpha}^{\bar{t}+\alpha}\Bigg] = -2i\pi\partial_{\bar{t}'}\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\,,
\end{aligned}
\tag{3.86}
$$

which must be computed by first performing the limit in ϵ and then in α.

Expanding the derivatives leaves four integrals, which can be combined as

$$
\begin{aligned}
I_2^\epsilon + I_3^\epsilon = \frac{v}{2\pi b}\lim_{\alpha\to 0^+}\Bigg\{&\int_{-\infty}^{\bar{t}-\alpha} d\bar{t}'\frac{i\pi}{\bar{t}'-\bar{t}}\partial_{\bar{t}'}\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\\
&-\int_{\bar{t}+\alpha}^{\infty} d\bar{t}'\frac{i\pi}{\bar{t}'-\bar{t}}\partial_{\bar{t}'}\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}')) - 2i\pi\partial_{\bar{t}'}\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\\
&+\lim_{\epsilon\to 0^+}\int_{\bar{t}-\alpha}^{\bar{t}+\alpha} d\bar{t}'\left[\frac{{}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))}{(\bar{t}'-\bar{t}+i\epsilon)^2} - \frac{{}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))}{(\bar{t}'-\bar{t}-i\epsilon)^2}\right]\\
&+\lim_{\epsilon\to 0^+}\int_{\bar{t}-\alpha}^{\bar{t}+\alpha} d\bar{t}'\left[\frac{\log\left(\epsilon - i\left(\bar{t}'-\bar{t}\right)\right)}{(\bar{t}'-\bar{t}+i\epsilon)} - \frac{\log\left(\epsilon + i\left(\bar{t}'-\bar{t}\right)\right)}{(\bar{t}'-\bar{t}-i\epsilon)}\right]\partial_{\bar{t}'}\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\Bigg\}\,,
\end{aligned}
\tag{3.87}
$$

where the third line vanishes when the limit for small ϵ is perform and the last term can be treated by adding and removing the divergent term when $\bar{t}' = \bar{t}$, thus isolating the divergence outside the integral sign:

$$
\begin{aligned}
I_2^\epsilon + I_3^\epsilon = \frac{v}{2\pi b}\lim_{\alpha\to 0^+}\Bigg\{&\int_{-\infty}^{\bar{t}-\alpha} d\bar{t}'\frac{i\pi}{\bar{t}'-\bar{t}}\partial_{\bar{t}'}\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\\
&-\int_{\bar{t}+\alpha}^{\infty} d\bar{t}'\frac{i\pi}{\bar{t}'-\bar{t}}\partial_{\bar{t}'}\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}')) - 2i\pi\partial_{\bar{t}'}\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))
\end{aligned}
$$

$$
\begin{aligned}
&+\lim_{\epsilon\to 0^+}\int_{t-\alpha}^{t+\alpha} d\bar{t}'\left[\frac{\log\left(\epsilon - i\left(\bar{t}'-\bar{t}\right)\right)}{\left(\bar{t}'-\bar{t}+i\epsilon\right)} - \frac{\log\left(\epsilon + i\left(\bar{t}'-\bar{t}\right)\right)}{\left(\bar{t}'-\bar{t}-i\epsilon\right)}\right]\partial_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\\
&\pm\partial_{\bar{t}}\, {}_s\tilde{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}))\lim_{\epsilon\to 0^+}\int_{t-\alpha}^{t+\alpha} d\bar{t}'\left[\frac{\log\left(\epsilon - i\left(\bar{t}'-\bar{t}\right)\right)}{\left(\bar{t}'-\bar{t}+i\epsilon\right)} - \frac{\log\left(\epsilon + i\left(\bar{t}'-\bar{t}\right)\right)}{\left(\bar{t}'-\bar{t}-i\epsilon\right)}\right]\Bigg\}\\
&=\frac{v}{2\pi b}\lim_{\alpha\to 0^+}\Bigg\{\int_{-\infty}^{\bar{t}-\alpha} d\bar{t}'\frac{i\pi}{\bar{t}'-\bar{t}}\partial_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\\
&\qquad -\int_{\bar{t}+\alpha}^{\infty} d\bar{t}'\frac{i\pi}{\bar{t}'-\bar{t}}\partial_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\\
&\qquad -2i\pi\partial_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}')) - 2i\pi\partial_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\log\alpha\\
&+i\int_{-\alpha}^{+\alpha} dy\left[\frac{\log(-iy)}{y}-\frac{\log(iy)}{y}\right]\left(\partial_y\, {}_s\tilde{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}+y)) - \partial_{\bar{t}}\, {}_s\tilde{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}))\right)\Bigg\}\,,
\end{aligned}
\tag{3.88}
$$

where the last integral has been defined by using the auxiliary variable $y = \bar{t}' - \bar{t}$. To extract a result, one must explicitly expand the coefficients of the Green function, which in general allows one to show that this integral vanishes. The final, more compact, expression is then

$$
\begin{aligned}
I_2 + I_3 = \frac{iv}{b}\lim_{\alpha\to 0^+}\Bigg\{&\int_{-\infty}^{\bar{t}-\alpha} d\bar{t}'\,\frac{\partial_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))}{2(\bar{t}'-\bar{t})} - \int_{\bar{t}+\alpha}^{\infty} d\bar{t}'\,\frac{\partial_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))}{2(\bar{t}'-\bar{t})}\\
&-\partial_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}')) - \partial_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))\log\alpha\Bigg\}\,,
\end{aligned}
\tag{3.89}
$$

By combining together this result with the one obtained for I_1 one gets:

$$
\begin{aligned}
{}_s\psi^{\rm NL}_{\ell m}(k=j=1;x^\mu) = \frac{iv}{b}\lim_{\alpha\to 0^+}\Bigg\{&\int_{-\infty}^{\bar{t}-\alpha} d\bar{t}'\,\frac{\partial_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}'))}{\bar{t}'-\bar{t}}\\
&-\left[\gamma_E + \log\alpha\right]\partial_{\bar{t}'}\, {}_s\tilde{\mathcal{G}}^{\rm NL}_{\ell m;11}(\bar{r},\bar{r}_p(\bar{t}))\Bigg\}\,,
\end{aligned}
\tag{3.90}
$$

where the divergence in the integral when $\bar{t}' \to \bar{t} - \alpha$ is canceled with the $\log\alpha$, providing a finite final result.

This result can be easily generalized for any j as

$$
\begin{aligned}
{}_s\psi^{\rm NL}_{\ell m}(k=1,j;x^\mu) = \frac{i^j v^j}{b}\lim_{\alpha\to 0^+}\Bigg\{&\int_{-\infty}^{\bar{t}-\alpha} d\bar{t}'\,\frac{\partial^{(j)}_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;j1}(\bar{r},\bar{r}_p(\bar{t}'))}{\bar{t}'-\bar{t}}\\
&-\left[\gamma_E + \log\alpha\right]\partial^{(j)}_{\bar{t}'}\, {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;j1}(\bar{r},\bar{r}_p(\bar{t}))\Bigg\}\,.
\end{aligned}
\tag{3.91}
$$

In order to validate this result, in App. B an alternative derivation for the computation of the non-local integral will be discussed by using distribution theory. By combining these distributional approaches with the coefficients presented in App. A, it is possible to write explicitly the integral for all k as

$$
{}_s\psi^{\rm NL}_{\ell m}(k,j;x^\mu) = \sum_{q=0}^{k-1} \frac{i^j v^j}{b} \left\{ a_q \left[\text{p.v.} \left(\frac{\log^q |\bar{t}' - \bar{t}|}{\bar{t}' - \bar{t}} \right) - \left(\frac{\log^q |\bar{t}' - \bar{t}|}{|\bar{t}' - \bar{t}|} \right)_{\bar{t}} \right] \right\}
$$

$$
+ B\, \partial_{\bar{t}}^{(j)}\; {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;jk}(\bar{r}, \bar{r}_p(\bar{t}))\,, \tag{3.92}
$$

where

$$
\text{p.v.} \left(\frac{\log^k |\bar{t}' - \bar{t}|}{\bar{t}' - \bar{t}} \right) := \int_{-\infty}^{\bar{t}} d\bar{t}' \log^k(|\bar{t}' - \bar{t}|) \frac{\partial_{\bar{t}'}^{(j)}\; {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;jk}(\bar{r}, \bar{r}_p(\bar{t}')) - \partial_{\bar{t}'}^{(j)}\; {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;jk}(\bar{r}, \bar{r}_p(-\bar{t}'))}{\bar{t}' - \bar{t}}\,, \tag{3.93a}
$$

$$
\left(\frac{\log^k |\bar{t}' - \bar{t}|}{|\bar{t}' - \bar{t}|} \right)_{\bar{t}} := \int_{|\bar{t}'| \le \bar{t}} d\bar{t}' \log^k(|\bar{t}' - \bar{t}|) \frac{\partial_{\bar{t}'}^{(j)}\; {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;jk}(\bar{r}, \bar{r}_p(\bar{t}')) - \partial_{\bar{t}}^{(j)}\; {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;jk}(\bar{r}, \bar{r}_p(\bar{t}))}{|\bar{t}' - \bar{t}|}
$$

$$
+ \int_{|\bar{t}'| \ge \bar{t}} d\bar{t}' \log^k(|\bar{t}' - \bar{t}|) \frac{\partial_{\bar{t}'}^{(j)}\; {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;jk}(\bar{r}, \bar{r}_p(\bar{t}'))}{|\bar{t}' - \bar{t}|}\,. \tag{3.93b}
$$

Last remarks

The computation can be heavily simplified by restricting the attention to bound orbits. While in the general case there are no specified orbital frequencies, for bound orbit these are already discrete and it is not necessary to perform the integrals as discussed above.

The field can then be expressed as

$$
{}_s\psi_{lm}(x^\mu) = \sum_j \frac{i^j v^j}{b} \left\{ \frac{\partial^{(j)}}{\partial \bar{t}'^{(j)}} \left[{}_s\hat{\mathcal{G}}^{\rm L}_{\ell m;j}(\bar{r}, \bar{r}_p(\bar{t}')) e^{-im\phi_p(\bar{t}')} \right] \bigg|_{\bar{t}'=\bar{t}} \right.
$$

$$
+ \lim_{\alpha \to 0^+} \left[\int_{-\infty}^{\bar{t}-\alpha} d\bar{t}' \frac{\partial_{\bar{t}'}^{(j)}\; {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;j1}(\bar{r}, \bar{r}_p(\bar{t}'))}{\bar{t}' - \bar{t}} - [\gamma_E + \log\alpha]\, \partial_{\bar{t}'}^{(j)}\; {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;j1}(\bar{r}, \bar{r}_p(\bar{t})) \right]
$$

$$
\left. + [k=2] + [k=3] + [...] \right\} {}_sY_{lm}(\theta, \phi) \tag{3.94}
$$

where the spherical harmonics have been factored out to make explicit the dependence of the field on the spacetime coordinates $x^\mu = (\bar{t}, \bar{r}, \theta, \phi)$. The higher order in k terms can be written in distributional sense by using Eq. (3.92).

3.4 Evaluation at the particle's position

At this stage, the main derivation is essentially complete, and Eq. (3.94) provides the inhomogeneous, first-order (in the self-force expansion) mode-decomposed solution of the Teukolsky equation, sourced by a point particle in a generic equatorial orbit around a Schwarzschild spacetime. The extension to Kerr is straightforward and requires only minor adjustments, namely accounting for the low-frequency expansion of the spheroidal harmonics, which affects some coefficients but it does not modify the structure in ω depicted above.

To obtain the physical solution, one must first sum over (l, m) and then evaluate the field at the particle's position. This step is nontrivial because the field is *divergent* at the particle location. Indeed, Eq. (3.94) corresponds to the retarded field ${}_s\psi^{\rm ret}(x^\mu)$, which is singular at the particle position by construction, even though each individual spherical-harmonic mode remains finite. The regularized field ${}_s\psi^{\rm R}(x^\mu)$ is obtained by subtracting from each l-mode a suitable regulator ${}_sB(t)$ that does not depend on l. This method is known as *mode-sum regularization* [147, 148].

The first step is to evaluate the retarded field for each (l, m) mode as

$$
{}_s\psi^{\rm ret}_{lm}(\bar{t}) = {}_s\psi_{lm}(x^\mu)\Big|_{x^\mu\to z^\mu(\bar{t})}\,, \tag{3.95}
$$

where $z^\mu(\bar{t})$ is generic equatorial geodesic, i.e. $z^\mu(\bar{t}) = (\bar{t}, \bar{r}_p(\bar{t}), \pi/2, \phi_p(\bar{t}))$. At this point, the sum over m can be performed using standard formulas (see Appendix F of [149]), yielding

$$
{}_s\psi^{\rm ret}_{l}(\bar{t}) = \sum_{m=-l}^{l} {}_s\psi^{\rm ret}_{lm}(\bar{t})\,. \tag{3.96}
$$

Before performing the final summation over l, one must subtract the regulator ${}_sB(t)$ from each l-mode. Exact expressions for these regulators are available in the literature, e.g. [44, 45].

After subtracting the regulator consistently for each mode, the remainder yields the regularized inhomogeneous solution to the Teukolsky equation. To summarize, the full procedure involves:

- Solving the mode-decomposed homogeneous equation using two sets of solutions: generic-l solutions and MST solutions valid for specific l values;
- Computing the Teukolsky source;
- Constructing the Green function from both generic-l and MST solutions, ensuring consistency with the desired PN order;
- Building the inhomogeneous solution from the Green function and source;
- Applying mode-sum regularization to obtain the resummed, regularized field.

These two chapters have introduced the methodology and discussed the subtleties that may arise in the calculation. In the following chapters, the procedure will be applied first to bound orbits—both circular and eccentric—and then to the hyperbolic case.

4 Self Force Theory for Unbound orbits

This Chapter is a reorganization of the discussions given in [100] where the scattering of a scalar charge off a Schwarzschild black hole has been investigated by using the formalism introduced previously specializing $s = 0$, i.e. scalar perturbation. This result was obtained by combining the post-Newtonian expansion with the post-Minkowskian one.

The geodesic hyperbolic motion on Schwarzschild has been discussed in 3.1.1, hence this Chapter will start directly by defining the corrections to geodesics, i.e. self-forced motion. From these definitions it will become automatic to define the *local* loss of energy and angular momentum[1] as the corrections to the scattering angle.

4.1 Self-forced motion

A point particle that scatters off the Schwarzschild black hole feels an acceleration due to its self-force, obeying the equation of motion

$$u^\beta \nabla_\beta u_\alpha = F_\alpha. \tag{4.1}$$

Here u^α is the particle's four-velocity, F_α is the self-force per unit mass, and ∇_β is the covariant derivative compatible with the Schwarzschild metric $g_{\alpha\beta}$. For the case of a scalar charged particle, the scalar self-force scales as $F_\alpha \propto q^2/(\mu M^2)$, where q and μ are the particle's charge and mass, respectively. It is worthwhile to mention that, for a gravitating point mass, the gravitational self-force scales as $F_\alpha \propto \mu/M^2$. Concretely, one can write

$$F_\alpha = O(\varepsilon/M), \tag{4.2}$$

with

$$\varepsilon = q^2/(\mu M) \ll 1 \tag{4.3}$$

or $\varepsilon = \mu/M \ll 1$, as appropriate and then expand all quantities to linear order in ε.

By expanding for small ε, one must specify what quantities are being held fixed: the initial energy E_- and angular momentum L_- that the particle has when it begins its orbit from infinite

[1]The term local is emphasizes because it keeps into account how much energy/angular momentum the point particle loses during the scattering event. This quantity could be split in two: energy (angular momentum) loss at infinity and at the horizon. This will be the subject of Chapter 5.

distance; the minus sign indicates that these are the values in the infinite past. In other words, the perturbed orbit is an expansion around a geodesic that has the same initial energy and angular momentum. The self-force correction to the scattering angle is then obtained as a function of E_- and L_-. Equivalently, one can parametrize the self-force correction in terms of b and v, defining those quantities from E_- and L_- using Eq. (3.11).

From the t and ϕ components of the force, the linear corrections to the scattering angle are of the form

$$\delta\chi = \int_{-\infty}^{\infty} d\tau \left\{\mathcal{G}_E(\tau)F_t(\tau) + \mathcal{G}_L(\tau)F_\phi(\tau)\right\}, \tag{4.4}$$

where $\mathcal{G}_{E/L}$ are two functions of geodesic quantities, derived also in [55]. Despite the compactness of this expression, it is possible to derive an alternative version of the scattering angle corrections which is more suitable for the purposes of this Chapter. In order to makes things more notationally transparent, hats will be used to denote zeroth-order, background quantities and δ to denote corrections that are linear in ε.

The starting point is the definition of the accelerated orbit's energy and angular momentum as $E = -u_t$ and $L = u_\phi$, which evolve according to

$$\frac{dE}{d\tau} = -F_t, \quad \frac{dL}{d\tau} = F_\phi. \tag{4.5}$$

These E and L, which are not the conserved energy and angular momentum used in Sec. 3.1.1, they could be defined as functions of τ as

$$E(\tau) = E_- - \int_{-\infty}^{0} F_t d\tau - \int_{0}^{\tau} F_t d\tau \equiv E_- + \delta E_0 + \delta\tilde{E}(\tau)\,, \tag{4.6a}$$

$$L(\tau) = L_- + \int_{-\infty}^{0} F_\phi d\tau + \int_{0}^{\tau} F_\phi d\tau \equiv L_- + \delta L_0 + \delta\tilde{L}(\tau)\,. \tag{4.6b}$$

The reason for splitting each correction into two terms will become clear below.

At this point, it is convenient to use r as a parameter along the orbit. In order to compute integrals along the perturbed orbit one has then to consider the incoming ($\hat{u}^r < 0$) and outgoing ($\hat{u}^r > 0$) branches separately, with r decreasing from infinity ($\tau \to -\infty$) up to a minimum value $r_{\rm min} = \hat{r}_{\rm min} + \delta r_{\rm min}$ ($\tau = \tau_0 = 0$), then increasing again to infinity ($\tau \to \infty$). The superscript $\pm$ will be used to distinguish a quantity taken in the incoming ($-$) and outgoing ($+$) branch, respectively, so that Eq. (3.20) can be slightly modified as

$$\chi = \sum_{\pm} \int_{r_{\rm min}}^{\infty} dr \left(\frac{d\phi_p}{dr}\right)^{\pm} - \pi\,. \tag{4.7}$$

The lower limit of integration is expanded at fixed (E_-, L_-), while the integrand is expanded to linear order in ε at fixed r (as well as fixed E_- and L_-), meaning $\phi_p(r,\varepsilon) = \hat{\phi}(r) + \delta\phi(r,\varepsilon)$. Explicitly,

$$\chi = \sum_{\pm} \int_{\hat{r}_{\rm min}+\delta r_{\rm min}}^{\infty} dr \left(\frac{d\hat{\phi}}{dr} + \frac{d\delta\phi}{dr}\right)^{\pm} - \pi\,, \tag{4.8}$$

which can be written to linear order as

$$\chi = \hat{\chi} + \delta\hat{\chi} + \delta\tilde{\chi}\,, \tag{4.9}$$

where

$$\delta\hat{\chi} = \delta r_{\min} \frac{\partial}{\partial \hat{r}_{\min}} \sum_{\pm} \int_{\hat{r}_{\min}}^{\infty} dr \left(\frac{d\hat{\phi}}{dr}\right)^{\pm} = \delta r_{\min} \frac{\partial \hat{\chi}}{\partial \hat{r}_{\min}} \tag{4.10}$$

is the $O(\varepsilon)$ correction due to the shift $\delta r_{\min}$, and

$$\delta\tilde{\chi} = \sum_{\pm} \int_{\hat{r}_{\min}}^{\infty} dr \left(\frac{d\delta\phi}{dr}\right)^{\pm}. \tag{4.11}$$

To evaluate Eq. (4.10), it is possible to leverage the fact that the equation for $\dot{r}_p$ in Eq. (3.8) remains true for the self-forced motion. The closest approach of the self-forced orbit, which occurs at $\tau = 0$, therefore satisfies

$$(E_- + \delta E_0)^2 = V(\hat{r}_{\min} + \delta r_{\min}; L_- + \delta L_0); \tag{4.12}$$

this was one reason for isolating the corrections to E_- and L_- at periastron in Eq. (4.6a). Equation (4.12) is identical to the equation for the closest approach of a geodesic with energy $E_- + \delta E_0$ and angular momentum $L_- + \delta L_0$. Therefore

$$\delta r_{\min} = \frac{\partial \hat{r}_{\min}}{\partial E_-}\delta E_0 + \frac{\partial \hat{r}_{\min}}{\partial L_-}\delta L_0. \tag{4.13}$$

Substituting this into Eq. (4.10) and appealing to the chain rule, a simple result can be obtained

$$\delta\hat{\chi} = \frac{\partial \hat{\chi}}{\partial E_-}\delta E_0 + \frac{\partial \hat{\chi}}{\partial L_-}\delta L_0\,. \tag{4.14}$$

By turning the attention to $\delta\tilde{\chi}$ in Eq. (4.11), one can express the integrand in terms of energy and angular momentum and use the simple relation $\frac{d\phi_p}{dr} = \frac{d\phi_p/d\tau}{dr_p/d\tau}$. Expanded to linear order in ε at fixed r, this becomes

$$\frac{d\phi_p}{dr} = \frac{\hat{u}^\phi + \delta u^\phi}{\hat{u}^r + \delta u^r} = \frac{\hat{u}^\phi}{\hat{u}^r}\left(1 + \frac{\delta u^\phi}{\hat{u}^\phi} - \frac{\delta u^r}{\hat{u}^r}\right), \tag{4.15}$$

which can be compared with $\frac{d\phi_p}{dr} = \frac{d\hat{\phi}}{dr} + \frac{d\,\delta\phi}{dr}$ to prove

$$\frac{d\,\delta\phi}{dr} = \frac{\hat{u}^\phi}{\hat{u}^r}\left(\frac{\delta u^\phi}{\hat{u}^\phi} - \frac{\delta u^r}{\hat{u}^r}\right). \tag{4.16}$$

The right-hand side can now be expressed in terms of δE and δL. All perturbations to the 4-velocity can be obtained from $\delta u_t = -\delta E(r)$ and $\delta u_\phi = \delta L(r)$ together with the normalization $g_{\mu\nu}u^\mu u^\nu = -1$, where

$$\delta E = \delta E_0 + \delta\tilde{E}, \tag{4.17}$$

and analogously for δL. Since the goal is to linearize in ε at fixed r, and since $g_{\mu\nu}$ is only a function of r, the linear term in the normalization condition becomes $g_{\mu\nu}\delta u^\mu \hat{u}^\nu = 0$, which one can use to obtain δu_r in terms of δE and δL. Substituting these results into Eq. (4.16), one obtains

$$\frac{d\,\delta\phi}{dr} = a_E(r)\delta E(r) + a_L(r)\delta L(r)\,, \tag{4.18}$$

with

$$a_E(r) = -\frac{L_- E_-}{r^2(\hat{u}^r)^3}\,, \qquad a_L(r) = \frac{E_-^2 - f(r)}{r^2(\hat{u}^r)^3}\,. \tag{4.19}$$

Equation (4.11) is then

$$\delta\tilde{\chi} = \sum_{\pm}\int_{\hat{r}_{\min}}^{\infty} dr\,\left[a_E^{\pm}(r)\delta E^{\pm}(r) + a_L^{\pm}(r)\delta L^{\pm}(r)\right]\,. \tag{4.20}$$

Interestingly, it is possible to note that

$$a_E^{-}(r) = -a_E^{+}(r)\,, \qquad a_L^{-}(r) = -a_L^{+}(r)\,, \tag{4.21}$$

hence δE_0 and δL_0 vanish from the integral; this was another reason for isolating these terms in Eq. (4.6a). In conclusion, one has

$$\begin{aligned}
\delta\tilde{\chi} &= \int_{\hat{r}_{\min}}^{\infty} dr\,\left\{a_E^{+}(r)\left[\delta\tilde{E}^{+}(r) - \delta\tilde{E}^{-}(r)\right] + a_L^{+}(r)\left[\delta\tilde{L}^{+}(r) - \delta\tilde{L}^{-}(r)\right]\right\} \\
&= \int_{\hat{r}_{\min}}^{\infty} dr\,\left\{-a_E^{+}(r)\int_{\hat{r}_{\min}}^{r}\frac{dr}{\hat{u}^r}\left[F_t^{+}(r) - F_t^{-}(r)\right] + a_L^{+}(r)\int_{\hat{r}_{\min}}^{r}\frac{dr}{\hat{u}^r}\left[F_\phi^{+}(r) - F_\phi^{-}(r)\right]\right\}\,,
\end{aligned} \tag{4.22}$$

having used

$$\delta\tilde{E}^{\pm}(r) = -\int_{\hat{r}_{\min}}^{r}\frac{dr}{\hat{u}^r}F_t^{\pm}(r)\,, \quad \delta\tilde{L}^{\pm}(r) = \int_{\hat{r}_{\min}}^{r}\frac{dr}{\hat{u}^r}F_\phi^{\pm}(r)\,. \tag{4.23}$$

At this point, it is possible to introduce a split in the forces that will be crucial in the following: the conservative-dissipative splitting. This procedure is trivial in this formalism, and it is given by:

$$\begin{aligned}
F_\alpha^{\mathrm{cons}\,+}(r) &= \frac{1}{2}[F_\alpha^{+}(r) - F_\alpha^{-}(r)] = -F_\alpha^{\mathrm{cons}\,-}(r)\,, \\
F_\alpha^{\mathrm{diss}\,+}(r) &= \frac{1}{2}[F_\alpha^{+}(r) + F_\alpha^{-}(r)] = F_\alpha^{\mathrm{diss}\,-}(r)\,,
\end{aligned} \tag{4.24}$$

which is a simple symmetrical or antisymmetric combination of the force's components in the incoming and outgoing branch. In particular, for $\alpha = t, \phi$, so that

$$\delta\tilde{\chi}^{\mathrm{cons}} = -2\int_{\hat{r}_{\min}}^{\infty} dr\,\left[a_E^{+}(r)\int_{\hat{r}_{\min}}^{r}\frac{dr}{\hat{u}^r}F_t^{\mathrm{cons}\,+} - a_L^{+}(r)\int_{\hat{r}_{\min}}^{r}\frac{dr}{\hat{u}^r}F_\phi^{\mathrm{cons}\,+}\right]\,.$$

Interestingly, $\delta\tilde{\chi}^{\mathrm{diss}} = 0$ because of the symmetries of F_α^{diss}. As a consequence, the dissipative correction to the scattering angle is simply given by

$$\delta\chi^{\mathrm{diss}} = \delta\hat{\chi}^{\mathrm{diss}} = -\frac{1}{2}\left(\frac{\partial\hat{\chi}}{\partial E_-}E_{\mathrm{rad}} + \frac{\partial\hat{\chi}}{\partial L_-}L_{\mathrm{rad}}\right)\,, \tag{4.25}$$

(a reminder of the linear response formula, [150, 151]). Here, E_{rad} and L_{rad} are the total radiated energy and angular momentum, which are related to the integrated dissipative self-force by

$$E_{\mathrm{rad}} = -2\delta E_0^{\mathrm{diss}}, \quad L_{\mathrm{rad}} = -2\delta L_0^{\mathrm{diss}}, \tag{4.26}$$

respectively. Recall that the local definitions of orbital E and L are not the conserved energy and angular momentum of the perturbed orbit in the conservative sector, as emphasized in Eq. (4.5). As a consequence, conservative self-force causes E and L to change with time. However, this change averages to zero over the complete orbit, and only the dissipative self-force is related to the emitted fluxes of energy and angular momentum in scalar radiation.

It is also worthwhile to mention that one can equivalently compute the scattering angle through a time parameterization in Eq. (4.8).

4.2 Scattering of a Scalar Charge off a Schwarzschild Black Hole

4.2.1 General set up for scalar perturbation

The procedure for $s = 0$ perturbations is identical to what is described in Chapters 2 and 3 where the value of s can be fixed from the beginning. The Teukolsky equation is then reduced to the massless Klein-Gordon (KG) equation with a scalar charge distribution source:

$$\Box_0 \psi = -4\pi\rho, \tag{4.27}$$

where $\Box = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the d'Alembertian and ${}_0T = -4\pi r^2\rho$. In this case ρ will be a point-particle (scalar) charge density,

$$\begin{aligned}\rho(x^\mu) &= q\int(-g)^{-1/2}\delta^4(x^\mu - x_p^\mu(\tau))d\tau \\ &= \frac{q}{u^t r^2}\delta(r - r_p(t))\delta(\theta - \pi/2)\delta(\phi - \phi_p(t)),\end{aligned} \tag{4.28}$$

where $g = -r^4\sin^2\theta$ is the determinant of the Schwarzschild metric. While this does not drastically simplify the homogeneous solutions, that must be obtained by using the same techniques introduced in previous Chapters, the source is instead much simpler than the gravitational source.

The aim is to describe this problem by combining a weak-field (PM) with a low-velocity (PN) limit. Specifically

$$b \gg M, \qquad v \ll 1. \tag{4.29}$$

The ultimate goal of this Section will be to calculate the scattering angle (4.9) in this limit. Doing so requires computing the self-force, which in turn requires knowledge of the field near the particle's worldline. Here the barred variables introduced in Eq. (3.51) will be intensively used.

It is worthwhile to remember that the Green function has the structure described in Eq. (3.55) and, as a consequence of the use of scaled coordinates, ${}_sG_{\ell m\omega}(\bar r, \bar r')$ depends on b starting at 1PM, and the coefficients ${}_sG^{(j,k)}_{\ell m}(\bar r, \bar r')$ here are also expansions in $1/b$. For the non-logarithmic contributions (i.e., $k = 0$), ${}_sG^{(j,0)}_{\ell m}(\bar r, \bar r')$ starts at order $1/b^0$ for all values of j, l, and m. Meanwhile, for $k \neq 0$, the first logarithm appears at $j = 2$ for $s = l = 0$. It is then possible to write explicitly the local and non-local contributions to the field by using the scalar source explicitly as

$$\rho(x^\mu) = \frac{q}{u^t \bar r^2 b^3}\delta(\bar r - \bar r_p(\bar t))\sum_{lm} Y^*_{lm}(\pi/2, 0)Y_{lm}(\theta,\phi)e^{-im\phi_p(\bar t)}, \tag{4.30}$$

by starting from Eq. (3.94)[2] :

$$
{}_0\psi^{\mathrm{L}}_{\ell m} = \frac{q}{b}\sum_{j\geq 0} i^j v^j \frac{\partial^j}{\partial \bar{t}^j}\left\{ {}_0G^{(j,0)}_{\ell m}[\bar{r},\bar{r}_p(\bar{t})]\frac{e^{-im\phi_p(\bar{t})}}{u^t(\bar{t})}\right\} Y^*_{\ell m}(\pi/2,0). \tag{4.31}
$$

$$
\begin{aligned}
{}_0\psi^{\mathrm{NL}}_{lm} = &-\frac{q}{b}\lim_{\epsilon\to 0}\sum_{j\geq 0} i^j v^j Y^*_{lm}(\pi/2,0)\Bigg\{(\gamma_E+\log\epsilon)\frac{\partial^{(j)}}{\partial\bar{t}^{(j)}}\left[G^{(j,1)}_{lm}(\bar{r},\bar{r}_p(\bar{t}))\frac{e^{-im\phi(\bar{t})}}{u^t(\bar{t})}\right] \\
&+\int_{-\infty}^{\bar{t}-\alpha}\frac{d\bar{t}'}{\bar{t}'-\bar{t}}\frac{\partial^{(j)}}{\partial\bar{t}'^{(j)}}\left[G^{(j,1)}_{lm}(\bar{r},\bar{r}_p(\bar{t}'))\frac{e^{-im\phi_p(\bar{t}')}}{u^t(\bar{t}')}\right]\Bigg\},
\end{aligned} \tag{4.32}
$$

where only $k=1$ has been taken into account. The reason is that, this problem, has been dealt up to 4.5PN, which contains only $k=0,1$. Because the PN expansion has been equipped also with a PM approximation, the results shown in the following are computed up to 5PM.

4.2.2 Scalar-field results

In order to compute the regularized field, it is first necessary to explicit the regularization parameter $B(\bar{t})$, which assumes the general expression:

$$
\psi^{\mathrm{R}}(t) = \sum_l [\psi^{\mathrm{ret}}_l(t) - B(t)]. \tag{4.33}
$$

$B(\bar{t})$ has been computed in Ref. [55, 148][3] as

$$
B(\bar{t}) = \frac{2}{\pi}\frac{q}{L}k(\bar{t})K[k(\bar{t})]\,, \qquad k(\bar{t}) = \sqrt{\frac{L^2}{L^2+r_p(\bar{t})^2}}\,, \tag{4.34}
$$

where $K(k)$ denotes the complete elliptic integral of the first kind. By expanding in PN-PM this quantity, one finds

$$
\begin{aligned}
B(\bar{t}) = &\frac{q}{b}\Bigg\{\frac{1}{\sqrt{1+\bar{t}^2}} - \frac{v^2}{4\left(1+\bar{t}^2\right)^{3/2}} - \frac{\left(7+16\bar{t}^2\right)v^4}{64\left(1+\bar{t}^2\right)^{5/2}} - \frac{\left(17+56\bar{t}^2+64\bar{t}^4\right)v^6}{256\left(1+\bar{t}^2\right)^{7/2}} - \frac{\left(759+3264\bar{t}^2+5376\bar{t}^4+4096\bar{t}^6\right)v^8}{16384\left(1+\bar{t}^2\right)^{9/2}}\Bigg\} \\
&+\frac{qM}{b^2}\Bigg\{\frac{1}{v^2}\left[\frac{1}{1+\bar{t}^2} - \frac{\bar{t}\,\mathrm{arcsinh}(\bar{t})}{\left(1+\bar{t}^2\right)^{3/2}}\right] - \frac{3}{4\left(1+\bar{t}^2\right)^2} + \frac{3\bar{t}\left(5+4\bar{t}^2\right)\mathrm{arcsinh}(\bar{t})}{4\left(1+\bar{t}^2\right)^{5/2}} - \frac{3v^2}{64}\left[\frac{1+16\bar{t}^2}{\left(1+\bar{t}^2\right)^3} + \frac{\bar{t}\left(47+32\bar{t}^2\right)\mathrm{arcsinh}(\bar{t})}{\left(1+\bar{t}^2\right)^{7/2}}\right] \\
&-\frac{v^4}{256}\left[\frac{7+24\bar{t}^2+192\bar{t}^4}{\left(1+\bar{t}^2\right)^4} + \frac{\bar{t}\left(29+588\bar{t}^2+384\bar{t}^4\right)\mathrm{arcsinh}(\bar{t})}{\left(1+\bar{t}^2\right)^{9/2}}\right] - \frac{3v^6}{16384}\left[\frac{101+448\bar{t}^2+768\bar{t}^4+4096\bar{t}^6}{\left(1+\bar{t}^2\right)^5}\right. \\
&\left.+\frac{\bar{t}\left(347+1536\bar{t}^2+13056\bar{t}^4+8192\bar{t}^6\right)\mathrm{arcsinh}(\bar{t})}{\left(1+\bar{t}^2\right)^{11/2}}\right]\Bigg\} \\
&+\frac{qM^2}{b^3}\Bigg\{\frac{1}{v^4}\left[\frac{1}{2\left(1+\bar{t}^2\right)^{3/2}} - \frac{3\bar{t}\,\mathrm{arcsinh}(\bar{t})}{\left(1+\bar{t}^2\right)^2} - \frac{\left(1-2\bar{t}^2\right)\mathrm{arcsinh}^2(\bar{t})}{2\left(1+\bar{t}^2\right)^{5/2}}\right] + \frac{1}{v^2}\left[\frac{7+16\bar{t}^2}{8\left(1+\bar{t}^2\right)^{5/2}} + \frac{3\bar{t}\left(21+16\bar{t}^2\right)\mathrm{arcsinh}(\bar{t})}{4\left(1+\bar{t}^2\right)^3}\right. \\
&\left.+\frac{3\left(9-12\bar{t}^2-16\bar{t}^4\right)\mathrm{arcsinh}^2(\bar{t})}{8\left(1+\bar{t}^2\right)^{7/2}}\right] - \frac{3\left(37+112\bar{t}^2\right)}{128\left(1+\bar{t}^2\right)^{7/2}} + \frac{15\bar{t}\,\mathrm{arctan}(\bar{t})}{2\left(1+\bar{t}^2\right)^{3/2}} - \frac{3\bar{t}\left(505+592\bar{t}^2+192\bar{t}^4\right)\mathrm{arcsinh}(\bar{t})}{64\left(1+\bar{t}^2\right)^4}
\end{aligned}
$$

[2] The apparent conflict between the powers of $\bar{r}$ and b in Eqs. (4.30) and (4.31) comes from the fact that ${}_0T = -4\pi r^2\rho = -4\pi b^2\bar{r}^2\rho$.

[3] Note that here L is the specific orbital angular momentum, while in Ref. [148] it denotes $l+1/2$.

$$-\frac{3\left(287-330\bar{t}^2-896\bar{t}^4-384\bar{t}^6\right)\operatorname{arcsinh}^2(\bar{t})}{128\left(1+\bar{t}^2\right)^{9/2}}+v^2\Bigg[-\frac{7\left(7+24\bar{t}^2+192\bar{t}^4\right)}{512\left(1+\bar{t}^2\right)^{9/2}}$$

$$+\frac{3\bar{t}\left(869+88\bar{t}^2-256\bar{t}^4\right)\operatorname{arcsinh}(\bar{t})}{256\left(1+\bar{t}^2\right)^5}-\frac{\left(-1663+6536\bar{t}^2+9696\bar{t}^4+3072\bar{t}^6\right)\operatorname{arcsinh}(\bar{t})^2}{512\left(1+\bar{t}^2\right)^{11/2}}-\frac{45\bar{t}\arctan(\bar{t})}{8\left(1+\bar{t}^2\right)^{5/2}}\Bigg]$$

$$-v^4\Bigg[\frac{1831+8128\bar{t}^2-6912\bar{t}^4+86016\bar{t}^6}{32768\left(1+\bar{t}^2\right)^{11/2}}+\frac{45\bar{t}\left(1+16\bar{t}^2\right)\arctan(\bar{t})}{128\left(1+\bar{t}^2\right)^{7/2}}-\frac{\left(3333\bar{t}+233664\bar{t}^3+59904\bar{t}^5-49152\bar{t}^7\right)\operatorname{arcsinh}(\bar{t})}{16384\left(1+\bar{t}^2\right)^6}$$

$$-\frac{3\left(1509+98766\bar{t}^2-57856\bar{t}^4-180224\bar{t}^6-65536\bar{t}^8\right)\operatorname{arcsinh}(\bar{t})^2}{32768\left(1+\bar{t}^2\right)^{13/2}}\Bigg]\Bigg]\Bigg\}$$

$$+\frac{qM^3}{b^4}\Bigg\{\frac{1}{v^6}\Bigg[\frac{\bar{t}^2}{2\left(1+\bar{t}^2\right)^2}-\frac{9\operatorname{arcsinh}(\bar{t})\,\bar{t}}{2\left(1+\bar{t}^2\right)^{5/2}}+\frac{\operatorname{arcsinh}(\bar{t})^2\left(-4+11\bar{t}^2\right)}{2\left(1+\bar{t}^2\right)^3}+\frac{\operatorname{arcsinh}(\bar{t})^3\left(3\bar{t}-2\bar{t}^3\right)}{2\left(1+\bar{t}^2\right)^{7/2}}\Bigg]$$

$$+\frac{1}{v^4}\Bigg[\frac{3\operatorname{arcsinh}(\bar{t})\,\bar{t}\left(77+52\bar{t}^2\right)}{8\left(1+\bar{t}^2\right)^{7/2}}+\frac{24+\bar{t}^2-28\bar{t}^4}{8\left(1+\bar{t}^2\right)^3}+\frac{\operatorname{arcsinh}(\bar{t})^3\,\bar{t}\left(-123-16\bar{t}^2+72\bar{t}^4\right)}{8\left(1+\bar{t}^2\right)^{9/2}}$$

$$-\frac{3\operatorname{arcsinh}(\bar{t})^2\left(-46+97\bar{t}^2+108\bar{t}^4\right)}{8\left(1+\bar{t}^2\right)^4}\Bigg]+\frac{1}{v^2}\Bigg[\frac{45\arctan(\bar{t})\,\bar{t}}{2\left(1+\bar{t}^2\right)^2}+\frac{3\operatorname{arcsinh}(\bar{t})^2\left(-2232+3961\bar{t}^2+9280\bar{t}^4+4032\bar{t}^6\right)}{128\left(1+\bar{t}^2\right)^5}$$

$$+\frac{\operatorname{arcsinh}(\bar{t})^3\left(7419\bar{t}+5642\bar{t}^3-4288\bar{t}^5-3456\bar{t}^7\right)}{128\left(1+\bar{t}^2\right)^{11/2}}+\frac{-24+2701\bar{t}^2+5280\bar{t}^4+2240\bar{t}^6}{128\left(1+\bar{t}^2\right)^4}$$

$$-\operatorname{arcsinh}(\bar{t})\left(\frac{15\arctan(\bar{t})\left(-1+2\bar{t}^2\right)}{2\left(1+\bar{t}^2\right)^{5/2}}+\frac{3\bar{t}\left(2623+2720\bar{t}^2+832\bar{t}^4\right)}{128\left(1+\bar{t}^2\right)^{9/2}}\right)\Bigg]-\frac{45\arctan(\bar{t})\,\bar{t}\left(9+4\bar{t}^2\right)}{8\left(1+\bar{t}^2\right)^3}$$

$$+\frac{-720-723\bar{t}^2+9724\bar{t}^4+21312\bar{t}^6+8960\bar{t}^8}{512\left(1+\bar{t}^2\right)^5}+\frac{\operatorname{arcsinh}(\bar{t})^3\,\bar{t}\left(-49371-47524\bar{t}^2+27304\bar{t}^4+45056\bar{t}^6+13824\bar{t}^8\right)}{512\left(1+\bar{t}^2\right)^{13/2}}$$

$$-\frac{\operatorname{arcsinh}(\bar{t})^2\left(-32782+75899\bar{t}^2+194844\bar{t}^4+138048\bar{t}^6+34560\bar{t}^8\right)}{512\left(1+\bar{t}^2\right)^6}+\operatorname{arcsinh}(\bar{t})\Bigg(\frac{45\arctan(\bar{t})\left(-5+8\bar{t}^2+8\bar{t}^4\right)}{8\left(1+\bar{t}^2\right)^{7/2}}$$

$$+\frac{9\bar{t}\left(2891+1508\bar{t}^2+448\bar{t}^4+256\bar{t}^6\right)}{512\left(1+\bar{t}^2\right)^{11/2}}\Bigg)+v^2\Bigg[-\frac{45\arctan(\bar{t})\,\bar{t}\left(-73+32\bar{t}^2\right)}{128\left(1+\bar{t}^2\right)^4}$$

$$+\frac{\operatorname{arcsinh}(\bar{t})^2\left(-811980+5748531\bar{t}^2+6584448\bar{t}^4+1502208\bar{t}^6-98304\bar{t}^8\right)}{32768\left(1+\bar{t}^2\right)^7}$$

$$-\frac{5\operatorname{arcsinh}(\bar{t})^3\,\bar{t}\left(-437055+118610\bar{t}^2+1001472\bar{t}^4+681984\bar{t}^6+131072\bar{t}^8\right)}{32768\left(1+\bar{t}^2\right)^{15/2}}$$

$$-\frac{8080+489967\bar{t}^2+1822976\bar{t}^4+2410496\bar{t}^6+786432\bar{t}^8}{32768\left(1+\bar{t}^2\right)^6}$$

$$+\operatorname{arcsinh}(\bar{t})\left(\frac{45\arctan(\bar{t})\left(47-186\bar{t}^2 128\bar{t}^4\right)}{128\left(1+\bar{t}^2\right)^{9/2}}+\frac{3\bar{t}\left(-156419+567808\bar{t}^2+181248\bar{t}^4-98304\bar{t}^6\right)}{32768\left(1+\bar{t}^2\right)^{13/2}}\right)\Bigg]\Bigg]\Bigg\}$$

$$+\frac{qM^4}{b^5}\Bigg\{\frac{1}{v^8}\Bigg[-\frac{\operatorname{arcsinh}(\bar{t})\,\bar{t}\left(7+2\bar{t}^2\right)}{2\left(1+\bar{t}^2\right)^3}+\frac{-1+12\bar{t}^2}{8\left(1+\bar{t}^2\right)^{5/2}}+\frac{\operatorname{arcsinh}(\bar{t})^2\left(-17+58\bar{t}^2\right)}{4\left(1+\bar{t}^2\right)^{7/2}}+\frac{\operatorname{arcsinh}(\bar{t})^3\left(55\bar{t}-50\bar{t}^3\right)}{6\left(1+\bar{t}^2\right)^4}$$

$$+\frac{\operatorname{arcsinh}(\bar{t})^4\left(3-24\bar{t}^2+8\bar{t}^4\right)}{8\left(1+\bar{t}^2\right)^{9/2}}\Bigg]+\frac{1}{v^6}\Bigg[\frac{\operatorname{arcsinh}(\bar{t})\,\bar{t}\left(173+148\bar{t}^2+80\bar{t}^4\right)}{8\left(1+\bar{t}^2\right)^4}$$

$$-\frac{3\left(-27+116\bar{t}^2+128\bar{t}^4\right)}{32\left(1+\bar{t}^2\right)^{7/2}}-\frac{3\operatorname{arcsinh}(\bar{t})^2\left(-237+664\bar{t}^2+656\bar{t}^4\right)}{16\left(1+\bar{t}^2\right)^{9/2}}+\frac{\operatorname{arcsinh}(\bar{t})^3\,\bar{t}\left(-905+114\bar{t}^2+704\bar{t}^4\right)}{8\left(1+\bar{t}^2\right)^5}$$

$$-\frac{3\operatorname{arcsinh}(\bar{t})^4\left(53-396\bar{t}^2-216\bar{t}^4+128\bar{t}^6\right)}{32\left(1+\bar{t}^2\right)^{11/2}}\Bigg]+\frac{1}{v^4}\Bigg[\frac{135\arctan(\bar{t})\,\bar{t}}{4\left(1+\bar{t}^2\right)^{5/2}}+\frac{3\left(1597+17428\bar{t}^2+26688\bar{t}^4+10752\bar{t}^6\right)}{512\left(1+\bar{t}^2\right)^{9/2}}$$

$$+\frac{\operatorname{arcsinh}(\bar{t})^3\left(70541\bar{t}+37522\bar{t}^3-66400\bar{t}^5-43776\bar{t}^7\right)}{128\left(1+\bar{t}^2\right)^6}+\frac{3\operatorname{arcsinh}(\bar{t})^4\left(4441-32096\bar{t}^2-44464\bar{t}^4-2176\bar{t}^6+9216\bar{t}^8\right)}{512\left(1+\bar{t}^2\right)^{13/2}}$$

$$+\operatorname{arcsinh}(\bar{t})\left(-\frac{15\arctan(\bar{t})\left(-4+11\bar{t}^2\right)}{2\left(1+\bar{t}^2\right)^3}-\frac{\bar{t}\left(3667+3678\bar{t}^2+11904\bar{t}^4+7168\bar{t}^6\right)}{128\left(1+\bar{t}^2\right)^5}\right)$$

$$
\begin{aligned}
&+\operatorname{arcsinh}(\bar{t})^2\left(\frac{45\arctan(\bar{t})\,\bar{t}\left(-3+2\bar{t}^2\right)}{4\left(1+\bar{t}^2\right)^{7/2}}+\frac{3\left(-14879+36634\bar{t}^2+74112\bar{t}^4+31104\bar{t}^6\right)}{256\left(1+\bar{t}^2\right)^{11/2}}\right)\Bigg]\\
&+\frac{1}{v^2}\left[\frac{15\arctan(\bar{t})\,\bar{t}\left(-67+64\bar{t}^2+56\bar{t}^4\right)}{16\left(1+\bar{t}^2\right)^{7/2}}+\frac{-9071-79332\bar{t}^2+23904\bar{t}^4+213760\bar{t}^6+107520\bar{t}^8}{2048\left(1+\bar{t}^2\right)^{11/2}}\right.\\
&+\frac{\operatorname{arcsinh}(\bar{t})^3\,\bar{t}\left(-2049091-1917418\bar{t}^2+1701648\bar{t}^4+2679936\bar{t}^6+884736\bar{t}^8\right)}{1536\left(1+\bar{t}^2\right)^7}\\
&-\frac{3\operatorname{arcsinh}(\bar{t})^4\left(46133-349340\bar{t}^2-699280\bar{t}^4-292480\bar{t}^6+110080\bar{t}^8+73728\bar{t}^{10}\right)}{2048\left(1+\bar{t}^2\right)^{15/2}}\\
&+\operatorname{arcsinh}(\bar{t})^2\left(-\frac{45\arctan(\bar{t})\,\bar{t}\left(-87-4\bar{t}^2+48\bar{t}^4\right)}{16\left(1+\bar{t}^2\right)^{9/2}}+\frac{320795-880276\bar{t}^2-2241600\bar{t}^4-1673472\bar{t}^6-442368\bar{t}^8}{1024\left(1+\bar{t}^2\right)^{13/2}}\right)\\
&\left.+\operatorname{arcsinh}(\bar{t})\left(\frac{45\arctan(\bar{t})\left(-30+69\bar{t}^2+64\bar{t}^4\right)}{8\left(1+\bar{t}^2\right)^4}+\frac{\bar{t}\left(-1407-42536\bar{t}^2+81984\bar{t}^4+118528\bar{t}^6+35840\bar{t}^8\right)}{512\left(1+\bar{t}^2\right)^6}\right)\right]\\
&+\frac{225\arctan(\bar{t})^2\left(-1+2\bar{t}^2\right)}{8\left(1+\bar{t}^2\right)^{5/2}}+\frac{45\arctan(\bar{t})\,\bar{t}\left(543-128\bar{t}^2+176\bar{t}^4+112\bar{t}^6\right)}{256\left(1+\bar{t}^2\right)^{9/2}}\\
&-\frac{3\left(122019+4057884\bar{t}^2+17571328\bar{t}^4+24161280\bar{t}^6+11272192\bar{t}^8+1433600\bar{t}^{10}\right)}{131072\left(1+\bar{t}^2\right)^{13/2}}\\
&-\frac{\operatorname{arcsinh}(\bar{t})^3\,\bar{t}\left(-54177749-39992282\bar{t}^2+65513856\bar{t}^4+103636480\bar{t}^6+55926784\bar{t}^8+11501568\bar{t}^{10}\right)}{32768\left(1+\bar{t}^2\right)^8}\\
&+\frac{3\operatorname{arcsinh}(\bar{t})^4\left(3888169-36423576\bar{t}^2-73918360\bar{t}^4-36623360\bar{t}^6+10260480\bar{t}^8+14188544\bar{t}^{10}+3538944\bar{t}^{12}\right)}{131072\left(1+\bar{t}^2\right)^{17/2}}\\
&+\operatorname{arcsinh}(\bar{t})\left(-\frac{45\arctan(\bar{t})\left(-792+2089\bar{t}^2+2896\bar{t}^4+960\bar{t}^6\right)}{128\left(1+\bar{t}^2\right)^5}\right.\\
&\left.+\frac{\bar{t}\left(-1635047+481430\bar{t}^2-2974208\bar{t}^4+7159296\bar{t}^6+10960896\bar{t}^8+3440640\bar{t}^{10}\right)}{32768\left(1+\bar{t}^2\right)^7}\right)\\
&-\operatorname{arcsinh}(\bar{t})^2\left(\frac{45\arctan(\bar{t})\,\bar{t}\left(3243+1274\bar{t}^2-2176\bar{t}^4-1152\bar{t}^6\right)}{256\left(1+\bar{t}^2\right)^{11/2}}\right.\\
&\left.\left.+\frac{3\left(5456971-26249862\bar{t}^2-49104896\bar{t}^4-38025216\bar{t}^6-17334272\bar{t}^8-3538944\bar{t}^{10}\right)}{65536\left(1+\bar{t}^2\right)^{15/2}}\right)\right\}.
\end{aligned}
\tag{4.35}
$$

This quantity is intrinsically local and can be straightforwardly extended to any order in PN-PM sense. It is now convenient to split the regularized scalar field between the local and the non-local sector.

Local field

In order to obtain the regularized scalar field in a PN-PM form it is then necessary to subtract the $B(\bar{t})$ term from it and then perform the infinite sum over the l-modes analytically. This gives

$$
\begin{aligned}
\psi^{\mathrm{R,L}}(\bar{t}) =\;& \frac{qM}{b^2}\left\{-\frac{v\bar{t}}{\left(1+\bar{t}^2\right)^{3/2}}+\frac{v^2\left(1-2\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^2}+\frac{v^3\left(7\bar{t}-5\bar{t}^3\right)}{2\left(1+\bar{t}^2\right)^{5/2}}+\frac{v^4\left(-1+13\bar{t}^2-6\bar{t}^4\right)}{2\left(1+\bar{t}^2\right)^3}+\frac{v^5\left(-11\bar{t}+82\bar{t}^3-27\bar{t}^5\right)}{8\left(1+\bar{t}^2\right)^{7/2}}\right.\\
&\left.-\frac{v^6\left(1+24\bar{t}^2-115\bar{t}^4+30\bar{t}^6\right)}{8\left(1+\bar{t}^2\right)^4}-\frac{v^7\,\bar{t}\left(5+83\bar{t}^2-305\bar{t}^4+65\bar{t}^6\right)}{16\left(1+\bar{t}^2\right)^{9/2}}\right\}\\
&+\frac{qM^2}{b^3}\left\{\frac{1}{v}\left[-\frac{3\bar{t}}{\left(1+\bar{t}^2\right)^2}+\frac{\operatorname{arcsinh}(\bar{t})\left(-1+2\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^{5/2}}\right]+\frac{4\operatorname{arcsinh}(\bar{t})\,\bar{t}\left(-2+\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^3}+\frac{2-10\bar{t}^2}{\left(1+\bar{t}^2\right)^{5/2}}\right.\\
&+v\left[\frac{31\bar{t}-29\bar{t}^3}{2\left(1+\bar{t}^2\right)^3}+\frac{\operatorname{arcsinh}(\bar{t})\left(13-49\bar{t}^2-2\bar{t}^4\right)}{2\left(1+\bar{t}^2\right)^{7/2}}\right]\\
&+v^2\left[\frac{\log\left(1+\bar{t}^2\right)\left(4-8\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^{5/2}}-\frac{8\log(2)\left(-1+2\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^{5/2}}-\frac{8\left(-1+2\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^{5/2}}\log\left(\frac{b}{M}\right)+\frac{\operatorname{arcsinh}(\bar{t})\left(40\bar{t}-26\bar{t}^3-6\bar{t}^5\right)}{\left(1+\bar{t}^2\right)^4}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{-3616 + 768\gamma_E + 207\pi^2 + \left(12736 - 768\gamma_E - 207\pi^2\right)\bar{t}^2 - 2\left(464 + 768\gamma_E + 207\pi^2\right)\bar{t}^4}{288\left(1+\bar{t}^2\right)^{7/2}}\Bigg] \\
&+ v^3\Bigg[\frac{-279\bar{t} + 1642\bar{t}^3 - 599\bar{t}^5}{24\left(1+\bar{t}^2\right)^4} - \frac{\operatorname{arcsinh}(\bar{t})\left(95 - 744\bar{t}^2 + 67\bar{t}^4 + 66\bar{t}^6\right)}{8\left(1+\bar{t}^2\right)^{9/2}}\Bigg] \\
&+ v^4\Bigg[-\frac{4\log(2)\left(7 - 61\bar{t}^2 + 22\bar{t}^4\right)}{3\left(1+\bar{t}^2\right)^{7/2}} - \frac{4\left(7 - 61\bar{t}^2 + 22\bar{t}^4\right)}{3\left(1+\bar{t}^2\right)^{7/2}}\log\left(\frac{b}{M}\right) \\
&- \frac{2\log\left(1+\bar{t}^2\right)\left(7 - 61\bar{t}^2 + 22\bar{t}^4\right)}{3\left(1+\bar{t}^2\right)^{7/2}} + \frac{\operatorname{arcsinh}(\bar{t})\left(-106\bar{t} + 283\bar{t}^3 + 32\bar{t}^5 - 21\bar{t}^7\right)}{2\left(1+\bar{t}^2\right)^5} + \frac{529792 - 215040\gamma - 20295\pi^2}{23040\left(1+\bar{t}^2\right)^{9/2}} \\
&- \frac{15\left(368000 - 110592\gamma_E - 12231\pi^2\right)\bar{t}^2 + 72\left(10928 - 16640\gamma_E - 1695\pi^2\right)\bar{t}^4 - 8\left(174032 - 84480\gamma_E - 10215\pi^2\right)\bar{t}^6}{23040\left(1+\bar{t}^2\right)^{9/2}}\Bigg] \\
&+ v^5\Bigg[-\frac{\bar{t}\left(41277 + 19291\bar{t}^2 - 174121\bar{t}^4 + 29305\bar{t}^6\right)}{720\left(1+\bar{t}^2\right)^5} + \frac{\operatorname{arcsinh}(\bar{t})\left(61 - 2015\bar{t}^2 + 2929\bar{t}^4 + 779\bar{t}^6 - 194\bar{t}^8\right)}{16\left(1+\bar{t}^2\right)^{11/2}}\Bigg]\Bigg\} \\
&+ \frac{qM^3}{b^4}\Bigg\{\frac{1}{v^3}\Bigg[-\frac{9\bar{t}}{2\left(1+\bar{t}^2\right)^{5/2}} + \frac{\operatorname{arcsinh}(\bar{t})\left(-4 + 11\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^3} + \frac{\operatorname{arcsinh}(\bar{t})^2\left(9\bar{t} - 6\bar{t}^3\right)}{2\left(1+\bar{t}^2\right)^{7/2}}\Bigg] \\
&+ \frac{1}{v^2}\Bigg[\frac{2 - 22\bar{t}^2}{\left(1+\bar{t}^2\right)^3} + \frac{\operatorname{arcsinh}(\bar{t})\left(-38\bar{t} + 34\bar{t}^3\right)}{\left(1+\bar{t}^2\right)^{7/2}} + \frac{\operatorname{arcsinh}(\bar{t})^2\left(-4 + 26\bar{t}^2 - 6\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^4}\Bigg] \\
&+ \frac{1}{v}\Bigg[\frac{119\bar{t} - 181\bar{t}^3}{4\left(1+\bar{t}^2\right)^{7/2}} + \frac{3\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(-81 + 73\bar{t}^2 + 14\bar{t}^4\right)}{4\left(1+\bar{t}^2\right)^{9/2}} + \frac{\operatorname{arcsinh}(\bar{t})\left(68 - 333\bar{t}^2 + 19\bar{t}^4\right)}{2\left(1+\bar{t}^2\right)^4}\Bigg] \\
&+ \frac{4\log(2)\left(7 - 23\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^3} + \frac{4\left(7 - 23\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^3}\log\left(\frac{b}{M}\right) + \frac{2\log\left(1+\bar{t}^2\right)\left(7 - 23\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^3} + \frac{\operatorname{arcsinh}(\bar{t})^2\left(32 - 245\bar{t}^2 - 10\bar{t}^4 + 27\bar{t}^6\right)}{\left(1+\bar{t}^2\right)^5} \\
&+ \frac{-12512 + 2688\gamma_E + 765\pi^2 + \left(44288 - 6144\gamma_E - 1575\pi^2\right)\bar{t}^2 - 4\left(-1240 + 2208\gamma_E + 585\pi^2\right)\bar{t}^4}{288\left(1+\bar{t}^2\right)^4} \\
&+ \operatorname{arcsinh}(\bar{t})\Bigg(\frac{8\log(2)\bar{t}\left(-3 + 2\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^{7/2}} + \frac{8\bar{t}\left(-3 + 2\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^{7/2}}\log\left(\frac{b}{M}\right) + \frac{4\log\left(1+\bar{t}^2\right)\bar{t}\left(-3 + 2\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^{7/2}} \\
&+ \frac{-9\left(-3552 + 256\gamma_E + 69\pi^2\right)\bar{t} - \left(24064 + 768\gamma_E + 207\pi^2\right)\bar{t}^3 + 2\left(-4976 + 768\gamma_E + 207\pi^2\right)\bar{t}^5}{96\left(1+\bar{t}^2\right)^{9/2}}\Bigg) \\
&+ v\Bigg[\frac{\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(4611 - 6052\bar{t}^2 - 2977\bar{t}^4 + 126\bar{t}^6\right)}{16\left(1+\bar{t}^2\right)^{11/2}} - \frac{\operatorname{arcsinh}(\bar{t})\left(716 - 6001\bar{t}^2 + 1330\bar{t}^4 + 487\bar{t}^6\right)}{8\left(1+\bar{t}^2\right)^5} \\
&+ \frac{-1441\bar{t} + 9718\bar{t}^3 - 6481\bar{t}^5 - 360\arctan(\bar{t})\left(1+\bar{t}^2\right)^2\left(-1 + 2\bar{t}^2\right)}{48\left(1+\bar{t}^2\right)^{9/2}}\Bigg] \\
&+ v^2\Bigg[-\frac{30\arctan(\bar{t})\,\bar{t}\left(-2 + \bar{t}^2\right)}{\left(1+\bar{t}^2\right)^3} + \frac{\left(-30 + 472\bar{t}^2 - 338\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^4}\log\left(\frac{b}{M}\right) + \frac{\log(2)\left(-46 + 488\bar{t}^2 - 306\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^4} \\
&+ \frac{\log\left(1+\bar{t}^2\right)\left(-15 + 236\bar{t}^2 - 169\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^4} + \frac{1088128 - 291840\gamma_E - 76635\pi^2}{7680\left(1+\bar{t}^2\right)^5} \\
&- \frac{3\left(3729792 - 1131520\gamma_E - 134295\pi^2\right)\bar{t}^2 + 24\left(70704 - 50560\gamma_E - 13425\pi^2\right)\bar{t}^4 - 8\left(677456 - 309120\gamma_E - 19665\pi^2\right)\bar{t}^6}{7680\left(1+\bar{t}^2\right)^5} \\
&- \frac{\operatorname{arcsinh}(\bar{t})^2\left(346 - 3711\bar{t}^2 + 273\bar{t}^4 + 1015\bar{t}^6 + 45\bar{t}^8\right)}{4\left(1+\bar{t}^2\right)^6} + \bar{t}\operatorname{arcsinh}(\bar{t})\Bigg(\frac{20\log(2)\left(15 - 25\bar{t}^2 + 2\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}} \\
&+ \frac{20\left(15 - 25\bar{t}^2 + 2\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}\log\left(\frac{b}{M}\right) + \frac{10\log\left(1+\bar{t}^2\right)\left(15 - 25\bar{t}^2 + 2\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}} - \frac{3597952 - 110745\pi^2 - 768000\gamma_E}{2560\left(1+\bar{t}^2\right)^{11/2}} \\
&- \frac{\left(14476672 - 1536000\gamma_E - 66645\pi^2\right)\bar{t}^2 + 8\left(1338544 - 441600\gamma_E - 52065\pi^2\right)\bar{t}^4 - 8\left(207472 - 38400\gamma_E + 2205\pi^2\right)\bar{t}^6}{7680\left(1+\bar{t}^2\right)^{11/2}}\Bigg)\Bigg] \\
&+ v^3\Bigg[\frac{218\log\left(1+\bar{t}^2\right)\bar{t}\left(3 - 2\bar{t}^2\right)}{15\left(1+\bar{t}^2\right)^{7/2}} + \frac{284\log(2)\bar{t}\left(-3 + 2\bar{t}^2\right)}{15\left(1+\bar{t}^2\right)^{7/2}} - \frac{436\bar{t}\left(-3 + 2\bar{t}^2\right)}{15\left(1+\bar{t}^2\right)^{7/2}}\log\left(\frac{b}{M}\right) - \frac{15\arctan(\bar{t})\left(7 - 43\bar{t}^2 + 10\bar{t}^4\right)}{4\left(1+\bar{t}^2\right)^{7/2}}
\end{aligned}
$$

$$-\frac{\bar{t}}{2400\left(1+\bar{t}^2\right)^{11/2}}\left(1249951-36480\gamma_E+9600\pi^2+\left(13993-48640\gamma_E+12800\pi^2\right)\bar{t}^2\right.$$

$$\left.+\left(-3433187+12160\gamma_E-3200\pi^2\right)\bar{t}^4+\left(524371+24320\gamma_E-6400\pi^2\right)\bar{t}^6\right)$$

$$+\frac{\operatorname{arcsinh}(\bar{t})\left(4076-182095\bar{t}^2+447999\bar{t}^4-52685\bar{t}^6-21575\bar{t}^8\right)}{240\left(1+\bar{t}^2\right)^6}$$

$$\left.\left.+\frac{\operatorname{arcsinh}(\bar{t})^2\left(-18759\bar{t}+48649\bar{t}^3+17393\bar{t}^5-6269\bar{t}^7-606\bar{t}^9\right)}{32\left(1+\bar{t}^2\right)^{13/2}}\right]\right\}$$

$$+\frac{qM^4}{b^5}\left\{\frac{1}{v^5}\left[-\frac{\bar{t}\left(7+2\bar{t}^2\right)}{2\left(1+\bar{t}^2\right)^3}+\frac{\operatorname{arcsinh}(\bar{t})\left(-17+58\bar{t}^2\right)}{2\left(1+\bar{t}^2\right)^{7/2}}+\frac{\operatorname{arcsinh}(\bar{t})^2\left(55\bar{t}-50\bar{t}^3\right)}{2\left(1+\bar{t}^2\right)^4}+\frac{\operatorname{arcsinh}(\bar{t})^3\left(3-24\bar{t}^2+8\bar{t}^4\right)}{2\left(1+\bar{t}^2\right)^{9/2}}\right]\right.$$

$$-\frac{1}{v^4}\left[\frac{2\operatorname{arcsinh}(\bar{t})\,\bar{t}\left(47-61\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^4}+\frac{\operatorname{arcsinh}(\bar{t})^2\left(23-191\bar{t}^2+74\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}-\frac{1-25\bar{t}^2-2\bar{t}^4}{\left(1+\bar{t}^2\right)^{7/2}}\right.$$

$$\left.-\frac{4\operatorname{arcsinh}(\bar{t})^3\,\bar{t}\left(7-15\bar{t}^2+2\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^5}\right]+\frac{1}{v^3}\left[\frac{3\bar{t}\left(33-101\bar{t}^2+6\bar{t}^4\right)}{4\left(1+\bar{t}^2\right)^4}+\frac{3\operatorname{arcsinh}(\bar{t})\left(119-723\bar{t}^2+138\bar{t}^4\right)}{4\left(1+\bar{t}^2\right)^{9/2}}\right.$$

$$\left.-\frac{\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(1783-2263\bar{t}^2-266\bar{t}^4\right)}{4\left(1+\bar{t}^2\right)^5}-\frac{\operatorname{arcsinh}(\bar{t})^3\left(99-993\bar{t}^2+272\bar{t}^4+104\bar{t}^6\right)}{4\left(1+\bar{t}^2\right)^{11/2}}\right]$$

$$+\frac{1}{v^2}\left[\frac{4\log(2)\left(13-62\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^{7/2}}+\frac{4\left(13-62\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^{7/2}}\log\left(\frac{b}{M}\right)+\frac{2\log\left(1+\bar{t}^2\right)\left(13-62\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^{7/2}}\right.$$

$$+\frac{-49696+9984\gamma_E+3015\pi^2+\left(160192-37632\gamma_E-9495\pi^2\right)\bar{t}^2-2\left(-15952+23808\gamma_E+6255\pi^2\right)\bar{t}^4+6336\bar{t}^6}{576\left(1+\bar{t}^2\right)^{9/2}}$$

$$+\frac{\operatorname{arcsinh}(\bar{t})^3\left(-354\bar{t}+736\bar{t}^3+230\bar{t}^5-60\bar{t}^7\right)}{\left(1+\bar{t}^2\right)^6}+\operatorname{arcsinh}(\bar{t})\left(\frac{8\log(2)\bar{t}\left(-53+52\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^4}+\frac{\left(8\bar{t}\left(-53+52\bar{t}^2\right)\right)}{3\left(1+\bar{t}^2\right)^4}\log\left(\frac{b}{M}\right)\right.$$

$$+\frac{4\log\left(1+\bar{t}^2\right)\bar{t}\left(-53+52\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^4}$$

$$\left.+\frac{\bar{t}\left(368480-40704\gamma_E-11133\pi^2-\left(348416+768\gamma_E+531\pi^2\right)\bar{t}^2+2\left(-81968+19968\gamma_E+5301\pi^2\right)\bar{t}^4\right)}{288\left(1+\bar{t}^2\right)^5}\right)$$

$$+\operatorname{arcsinh}(\bar{t})^2\left(-\frac{4\log(2)\left(3-24\bar{t}^2+8\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}-\frac{4\left(3-24\bar{t}^2+8\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}\log\left(\frac{b}{M}\right)-\frac{2\log\left(1+\bar{t}^2\right)\left(3-24\bar{t}^2+8\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}\right.$$

$$+\frac{1}{192\left(1+\bar{t}^2\right)^{11/2}}\left(51360-2304\gamma_E-621\pi^2+9\left(-52672+1792\gamma_E+483\pi^2\right)\bar{t}^2+16\left(1534+768\gamma_E+207\pi^2\right)\bar{t}^4\right.$$

$$\left.\left.\left.-8\left(-11144+768\gamma_E+207\pi^2\right)\bar{t}^6\right)\right)\right]+\frac{1}{v}\left[\frac{15\arctan(\bar{t})\left(4-11\bar{t}^2\right)}{2\left(1+\bar{t}^2\right)^3}+\frac{\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(41209-69460\bar{t}^2-25027\bar{t}^4+2482\bar{t}^6\right)}{16\left(1+\bar{t}^2\right)^6}\right.$$

$$+\frac{133\bar{t}+11624\bar{t}^3-27311\bar{t}^5-1002\bar{t}^7}{48\left(1+\bar{t}^2\right)^5}+\frac{\operatorname{arcsinh}(\bar{t})^3\left(2509-30854\bar{t}^2+5189\bar{t}^4+11336\bar{t}^6+504\bar{t}^8\right)}{16\left(1+\bar{t}^2\right)^{13/2}}$$

$$\left.-\operatorname{arcsinh}(\bar{t})\left(\frac{45\arctan(\bar{t})\,\bar{t}\left(3-2\bar{t}^2\right)}{2\left(1+\bar{t}^2\right)^{7/2}}+\frac{14557-136736\bar{t}^2+57977\bar{t}^4+5150\bar{t}^6}{48\left(1+\bar{t}^2\right)^{11/2}}\right)\right]+$$

$$+\frac{\arctan(\bar{t})\left(285\bar{t}-255\bar{t}^3\right)}{\left(1+\bar{t}^2\right)^{7/2}}+\frac{\log\left(1+\bar{t}^2\right)\left(-13+655\bar{t}^2-802\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}-\frac{2\log(2)\left(53-719\bar{t}^2+698\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}$$

$$-\frac{2\left(13-655\bar{t}^2+802\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}\log\left(\frac{b}{M}\right)+\frac{1}{15360\left(1+\bar{t}^2\right)^{11/2}}\left(3\left(-22130816+6696960\gamma_E+926865\pi^2\right)\bar{t}^2\right.$$

$$\left.-24\left(-205168+80640\gamma_E-91575\pi^2\right)\bar{t}^4-8\left(-6282608+2880000\gamma_E+140805\pi^2\right)\bar{t}^6-810240\bar{t}^8\right)$$

$$+\frac{5559424-1013760\gamma_E-543645\pi^2}{15360\left(1+\bar{t}^2\right)^{11/2}}+\frac{\operatorname{arcsinh}(\bar{t})^3\,\bar{t}\left(3549-8587\bar{t}^2-4935\bar{t}^4+727\bar{t}^6+246\bar{t}^8\right)}{2\left(1+\bar{t}^2\right)^7}$$

$$+\operatorname{arcsinh}(\bar{t})\left(\frac{30\arctan(\bar{t})\left(2-13\bar{t}^2+3\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^4}+\frac{4\log(2)\bar{t}\left(517-1161\bar{t}^2+212\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^5}+\frac{4\bar{t}\left(477-1169\bar{t}^2+244\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^5}\log\left(\frac{b}{M}\right)\right.$$

$$
\begin{aligned}
&+\frac{\log\left(1+\bar{t}^2\right)\left(954\bar{t}-2338\bar{t}^3+488\bar{t}^5\right)}{\left(1+\bar{t}^2\right)^5}+\frac{\bar{t}}{7680\left(1+\bar{t}^2\right)^6}\Bigg(\left(104246656-20520960\gamma_E-1067805\pi^2\right)\bar{t}^2\\
&-8\left(-10632112+3598080\gamma_E+512415\pi^2\right)\bar{t}^4+8\left(-2685296+875520\gamma_E-29565\pi^2\right)\bar{t}^6\Bigg)\\
&-\frac{\bar{t}(66804608-15267840\gamma_E-2794995\pi^2)}{7680\left(1+\bar{t}^2\right)^6}\Bigg)+\operatorname{arcsinh}(\bar{t})^2\Bigg(\frac{2\log(2)\left(93-1101\bar{t}^2+704\bar{t}^4+8\bar{t}^6\right)}{\left(1+\bar{t}^2\right)^{11/2}}\\
&+\frac{2\left(93-1101\bar{t}^2+704\bar{t}^4+8\bar{t}^6\right)}{\left(1+\bar{t}^2\right)^{11/2}}\log\left(\frac{b}{M}\right)+\frac{\log\left(1+\bar{t}^2\right)\left(93-1101\bar{t}^2+704\bar{t}^4+8\bar{t}^6\right)}{\left(1+\bar{t}^2\right)^{11/2}}\\
&+\frac{3(-2363264+53475\pi^2+317440\gamma_E}{5120\left(1+\bar{t}^2\right)^{13/2}}+\frac{-3\left(-83107712+10321920\gamma_E+1472175\pi^2\right)\bar{t}^2}{15360\left(1+\bar{t}^2\right)^{13/2}}\\
&+\frac{1}{15360\left(1+\bar{t}^2\right)^{13/2}}\Bigg(-16\left(-226664+762240\gamma_E+211725\pi^2\right)\bar{t}^4+8\left(-14566288+2734080\gamma_E+247275\pi^2\right)\bar{t}^6\\
&+32\left(-137288+7680\gamma_E+14625\pi^2\right)\bar{t}^8\Bigg)\Bigg)+v\Bigg[\frac{4\bar{t}\left(387-376\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^4}\log\left(\frac{b}{M}\right)+\frac{28\log(2)\bar{t}\left(-39+32\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^4}\\
&+\frac{\log\left(1+\bar{t}^2\right)\left(774\bar{t}-752\bar{t}^3\right)}{3\left(1+\bar{t}^2\right)^4}-\frac{15\arctan(\bar{t})\left(44-291\bar{t}^2+85\bar{t}^4\right)}{4\left(1+\bar{t}^2\right)^4}-\bar{t}\frac{3218233-109440\gamma_E+28800\pi^2}{1440\left(1+\bar{t}^2\right)^6}\\
&+\frac{\bar{t}}{1440\left(1+\bar{t}^2\right)^6}\Bigg(3\left(601841+24320\gamma_E-6400\pi^2\right)\bar{t}^2+\left(8858883-182400\gamma_E+48000\pi^2\right)\bar{t}^4\\
&+\left(-970063-145920\gamma_E+38400\pi^2\right)\bar{t}^6-148230\bar{t}^8\Bigg)+\operatorname{arcsinh}(\bar{t})\Bigg(-\frac{45\arctan(\bar{t})\,\bar{t}\left(-63+79\bar{t}^2+2\bar{t}^4\right)}{4\left(1+\bar{t}^2\right)^{9/2}}\\
&-\frac{284\log(2)\left(3-24\bar{t}^2+8\bar{t}^4\right)}{15\left(1+\bar{t}^2\right)^{9/2}}+\frac{436\left(3-24\bar{t}^2+8\bar{t}^4\right)}{15\left(1+\bar{t}^2\right)^{9/2}}\log\left(\frac{b}{M}\right)+\frac{218\log\left(1+\bar{t}^2\right)\left(3-24\bar{t}^2+8\bar{t}^4\right)}{15\left(1+\bar{t}^2\right)^{9/2}}\\
&+\frac{9\left(45579-24320\gamma_E+6400\pi^2\right)\bar{t}^2+\left(26120659-449920\gamma_E+118400\pi^2\right)\bar{t}^4-\left(13035679+97280\gamma_E-25600\pi^2\right)\bar{t}^6}{2400\left(1+\bar{t}^2\right)^{13/2}}\\
&-\frac{\left(845043-48640\gamma_E+12800\pi^2\right)\bar{t}^8}{1200\left(1+\bar{t}^2\right)^{13/2}}-\frac{465641-36480\gamma_E+9600\pi^2}{2400\left(1+\bar{t}^2\right)^{13/2}}\Bigg)\\
&-\frac{\operatorname{arcsinh}(\bar{t})^2\left(2834057\bar{t}-7547311\bar{t}^3-1517703\bar{t}^5+324475\bar{t}^7+109450\bar{t}^9\right)}{480\left(1+\bar{t}^2\right)^7}\\
&-\frac{\operatorname{arcsinh}(\bar{t})^3\left(15475-242995\bar{t}^2+71705\bar{t}^4-165559\bar{t}^6+25760\bar{t}^8-1816\bar{t}^{10}\right)}{32\left(1+\bar{t}^2\right)^{15/2}}\Bigg]\Bigg\}.
\end{aligned}
\tag{4.36}
$$

Non-local terms

The non-local terms are, on the other hand, extremely complicated and the integrals are highly non-trivial. The reason is the appearance of this PolyLog structure that is well known from computations at very high order in perturbation theory performed in high-energy physics. It is more instructive to look at the explicit result for the non-local field and then provide some comments:

$$
\begin{aligned}
\psi^{\mathrm{R,NL}}(\bar{t})=\frac{qM^2}{b^3}\Bigg\{&v^2\Bigg[-\frac{40\log(2)\bar{t}^2}{3\left(1+\bar{t}^2\right)^{5/2}}-\frac{8\operatorname{arcsinh}(\bar{t})\left(-1+2\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^{5/2}}+\frac{8\left(-1+2\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^{5/2}}\log\left(\frac{b}{M}\right)\\
&-\frac{8\log(v)\left(-1+2\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^{5/2}}-\frac{8\log\left(1+\bar{t}^2\right)\left(-1+2\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^{5/2}}+\frac{8\log(2)\left(1+3\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^{5/2}}+\frac{4\left(-3-2\gamma_E+2(5+2\gamma_E)\bar{t}^2+6\bar{t}\sqrt{1+\bar{t}^2}\right)}{3\left(1+\bar{t}^2\right)^{5/2}}\Bigg]\\
&+v^4\Bigg[-\frac{4\operatorname{arcsinh}(\bar{t})\left(7-61\bar{t}^2+22\bar{t}^4\right)}{3\left(1+\bar{t}^2\right)^{7/2}}+\frac{4\left(7-61\bar{t}^2+22\bar{t}^4\right)}{3\left(1+\bar{t}^2\right)^{7/2}}\log\left(\frac{b}{M}\right)-\frac{4\log(2v)\left(7-61\bar{t}^2+22\bar{t}^4\right)}{3\left(1+\bar{t}^2\right)^{7/2}}\\
&-\frac{4\log\left(1+\bar{t}^2\right)\left(7-61\bar{t}^2+22\bar{t}^4\right)}{3\left(1+\bar{t}^2\right)^{7/2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2\left(169+70\gamma_E-(1749+610\gamma_E)\bar{t}^2+(842+220\gamma_E)\bar{t}^4-450\bar{t}\sqrt{1+\bar{t}^2}+450\bar{t}^3\sqrt{1+\bar{t}^2}\right)}{15\left(1+\bar{t}^2\right)^{7/2}}\Bigg]\Bigg\}\\
&+\frac{qM^3}{b^4}\Bigg\{\frac{\log\left(1+\bar{t}^2\right)\left(4-16\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^3}-\frac{4\log(2)}{3\left(1+\bar{t}^2\right)^2}+\frac{8\text{arcsinh}\left(\bar{t}\right)^2\bar{t}\left(-3+2\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^{7/2}}+\frac{4\gamma_E\left(-7+23\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^3}\\
&+\frac{4\left(-7+23\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^3}\log\left(\frac{b}{M}\right)-\frac{4\log(v)\left(-7+23\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^3}+\frac{52\bar{t}-8\bar{t}^3}{3\left(1+\bar{t}^2\right)^{5/2}}+\frac{\pi^2\left(6\bar{t}-4\bar{t}^3\right)}{3\left(1+\bar{t}^2\right)^{7/2}}-\frac{4\left(7-38\bar{t}^2+2\bar{t}^4\right)}{3\left(1+\bar{t}^2\right)^3}\\
&+\frac{\pi\bar{t}\left(2679-2338\bar{t}^2-793\bar{t}^4+324\bar{t}^6+300\bar{t}^8\right)}{420\left(1+\bar{t}^2\right)^3}+\frac{\arctan\left(\bar{t}\right)\bar{t}\left(2679-2338\bar{t}^2-793\bar{t}^4+324\bar{t}^6+300\bar{t}^8\right)}{210\left(1+\bar{t}^2\right)^3}\\
&+\frac{\arctan\left(\bar{t}+\sqrt{1+\bar{t}^2}\right)\left(-2679\bar{t}+2338\bar{t}^3+793\bar{t}^5-324\bar{t}^7-300\bar{t}^9\right)}{105\left(1+\bar{t}^2\right)^3}+\text{arcsinh}\left(\bar{t}\right)\Bigg(\frac{4\bar{t}\left(39-38\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^{7/2}}\\
&+\frac{8\gamma_E\bar{t}\left(3-2\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^{7/2}}+\frac{8\bar{t}\left(3-2\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^{7/2}}\log\left(\frac{b}{M}\right)-\frac{8\log(2)\bar{t}\left(3-2\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^{7/2}}-\frac{8\log(v)\bar{t}\left(3-2\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^{7/2}}-\frac{8\log\left(1+\bar{t}^2\right)\bar{t}\left(3-2\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^{7/2}}\\
&-\frac{4\left(-7+23\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^3}\Bigg)+\frac{4\bar{t}\left(-3+2\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^{7/2}}\text{Li}_2\left(\frac{\bar{t}+\sqrt{1+\bar{t}^2}}{\bar{t}-\sqrt{1+\bar{t}^2}}\right)+v^2\Bigg[\frac{\log\left(1+\bar{t}^2\right)\left(26-96\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^3}-\frac{8\pi\bar{t}\left(-3+\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^3}\\
&-\frac{16\arctan\left(\bar{t}\right)\bar{t}\left(-3+\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^3}+\frac{78\bar{t}-12\bar{t}^3}{\left(1+\bar{t}^2\right)^{5/2}}+\frac{\log(v)\left(-38+480\bar{t}^2-322\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^4}+\frac{20\text{arcsinh}\left(\bar{t}\right)^2\bar{t}\left(15-25\bar{t}^2+2\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}\\
&+\frac{10\log(2)\left(9-62\bar{t}^2+13\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^4}+\frac{\gamma_E\left(38-480\bar{t}^2+322\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^4}+\frac{\left(38-480\bar{t}^2+322\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^4}\log\left(\frac{b}{M}\right)-\frac{\pi^2\left(39\bar{t}-87\bar{t}^3+14\bar{t}^5\right)}{\left(1+\bar{t}^2\right)^{9/2}}\\
&-\frac{2\left(-8+253\bar{t}^2-301\bar{t}^4+6\bar{t}^6\right)}{\left(1+\bar{t}^2\right)^4}+\text{arcsinh}\left(\bar{t}\right)\Bigg(\frac{2\left(-17+13\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^3}-\frac{20\gamma_E\bar{t}\left(15-25\bar{t}^2+2\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}\\
&+\frac{20\log(2)\bar{t}\left(15-25\bar{t}^2+2\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}-\frac{20\bar{t}\left(15-25\bar{t}^2+2\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}\log\left(\frac{b}{M}\right)+\frac{20\log(v)\bar{t}\left(15-25\bar{t}^2+2\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}\\
&+\frac{20\log\left(1+\bar{t}^2\right)\bar{t}\left(15-25\bar{t}^2+2\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}-\frac{2\bar{t}\left(5991-13303\bar{t}^2+1826\bar{t}^4\right)}{15\left(1+\bar{t}^2\right)^{9/2}}\Bigg)+\frac{6\bar{t}\left(39-87\bar{t}^2+14\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}\text{Li}_2\left(\frac{\bar{t}+\sqrt{1+\bar{t}^2}}{\bar{t}-\sqrt{1+\bar{t}^2}}\right)\Bigg]\\
&+v^3\Bigg[\frac{76\gamma_E\bar{t}\left(-3+2\bar{t}^2\right)}{15\left(1+\bar{t}^2\right)^{7/2}}-\frac{76\text{arcsinh}\left(\bar{t}\right)\bar{t}\left(-3+2\bar{t}^2\right)}{15\left(1+\bar{t}^2\right)^{7/2}}-\frac{76\log(2)\bar{t}\left(-3+2\bar{t}^2\right)}{15\left(1+\bar{t}^2\right)^{7/2}}+\frac{76\bar{t}\left(-3+2\bar{t}^2\right)}{15\left(1+\bar{t}^2\right)^{7/2}}\log\left(\frac{b}{M}\right)\\
&-\frac{76\log(v)\bar{t}\left(-3+2\bar{t}^2\right)}{15\left(1+\bar{t}^2\right)^{7/2}}-\frac{76\log\left(1+\bar{t}^2\right)\bar{t}\left(-3+2\bar{t}^2\right)}{15\left(1+\bar{t}^2\right)^{7/2}}+\frac{76\left(-4+11\bar{t}^2\right)}{45\left(1+\bar{t}^2\right)^3}+\frac{76\bar{t}\left(-15+16\bar{t}^2\right)}{45\left(1+\bar{t}^2\right)^{7/2}}\Bigg]\Bigg\}+O(b^{-5}),
\end{aligned}
$$

where the expressions at b^{-5} have not been included because they are extremely large. Besides, the structure of the b^{-3} and b^{-4} terms is already sufficient for discussing the analytical structure of these functions. It is however possible to see these expressions in the ancillary file of [100].

By looking at the explicit result the special functions hierarchy becomes clear and it can be categorized by looking at each PN-PM order. Namely, one has

- 3PM: rational functions in $(1+\bar{t}^2)^{n/2}$, $\text{arcsinh}(\bar{t})$ and $\log\big(1+\bar{t}^2\big)$;
- 4PM: rational functions in $(1+\bar{t}^2)^{n/2}$, $\log\big(1+\bar{t}^2\big)$, $\text{arcsinh}(\bar{t})$ and $\text{arcsinh}(\bar{t})^2$, but also $\arctan(\bar{t})$ and $\text{Li}_2\left(\frac{\bar{t}+\sqrt{1+\bar{t}^2}}{\bar{t}-\sqrt{1+\bar{t}^2}}\right)$. Together with these functions also cross products between the functions already present at 3PM are present, in particular products between logarithms and arcsinh;
- 5PM: all the functions that are present at 4PM are also at the successive order, in addition one could find $\text{arcsinh}(\bar{t})^3$, cross products between all the functions arcsinh and $\arctan$, but also between Li_2 and arcsinh or $\log$. Also Li_3 starts to appear at this order;

thanks to this observation it is possible to understand that the PolyLog structure becomes richer and richer as one goes at higher order. These structure will present, at some point, also products of Li_2 with Li_3 as also $(\mathrm{Li}_2)^2$. The presence of such class of functions is very new to SF calculations, that never encountered such special functions in the analytical calculations for bound orbits.

4.2.3 Scalar Self-Force

The Scalar Self-Force (per unit mass) acting on a particle with scalar charge q and mass μ is given by [26]

$$F_\alpha = \frac{q}{\mu} P_\alpha{}^\beta \nabla_\beta \psi^{\mathrm{R}}, \tag{4.37}$$

where ψ^{R} is the Detweiler-Whiting regular field defined before and $P_\alpha{}^\beta = \delta_\alpha^\beta + u_\alpha u^\beta$ projects orthogonally to the particle's worldline. At leading-order, i.e. linear corrections due to the force, the projector can be approximated as $P_\alpha{}^\beta$ by $\hat{P}_\alpha{}^\beta = \delta_\alpha^\beta + \hat{u}_\alpha \hat{u}^\beta$. Similarly, although μ is not constant [26], it can be treated as constant in this leading-order analysis.

For equatorial scattering, the non-vanishing components of F_α are

$$F_t = \frac{q}{\mu}\left(1 - \hat{u}_t^2 f(r)\right)\partial_t\psi^{\mathrm{R}} - \frac{q}{\mu} f(r)\hat{u}_t\left(\hat{u}_r\partial_r\psi^{\mathrm{R}} + \hat{u}_\phi\partial_\phi\psi^{\mathrm{R}}\right), \tag{4.38}$$

$$F_r = \frac{q}{\mu}(1 + \hat{u}_r^2 f^{-1}(r))\partial_r\psi^{\mathrm{R}} + \frac{q}{\mu}\hat{u}_r f^{-1}(r)\left(\hat{u}_t\partial_t\psi^{\mathrm{R}} + \hat{u}_\phi\partial_\phi\psi^{\mathrm{R}}\right), \tag{4.39}$$

$$F_\phi = \frac{q}{\mu}(1 + r^2\hat{u}_\phi^2)\partial_\phi\psi^{\mathrm{R}} + \frac{q}{\mu}\hat{u}_\phi r^2\left(\hat{u}_t\partial_t\psi^{\mathrm{R}} + \hat{u}_r\partial_r\psi^{\mathrm{R}}\right). \tag{4.40}$$

Practically speaking, only the t and ϕ components will be used for the computation of the conservative and dissipative dynamics.

To compute F_α from the retarded field modes ψ_{lm}, one should use again the method of mode-sum regularization, as explained above Eq. (4.33). The derivatives of the retarded field possess a linear-in-l divergence on the particle, and the mode-sum formula for the force is

$$F_\alpha(t) = \sum_l \left\{F_\alpha^{l\pm}(t) - (l + 1/2)A_\alpha^{l\pm}(t) - B_\alpha(t)\right\}, \tag{4.41}$$

where $F_\alpha^{l\pm}$ denotes the result of substituting a retarded-field mode $\psi_{lm}Y_{lm}$ into Eqs. (4.38)–(4.40), evaluating on the particle, and summing over m. The $\pm$ indicates[4] whether the particle is approached from the right or left, i.e. $r_p^+(t)$ or $r_p^-(t)$, when the limit $r \to r_p(t)$ is performed. It is well known [148] that by computing the average mode contribution from the two directions, as in

$$F_\alpha^l(t) = \frac{1}{2}\left[F_\alpha^{l+}(t) + F_\alpha^{l-}(t)\right], \tag{4.42}$$

then the contribution from $A_\alpha^{l\pm}(t)$ vanish and only the $B_\alpha(t)$ terms are required in the mode-sum formula. For this reason, the averaged components of the force will be presented in the following.

As already mentioned, these calculations require to compute derivatives of the retarded field modes. The derivatives that come from the local sector are trivial to compute. However, the ones that comes from the non-local sector could be problematic, in particular the derivative with respect

[4] It is important to remind that this $\pm$ is not used to indicate if the particle is approaching/departing from the primary.

to t when the field is written in the form of (superficially) divergent integrals. However, it is easy to realize that for an integral of the form

$$\frac{\partial I}{\partial t} = \frac{\partial}{\partial t} \int dt' f(r,t') \int \frac{d\omega}{2\pi} e^{i\omega(t'-t)} \omega^j \log(-i\omega), \tag{4.43}$$

where $f(r,t') = {}_0G^{(j,1)}_{\ell m}(r, r(t'))e^{-im\phi(t')}$, and hence

$$\partial_t I = -i \int dt' f(r,t') \int \frac{d\omega}{2\pi} e^{i\omega(t'-t)} \omega^{j+1} \log(-i\omega)\,, \tag{4.44}$$

which ensures that the formalism employed for dealing with the non-local sector of the field could be easily readapted.

As done for the field, the following explicit expressions will be presented: regularization parameters for $B_t(\bar{t})$ and $B_\phi(\bar{t})$, local and non-local sector for both $F_t(\bar{t})$ and $F_\phi(\bar{t})$.

Local terms

The regularization parameters are

$$\begin{aligned}
B_t = {} & \frac{\varepsilon M}{b^2} \Bigg\{ \frac{v\bar{t}}{2\left(1+\bar{t}^2\right)^{3/2}} + \frac{v^3\bar{t}\left(7+4\bar{t}^2\right)}{8\left(1+\bar{t}^2\right)^{5/2}} + \frac{v^5\bar{t}\left(25+224\bar{t}^2+64\bar{t}^4\right)}{128\left(1+\bar{t}^2\right)^{7/2}} + \frac{v^7\bar{t}\left(87+300\bar{t}^2+1344\bar{t}^4+256\bar{t}^6\right)}{512\left(1+\bar{t}^2\right)^{9/2}} \\
& + \frac{v^9\bar{t}\left(5001+22272\bar{t}^2+38400\bar{t}^4+114688\bar{t}^6+16384\bar{t}^8\right)}{32768\left(1+\bar{t}^2\right)^{11/2}} \Bigg\} + \frac{\varepsilon M^2}{b^3} \Bigg\{ \frac{1}{v} \Bigg[\frac{\operatorname{arcsinh}(\bar{t})\left(1-2\bar{t}^2\right)}{2\left(1+\bar{t}^2\right)^{5/2}} + \frac{3\bar{t}}{2\left(1+\bar{t}^2\right)^2} \Bigg] \\
& + v \Bigg[\frac{\bar{t}\left(23+8\bar{t}^2\right)}{8\left(1+\bar{t}^2\right)^3} - \frac{\operatorname{arcsinh}(\bar{t})\left(5+4\bar{t}^2-16\bar{t}^4\right)}{8\left(1+\bar{t}^2\right)^{7/2}} \Bigg] \\
& + v^3 \Bigg[\frac{\bar{t}\left(-385+688\bar{t}^2+128\bar{t}^4\right)}{128\left(1+\bar{t}^2\right)^4} + \frac{\operatorname{arcsinh}(\bar{t})\left(-311+954\bar{t}^2+576\bar{t}^4+256\bar{t}^6\right)}{128\left(1+\bar{t}^2\right)^{9/2}} \Bigg] \\
& + v^5 \Bigg[\frac{\bar{t}\left(83-4272\bar{t}^2+4032\bar{t}^4+512\bar{t}^6\right)}{512\left(1+\bar{t}^2\right)^5} + \frac{\operatorname{arcsinh}(\bar{t})\left(-213-6360\bar{t}^2+5568\bar{t}^4+4864\bar{t}^6+1024\bar{t}^8\right)}{512\left(1+\bar{t}^2\right)^{11/2}} \Bigg] \\
& + v^7 \Bigg[\frac{\bar{t}\left(4899+22080\bar{t}^2-524544\bar{t}^4+339968\bar{t}^6+32768\bar{t}^8\right)}{32768\left(1+\bar{t}^2\right)^6} \\
& + \frac{\operatorname{arcsinh}(\bar{t})\left(-11703-39066\bar{t}^2-969984\bar{t}^4+315904\bar{t}^6+475136\bar{t}^8+65536\bar{t}^{10}\right)}{32768\left(1+\bar{t}^2\right)^{13/2}} \Bigg] \Bigg\} \\
& + \frac{\varepsilon M^3}{b^4} \Bigg\{ \frac{1}{v^3} \Bigg[\frac{\operatorname{arcsinh}(\bar{t})\left(4-11\bar{t}^2\right)}{2\left(1+\bar{t}^2\right)^3} + \frac{9\bar{t}}{4\left(1+\bar{t}^2\right)^{5/2}} + \frac{3\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(-3+2\bar{t}^2\right)}{4\left(1+\bar{t}^2\right)^{7/2}} \Bigg] \\
& + \frac{1}{v} \Bigg[\frac{\bar{t}\left(127+52\bar{t}^2\right)}{16\left(1+\bar{t}^2\right)^{7/2}} - \frac{15\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(-9-8\bar{t}^2+8\bar{t}^4\right)}{16\left(1+\bar{t}^2\right)^{9/2}} + \frac{\operatorname{arcsinh}(\bar{t})\left(-30-11\bar{t}^2+124\bar{t}^4\right)}{8\left(1+\bar{t}^2\right)^4} \Bigg] \\
& + v \Bigg[\frac{15\arctan(\bar{t})\left(-1+2\bar{t}^2\right)}{4\left(1+\bar{t}^2\right)^{5/2}} + \frac{3\bar{t}\left(-1021+1440\bar{t}^2+256\bar{t}^4\right)}{256\left(1+\bar{t}^2\right)^{9/2}} + \frac{\operatorname{arcsinh}(\bar{t})\left(-1560+9409\bar{t}^2+2720\bar{t}^4+256\bar{t}^6\right)}{128\left(1+\bar{t}^2\right)^5} \\
& + \frac{3\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(1137-3234\bar{t}^2-1024\bar{t}^4+512\bar{t}^6\right)}{256\left(1+\bar{t}^2\right)^{11/2}} \Bigg] + v^3 \Bigg[\frac{15\arctan(\bar{t})\left(-7+16\bar{t}^2+8\bar{t}^4\right)}{16\left(1+\bar{t}^2\right)^{7/2}} \\
& + \frac{3\bar{t}\left(1789-14028\bar{t}^2+8832\bar{t}^4+1024\bar{t}^6\right)}{1024\left(1+\bar{t}^2\right)^{11/2}} + \frac{\operatorname{arcsinh}(\bar{t})\left(8222-80127\bar{t}^2+13764\bar{t}^4+16512\bar{t}^6+1024\bar{t}^8\right)}{512\left(1+\bar{t}^2\right)^6} \\
& + \frac{3\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(-22287+25172\bar{t}^2+19608\bar{t}^4+3072\bar{t}^6+2048\bar{t}^8\right)}{1024\left(1+\bar{t}^2\right)^{13/2}} \Bigg] \\
& + v^5 \Bigg[\frac{\bar{t}\left(27329+1583360\bar{t}^2-5693184\bar{t}^4+2285568\bar{t}^6+196608\bar{t}^8\right)}{65536\left(1+\bar{t}^2\right)^{13/2}} + \frac{15\arctan(\bar{t})\left(-25-522\bar{t}^2+576\bar{t}^4+128\bar{t}^6\right)}{256\left(1+\bar{t}^2\right)^{9/2}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\operatorname{arcsinh}(\bar t)\left(18156+3823461\bar t^{2}-10647552\bar t^{4}-2065152\bar t^{6}+1417216\bar t^{8}+65536\bar t^{10}\right)}{32768\left(1+\bar t^{2}\right)^{7}}\\
&+\frac{3\operatorname{arcsinh}(\bar t)^{2}\,\bar t\left(688797-5574710\bar t^{2}-920320\bar t^{4}+2188800\bar t^{6}+655360\bar t^{8}+131072\bar t^{10}\right)}{65536\left(1+\bar t^{2}\right)^{15/2}}\Bigg]\Bigg\}\\
&+\frac{\varepsilon M^{4}}{b^{5}}\Bigg\{\frac{1}{v^{5}}\Bigg[\frac{\operatorname{arcsinh}(\bar t)\left(17-58\bar t^{2}\right)}{4\left(1+\bar t^{2}\right)^{7/2}}+\frac{\bar t\left(7+2\bar t^{2}\right)}{4\left(1+\bar t^{2}\right)^{3}}+\frac{5\operatorname{arcsinh}(\bar t)^{2}\,\bar t\left(-11+10\bar t^{2}\right)}{4\left(1+\bar t^{2}\right)^{4}}+\frac{\operatorname{arcsinh}(\bar t)^{3}\left(-3+24\bar t^{2}-8\bar t^{4}\right)}{4\left(1+\bar t^{2}\right)^{9/2}}\Bigg]\\
&+\frac{1}{v^{3}}\Bigg[\frac{\operatorname{arcsinh}(\bar t)\left(-137-136\bar t^{2}+736\bar t^{4}\right)}{16\left(1+\bar t^{2}\right)^{9/2}}+\frac{\operatorname{arcsinh}(\bar t)^{2}\left(1013\bar t+742\bar t^{3}-1216\bar t^{5}\right)}{16\left(1+\bar t^{2}\right)^{5}}+\frac{301\bar t+148\bar t^{3}-48\bar t^{5}}{16\left(1+\bar t^{2}\right)^{4}}\\
&+\frac{\operatorname{arcsinh}(\bar t)^{3}\left(81-492\bar t^{2}-632\bar t^{4}+256\bar t^{6}\right)}{16\left(1+\bar t^{2}\right)^{11/2}}\Bigg]+\frac{1}{v}\Bigg[\frac{15\arctan(\bar t)\left(-4+11\bar t^{2}\right)}{4\left(1+\bar t^{2}\right)^{3}}+\frac{\bar t\left(-5977+11846\bar t^{2}+7360\bar t^{4}+3712\bar t^{6}\right)}{256\left(1+\bar t^{2}\right)^{5}}\\
&+\frac{\operatorname{arcsinh}(\bar t)^{3}\left(-1023-11712\bar t^{2}+53392\bar t^{4}+23168\bar t^{6}-9728\bar t^{8}\right)}{256\left(1+\bar t^{2}\right)^{13/2}}+\operatorname{arcsinh}(\bar t)\Bigg(-\frac{45\arctan(\bar t)\,\bar t\left(-3+2\bar t^{2}\right)}{4\left(1+\bar t^{2}\right)^{7/2}}\\
&+\frac{-10359+83210\bar t^{2}+14464\bar t^{4}-2560\bar t^{6}}{256\left(1+\bar t^{2}\right)^{11/2}}\Bigg)+\frac{\operatorname{arcsinh}(\bar t)^{2}\,\bar t\left(20885-123774\bar t^{2}-21024\bar t^{4}+30080\bar t^{6}\right)}{256\left(1+\bar t^{2}\right)^{6}}\Bigg]\\
&+v\Bigg[\frac{15\arctan(\bar t)\left(-18+95\bar t^{2}+8\bar t^{4}\right)}{16\left(1+\bar t^{2}\right)^{4}}+\frac{\bar t\left(-14091-110904\bar t^{2}+183648\bar t^{4}+111104\bar t^{6}+32768\bar t^{8}\right)}{1024\left(1+\bar t^{2}\right)^{6}}\\
&+\frac{\operatorname{arcsinh}(\bar t)^{2}\left(-779727\bar t+1367022\bar t^{3}+1151184\bar t^{5}+126464\bar t^{7}-4096\bar t^{9}\right)}{1024\left(1+\bar t^{2}\right)^{7}}\\
&+\frac{\operatorname{arcsinh}(\bar t)^{3}\left(-35931+582180\bar t^{2}-149840\bar t^{4}-501120\bar t^{6}-92160\bar t^{8}+16384\bar t^{10}\right)}{1024\left(1+\bar t^{2}\right)^{15/2}}\\
&+\operatorname{arcsinh}(\bar t)\left(\frac{45\arctan(\bar t)\,\bar t\left(-9-28\bar t^{2}+16\bar t^{4}\right)}{16\left(1+\bar t^{2}\right)^{9/2}}+\frac{100807-927924\bar t^{2}+18096\bar t^{4}+99072\bar t^{6}+5120\bar t^{8}}{1024\left(1+\bar t^{2}\right)^{13/2}}\right)\Bigg]\\
&+\frac{1}{v^{3}}\Bigg[\frac{\bar t\left(-1202385+495770\bar t^{2}-17506048\bar t^{4}+24160256\bar t^{6}+12353536\bar t^{8}+2097152\bar t^{10}\right)}{65536\left(1+\bar t^{2}\right)^{7}}\\
&+\frac{15\arctan(\bar t)\left(696-5713\bar t^{2}+2224\bar t^{4}+128\bar t^{6}\right)}{256\left(1+\bar t^{2}\right)^{5}}+\operatorname{arcsinh}(\bar t)\Bigg(\frac{45\arctan(\bar t)\,\bar t\left(-1569+1458\bar t^{2}+448\bar t^{4}+256\bar t^{6}\right)}{256\left(1+\bar t^{2}\right)^{11/2}}\\
&+\frac{-2929831+71920254\bar t^{2}-103987712\bar t^{4}-26964480\bar t^{6}+8732672\bar t^{8}+327680\bar t^{10}}{65536\left(1+\bar t^{2}\right)^{15/2}}\Bigg)\\
&-\frac{\operatorname{arcsinh}(\bar t)^{2}\,\bar t\left(-72008697+221559966\bar t^{2}+96764544\bar t^{4}-40930304\bar t^{6}-9404416\bar t^{8}+262144\bar t^{10}\right)}{65536\left(1+\bar t^{2}\right)^{8}}\\
&+\frac{\operatorname{arcsinh}(\bar t)^{3}\left(4967901-87936504\bar t^{2}+25953160\bar t^{4}+80020480\bar t^{6}+14376960\bar t^{8}-917504\bar t^{10}+1048576\bar t^{12}\right)}{65536\left(1+\bar t^{2}\right)^{17/2}}\Bigg]\Bigg\},
\end{aligned}
\tag{4.45}
$$

and

$$
\begin{aligned}
B_{\phi}=\;&\frac{\varepsilon M}{b}\Bigg\{-\frac{3v^{2}\bar t}{4\left(1+\bar t^{2}\right)^{3/2}}-\frac{3v^{4}\bar t\left(3+8\bar t^{2}\right)}{32\left(1+\bar t^{2}\right)^{5/2}}-\frac{3v^{6}\bar t\left(19+48\bar t^{2}+64\bar t^{4}\right)}{256\left(1+\bar t^{2}\right)^{7/2}}-\frac{3v^{8}\bar t\left(259+912\bar t^{2}+1152\bar t^{4}+1024\bar t^{6}\right)}{4096\left(1+\bar t^{2}\right)^{9/2}}\Bigg\}\\
&+\frac{\varepsilon M^{2}}{b^{2}}\Bigg\{-\frac{9\bar t}{4\left(1+\bar t^{2}\right)^{2}}+\frac{3\operatorname{arcsinh}(\bar t)\left(-1+2\bar t^{2}\right)}{4\left(1+\bar t^{2}\right)^{5/2}}+v^{2}\Bigg[-\frac{3\bar t\left(-9+16\bar t^{2}\right)}{32\left(1+\bar t^{2}\right)^{3}}-\frac{3\operatorname{arcsinh}(\bar t)\left(-21+36\bar t^{2}+32\bar t^{4}\right)}{32\left(1+\bar t^{2}\right)^{7/2}}\Bigg]\\
&+v^{4}\Bigg[-\frac{3\bar t\left(13-104\bar t^{2}+128\bar t^{4}\right)}{256\left(1+\bar t^{2}\right)^{4}}-\frac{3\operatorname{arcsinh}(\bar t)\left(-53-330\bar t^{2}+224\bar t^{4}+256\bar t^{6}\right)}{256\left(1+\bar t^{2}\right)^{9/2}}\Bigg]\\
&+v^{6}\Bigg[-\frac{3\bar t\left(203+704\bar t^{2}-2176\bar t^{4}+2048\bar t^{6}\right)}{4096\left(1+\bar t^{2}\right)^{5}}+\frac{3\operatorname{arcsinh}(\bar t)\left(653+1688\bar t^{2}+7296\bar t^{4}-2560\bar t^{6}-4096\bar t^{8}\right)}{4096\left(1+\bar t^{2}\right)^{11/2}}\Bigg]\Bigg\}\\
&+\frac{\varepsilon M^{3}}{b^{3}}\Bigg\{\frac{1}{v^{2}}\Bigg[-\frac{27\bar t}{8\left(1+\bar t^{2}\right)^{5/2}}-\frac{9\operatorname{arcsinh}(\bar t)^{2}\,\bar t\left(-3+2\bar t^{2}\right)}{8\left(1+\bar t^{2}\right)^{7/2}}+\frac{3\operatorname{arcsinh}(\bar t)\left(-4+11\bar t^{2}\right)}{4\left(1+\bar t^{2}\right)^{3}}\Bigg]-\frac{3\bar t\left(-21+104\bar t^{2}\right)}{64\left(1+\bar t^{2}\right)^{7/2}}\\
&+\frac{15\operatorname{arcsinh}(\bar t)^{2}\,\bar t\left(-87-4\bar t^{2}+48\bar t^{4}\right)}{64\left(1+\bar t^{2}\right)^{9/2}}-\frac{3\operatorname{arcsinh}(\bar t)\left(-126+297\bar t^{2}+248\bar t^{4}\right)}{32\left(1+\bar t^{2}\right)^{4}}
\end{aligned}
$$

$$+ v^2 \Bigg[- \frac{45\arctan(\bar{t})\left(-1+2\bar{t}^2\right)}{8\left(1+\bar{t}^2\right)^{5/2}} - \frac{3\bar{t}\left(163-784\bar{t}^2+768\bar{t}^4\right)}{512\left(1+\bar{t}^2\right)^{9/2}} - \frac{3\operatorname{arcsinh}(\bar{t})\left(680-3109\bar{t}^2-1328\bar{t}^4+256\bar{t}^6\right)}{256\left(1+\bar{t}^2\right)^{5}}$$

$$- \frac{3\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(-5439-802\bar{t}^2+3968\bar{t}^4+1536\bar{t}^6\right)}{512\left(1+\bar{t}^2\right)^{11/2}} \Bigg] + v^4 \Bigg[\frac{45\arctan(\bar{t})\left(3+12\bar{t}^2-16\bar{t}^4\right)}{64\left(1+\bar{t}^2\right)^{7/2}}$$

$$- \frac{9\bar{t}\left(401+3568\bar{t}^2-6912\bar{t}^4+4096\bar{t}^6\right)}{8192\left(1+\bar{t}^2\right)^{11/2}} - \frac{3\operatorname{arcsinh}(\bar{t})\left(1470+35701\bar{t}^2-54352\bar{t}^4-32512\bar{t}^6+4096\bar{t}^8\right)}{4096\left(1+\bar{t}^2\right)^{6}}$$

$$- \frac{9\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(4197-53032\bar{t}^2-35808\bar{t}^4+12288\bar{t}^6+8192\bar{t}^8\right)}{8192\left(1+\bar{t}^2\right)^{13/2}} \Bigg] \Bigg\}$$

$$+ \frac{\varepsilon M^4}{b^4} \Bigg\{ \frac{1}{v^4} \Bigg[- \frac{3\bar{t}\left(7+2\bar{t}^2\right)}{8\left(1+\bar{t}^2\right)^{3}} - \frac{15\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(-11+10\bar{t}^2\right)}{8\left(1+\bar{t}^2\right)^{4}} + \frac{3\operatorname{arcsinh}(\bar{t})\left(-17+58\bar{t}^2\right)}{8\left(1+\bar{t}^2\right)^{7/2}} + \frac{3\operatorname{arcsinh}(\bar{t})^3\left(3-24\bar{t}^2+8\bar{t}^4\right)}{8\left(1+\bar{t}^2\right)^{9/2}} \Bigg]$$

$$+ \frac{1}{v^2} \Bigg[\frac{3\bar{t}\left(-173-252\bar{t}^2+96\bar{t}^4\right)}{64\left(1+\bar{t}^2\right)^{4}} - \frac{3\operatorname{arcsinh}(\bar{t})\left(-681+2016\bar{t}^2+1472\bar{t}^4\right)}{64\left(1+\bar{t}^2\right)^{9/2}} + \frac{3\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(-3357+650\bar{t}^2+2432\bar{t}^4\right)}{64\left(1+\bar{t}^2\right)^{5}}$$

$$+ \frac{3\operatorname{arcsinh}(\bar{t})^3\left(-217+1644\bar{t}^2+824\bar{t}^4-512\bar{t}^6\right)}{64\left(1+\bar{t}^2\right)^{11/2}} \Bigg] - \frac{45\arctan(\bar{t})\left(-4+11\bar{t}^2\right)}{8\left(1+\bar{t}^2\right)^{3}} - \frac{3\bar{t}\left(-1979-246\bar{t}^2+9120\bar{t}^4+3712\bar{t}^6\right)}{512\left(1+\bar{t}^2\right)^{5}}$$

$$- \frac{15\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(-13469-2330\bar{t}^2+12304\bar{t}^4+6016\bar{t}^6\right)}{512\left(1+\bar{t}^2\right)^{6}} + \frac{3\operatorname{arcsinh}(\bar{t})^3\left(5293-41208\bar{t}^2-46672\bar{t}^4+1472\bar{t}^6+9728\bar{t}^8\right)}{512\left(1+\bar{t}^2\right)^{13/2}}$$

$$+ \operatorname{arcsinh}(\bar{t}) \left(\frac{135\arctan(\bar{t})\,\bar{t}\left(-3+2\bar{t}^2\right)}{8\left(1+\bar{t}^2\right)^{7/2}} + \frac{15\left(-1423+5618\bar{t}^2+3584\bar{t}^4+512\bar{t}^6\right)}{512\left(1+\bar{t}^2\right)^{11/2}} \right)$$

$$+ v^2 \Bigg[- \frac{45\arctan(\bar{t})\left(30-129\bar{t}^2+16\bar{t}^4\right)}{64\left(1+\bar{t}^2\right)^{4}} + \frac{3\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(-952289+747962\bar{t}^2+1414720\bar{t}^4+406528\bar{t}^6+16384\bar{t}^8\right)}{8192\left(1+\bar{t}^2\right)^{7}}$$

$$+ \frac{3\bar{t}\left(42723+85032\bar{t}^2-14592\bar{t}^4-309248\bar{t}^6-131072\bar{t}^8\right)}{8192\left(1+\bar{t}^2\right)^{6}}$$

$$- \frac{3\operatorname{arcsinh}(\bar{t})^3\left(91221-954180\bar{t}^2-900560\bar{t}^4+161280\bar{t}^6+307200\bar{t}^8+65536\bar{t}^{10}\right)}{8192\left(1+\bar{t}^2\right)^{15/2}}$$

$$+ \operatorname{arcsinh}(\bar{t}) \left(- \frac{45\arctan(\bar{t})\,\bar{t}\left(-219+52\bar{t}^2+96\bar{t}^4\right)}{64\left(1+\bar{t}^2\right)^{9/2}} + \frac{85923-1511556\bar{t}^2+798144\bar{t}^4+619008\bar{t}^6-61440\bar{t}^8}{8192\left(1+\bar{t}^2\right)^{13/2}} \right) \Bigg] \Bigg\}$$

$$+ \frac{\varepsilon M^5}{b^5} \Bigg\{ \frac{1}{v^6} \Bigg[\frac{45\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(23-26\bar{t}^2\right)}{16\left(1+\bar{t}^2\right)^{9/2}} + \frac{3\bar{t}\left(1-44\bar{t}^2\right)}{32\left(1+\bar{t}^2\right)^{7/2}} - \frac{9\operatorname{arcsinh}(\bar{t})\left(8-29\bar{t}^2-2\bar{t}^4\right)}{8\left(1+\bar{t}^2\right)^{4}}$$

$$- \frac{15\operatorname{arcsinh}(\bar{t})^4\,\bar{t}\left(15-40\bar{t}^2+8\bar{t}^4\right)}{32\left(1+\bar{t}^2\right)^{11/2}} + \frac{\operatorname{arcsinh}(\bar{t})^3\left(64-607\bar{t}^2+274\bar{t}^4\right)}{8\left(1+\bar{t}^2\right)^{5}} \Bigg] + \frac{1}{v^4} \Bigg[\frac{3\bar{t}\left(-2501-276\bar{t}^2+2400\bar{t}^4\right)}{256\left(1+\bar{t}^2\right)^{9/2}}$$

$$+ \frac{3\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(-25155+10604\bar{t}^2+21584\bar{t}^4\right)}{128\left(1+\bar{t}^2\right)^{11/2}} - \frac{3\operatorname{arcsinh}(\bar{t})\left(-1084+3073\bar{t}^2+1964\bar{t}^4+432\bar{t}^6\right)}{64\left(1+\bar{t}^2\right)^{5}}$$

$$+ \frac{15\operatorname{arcsinh}(\bar{t})^4\,\bar{t}\left(1495-2900\bar{t}^2-2536\bar{t}^4+704\bar{t}^6\right)}{256\left(1+\bar{t}^2\right)^{13/2}} - \frac{7\operatorname{arcsinh}(\bar{t})^3\left(792-7161\bar{t}^2-2582\bar{t}^4+2896\bar{t}^6\right)}{64\left(1+\bar{t}^2\right)^{6}} \Bigg]$$

$$+ \frac{1}{v^2} \Bigg[- \frac{45\arctan(\bar{t})\left(-17+58\bar{t}^2\right)}{16\left(1+\bar{t}^2\right)^{7/2}} - \frac{3\bar{t}\left(-34195+75748\bar{t}^2+222336\bar{t}^4+95488\bar{t}^6\right)}{2048\left(1+\bar{t}^2\right)^{11/2}}$$

$$- \frac{15\operatorname{arcsinh}(\bar{t})^4\,\bar{t}\left(57485-81280\bar{t}^2-183408\bar{t}^4-43648\bar{t}^6+22016\bar{t}^8\right)}{2048\left(1+\bar{t}^2\right)^{15/2}} + \operatorname{arcsinh}(\bar{t})^2 \Bigg(- \frac{135\arctan(\bar{t})\left(3-24\bar{t}^2+8\bar{t}^4\right)}{16\left(1+\bar{t}^2\right)^{9/2}}$$

$$- \frac{3\bar{t}\left(-665495-7074\bar{t}^2+752192\bar{t}^4+360576\bar{t}^6\right)}{1024\left(1+\bar{t}^2\right)^{13/2}} \Bigg) + \frac{\operatorname{arcsinh}(\bar{t})^3\left(179416-1623557\bar{t}^2-1684714\bar{t}^4+311232\bar{t}^6+508288\bar{t}^8\right)}{512\left(1+\bar{t}^2\right)^{7}}$$

$$+ \operatorname{arcsinh}(\bar{t}) \left(\frac{225\arctan(\bar{t})\,\bar{t}\left(-11+10\bar{t}^2\right)}{8\left(1+\bar{t}^2\right)^{4}} + \frac{3\left(-13488+33409\bar{t}^2-15650\bar{t}^4+12096\bar{t}^6+18048\bar{t}^8\right)}{512\left(1+\bar{t}^2\right)^{6}} \right) \Bigg]$$

$$- \frac{45\arctan(\bar{t})\left(161-1032\bar{t}^2+256\bar{t}^4+224\bar{t}^6\right)}{128\left(1+\bar{t}^2\right)^{9/2}} - \frac{3\bar{t}\left(-430875-2039012\bar{t}^2+515264\bar{t}^4+5681664\bar{t}^6+2674688\bar{t}^8\right)}{32768\left(1+\bar{t}^2\right)^{13/2}}$$

$$
\begin{aligned}
&+\frac{\operatorname{arcsinh}(\bar t)^3\left(-5220704+53305943\bar t^2+67880578\bar t^4+5371936\bar t^6-23015680\bar t^8-8898560\bar t^{10}\right)}{8192\left(1+\bar t^2\right)^8}\\
&+\frac{15\operatorname{arcsinh}(\bar t)^4\,\bar t\left(2107095-2868020\bar t^2-8851376\bar t^4-4966656\bar t^6+117760\bar t^8+532480\bar t^{10}\right)}{32768\left(1+\bar t^2\right)^{17/2}}\\
&+\operatorname{arcsinh}(\bar t)^2\Bigg(\frac{675\arctan(\bar t)\left(29-228\bar t^2-88\bar t^4+64\bar t^6\right)}{128\left(1+\bar t^2\right)^{11/2}}\\
&-\frac{3\bar t\left(15367895-4468648\bar t^2-20889824\bar t^4-13013504\bar t^6-3176448\bar t^8\right)}{16384\left(1+\bar t^2\right)^{15/2}}\Bigg)+\operatorname{arcsinh}(\bar t)\Bigg(\frac{315\arctan(\bar t)\,\bar t\left(291-110\bar t^2-176\bar t^4\right)}{64\left(1+\bar t^2\right)^5}\\
&+\frac{3\left(75972-914597\bar t^2+1685000\bar t^4+13152\bar t^6-776704\bar t^8-141312\bar t^{10}\right)}{8192\left(1+\bar t^2\right)^7}\Bigg)\Bigg\}\,,
\end{aligned}
\tag{4.46}
$$

where $\varepsilon = q^2/\mu M^2$ has been reintroduced.

The local terms in the force, as computed with mode-sum regularization, are then

$$
\begin{aligned}
F_t^{\mathrm{L}}(\bar t) ={}& \frac{\varepsilon M^2}{b^3}\Bigg\{\frac{2v^3\bar t}{\left(1+\bar t^2\right)^3}+\frac{v^2\left(1-2\bar t^2\right)}{3\left(1+\bar t^2\right)^{5/2}}+\frac{v^4\left(-1+2\bar t^2\right)}{6\left(1+\bar t^2\right)^{5/2}}+\frac{v^5\bar t\left(15-11\bar t^2+4\bar t^4\right)}{3\left(1+\bar t^2\right)^4}+\frac{v^7\bar t\left(-87+574\bar t^2-307\bar t^4+40\bar t^6\right)}{12\left(1+\bar t^2\right)^5}\\
&+\frac{v^6\left(-49+432\bar t^2-309\bar t^4+50\bar t^6\right)}{24\left(1+\bar t^2\right)^{9/2}}+\frac{7v^8\left(-7-151\bar t^2+669\bar t^4-309\bar t^6+30\bar t^8\right)}{48\left(1+\bar t^2\right)^{11/2}}\Bigg\}\\
&+\frac{\varepsilon M^3}{b^4}\Bigg\{\frac{\operatorname{arcsinh}(\bar t)\,\bar t\left(-3+2\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}-\frac{5\left(-1+2\bar t^2\right)}{3\left(1+\bar t^2\right)^3}+v\left[\frac{\operatorname{arcsinh}(\bar t)\left(2-10\bar t^2\right)}{\left(1+\bar t^2\right)^4}+\frac{12\bar t}{\left(1+\bar t^2\right)^{7/2}}\right]\\
&+v^2\left[-\frac{7\operatorname{arcsinh}(\bar t)\,\bar t\left(-3+2\bar t^2\right)}{2\left(1+\bar t^2\right)^{7/2}}+\frac{5\left(-1+6\bar t^2\right)}{2\left(1+\bar t^2\right)^3}\right]+v^3\left[-\frac{21\pi^2\bar t\left(-3+2\bar t^2\right)}{64\left(1+\bar t^2\right)^{7/2}}+\frac{\gamma_E\left(12\bar t-8\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{\log(2)\left(12\bar t-8\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}\right.\\
&\left.+\frac{\left(12\bar t-8\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}\log\left(\frac{b}{M}\right)+\frac{\log\left(1+\bar t^2\right)\left(6\bar t-4\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{2\bar t\left(-9-62\bar t^2+67\bar t^4\right)}{3\left(1+\bar t^2\right)^{9/2}}+\frac{\operatorname{arcsinh}(\bar t)\left(-1-22\bar t^2+55\bar t^4-4\bar t^6\right)}{\left(1+\bar t^2\right)^5}\right]\\
&+v^4\left[\frac{-131+1408\bar t^2-727\bar t^4+254\bar t^6}{8\left(1+\bar t^2\right)^5}+\frac{\operatorname{arcsinh}(\bar t)\left(399\bar t-1468\bar t^3+627\bar t^5-26\bar t^7\right)}{8\left(1+\bar t^2\right)^{11/2}}\right]+v^5\left[\frac{\log\left(1+\bar t^2\right)\bar t\left(-3+2\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}\right.\\
&+\frac{\gamma_E\left(-6\bar t+4\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{\log(2)\left(-6\bar t+4\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{\left(-6\bar t+4\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}\log\left(\frac{b}{M}\right)+\frac{3\pi^2\bar t\left(1797-3376\bar t^2+392\bar t^4\right)}{1024\left(1+\bar t^2\right)^{9/2}}\\
&\left.+\frac{\bar t\left(-723+2906\bar t^2-1169\bar t^4+242\bar t^6\right)}{6\left(1+\bar t^2\right)^{11/2}}+\frac{\operatorname{arcsinh}(\bar t)\left(-89+1327\bar t^2-1599\bar t^4+353\bar t^6+8\bar t^8\right)}{4\left(1+\bar t^2\right)^6}\right]\\
&+v^6\left[\frac{\operatorname{arcsinh}(\bar t)\,\bar t\left(-3135+15325\bar t^2-10423\bar t^4+775\bar t^6+90\bar t^8\right)}{16\left(1+\bar t^2\right)^{13/2}}+\frac{961-62387\bar t^2+249153\bar t^4-115513\bar t^6+15506\bar t^8}{240\left(1+\bar t^2\right)^6}\right]\Bigg\}\\
&+\frac{\varepsilon M^4}{b^5}\Bigg\{\frac{1}{v^2}\left[-\frac{25\left(-1+2\bar t^2\right)}{6\left(1+\bar t^2\right)^{7/2}}+\frac{\operatorname{arcsinh}(\bar t)\,\bar t\left(-59+46\bar t^2\right)}{3\left(1+\bar t^2\right)^4}+\frac{\operatorname{arcsinh}(\bar t)^2\left(-3+24\bar t^2-8\bar t^4\right)}{2\left(1+\bar t^2\right)^{9/2}}\right]\\
&+\frac{1}{v}\left[\frac{\operatorname{arcsinh}(\bar t)\left(14-82\bar t^2\right)}{\left(1+\bar t^2\right)^{9/2}}+\frac{36\bar t}{\left(1+\bar t^2\right)^4}+\frac{6\operatorname{arcsinh}(\bar t)^2\,\bar t\left(-3+5\bar t^2\right)}{\left(1+\bar t^2\right)^5}\right]+\frac{5\left(-7+62\bar t^2\right)}{4\left(1+\bar t^2\right)^{7/2}}+\frac{\operatorname{arcsinh}(\bar t)\left(229\bar t-226\bar t^3\right)}{2\left(1+\bar t^2\right)^4}\\
&+\frac{13\operatorname{arcsinh}(\bar t)^2\left(3-24\bar t^2+8\bar t^4\right)}{4\left(1+\bar t^2\right)^{9/2}}+v\left[\frac{7\pi^2\bar t\left(59-46\bar t^2\right)}{64\left(1+\bar t^2\right)^4}+\frac{4\gamma_E\bar t\left(61-44\bar t^2\right)}{3\left(1+\bar t^2\right)^4}+\frac{4\log(2)\bar t\left(61-44\bar t^2\right)}{3\left(1+\bar t^2\right)^4}\right.\\
&+\frac{4\bar t\left(61-44\bar t^2\right)}{3\left(1+\bar t^2\right)^4}\log\left(\frac{b}{M}\right)-\frac{2\log\left(1+\bar t^2\right)\bar t\left(-61+44\bar t^2\right)}{3\left(1+\bar t^2\right)^4}+\frac{4\bar t\left(-310-353\bar t^2+677\bar t^4\right)}{9\left(1+\bar t^2\right)^5}\\
&+\frac{\operatorname{arcsinh}(\bar t)^2\,\bar t\left(37+162\bar t^2-267\bar t^4+8\bar t^6\right)}{\left(1+\bar t^2\right)^6}+\operatorname{arcsinh}(\bar t)\Bigg(\frac{4\gamma_E\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}+\frac{21\pi^2\left(3-24\bar t^2+8\bar t^4\right)}{64\left(1+\bar t^2\right)^{9/2}}\\
&+\frac{4\log(2)\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}+\frac{4\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}\log\left(\frac{b}{M}\right)+\frac{2\log\left(1+\bar t^2\right)\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}\\
&\left.-\frac{147-354\bar t^2-2329\bar t^4+572\bar t^6}{3\left(1+\bar t^2\right)^{11/2}}\Bigg)\right]+v^2\left[\frac{15\arctan(\bar t)\,\bar t\left(3-2\bar t^2\right)}{2\left(1+\bar t^2\right)^{7/2}}+\frac{\operatorname{arcsinh}(\bar t)\,\bar t\left(3193-16696\bar t^2+8397\bar t^4+566\bar t^6\right)}{8\left(1+\bar t^2\right)^6}\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{-3337+38224\bar{t}^2-19309\bar{t}^4+7170\bar{t}^6}{48\left(1+\bar{t}^2\right)^{11/2}}+\frac{\operatorname{arcsinh}(\bar{t})^2\left(147-6882\bar{t}^2+17987\bar{t}^4-3272\bar{t}^6-568\bar{t}^8\right)}{16\left(1+\bar{t}^2\right)^{13/2}}\Bigg]\\
&+v^3\Bigg[\frac{15\arctan(\bar{t})\left(-1+5\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^4}+\frac{2\bar{t}\left(-223+284\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^4}\log\left(\frac{b}{M}\right)+\frac{\log\left(1+\bar{t}^2\right)\bar{t}\left(-223+284\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^4}+\frac{2\gamma_E\bar{t}\left(-235+308\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^4}\\
&+\frac{2\log(2)\bar{t}\left(-247+332\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^4}+\frac{\pi^2\bar{t}\left(29687-87008\bar{t}^2+33560\bar{t}^4\right)}{1024\left(1+\bar{t}^2\right)^5}-\frac{\bar{t}\left(44635-226488\bar{t}^2+119169\bar{t}^4+12292\bar{t}^6\right)}{90\left(1+\bar{t}^2\right)^6}\\
&-\frac{7\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(-295+1715\bar{t}^2-1029\bar{t}^4-143\bar{t}^6+16\bar{t}^8\right)}{4\left(1+\bar{t}^2\right)^7}-\operatorname{arcsinh}(\bar{t})\Bigg(\frac{14\gamma_E\left(3-24\bar{t}^2+8\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}\\
&+\frac{14\log(2)\left(3-24\bar{t}^2+8\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}+\frac{14\left(3-24\bar{t}^2+8\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}\log\left(\frac{b}{M}\right)+\frac{7\log\left(1+\bar{t}^2\right)\left(3-24\bar{t}^2+8\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}}\\
&-\frac{3\pi^2\left(-789+17448\bar{t}^2-27592\bar{t}^4+4256\bar{t}^6\right)}{1024\left(1+\bar{t}^2\right)^{11/2}}+\frac{-1461+39153\bar{t}^2-79075\bar{t}^4+6271\bar{t}^6+5000\bar{t}^8}{12\left(1+\bar{t}^2\right)^{13/2}}\Bigg)\Bigg]\\
&+v^4\Bigg[\frac{15\arctan(\bar{t})\,\bar{t}\left(-3+2\bar{t}^2\right)}{4\left(1+\bar{t}^2\right)^{7/2}}-\frac{76\gamma_E\left(3-24\bar{t}^2+8\bar{t}^4\right)}{15\left(1+\bar{t}^2\right)^{9/2}}+\frac{4\pi^2\left(3-24\bar{t}^2+8\bar{t}^4\right)}{3\left(1+\bar{t}^2\right)^{9/2}}-\frac{76\log(2)\left(3-24\bar{t}^2+8\bar{t}^4\right)}{15\left(1+\bar{t}^2\right)^{9/2}}\\
&-\frac{76\left(3-24\bar{t}^2+8\bar{t}^4\right)}{15\left(1+\bar{t}^2\right)^{9/2}}\log\left(\frac{b}{M}\right)-\frac{38\log\left(1+\bar{t}^2\right)\left(3-24\bar{t}^2+8\bar{t}^4\right)}{15\left(1+\bar{t}^2\right)^{9/2}}\\
&-\frac{\operatorname{arcsinh}(\bar{t})\,\bar{t}\left(590581-2850707\bar{t}^2+2021217\bar{t}^4-336361\bar{t}^6-33106\bar{t}^8\right)}{240\left(1+\bar{t}^2\right)^7}\\
&+\frac{394671-5267121\bar{t}^2+8924731\bar{t}^4-7843491\bar{t}^6+1963586\bar{t}^8}{2400\left(1+\bar{t}^2\right)^{13/2}}\\
&-\frac{3\operatorname{arcsinh}(\bar{t})^2\left(1843-43855\bar{t}^2+81425\bar{t}^4-7733\bar{t}^6-6400\bar{t}^8+328\bar{t}^{10}\right)}{32\left(1+\bar{t}^2\right)^{15/2}}\Bigg]\Bigg\}, \qquad (4.47)
\end{aligned}
$$

and

$$
\begin{aligned}
F^{\mathrm{L}}_{\phi}(\bar{t}) =\;& \frac{\varepsilon M^2}{b^2}\Bigg\{-\frac{v}{3\left(1+\bar{t}^2\right)^{3/2}}-\frac{2v^2\bar{t}}{\left(1+\bar{t}^2\right)^2}+\frac{v^3\left(1-17\bar{t}^2\right)}{6\left(1+\bar{t}^2\right)^{5/2}}-\frac{v^4\bar{t}\left(9+13\bar{t}^2\right)}{3\left(1+\bar{t}^2\right)^3}+\frac{v^5\left(49-202\bar{t}^2-131\bar{t}^4\right)}{24\left(1+\bar{t}^2\right)^{7/2}}\\
&-\frac{v^6\bar{t}\left(-75+226\bar{t}^2+85\bar{t}^4\right)}{12\left(1+\bar{t}^2\right)^4}+\frac{v^7\left(49+789\bar{t}^2-1569\bar{t}^4-405\bar{t}^6\right)}{48\left(1+\bar{t}^2\right)^{9/2}}\Bigg\}\\
&+\frac{\varepsilon M^3}{b^3}\Bigg\{\frac{1}{v}\Bigg[\frac{\operatorname{arcsinh}(\bar{t})\,\bar{t}}{\left(1+\bar{t}^2\right)^{5/2}}-\frac{1}{\left(1+\bar{t}^2\right)^2}\Bigg]-\frac{8\bar{t}}{\left(1+\bar{t}^2\right)^{5/2}}+\frac{\operatorname{arcsinh}(\bar{t})\left(-2+6\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^3}+v\Bigg[\frac{3\left(1-9\bar{t}^2\right)}{2\left(1+\bar{t}^2\right)^3}\\
&-\frac{\operatorname{arcsinh}(\bar{t})\,\bar{t}\left(19-11\bar{t}^2\right)}{2\left(1+\bar{t}^2\right)^{7/2}}\Bigg]+v^2\Bigg[-\frac{8\gamma_E\bar{t}}{\left(1+\bar{t}^2\right)^{5/2}}-\frac{21\pi^2\bar{t}}{32\left(1+\bar{t}^2\right)^{5/2}}-\frac{8\log(2)\bar{t}}{\left(1+\bar{t}^2\right)^{5/2}}-\frac{8\bar{t}}{\left(1+\bar{t}^2\right)^{5/2}}\log\left(\frac{b}{M}\right)\\
&-\frac{4\log\left(1+\bar{t}^2\right)\bar{t}}{\left(1+\bar{t}^2\right)^{5/2}}+\frac{2\bar{t}\left(5+\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^{7/2}}+\frac{\operatorname{arcsinh}(\bar{t})\left(3-10\bar{t}^2-5\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^4}\Bigg]+v^3\Bigg[\frac{101-378\bar{t}^2-199\bar{t}^4}{8\left(1+\bar{t}^2\right)^4}\\
&+\frac{\operatorname{arcsinh}(\bar{t})\left(-93\bar{t}+114\bar{t}^3-73\bar{t}^5\right)}{8\left(1+\bar{t}^2\right)^{9/2}}\Bigg]+v^4\Bigg[-\frac{3\pi^2\bar{t}\left(257+72\bar{t}^2\right)}{256\left(1+\bar{t}^2\right)^{7/2}}+\frac{\gamma_E\left(20\bar{t}-60\bar{t}^3\right)}{\left(1+\bar{t}^2\right)^{7/2}}+\frac{\log(2)\left(20\bar{t}-60\bar{t}^3\right)}{\left(1+\bar{t}^2\right)^{7/2}}\\
&+\frac{\left(20\bar{t}-60\bar{t}^3\right)}{\left(1+\bar{t}^2\right)^{7/2}}\log\left(\frac{b}{M}\right)+\frac{\log\left(1+\bar{t}^2\right)\left(10\bar{t}-30\bar{t}^3\right)}{\left(1+\bar{t}^2\right)^{7/2}}+\frac{2\bar{t}\left(251+92\bar{t}^2+921\bar{t}^4\right)}{15\left(1+\bar{t}^2\right)^{9/2}}+\frac{\operatorname{arcsinh}(\bar{t})\left(61-389\bar{t}^2+55\bar{t}^4-71\bar{t}^6\right)}{4\left(1+\bar{t}^2\right)^5}\Bigg]\\
&+v^5\Bigg[-\frac{157-13599\bar{t}^2+17251\bar{t}^4+2447\bar{t}^6}{80\left(1+\bar{t}^2\right)^5}+\frac{\operatorname{arcsinh}(\bar{t})\left(1873\bar{t}-3411\bar{t}^3+47\bar{t}^5-381\bar{t}^7\right)}{16\left(1+\bar{t}^2\right)^{11/2}}\Bigg]\Bigg\}\\
&+\frac{\varepsilon M^4}{b^4}\Bigg\{\frac{1}{v^3}\Bigg[\frac{\operatorname{arcsinh}(\bar{t})^2\left(1-4\bar{t}^2\right)}{2\left(1+\bar{t}^2\right)^{7/2}}+\frac{5\operatorname{arcsinh}(\bar{t})\,\bar{t}}{\left(1+\bar{t}^2\right)^3}-\frac{3}{2\left(1+\bar{t}^2\right)^{5/2}}\Bigg]\\
&+\frac{1}{v^2}\Bigg[\frac{12\operatorname{arcsinh}(\bar{t})^2\,\bar{t}\left(1-\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^4}-\frac{16\bar{t}}{\left(1+\bar{t}^2\right)^3}-\frac{2\operatorname{arcsinh}(\bar{t})\left(5-19\bar{t}^2\right)}{\left(1+\bar{t}^2\right)^{7/2}}\Bigg]\\
&+\frac{1}{v}\Bigg[\frac{11-139\bar{t}^2}{4\left(1+\bar{t}^2\right)^{7/2}}+\frac{\operatorname{arcsinh}(\bar{t})\,\bar{t}\left(-121+89\bar{t}^2\right)}{2\left(1+\bar{t}^2\right)^4}-\frac{5\operatorname{arcsinh}(\bar{t})^2\left(5-33\bar{t}^2+4\bar{t}^4\right)}{4\left(1+\bar{t}^2\right)^{9/2}}\Bigg]-\frac{40\gamma_E\bar{t}}{\left(1+\bar{t}^2\right)^3}-\frac{105\pi^2\bar{t}}{32\left(1+\bar{t}^2\right)^3}
\end{aligned}
$$

$$-\frac{40\log(2)\bar t}{(1+\bar t^2)^3} - \frac{40\bar t}{(1+\bar t^2)^3}\log\left(\frac{b}{M}\right) - \frac{20\log(1+\bar t^2)\,\bar t}{(1+\bar t^2)^3} + \frac{6\bar t(13+9\bar t^2)}{(1+\bar t^2)^4} + \frac{\mathrm{arcsinh}(\bar t)^2(-58\bar t+20\bar t^3+46\bar t^5)}{(1+\bar t^2)^5}$$

$$+\mathrm{arcsinh}(\bar t)\Bigg(\frac{8\gamma_E(-1+4\bar t^2)}{(1+\bar t^2)^{7/2}} + \frac{21\pi^2(-1+4\bar t^2)}{32(1+\bar t^2)^{7/2}} + \frac{8\log(2)(-1+4\bar t^2)}{(1+\bar t^2)^{7/2}} + \frac{8(-1+4\bar t^2)}{(1+\bar t^2)^{7/2}}\log\left(\frac{b}{M}\right)$$

$$+\frac{4\log(1+\bar t^2)(-1+4\bar t^2)}{(1+\bar t^2)^{7/2}} + \frac{43-156\bar t^2-135\bar t^4}{(1+\bar t^2)^{9/2}}\Bigg) + v\Bigg[\frac{2089-7642\bar t^2-3851\bar t^4}{48(1+\bar t^2)^{9/2}} - \frac{15\arctan(\bar t)\,\bar t}{2(1+\bar t^2)^{5/2}}$$

$$-\frac{5\mathrm{arcsinh}(\bar t)\,\bar t(113-250\bar t^2+141\bar t^4)}{8(1+\bar t^2)^5} + \frac{5\mathrm{arcsinh}(\bar t)^2(27-90\bar t^2-457\bar t^4+164\bar t^6)}{16(1+\bar t^2)^{11/2}}\Bigg]$$

$$+v^2\Bigg[\frac{\arctan(\bar t)(15-45\bar t^2)}{(1+\bar t^2)^3} - \frac{4\gamma_E\bar t(-121+299\bar t^2)}{3(1+\bar t^2)^4} - \frac{4\log(2)\bar t(-121+299\bar t^2)}{3(1+\bar t^2)^4} - \frac{4\bar t(-121+299\bar t^2)}{3(1+\bar t^2)^4}\log\left(\frac{b}{M}\right)$$

$$+\frac{\pi^2\bar t(-3517+368\bar t^2)}{256(1+\bar t^2)^4} + \frac{\log(1+\bar t^2)(242\bar t-598\bar t^3)}{3(1+\bar t^2)^4} + \frac{4\bar t(56+1087\bar t^2+2327\bar t^4)}{9(1+\bar t^2)^5}$$

$$+\frac{\mathrm{arcsinh}(\bar t)^2\,\bar t(-215+845\bar t^2-369\bar t^4+11\bar t^6)}{2(1+\bar t^2)^6} + \mathrm{arcsinh}(\bar t)\Bigg(\frac{4\gamma_E(11-93\bar t^2+36\bar t^4)}{(1+\bar t^2)^{9/2}} + \frac{4\log(2)(11-93\bar t^2+36\bar t^4)}{(1+\bar t^2)^{9/2}}$$

$$+\frac{4(11-93\bar t^2+36\bar t^4)}{(1+\bar t^2)^{9/2}}\log\left(\frac{b}{M}\right) + \frac{2\log(1+\bar t^2)(11-93\bar t^2+36\bar t^4)}{(1+\bar t^2)^{9/2}} - \frac{3\pi^2(89-822\bar t^2+384\bar t^4)}{256(1+\bar t^2)^{9/2}}$$

$$+\frac{583-7871\bar t^2+47589\bar t^4-30357\bar t^6}{60(1+\bar t^2)^{11/2}}\Bigg)\Bigg] + v^3\Bigg[\frac{76\gamma_E(1-4\bar t^2)}{5(1+\bar t^2)^{7/2}} + \frac{4\pi^2(-1+4\bar t^2)}{(1+\bar t^2)^{7/2}} - \frac{76\log(2)(-1+4\bar t^2)}{5(1+\bar t^2)^{7/2}}$$

$$-\frac{76(-1+4\bar t^2)}{5(1+\bar t^2)^{7/2}}\log\left(\frac{b}{M}\right) - \frac{38\log(1+\bar t^2)(-1+4\bar t^2)}{5(1+\bar t^2)^{7/2}} - \frac{15\arctan(\bar t)\,\bar t(-13+17\bar t^2)}{4(1+\bar t^2)^{7/2}}$$

$$+\frac{-319751+2408077\bar t^2-112393\bar t^4+1015379\bar t^6}{2400(1+\bar t^2)^{11/2}} + \frac{\mathrm{arcsinh}(\bar t)(87843\bar t-192721\bar t^3+19789\bar t^5-13807\bar t^7)}{80(1+\bar t^2)^6}$$

$$+\frac{\mathrm{arcsinh}(\bar t)^2(2431-34921\bar t^2+27301\bar t^4+1593\bar t^6-228\bar t^8)}{32(1+\bar t^2)^{13/2}}\Bigg]\Bigg\}$$

$$+\frac{\varepsilon M^5}{b^5}\Bigg\{\frac{1}{v^5}\Bigg[\frac{\mathrm{arcsinh}(\bar t)^2(6-29\bar t^2)}{2(1+\bar t^2)^4} + \frac{25\mathrm{arcsinh}(\bar t)\,\bar t}{2(1+\bar t^2)^{7/2}} - \frac{8+3\bar t^2}{6(1+\bar t^2)^3} + \frac{5\mathrm{arcsinh}(\bar t)^3\,\bar t(-3+4\bar t^2)}{6(1+\bar t^2)^{9/2}}\Bigg]$$

$$+\frac{1}{v^4}\Bigg[-\frac{\bar t(19+3\bar t^2)}{(1+\bar t^2)^{7/2}} + \frac{2\mathrm{arcsinh}(\bar t)(-13+59\bar t^2)}{(1+\bar t^2)^4} + \frac{\mathrm{arcsinh}(\bar t)^2(85\bar t-107\bar t^3)}{(1+\bar t^2)^{9/2}} + \frac{4\mathrm{arcsinh}(\bar t)^3(1-10\bar t^2+5\bar t^4)}{(1+\bar t^2)^5}\Bigg]$$

$$+\frac{1}{v^3}\Bigg[\frac{3\mathrm{arcsinh}(\bar t)\,\bar t(-249+241\bar t^2)}{4(1+\bar t^2)^{9/2}} + \frac{\mathrm{arcsinh}(\bar t)^2(-182+1417\bar t^2-291\bar t^4)}{4(1+\bar t^2)^5} + \frac{8-631\bar t^2-9\bar t^4}{12(1+\bar t^2)^4}$$

$$-\frac{5\mathrm{arcsinh}(\bar t)^3\,\bar t(-129+253\bar t^2+4\bar t^4)}{12(1+\bar t^2)^{11/2}}\Bigg] + \frac{1}{v^2}\Bigg[-\frac{100\gamma_E\bar t}{(1+\bar t^2)^{7/2}} - \frac{525\pi^2\bar t}{64(1+\bar t^2)^{7/2}} - \frac{100\log(2)\bar t}{(1+\bar t^2)^{7/2}} - \frac{100\bar t}{(1+\bar t^2)^{7/2}}\log\left(\frac{b}{M}\right)$$

$$-\frac{50\log(1+\bar t^2)\,\bar t}{(1+\bar t^2)^{7/2}} + \frac{\bar t(1197+1028\bar t^2+87\bar t^4)}{6(1+\bar t^2)^{9/2}} + \frac{\mathrm{arcsinh}(\bar t)^3(-94+906\bar t^2+270\bar t^4-410\bar t^6)}{3(1+\bar t^2)^6}$$

$$+\mathrm{arcsinh}(\bar t)\Bigg(\frac{8\gamma_E(-6+29\bar t^2)}{(1+\bar t^2)^4} + \frac{21\pi^2(-6+29\bar t^2)}{32(1+\bar t^2)^4} + \frac{8\log(2)(-6+29\bar t^2)}{(1+\bar t^2)^4} + \frac{8(-6+29\bar t^2)}{(1+\bar t^2)^4}\log\left(\frac{b}{M}\right)$$

$$+\frac{4\log(1+\bar t^2)(-6+29\bar t^2)}{(1+\bar t^2)^4} + \frac{199-856\bar t^2-799\bar t^4}{(1+\bar t^2)^5}\Bigg) + \bar t\,\mathrm{arcsinh}(\bar t)^2\Bigg(\frac{105\pi^2(3-4\bar t^2)}{64(1+\bar t^2)^{9/2}} + \frac{\gamma_E(60-80\bar t^2)}{(1+\bar t^2)^{9/2}}$$

$$+\frac{\log(2)(60-80\bar t^2)}{(1+\bar t^2)^{9/2}} + \frac{(60-80\bar t^2)}{(1+\bar t^2)^{9/2}}\log\left(\frac{b}{M}\right) + \frac{\log(1+\bar t^2)(30-40\bar t^2)}{(1+\bar t^2)^{9/2}} - \frac{(1333-748\bar t^2-1441\bar t^4)}{2(1+\bar t^2)^{11/2}}\Bigg)\Bigg]$$

$$+\frac{1}{v}\Bigg[-\frac{75\arctan(\bar t)\,\bar t}{2(1+\bar t^2)^3} + \frac{5\mathrm{arcsinh}(\bar t)^3\,\bar t(-1809+614\bar t^2+6907\bar t^4-1060\bar t^6)}{48(1+\bar t^2)^{13/2}} + \frac{5032-20401\bar t^2-12638\bar t^4+195\bar t^6}{48(1+\bar t^2)^5}$$

$$+\frac{\mathrm{arcsinh}(\bar t)^2(1322-5211\bar t^2-24328\bar t^4+9925\bar t^6)}{16(1+\bar t^2)^6} - \mathrm{arcsinh}(\bar t)\Bigg(\frac{15\arctan(\bar t)(1-4\bar t^2)}{2(1+\bar t^2)^{7/2}} + \frac{16343\bar t-34934\bar t^3+16763\bar t^5}{48(1+\bar t^2)^{11/2}}\Bigg)\Bigg]$$

$$
\begin{aligned}
&+\frac{\operatorname{arctan}(\bar t)\left(75-285\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{3\pi^2\bar t\left(-6193+2872\bar t^2\right)}{512\left(1+\bar t^2\right)^{9/2}}+\frac{\gamma_E\left(538\bar t-1422\bar t^3\right)}{\left(1+\bar t^2\right)^{9/2}}+\frac{\log(2)\left(538\bar t-1422\bar t^3\right)}{\left(1+\bar t^2\right)^{9/2}}\\
&+\frac{\left(538\bar t-1422\bar t^3\right)}{\left(1+\bar t^2\right)^{9/2}}\log\left(\frac{b}{M}\right)+\frac{\log\left(1+\bar t^2\right)\left(269\bar t-711\bar t^3\right)}{\left(1+\bar t^2\right)^{9/2}}+\frac{-679\bar t+60727\bar t^3+94283\bar t^5-1683\bar t^7}{24\left(1+\bar t^2\right)^{11/2}}\\
&+\frac{\operatorname{arcsinh}(\bar t)^3\left(133+1756\bar t^2-13482\bar t^4+3500\bar t^6+1325\bar t^8\right)}{6\left(1+\bar t^2\right)^7}+\operatorname{arcsinh}(\bar t)\Bigg(\frac{180\operatorname{arctan}(\bar t)\,\bar t\left(-1+\bar t^2\right)}{\left(1+\bar t^2\right)^4}\\
&+\frac{\pi^2\left(-760+16597\bar t^2-17608\bar t^4\right)}{256\left(1+\bar t^2\right)^5}+\frac{4\gamma_E\left(262-2437\bar t^2+1081\bar t^4\right)}{3\left(1+\bar t^2\right)^5}+\frac{4\log(2)\left(262-2437\bar t^2+1081\bar t^4\right)}{3\left(1+\bar t^2\right)^5}\\
&+\frac{4\left(262-2437\bar t^2+1081\bar t^4\right)}{3\left(1+\bar t^2\right)^5}\log\left(\frac{b}{M}\right)+\frac{2\log\left(1+\bar t^2\right)\left(262-2437\bar t^2+1081\bar t^4\right)}{3\left(1+\bar t^2\right)^5}\\
&+\frac{-56411+471147\bar t^2+962127\bar t^4-861431\bar t^6}{180\left(1+\bar t^2\right)^6}\Bigg)+\operatorname{arcsinh}(\bar t)^2\Bigg(-\frac{30\gamma_E\bar t\left(25-55\bar t^2+4\bar t^4\right)}{\left(1+\bar t^2\right)^{11/2}}-\frac{30\log(2)\bar t\left(25-55\bar t^2+4\bar t^4\right)}{\left(1+\bar t^2\right)^{11/2}}\\
&-\frac{30\bar t\left(25-55\bar t^2+4\bar t^4\right)}{\left(1+\bar t^2\right)^{11/2}}\log\left(\frac{b}{M}\right)-\frac{15\log\left(1+\bar t^2\right)\bar t\left(25-55\bar t^2+4\bar t^4\right)}{\left(1+\bar t^2\right)^{11/2}}+\frac{45\pi^2\bar t\left(-5-430\bar t^2+352\bar t^4\right)}{512\left(1+\bar t^2\right)^{11/2}}\\
&+\frac{\bar t\left(7715+21965\bar t^2-53719\bar t^4+1151\bar t^6\right)}{8\left(1+\bar t^2\right)^{13/2}}\Bigg)+v\Bigg[\frac{105\operatorname{arctan}(\bar t)\,\bar t\left(13-17\bar t^2\right)}{4\left(1+\bar t^2\right)^4}-\frac{76\gamma_E\left(-11+52\bar t^2\right)}{9\left(1+\bar t^2\right)^4}+\frac{20\pi^2\left(-11+52\bar t^2\right)}{9\left(1+\bar t^2\right)^4}\\
&-\frac{76\log(2)\left(-11+52\bar t^2\right)}{9\left(1+\bar t^2\right)^4}-\frac{76\left(-11+52\bar t^2\right)}{9\left(1+\bar t^2\right)^4}\log\left(\frac{b}{M}\right)-\frac{38\log\left(1+\bar t^2\right)\left(-11+52\bar t^2\right)}{9\left(1+\bar t^2\right)^4}\\
&+\frac{-1054088+5908347\bar t^2+5262063\bar t^4+4721893\bar t^6-175095\bar t^8}{1440\left(1+\bar t^2\right)^6}\\
&+\frac{\operatorname{arcsinh}(\bar t)^2\left(112898-1957691\bar t^2+2155189\bar t^4+79243\bar t^6-62455\bar t^8\right)}{160\left(1+\bar t^2\right)^7}\\
&+\frac{\operatorname{arcsinh}(\bar t)^3\,\bar t\left(-29045+179395\bar t^2-110367\bar t^4-37955\bar t^6+8580\bar t^8\right)}{32\left(1+\bar t^2\right)^{15/2}}+\operatorname{arcsinh}(\bar t)\Bigg(\frac{76\gamma_E\bar t\left(-3+4\bar t^2\right)}{\left(1+\bar t^2\right)^{9/2}}\\
&+\frac{76\log(2)\bar t\left(-3+4\bar t^2\right)}{\left(1+\bar t^2\right)^{9/2}}+\frac{76\bar t\left(-3+4\bar t^2\right)}{\left(1+\bar t^2\right)^{9/2}}\log\left(\frac{b}{M}\right)+\frac{38\log\left(1+\bar t^2\right)\bar t\left(-3+4\bar t^2\right)}{\left(1+\bar t^2\right)^{9/2}}+\frac{\pi^2\left(60\bar t-80\bar t^3\right)}{\left(1+\bar t^2\right)^{9/2}}\\
&+\frac{15\operatorname{arctan}(\bar t)\left(19-147\bar t^2+44\bar t^4\right)}{4\left(1+\bar t^2\right)^{9/2}}-\frac{\bar t\left(-3359487+5790221\bar t^2+380383\bar t^4+1597955\bar t^6\right)}{480\left(1+\bar t^2\right)^{13/2}}\Bigg)\Bigg]\Bigg\}.
\end{aligned}
\tag{4.48}
$$

It is important to mention that these two components of the force have different leading orders in $1/b$.

Non-local terms

The non-local time and ϕ components of the Self-Force are

$$
\begin{aligned}
F_t^{\mathrm{NL}}(\bar t)=\;&\frac{\varepsilon M^3}{b^4}\Bigg\{v^3\Bigg[\frac{4\gamma_E\bar t\left(-3+2\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{4\bar t\left(-3+2\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}\log\left(\frac{b}{M}\right)+\frac{\operatorname{arcsinh}(\bar t)\left(12\bar t-8\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{\log(2)\left(12\bar t-8\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}\\
&+\frac{\log(v)\left(12\bar t-8\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{\log\left(1+\bar t^2\right)\left(12\bar t-8\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{8\bar t\left(-12+11\bar t^2\right)}{3\left(1+\bar t^2\right)^{7/2}}+\frac{4\left(-4+11\bar t^2\right)}{3\left(1+\bar t^2\right)^3}\Bigg]\\
&+v^5\Bigg[\frac{\gamma_E\left(6\bar t-4\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{\left(6\bar t-4\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}\log\left(\frac{b}{M}\right)+\frac{\operatorname{arcsinh}(\bar t)\left(-6\bar t+4\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{\log(2)\left(-6\bar t+4\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{\log(v)\left(-6\bar t+4\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}\\
&+\frac{\log\left(1+\bar t^2\right)\left(-6\bar t+4\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{2\bar t\left(93-92\bar t^2\right)}{15\left(1+\bar t^2\right)^{7/2}}+\frac{8-22\bar t^2}{3\left(1+\bar t^2\right)^3}\Bigg]\Bigg\}\\
&+\frac{\varepsilon M^4}{b^5}\Bigg\{v\Bigg[\frac{4\log(v)\bar t\left(61-44\bar t^2\right)}{3\left(1+\bar t^2\right)^4}+\frac{8\log\left(1+\bar t^2\right)\bar t}{\left(1+\bar t^2\right)^3}+\frac{28\log(2)\bar t\left(-7+8\bar t^2\right)}{3\left(1+\bar t^2\right)^4}+\frac{4\gamma_E\bar t\left(-61+44\bar t^2\right)}{3\left(1+\bar t^2\right)^4}+\frac{8\left(-2+\bar t^2\right)}{3\left(1+\bar t^2\right)^{5/2}}\\
&+\frac{4\bar t\left(-61+44\bar t^2\right)}{3\left(1+\bar t^2\right)^4}\log\left(\frac{b}{M}\right)-\frac{2\pi^2\left(3-24\bar t^2+8\bar t^4\right)}{3\left(1+\bar t^2\right)^{9/2}}+\frac{4\operatorname{arcsinh}(\bar t)^2\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}+\frac{4\bar t\left(-89+55\bar t^2+2\bar t^4\right)}{3\left(1+\bar t^2\right)^4}
\end{aligned}
$$

$$
\begin{aligned}
&+\operatorname{arcsinh}(\bar t)\Bigg(-\frac{4\gamma_E\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}+\frac{4\log(2)\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}-\frac{4\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}\log\left(\frac{b}{M}\right)+\frac{8\bar t}{\left(1+\bar t^2\right)^{3}}\\
&+\frac{4\log(v)\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}+\frac{4\log\left(1+\bar t^2\right)\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}-\frac{16\left(6-57\bar t^2+25\bar t^4\right)}{3\left(1+\bar t^2\right)^{9/2}}\Bigg)\\
&+\frac{4\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}\mathrm{Li}_2\left(\frac{\bar t+\sqrt{1+\bar t^2}}{\bar t-\sqrt{1+\bar t^2}}\right)\Bigg]\Bigg]+v^3\Bigg[\frac{2\log(v)\bar t\left(-235+308\bar t^2\right)}{3\left(1+\bar t^2\right)^{4}}+\frac{\gamma_E\left(470\bar t-616\bar t^3\right)}{3\left(1+\bar t^2\right)^{4}}\\
&+\frac{\left(470\bar t-616\bar t^3\right)}{3\left(1+\bar t^2\right)^{4}}\log\left(\frac{b}{M}\right)+\frac{\log(2)\left(518\bar t-256\bar t^3\right)}{3\left(1+\bar t^2\right)^{4}}+\frac{\log\left(1+\bar t^2\right)\left(8\bar t+60\bar t^3\right)}{\left(1+\bar t^2\right)^{4}}-\frac{12\pi\left(1-6\bar t^2+\bar t^4\right)}{\left(1+\bar t^2\right)^{4}}\\
&-\frac{24\arctan(\bar t)\left(1-6\bar t^2+\bar t^4\right)}{\left(1+\bar t^2\right)^{4}}+\frac{29\pi^2\left(3-24\bar t^2+8\bar t^4\right)}{15\left(1+\bar t^2\right)^{9/2}}-\frac{14\operatorname{arcsinh}(\bar t)^2\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}+\frac{4\left(26-468\bar t^2+61\bar t^4\right)}{15\left(1+\bar t^2\right)^{7/2}}\\
&+\frac{2\bar t\left(1222-3133\bar t^2+122\bar t^4\right)}{15\left(1+\bar t^2\right)^{4}}+\operatorname{arcsinh}(\bar t)\Bigg(\frac{14\gamma_E\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}-\frac{14\log(2)\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}\\
&+\frac{14\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}\log\left(\frac{b}{M}\right)-\frac{14\log(v)\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}-\frac{14\log\left(1+\bar t^2\right)\left(3-24\bar t^2+8\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}\\
&+\frac{2\left(813-7764\bar t^2+3428\bar t^4\right)}{15\left(1+\bar t^2\right)^{9/2}}+\frac{4\bar t\left(-21+37\bar t^2\right)}{\left(1+\bar t^2\right)^{4}}\Bigg)-\frac{58\left(3-24\bar t^2+8\bar t^4\right)}{5\left(1+\bar t^2\right)^{9/2}}\mathrm{Li}_2\left(\frac{\bar t+\sqrt{1+\bar t^2}}{\bar t-\sqrt{1+\bar t^2}}\right)\Bigg]\\
&+v^4\Bigg[\frac{76\gamma_E\left(3-24\bar t^2+8\bar t^4\right)}{15\left(1+\bar t^2\right)^{9/2}}-\frac{76\operatorname{arcsinh}(\bar t)\left(3-24\bar t^2+8\bar t^4\right)}{15\left(1+\bar t^2\right)^{9/2}}-\frac{76\log(2)\left(3-24\bar t^2+8\bar t^4\right)}{15\left(1+\bar t^2\right)^{9/2}}\\
&+\frac{76\left(3-24\bar t^2+8\bar t^4\right)}{15\left(1+\bar t^2\right)^{9/2}}\log\left(\frac{b}{M}\right)-\frac{76\log(v)\left(3-24\bar t^2+8\bar t^4\right)}{15\left(1+\bar t^2\right)^{9/2}}-\frac{76\log\left(1+\bar t^2\right)\left(3-24\bar t^2+8\bar t^4\right)}{15\left(1+\bar t^2\right)^{9/2}}\\
&+\frac{304\left(6-57\bar t^2+25\bar t^4\right)}{45\left(1+\bar t^2\right)^{9/2}}+\frac{76\bar t\left(-11+10\bar t^2\right)}{9\left(1+\bar t^2\right)^{4}}\Bigg]\Bigg]\Bigg\},
\end{aligned}
\tag{4.49}
$$

and

$$
\begin{aligned}
F_\phi^{\mathrm{NL}}(\bar t)=\ &\frac{\varepsilon M^3}{b^3}\Bigg\{v^2\Bigg[\frac{8\gamma_E\bar t}{\left(1+\bar t^2\right)^{5/2}}+\frac{8\bar t}{\left(1+\bar t^2\right)^{5/2}}\log\left(\frac{b}{M}\right)-\frac{8\log(v)\bar t}{\left(1+\bar t^2\right)^{5/2}}-\frac{8\log\left(2\left(1+\bar t^2\right)\right)\bar t}{\left(1+\bar t^2\right)^{5/2}}-\frac{8\operatorname{arcsinh}(\bar t)\bar t}{\left(1+\bar t^2\right)^{5/2}}\\
&+\frac{68\bar t-8\bar t^3}{3\left(1+\bar t^2\right)^{5/2}}-\frac{8\left(-2+\bar t^2\right)}{3\left(1+\bar t^2\right)^{2}}\Bigg]+v^4\Bigg[\frac{20\operatorname{arcsinh}(\bar t)\bar t\left(1-3\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}\ \frac{20\log\left(2\left(1+\bar t^2\right)\right)\bar t\left(1-3\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{20\gamma_E\bar t\left(-1+3\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}\\
&+\frac{20\bar t\left(-1+3\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}\log\left(\frac{b}{M}\right)+\frac{\log(v)\left(20\bar t-60\bar t^3\right)}{\left(1+\bar t^2\right)^{7/2}}-\frac{14\bar t\left(53-217\bar t^2+10\bar t^4\right)}{15\left(1+\bar t^2\right)^{7/2}}-\frac{4\left(2-51\bar t^2+7\bar t^4\right)}{3\left(1+\bar t^2\right)^{3}}\Bigg]\Bigg\}\\
&+\frac{\varepsilon M^4}{b^4}\Bigg\{\frac{\pi^2\left(2-8\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{40\gamma_E\bar t}{\left(1+\bar t^2\right)^{3}}+\frac{40\bar t}{\left(1+\bar t^2\right)^{3}}\log\left(\frac{b}{M}\right)-\frac{40\log(v)\bar t}{\left(1+\bar t^2\right)^{3}}-\frac{16\log(2)\bar t\left(-4+\bar t^2\right)}{\left(1+\bar t^2\right)^{3}}\\
&+\frac{8\operatorname{arcsinh}(\bar t)^2\left(-1+4\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{\log\left(1+\bar t^2\right)\left(12\bar t-8\bar t^3\right)}{\left(1+\bar t^2\right)^{3}}+\frac{4\left(-4+11\bar t^2\right)}{3\left(1+\bar t^2\right)^{5/2}}+\frac{4\bar t\left(28+11\bar t^2\right)}{3\left(1+\bar t^2\right)^{3}}\\
&+\operatorname{arcsinh}(\bar t)\Bigg(\frac{\gamma_E\left(8-32\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{\left(8-32\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}\log\left(\frac{b}{M}\right)+\frac{8\log(2)\left(-1+4\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{8\log(v)\left(-1+4\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}\\
&+\frac{8\log\left(1+\bar t^2\right)\left(-1+4\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}+\frac{4\left(17-86\bar t^2+4\bar t^4\right)}{3\left(1+\bar t^2\right)^{7/2}}-\frac{8\bar t\left(-11+4\bar t^2\right)}{3\left(1+\bar t^2\right)^{3}}\Bigg)+\frac{12\left(-1+4\bar t^2\right)}{\left(1+\bar t^2\right)^{7/2}}\mathrm{Li}_2\left(\frac{\bar t+\sqrt{1+\bar t^2}}{\bar t-\sqrt{1+\bar t^2}}\right)\\
&+v^2\Bigg[\frac{4\gamma_E\bar t\left(-121+299\bar t^2\right)}{3\left(1+\bar t^2\right)^{4}}+\frac{4\bar t\left(-121+299\bar t^2\right)}{3\left(1+\bar t^2\right)^{4}}\log\left(\frac{b}{M}\right)-\frac{4\log(v)\bar t\left(-121+299\bar t^2\right)}{3\left(1+\bar t^2\right)^{4}}+\frac{\pi^2\left(-87+951\bar t^2-572\bar t^4\right)}{15\left(1+\bar t^2\right)^{9/2}}\\
&-\frac{4\pi\left(-3+6\bar t^2+\bar t^4\right)}{\left(1+\bar t^2\right)^{3}}-\frac{8\arctan(\bar t)\left(-3+6\bar t^2+\bar t^4\right)}{\left(1+\bar t^2\right)^{3}}-\frac{2\log\left(1+\bar t^2\right)\bar t\left(139-37\bar t^2+34\bar t^4\right)}{5\left(1+\bar t^2\right)^{4}}\\
&+\frac{4\operatorname{arcsinh}(\bar t)^2\left(11-93\bar t^2+36\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}-\frac{8\log(2)\bar t\left(511-803\bar t^2+51\bar t^4\right)}{15\left(1+\bar t^2\right)^{4}}+\frac{2\left(68-359\bar t^2+203\bar t^4\right)}{15\left(1+\bar t^2\right)^{7/2}}\\
&+\frac{2\bar t\left(-1002+4401\bar t^2+143\bar t^4\right)}{15\left(1+\bar t^2\right)^{4}}+\operatorname{arcsinh}(\bar t)\Bigg(-\frac{4\gamma_E\left(11-93\bar t^2+36\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}+\frac{4\log(2)\left(11-93\bar t^2+36\bar t^4\right)}{\left(1+\bar t^2\right)^{9/2}}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{4\left(11-93\bar{t}^2+36\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}} \log\left(\frac{b}{M}\right) + \frac{4\log(v)\left(11-93\bar{t}^2+36\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}} + \frac{4\log\left(1+\bar{t}^2\right)\left(11-93\bar{t}^2+36\bar{t}^4\right)}{\left(1+\bar{t}^2\right)^{9/2}} \\
&+ \frac{2\left(-881+9153\bar{t}^2-4866\bar{t}^4+20\bar{t}^6\right)}{15\left(1+\bar{t}^2\right)^{9/2}} + \frac{4\bar{t}\left(13+551\bar{t}^2-92\bar{t}^4\right)}{15\left(1+\bar{t}^2\right)^4}\Bigg) + \frac{2\left(87-951\bar{t}^2+572\bar{t}^4\right)}{5\left(1+\bar{t}^2\right)^{9/2}} \mathrm{Li}_2\left(\frac{\bar{t}+\sqrt{1+\bar{t}^2}}{\bar{t}-\sqrt{1+\bar{t}^2}}\right)\Bigg] \\
&+ v^3\Bigg[\frac{76\gamma_E\left(-1+4\bar{t}^2\right)}{5\left(1+\bar{t}^2\right)^{7/2}} - \frac{76\mathrm{arcsinh}\left(\bar{t}\right)\left(-1+4\bar{t}^2\right)}{5\left(1+\bar{t}^2\right)^{7/2}} - \frac{76\log(2)\left(-1+4\bar{t}^2\right)}{5\left(1+\bar{t}^2\right)^{7/2}} + \frac{76\left(-1+4\bar{t}^2\right)}{5\left(1+\bar{t}^2\right)^{7/2}} \log\left(\frac{b}{M}\right) \\
&- \frac{76\log(v)\left(-1+4\bar{t}^2\right)}{5\left(1+\bar{t}^2\right)^{7/2}} - \frac{76\log\left(1+\bar{t}^2\right)\left(-1+4\bar{t}^2\right)}{5\left(1+\bar{t}^2\right)^{7/2}} - \frac{304\left(44-281\bar{t}^2+315\bar{t}^4+364\bar{t}^6+212\bar{t}^8+48\bar{t}^{10}\right)}{525\left(1+\bar{t}^2\right)^{7/2}} \\
&- \frac{76\bar{t}\left(-279+790\bar{t}^2+1104\bar{t}^4+752\bar{t}^6+192\bar{t}^8\right)}{525\left(1+\bar{t}^2\right)^3}\Bigg]\Bigg\} + O(b^{-5}) \,,
\end{aligned}
\tag{4.50}
$$

where the same hierarchy already discussed for the field can be observed here. The crucial difference is that, at b^{-5} the t-component of the force only develops, at best, polylogarithm of order 2, while the ϕ-component contains also Li_3. This happens because the time derivatives is itself an operator that goes as $1/b$, which indeed shifts at one order higher all the special functions that were observed in the field, also the projection along the orthogonal direction to the particle motion maintains this feature. Similarly, the derivative with respect to ϕ goes as b^0 and it maintains the analytical structure of the field. For this reason, the $O(b^{-5})$ terms are not included in Eq. (4.50) for the ϕ-component of the force, but they are needed for computing the scattering angle and angular momentum loss up to b^{-5}.

Once the self-force is computed, one can finally compute observables. In particular the loss of energy and angular momentum and the self-force correction to the scattering angle. The first two quantities are purely dissipative effects, hence associated with radiation. The scattering angle, however, contains also a contribution from the conservative sector.

Leveraging the formal results obtained at the beginning of this Chapter, now the procedure for extracting the interesting observables is then conceptually easy, though it can be technically challenging when evaluating integrals from the non-local sector.

4.2.4 Dissipative Sector

The total radiated energy and angular momentum, can be easily computed by integrating the dissipative piece of the self-force along the worldline according to Eq. (4.26) with Eqs. (4.6a) and (4.24):

$$E^{\mathrm{rad}}_{1\mathrm{PM}} = E^{\mathrm{rad}}_{2\mathrm{PM}} = 0, \tag{4.51}$$

$$E^{\mathrm{rad}}_{3\mathrm{PM}} = \frac{\pi\varepsilon M^3}{b^3 v}\left[\frac{1}{6} + \frac{11v^2}{12} + \frac{25v^4}{24} + \frac{311v^6}{240} + O(v^7)\right], \tag{4.52}$$

$$E^{\mathrm{rad}}_{4\mathrm{PM}} = \frac{\varepsilon M^4}{b^4}\frac{4}{3v^3}\left[1 + \frac{23v^2}{3} - 2v^3 + \frac{28v^4}{5} - \frac{16v^5}{3} + \frac{4324v^6}{525} + O(v^7)\right], \tag{4.53}$$

$$
\begin{aligned}
E^{\mathrm{rad}}_{5\mathrm{PM}} = \frac{\pi\varepsilon M^5}{b^5}\frac{1}{2v^5}\Bigg\{&1 + 14v^2 - \frac{3\pi^2 v^3}{4} + \frac{89v^4}{8} - \left(\frac{96}{5} + \frac{9\pi^2}{40}\right)v^5 \\
&+ \left[\frac{20033}{600} + \frac{\pi^2}{2} - \frac{19\log(2)}{2} - \frac{19\log(b/M)}{5} + \frac{19\log(v)}{10}\right]v^6 + O(v^7)\Bigg\},
\end{aligned}
\tag{4.54}
$$

and

$$L^{\mathrm{rad}}_{1\mathrm{PM}} = 0, \tag{4.55}$$

$$L_{2\mathrm{PM}}^{\mathrm{rad}} = \frac{\varepsilon M^2}{b}\left[\frac{2}{3}+\frac{4v^2}{3}+\frac{4v^4}{3}+\frac{4v^6}{3}+O(v^7)\right], \tag{4.56}$$

$$L_{3\mathrm{PM}}^{\mathrm{rad}} = \frac{\pi\varepsilon M^3}{b^2v^2}\left[\frac{1}{3}+\frac{5v^2}{6}+\frac{v^4}{12}-\frac{31v^6}{120}+O(v^7)\right], \tag{4.57}$$

$$L_{4\mathrm{PM}}^{\mathrm{rad}} = \frac{\varepsilon M^4}{b^3}\frac{2}{3v^4}\left[1+8v^2-4v^3-\frac{23v^4}{3}-\frac{16v^5}{5}-\frac{1304v^6}{75}+O(v^7)\right], \tag{4.58}$$

$$\begin{aligned} L_{5\mathrm{PM}}^{\mathrm{rad}} = \frac{\pi\varepsilon M^5}{b^4}\frac{5}{2v^4}&\left\{1-\frac{\pi^2 v}{10}-\frac{11v^2}{10}-\left(\frac{16}{5}-\frac{\pi^2}{100}\right)v^3\right. \\ &\left.-\left[\frac{9883}{3000}-\frac{\pi^2}{15}+\frac{152\log(2)}{75}+\frac{19\log(b/M)}{25}-\frac{38\log(v)}{75}\right]v^4+O(v^5)\right\}, \end{aligned} \tag{4.59}$$

where $\varepsilon = q^2/(\mu M) \ll 1$ as defined in (4.3), and note the leading PN order is 1.5 for both δE^{diss} and δL^{diss}, except for the 5PM radiated angular momentum, which has its first contribution at 2.5PN.

It is also worth commenting on the absence of the Euler-Mascheroni constant γ_E in the final results. The presence of γ_E is usually tied up, in PN theory, to the presence of logarithms in the observables. It is then surprising that, for the scattering motion, the PN expansions of these quantities do contain logarithms but not γ_E. It is possible, however, that at higher orders γ_E will appear. This feature will be present also in the dissipative scattering angle.

Comparing these energy and angular momentum loss against the one obtained with amplitudes methods in Ref. [56], it is interesting to see a mismatch at 3PM-2.5PN,

$$E_{3\mathrm{PM}}^{\mathrm{rad}} - E_{3\mathrm{PM}}^{\mathrm{rad,Ref.[56]}} = \frac{\varepsilon M^3}{b^3}\frac{\pi}{2}v\left[1+\frac{v^2}{2}+O(v^5)\right], \tag{4.60}$$

which will propagate to the comparison between the scattering angle. This can be attributed to the fact that it is purely the loss of energy away from the binary system that is being computed in [56], whereas the results here, coming from the local dissipative force, also include energy loss into the black hole horizon. In this sense, this is a generalization of their result, which includes the full dissipative contributions on the scattering angle at each PM order.

Nonetheless, it is still possible to provide some verification of these results. In [152], the 3PM mass shift of the Schwarzschild black hole in the case of scalar absorption was computed. The horizon flux that they obtained is

$$E_H = \frac{\varepsilon M^3}{b^3}\frac{\pi}{2}\frac{v}{\sqrt{1-v^2}}, \tag{4.61}$$

which is precisely the resummation of Eq. (4.60).

The last observable that one can compute in the dissipative sector is the radiative correction to the scattering angle. Starting from Eq. (4.25), one find

$$\delta\chi_{1\mathrm{PM}}^{\mathrm{diss}} = \delta\chi_{2\mathrm{PM}}^{\mathrm{diss}} = 0, \tag{4.62}$$

$$\delta\chi_{3\mathrm{PM}}^{\mathrm{diss}} = \frac{\varepsilon M^3}{b^3v^3}\left[\frac{2}{3}+\frac{5v^2}{3}+\frac{19v^4}{12}+\frac{25v^6}{24}+O(v^7)\right], \tag{4.63}$$

$$\delta\chi_{4\mathrm{PM}}^{\mathrm{diss}} = \frac{\pi\varepsilon M^4}{b^4v^5}\left[\frac{1}{2}+\frac{10v^2}{3}+\frac{89v^4}{48}+\frac{53v^6}{80}+O(v^7)\right], \tag{4.64}$$

$$\delta\chi^{\rm diss}_{\rm 5PM} = \frac{4\varepsilon M^5}{3b^5v^7}\left[1+\left(\frac{91}{6}+\frac{3\pi^2}{4}\right)v^2 - 4v^3 + \left(\frac{1357}{120}+\frac{39\pi^2}{32}\right)v^4 - \frac{14v^5}{15}\right.$$
$$\left. + \left(\frac{16103}{560}-\frac{45\pi^2}{16}\right)v^6 + O(v^7)\right]. \tag{4.65}$$

It is interesting how the analytical features of the energy and angular momentum loss are reflected in the scattering angle. First of all, the 3PM energy and angular momentum loss and the 4PM scattering angle present only (relative) integer PN orders (corresponding to even powers of v). This is because only the local sector contributes at this PM order. However, at the next PM order, half-integer PN terms (odd powers of v) start to appear, which are usually related to tail effects in classical PN theory. This structure between local and non-local terms become apparent after the final integration over the orbit, and not, for instance in the self-force or in the regular field. Another smoking gun for tail contributions is the presence of $\log(b/M)$ and $\log v$, which are related to the physical scales of the problem. These log contributions are present in the 5PM losses but they cancel in the 5PM scattering angle, suggesting that they should appear in the dissipative sector at higher order in both PM and PN.

4.2.5 Conservative Sector

The conservative dynamics affects the shape of the hyperbolic orbit, by changing the relationships between orbital parameters and, more importantly, the scattering angle. In the conservative sector the $\delta\chi_{\rm cons}$ takes the form

$$\delta\chi_{\rm cons} = \frac{\partial\bar{\chi}}{\partial E_-}\delta E_0^{\rm cons} + \frac{\partial\bar{\chi}}{\partial L_-}\delta L_0^{\rm cons}$$
$$-2\int_{\bar{r}_{\rm min}}^{\infty} dr\left[a_E^+(r)\int_{\bar{r}_{\rm min}}^{r}\frac{dr}{\bar{u}^r}F_t^{\rm cons\,+} - a_L^+(r)\int_{\bar{r}_{\rm min}}^{r}\frac{dr}{\bar{u}^r}F_\phi^{\rm cons\,+}\right], \tag{4.66}$$

where

$$\delta E_0^{\rm cons} = -\int_{-\infty}^{0} d\tau F_t^{\rm cons}, \qquad \delta L_0^{\rm cons} = \int_{-\infty}^{0} d\tau F_\phi^{\rm cons}, \tag{4.67}$$

and for the conservative dynamics, one clearly has

$$\int_{-\infty}^{+\infty} d\tau F^{\rm cons}_{t/\phi} = 0. \tag{4.68}$$

Evaluating these formulas, the conservative scattering angle is then

$$\delta\chi^{\rm cons}_{\rm 2PM} = -\frac{\pi\varepsilon M^2}{4b^2}, \tag{4.69}$$

$$\delta\chi^{\rm cons}_{\rm 3PM} = -\frac{\varepsilon M^3}{b^3v^2}\left[4+\frac{2v^2}{3}+\frac{5v^4}{6}+O(v^5)\right], \tag{4.70}$$

$$\delta\chi^{\rm cons}_{\rm 4PM} = -\frac{\pi\varepsilon M^4}{b^4v^4}\left\{\frac{9}{4}+v^2\left[\frac{91}{24}+\frac{21\pi^2}{128}+\log(v/2)\right]\right.$$
$$\left. + v^4\left[\frac{493}{480}+\frac{4335\pi^2}{8192}-\frac{3\log(b/M)}{2}+2\log\left(\frac{v}{2}\right)\right]+O(v^5)\right\}, \tag{4.71}$$

$$\delta\chi^{\rm cons}_{\rm 5PM} = -\frac{16\varepsilon M^5}{3b^5 v^6}\left\{1 + v^2\left[\frac{47}{6} + \frac{21\pi^2}{64} + 2\log(2v)\right]\right.$$
$$\left. - v^4\left[\frac{797}{72} - \frac{205\pi^2}{128} - \frac{43\log(2)}{3} + 4\log\left(\frac{b}{M}\right) - \frac{19\log(v)}{3}\right] + \frac{19v^5}{15} + O(v^6)\right\}, \tag{4.72}$$

where the leading term is at 2PN. Consistent with the dissipative sector, no γ_E appears in these final results.

Another interesting feature is that both the energy and the angular momentum contribute to the leading 2PM term of the conservative scattering angle, displayed in Eq. (4.69), and this leading term is totally determined by the local sector of the forces.

The non-local sector starts to contribute at 4PM, in Eq. (4.71), introducing a transcendental structure to the result. In this case, one get first the logarithmic contributions and then at 5PM the odd powers of the velocity. However, by looking at the non-local expressions for the forces in Eqs. (4.49)-(4.50), the leading terms are at lower powers in $1/b$. It is a non-trivial statement that these leading terms in the non-local components of the force are integrated to zero once they are multiplied by $a^+_{E/L}$.

As for the dissipative sector, it is possible to compare these results with the conservative scattering angle computed using amplitudes methods in Ref. [56]. In their framework, the scattering angle up to 4PM was obtained to *all orders in* v, but with the presence of two free parameters, namely c_1 and $c_2(\bar\mu)$, that appear precisely at 4PM. There is an immediate agreement at 2PM and 3PM (restricted to the precision in the velocity expansion), while the 4PM scattering angle is [56]

$$\chi^{\rm 4PM}_{\rm cons} = \frac{3\pi\varepsilon M^4}{8b^4}\frac{1}{\sigma^2-1}\left\{-\left[4\mathcal{M}_4^t\log\left(\frac{\sqrt{\sigma^2-1}}{2}\right) + \mathcal{M}_4^{\pi^2} - \mathcal{M}_4^{\rm rem}\right]\right.$$
$$\left. + (\sigma^2-5)c_1 + \left[c_2(\bar\mu) - \frac{31}{3} + 2\log\left(2b^2 e^{2\gamma_E}\bar\mu^2\right)\right](\sigma^2-1)\right\}, \tag{4.73}$$

where $\sigma = (1-v^2)^{-1/2}$ and the functions $\mathcal{M}_4^t$, $\mathcal{M}_4^{\pi^2}$ and $\mathcal{M}_4^{\rm rem}$ are complicated functions of σ given in Eqs. (5.7)–(5.8) of Ref. [56].

The low-velocity expansion of the 4PM scattering angle (4.73) gives

$$\chi^{\rm 4PM}_{\rm cons} = -\frac{9\pi\varepsilon M^4}{4b^4 v^4}\left\{1 + v^2\left[\frac{85}{54} + \frac{2c_1}{3} + \frac{7\pi^2}{96} - \frac{4\log 2}{9} + \frac{4\log v}{9}\right]\right.$$
$$+ v^4\left[\frac{313}{1080} - \frac{5c_1}{6} - \frac{c_2(\bar\mu)}{6} - \frac{2\gamma_E}{3} + \frac{1445\pi^2}{6144} - \frac{8\log 2}{9} + \frac{8\log v}{9} - \frac{1}{3}\log\left(2b^2\bar\mu^2\right)\right]$$
$$\left. + O(v^5)\right\}. \tag{4.74}$$

An important feature of this expression is that c_1 and $c_2(\bar\mu)$ appear independently and at successive PN orders. It is thus clear from this expression that, by matching the powers of v with the 4PM scattering angle in Eq. (4.71), there are enough conditions to uniquely determine the two coefficients, namely

$$c_1 = \frac{1}{6}, \quad c_2(\bar\mu) = -\frac{11}{6} - 4\gamma_E - 2\log(2) - 4\log(\bar\mu M). \tag{4.75}$$

It was suggested at the end of Sec. 3.2 of Ref. [56] that c_1 could be associated with a static Love number, which vanishes for a black hole in general relativity, whereas $c_2(\bar{\mu})$ arises from a regularization procedure. In Ref. [56] some constraints on these parameters were found by comparing the analytical PM scattering angle to fully relativistic, numerical self-force calculations, with the scale fixed to $\bar{\mu} = (2M)^{-1}$. Specifically they found for the first coefficient $c_1 = 0.31 \pm 0.38$, whereas for the second an interval was presented in the approximate range $(-25, -35)$. Clearly, this value for c_1 is compatible with the numerical values they find inside their error bars. If c_1 is proportional to a static Love number it must be zero, but the case of a non-vanishing c_1 was already considered to be possible in Ref. [56] because of the field redefinitions used in the construction of the leading tidal action. For c_2 there is no particular theoretical value one might expect.

5 Asymptotic solutions: Gravitational waveform and fluxes

This Chapter is based on ongoing and unpublished projects on the calculation of self-force gravitational waveform and fluxes from a hyperbolic encounter. This was already done in [101] by computing the fluxes at infinity for the energy and the angular momentum at 7PN and at 5PM and 4PM, respectively. Here the main focus will be on the computation of the so-called *amplitudes*, which are the basic building blocks for both the waveforms and the fluxes.

5.1 Introduction

In Chapter 2 the boundary conditions for the radial Teukolsky equation were presented in Eq. (2.52). In these expressions some constant in r coefficients appears:

$$ {}_sR^{-,\mathrm{trans}}_{lm\omega}\,, \quad {}_sR^{-,\mathrm{inc}}_{lm\omega}\,, \quad {}_sR^{-,\mathrm{ref}}_{lm\omega}\,, \tag{5.1} $$

which are associated respectively with the transmitted, incident, and reflected wave at the boundary, i.e. the future null infinity and the horizon. It is then possible to write the asymptotic solution of the radial Teukolsky equation by starting from Eq. (3.46), taking the limit for r that goes to $+\infty$ or r_+, by expliciting these coefficients. Hence, one gets

$$ \lim_{r\to+\infty} {}_s\Psi_{lm\omega}(r) = r^{-1-2s} e^{i\omega r_*}\, {}_sR^{\mathrm{up,trans}}_{lm\omega} \int_{r_+}^{+\infty} dr'\, \frac{{}_sR^{\mathrm{in}}_{lm\omega}(r')}{W(r')\Delta}\, {}_sT_{lm\omega}(r')\,, \tag{5.2a} $$

$$ \lim_{r\to r_+} {}_s\Psi_{lm\omega}(r) = \Delta^{-s} e^{-ikr_*}\, {}_sR^{\mathrm{in,trans}}_{lm\omega} \int_{r_+}^{+\infty} dr'\, \frac{{}_sR^{\mathrm{up}}_{lm\omega}(r')}{W(r')\Delta}\, {}_sT_{lm\omega}(r')\,, \tag{5.2b} $$

where a common choice is to define the amplitudes $\mathcal{Z}$ as

$$ {}_s\mathcal{Z}^{\infty}_{lm\omega} = {}_sR^{\mathrm{up,trans}}_{lm\omega} \int_{r_+}^{+\infty} dr'\, \frac{{}_sR^{\mathrm{in}}_{lm\omega}(r')}{W(r')\Delta}\, {}_sT_{lm\omega}(r')\,, \tag{5.3a} $$

$$ {}_s\mathcal{Z}^{H}_{lm\omega} = {}_sR^{\mathrm{in,trans}}_{lm\omega} \int_{r_+}^{+\infty} dr'\, \frac{{}_sR^{\mathrm{up}}_{lm\omega}(r')}{W(r')\Delta}\, {}_sT_{lm\omega}(r')\,, \tag{5.3b} $$

which are respectively the amplitude at infinity and at the horizon. It is possible to prove that the amplitudes are the basic building blocks that are needed to compute both the waveform and the flux of energy and angular momentum.

In the following the focus will be only on evaluating the asymptotic solution at infinity. In the general case of a spin-s perturbation, in a bound or unbound orbit, the procedure starts from evaluating the time domain field

$$\begin{aligned}
\lim_{r\to+\infty} {}_s\Psi_{lm}(r) &= r^{-1-2s}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,e^{-i\omega(t-r_*)}\,{}_sR^{\text{up,trans}}_{lm\omega}\\
&\times\int_{r_+}^{+\infty}dr'\,\frac{{}_sR^{\text{in}}_{lm\omega}(r')}{W(r')\Delta}\int_{-\infty}^{\infty}dt'\,e^{i\omega t'}\,{}_sT_{lm}(t',r')\,{}_sS_{lm}(\theta,\phi;a\omega)\\
&= r^{-1-2s}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,e^{-i\omega(t-r_*)}\,{}_s\mathcal{Z}^{\infty}_{lm\omega}\,,
\end{aligned}\tag{5.4}$$

where:

- The source ${}_sT_{lm}(t',r')$ assumes the same generic form highlighted in Eq. (3.43);
- The transmission coefficient ${}_sR^{\text{up,trans}}_{lm\omega}$ and the homogeneous solution inside the radial integral both have the polynomial and logarithmic in ω behaviour that was leveraged in Chapter 3 for defining local and non-local contributions;

which allows the use of the same exact methodology developed for the retarded field in Chapter 3. It is worthwhile to notice that the time domain amplitudes can then be defined as

$${}_s\mathcal{Z}^{\infty}_{lm}(T) := \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,e^{-i\omega T}\,{}_s\mathcal{Z}^{\infty}_{lm\omega}\,,\tag{5.5}$$

where $T := t - r_*$ is the retarded time.

Once the amplitudes are computed, one can use them to obtain the flux. It is more convenient, however, to perform this last step by specifying the spin s of the perturbation. The reason is that, for different values of the spin of the perturbations, there are different relation that could be leveraged. For instance, for $s = 0$ it is convenient to define the stress energy tensor

$$T_{\mu\nu} = \frac{\mu^2}{8\pi}\left\{\nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}\nabla^\lambda\psi\nabla_\lambda\psi\right\},\tag{5.6}$$

where ψ is the time-domain solution to the KG equation and μ is the mass of the scalar charge that sourced the field ψ. The loss of energy at infinity per unit volume is then defined as

$$\frac{d^2E^\infty}{d\Omega dt} = \lim_{r\to\infty} r^2\,T^r{}_t\,,\tag{5.7}$$

and similarly, by looking at the (r,ϕ) component of the stress energy tensor, it is possible to define the angular momentum flux at infinity.

For gravitational perturbations a similar procedure can be devised by defining a gravitational stress energy tensor, as was done by Isaacson in [153], but this procedure is rather impractical, as pointed out in [110]. Instead, one can use the simple relation between the Weyl scalar ψ_4 with the gravitational wave strain

$$\zeta^4\psi_4 \sim -\frac{1}{2}\ddot{h}_{\bar{m}\bar{m}}\,, \tag{5.8}$$

where $h_{\bar{m}\bar{m}} = h_+ - ih_\times$ and the dot indicates the derivative with respect to the coordinate time. The energy flux can be written as

$$\left(\frac{dE}{d\Omega dt}\right)^{\text{rad}}_{r\to\infty} = \frac{r^2}{16\pi}\left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle = \frac{r^2}{16\pi}\left\langle \dot{h}_{mm}\dot{h}_{\bar{m}\bar{m}} \right\rangle\,, \tag{5.9}$$

where the square brackets indicate the orbit average if the motion is bound (a integral over an orbital cycle) or, if the orbit is unbound a integral over the full scattering event. The flux of angular momentum along the z axis can also be computed by using a similar definition as

$$\left(\frac{dL_z}{d\Omega dt}\right)^{\text{rad}}_{r\to\infty} = \frac{r^2}{16\pi}\left\langle \partial_\phi h_{mm}\dot{h}_{\bar{m}\bar{m}}\,, \right\rangle\,, \tag{5.10}$$

however this calculation will not be discussed in this Chapter.

In the following all the discussion will be specialized to the case of a Schwarzschild black hole but, as before, there are no conceptual issues in generalizing to Kerr.

5.2 Gravitational Flux

From the asymptotic relation given in Eq. (5.8), one can easily extract the expression for the gravitational strain integrating two times the Weyl scalar with respect to the coordinate time t.

In order to get there, one should first prepare the ground by manipulating the asymptotic expressions for ψ_4, starting from

$$\lim_{r\to+\infty}\zeta^4\psi_{4,lm\omega}(r) = r^3 e^{i\omega r_*}\,{}_{-2}R^{\text{up,trans}}_{lm\omega}\int_{r_+}^{+\infty} dr'\,\frac{{}_{-2}R^{\text{in}}_{lm\omega}(r')}{W_{lm\omega}}\Delta^{-2}\,{}_{-2}T_{lm\omega}(r')\,, \tag{5.11}$$

where the subscript -2 will be dropped in following computation to lighten the notation and the invariant Wronskian has been introduced again. This can be again simplified by introducing the rescaling (3.51) to get

$$\lim_{r\to+\infty}\psi_{4,lm\omega}(r) = \frac{1}{\bar{r}}e^{i\bar{\omega}\bar{r}_*}R^{\text{up,trans}}_{lm\bar{\omega}}\int_{\bar{r}_+}^{+\infty} d\bar{r}'\,\frac{R^{\text{in}}_{lm\omega}(\bar{r}')}{W_{lm\omega}}\Delta^{-2}T_{lm\bar{\omega}}(\bar{r}')\,. \tag{5.12}$$

and by using Eq. (3.43) one can easily perform the radial integral and extract the t' integral that it is hidden inside the source term as

$$\begin{aligned}\lim_{r\to+\infty}\psi_{4,lm\omega}(r) = \frac{b}{v\bar{r}}e^{i\bar{\omega}\bar{r}_*}R^{\text{up,trans}}_{lm\bar{\omega}}\int d\bar{t}'\,e^{i\bar{\omega}\bar{t}'}\int_{\bar{r}_+}^{+\infty} d\bar{r}'\,\frac{R^{\text{in}}_{lm\bar{\omega}}(\bar{r}')}{W_{lm\bar{\omega}}}\Delta^{-2}\Big\{T^{(0)}_{lm}(\bar{t}')\delta(\bar{r}'-\bar{r}(\bar{t}'))\\ + \frac{1}{b}T^{(1)}_{lm}(\bar{t}')\delta'(\bar{r}'-\bar{r}(\bar{t}')) + \frac{1}{b^2}T^{(2)}_{lm}(\bar{t}')\delta''(\bar{r}'-\bar{r}(\bar{t}'))\Big\}\,,\end{aligned} \tag{5.13}$$

where the radial integral is trivial and, by defining

$$R_{lm\bar{\omega}}(\bar{r}_p(\bar{t}')) = R^{\text{in}}_{lm\bar{\omega}}(\bar{r}_p(\bar{t}'))\Delta^{-2}T^{(0)}_{lm}(\bar{t}') - \frac{1}{b}\frac{\partial}{\partial \bar{r}'}\left[R^{\text{in}}_{lm\bar{\omega}}(\bar{r}')\Delta^{-2}T^{(1)}_{lm}(\bar{t}')\right]\Bigg|_{\bar{r}'=\bar{r}_p(\bar{t}')}$$
$$+\frac{1}{b^2}\frac{\partial^2}{\partial \bar{r}'^2}\left[R^{\text{in}}_{lm\bar{\omega}}(\bar{r}')\Delta^{-2}T^{(2)}_{lm}(\bar{t}')\right]\Bigg|_{\bar{r}'=\bar{r}_p(\bar{t}')}, \tag{5.14}$$

it leads immediately to the following definition for the frequency domain asymptotic ψ_4

$$\lim_{r\to+\infty}\psi_{4,lm\omega}(r) = \frac{b}{v\bar{r}}e^{i\bar{\omega}\bar{r}_*}R^{\text{up,trans}}_{lm\bar{\omega}}\int d\bar{t}'\, e^{i\bar{\omega}\bar{t}'}\frac{R_{lm\bar{\omega}}(\bar{r}_p(\bar{t}'))}{W_{lm\bar{\omega}}}, \tag{5.15}$$

and by returning to the time domain one has

$$\lim_{r\to+\infty}\psi_{4,lm}(\bar{T}) = \frac{1}{\bar{r}}\int_{-\infty}^{+\infty}\frac{d\bar{\omega}}{2\pi}e^{-i\bar{\omega}\bar{T}}R^{\text{up,trans}}_{lm\bar{\omega}}\int d\bar{t}'\, e^{i\bar{\omega}\bar{t}'}\frac{R_{lm\bar{\omega}}(\bar{r}_p(\bar{t}'))}{W_{lm\bar{\omega}}}, \tag{5.16}$$

where $\bar{T}$ is the rescaled retarded time. It is easy to recognize that the right hand side is simply $\bar{r}^{-1}$ times the time domain amplitude of Eq. (5.5).

Finally, one can use Eq. (5.8) and write

$$\ddot{h}_+ - i\ddot{h}_\times = -\frac{2}{\bar{r}}\sum_{l,m}\int_{-\infty}^{+\infty}\frac{d\bar{\omega}}{2\pi}e^{-i\bar{\omega}\bar{T}}R^{\text{up,trans}}_{lm\bar{\omega}}\int d\bar{t}'\, e^{i\bar{\omega}\bar{t}'}\frac{R_{lm\bar{\omega}}(\bar{r}_p(\bar{t}'))}{W_{lm\bar{\omega}}}\,{}_{-2}Y_{lm}(\theta,\phi), \tag{5.17}$$

where the sum over the modes has been reintroduced and, by integrating two times with respect to $\bar{t}$ (or equivalently $\bar{T}$), one gets

$$h_+ - ih_\times = \frac{2}{\bar{r}}\sum_{l,m}\int_{-\infty}^{+\infty}\frac{d\bar{\omega}}{2\pi}e^{-i\bar{\omega}\bar{T}}\frac{R^{\text{up,trans}}_{lm\bar{\omega}}}{\bar{\omega}^2}\int d\bar{t}'\, e^{i\bar{\omega}\bar{t}'}\frac{R_{lm\bar{\omega}}(\bar{r}_p(\bar{t}'))}{W_{lm\bar{\omega}}}\,{}_{-2}Y_{lm}(\theta,\phi), \tag{5.18}$$

which can be written in a more compact form as

$$(h_+ - ih_\times)(\bar{T}) = \frac{2}{\bar{r}}\sum_{l,m}\int_{-\infty}^{+\infty}\frac{d\bar{\omega}}{2\pi}e^{-i\bar{\omega}\bar{T}}\frac{\mathcal{Z}^\infty_{lm\bar{\omega}}}{\bar{\omega}^2}\,{}_{-2}Y_{lm}(\theta,\phi), \tag{5.19}$$

where the explicit dependence of the strain from the angles (θ,ϕ) is completely encoded inside the spin-2 spherical harmonics and also

$$\mathcal{Z}^\infty_{lm\bar{\omega}} := \int d\bar{t}' e^{i\bar{\omega}\bar{t}'}\tilde{\mathcal{Z}}^\infty_{lm\bar{\omega}}(\bar{t}') = R^{\text{up,trans}}_{lm\bar{\omega}}\int d\bar{t}'\, e^{i\bar{\omega}\bar{t}'}\frac{R_{lm\bar{\omega}}(\bar{r}_p(\bar{t}'))}{W_{lm\bar{\omega}}}. \tag{5.20}$$

In addition to this last expression, the structure in $\bar{\omega}$ of the $\mathcal{Z}^\infty_{lm\bar{\omega}}$ is known and identical to Eq. (3.55) and hence

$$(h_+ - ih_\times)(\bar{T}) = \frac{2}{\bar{r}}\sum_{l,m}\int_{-\infty}^{+\infty}\frac{d\bar{\omega}}{2\pi}e^{-i\bar{\omega}\bar{T}}\left\{\sum_j \bar{\omega}^{j-2}\mathcal{Z}^\infty_{lm;j}\right.$$

$$+\sum_{j=2}\sum_{k=1}\bar{\omega}^{j-2}\log^k(-i\bar{\omega})\mathcal{Z}^{\infty}_{lm;jk}\Bigg\}\,{}_{-2}Y_{lm}(\theta,\phi)\,, \tag{5.21}$$

where, crucially, $j \geq 2$. From this expression it is possible to separate the local and non-local contributions to the strain and, by combining the strain with its complex conjugate, it is possible to extract the two polarizations of the strain.

Once the strain is computed, the flux can be rapidly obtained by using Eq. (5.9) as

$$(\dot{h}_+ - i\dot{h}_\times)(\bar{T}) = \frac{-2i}{\bar{r}}\sum_{l,m}\int_{-\infty}^{+\infty}\frac{d\bar{\omega}}{2\pi}e^{-i\bar{\omega}\bar{T}}\Bigg\{\sum_j \bar{\omega}^{j-1}\mathcal{Z}^{\infty}_{lm;j}$$

$$+\sum_{j=2}\sum_{k=1}\bar{\omega}^{j-1}\log^k(-i\bar{\omega})\mathcal{Z}^{\infty}_{lm;jk}\Bigg\}\,{}_{-2}Y_{lm}(\theta,\phi)\,, \tag{5.22}$$

which must be multiplied by its complex conjugate in order to obtain the following expression for the flux

$$\left(\frac{dE}{d\Omega d\bar{T}}\right)^{\rm rad}_{r\to\infty} = \sum_{l,m}\sum_{l',m'}\int d\bar{T}\,\frac{\bar{r}^2}{16\pi}\frac{-2i}{\bar{r}}\frac{2i}{\bar{r}}\int\frac{d\bar{\omega}}{2\pi}e^{-i\bar{\omega}\bar{T}}\Bigg\{\sum_j\bar{\omega}^{j-1}\mathcal{Z}^{\infty}_{lm;j}$$

$$+\sum_{j=2}\sum_{k=1}\bar{\omega}^{j-1}\log^k(-i\bar{\omega})\mathcal{Z}^{\infty}_{lm;jk}\Bigg\}\,{}_{-2}Y_{lm}(\theta,\phi)$$

$$\times\int\frac{d\bar{\omega}'}{2\pi}e^{i\bar{\omega}'\bar{T}}\Bigg\{\sum_{j'}\bar{\omega}'^{j'-1}\mathcal{Z}^{*,\infty}_{l'm';j'} + \sum_{j'=2}\sum_{k'=1}\bar{\omega}'^{j'-1}\log^{k'}(i\bar{\omega}')\mathcal{Z}^{*,\infty}_{l'm';j'k'}\Bigg\}\,{}_{-2}Y^*_{l'm'}(\theta,\phi)\,, \tag{5.23}$$

this expression can be easily simplified by exploiting the integration over the surface element as

$$\left(\frac{dE}{d\bar{T}}\right)^{\rm rad}_{r\to\infty} = \sum_{l,m}\sum_{l',m'}\int d\bar{T}\,\frac{1}{4\pi}\int d\Omega\,{}_{-2}Y^*_{l'm'}(\theta,\phi)\,{}_{-2}Y_{lm}(\theta,\phi)$$

$$\times\int\frac{d\bar{\omega}}{2\pi}e^{-i\bar{\omega}\bar{T}}\Bigg\{\sum_j\bar{\omega}^{j-1}\mathcal{Z}^{\infty}_{lm;j} + \sum_{j=2}\sum_{k=1}\bar{\omega}^{j-1}\log^k(-i\bar{\omega})\mathcal{Z}^{\infty}_{lm;jk}\Bigg\}$$

$$\times\int\frac{d\bar{\omega}'}{2\pi}e^{i\bar{\omega}'\bar{T}}\Bigg\{\sum_{j'}\bar{\omega}'^{j'-1}\mathcal{Z}^{*,\infty}_{l'm';j'} + \sum_{j'=2}\sum_{k'=1}\bar{\omega}'^{j'-1}\log^{k'}(i\bar{\omega}')\mathcal{Z}^{*,\infty}_{l'm';j'k'}\Bigg\}\,, \tag{5.24}$$

where $d\Omega = \sin\theta d\theta d\phi$ by using the identity for the spheroidal harmonics

$$\int d\Omega\,{}_{-2}Y^*_{l'm'}(\theta,\phi)\,{}_{-2}Y_{lm}(\theta,\phi) = \delta_{l,l'}\delta_{m,m'}\,, \tag{5.25}$$

which simplifies the overall expression by removing the summation over the primed indices

$$\left(\frac{dE}{d\bar{T}}\right)^{\rm rad}_{r\to\infty} = \sum_{l,m}\int d\bar{T}\,\frac{1}{4\pi}\int\frac{d\bar{\omega}}{2\pi}e^{-i\bar{\omega}\bar{T}}\Bigg\{\sum_j\bar{\omega}^{j-1}\mathcal{Z}^{\infty}_{lm;j} + \sum_{j=2}\sum_{k=1}\bar{\omega}^{j-1}\log^k(-i\bar{\omega})\mathcal{Z}^{\infty}_{lm;jk}\Bigg\}$$

$$\times \int \frac{d\bar{\omega}'}{2\pi} e^{i\bar{\omega}'\bar{T}} \left\{ \sum_{j'} \bar{\omega}'^{j'-1} \mathcal{Z}^{*,\infty}_{lm;j'} + \sum_{j'=2} \sum_{k'=1} \bar{\omega}'^{j'-1} \log^{k'}(i\bar{\omega}') \mathcal{Z}^{*,\infty}_{lm;j'k'} \right\} , \tag{5.26}$$

and the $\bar{T}$ integral can be used to send $\bar{\omega}' \to \bar{\omega}$ and write

$$\left(\frac{dE}{d\bar{T}}\right)^{\rm rad}_{r\to\infty} = \sum_{l,m} \frac{1}{4\pi} \int \frac{d\bar{\omega}}{2\pi} \left\{ \sum_{j} \bar{\omega}^{j-1} \mathcal{Z}^{\infty}_{lm;j} + \sum_{j=2} \sum_{k=1} \bar{\omega}^{j-1} \log^{k}(-i\bar{\omega}) \mathcal{Z}^{\infty}_{lm;jk} \right\}$$
$$\times \left\{ \sum_{j'} \bar{\omega}^{j'-1} \mathcal{Z}^{*,\infty}_{lm;j'} + \sum_{j'=2} \sum_{k'=1} \bar{\omega}^{j'-1} \log^{k'}(i\bar{\omega}) \mathcal{Z}^{*,\infty}_{lm;j'k'} \right\} . \tag{5.27}$$

Hence by expliciting the product one can split between the local and the non-local flux as

$$\left(\frac{dE}{d\bar{T}}\right)^{\rm rad}_{r\to\infty} = \sum_{l,m} \frac{1}{4\pi} \int \frac{d\bar{\omega}}{2\pi} \left\{ \sum_{j,j'} \bar{\omega}^{j+j'-2} \mathcal{Z}^{\infty}_{lm;j} \mathcal{Z}^{*,\infty}_{lm;j'} \right.$$
$$+ \sum_{j=2,j'} \sum_{k=1} \bar{\omega}^{j+j'-2} \log^{k}(-i\bar{\omega}) \mathcal{Z}^{\infty}_{lm;jk} \mathcal{Z}^{*,\infty}_{lm;j'} + \sum_{j'=2,j} \sum_{k'=1} \bar{\omega}^{j+j'-2} \log^{k'}(i\bar{\omega}) \mathcal{Z}^{\infty}_{lm;j} \mathcal{Z}^{*,\infty}_{lm;j'k'}$$
$$\left. + \sum_{j'=2,j=2} \sum_{k'=1,k=1} \bar{\omega}^{j+j'-2} \log^{k'}(i\bar{\omega}) \log^{k}(-i\bar{\omega}) \mathcal{Z}^{*,\infty}_{lm;jk} \mathcal{Z}^{*,\infty}_{lm;j'k'} \right\} , \tag{5.28}$$

where

$$\left(\frac{dE}{d\bar{T}}\right)^{\rm rad,L}_{r\to\infty} = \sum_{l,m} \frac{1}{4\pi} \int \frac{d\bar{\omega}}{2\pi} \sum_{j,j'} \bar{\omega}^{j+j'-2} \mathcal{Z}^{\infty}_{lm;j} \mathcal{Z}^{*,\infty}_{lm;j'} , \tag{5.29a}$$

$$\left(\frac{dE}{d\bar{T}}\right)^{\rm rad,NL}_{r\to\infty} = \sum_{l,m} \frac{1}{4\pi} \int \frac{d\bar{\omega}}{2\pi} \left\{ \sum_{j'=2,j=2} \sum_{k'=1,k=1} \bar{\omega}^{j+j'-2} \log^{k'}(i\bar{\omega}) \log^{k}(-i\bar{\omega}) \mathcal{Z}^{*,\infty}_{lm;jk} \mathcal{Z}^{*,\infty}_{lm;j'k'} \right.$$
$$\left. + \sum_{j=2,j'} \sum_{k=1} \bar{\omega}^{j+j'-2} \log^{k}(-i\bar{\omega}) \mathcal{Z}^{\infty}_{lm;jk} \mathcal{Z}^{*,\infty}_{lm;j'} + \sum_{j'=2,j} \sum_{k'=1} \bar{\omega}^{j+j'-2} \log^{k'}(i\bar{\omega}) \mathcal{Z}^{\infty}_{lm;j} \mathcal{Z}^{*,\infty}_{lm;j'k'} \right\} . \tag{5.29b}$$

The Eqs. (5.29) are the general expressions, in terms of amplitudes, for the local and non-local flux that could be used for both bound or unbound orbits. For practical uses, however, this expression is not very suitable for analytical expressions. The reason is twofold: inside the amplitudes $\mathcal{Z}^{\infty}_{lm;j}$ is always hidden an integral with respect to the rescaled coordinate time, hence each term should always be intended as a triplets of integral, for instance in the local sector the full expression is

$$\left(\frac{dE}{d\bar{T}}\right)^{\rm rad,L}_{r\to\infty} = \sum_{l,m} \frac{1}{4\pi} \int \frac{d\bar{\omega}}{2\pi} \sum_{j,j'} \bar{\omega}^{j+j'-2} R^{\rm up,trans}_{lm;j} R^{*,\rm up,trans}_{lm;j'} \int d\bar{t}' e^{i\bar{\omega}\bar{t}'} \frac{R_{lm;j}(\bar{r}_p(\bar{t}'))}{W_{lm;j}}$$
$$\times \int d\bar{t}'' e^{-i\bar{\omega}\bar{t}''} \frac{R^{*}_{lm;j'}(\bar{r}_p(\bar{t}''))}{W^{*}_{lm;j'}} \left| {}_{-2}Y_{lm}\left(\frac{\pi}{2}, 0\right) \right|^2 , \tag{5.30}$$

and the by combining the exponentials, the $\bar{\omega}$ integral becomes trivial and one gets

$$\left(\frac{dE}{d\bar{T}}\right)^{\text{rad,L}}_{r\to\infty} = \sum_{l,m}\frac{1}{4\pi}\sum_{j,j'} R^{\text{up,trans}}_{lm;j} R^{*,\text{up,trans}}_{lm;j'}\, i^{-(j+j'-2)} \\ \int d\bar{t}' \int d\bar{t}''\delta^{(j+j'-2)}(\bar{t}'-\bar{t}'')\frac{R_{lm;j}(\bar{r}_p(\bar{t}'))}{W_{lm;j}}\frac{R^*_{lm;j'}(\bar{r}_p(\bar{t}''))}{W^*_{lm;j'}}\left|{}_{-2}Y_{lm}\left(\frac{\pi}{2},0\right)\right|^2 , \tag{5.31}$$

the delta function that allows to perform one of the two time integrals trivially.

The problem with this delta function is that, the exponent of $\bar{\omega}$ grows very rapidly with the PN order: the leading, Newtonian, contribution to the flux goes as ω^6, which means that six derivatives of $R_{lm;j}(\bar{r}_p(\bar{t}'))$ with respect to $\bar{t}'$ must be computed. Going beyond the Newtonian case, at each specific PN order, more powers of ω will enter and, as an example, at 6PN one should compute up to 22 derivatives of a single coefficients of $R_{lm;j}(\bar{r}_p(\bar{t}'))$ together with lower order derivatives that hits different $R_{lm;j}(\bar{r}_p(\bar{t}'))$. In addition to that, the remaining integral is indeed particularly complicated, even in the local sector. In order to reduce the number of derivatives, one could perform a PM expansion, in order to see if this large number of derivatives is actually necessary. Unfortunately, the PM expansion does not significantly reduce the computational time.

For this reason, another approach could be devised. In particular, going back to Eq. (5.21), it is more convenient to perform the integrations in order to get an explicit expression for the gravitational wave strain in terms of the retarded time. Simbolically, one can write

$$(h_+ - ih_\times)(\bar{T}) = \frac{1}{\bar{r}}\sum_{l,m} H_{lm}(\bar{T})_{-2}Y_{lm}(\theta,\phi)\,, \tag{5.32}$$

where the function $H_{lm}(\bar{T})$ contains the explicit expression obtained once the integration, for both the local and non-local sector, are exploited. Integrating at this level is particularly convenient because the integrand of $(h_+ - ih_\times)(\bar{T})$ that are schematically written in Eq. (5.21) contains lower powers in $\bar{\omega}$, which can be immediately translated in less derivatives with respect to $\bar{t}'$. For instance, the Newtonian waveform requires only three derivatives of the amplitude $\mathcal{Z}^\infty_{lm;j}$. This allows to compute higher PN terms of the waveform, up to 9PN, where the maximum number of derivatives is still quite large, i.e. 24, but the overall expressions are generically simpler and easier to handle.

Once the time domain waveform is computed, by using the definition of the energy flux one gets directly

$$\left(\frac{dE}{d\bar{T}}\right)^{\text{rad}}_{r\to\infty} = \frac{1}{16\pi}\sum_{l,m}\int_{-\infty}^{+\infty} d\bar{T}\,\left|\dot{H}_{lm}(\bar{T})\right|^2 , \tag{5.33}$$

which is still a complicated integral.

In the following Section, explicit expressions regarding the local waveform and energy flux at infinity is presented for the sole $l = 2$ mode. The non-local sector will also be discussed in much details, without providing explicit results in time domain for the non-local contributions to the waveform but showing the families of integrands that appears that needs to be computed.

5.2.1 Explicit results

Local contributions for l=2

In units where $G = c = 1$ and also the masses of the Schwarzschild black hole and the light particle have been set to 1. The explicit expressions are presented here up to b^{-9}, i.e. 9PM, with few power in velocities representing the PN expansion. Hence, for the plus and cross polarization one has

$$
\begin{aligned}
r\,h_{+}^{\mathrm{L}}(\bar{T},\theta,\phi) ={}& \frac{GM\nu E}{c^2}\left\{v^2+\frac{v^4}{7}+\frac{v^6}{21}+\frac{5v^8}{231}+\frac{5v^{10}}{429}+\frac{v^{12}}{143}+\frac{v^{14}}{221}+\frac{v^{16}}{323}+\frac{5v^{18}}{2261}+\frac{5v^{20}}{3059}+\frac{v^{22}}{805}\right\}f_{+}(\bar{T},\theta,\phi)\\
&+\frac{G^2M^2\nu E}{bc^2}\left\{\frac{1}{4}u_p\left(\bar{T}\right)\left[4\sin^2(\theta)-2(3+\cos(2\theta))\cos(2\phi)\cos\left(2\phi_p\left(\bar{T}\right)\right)-2(3+\cos(2\theta))\sin(2\phi)\sin\left(2\phi_p\left(\bar{T}\right)\right)\right.\right.\\
&\left.-\frac{4(3+\cos(2\theta))\sin\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p\left(\bar{T}\right)}{\sqrt{1-u_p\left(\bar{T}\right)^2}}\right]+\frac{4v}{3}\sin(\theta)\sin\left(\phi-\phi_p\left(\bar{T}\right)\right)u_p^2\left(\bar{T}\right)\\
&+v^2\left[\frac{(3+\cos(2\theta))\sin\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^2\left(\bar{T}\right)\left(167-173u_p^2\left(\bar{T}\right)\right)}{42\sqrt{1-u_p^2\left(\bar{T}\right)}}+\frac{u_p\left(\bar{T}\right)}{21}\left(-6\sin^2(\theta)\left(10+13u_p^2\left(\bar{T}\right)\right)\right.\right.\\
&\left.\left.-5(3+\cos(2\theta))\cos(2\phi)\cos\left(2\phi_p\left(\bar{T}\right)\right)\left(-6+17u_p^2\left(\bar{T}\right)\right)-5(3+\cos(2\theta))\sin(2\phi)\sin\left(2\phi_p\left(\bar{T}\right)\right)\left(-6+17u_p^2\left(\bar{T}\right)\right)\right)\right]\\
&+v^3\left[\frac{166}{21}\cos\left(\phi-\phi_p\left(\bar{T}\right)\right)\sin(\theta)u_p^3\left(\bar{T}\right)\sqrt{1-u_p^2\left(\bar{T}\right)}+\frac{5}{21}\sin(\theta)\sin\left(\phi-\phi_p\left(\bar{T}\right)\right)u_p^2\left(\bar{T}\right)\left(-16+17u_p^2\left(\bar{T}\right)\right)\right]\\
&+v^4\left[\frac{(3+\cos(2\theta))\sin\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^2\left(\bar{T}\right)\left(104+299u_p^2\left(\bar{T}\right)-427u_p^4\left(\bar{T}\right)\right)}{504\sqrt{1-u_p^2\left(\bar{T}\right)}}\right.\\
&+\frac{u_p\left(\bar{T}\right)}{1008}\left((3+\cos(2\theta))\cos(2(\phi-\phi_p(\bar{T})))\left(192-204u_p^2\left(\bar{T}\right)-199u_p^4\left(\bar{T}\right)\right)\right.\\
&\left.\left.-6\sin^2(\theta)\left(64+31u_p^2\left(\bar{T}\right)\left(4-7u_p^2\left(\bar{T}\right)\right)\right)\right)\right]+\frac{\sin(\theta)u_p^2\left(\bar{T}\right)v^5}{63}\left[\sin\left(\phi-\phi_p\left(\bar{T}\right)\right)\left(-32+u_p^2\left(\bar{T}\right)\right.\right.\\
&\left.\left.\left.+68u_p^4\left(\bar{T}\right)\right)+\cos\left(\phi-\phi_p\left(\bar{T}\right)\right)u_p\left(\bar{T}\right)\sqrt{1-u_p^2\left(\bar{T}\right)}\left(16+43u_p^2\left(\bar{T}\right)\right)\right]+O(v^6)\right\}+\\
&+\frac{G^3M^3\nu E}{b^2v^2c^2}\left\{\frac{(3+\cos(2\theta))\sin\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^3\left(\bar{T}\right)}{2\left(1-u_p^2\left(\bar{T}\right)\right)^{3/2}}+v^2\left[\frac{u_p(\bar{T})}{84}\left(60\sin^2(\theta)-30(3+\cos(2\theta))\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)\right.\right.\right.\\
&\left.\left.+\frac{2(3+\cos(2\theta))\sin\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p\left(\bar{T}\right)\left(92-89u_p^2\left(\bar{T}\right)\right)}{\left(1-u_p^2\left(\bar{T}\right)\right)^{3/2}}\right)\right]-v^3\left[\frac{1}{3}\left(32\sin(\theta)\sin\left(\phi-\phi_p\left(\bar{T}\right)\right)\right.\right.\\
&\left.-(3+\cos(2\theta))(17-12\gamma_E-24\log(2))\sin\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)\right)u_p^3\left(\bar{T}\right)\\
&+\frac{u_p^2\left(\bar{T}\right)}{168\sqrt{1-u_p^2\left(\bar{T}\right)}}\left(7(-17+12\gamma_E+24\log(2))\left(2(3+\cos(2\theta))\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)-4\sin^2(\theta)\right)\right.\\
&+2\left(7\left((3+\cos(2\theta))\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)-2\sin^2(\theta)\right)(17-12\gamma_E-24\log(2))\right.\\
&\left.\left.\left.\left.-664\cos\left(\phi-\phi_p\left(\bar{T}\right)\right)\sin(\theta)\right)u_p^2\left(\bar{T}\right)\right)\right]+O(v^4)\right\}+\\
&-\frac{G^4M^4\nu E}{b^3v^4c^2}\left\{\frac{(3+\cos(2\theta))\sin\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^4\left(\bar{T}\right)}{2\left(1-u_p^2\left(\bar{T}\right)\right)^{5/2}}+\frac{(3+\cos(2\theta))v^2\sin\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^4\left(\bar{T}\right)\left(11-5u_p^2\left(\bar{T}\right)\right)}{84\left(1-u_p^2\left(\bar{T}\right)\right)^{5/2}}\right.\\
&+v^3\left[\frac{(17-12\gamma_E-24\log(2))\left((3+\cos(2\theta))\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)-2\sin^2(\theta)\right)u_p^3\left(\bar{T}\right)}{12\sqrt{1-u_p^2\left(\bar{T}\right)}}\right.\\
&\left.\left.+\frac{83\cos\left(\phi-\phi_p\left(\bar{T}\right)\right)\sin(\theta)u_p^5\left(\bar{T}\right)}{21\left(1-u_p^2\left(\bar{T}\right)\right)^{3/2}}\right]+O(v^4)\right\}+\frac{G^5M^5\nu E}{b^4v^6c^2}\left\{\frac{5(3+\cos(2\theta))\sin\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^5\left(\bar{T}\right)}{8\left(1-u_p^2\left(\bar{T}\right)\right)^{7/2}}\right.\\
&-v^2\frac{(3+\cos(2\theta))\sin\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^5\left(\bar{T}\right)\left(143-158u_p^2\left(\bar{T}\right)\right)}{168\left(1-u_p^2\left(\bar{T}\right)\right)^{7/2}}+v^3\left[\frac{83\cos\left(\phi-\phi_p\left(\bar{T}\right)\right)\sin(\theta)u_p^6\left(\bar{T}\right)}{21\left(1-u_p^2\left(\bar{T}\right)\right)^{5/2}}\right.\\
&\left.\left.-\frac{(17-12\gamma_E-24\log(2))\left((3+\cos(2\theta))\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)-2\sin^2(\theta)\right)u_p^4\left(\bar{T}\right)}{24\left(1-u_p^2\left(\bar{T}\right)\right)^{3/2}}\right]+O(v^4)\right\}+\\
&-\frac{G^6M^6\nu E}{b^5c^2v^8}\left\{\frac{7(3+\cos(2\theta))\sin\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^6\left(\bar{T}\right)}{8\left(1-u_p^2\left(\bar{T}\right)\right)^{9/2}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(3+\cos(2\theta))v^2\sin\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_p^6\left(\bar{T}\right)\left(773-815u_p^2\left(\bar{T}\right)\right)}{336\left(1-u_p^2\left(\bar{T}\right)\right)^{9/2}}+v^3\left[\frac{415\cos\left(\phi-\phi_P\left(\bar{T}\right)\right)\sin(\theta)u_p^7\left(\bar{T}\right)}{84\left(1-u_p^2\left(\bar{T}\right)\right)^{7/2}}\right.\\
&\left.-\frac{(17-12\gamma_E-24\log(2))\left((3+\cos(2\theta))\cos\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)-2\sin^2(\theta)\right)u_p^5\left(\bar{T}\right)}{24\left(1-u_p^2\left(\bar{T}\right)\right)^{5/2}}\right]+O(v^4)\Bigg\}\\
&+\frac{G^7M^7\nu E}{b^6c^2v^{10}}\Bigg\{\frac{21(3+\cos(2\theta))\sin\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_p^7\left(\bar{T}\right)}{16\left(1-u_p^2\left(\bar{T}\right)\right)^{11/2}}+\\
&+-\frac{(3+\cos(2\theta))v^2\sin\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_p^7\left(\bar{T}\right)\left(238-247u_p^2\left(\bar{T}\right)\right)}{48\left(1-u_p^2\left(\bar{T}\right)\right)^{11/2}}+v^3\left[\frac{83\cos\left(\phi-\phi_P\left(\bar{T}\right)\right)\sin(\theta)u_p^8\left(\bar{T}\right)}{12\left(1-u_p^2\left(\bar{T}\right)\right)^{9/2}}\right.\\
&\left.+\frac{5(-17+12\gamma_E+24\log(2))\left((3+\cos(2\theta))\cos\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)-2\sin^2(\theta)\right)u_p^6\left(\bar{T}\right)}{96\left(1-u_p^2\left(\bar{T}\right)\right)^{7/2}}\right]+O(v^4)\Bigg\}\\
&+\frac{G^8M^8\nu E}{b^7c^2v^{12}}\Bigg\{-\frac{33(3+\cos(2\theta))\sin\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_p^8\left(\bar{T}\right)}{16\left(1-u_p^2\left(\bar{T}\right)\right)^{13/2}}\\
&+\frac{(3+\cos(2\theta))v^2\sin\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_p^8\left(\bar{T}\right)\left(2251-2317u_p^2\left(\bar{T}\right)\right)}{224\left(1-u_p^2\left(\bar{T}\right)\right)^{13/2}}-v^3\left[\frac{83\cos\left(\phi-\phi_P\left(\bar{T}\right)\right)\sin(\theta)u_p^9\left(\bar{T}\right)}{8\left(1-u_p^2\left(\bar{T}\right)\right)^{11/2}}\right.\\
&\left.-\frac{7(17-12\gamma_E-24\log(2))\left((3+\cos(2\theta))\cos\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)-2\sin^2(\theta)\right)u_p^7\left(\bar{T}\right)}{96\left(1-u_p^2\left(\bar{T}\right)\right)^{9/2}}\right]+O(v^4)\Bigg\}+\\
&+\frac{G^9M^9\nu E}{b^8c^2v^{14}}\Bigg\{\frac{429(3+\cos(2\theta))\sin\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_p^9\left(\bar{T}\right)}{128\left(1-u_p^2\left(\bar{T}\right)\right)^{15/2}}\\
&-\frac{11(3+\cos(2\theta))v^2\sin\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_p^9\left(\bar{T}\right)\left(1621-1660u_p^2\left(\bar{T}\right)\right)}{896\left(1-u_p^2\left(\bar{T}\right)\right)^{15/2}}+v^3\left[\frac{913\cos\left(\phi-\phi_P\left(\bar{T}\right)\right)\sin(\theta)u_p^{10}\left(\bar{T}\right)}{56\left(1-u_p^2\left(\bar{T}\right)\right)^{13/2}}\right.\\
&\left.-\frac{7(17-12\gamma_E-24\log(2))\left((3+\cos(2\theta))\cos\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)-2\sin^2(\theta)\right)u_p^8\left(\bar{T}\right)}{64\left(1-u_p^2\left(\bar{T}\right)\right)^{11/2}}\right]+O(v^4)\Bigg\}+\\
&+\frac{G^{10}M^{10}\nu E}{b^9c^2v^{16}}\Bigg\{-\frac{715(3+\cos(2\theta))\sin\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_p^{10}\left(\bar{T}\right)}{128\left(1-u_p^2\left(\bar{T}\right)\right)^{17/2}}\\
&+\frac{143(3+\cos(2\theta))v^2\sin\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_p^{10}\left(\bar{T}\right)\left(489-499u_p^2\left(\bar{T}\right)\right)}{1792\left(1-u_p^2\left(\bar{T}\right)\right)^{17/2}}-v^3\left[\frac{11869\cos\left(\phi-\phi_P\left(\bar{T}\right)\right)\sin(\theta)u_p^{11}\left(\bar{T}\right)}{448\left(1-u_p^2\left(\bar{T}\right)\right)^{15/2}}\right.\\
&\left.-\frac{11(17-12\gamma_E-24\log(2))\left((3+\cos(2\theta))\cos\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)-2\sin^2(\theta)\right)u_p^9\left(\bar{T}\right)}{64\left(1-u_p^2\left(\bar{T}\right)\right)^{13/2}}\right]+O(v^4)\Bigg\},
\end{aligned}
\tag{5.34a}
$$

$$
\begin{aligned}
r\,h_\times^{\rm L}(\bar{T},\theta,\phi)=&\frac{GM\nu E}{c^2}\left\{v^2+\frac{v^4}{7}+\frac{v^6}{21}+\frac{5v^8}{231}+\frac{5v^{10}}{429}+\frac{v^{12}}{143}+\frac{v^{14}}{221}+\frac{v^{16}}{323}+\frac{5v^{18}}{2261}+\frac{5v^{20}}{3059}+\frac{v^{22}}{805}\right\}f_\times(\bar{T},\theta,\phi)\\
&\frac{G^2M^2\nu E}{bc^2}\Bigg\{2\cos(\theta)u_P\left(\bar{T}\right)\left[\sin\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)-\frac{2\cos\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_P\left(\bar{T}\right)}{\sqrt{1-u_p^2\left(\bar{T}\right)}}\right]+\frac{2v}{3}\cos\left(\phi-\phi_P\left(\bar{T}\right)\right)\sin(2\theta)u_p^2\left(\bar{T}\right)\\
&\frac{2v^2}{21}\cos(\theta)u_P\left(\bar{T}\right)\left[\frac{\cos\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_P\left(\bar{T}\right)\left(167-173u_p^2\left(\bar{T}\right)\right)}{\sqrt{1-u_p^2\left(\bar{T}\right)}}-10\sin\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)\left(6-17u_p^2\left(\bar{T}\right)\right)\right]\\
&-\frac{v^3}{42}\sin(2\theta)u_p^2\left(\bar{T}\right)\left[5\cos\left(\phi-\phi_P\left(\bar{T}\right)\right)\left(16-17u_p^2\left(\bar{T}\right)\right)+166\sin\left(\phi-\phi_P\left(\bar{T}\right)\right)u_P\left(\bar{T}\right)\sqrt{1-u_p^2\left(\bar{T}\right)}\right]+O(v^4)\Bigg\}\\
&+\frac{G^3M^3\nu E}{b^2c^2v^2}\Bigg\{\frac{2\cos(\theta)\cos\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_p^3\left(\bar{T}\right)}{\left(1-u_p^2\left(\bar{T}\right)\right)^{3/2}}+\frac{2v^2}{21}\cos(\theta)u_p^2\left(\bar{T}\right)\left[15\sin\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)\right.\\
&\left.-\frac{\cos\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_P\left(\bar{T}\right)\left(-92+89u_p^2\left(\bar{T}\right)\right)}{\left(1-u_p^2\left(\bar{T}\right)\right)^{3/2}}\right]-\frac{2v^3}{21}\cos(\theta)u_p^2\left(\bar{T}\right)\left[\frac{83\sin(\theta)\sin\left(\phi-\phi_P\left(\bar{T}\right)\right)u_p^2\left(\bar{T}\right)}{\sqrt{1-u_p^2\left(\bar{T}\right)}}\right.\\
&-\frac{7}{2}(17-12\gamma_E-24\log(2))\sin\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)\sqrt{1-u_p^2\left(\bar{T}\right)}+14\Big(\cos\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)(17-12\gamma_E-24\log(2))\\
&\left.+8\cos\left(\phi-\phi_P\left(\bar{T}\right)\right)\sin(\theta)\Big)u_P\left(\bar{T}\right)\right]+O(v^4)\Bigg\}\\
&-\frac{G^4M^4\nu E}{b^3c^2v^4}\Bigg\{\frac{2\cos(\theta)\cos\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_p^4\left(\bar{T}\right)}{\left(1-u_p^2\left(\bar{T}\right)\right)^{5/2}}+\frac{v^2\cos(\theta)\cos\left(2\left(\phi-\phi_P\left(\bar{T}\right)\right)\right)u_p^4\left(\bar{T}\right)\left(11-5u_p^2\left(\bar{T}\right)\right)}{21\left(1-u_p^2\left(\bar{T}\right)\right)^{5/2}}\\
&+\frac{2v^3\left(\cos(\theta)\sin\left(\phi-\phi_P\left(\bar{T}\right)\right)u_p^3\left(\bar{T}\right)\right)}{3\sqrt{1-u_p^2\left(\bar{T}\right)}}\left[\cos\left(\phi-\phi_P\left(\bar{T}\right)\right)(17-12\gamma_E-24\log(2))-\frac{83\sin(\theta)u_p^2\left(\bar{T}\right)}{14\left(1-u_p^2\left(\bar{T}\right)\right)}\right]+O(v^4)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{G^5M^5\nu E}{b^4c^2v^6}\Bigg\{\frac{5\cos(\theta)\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^5\left(\bar{T}\right)}{2\left(1-u_p^2\left(\bar{T}\right)\right)^{7/2}}-\frac{\cos(\theta)v^2\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^5\left(\bar{T}\right)\left(143-158u_p^2\left(\bar{T}\right)\right)}{42\left(1-u_p^2\left(\bar{T}\right)\right)^{7/2}}\\
&+\frac{\cos(\theta)\sin\left(\phi-\phi_p\left(\bar{T}\right)\right)u_p^4\left(\bar{T}\right)}{3\left(1-u_p^2\left(\bar{T}\right)\right)^{3/2}}\left[\cos\left(\phi-\phi_p\left(\bar{T}\right)\right)(17-12\gamma-24\log(2))-\frac{83\sin(\theta)u_p^4\left(\bar{T}\right)}{7\left(1-u_p^2\left(\bar{T}\right)\right)}\right]v^3+O(v^4)\Bigg\}\\
&-\frac{G^6M^6\nu E}{b^5c^2v^8}\Bigg\{\frac{7\cos(\theta)\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^6\left(\bar{T}\right)}{2\left(1-u_p^2\left(\bar{T}\right)\right)^{9/2}}-\frac{\cos(\theta)v^2\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^6\left(\bar{T}\right)\left(773-815u_p^2\left(\bar{T}\right)\right)}{84\left(1-u_p^2\left(\bar{T}\right)\right)^{9/2}}\\
&+\frac{\cos(\theta)\sin\left(\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^5\left(\bar{T}\right)}{84\left(1-u_p^2\left(\bar{T}\right)\right)^{5/2}}\left[28\cos(\phi-\phi_P(\bar{T}))(17-12\gamma_E-24\log(2))-\frac{415\sin(\theta)u_p^2\left(\bar{T}\right)}{\left(1-u_p^2\left(\bar{T}\right)\right)}\right]v^3+O(v^4)\Bigg\}\\
&+\frac{G^7M^7\nu E}{b^6c^2v^{10}}\Bigg\{\frac{21\cos(\theta)\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^7\left(\bar{T}\right)}{4\left(1-u_p^2\left(\bar{T}\right)\right)^{11/2}}-\frac{\cos(\theta)v^2\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^7\left(\bar{T}\right)\left(238-247u_p^2\left(\bar{T}\right)\right)}{12\left(1-u_p^2\left(\bar{T}\right)\right)^{11/2}}\\
&+\frac{\cos(\theta)\sin\left(\phi-\phi_p\left(\bar{T}\right)\right)u_p^6\left(\bar{T}\right)}{12\left(1-u_p^2\left(\bar{T}\right)\right)^{7/2}}\left[5\cos\left(\phi-\phi_p\left(\bar{T}\right)\right)(17-12\gamma_E-24\log(2))-\frac{83\sin(\theta)u_p^2\left(\bar{T}\right)}{1-u_p^2\left(\bar{T}\right)}\right]v^3+O(v^4)\Bigg\}\\
&-\frac{G^8M^8\nu E}{b^7c^2v^{12}}\Bigg\{\frac{33\cos(\theta)\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^8\left(\bar{T}\right)}{4\left(1-u_p^2\left(\bar{T}\right)\right)^{13/2}}-\frac{\cos(\theta)\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^8\left(\bar{T}\right)\left(2251-2317u_p^2\left(\bar{T}\right)\right)}{56\left(1-u_p^2\left(\bar{T}\right)\right)^{13/2}}\\
&+\frac{\left(7\cos(\theta)\sin\left(\phi-\phi_p\left(\bar{T}\right)\right)u_p^7\left(\bar{T}\right)\right)}{12\left(1-u_p^2\left(\bar{T}\right)\right)^{9/2}}\left[\cos\left(\phi-\phi_p\left(\bar{T}\right)\right)(17-12\gamma_E-24\log(2))-\frac{249\sin(\theta)u_p^2\left(\bar{T}\right)}{14\left(1-u_p^2\left(\bar{T}\right)\right)}\right]v^3+O(v^4)\Bigg\}\\
&+\frac{G^9M^9\nu E}{b^8c^2v^{14}}\Bigg\{\frac{429\cos(\theta)\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^9\left(\bar{T}\right)}{32\left(1-u_p^2\left(\bar{T}\right)\right)^{15/2}}-\frac{11\cos(\theta)v^2\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^9\left(\bar{T}\right)\left(1621-1660u_p^2\left(\bar{T}\right)\right)}{224\left(1-u_p^2\left(\bar{T}\right)\right)^{15/2}}\\
&+\frac{7\cos(\theta)\sin\left(\phi-\phi_p\left(\bar{T}\right)\right)u_p^8\left(\bar{T}\right)}{8\left(1-u_p^2\left(\bar{T}\right)\right)^{11/2}}\left[\cos\left(\phi-\phi_p\left(\bar{T}\right)\right)(17-12\gamma_E-24\log(2))-\frac{913\sin(\theta)u_p^2\left(\bar{T}\right)}{49\left(1-u_p^2\left(\bar{T}\right)\right)}\right]v^3+O(v^4)\Bigg\}\\
&+\frac{G^{10}M^{10}\nu E}{b^9c^2v^{16}}\Bigg\{\frac{715\cos(\theta)\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^{10}\left(\bar{T}\right)}{32\left(1-u_p^2\left(\bar{T}\right)\right)^{17/2}}-\frac{143\cos(\theta)\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p^{10}\left(\bar{T}\right)\left(489-499u_p^2\left(\bar{T}\right)\right)}{448\left(1-u_p^2\left(\bar{T}\right)\right)^{17/2}}\\
&+\frac{11\cos(\theta)\sin\left(\phi-\phi_p\left(\bar{T}\right)\right)u_p^9\left(\bar{T}\right)}{8\left(1-u_p^2\left(\bar{T}\right)\right)^{13/2}}\left[\cos\left(\phi-\phi_p\left(\bar{T}\right)\right)(17-12\gamma-24\log(2))-\frac{1079\sin(\theta)u_p^2\left(\bar{T}\right)}{56\left(1-u_p^2\left(\bar{T}\right)\right)}\right]v^3+O(v^4)\Bigg\},
\end{aligned}
\tag{5.34b}
$$

where the time dependence is inside $u_p(\bar{T}) = M/\bar{r}_p(\bar{T})$ and $\phi_p(\bar{T})$, which is the geodesic trajectory of the incoming particle, with energy $E = 1/\sqrt{1-v^2}$ and $\nu = mM/(m+M)^2$ where M is the mass of the Schwarzschild black hole and m is the mass of the incoming particle. The dependence on θ and ϕ, that come from the spherical harmonics, is also made explicit. Besides, for the b^0 the following functions have been defined

$$
\begin{aligned}
f_+(\bar{T},\theta,\phi) = \sin^2(\theta) - \frac{1}{2}(3+\cos(2\theta))\Bigg\{&\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)\\
&-2u_p\left(\bar{T}\right)\left[\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p\left(\bar{T}\right)-\sin\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)\sqrt{1-u_p^2\left(\bar{T}\right)}\right]\Bigg\},
\end{aligned}
\tag{5.35a}
$$

$$
\begin{aligned}
f_\times(\bar{T},\theta,\phi) = 2\cos(\theta)\Bigg\{&\sin\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)\left(1-2u_p^2\left(\bar{T}\right)\right)\\
&-2\cos\left(2\left(\phi-\phi_p\left(\bar{T}\right)\right)\right)u_p\left(\bar{T}\right)\sqrt{1-u_p^2\left(\bar{T}\right)}\Bigg\},
\end{aligned}
\tag{5.35b}
$$

this rather simple expression suggest that a PN resummation, for the leading PM contribution should be possible, once all the other l-modes would be computed.

The corresponding local flux at the 8PM and 2PN order for the $l=2$ mode is

$$
\left(\frac{dE}{d\bar{T}}\right)^{\mathrm{rad,L}}_{r\to\infty} = G^3M^2m\Bigg\{\frac{\pi v}{b^3}\left(\frac{37}{15}+\frac{29v^2}{35}+\frac{13667v^4}{2940}\right)+\frac{GM}{b^4v}\left(\frac{1568}{45}-\frac{119488v^2}{4725}+\frac{2024096v^4}{33075}\right)
$$

$$
+\frac{G^2M^2\pi}{b^5v^3}\left(\frac{122}{5}-\frac{3484v^2}{105}+\frac{168103v^4}{2940}\right)+\frac{G^3M^3}{b^6v^5}\left(\frac{4672}{45}-\frac{36352v^2}{315}+\frac{4074944v^4}{19845}\right)
+\frac{G^4M^4\pi}{b^7v^7}\left(\frac{85}{3}+\frac{347v^2}{9}-\frac{109001v^4}{1764}\right)+\frac{G^5M^5}{b^8v^9}\left(\frac{3104}{75}+\frac{830912v^2}{1575}-\frac{16926496v^4}{33075}\right)\Bigg\}\,, \tag{5.36}
$$

which does not contains any non-local terms and $c=1$. Despite the purely local nature of this flux, the 9PM terms contain already at 2PN integrands that needs to be handled carefully, in particular products between $\operatorname{arcsinh}^n(\bar{T})$ and $\arctan^m(\bar{T})$, where $n=0...6$ and $m=0...3$. These integrals can be managed by a suitable integration by parts, but, there are more robust techniques that can be employed. In particular, by using the logarithmic representation of these functions, it is possible to rewrite all these integrands as multiple polylogarithm, see [154] or [155], and by borrowing tools from high energy physics, integrate them more solidly.

Non-local contributions for l=2

The non-local contributions must be handled with great care. First of all, at the PN-PM order under investigation the index k, that keeps into account the value of the exponent of the $\log(-i\omega)$, goes from $k=1,...,6$, which means that most of the results listed in App. A needs to be used. The non-local contributions to the waveform are

$$
(h_+ - ih_\times)^{\rm NL}(\bar{T}) = \sum_{j=2}\sum_{k=1}\frac{2}{\bar{r}}\sum_{l,m}\int_{-\infty}^{+\infty}\frac{d\bar{\omega}}{2\pi}e^{-i\bar{\omega}\bar{T}}\bar{\omega}^{j-2}\log^k(-i\bar{\omega})\mathcal{Z}^{\infty}_{lm;jk}\,{}_{-2}Y_{lm}(\theta,\phi)\,,
$$

and each j,k of $\mathcal{Z}^{\infty}_{lm;jk}$ can be removed in favor of its definition from Eq. (5.2) as

$$
(h_+ - ih_\times)^{\rm NL}(\bar{T}) = \sum_{j=2}\sum_{k=1}\frac{2}{\bar{r}}\sum_{l,m}\int_{-\infty}^{+\infty}\frac{d\bar{\omega}}{2\pi}e^{-i\bar{\omega}\bar{T}}\bar{\omega}^{j-2}\log^k(-i\bar{\omega})
\times\int d\bar{t}'\,e^{i\bar{\omega}\bar{t}'}\,\tilde{\mathcal{Z}}^{\infty,\rm NL}_{lm;jk}(\bar{t}')\,{}_{-2}Y_{lm}(\theta,\phi)\,, \tag{5.37}
$$

and by using the results given in Chap. 3, it is possible write directly these contributions as

$$
(h_+ - ih_\times)^{\rm NL}(\bar{T}) = \sum_{j=2}\sum_{k=1}\frac{2}{\bar{r}}\sum_{l,m}{}_{-2}Y_{lm}(\theta,\phi)\left\{\sum_{q=0}^{k-1}i^j\left\{a_q\left[\text{p.v.}\left(\frac{\log^q|\bar{t}'-\bar{T}|}{\bar{t}'-\bar{T}}\right)\right.\right.\right.
\left.\left.\left.-\left(\frac{\log^q|\bar{t}'-\bar{T}|}{|\bar{t}'-\bar{T}|}\right)_{\bar{T}}\right]\right\}+B\,\partial^{(j)}_{\bar{T}}\,\tilde{\mathcal{Z}}^{\infty,\rm NL}_{lm;jk}(\bar{T})\right\}, \tag{5.38}
$$

where

$$
\text{p.v.}\left(\frac{\log^k|\bar{t}'-\bar{T}|}{\bar{t}'-\bar{T}}\right) := \int_{-\infty}^{\bar{T}}d\bar{t}'\log^k(|\bar{t}'-\bar{T}|)\frac{\partial^{(j)}_{\bar{t}'}\,\tilde{\mathcal{Z}}^{\infty,\rm NL}_{lm;jk}(\bar{t}')-\partial^{(j)}_{\bar{t}'}\,\tilde{\mathcal{Z}}^{\infty,\rm NL}_{lm;jk}(-\bar{t}')}{\bar{t}'-\bar{T}}\,, \tag{5.39a}
$$

$$
\left(\frac{\log^k|\bar{t}'-\bar{T}|}{|\bar{t}'-\bar{T}|}\right)_{\bar{T}} := \int_{|\bar{t}'|\le\bar{T}}d\bar{t}'\log^k(|\bar{t}'-\bar{T}|)\frac{\partial^{(j)}_{\bar{t}'}\,\tilde{\mathcal{Z}}^{\infty,\rm NL}_{lm;jk}(\bar{t}')-\partial^{(j)}_{\bar{T}}\,\tilde{\mathcal{Z}}^{\infty,\rm NL}_{lm;jk}(\bar{T})}{|\bar{t}'-\bar{T}|}
$$

$$+\int_{|\bar{t}'|\geq\bar{T}} d\bar{t}' \log^k(|\bar{t}'-\bar{T}|)\frac{\partial^{(j)}_{\bar{t}'}\tilde{\mathcal{Z}}^{\infty,\mathrm{NL}}_{lm;jk}(\bar{t}')}{|\bar{t}'-\bar{T}|}\,. \tag{5.39b}$$

The integrals that need to be performed then are strongly dependent on the functional form of $\tilde{\mathcal{Z}}^{\infty,\mathrm{NL}}_{lm\bar{\omega}}(\bar{t}')$ and its derivative. This can be easily investigated for $l=2$ at each order in both PM and PN. The leading non-local effect appears in the waveform at b^{-2}, i.e. 2PM and at 1.5PN. It is however interesting that, for $\theta=\pi/2$ (equatorial plane), the first non-local effect shifts of half PN order, namely appears at 2PN.

In order to look at the structure of the $\tilde{\mathcal{Z}}^{\infty,\mathrm{NL}}_{lm}(\bar{t}')$, it is convenient to focus at each PM order separately, then discuss how the analytical structure changes at successive PN orders. This is because the structure present at a certain n-PM order will be completely inherited by the $n+1$-order. By restricting to the equatorial plane for simplicity, one can then observe the following structure up to 9PM:

- 2PM: Only $k=1$ is involved at this order. All the functions that appear are of the form $\dfrac{P(\bar{t}')}{(1+\bar{t}')^{n/2}}$, where $P(\bar{t}')$ is a polynomial in $\bar{t}'$ and n can be any positive integer. By evaluating the integrals of Eq. (5.39) the resulting expressions will contain other rational functions of $(1+\bar{t}'^2)$ and also inverse hyperbolic functions;

- 3PM: For $k=1$ the integrands contains now the following functions $\dfrac{P(\bar{t}')\mathrm{arcsinh}(\bar{t}')}{(1+\bar{t}')^{n/2}}$, which originates, once integrated, other $\mathrm{arcsinh}(\bar{t}')$ and $\mathrm{arcsinh}^2(\bar{t}')$, together with $\arctan(\bar{t}')$, $\log(1+\bar{t}')$ and, more interestingly, $\mathrm{Li}_2(\bar{t}')$ which is always tied to $\zeta(2)=\pi^2/6$ which enriches the transcendental structure of the result;

- 4PM: For $k=1$ the integrands contains now also $\mathrm{arcsinh}^2(\bar{t}')$ which produces, once integrated, $\mathrm{Li}_2(\bar{t}')$ and $\mathrm{Li}_3(\bar{t}')$. At 4PN at this order also integrands of the form $\dfrac{P(\bar{t}')\arctan(\bar{t}')}{(1+\bar{t}')^{n/2}}$ make their first appearance, producing other $\mathrm{Li}_2(\bar{t}')$ and $\mathrm{Li}_3(\bar{t}')$. To further enrich the structure, at 4.5 PN order, also functions as $\dfrac{P(\bar{t}')\log\left(1+\bar{t}'^2\right)}{(1+\bar{t}')^{n/2}}$ starts to appear. It is worthwhile to mention that the exponent of the $\mathrm{arcsinh}(\bar{t}')$ is related to the PM order and, in general, at a given n-PM order the $\mathrm{arcsinh}(\bar{t}')$ will appear with exponents up to $n-2$ and, similarly, polylogarithms will appears with weight up to $n-1$. The presence of these inverse hyperbolic functions will be ignored from now on;

- 5PM: For $k=1$ the structure does not contain new functions *but* at this order the structure that where always treated separately, now products between $\mathrm{arcsinh}(\bar{t}')$, $\arctan(\bar{t}')$ and $\log\left(1+\bar{t}'^2\right)$ appears. This is the first signal that a far richer analytical structure than polylogarithms of increasing weight are present in the waveform. This structure is well known in high energy physics and it falls under the name of *cyclotomic polynomials* [156] and, from now on, all the integrals of the new functions that will appear must be written in terms of this cyclotomic polynomials;

- 6PM: For $k=1$ two new functions appears: at 6PN $\dfrac{P(\bar{t}')\arctan^2(\bar{t}')}{(1+\bar{t}')^{n/2}}$ and at 7.5PN also $\dfrac{P(\bar{t}')\log^2(1+\bar{t}'^2)}{(1+\bar{t}')^{n/2}}$;

- 7PM: For $k = 1$ no new functions appears but more cross products between previously mentioned functions appear;
- 8PM: For $k = 1$ the last new function appears, namely $\dfrac{P(\bar{t}')\text{arctan}^3(\bar{t}')}{(1+\bar{t}')^{n/2}}$.;
- 9PM: There are no new integrands for $k = 1$.;

It is then clear that this structure can be generalized and written in terms of 4 parameters

$$I_{m,n,p,q}(\bar{t}') = \frac{\log^m(1+\bar{t}'^2)\text{arctan}^n(\bar{t}')\text{arcsinh}^q(\bar{t}')}{(1+\bar{t}'^2)^{q/2}}\,. \tag{5.40}$$

These integrands needs to be then inserted inside Eqs. (5.39) and integrated consistently in order to unravel the non-linear structure of the waveform and, consequently, of the flux.

6 Quasi Keplerian parameterization for Scattering in Scalar-Tensor Theories

This last Chapter covers a topic that it is rather different from Self Force but its subject is another approach for extracting analytical information on Scattering and it is mainly based on [157]. Here, by using standard PN theory, it was possible to deduce a 2PN accurate expression for the Quasi-Keplerian parameters describing an hyperbolic encounter in a class of modified theories of gravity called *Scalar-Tensor Theories*. These theories have a broad range of applications: from the investigation of effects beyond-GR, to the parametrization of environmental effects on binary systems.

6.1 Introduction

In order to make full use of the scientific potential of gravitational wave detectors, there has been a push towards improving waveform models. Experimentally, the current and future gravitational detectors are mostly sensitive to gravitational waves generated by compact binaries on bound orbits. One of the most relevant approaches to model these systems is the post-Newtonian (PN) approximation, which yields fully analytical waveforms in the regime of slow velocities and weak fields. As already described, this approximation is well adapted to the inspiral phase of a compact binary on a bound orbit: indeed, the virial theorem guarantees that a largely separated system is automatically slowly orbiting.

However, the PN expansion can be also adapted to the unbound, scattering case: here, the absence of a virial theorem in this case means that the slow velocity (post-Newtonian) expansion is orthogonal to the large separation (post-Minkowskian) expansion. Indeed, although post-Minkowksian produce a complete resummation of the PN series at each order in G, they are typically harder to obtain, and run into special functions like elliptic integrals, harmonic polylogarithms , and even Calabi-Yau manifolds entering at 5PM [92, 93, 158]. Thus, PN methods can be precious both for cross-checking PM results and completing analytical knowledge of coefficients inaccessible with PM methods.

In PN theory, the hyperbolic problem has mostly been studied as a doppelgänger of the elliptic problem. Indeed, the quasielliptic motion was studied using an elegant generalization of the Kepler solution to first post-Newtonian (1PN) order, dubbed the quasi-Keplerian (QK) parametrization, which was introduced by Damour and Deruelle [159], and generalized up to 3PN order [160–162] and partially to 4PN order [163]. At 1PN order, Damour and Deruelle showed that the quasihyperbolic motion could be obtained through analytical continuation of the elliptic motion [159], and Blanchet and Schäfer [164] used this result to obtain the energy and angular momentum losses at 1PN order beyond Einstein's quadrupole formula. This statement seems to break down after 3PN order, and the QK parametrization for the quasihyperbolic case [95, 165] was developed independently from the quasielliptic case. Using this parametrization, it was then possible to compute the fluxes at 3PN order [166, 167], and obtain dissipative corrections to, e.g., the scattering angle at 3PN order [74, 168].

All these waveform models have mainly been developed assuming that general relativity (GR) is the underlying theory of gravity, but future detectors might unveil very small deviations to GR, which need to be modeled. One approach to modeling alternative theories is to take a theory-agnostic, parametrized approach [169–172]. In this Chapter, the starting point is to adopt a theory-dependent approach, and derive predictions for the gravitational waveform within the class of massless scalar-tensor (ST) theories of gravity, which were first introduced by Jordan [173], Fierz [174], and Brans and Dicke [175], and later generalized in [176, 177]. At relatively low PN orders, other popular theories such as scalar Gauss-Bonnet reduce to the theory under scrutiny in this work [178]; see also Refs. [179–183] for studies of compact binaries in extensions of these theories and [184] for dark matter effects in GR.

Previous results in massless ST theories using the PN expansion were derived using (i) effective field theory methods [185–189], (ii) the direct integration of the relaxed field equations (DIRE) [190–194], and (iii) the post-Newtonian, multipolar post-Minkowskian (PN-MPM) formalism [195–199]. It is the latter that will be used hereafter, as it has been successfully used in GR to compute the energy flux and GW phasing at very high 4.5PN order [200].

Gravitational radiation in ST theories was first studied by Refs. [185, 186, 188, 189, 201] in an alternative (but equivalent) formulation of the action. The equations of motion and conserved energy and angular momentum were first obtained at 2.5PN order in [192], then at 3PN order in [195, 196], including the contribution of the (dipolar) tail term. The waveform and flux were obtained in terms of positions and velocities at 1.5PN beyond the leading quadrupolar radiation, i.e. at 2.5PN beyond the leading dipolar radiation [193, 194, 197]. The fluxes, the phase as well as the gravitational and scalar waveform amplitudes were then obtained in the case of circular orbits at 1.5PN [197, 202]. In order to extend these results to the case of quasielliptic orbits, the QK parametrization was established at 2PN order in these theories [203], and used to compute the fluxes and secular evolution of the orbital parameters at 1.5PN beyond the quadrupolar radiation [204].

Here the aims is to extend the QK parametrization for ST theories of Ref. [203] to the case of hyperbolic orbits. The main application will be completion of the full scattering angle at 2.5PN order in scalar tensor theory, including both conservative and dissipative contributions. The conservative sector will be in agreement with Ref. [205], and the dissipative is new.

This Chapter is organized as follows. After a brief notational reminder, the specific theory under consideration will be defined in Sec. 6.2 together with a brief recap on how PN-MPM computation are done in these theories. In Sec. 6.3, the derivation of the QK parametrization for hyperbolic orbits at 2PN order, obtain the gauge invariant scattering angle and impact parameter, and perform a check of the scatter-to-bound map between the scattering angle and periastron advance. In Sec. 6.4, the total radiated energy and angular momentum of the system are computed at Newtonian order beyond Einstein's quadrupole formula in GR, or equivalently, at 1PN order beyond the -1PN order dominant dipolar radiation of the scalar field. In Sec. 6.4.1, the dissipative contributions to a few orbitals elements are presented, namely the scattering angle, the impact parameter, the time eccentricity and the asymptotic velocity. In Sec. 6.5, some limit of the previous results are presented.

6.1.1 Notation conventions and parameter overview

Throughout this Chapter, Post-Newtonian (PN) orders are defined relative to the Newtonian limit and the standard quadrupolar gravitational radiation of General Relativity (GR). Accordingly, leading-order dipolar radiation appears at -0.5PN order in the waveform and at -1PN order in the energy flux. The 2PN QK parametrization developed here corresponds to a next-to-next-to-leading-order correction, while Newtonian fluxes and waveforms are classified as next-to-leading-order contributions.

A 3+1 decomposition of spacetime is adopted and three-dimensional Euclidean vectors will be denoted using bold symbols. Unless explicitly stated, all the quantities will be defined in the center-of-mass (CM) frame. The position of the field point is written as $\boldsymbol{X} = R\,\boldsymbol{N}$, where $\boldsymbol{N}$ is a unit vector. In spherical coordinates, this becomes (R, Θ, Φ). The coordinate time is denoted by t, and the retarded time is given by $U = t - R/c$; this is distinguished from the QK motion's eccentric anomaly, u. Both scalar and spin-weighted (with weight $s = -2$) spherical harmonics are used, denoted by $Y^{\ell\mathrm{m}}(\Theta, \Phi)$ and $Y^{\ell\mathrm{m}}_{-2}(\Theta, \Phi)$ respectively. The index m should not be mistaken for the total mass m.

The individual positions of the two compact objects are denoted $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, and their relative separation is given by $\boldsymbol{x} = \boldsymbol{y}_1 - \boldsymbol{y}_2$, with magnitude $r = |\boldsymbol{x}|$ and direction $\boldsymbol{n} = \boldsymbol{x}/r$. The relative velocity is defined as $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$. For a non-precessing eccentric binary, the orthonormal basis is defined as $(\boldsymbol{n}, \boldsymbol{\lambda}, \boldsymbol{\ell})$, where $\boldsymbol{\lambda}$ lies within the orbital plane, satisfies $\boldsymbol{\lambda}\cdot\boldsymbol{v} > 0$, and $\boldsymbol{\ell} = \boldsymbol{n}\times\boldsymbol{\lambda}$. Introducing a fixed inertial triad $(\boldsymbol{n}_0, \boldsymbol{\lambda}_0, \boldsymbol{\ell})$, the orbital motion can be expressed in polar coordinates (r, ϕ) such that $\boldsymbol{n} = \cos\phi\,\boldsymbol{n}_0 + \sin\phi\,\boldsymbol{\lambda}_0$. The velocity then becomes $\boldsymbol{v} = \dot{r}\,\boldsymbol{n} + r\dot{\phi}\,\boldsymbol{\lambda}$. This implies the relations: $\boldsymbol{n}\cdot\boldsymbol{v} = \dot{r}$, $v^2 = \dot{r}^2 + r^2\dot{\phi}^2$, and $\boldsymbol{n}\times\boldsymbol{v} = r\dot{\phi}\boldsymbol{\ell}$.

The multi-index notation is used, where $L = i_1 \cdots i_\ell$ indicates ℓ spatial indices (with K indicating k indices similarly). For derivatives and vector components, one hase $\partial_L = \partial_{i_1}\cdots\partial_{i_\ell}$, $\partial_{aL-1} = \partial_a\partial_{i_1}\cdots\partial_{i_{\ell-1}}$, and likewise for products: $n_L = n_{i_1}\cdots n_{i_\ell}$, $n_{aL-1} = n_a n_{i_1}\cdots n_{i_{\ell-1}}$. The symmetric trace-free (STF) part of a tensor is denoted with either a hat or angle brackets: for instance, $\mathrm{STF}_L[\partial_L] = \hat{\partial}L = \partial_{\langle i_1}\cdots\partial_{i_\ell\rangle}$, $\mathrm{STF}_L[n_L] = \hat{n}L = n_{\langle i_1}\cdots n_{i_\ell\rangle}$, and $\mathrm{STF}_L[x_L] = \hat{x}L = r^\ell n_{\langle i_1}\cdots n_{i_\ell\rangle}$. Higher-order time derivatives of a function $F(t)$ are denoted as $F^{(n)}(t) = \mathrm{d}^n F/\mathrm{d}t^n$.

The scalar field asymptotically approaches the constant value ϕ_0 at spatial infinity, and the normalized scalar field is defined as $\varphi \equiv \phi/\phi_0$. The scalar coupling function $\omega(\phi)$ is expanded around ϕ_0, with $\omega_0 \equiv \omega(\phi_0)$. Similarly, the scalar-dependent masses $m_A(\phi)$ (defined in Section 6.2) are expanded about ϕ_0, yielding $m_A \equiv m_A(\phi_0)$ for $A \in 1, 2$. The definition of the total mass is $m = m_1 + m_2$, the reduced mass $\mu = m_1 m_2/m$, the symmetric mass ratio $\nu = \mu/m \in\ [0, 1/4]$, and the fractional mass difference $\delta = (m_1 - m_2)/m \in\ [0, 1)$. These two mass parameters are related via $\delta^2 = 1 - 4\nu$.

Following [195, 197], in Table 6.1 a list of scalar-tensor (ST) and post-Newtonian (PN) parameters derived from the expansions of $\omega(\phi)$ and $m_A(\phi)$ is presented. The ST parameters arise directly from these expansions, while the PN parameters are specific combinations of the former that extend the standard Parametrized Post-Newtonian (PPN) formalism to a general ST framework [169, 206].

	ST parameters
general	$\omega_0 = \omega(\phi_0)$, $\omega_0' = \frac{\mathrm{d}\omega}{\mathrm{d}\phi}\vert_{\phi=\phi_0}$, $\omega_0'' = \frac{\mathrm{d}^2\omega}{\mathrm{d}\phi^2}\vert_{\phi=\phi_0}$, $\varphi = \frac{\phi}{\phi_0}$, $\tilde{g}_{\mu\nu} = \varphi\, g_{\mu\nu}$, $\tilde{G} = \frac{G(4+2\omega_0)}{\phi_0(3+2\omega_0)}$, $\zeta = \frac{1}{4+2\omega_0}$, $\lambda_1 = \frac{\zeta^2}{(1-\zeta)} \frac{\mathrm{d}\omega}{\mathrm{d}\varphi}\Big\vert_{\varphi=1}$, $\lambda_2 = \frac{\zeta^3}{(1-\zeta)} \frac{\mathrm{d}^2\omega}{\mathrm{d}\varphi^2}\Big\vert_{\varphi=1}$, $\lambda_3 = \frac{\zeta^4}{(1-\zeta)} \frac{\mathrm{d}^3\omega}{\mathrm{d}\varphi^3}\Big\vert_{\varphi=1}$.
sensitivities	$s_A = \frac{\mathrm{d}\ln m_A(\phi)}{\mathrm{d}\ln\phi}\vert_{\phi=\phi_0}$, $s_A^{(k)} = \frac{\mathrm{d}^{k+1}\ln m_A(\phi)}{\mathrm{d}(\ln\phi)^{k+1}}\vert_{\phi=\phi_0}$, $(A = 1, 2)$ $s_A' = s_A^{(1)}$, $s_A'' = s_A^{(2)}$, $s_A''' = s_A^{(3)}$, $\mathcal{S}_+ = \frac{1-s_1-s_2}{\sqrt{\alpha}}$, $\mathcal{S}_- = \frac{s_2-s_1}{\sqrt{\alpha}}$.
Order	**PN parameters**
N	$\alpha = 1 - \zeta + \zeta\,(1-2s_1)\,(1-2s_2)$
1PN	$\overline{\gamma} = -\frac{2\zeta}{\alpha}\,(1-2s_1)\,(1-2s_2)$, $\overline{\beta}_1 = \frac{\zeta}{\alpha^2}\,(1-2s_2)^2\,(\lambda_1\,(1-2s_1) + 2\zeta s_1')$, $\overline{\beta}_2 = \frac{\zeta}{\alpha^2}\,(1-2s_1)^2\,(\lambda_1\,(1-2s_2) + 2\zeta s_2')$, $\overline{\beta}_+ = \frac{\overline{\beta}_1+\overline{\beta}_2}{2}$, $\overline{\beta}_- = \frac{\overline{\beta}_1-\overline{\beta}_2}{2}$. Degeneracy $\alpha(2+\overline{\gamma}) = 2(1-\zeta)$
2PN	$\overline{\delta}_1 = \frac{\zeta(1-\zeta)}{\alpha^2}\,(1-2s_1)^2$, $\overline{\delta}_2 = \frac{\zeta(1-\zeta)}{\alpha^2}\,(1-2s_2)^2$, $\overline{\delta}_+ = \frac{\overline{\delta}_1+\overline{\delta}_2}{2}$, $\overline{\delta}_- = \frac{\overline{\delta}_1-\overline{\delta}_2}{2}$, $\overline{\chi}_1 = \frac{\zeta}{\alpha^3}\,(1-2s_2)^3\left[\left(\lambda_2 - 4\lambda_1^2 + \zeta\lambda_1\right)(1-2s_1) - 6\zeta\lambda_1 s_1' + 2\zeta^2 s_1''\right]$, $\overline{\chi}_2 = \frac{\zeta}{\alpha^3}\,(1-2s_1)^3\left[\left(\lambda_2 - 4\lambda_1^2 + \zeta\lambda_1\right)(1-2s_2) - 6\zeta\lambda_1 s_2' + 2\zeta^2 s_2''\right]$, $\overline{\chi}_+ = \frac{\overline{\chi}_1+\overline{\chi}_2}{2}$, $\overline{\chi}_- = \frac{\overline{\chi}_1-\overline{\chi}_2}{2}$. Degeneracy $16\overline{\delta}_1\overline{\delta}_2 = \overline{\gamma}^2(2+\overline{\gamma})^2$

Table 6.1: Summary of parameters for the general ST theory and notations for PN parameters.

6.2 Massless scalar-tensor theories

Following Refs.[193–197, 203, 204, 207], one can consider a general class of scalar-tensor (ST) theories involving a single massless scalar field ϕ minimally coupled to the metric $g_{\mu\nu}$. The dynamics are governed by the action

$$S_{\rm ST} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right] + S_{\rm m}\left(\mathfrak{m}, g_{\alpha\beta}\right)\,, \tag{6.1}$$

where R is the Ricci scalar, g is the determinant of the metric, ω is a function of the scalar field, and $\mathfrak{m}$ collectively denotes the matter fields.

This is the so-called Jordan frame formulation, in which matter is coupled only to the “physical" metric $g_{\alpha\beta}$, see [208].

In this context, the matter action $S_{\rm m}$ depends solely on the matter fields and the metric. However, since the strong equivalence principle is violated in ST theories, internal gravitational binding energy must be taken into account. Specifically, the scalar field controls the effective gravitational constant, thereby influencing the balance between gravitational and non-gravitational forces within a body. Consequently, the scalar field indirectly affects the body's structure and internal gravity.

To incorporate these effects, it will be adopted the approach introduced by Eardley [209] (see also [210]), modeling $S_{\rm m}$ as the effective action of N nonspinning point particles, whose masses $m_A(\phi)$ depend on the local value of the scalar field:

$$S_{\rm m} = -c\sum_A\int\, m_A(\phi)\sqrt{-\left(g_{\alpha\beta}\right)_A\mathrm{d}y_A^\alpha\,\mathrm{d}y_A^\beta}\,, \tag{6.2}$$

where y_A^α are the spacetime coordinates of particle A, and $\left(g_{\alpha\beta}\right)_A$ denotes the metric evaluated[1] at that location. Through the mass dependence $m_A(\phi)$, the matter action acquires an implicit dependence on the scalar field. One then introduces the sensitivities of each particle to variations in the scalar field via

$$s_A^{(k)} \equiv \left.\frac{\mathrm{d}^{k+1}\ln m_A(\phi)}{\mathrm{d}(\ln\phi)^{k+1}}\right|_{\phi=\phi_0}\,, \tag{6.3}$$

with $s_A \equiv s_A^{(1)}$ and ϕ_0 the asymptotic value of the scalar field, assumed constant in time to neglect cosmological evolution.

For stationary, ordinary, black holes, all information regarding the matter which formed the black hole has disappeared behind the horizon, hence the mass can depend only on the Planck scale: $m_A \propto M_{\rm Planck} \propto G^{-1/2} \propto \phi^{1/2}$, see [175]. This uniquely fixes the sensitivities to $s_A^{\rm BH} = 1/2$.

To analyze perturbations in these theories, it is convenient to define a rescaled scalar field and a conformally related metric,

$$\varphi \equiv \frac{\phi}{\phi_0} \qquad \text{and} \qquad \tilde{g}_{\alpha\beta} \equiv \varphi\, g_{\alpha\beta}\,, \tag{6.4}$$

such that both $\tilde{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ share the same asymptotic behavior at spatial infinity. Quantities expressed in terms of $(\varphi, \tilde{g}_{\alpha\beta})$ belong to the so-called Einstein frame.

Perturbations are defined via $\psi \equiv \varphi - 1$ and $h^{\mu\nu} \equiv \sqrt{-\tilde{g}}\tilde{g}^{\mu\nu} - \eta^{\mu\nu}$, where $\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ denotes the Minkowski metric. The field equations in this frame become:

$$\Box_\eta\, h^{\mu\nu} = \frac{16\pi G}{c^4}\tau^{\mu\nu}\,, \tag{6.5a}$$

[1] Divergences are regularized via Hadamard regularization [211], which is equivalent at this order to dimensional regularization [212, 213].

$$\Box_\eta \, \psi = -\frac{8\pi G}{c^4}\tau_s \,, \tag{6.5b}$$

where $\Box_\eta$ is the flat-space d'Alembertian operator. The source terms are:

$$\tau^{\mu\nu} = \frac{\varphi}{\phi_0}(-g)T^{\mu\nu} + \frac{c^4}{16\pi G}\Lambda^{\mu\nu}[h,\psi] \,, \tag{6.6a}$$

$$\tau_s = -\frac{\varphi}{\phi_0(3+2\omega)}\sqrt{-g}\left(T - 2\varphi\frac{\partial T}{\partial\varphi}\right) - \frac{c^4}{8\pi G}\Lambda_s[h,\psi] \,. \tag{6.6b}$$

Here $T^{\mu\nu} = 2(-g)^{-1/2}\delta S_\mathrm{m}/\delta g_{\mu\nu}$ is the stress-energy tensor, $T = g_{\mu\nu}T^{\mu\nu}$ its trace, and $\partial T/\partial\varphi$ is the partial derivative of $T(g_{\mu\nu},\varphi)$ with respect to φ at fixed $g_{\mu\nu}$. The nonlinear source terms $\Lambda^{\mu\nu}$ and Λ^s are functionals of $h^{\mu\nu}$ and ψ, defined in Eqs.(2.8) and (2.9) of [197], and start at quadratic order in the fields.

The system (6.5) is solved using the PN-MPM algorithm [14, 214]. For a complete treatment in scalar-tensor theories, see [197]. In this work, the only requirement is that the expressions of the source multipole moments I_L, J_L, and I^s_L, which are STF functions of retarded time U and fully determine the linearized fields $h_1^{\mu\nu}$ and ψ_1. For example, the scalar field reads:

$$\psi_1 = -\frac{2}{c^2}\sum_{\ell=0}^{+\infty}\frac{(-)^\ell}{\ell!}\partial_L\left[\frac{\mathrm{I}^s_L(U)}{R}\right]. \tag{6.7}$$

In scalar-tensor theories, the scalar monopole I^s is not constant. However, its variation in time is suppressed by two PN orders: $\mathrm{dI}^s/\mathrm{d}t = \mathcal{O}(c^{-2})$, implying that dipole radiation dominates at leading order. As shown in [197], a convenient definition is

$$\mathrm{I}^s(U) = \frac{1}{\phi_0}\left[m^s + \frac{E^s(U)}{c^2}\right], \tag{6.8a}$$

with

$$m^s = -\frac{1}{3+2\omega_0}\sum_A m_A(1-s_A) \tag{6.8b}$$

where m^s is constant, and $E^s(U)$ encapsulates the PN time-dependence.

Far from the source, where $Gm/(c^2 R_\mathrm{obs}) \ll 1$, gravitational and scalar radiation can also be expanded in multipoles. Introducing radiative coordinates $(T, R, \boldsymbol{N})$ [or equivalently (T, R, Θ, Φ)], the asymptotic waveform can be expressed in terms of three sets of radiative moments [14, 197]: $\mathcal{U}_L$, $\mathcal{V}_L$, and $\mathcal{U}^s_L$. In transverse-traceless gauge, the waveforms are

$$h^\mathrm{TT}_{ij} = -\frac{4G}{c^2R}\perp^\mathrm{TT}_{ijab}\sum_{\ell=2}^{+\infty}\frac{1}{c^\ell\ell!}\Bigg[N_{L-2}\,\mathcal{U}_{abL-2}(U) - \frac{2\ell}{c(\ell+1)}N_{cL-2}\epsilon_{cd(a}\mathcal{V}_{b)dL-2}(U)\Bigg] + \mathcal{O}\left(\frac{1}{R^2}\right), \tag{6.9a}$$

$$\psi = -\frac{2G}{c^2R}\sum_{\ell=0}^{+\infty}\frac{1}{c^\ell\ell!}N_L\,\mathcal{U}^s_L(U) + \mathcal{O}\left(\frac{1}{R^2}\right), \tag{6.9b}$$

where $\perp^{\rm TT}_{ijab}{=}\perp_{a(i}\perp_{j)b}-\frac{1}{2}\perp_{ij}\perp_{ab}$ with $\perp_{ij}{=}\ \delta_{ij}-N_iN_j$. The radiative moments are related to the source moments by the relations

$$\mathcal{U}_L(U)=\overset{(\ell)}{\mathrm{I}}_L(U)+\mathcal{O}\left(\frac{1}{c^3}\right), \tag{6.10}$$

$$\mathcal{V}_L(U)=\overset{(\ell)}{\mathrm{J}}_L(U)+\mathcal{O}\left(\frac{1}{c^3}\right), \tag{6.11}$$

$$\mathcal{U}^s_L(U)=\overset{(\ell)}{\mathrm{I}^s}_L(U)+\mathcal{O}\left(\frac{1}{c^3}\right), \tag{6.12}$$

where the $\mathcal{O}(1/c^3)$ corrections include nonlinear effects from the MPM scheme, such as tails and memory terms (see [197] for details). Since this work focuses only on linear radiation, these corrections will not be considered.

6.3 Motion for hyperbolic orbits

The first step is the computation of the QK parametrization for hyperbolic orbits. One can follow essentially the same steps as Ref. [203], and obtain a closed form, 2PN-accurate, unbound solution to the equations of motion for nonspinning particles. This solution will be parametrized by only two independent variables, the conserved energy E and angular momentum $J=|\boldsymbol{J}|$. In practice, the *reduced* energy $\bar{\varepsilon}$ and angular momentum $\bar{\jmath}$ are the two quantities that could be used, which are defined as

$$\bar{\varepsilon}=\frac{2E}{\mu c^2}\qquad\text{and}\qquad\bar{\jmath}=\frac{2J^2E}{\mu^3(\tilde{G}\alpha m)}\,. \tag{6.13}$$

Note that these reduced variables are decorated with a bar, to distinguish them from their counterparts in the bound case; see [203]. The barred and unbarred energy and angular momentum differ only by a sign, namely $\bar{\varepsilon}=-\varepsilon$ and $\bar{\jmath}=-j$, such that $\varepsilon>0$ and $j>0$ for bound systems and $\bar{\varepsilon}>0$ and $\bar{\jmath}>0$ for unbound systems.

In Sec. 6.3.1, the expressions for the 2PN QK parameters for an hyperbolic encounter in a generic theory of gravity are worked out, by expressing the results in terms of the coefficients (denoted A, B, C, ...) parameterizing the polynomial relation linking the energy and angular momentum of the binary system to the relative trajectories and velocities. In Sec. 6.3.2, it will be discussed the derivation of two special, gauge invariant QK parameters: the scattering angle and impact parameter. The scattering angle will be then related to the periastron advance *via* a scatter-to-bound map in Sec. 6.3.3.

These coefficients $(A, B, C, \ldots)$ can then be specialized to the case of ST theory by replacing them by their expressions in terms of $(\bar{\varepsilon},\bar{\jmath})$, which were given in Ref. [203]. Thanks to the values of these coefficients, the explicit expressions of the various QK parameters in terms of $(\bar{\varepsilon},\bar{\jmath})$ will be presented.

6.3.1 The quasi-Keplerian parametrization for hyperbolic orbits

Consider the conservative dynamics of a spinless binary system, characterized by E and J. The dynamics are thus planar, and can be described by the two equations of motion for respectively the radial and angular motion, namely

$$\dot{r}^2 = s^{-4}\dot{s}^2 = \mathcal{R}(s, E, J), \tag{6.14a}$$

$$\dot{\phi} = \mathcal{S}(s, E, J), \tag{6.14b}$$

where $s \equiv 1/r$ is the inverse of the radial distance between the two compact objects. At this stage, a generic theory of gravity is considered, and one can treat it perturbatively as a PN expansion. In this sense, one can write both $\mathcal{R}$ and $\mathcal{S}$ for Eq. (6.14) as fifth order polynomials in s, with the following structure

$$\mathcal{R}(s, E, J) = A + 2Bs + Cs^2 + D_1 s^3 + D_2 s^4 + D_3 s^5, \tag{6.15a}$$

$$\mathcal{S}(s, E, J) = Fs^2 + I_1 s^3 + I_2 s^4 + I_3 s^5, \tag{6.15b}$$

where A, B, C and F are of order $\mathcal{O}(1)$, D_1 and I_1 are of order $\mathcal{O}(c^{-2})$ and D_2, D_3, I_2 and I_3 are of order $\mathcal{O}(c^{-4})$. This structure for the equations of motion is satisfied in GR, in ST theories, and potentially in other alternative theories of gravity. Notice the absence of logarithmic terms, which appear at 3PN in GR, and of non-local tails, which appear at 4PN in GR but only at 3PN in ST theories.

There are some properties of the coefficients in the radial equation that are universal in both GR and ST theories:

- $A < 0$, $B > 0$ and $C < 0$ for quasielliptic orbits;
- $A > 0$, $B > 0$ and $C < 0$ for quasihyperbolic orbits;
- $A = 0$, $B > 0$ and $C < 0$ for quasiparabolic orbits.

The polynomial $\mathcal{R}(s)$ has exactly two roots that are nonvanishing in the $c \to \infty$ limit, which is denoted as $s_+ \geq s_-$. In the quasielliptic case studied in Ref. [203], the two roots were positive, namely $0 < s_- < s_+$, such that the accessible radii were $r_p < r < r_a$, where the periastron and apastron were defined as $r_p = 1/s_+$ and $r_a = 1/s_-$, respectively. Here, in the quasihyperbolic case, $s_- < 0 < s_+$, so the negative root does not correspond to anything physical, and the accessible radii correspond to $r > 1/s_+$. Moreover, in the quasihyperbolic case, it is useful to introduce the excess velocity $v_\infty \equiv \lim_{s\to 0} \sqrt{\mathcal{R}(s)} = \sqrt{A}$.

The QK representation of the motion has already been derived in the case of bound orbits in terms of A, B, etc. in Ref. [203]. There, the starting point of the parametrization was the computation of the two physically-meaningful gauge-invariant periods of the system, namely the radial period P (associated to the mean motion $n = 2\pi/P$) and the increase in true anomaly per radial period, denoted Φ (associated to the periastron advance $K = \Phi/(2\pi)$). The unbound motion is intrinsically not periodic, so physical periods cannot be naturally defined anymore. Thus, the definitions of the analogous unbound quantities $\bar{n}$ and $\bar{K}$, will exhibit a high degree of arbitrariness, and they will be defined in order to: (i) simplify the expression of the QK parametrization and (ii) immediately recover results from the literature in the GR case [95, 165].

As already stated, the polynomial $\mathcal{R}(s)$ has, just like in the bound case, only two real roots which are nonvanishing in the $c \to \infty$, denoted as $s_- < 0 < s_+$. Thus, s_- is nonphysical, whereas

$r_+ = 1/s_+$ is the radius of closest approach of the encounter. These roots are obtained at leading order by factorizing the $c \to \infty$ limit of $\mathcal{R}(s)$, which is of degree 2. Then, PN corrections to these roots are obtained iteratively by requiring that they cancel $\mathcal{R}(s)$, ignoring higher-order PN corrections. One can then factorize the polynomial as

$$\mathcal{R}(s) = (s_+ - s)(s - s_-)\tilde{\mathcal{R}}(s)\,. \tag{6.16}$$

The first two QK parameters that could be defined are

$$\bar{a}_r = -\frac{s_+ + s_-}{2s_+ s_-} > 0, \quad \bar{e}_r = \frac{s_+ - s_-}{s_+ + s_-} > 1 \tag{6.17}$$

are then trivial to obtain. Following [165], it is now very useful to introduce the auxiliary variable

$$\bar{v}' = 2\arctan\left(\sqrt{\frac{\bar{e}_r + 1}{\bar{e}_r - 1}}\tanh\frac{\bar{u}}{2}\right)\,. \tag{6.18}$$

To determine the Kepler equation, one should first integrate Eq. (6.14a) to find

$$|t - t_0| = \int_s^{s_+} \frac{\mathrm{d}s}{s^2\sqrt{\mathcal{R}(s)}} = \int_s^{s_+} ds\, \frac{\overline{P}(s)}{s^2\sqrt{(s_+ - s)(s - s_-)}}, \tag{6.19}$$

where t_0 is the time of closest approach and where $\overline{P}(s)$ arises from the PN expansion of $1/\sqrt{\tilde{R}(s)}$.

The problem is now reduced to computing a small set of master integrals of the form

$$\bar{\mathcal{J}}_i = \int_s^{s_+} ds \frac{s^{i-2}}{\sqrt{(s_+ - s)(s - s_-)}}\,. \tag{6.20}$$

That are given by

$$\bar{\mathcal{J}}_0 = \frac{\sinh(\bar{u})\,(s_+ - s_-) - \bar{u}\,(s_- + s_+)}{2\,(-s_- s_+)^{3/2}}, \tag{6.21a}$$

$$\bar{\mathcal{J}}_1 = \frac{\bar{u}}{\sqrt{-s_- s_+}}, \tag{6.21b}$$

$$\bar{\mathcal{J}}_2 = \bar{v}', \tag{6.21c}$$

$$\bar{\mathcal{J}}_3 = \frac{1}{2}\sin(\bar{v}')\,(s_+ - s_-) + \frac{1}{2}(s_+ + s_-)\,\bar{v}', \tag{6.21d}$$

$$\bar{\mathcal{J}}_4 = \frac{1}{16}\sin(2\bar{v}')\,(s_+ - s_-)^2 + \frac{1}{2}\sin(\bar{v}')\left(s_+^2 - s_-^2\right) + \frac{1}{8}\left(3s_+^2 + 2s_+s_- + 3s_-^2\right)\bar{v}', \tag{6.21e}$$

$$\begin{aligned}\bar{\mathcal{J}}_5 = {} & \frac{1}{96}\sin(3\bar{v}')\,(s_+ - s_-)^3 + \frac{3}{32}\sin(2\bar{v}')\,(s_+ - s_-)^2\,(s_+ + s_-) \\ & + \frac{3}{32}\sin(\bar{v}')\,(s_+ - s_-)\left(5s_+^2 + 6s_+s_- + 5s_-^2\right) + \frac{1}{16}(s_+ + s_-)\left(5s_+^2 - 2s_+s_- + 5s_-^2\right)\bar{v}'\,.\end{aligned} \tag{6.21f}$$

Replacing s_+ and s_- by their 2PN-accurate expressions, an intermediate form for the Kepler equation can be found,

$$t - t_0 = -\bar{\kappa}_0\,\bar{u} + \bar{\kappa}_1\sinh(\bar{u}) + \bar{\kappa}_2\,\bar{v}' + \bar{\kappa}_3\,\sin(\bar{v}')\,. \tag{6.22}$$

Similarly, the computation of the angular equation starts by integrating (6.14b) to find

$$|\phi - \phi_0| = \int_s^{s_+} \frac{\mathrm{d}s\, \mathcal{S}(s)}{s^2\sqrt{\mathcal{R}(s)}} = \int_s^{s_+} \frac{\mathrm{d}s\, \overline{Q}(s)}{s^2\sqrt{(s_+ - s)(s - s_-)}}\,, \tag{6.23}$$

where ϕ_0 is the reference angle at t_0 and where $\overline{Q}(s)$ arises from the PN expansion of $\mathcal{S}(s)/\sqrt{\tilde{R}(s)}$. The master integrals of (6.20) apply, and also an intermediate form of the angular equation can be obtained

$$\phi - \phi_0 = \bar{\lambda}_0 \bar{v}' + \bar{\lambda}_1 \sin(\bar{v}') + \bar{\lambda}_2 \sin(2\bar{v}') + \bar{\lambda}_3 \sin(3\bar{v}')\,. \tag{6.24}$$

It is possible to simplify these expressions by defining a different eccentricity, namely $\bar{e}_\phi$ as

$$\bar{e}_r = \bar{e}_\phi \left(1 + \frac{\alpha}{c^2} + \frac{\beta}{c^4}\right), \tag{6.25}$$

that equates $\bar{e}_r$ at the Newtonian level but they differs at higher-PN. This eccentricity can be used to defined another angle variables

$$\bar{v} = 2\arctan\left(\sqrt{\frac{\bar{e}_\phi + 1}{\bar{e}_\phi - 1}}\tanh\frac{\bar{u}}{2}\right), \tag{6.26}$$

which is called *true anomaly*.

The goal is now to express the Kepler and angular equations in terms of $\bar{v}$, by finding a value for the coefficients α and β, which are uniquely determined by the condition that the term proportional to $\sin(\bar{v})$ in the 2PN-accurate angular equation must vanish. This request is consistent with the procedure already employed in GR in [95, 165].

Thus, after some simplifications one finds the following parametrization:

$$t - t_0 = -\bar{\kappa}_0'\,\bar{u} + \bar{\kappa}_1 \sinh(\bar{u}) + \bar{\kappa}_2'\,\bar{v} + \bar{\kappa}_3' \sin(\bar{v}), \tag{6.27a}$$

$$\phi - \phi_0 = \bar{\lambda}_0'\bar{v} + \bar{\lambda}_2' \sin(2\bar{v}) + \bar{\lambda}_3' \sin(3\bar{v}), \tag{6.27b}$$

where the primed coefficients are defined because the transformation between $\bar{v}'$ and $\bar{v}$ modifies the previous coefficients. Recalling that there is no natural definition of $\bar{n}$ and $\bar{K}$ in the unbound case, one can finalize the Kepler and angular equations by requiring that the lower order PN-terms are factored in an analogous way as the bound case. Hence, one has

$$\bar{n} = \frac{1}{\bar{\kappa}_0'}, \quad \text{and} \quad \bar{K} = \bar{\lambda}_0'\,, \tag{6.28}$$

and obtain the final parametrization

$$r = \bar{a}_r(\bar{e}_r \cosh(\bar{u}) - 1)\,, \tag{6.29a}$$

$$\bar{n}(t - t_0) = -\bar{u} + \bar{e}_t \sinh(\bar{u}) + \bar{g}_t\,\bar{v} + \bar{f}_t \sin(\bar{v})\,, \tag{6.29b}$$

$$\frac{\phi - \phi_0}{\bar{K}} = \bar{v} + \bar{f}_\phi \sin(2\bar{v}) + \bar{g}_\phi \sin(3\bar{v})\,. \tag{6.29c}$$

It is important to mention that $\bar{u}$ is a parameter ranging from $-\infty$ to $+\infty$ called *eccentric anomaly*.

Contrary to the bound case, all three eccentricities $\bar{e}_r$, $\bar{e}_t$ and $\bar{e}_\phi$ are greater than 1. Importantly, note that here, the naming convention for $\bar{f}_t$ and $\bar{g}_t$ is in close analogy to the usual notations in the bound case [161–163]. Some previous papers [95, 165] have swapped the two, which can lead to confusion.

The various QK parameters are given in terms of A, B, etc., as follows :

$$\bar{n} = \frac{A^{3/2}}{B}\,, \tag{6.30a}$$

$$\begin{aligned}\bar{K} =& \frac{F}{\sqrt{-C}} + \frac{B\,(3D_1F - 2CI_1)}{2(-C)^{5/2}}\\ &+ \frac{1}{16(-C)^{9/2}}\Big[105B^2D_1^2F - 15ACD_1^2F - 60B^2CD_2F + 12AC^2D_2F + 140B^3D_3F\\ &- 60ABCD_3F - 60B^2CD_1I_1 + 12AC^2D_1I_1 + 24B^2C^2I_2 - 8AC^3I_2 - 40B^3CI_3 + 24ABC^2I_3\Big]\,,\end{aligned} \tag{6.30b}$$

$$\bar{a}_r = \frac{B}{A} - \frac{D_1}{2C} - \frac{\left(2BD_1^2 - 2BCD_2 + 4B^2D_3 - ACD_3\right)}{2C^3}\,, \tag{6.30c}$$

$$\begin{aligned}\bar{e}_r =& \frac{\sqrt{B^2 - AC}}{B} + \frac{A\left(2B^2 - AC\right)D_1}{2B^2C\sqrt{B^2 - AC}} + \frac{A}{8B^3C^3\left(B^2 - AC\right)^{3/2}}\Big[16B^6D_1^2 - 24AB^4CD_1^2 + 5A^2B^2C^2D_1^2 + 2A^3C^3D_1^2\\ &- 16B^6CD_2 + 28AB^4C^2D_2 - 12A^2B^2C^3D_2 + 32B^7D_3 - 64AB^5CD_3 + 36A^2B^3C^2D_3 - 4A^3BC^3D_3\Big]\,,\end{aligned} \tag{6.30d}$$

$$\begin{aligned}\bar{e}_t =& \frac{\sqrt{B^2 - AC}}{B} + \frac{AD_1}{2C\sqrt{B^2 - AC}} + \frac{A}{8BC^3\left(B^2 - AC\right)^{3/2}}\Big[8B^4D_1^2 - 12AB^2CD_1^2 + 3A^2C^2D_1^2 - 8B^4CD_2\\ &+ 12AB^2C^2D_2 - 4A^2C^3D_2 + 16B^5D_3 - 28AB^3CD_3 + 12A^2BC^2D_3\Big]\,,\end{aligned} \tag{6.30e}$$

$$\begin{aligned}\bar{e}_\phi =& \frac{\sqrt{B^2 - AC}}{B} + \frac{A}{2B^2C\sqrt{B^2 - AC}F}\left[3B^2D_1F - 2ACD_1F - 2B^2CI_1 + 2AC^2I_1\right]\\ &+ \frac{A}{8B^3C^3\left(B^2 - AC\right)^{3/2}F^2}\Big[26B^6D_1^2F^2 - 36AB^4CD_1^2F^2 + A^2B^2C^2D_1^2F^2 + 8A^3C^3D_1^2F^2 - 32B^6CD_2F^2\\ &+ 60AB^4C^2D_2F^2 - 28A^2B^2C^3D_2F^2 + 79B^7D_3F^2 - 165AB^5CD_3F^2 + 97A^2B^3C^2D_3F^2 - 11A^3BC^3D_3F^2\\ &+ 8B^6CD_1FI_1 - 36AB^4C^2D_1FI_1 + 44A^2B^2C^3D_1FI_1 - 16A^3C^4D_1FI_1 - 16B^6C^2I_1^2 + 40AB^4C^3I_1^2\\ &- 32A^2B^2C^4I_1^2 + 8A^3C^5I_1^2 + 16B^6C^2FI_2 - 32AB^4C^3FI_2 + 16A^2B^2C^4FI_2 - 30B^7CFI_2 + 66AB^5C^2FI_3\\ &- 42A^2B^3C^3FI_3 + 6A^3BC^4FI_3\Big]\,,\end{aligned} \tag{6.30f}$$

$$\bar{f}_{4t} = -\frac{A^{3/2}\sqrt{B^2 - AC}D_3}{2B(-C)^{5/2}}\,, \tag{6.30g}$$

$$\bar{g}_{4t} = \frac{A^{3/2}\left(3D_1^2 - 4CD_2 + 12BD_3\right)}{8B(-C)^{5/2}}\,, \tag{6.30h}$$

$$\bar{f}_{4\phi} = \frac{B^2 - AC}{32C^4F^2}\left(D_1^2F^2 - 4CD_2F^2 + 20BD_3F^2 + 4CD_1FI_1 - 8C^2I_1^2 + 8C^2FI_2 - 24BCFI_3\right)\,, \tag{6.30i}$$

$$\bar{g}_{4\phi} = \frac{\left(B^2 - AC\right)^{3/2}(D_3F - 2CI_3)}{24C^4F}\,. \tag{6.30j}$$

It has been checked explicitly that replacing A, B, etc. by their GR values yields exactly the QK parametrization of [95, 165]; see Table VIII of Ref. [95] . In the case of ST theories, the explicit expressions of A, B, etc. in terms of the reduced energy $\varepsilon = -\bar{\varepsilon}$ and angular momentum $j = -\bar{\jmath}$ are given in (3.10) of [203]. Replacing these coefficients by their explicit values in Eq. (6.30),the explicit expressions of the QK parameters in terms of $(\bar{\varepsilon}, \bar{\jmath})$ for ST theories are

$$\bar{n} = \frac{c^3\bar{\varepsilon}^{3/2}}{\tilde{G}\alpha m}\left\{1 + \bar{\varepsilon}\left(\frac{15}{8} + \bar{\gamma} - \frac{1}{8}\nu\right) + \bar{\varepsilon}^2\left(\frac{555}{128} + \frac{33}{8}\bar{\gamma} + \bar{\gamma}^2 + \frac{15}{64}\nu + \frac{1}{8}\bar{\gamma}\nu + \frac{11}{128}\nu^2\right)\right\}\,, \tag{6.31a}$$

$$\begin{aligned}\bar{K} =& 1 + \frac{\bar{\varepsilon}}{\bar{\jmath}}\left(3 - \bar{\beta}_+ + 2\bar{\gamma} + \bar{\beta}_-\delta\right)\\ &+ \frac{\bar{\varepsilon}^2}{\bar{\jmath}^2}\Bigg\{\frac{105}{4} + \frac{3}{2}\bar{\beta}_-^2 - 15\bar{\beta}_+ + \frac{3}{2}\bar{\beta}_+^2 - \frac{5}{4}\bar{\delta}_+ + \frac{139}{4}\bar{\gamma} - 12\bar{\beta}_+\bar{\gamma} + \frac{187}{16}\bar{\gamma}^2 - 2\bar{\chi}_+\\ &+ \delta\left(15\bar{\beta}_- - 3\bar{\beta}_-\bar{\beta}_+ - \frac{5}{4}\bar{\delta}_- + 12\bar{\beta}_-\bar{\gamma} + 2\bar{\chi}_-\right)\end{aligned}$$

$$
\begin{aligned}
&+\nu\Big(-\frac{15}{2}-6\bar{\beta}_-^2+\frac{21}{2}\bar{\beta}_+-2\bar{\delta}_+-8\bar{\gamma}-\frac{1}{2}\bar{\gamma}^2+24\bar{\beta}_-^2\bar{\gamma}^{-1}-24\bar{\beta}_+^2\bar{\gamma}^{-1}+4\bar{\chi}_+-\frac{3}{2}\bar{\beta}_-\delta\Big)\\
&+\bar{\jmath}\Big[\frac{15}{4}-\frac{1}{4}\bar{\delta}_++\frac{15}{4}\bar{\gamma}+\frac{15}{16}\bar{\gamma}^2-\frac{1}{4}\bar{\delta}_-\delta+\nu\Big(-\frac{3}{2}+\frac{1}{2}\bar{\beta}_+-\bar{\gamma}-\frac{1}{2}\bar{\beta}_-\delta\Big)\Big]\Big\}\,,
\end{aligned}
\tag{6.31b}
$$

$$
\begin{aligned}
\bar{a}_r = {}& \frac{\tilde{G}\alpha m}{c^2\bar{\varepsilon}}\Big(1+\bar{\varepsilon}\Big(\frac{7}{4}+\bar{\gamma}-\frac{1}{4}\nu\Big)\\
&+\bar{\varepsilon}^2\Big\{\frac{1}{16}+\frac{1}{16}\nu^2+\frac{1}{\bar{\jmath}}\Big[4-2\bar{\beta}_+-\frac{2}{3}\bar{\delta}_++\frac{16}{3}\bar{\gamma}-2\bar{\beta}_+\bar{\gamma}+\frac{11}{6}\bar{\gamma}^2-\frac{2}{3}\bar{\chi}_++\delta\Big(2\bar{\beta}_--\frac{2}{3}\bar{\delta}_-+2\bar{\beta}_-\bar{\gamma}+\frac{2}{3}\bar{\chi}_-\Big)\\
&\qquad+\nu\Big(-7+5\bar{\beta}_+-\frac{2}{3}\bar{\delta}_+-\frac{17}{3}\bar{\gamma}-\frac{1}{6}\bar{\gamma}^2+8\bar{\beta}_-^2\bar{\gamma}^{-1}-8\bar{\beta}_+^2\bar{\gamma}^{-1}+\frac{4}{3}\bar{\chi}_+-\bar{\beta}_-\delta\Big)\Big]\Big\}\Big)\,,
\end{aligned}
\tag{6.31c}
$$

$$
\begin{aligned}
\bar{e}_r = {}& \sqrt{1+\bar{\jmath}}+\frac{\bar{\varepsilon}}{\sqrt{1+\bar{\jmath}}}\Big\{-3+\bar{\beta}_+-2\bar{\gamma}-\bar{\beta}_-\delta+\bar{\jmath}\Big(-\frac{15}{8}-\bar{\gamma}+\frac{5}{8}\nu\Big)+\frac{1}{2}\nu\Big\}\\
&+\frac{\bar{\varepsilon}^2}{(1+\bar{\jmath})^{3/2}}\Big\{-\frac{35}{4}-\frac{1}{2}\bar{\beta}_-^2+\frac{21}{4}\bar{\beta}_+-\frac{1}{2}\bar{\beta}_+^2+\frac{5}{2}\bar{\delta}_+-12\bar{\gamma}+6\bar{\beta}_+\bar{\gamma}-\frac{35}{8}\bar{\gamma}^2+2\bar{\chi}_+\\
&\quad+\delta\Big(-\frac{21}{4}\bar{\beta}_-+\bar{\beta}_-\bar{\beta}_++\frac{5}{2}\bar{\delta}_--6\bar{\beta}_-\bar{\gamma}-2\bar{\chi}_-\Big)\\
&\quad+\nu\Big(\frac{99}{4}+2\bar{\beta}_-^2-\frac{65}{4}\bar{\beta}_++2\bar{\delta}_++\frac{39}{2}\bar{\gamma}+\frac{1}{2}\bar{\gamma}^2-24\bar{\beta}_-^2\bar{\gamma}^{-1}+24\bar{\beta}_+^2\bar{\gamma}^{-1}-4\bar{\chi}_++\frac{13}{4}\bar{\beta}_-\delta\Big)\\
&\quad+\frac{1}{\bar{\jmath}}\Big[-8+4\bar{\beta}_++\frac{4}{3}\bar{\delta}_+-\frac{32}{3}\bar{\gamma}+4\bar{\beta}_+\bar{\gamma}-\frac{11}{3}\bar{\gamma}^2+\frac{4}{3}\bar{\chi}_++\delta\Big(-4\bar{\beta}_-+\frac{4}{3}\bar{\delta}_--4\bar{\beta}_-\bar{\gamma}-\frac{4}{3}\bar{\chi}_-\Big)\\
&\qquad+\nu\Big(14-10\bar{\beta}_++\frac{4}{3}\bar{\delta}_++\frac{34}{3}\bar{\gamma}+\frac{1}{3}\bar{\gamma}^2-16\bar{\beta}_-^2\bar{\gamma}^{-1}+16\bar{\beta}_+^2\bar{\gamma}^{-1}-\frac{8}{3}\bar{\chi}_++2\bar{\beta}_-\delta\Big)\Big]\\
&\quad+\bar{\jmath}\Big[\frac{25}{8}+\frac{1}{8}\bar{\beta}_++\frac{7}{6}\bar{\delta}_++\frac{41}{12}\bar{\gamma}+\bar{\beta}_+\bar{\gamma}+\frac{19}{24}\bar{\gamma}^2+\frac{2}{3}\bar{\chi}_++\delta\Big(-\frac{1}{8}\bar{\beta}_-+\frac{7}{6}\bar{\delta}_--\bar{\beta}_-\bar{\gamma}-\frac{2}{3}\bar{\chi}_-\Big)\\
&\qquad+\nu\Big(\frac{37}{4}-\frac{51}{8}\bar{\beta}_++\frac{2}{3}\bar{\delta}_++\frac{89}{12}\bar{\gamma}+\frac{1}{6}\bar{\gamma}^2-8\bar{\beta}_-^2\bar{\gamma}^{-1}+8\bar{\beta}_+^2\bar{\gamma}^{-1}-\frac{4}{3}\bar{\chi}_++\frac{11}{8}\bar{\beta}_-\delta\Big)+\frac{1}{16}\nu^2\Big]\\
&\quad+\bar{\jmath}^2\Big[\frac{415}{128}+\frac{29}{8}\bar{\gamma}+\bar{\gamma}^2-\nu\Big(\frac{105}{64}+\frac{7}{8}\bar{\gamma}\Big)+\frac{7}{128}\nu^2\Big]\Big\}\,,
\end{aligned}
\tag{6.31d}
$$

$$
\begin{aligned}
\bar{e}_t = {}& \sqrt{1+\bar{\jmath}}+\frac{\bar{\varepsilon}}{\sqrt{1+\bar{\jmath}}}\Big[1+\bar{\beta}_+-\bar{\beta}_-\delta+\bar{\jmath}\Big(\frac{17}{8}+\bar{\gamma}-\frac{7}{8}\nu\Big)-\nu\Big]\\
&+\frac{\bar{\varepsilon}^2}{(1+\bar{\jmath})^{3/2}}\Big\{\frac{21}{4}\bar{\beta}_+-\frac{15}{4}-\frac{1}{2}\bar{\beta}_-^2-\frac{1}{2}\bar{\beta}_+^2+\frac{7}{6}\bar{\delta}_+-\frac{41}{6}\bar{\gamma}+4\bar{\beta}_+\bar{\gamma}-\frac{65}{24}\bar{\gamma}^2+\frac{2}{3}\bar{\chi}_+\\
&\quad+\delta\Big(-\frac{21}{4}\bar{\beta}_-+\bar{\beta}_-\bar{\beta}_++\frac{7}{6}\bar{\delta}_--4\bar{\beta}_-\bar{\gamma}-\frac{2}{3}\bar{\chi}_-\Big)+\frac{3}{4}\nu^2\\
&\quad+\nu\Big(\frac{25}{2}+2\bar{\beta}_-^2-\frac{31}{4}\bar{\beta}_++\frac{2}{3}\bar{\delta}_++\frac{29}{3}\bar{\gamma}+\frac{1}{6}\bar{\gamma}^2-8\bar{\beta}_-^2\bar{\gamma}^{-1}+8\bar{\beta}_+^2\bar{\gamma}^{-1}-\frac{4}{3}\bar{\chi}_++\frac{11}{4}\bar{\beta}_-\delta\Big)\\
&\quad+\frac{1}{\bar{\jmath}}\Big[-4+2\bar{\beta}_++\frac{2}{3}\bar{\delta}_+-\frac{16}{3}\bar{\gamma}+2\bar{\beta}_+\bar{\gamma}-\frac{11}{6}\bar{\gamma}^2+\frac{2}{3}\bar{\chi}_++\delta\Big(-2\bar{\beta}_-+\frac{2}{3}\bar{\delta}_--2\bar{\beta}_-\bar{\gamma}-\frac{2}{3}\bar{\chi}_-\Big)\\
&\qquad+\nu\Big(7-5\bar{\beta}_++\frac{2}{3}\bar{\delta}_++\frac{17}{3}\bar{\gamma}+\frac{1}{6}\bar{\gamma}^2-8\bar{\beta}_-^2\bar{\gamma}^{-1}+8\bar{\beta}_+^2\bar{\gamma}^{-1}-\frac{4}{3}\bar{\chi}_++\bar{\beta}_-\delta\Big)\Big]\\
&\quad+\bar{\jmath}\Big[\frac{45}{8}+\frac{17}{8}\bar{\beta}_++\frac{1}{2}\bar{\delta}_++4\bar{\gamma}+\bar{\beta}_+\bar{\gamma}+\frac{5}{8}\bar{\gamma}^2+\delta\Big(-\frac{17}{8}\bar{\beta}_-+\frac{1}{2}\bar{\delta}_--\bar{\beta}_-\bar{\gamma}\Big)\\
&\qquad+\nu\Big(\frac{73}{16}-\frac{23}{8}\bar{\beta}_++\frac{7}{2}\bar{\gamma}+\frac{15}{8}\bar{\beta}_-\delta\Big)+\frac{11}{8}\nu^2\Big]\\
&\quad+\bar{\jmath}^2\Big[\frac{607}{128}+\frac{35}{8}\bar{\gamma}+\bar{\gamma}^2-\nu\Big(\frac{69}{64}+\frac{5}{8}\bar{\gamma}\Big)+\frac{79}{128}\nu^2\Big]\Big\}\,,
\end{aligned}
\tag{6.31e}
$$

$$
\begin{aligned}
\bar{e}_\phi = {}& \sqrt{1+\bar{\jmath}}+\frac{\bar{\varepsilon}}{\sqrt{1+\bar{\jmath}}}\Big[-3+\bar{\beta}_+-2\bar{\gamma}-\bar{\beta}_-\delta-\bar{\jmath}\Big(\frac{15}{8}+\bar{\gamma}+\frac{1}{8}\nu\Big)\Big]\\
&+\frac{\bar{\varepsilon}^2}{(1+\bar{\jmath})^{3/2}}\Big\{\frac{37}{4}\bar{\beta}_+-\frac{75}{4}-\frac{1}{2}\bar{\beta}_-^2-\frac{1}{2}\bar{\beta}_+^2+\frac{11}{6}\bar{\delta}_+-\frac{74}{3}\bar{\gamma}+10\bar{\beta}_+\bar{\gamma}-\frac{205}{24}\bar{\gamma}^2+\frac{10}{3}\bar{\chi}_+\\
&\quad+\delta\Big(-\frac{37}{4}\bar{\beta}_-+\bar{\beta}_-\bar{\beta}_++\frac{11}{6}\bar{\delta}_--10\bar{\beta}_-\bar{\gamma}-\frac{10}{3}\bar{\chi}_-\Big)+\frac{33}{32}\nu^2\\
&\quad+\nu\Big(\frac{125}{32}+2\bar{\beta}_-^2-\frac{59}{4}\bar{\beta}_++\frac{10}{3}\bar{\delta}_++\frac{47}{6}\bar{\gamma}+\frac{5}{6}\bar{\gamma}^2-40\bar{\beta}_-^2\bar{\gamma}^{-1}+40\bar{\beta}_+^2\bar{\gamma}^{-1}-\frac{20}{3}\bar{\chi}_++\frac{7}{4}\bar{\beta}_-\delta\Big)\\
&\quad+\frac{1}{\bar{\jmath}}\Big[-13+6\bar{\beta}_++\bar{\delta}_+-17\bar{\gamma}+6\bar{\beta}_+\bar{\gamma}-\frac{23}{4}\bar{\gamma}^2+2\bar{\chi}_++\delta\Big(-6\bar{\beta}_-+\bar{\delta}_--6\bar{\beta}_-\bar{\gamma}-2\bar{\chi}_-\Big)\\
&\qquad+\nu\Big(\frac{91}{32}-9\bar{\beta}_++2\bar{\delta}_++5\bar{\gamma}+\frac{1}{2}\bar{\gamma}^2-24\bar{\beta}_-^2\bar{\gamma}^{-1}+24\bar{\beta}_+^2\bar{\gamma}^{-1}-4\bar{\chi}_++\bar{\beta}_-\delta\Big)+\frac{15}{32}\nu^2\Big]\\
&\quad+\bar{\jmath}\Big[\frac{17}{8}\bar{\beta}_+-\frac{15}{8}+\frac{5}{6}\bar{\delta}_+-\frac{35}{12}\bar{\gamma}+3\bar{\beta}_+\bar{\gamma}-\frac{31}{24}\bar{\gamma}^2+\frac{4}{3}\bar{\chi}_++\delta\Big(-\frac{17}{8}\bar{\beta}_-+\frac{5}{6}\bar{\delta}_--3\bar{\beta}_-\bar{\gamma}-\frac{4}{3}\bar{\chi}_-\Big)\\
&\qquad+\nu\Big(\frac{15}{32}-\frac{47}{8}\bar{\beta}_++\frac{4}{3}\bar{\delta}_++\frac{31}{12}\bar{\gamma}+\frac{1}{3}\bar{\gamma}^2-16\bar{\beta}_-^2\bar{\gamma}^{-1}+16\bar{\beta}_+^2\bar{\gamma}^{-1}-\frac{8}{3}\bar{\chi}_++\frac{7}{8}\bar{\beta}_-\delta\Big)+\frac{21}{32}\nu^2\Big]
\end{aligned}
$$

$$+\bar{\jmath}^2\left[\frac{415}{128}+\frac{29}{8}\bar{\gamma}+\bar{\gamma}^2+\nu\left(-\frac{47}{64}-\frac{3}{8}\bar{\gamma}\right)+\frac{11}{128}\nu^2\right]\Bigg\}, \tag{6.31f}$$

$$\bar{f}_t = \frac{\bar{\varepsilon}^2\sqrt{1+\bar{\jmath}}\,\nu\,(15+8\bar{\gamma}-\nu)}{8\sqrt{\bar{\jmath}}}, \tag{6.31g}$$

$$\bar{f}_\phi = \frac{\bar{\varepsilon}^2(1+\bar{\jmath})}{8\bar{\jmath}^2}\left[1+\bar{\delta}_++\bar{\gamma}+\frac{1}{4}\bar{\gamma}^2+\bar{\delta}_-\delta+\nu\Big(19-6\bar{\beta}_++12\bar{\gamma}+2\bar{\beta}_-\delta\Big)-3\nu^2\right], \tag{6.31h}$$

$$\bar{g}_t = \frac{\bar{\varepsilon}^2}{8\sqrt{\bar{\jmath}}}\left[60-4\bar{\delta}_++60\bar{\gamma}+15\bar{\gamma}^2-4\bar{\delta}_-\delta+\nu\Big(-24+8\bar{\beta}_+-16\bar{\gamma}-8\bar{\beta}_-\delta\Big)\right], \tag{6.31i}$$

$$\bar{g}_\phi = \frac{\bar{\varepsilon}^2(1+\bar{\jmath})^{3/2}\nu(1-3\nu)}{32\bar{\jmath}^2}. \tag{6.31j}$$

6.3.2 Gauge invariant quantities: conservative scattering angle and impact parameter

From this parametrization, one can obtain two physical quantities of interest for the scattering problem: the scattering angle χ and the impact parameter b. The full 2PN expressions of the scattering angle χ and the impact parameter b in the case of ST theories, expressed explicitly in terms of the energy and angular momentum are presented in the following Subsections.

The scattering angle χ

The scattering angle is defined as $\chi = \Delta\phi - \pi$, where the accumulated azimuthal angle is defined as $\Delta\phi \equiv \lim_{t\to+\infty}\phi(t) - \lim_{t\to-\infty}\phi(t)$, as shown in [74, 151]. It reads at 2PN order:

$$\begin{aligned}
\chi = -\pi &+ \frac{\sqrt{A}}{C^2}\left(\frac{\left(3B^2-2AC\right)D_1F}{B^2-AC}-2CI_1\right)+\frac{\sqrt{A}}{24C^4\left(B^2-AC\right)^2}\Big[315B^5D_1^2F-570AB^3CD_1^2F+243A^2BC^2D_1^2F\\
&-180B^5CD_2F+336AB^3C^2D_2F-156A^2BC^3D_2F+420B^6D_3F-880AB^4CD_3F+524A^2B^2C^2D_3F\\
&-64A^3C^3D_3F-180B^5CD_1I_1+336AB^3C^2D_1I_1-156A^2BC^3D_1I_1+72B^5C^2I_2-144AB^3C^3I_2\\
&+72A^2BC^4I_2-120B^6CI_3+272AB^4C^2I_3-184A^2B^2C^3I_3+32A^3C^4I_3\Big]\\
&+\frac{1}{\sqrt{-C}}\arctan\left(\frac{B+\sqrt{B^2-AC}}{\sqrt{-AC}}\right)\Bigg[4F+\frac{B}{C^2}\left(6D_1F-4CI_1\right)\\
&+\frac{1}{4C^4}\Big(105B^2D_1^2F-15ACD_1^2F-60B^2CD_2F+12AC^2D_2F+140B^3D_3F-60ABCD_3F\\
&-60B^2CD_1I_1+12AC^2D_1I_1+24B^2C^2I_2-8AC^3I_2-40B^3CI_3+24ABC^2I_3\Big)\Bigg].
\end{aligned} \tag{6.32}$$

This expression agrees[2] at 2PN, in the GR case, with (5.49) of [151] and with (45) of [74]. The agreement[3] is also at 2PN, in the case of ST theories, with Eq. (3.14-16) of [205].

The full scattering angle in terms of energy and angular momentum is

$$\begin{aligned}
\frac{\chi}{2} = \arctan\left(\frac{1}{\sqrt{\bar{\jmath}}}\right)&\left(1+\frac{\bar{\varepsilon}}{\bar{\jmath}}\left(3-\bar{\beta}_++2\bar{\gamma}+\bar{\beta}_-\delta\right)\right.\\
&+\frac{\bar{\varepsilon}^2}{\bar{\jmath}}\left\{\frac{15}{4}-\frac{1}{4}\bar{\delta}_++\frac{15}{4}\bar{\gamma}+\frac{15}{16}\bar{\gamma}^2-\frac{1}{4}\bar{\delta}_-\delta+\nu\left(-\frac{3}{2}+\frac{1}{2}\bar{\beta}_+-\bar{\gamma}-\frac{1}{2}\bar{\beta}_-\delta\right)\right.
\end{aligned}$$

[2] The comparison with [74] is non trivial because of different definitions for the energy. In that paper, $\bar{E}$ is defined in an EOB setting, hence one should the E used here in terms of the $\bar{E}$ used in that context as follows:

$$E = mc^2\left(\sqrt{1+2\nu(\sqrt{2\bar{E}+1}-1)}-1\right).$$

When performing the PN expansion of this relation and the required order, it is possible to find a perfect agreement.

[3] The agreement is obtained after a typo is corrected in Ref. [205]: the global prefactor of the C^0_{2PN} coefficient in Eq. (3.18) of the published version of Ref. [205] should be $\frac{1}{4\alpha^2(1+\alpha^2)^2}$. It is also necessary to take into account the different definitions of energy described in Footnote 2.

$$
\begin{aligned}
&\qquad + \frac{1}{\bar{\jmath}}\Bigg[\frac{105}{4} + \frac{3}{2}\bar{\beta}_-^2 - 15\bar{\beta}_+ + \frac{3}{2}\bar{\beta}_+^2 - \frac{5}{4}\bar{\delta}_+ + \frac{139}{4}\bar{\gamma} - 12\bar{\beta}_+\bar{\gamma} + \frac{187}{16}\bar{\gamma}^2 - 2\bar{\chi}_+ \\
&\qquad\quad + \delta\left(15\bar{\beta}_- - 3\bar{\beta}_-\bar{\beta}_+ - \frac{5}{4}\bar{\delta}_- + 12\bar{\beta}_-\bar{\gamma} + 2\bar{\chi}_-\right) \\
&\qquad\quad + \nu\left(-\frac{15}{2} - 6\bar{\beta}_-^2 + \frac{21}{2}\bar{\beta}_+ - 2\bar{\delta}_+ - 8\bar{\gamma} - \frac{1}{2}\bar{\gamma}^2 + 24\bar{\beta}_-^2\bar{\gamma}^{-1} - 24\bar{\beta}_+^2\bar{\gamma}^{-1} + 4\bar{\chi}_+ - \frac{3}{2}\bar{\beta}_-\delta\right)\Bigg]\Bigg\}\Bigg) \\
&+ \frac{\bar{\varepsilon}}{\bar{\jmath}}\left\{\pi\left[\frac{3}{2} - \frac{1}{2}\bar{\beta}_+ + \bar{\gamma} + \frac{1}{2}\bar{\beta}_-\delta\right] + \frac{\sqrt{\bar{\jmath}}}{1+\bar{\jmath}}\left(3 - \bar{\beta}_+ + 2\bar{\gamma} + \bar{\beta}_-\delta\right) + \frac{\bar{\jmath}^{3/2}}{1+\bar{\jmath}}\left(\frac{15}{8} + \bar{\gamma} - \frac{1}{8}\nu\right)\right\} \\
&+ \frac{\bar{\varepsilon}^2}{\bar{\jmath}^2(1+\bar{\jmath})^2}\Bigg\{\pi\Bigg[\frac{105}{8} + \frac{3}{4}\bar{\beta}_-^2 - \frac{15}{2}\bar{\beta}_+ + \frac{3}{4}\bar{\beta}_+^2 - \frac{5}{8}\bar{\delta}_+ + \frac{139}{8}\bar{\gamma} - 6\bar{\beta}_+\bar{\gamma} + \frac{187}{32}\bar{\gamma}^2 - \bar{\chi}_+ \\
&\qquad\qquad + \delta\left(\frac{15}{2}\bar{\beta}_- - \frac{3}{2}\bar{\beta}_-\bar{\beta}_+ - \frac{5}{8}\bar{\delta}_- + 6\bar{\beta}_-\bar{\gamma} + \bar{\chi}_-\right) \\
&\qquad\qquad + \nu\left(-\frac{15}{4} - 3\bar{\beta}_-^2 + \frac{21}{4}\bar{\beta}_+ - \bar{\delta}_+ - 4\bar{\gamma} - \frac{1}{4}\bar{\gamma}^2 + 12\bar{\beta}_-^2\bar{\gamma}^{-1} - 12\bar{\beta}_+^2\bar{\gamma}^{-1} + 2\bar{\chi}_+ - \frac{3}{4}\bar{\beta}_-\delta\right)\Bigg] \\
&\qquad + \sqrt{\bar{\jmath}}\Bigg[\frac{105}{4} + \frac{3}{2}\bar{\beta}_-^2 - 15\bar{\beta}_+ + \frac{3}{2}\bar{\beta}_+^2 - \frac{5}{4}\bar{\delta}_+ + \frac{139}{4}\bar{\gamma} - 12\bar{\beta}_+\bar{\gamma} + \frac{187}{16}\bar{\gamma}^2 - 2\bar{\chi}_+ \\
&\qquad\quad + \delta\left(15\bar{\beta}_- - 3\bar{\beta}_-\bar{\beta}_+ - \frac{5}{4}\bar{\delta}_- + 12\bar{\beta}_-\bar{\gamma} + 2\bar{\chi}_-\right) \\
&\qquad\quad + \nu\left(-\frac{15}{2} - 6\bar{\beta}_-^2 + \frac{21}{2}\bar{\beta}_+ - 2\bar{\delta}_+ - 8\bar{\gamma} - \frac{1}{2}\bar{\gamma}^2 + 24\bar{\beta}_-^2\bar{\gamma}^{-1} - 24\bar{\beta}_+^2\bar{\gamma}^{-1} + 4\bar{\chi}_+ - \frac{3}{2}\bar{\beta}_-\delta\right)\Bigg] \\
&\qquad + \bar{\jmath}\,\pi\Bigg[\frac{225}{8} + \frac{3}{2}\bar{\beta}_-^2 - 15\bar{\beta}_+ + \frac{3}{2}\bar{\beta}_+^2 - \frac{11}{8}\bar{\delta}_+ + \frac{293}{8}\bar{\gamma} - 12\bar{\beta}_+\bar{\gamma} + \frac{389}{32}\bar{\gamma}^2 - 2\bar{\chi}_+ \\
&\qquad\quad + \delta\left(15\bar{\beta}_- - 3\bar{\beta}_-\bar{\beta}_+ - \frac{11}{8}\bar{\delta}_- + 12\bar{\beta}_-\bar{\gamma} + 2\bar{\chi}_-\right) \\
&\qquad\quad + \nu\left(-\frac{33}{4} - 6\bar{\beta}_-^2 + \frac{43}{4}\bar{\beta}_+ - 2\bar{\delta}_+ - \frac{17}{2}\bar{\gamma} - \frac{1}{2}\bar{\gamma}^2 + 24\bar{\beta}_-^2\bar{\gamma}^{-1} - 24\bar{\beta}_+^2\bar{\gamma}^{-1} + 4\bar{\chi}_+ - \frac{7}{4}\bar{\beta}_-\delta\right)\Bigg] \\
&\qquad + \bar{\jmath}^{3/2}\Bigg[\frac{95}{2} + \frac{5}{2}\bar{\beta}_-^2 - 25\bar{\beta}_+ + \frac{5}{2}\bar{\beta}_+^2 - \frac{7}{3}\bar{\delta}_+ + \frac{185}{3}\bar{\gamma} - 20\bar{\beta}_+\bar{\gamma} + \frac{245}{12}\bar{\gamma}^2 - \frac{10}{3}\bar{\chi}_+ \\
&\qquad\quad + \delta\left(25\bar{\beta}_- - 5\bar{\beta}_-\bar{\beta}_+ - \frac{7}{3}\bar{\delta}_- + 20\bar{\beta}_-\bar{\gamma} + \frac{10}{3}\bar{\chi}_-\right) \\
&\qquad\quad + \nu\left(-14 - 10\bar{\beta}_-^2 + 18\bar{\beta}_+ - \frac{10}{3}\bar{\delta}_+ - \frac{43}{3}\bar{\gamma} - \frac{5}{6}\bar{\gamma}^2 + 40\bar{\beta}_-^2\bar{\gamma}^{-1} - 40\bar{\beta}_+^2\bar{\gamma}^{-1} + \frac{20}{3}\bar{\chi}_+ - 3\bar{\beta}_-\delta\right)\Bigg] \\
&\qquad + \bar{\jmath}^2\pi\Bigg[\frac{135}{8} + \frac{3}{4}\bar{\beta}_-^2 - \frac{15}{2}\bar{\beta}_+ + \frac{3}{4}\bar{\beta}_+^2 - \frac{7}{8}\bar{\delta}_+ + \frac{169}{8}\bar{\gamma} - 6\bar{\beta}_+\bar{\gamma} + \frac{217}{32}\bar{\gamma}^2 - \bar{\chi}_+ \\
&\qquad\quad + \delta\left(\frac{15}{2}\bar{\beta}_- - \frac{3}{2}\bar{\beta}_-\bar{\beta}_+ - \frac{7}{8}\bar{\delta}_- + 6\bar{\beta}_-\bar{\gamma} + \bar{\chi}_-\right) \\
&\qquad\quad + \nu\left(-\frac{21}{4} - 3\bar{\beta}_-^2 + \frac{23}{4}\bar{\beta}_+ - \bar{\delta}_+ - 5\bar{\gamma} - \frac{1}{4}\bar{\gamma}^2 + 12\bar{\beta}_-^2\bar{\gamma}^{-1} - 12\bar{\beta}_+^2\bar{\gamma}^{-1} + 2\bar{\chi}_+ - \frac{5}{4}\bar{\beta}_-\delta\right)\Bigg] \\
&\qquad + \bar{\jmath}^{5/2}\Bigg[\frac{2593}{128} - \frac{31}{4}\bar{\beta}_+ - \frac{13}{12}\bar{\delta}_+ + \frac{595}{24}\bar{\gamma} - 6\bar{\beta}_+\bar{\gamma} + \frac{371}{48}\bar{\gamma}^2 - \frac{4}{3}\bar{\chi}_+ \\
&\qquad\quad + \delta\left(\frac{31}{4}\bar{\beta}_- - \frac{13}{12}\bar{\delta}_- + 6\bar{\beta}_-\bar{\gamma} + \frac{4}{3}\bar{\chi}_-\right) + \frac{1}{128}\nu^2 \\
&\qquad\quad + \nu\left(-\frac{419}{64} + \frac{31}{4}\bar{\beta}_+ - \frac{4}{3}\bar{\delta}_+ - \frac{155}{24}\bar{\gamma} - \frac{1}{3}\bar{\gamma}^2 + 16\bar{\beta}_-^2\bar{\gamma}^{-1} - 16\bar{\beta}_+^2\bar{\gamma}^{-1} + \frac{8}{3}\bar{\chi}_+ - \frac{7}{4}\bar{\beta}_-\delta\right)\Bigg] \\
&\qquad + \bar{\jmath}^3\pi\left[\frac{15}{8} - \frac{1}{8}\bar{\delta}_+ + \frac{15}{8}\bar{\gamma} + \frac{15}{32}\bar{\gamma}^2 - \frac{1}{8}\bar{\delta}_-\delta + \nu\left(-\frac{3}{4} + \frac{1}{4}\bar{\beta}_+ - \frac{1}{2}\bar{\gamma} - \frac{1}{4}\bar{\beta}_-\delta\right)\right] \\
&\qquad + \bar{\jmath}^{7/2}\left[\frac{35}{128} + \frac{1}{8}\bar{\gamma} + \nu\left(\frac{15}{64} + \frac{1}{8}\bar{\gamma}\right) + \frac{3}{128}\nu^2\right]\Bigg\}\,.
\end{aligned}
\tag{6.33}
$$

The impact parameter b

The impact parameter was defined in Refs. [165, 215] as

$$b = \lim_{r\to\infty} \frac{|\mathbf{r}\times\mathbf{v}|}{v_\infty}\,, \tag{6.34}$$

which is not a manifestly gauge-invariant definition. The structure of Eqs (6.14a) and (6.15a), together with Eq. (6.34) means that, at 2PN, the impact parameter then simply reads

$$b = \frac{F}{\sqrt{A}}\,. \tag{6.35}$$

In fact, from the structure of the QK parametrization in GR [162, 165], it is clear that Eq. (6.35) even holds at 3PN in GR (in ST theories, this structure is spoiled by the tails appearing at 3PN in the equations of motion).

In GR, Eq. (6.35) yields the same results in both ADM or modified harmonic coordinates at 3PN, suggesting that Eq. (6.35) is indeed gauge-invariant, even if this is not manifest.

It is worthile to mention that, in GR, the 3PN impact parameter that was found in the literature contains typos. In GR, the impact parameter reads at 3PN:

$$\begin{aligned} b_{\mathrm{GR}} = \frac{Gm\sqrt{\bar{\jmath}}}{\bar{\varepsilon}}\Bigg\{ & 1 + \left(-\frac{1}{8}+\frac{3}{8}\nu\right)\bar{\varepsilon} + \bar{\varepsilon}^2\left(\frac{3}{128}-\frac{5}{64}\nu-\frac{5}{128}\nu^2\right) \\ & + \bar{\varepsilon}^3\left(-\frac{5}{1024}+\frac{21}{1024}\nu-\frac{7}{1024}\nu^2-\frac{7}{1024}\nu^3\right)\Bigg\}, \end{aligned} \tag{6.36}$$

where here the reduced energy and angular momentum are defined as in GR, namely

$$\bar{\varepsilon} = \frac{2E}{\mu c^2}, \quad \text{and} \quad \bar{\jmath} = \frac{2J^2E}{\mu^3(Gm)^2}\,. \tag{6.37}$$

This result is in agreement at 1PN with (40) of [216] and (2.18) of [215], but it disagrees already at 2PN with the 3PN-accurate result of (3.8) of [165]. In more details, the agreement with [216] is immediate, but the agreement with [215] is obtained after a typo is corrected in that work: an extra minus sign is needed in front of the 1PN correction. The result obtained by [165] also has a missing minus sign at 1PN, but starting at 2PN, there is an unresolvable disagreement with the result claimed there. This disagreement should propagate to [166], which uses this result, see their Eq. (32).

In ST theories, the expression for the impact parameter had never been derived. By using the expressions for A and F in harmonic coordinates [203] (the corresponding expressions in ADM coordinates are not known in ST theories), it is possible to write the explicit expression of the impact parameter as

$$b = \frac{\tilde{G}\alpha m\sqrt{\bar{\jmath}}}{c^2\bar{\varepsilon}}\left\{1+\bar{\varepsilon}\left(-\frac{1}{8}+\frac{3}{8}\nu\right)+\bar{\varepsilon}^2\left(\frac{3}{128}-\frac{5}{64}\nu-\frac{5}{128}\nu^2\right)\right\}. \tag{6.38}$$

The impact parameter can be computed following another procedure, in both GR and ST theories, using the manifestly gauge-invariant definition of Refs. [168, 217, 218], which reads

$$b = \frac{J}{p_{\mathrm{cm}}}\,, \tag{6.39}$$

where $p_{\rm cm}$ is defined in an EOB context as the incoming 3-momentum of either particle in the center-of-mass frame before the scattering event. The result obtained with this procedure is identical to the one presented above, computed starting from Eq. (6.34). Since $p_{\rm cm}$ can be related to E in a 'special relativistic', theory-independent way, the definition (6.39) clarifies why the expression of b in terms of (E, J) is identical in GR and ST theories [compare (6.36) and (6.38)].

6.3.3 Explicit check of the scatter-to-bound map

It has been proven [67, 68] on very general grounds that for the conservative dynamics, in the absence of hereditary terms, there is a very simple formula mapping the scattering angle in the unbound configuration to the pericenter advance in the bound configuration,

$$\Delta\Phi(E, J) = \chi(E, J) + \chi(E, -J)\,, \tag{6.40}$$

where $\Delta\Phi = 2\pi(K-1)$. This result is independent of the particular theory under consideration or the perturbation scheme. However, it requires that the motion derives from a Hamiltonian that is local in the positions and velocities. In particular, at 3PN in ST theories or at 4PN in GR, this map has not yet been fully proven due to the presence of hereditary tails in the equations of motion [195, 219], except by adding some additional hypothesis, such as a large eccentricity expansion, to include non-local-in-time effects.

In GR, this map was also verified at 3PN order in Section 5.5 of [69] for the dissipative dynamics. It is possible to check that constructing the aforementioned combination, it is possible to compute the periastron advance from the scattering angle given by (6.33) and the periastron advance listed in Eq. (C1b) of [203].

6.4 Radiated energy and angular momentum

In this Section, the instantaneous fluxes of energy and angular momentum for quasihyperbolic motion at Newtonian order will be computed, i.e. at 1PN relative order with respect to the leading -1PN dipolar contribution [197]. By integrating over the entire orbit one can compute the total energy and angular momentum lost by the system after the scattering process.

It was shown in [203] that the Newtonian fluxes can be expressed in terms of the source moments as follows (neglecting $\mathcal{O}(c^{-3})$ terms):

$$\mathcal{F} = \frac{G\phi_0}{5c^5}\overset{(3)}{\mathrm{I}}_{ij}\,\overset{(3)}{\mathrm{I}}_{ij}\,, \tag{6.41a}$$

$$\mathcal{G}_i = \frac{2G\phi_0}{5c^5}\epsilon_{iab}\,\overset{(2)}{\mathrm{I}}_{ak}\,\overset{(3)}{\mathrm{I}}_{bk}\,, \tag{6.41b}$$

$$\mathcal{F}^s = \frac{G\phi_0(3+2\omega_0)}{c^3}\left[\frac{\overset{(2)}{\mathrm{I}^s_a}\,\overset{(2)}{\mathrm{I}^s_a}}{3} + \frac{\overset{(1)}{E^s}\,\overset{(1)}{E^s}}{\phi_0^2c^2} + \frac{\overset{(3)}{\mathrm{I}^s_{ab}}\,\overset{(3)}{\mathrm{I}^s_{ab}}}{30c^2}\right], \tag{6.41c}$$

$$\mathcal{G}^s_i = \frac{G\phi_0(3+2\omega_0)}{c^3}\epsilon_{iab}\left[\frac{1}{3}\overset{(1)}{\mathrm{I}^s_a}\,\overset{(2)}{\mathrm{I}^s_b} + \frac{1}{15c^2}\overset{(2)}{\mathrm{I}^s_{ak}}\,\overset{(3)}{\mathrm{I}^s_{bk}}\right]. \tag{6.41d}$$

The source moments are in turn expressed in terms of the relative positions and velocities of the two particles in (4.6) of [197], whereas the acceleration needed to compute the time derivatives is given by (3.10) of [196] (but see also [207]). Since only nonspinning particles are considered, there is no precession of the orbital plane, so it is possible to write $\mathcal{G}_i = \mathcal{G}\ell_i$ and $\mathcal{G}_i^s = \mathcal{G}^s\ell_i$, where $\boldsymbol{\ell}$ is the constant unit vector orthogonal to the orbital plane. Finally, only the 1PN truncation of the QK parametrization derived in Sec. 6.3.1 is required, and allows to express the fluxes as pure functions of the eccentric anomaly $\bar{u}$.

The total energy and angular momentum lost by the binary after the scattering event are defined as

$$\Delta E = -\int_{-\infty}^{+\infty} \mathrm{d}t\, [\mathcal{F}(t) + \mathcal{F}_s(t)] \,, \tag{6.42a}$$

$$\Delta J = -\int_{-\infty}^{+\infty} \mathrm{d}t\, [\mathcal{G}(t) + \mathcal{G}_s(t)] \,, \tag{6.42b}$$

where it has been used the convention where both ΔE and ΔJ are negative. In practice, these are computed by performing the following change of variables in the integral:

$$\bar{n}\,\mathrm{d}t = \bar{e}_t \sinh{(\bar{u})}\,\mathrm{d}u + \mathcal{O}(c^{-4}) \,, \tag{6.43}$$

where Eq. (6.29b) has been leveraged. The fluxes are re-expressed in terms of the pair or variable $(\bar{x}, \bar{e}_t)$, where

$$\bar{x} = \left(\frac{\tilde{G}\alpha m \bar{K}\bar{n}}{c^3}\right)^{2/3} \tag{6.44}$$

and used the following 1PN-accurate relations, which can be directly obtained from Eq. (6.31):

$$\bar{\jmath} = \bar{e}_t^2 - 1 + \frac{1}{4}\big[9 + \nu - 8(\beta_+ - \beta_-\delta - \bar{\gamma}) - \bar{e}_t^2(17 - 7\nu + 8\bar{\gamma})\big]\bar{x} + O(\bar{x}^2) \,, \tag{6.45a}$$

$$\bar{\varepsilon} = \bar{x} - \frac{1}{12(\bar{e}_t^2 - 1)}\big[9 + \nu - 8(\beta_+ - \beta_-\delta - \bar{\gamma}) + \bar{e}_t^2(15 - \nu + 8\bar{\gamma})\big]\bar{x}^2 + O(\bar{x}^3) \,. \tag{6.45b}$$

Note that, contrary to the bound case, $\bar{x}$ is not related to any physical frequency, and is simply a convenient parametrization.

The expressions of the integrands can then be expressed, up to neglected 2PN terms, as linear combinations of terms $\propto (\bar{e}_t \cosh \bar{u} - 1)^{-N}$ and $\propto \sinh(\bar{u})(\bar{e}_t \cosh \bar{u} - 1)^{-N}$. The integrals are finally performed thanks to the formulas [164]:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\bar{u}}{(\bar{e}_t \cosh \bar{u} - 1)^{N+1}} = \frac{2}{N!}\frac{\mathrm{d}^N}{\mathrm{d}x^N}\left(\frac{\arccos(-x/\bar{e}_r)}{\sqrt{\bar{e}_t^2 - x^2}}\right)\Bigg|_{x=1} \,, \tag{6.46a}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\bar{u}\sinh(\bar{u})}{(\bar{e}_t \cosh \bar{u} - 1)^{N+1}} = 0 \,. \tag{6.46b}$$

Finally, using Eq. (6.31) one can re-express the losses of energy in terms of $(\bar{\varepsilon}, \bar{\jmath})$. The total losses of energy and angular momentum are decomposed into scalar and gravitational contributions, namely $\Delta E = [\Delta E]_{\mathrm{grav}} + [\Delta E]_{\mathrm{scalar}}$ and $\Delta J = [\Delta J]_{\mathrm{grav}} + [\Delta J]_{\mathrm{scalar}}$. These contributions read:

$$[\Delta E]_{\mathrm{grav}} = -\frac{170c^2(1 + \bar{\gamma}/2)m\nu^2\bar{\varepsilon}^{7/2}}{3\bar{\jmath}^{7/2}}\left[\bar{\jmath}^{1/2}\left(1 + \frac{673}{1275}\bar{\jmath}\right) + \left(1 + \frac{366}{425}\bar{\jmath} + \frac{37}{425}\bar{\jmath}^2\right)(\pi - \arctan\sqrt{\bar{\jmath}})\right] , \tag{6.47a}$$

$$[\Delta J]_{\rm grav} = -\frac{24(1+\bar\gamma/2)\tilde G \alpha m^2 \nu^2 \bar\varepsilon^2}{c\bar\jmath^2}\left[\bar\jmath^{1/2}\left(1+\frac{2}{15}\bar\jmath\right)+\left(1+\frac{7}{15}\bar\jmath\right)\left(\pi-\arctan\sqrt{\bar\jmath}\right)\right], \tag{6.47b}$$

$$\begin{aligned}
[\Delta E]_{\rm scalar} = -\frac{c^2 m\nu^2\bar\varepsilon^{5/2}}{\bar\jmath^2}\Bigg(&4\zeta\mathcal{S}_-^2\left\{1+\frac{1}{\sqrt{\bar\jmath}}\left(1+\frac{1}{3}\bar\jmath\right)\left(\pi-\arctan\sqrt{\bar\jmath}\right)\right\}\\
&+\bar\varepsilon\Bigg\{\frac{208}{9}\bar\beta_+ - \frac{571}{90}\bar\gamma - \frac{104}{3}\bar\beta_+^2\bar\gamma^{-1} + \frac{481}{18}\zeta\mathcal{S}_-^2 - \frac{64}{9}\bar\beta_+\zeta\mathcal{S}_-^2 + \frac{112}{9}\zeta\bar\gamma\mathcal{S}_-^2 - \frac{196}{9}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\quad + \frac{208}{3}\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 + \frac{208}{3}\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 - \frac{196}{9}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ + \frac{416}{3}\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\quad + \delta\Big(\frac{64}{9}\bar\beta_-\zeta\mathcal{S}_-^2 - \frac{116}{9}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - \frac{116}{9}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\Big) - \frac{239}{18}\zeta\mathcal{S}_-^2\nu\\
&\quad + \frac{1}{\bar\jmath}\Bigg[\frac{80}{3}\bar\beta_+ - \frac{55}{6}\bar\gamma - 40\bar\beta_+^2\bar\gamma^{-1} + \frac{202}{3}\zeta\mathcal{S}_-^2 - \frac{128}{3}\bar\beta_+\zeta\mathcal{S}_-^2 + \frac{140}{3}\zeta\bar\gamma\mathcal{S}_-^2 - \frac{20}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\qquad + 80\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 + 80\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 - \frac{20}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ + 160\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\qquad + \delta\left(\frac{128}{3}\bar\beta_-\zeta\mathcal{S}_-^2 - \frac{100}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - \frac{100}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right) - \frac{62}{3}\zeta\mathcal{S}_-^2\nu\Bigg]\\
&\quad - \frac{1}{1+\bar\jmath}\Bigg[\frac{59}{6}\zeta\mathcal{S}_-^2 + \frac{4}{3}\bar\beta_+\zeta\mathcal{S}_-^2 + 4\zeta\bar\gamma\mathcal{S}_-^2 - \frac{4}{3}\bar\beta_-\zeta\mathcal{S}_-^2\delta - \frac{29}{6}\zeta\mathcal{S}_-^2\nu\\
&\qquad + \bar\jmath\left(\frac{17}{6}\zeta\mathcal{S}_-^2 + \frac{4}{3}\zeta\bar\gamma\mathcal{S}_-^2 - \frac{7}{6}\zeta\mathcal{S}_-^2\nu\right) + \frac{4\zeta\mathcal{S}_-^2}{\bar\jmath}\left(1+\bar\beta_+ - \bar\beta_-\delta - \nu\right)\Bigg]\\
&\quad + \left(\pi-\arctan\sqrt{\bar\jmath}\right)\Bigg[\frac{1}{\sqrt{\bar\jmath}}\Big(32\bar\beta_+ - \frac{47}{5}\bar\gamma - 48\bar\beta_+^2\bar\gamma^{-1} + 42\zeta\mathcal{S}_-^2 - 20\bar\beta_+\zeta\mathcal{S}_-^2 + 24\zeta\bar\gamma\mathcal{S}_-^2 - 24\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\qquad + 96\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 + 96\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 - 24\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ + 192\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\qquad + \delta\Big(20\bar\beta_-\zeta\mathcal{S}_-^2 - 24\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - 24\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\Big) - 18\zeta\mathcal{S}_-^2\nu\Big)\\
&\quad + \frac{1}{\bar\jmath^{3/2}}\left(\frac{80}{3}\bar\beta_+ - \frac{55}{6}\bar\gamma - 40\bar\beta_+^2\bar\gamma^{-1} + \frac{190}{3}\zeta\mathcal{S}_-^2 - \frac{140}{3}\bar\beta_+\zeta\mathcal{S}_-^2 + \frac{140}{3}\zeta\bar\gamma\mathcal{S}_-^2 - \frac{20}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\right.\\
&\qquad + 80\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 + 80\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 - \frac{20}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ + 160\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\qquad \left. + \delta\Big(\frac{140}{3}\bar\beta_-\zeta\mathcal{S}_-^2 - \frac{100}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - \frac{100}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\Big) - \frac{50}{3}\zeta\mathcal{S}_-^2\nu\right)\\
&\quad + \sqrt{\bar\jmath}\Big(\frac{16}{3}\bar\beta_+ - \frac{13}{10}\bar\gamma - 8\bar\beta_+^2\bar\gamma^{-1} + \frac{13}{3}\zeta\mathcal{S}_-^2 + \frac{4}{3}\zeta\bar\gamma\mathcal{S}_-^2 - \frac{20}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\qquad + 16\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 + 16\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 - \frac{20}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ + 32\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\qquad - \delta\Big(\frac{4}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 + \frac{4}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\Big) - \frac{7}{3}\zeta\mathcal{S}_-^2\nu\Big)\Bigg]\Bigg\}\Bigg),
\end{aligned} \tag{6.47c}$$

$$\begin{aligned}
[\Delta J]_{\rm scalar} = -\frac{2\tilde G\alpha m^2\nu^2\bar\varepsilon}{3c\bar\jmath}\Bigg(&4\zeta\mathcal{S}_-^2\left\{\bar\jmath^{1/2} + \left(\pi-\arctan\sqrt{\bar\jmath}\right)\right\}\\
&-\bar\varepsilon\Bigg\{\frac{1}{\sqrt{\bar\jmath}}\Big[16\bar\beta_+\zeta\mathcal{S}_-^2 + 3\bar\gamma - 38\zeta\mathcal{S}_-^2 - 24\zeta\bar\gamma\mathcal{S}_-^2 - 24\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - 24\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ - 20\zeta\mathcal{S}_-^2\nu\\
&\qquad - \delta\Big(16\bar\beta_-\zeta\mathcal{S}_-^2 - 24\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - 24\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\Big)\Big]\\
&\quad + \bar\jmath^{1/2}\left[\frac{2}{5}\bar\gamma - 17\zeta\mathcal{S}_-^2 - 8\zeta\bar\gamma\mathcal{S}_-^2 + 7\zeta\mathcal{S}_-^2\nu\right]\\
&\quad + \frac{1}{1+\bar\jmath}\Big[\bar\jmath^{3/2}\Big(\frac{17}{2}\zeta\mathcal{S}_-^2 + 4\zeta\bar\gamma\mathcal{S}_-^2 - \frac{7}{2}\zeta\mathcal{S}_-^2\nu\Big) + \frac{8\zeta\mathcal{S}_-^2}{\bar\jmath^{1/2}}\left(1+\bar\beta_+ - \bar\beta_-\delta - \nu\right)\\
&\qquad + \bar\jmath^{1/2}\Big(21\zeta\mathcal{S}_-^2 + 4\bar\beta_+\zeta\mathcal{S}_-^2 + 8\zeta\bar\gamma\mathcal{S}_-^2 - 4\bar\beta_-\zeta\mathcal{S}_-^2\delta - 11\zeta\mathcal{S}_-^2\nu\Big)\Big]\\
&\quad - \left(\pi-\arctan\sqrt{\bar\jmath}\right)\Big[31\zeta\mathcal{S}_-^2 - \frac{7}{5}\bar\gamma - 4\bar\beta_+\zeta\mathcal{S}_-^2 + 16\zeta\bar\gamma\mathcal{S}_-^2 + 8\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2 + 8\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\\
&\qquad + \delta\Big(4\bar\beta_-\zeta\mathcal{S}_-^2 - 8\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - 8\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\Big) - 15\zeta\mathcal{S}_-^2\nu\\
&\qquad - \frac{1}{\bar\jmath}\Big(3\bar\gamma - 38\zeta\mathcal{S}_-^2 + 16\bar\beta_+\zeta\mathcal{S}_-^2 - 24\zeta\bar\gamma\mathcal{S}_-^2 - 24\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - 24\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\\
&\qquad\quad - \delta(16\bar\beta_-\zeta\mathcal{S}_-^2 - 24\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - 24\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+) + 20\zeta\mathcal{S}_-^2\nu\Big)\\
&\qquad - \frac{1}{1+\bar\jmath}\Big(25\zeta\mathcal{S}_-^2 + 8\bar\beta_+\zeta\mathcal{S}_-^2 + 8\zeta\bar\gamma\mathcal{S}_-^2 - 8\bar\beta_-\zeta\mathcal{S}_-^2\delta - 15\zeta\mathcal{S}_-^2\nu
\end{aligned}$$

$$+\,\bar{\jmath}\,(17+8\bar\gamma-7\nu)\,\zeta\mathcal{S}_-^2+\frac{8\zeta\mathcal{S}_-^2}{\bar{\jmath}}\left(1+\bar\beta_+-\bar\beta_-\delta-\nu\right)\Big)\Big]\Big\}\Big). \tag{6.47d}$$

6.4.1 Dissipative contributions to orbital elements

In the full dissipative problem, the radiation reaction makes the QK parameters vary throughout the motion by a small 1.5PN correction. For any QK parameter ξ, the initial value (before the scattering event) will be denoted by ξ^-, and its final value (after the scattering event) by ξ^+. The difference will be $\Delta\xi=\xi^+-\xi^-$. Since a system now has evolving energy and angular momentum, one needs to choose a convention in order to uniquely characterize a scattering configuration. A possible choice is to systematically describe a system using the initial energy and angular momentum (E^-,J^-). Note, however, that ΔE and ΔJ in (6.47) are only given to relative 1PN accuracy, so it doesn't matter whether they are computed with the initial or final QK parameters.

The evolution of the QK parameters can be obtained at low orders using the linear response formula [78, 151, 168]:

$$\Delta\xi=\frac{\partial\xi^{\rm cons}}{\partial E}\Delta E+\frac{\partial\xi^{\rm cons}}{\partial J}\Delta J. \tag{6.48}$$

Applying this formula, one can compute Δb, Δv_∞ and $\Delta\bar e_t$, whose explicit expressions are

$$
\begin{aligned}
\Delta b={}&\frac{\alpha\tilde G m^2\bar\varepsilon^{1/2}\nu^2}{2c^2}\Bigg\{4\zeta\mathcal{S}_-^2\left[\frac{2}{\bar\jmath^{3/2}}-\frac{4}{3\bar\jmath^{1/2}}+\left(\frac{2}{\bar\jmath^2}-\frac{2}{3\bar\jmath}\right)\left(\pi-\arctan\sqrt{\bar\jmath}\right)\right]\\
&+\frac{\bar\varepsilon}{1+\bar\jmath}\Bigg[\bar\jmath^{1/2}\left(-\frac{32}{5}-\frac{8}{3}\bar\gamma-\frac{32}{3}\zeta\mathcal{S}_-^2-\frac{16}{3}\zeta\bar\gamma\mathcal{S}_-^2+\frac{8}{3}\zeta\mathcal{S}_-^2\nu\right)\\
&\quad+\frac{1}{\bar\jmath^{1/2}}\Bigg(\frac{244}{45}+\frac{416}{9}\bar\beta_+-\frac{49}{9}\bar\gamma-\frac{208}{3}\bar\beta_+^2\bar\gamma^{-1}+\frac{37}{9}\zeta\mathcal{S}_-^2+\frac{112}{9}\bar\beta_+\zeta\mathcal{S}_-^2-\frac{88}{9}\zeta\bar\gamma\mathcal{S}_-^2\\
&\qquad-\frac{680}{9}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2+\frac{416}{3}\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2+\frac{416}{3}\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2+\frac{832}{3}\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\qquad-\frac{680}{9}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+-\frac{71}{9}\zeta\mathcal{S}_-^2\nu+\delta\left(-\frac{112}{9}\bar\beta_-\zeta\mathcal{S}_-^2+\frac{56}{9}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2+\frac{56}{9}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\Bigg)\\
&\quad+\frac{1}{\bar\jmath^{5/2}}\Bigg(\frac{340}{3}+\frac{160}{3}\bar\beta_++\frac{115}{3}\bar\gamma-80\bar\beta_+^2\bar\gamma^{-1}+\frac{380}{3}\zeta\mathcal{S}_-^2-\frac{280}{3}\bar\beta_+\zeta\mathcal{S}_-^2+\frac{280}{3}\zeta\bar\gamma\mathcal{S}_-^2-\frac{40}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\qquad+160\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2+160\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2-\frac{40}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_++320\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_++\frac{100}{3}\zeta\mathcal{S}_-^2\nu\\
&\qquad+\delta\left(\frac{280}{3}\bar\beta_-\zeta\mathcal{S}_-^2-\frac{200}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2-\frac{200}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\Bigg)\\
&\quad+\frac{1}{\bar\jmath^{3/2}}\Bigg(\frac{5632}{45}+\frac{896}{9}\bar\beta_++\frac{320}{9}\bar\gamma-\frac{448}{3}\bar\beta_+^2\bar\gamma^{-1}+\frac{1165}{9}\zeta\mathcal{S}_-^2-\frac{632}{9}\bar\beta_+\zeta\mathcal{S}_-^2+\frac{704}{9}\zeta\bar\gamma\mathcal{S}_-^2\\
&\qquad-\frac{800}{9}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2+\frac{896}{3}\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2+\frac{896}{3}\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2+\frac{1792}{3}\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+-\frac{407}{9}\zeta\mathcal{S}_-^2\nu\\
&\qquad-\frac{800}{9}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_++\delta\left(\frac{632}{9}\bar\beta_-\zeta\mathcal{S}_-^2-\frac{544}{9}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2-\frac{544}{9}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\Bigg)\Bigg]\\
&\quad+\frac{\bar\varepsilon}{1+\bar\jmath}\left(\pi-\arctan\sqrt{\bar\jmath}\right)\Bigg[-\frac{188}{15}+\frac{32}{3}\bar\beta_+-7\bar\gamma-16\bar\beta_+^2\bar\gamma^{-1}-9\zeta\mathcal{S}_-^2+\frac{16}{3}\bar\beta_+\zeta\mathcal{S}_-^2-8\zeta\bar\gamma\mathcal{S}_-^2\\
&\qquad-24\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2+32\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2+32\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2-24\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_++64\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\qquad+\delta\left(-\frac{16}{3}\bar\beta_-\zeta\mathcal{S}_-^2+8\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2+8\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)+3\zeta\mathcal{S}_-^2\nu\\
&\quad+\frac{1}{\bar\jmath}\Bigg(\frac{556}{15}+\frac{224}{3}\bar\beta_++3\bar\gamma-112\bar\beta_+^2\bar\gamma^{-1}+36\zeta\mathcal{S}_-^2-\frac{8}{3}\bar\beta_+\zeta\mathcal{S}_-^2+8\zeta\bar\gamma\mathcal{S}_-^2-104\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\qquad+224\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2+224\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2-104\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_++448\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+-20\zeta\mathcal{S}_-^2\nu\\
&\qquad+\delta\left(\frac{8}{3}\bar\beta_-\zeta\mathcal{S}_-^2-8\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2-8\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\Bigg)\\
&\quad+\frac{1}{\bar\jmath^3}\Bigg(\frac{340}{3}+\frac{160}{3}\bar\beta_++\frac{115}{3}\bar\gamma-80\bar\beta_+^2\bar\gamma^{-1}+\frac{380}{3}\zeta\mathcal{S}_-^2-\frac{280}{3}\bar\beta_+\zeta\mathcal{S}_-^2+\frac{280}{3}\zeta\bar\gamma\mathcal{S}_-^2-\frac{40}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\qquad+160\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2+160\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2-\frac{40}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_++320\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+-\frac{100}{3}\zeta\mathcal{S}_-^2\nu
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \delta\left(\frac{280}{3}\bar\beta_-\zeta\mathcal{S}_-^2 - \frac{200}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - \frac{200}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\Bigg)\\
&+ \frac{1}{\bar\jmath^2}\Bigg(\frac{2444}{15} + \frac{352}{3}\bar\beta_+ + \frac{145}{3}\bar\gamma - 176\bar\beta_+^2\bar\gamma^{-1} + \frac{515}{3}\zeta\mathcal{S}_-^2 - \frac{304}{3}\bar\beta_+\zeta\mathcal{S}_-^2 + \frac{328}{3}\zeta\bar\gamma\mathcal{S}_-^2 - \frac{280}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\quad + 352\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 + 352\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 - \frac{280}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ + 704\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\quad + \delta\left(\frac{304}{3}\bar\beta_-\zeta\mathcal{S}_-^2 - \frac{248}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - \frac{248}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right) - \frac{169}{3}\zeta\mathcal{S}_-^2\nu\Bigg)\Bigg]\Bigg\}, \qquad (6.49a)\\
\Delta v_\infty &= -cm\bar\varepsilon^2\nu^2\Bigg\{\frac{4\zeta\mathcal{S}_-^2}{3\bar\jmath^{5/2}}\left(3\bar\jmath^{1/2} + (3+\bar\jmath)\left(\pi - \arctan\sqrt{\bar\jmath}\right)\right)\\
&+ \frac{\bar\varepsilon}{1+\bar\jmath}\Bigg[\frac{1}{\bar\jmath^3}\Bigg(\frac{170}{3} + \frac{80}{3}\bar\beta_+ + \frac{115}{6}\bar\gamma - 40\bar\beta_+^2\bar\gamma^{-1} + \frac{190}{3}\zeta\mathcal{S}_-^2 - \frac{140}{3}\bar\beta_+\zeta\mathcal{S}_-^2 + \frac{140}{3}\zeta\bar\gamma\mathcal{S}_-^2 - \frac{20}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\quad + 80\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 + 80\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 - \frac{20}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ + 160\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+ - \frac{50}{3}\zeta\mathcal{S}_-^2\nu\\
&\quad + \delta\left(\frac{140}{3}\bar\beta_-\zeta\mathcal{S}_-^2 - \frac{100}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - \frac{100}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\Bigg)\\
&\quad + \frac{1}{\bar\jmath^2}\Bigg(\frac{3896}{45} + \frac{448}{9}\bar\beta_+ + \frac{250}{9}\bar\gamma - \frac{224}{3}\bar\beta_+^2\bar\gamma^{-1} + \frac{1435}{18}\zeta\mathcal{S}_-^2 - \frac{460}{9}\bar\beta_+\zeta\mathcal{S}_-^2 + \frac{496}{9}\zeta\bar\gamma\mathcal{S}_-^2\\
&\qquad + \frac{448}{3}\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 + \frac{448}{3}\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 - \frac{256}{9}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ + \frac{896}{3}\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\qquad - \frac{281}{18}\zeta\mathcal{S}_-^2\nu - \frac{256}{9}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2 + \delta\left(\frac{460}{9}\bar\beta_-\zeta\mathcal{S}_-^2 - \frac{416}{9}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - \frac{416}{9}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\Bigg)\\
&\quad + \frac{1}{\bar\jmath}\Bigg(\frac{1346}{45} + \frac{208}{9}\bar\beta_+ + \frac{155}{18}\bar\gamma - \frac{104}{3}\bar\beta_+^2\bar\gamma^{-1} + \frac{349}{18}\zeta\mathcal{S}_-^2 - \frac{64}{9}\bar\beta_+\zeta\mathcal{S}_-^2 + \frac{100}{9}\zeta\bar\gamma\mathcal{S}_-^2 - \frac{196}{9}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\qquad + \frac{208}{3}\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 + \frac{208}{3}\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 - \frac{196}{9}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ + \frac{416}{3}\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+ + \frac{25}{18}\zeta\mathcal{S}_-^2\nu\\
&\qquad + \delta\left(\frac{64}{9}\bar\beta_-\zeta\mathcal{S}_-^2 - \frac{116}{9}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - \frac{116}{9}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\Bigg)\Bigg]\\
&+ \frac{\bar\varepsilon}{1+\bar\jmath}\left(\pi - \arctan\sqrt{\bar\jmath}\right)\Bigg[\frac{1}{\bar\jmath^{5/2}}\Bigg(\frac{1582}{15} + \frac{176}{3}\bar\beta_+ + \frac{205}{6}\bar\gamma - 88\bar\beta_+^2\bar\gamma^{-1} + \frac{605}{6}\zeta\mathcal{S}_-^2 - \frac{200}{3}\bar\beta_+\zeta\mathcal{S}_-^2 + \frac{212}{3}\zeta\bar\gamma\mathcal{S}_-^2\\
&\quad - \frac{92}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2 + 176\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 + 176\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 - \frac{92}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ + 352\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\quad + \delta\left(\frac{200}{3}\bar\beta_-\zeta\mathcal{S}_-^2 - \frac{172}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - \frac{172}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right) - \frac{127}{6}\zeta\mathcal{S}_-^2\nu\Bigg)\\
&\quad + \frac{1}{\bar\jmath^{7/2}}\Bigg(\frac{170}{3} + \frac{80}{3}\bar\beta_+ + \frac{115}{6}\bar\gamma - 40\bar\beta_+^2\bar\gamma^{-1} + \frac{190}{3}\zeta\mathcal{S}_-^2 - \frac{140}{3}\bar\beta_+\zeta\mathcal{S}_-^2 + \frac{140}{3}\zeta\bar\gamma\mathcal{S}_-^2 - \frac{20}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\qquad + 80\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 + 80\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 - \frac{20}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ + 160\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+ - \frac{50}{3}\zeta\mathcal{S}_-^2\nu\\
&\qquad + \delta\left(\frac{140}{3}\bar\beta_-\zeta\mathcal{S}_-^2 - \frac{100}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - \frac{100}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\Bigg)\\
&\quad + \frac{1}{\bar\jmath^{3/2}}\Bigg(\frac{806}{15} + \frac{112}{3}\bar\beta_+ + \frac{97}{6}\bar\gamma - 56\bar\beta_+^2\bar\gamma^{-1} + \frac{121}{3}\zeta\mathcal{S}_-^2 - 20\bar\beta_+\zeta\mathcal{S}_-^2 + \frac{76}{3}\zeta\bar\gamma\mathcal{S}_-^2 - \frac{92}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\qquad + 112\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 + 112\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 - \frac{92}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ + 224\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+ - \frac{7}{3}\zeta\mathcal{S}_-^2\nu\\
&\qquad + \delta\left(20\bar\beta_-\zeta\mathcal{S}_-^2 - \frac{76}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - \frac{76}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\Bigg)\\
&\quad + \frac{1}{\bar\jmath^{1/2}}\Bigg(\frac{74}{15} + \frac{16}{3}\bar\beta_+ + \frac{7}{6}\bar\gamma - 8\bar\beta_+^2\bar\gamma^{-1} + \frac{17}{6}\zeta\mathcal{S}_-^2 + \frac{4}{3}\zeta\bar\gamma\mathcal{S}_-^2 - \frac{20}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2 + 16\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2\\
&\qquad + \frac{13}{6}\zeta\mathcal{S}_-^2\nu + 16\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 - \frac{20}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ + 32\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\qquad + \delta\left(-\frac{4}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - \frac{4}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\Bigg)\Bigg]\Bigg\}, \qquad (6.49b)\\
\Delta\bar e_t &= -m\bar\varepsilon^{3/2}\nu^2\Bigg\{\frac{4\zeta\mathcal{S}_-^2}{(1+\bar\jmath)^{1/2}\bar\jmath^{3/2}}\left[\bar\jmath^{1/2} + \frac{2}{3}\bar\jmath^{3/2} + (1+\bar\jmath)\left(\pi - \arctan\sqrt{\bar\jmath}\right)\right]\\
&+ \frac{\bar\varepsilon}{(1+\bar\jmath)^{3/2}}\Bigg[\frac{514}{9} + \frac{208}{9}\bar\beta_+ + \frac{359}{18}\bar\gamma - \frac{104}{3}\bar\beta_+^2\bar\gamma^{-1} + \frac{1453}{18}\zeta\mathcal{S}_-^2 - \frac{208}{9}\bar\beta_+\zeta\mathcal{S}_-^2 + \frac{400}{9}\zeta\bar\gamma\mathcal{S}_-^2 - \frac{52}{9}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\quad + \frac{208}{3}\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 + \frac{208}{3}\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2 - \frac{52}{9}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ + \frac{416}{3}\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+ - \frac{635}{18}\zeta\mathcal{S}_-^2\nu\\
&\quad + \delta\left(\frac{208}{9}\bar\beta_-\zeta\mathcal{S}_-^2 - \frac{260}{9}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2 - \frac{260}{9}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\jmath}\bigg(\frac{4976}{45}+\frac{448}{9}\bar\beta_+ +\frac{340}{9}\bar\gamma-\frac{224}{3}\bar\beta_+^2\bar\gamma^{-1}+\frac{1280}{9}\zeta\mathcal{S}_-^2-\frac{568}{9}\bar\beta_+\zeta\mathcal{S}_-^2+\frac{784}{9}\zeta\bar\gamma\mathcal{S}_-^2\\
&\qquad-\frac{112}{9}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2+\frac{448}{3}\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2-\frac{112}{9}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ +\frac{896}{3}\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\qquad+\frac{448}{3}\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2-\frac{496}{9}\zeta\mathcal{S}_-^2\nu+\delta\left(\frac{568}{9}\bar\beta_-\zeta\mathcal{S}_-^2-\frac{560}{9}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2-\frac{560}{9}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\bigg)\\
&+\frac{1}{\jmath^2}\bigg(\frac{170}{3}+\frac{80}{3}\bar\beta_+ +\frac{115}{6}\bar\gamma-40\bar\beta_+^2\bar\gamma^{-1}+\frac{214}{3}\zeta\mathcal{S}_-^2-\frac{116}{3}\bar\beta_+\zeta\mathcal{S}_-^2+\frac{140}{3}\zeta\bar\gamma\mathcal{S}_-^2-\frac{20}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\qquad+80\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2+80\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2-\frac{20}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ +160\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\qquad-\frac{74}{3}\zeta\mathcal{S}_-^2\nu+\delta\left(\frac{116}{3}\bar\beta_-\zeta\mathcal{S}_-^2-\frac{100}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2-\frac{100}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\bigg)\\
&+\jmath\left(\frac{16}{5}+\frac{4}{3}\bar\gamma+\frac{34}{3}\zeta\mathcal{S}_-^2+\frac{16}{3}\zeta\bar\gamma\mathcal{S}_-^2-\frac{14}{3}\zeta\mathcal{S}_-^2\nu\right)\bigg]\\
+\frac{\bar\varepsilon}{(1+\jmath)^{3/2}}\left(\pi-\arctan\sqrt{\jmath}\right)&\bigg[\frac{1}{\jmath^{3/2}}\bigg(\frac{1942}{15}+\frac{176}{3}\bar\beta_+ +\frac{265}{6}\bar\gamma-88\bar\beta_+^2\bar\gamma^{-1}+166\zeta\mathcal{S}_-^2-76\bar\beta_+\zeta\mathcal{S}_-^2+\frac{308}{3}\zeta\bar\gamma\mathcal{S}_-^2\\
&\qquad-\frac{44}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2+176\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2+176\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2-\frac{44}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ -\frac{190}{3}\zeta\mathcal{S}_-^2\nu\\
&\qquad+352\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\nu+\delta\left(76\bar\beta_-\zeta\mathcal{S}_-^2-\frac{220}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2-\frac{220}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\bigg)\\
&+\frac{1}{\jmath^{1/2}}\bigg(\frac{1334}{15}+\frac{112}{3}\bar\beta_+ +\frac{185}{6}\bar\gamma-56\bar\beta_+^2\bar\gamma^{-1}+\frac{245}{2}\zeta\mathcal{S}_-^2-40\bar\beta_+\zeta\mathcal{S}_-^2+\frac{208}{3}\zeta\bar\gamma\mathcal{S}_-^2\\
&\qquad+112\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2+112\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2-\frac{28}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ +224\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\qquad-\frac{313}{6}\zeta\mathcal{S}_-^2\nu-\frac{28}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2+\delta\left(40\bar\beta_-\zeta\mathcal{S}_-^2-\frac{140}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2-\frac{140}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\bigg)\\
&+\frac{1}{\jmath^{5/2}}\bigg(\frac{170}{3}+\frac{80}{3}\bar\beta_+ +\frac{115}{6}\bar\gamma-40\bar\beta_+^2\bar\gamma^{-1}+\frac{214}{3}\zeta\mathcal{S}_-^2-\frac{116}{3}\bar\beta_+\zeta\mathcal{S}_-^2+\frac{140}{3}\zeta\bar\gamma\mathcal{S}_-^2\\
&\qquad-\frac{20}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2+80\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2+80\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2-\frac{20}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ -\frac{74}{3}\zeta\mathcal{S}_-^2\nu\\
&\qquad+160\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+ +\delta\left(\frac{116}{3}\bar\beta_-\zeta\mathcal{S}_-^2-\frac{100}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2-\frac{100}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)\bigg)\\
&+\jmath^{1/2}\bigg(\frac{242}{15}+\frac{16}{3}\bar\beta_+ +\frac{35}{6}\bar\gamma-8\bar\beta_+^2\bar\gamma^{-1}+\frac{167}{6}\zeta\mathcal{S}_-^2-\frac{8}{3}\bar\beta_+\zeta\mathcal{S}_-^2+\frac{40}{3}\zeta\bar\gamma\mathcal{S}_-^2-\frac{4}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-^2\\
&\qquad+16\bar\beta_-^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2+16\bar\beta_+^2\zeta\bar\gamma^{-2}\mathcal{S}_-^2-\frac{4}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+ +32\bar\beta_-\bar\beta_+\zeta\bar\gamma^{-2}\mathcal{S}_-\mathcal{S}_+\\
&\qquad+\delta\left(\frac{8}{3}\bar\beta_-\zeta\mathcal{S}_-^2-\frac{20}{3}\bar\beta_-\zeta\bar\gamma^{-1}\mathcal{S}_-^2-\frac{20}{3}\bar\beta_+\zeta\bar\gamma^{-1}\mathcal{S}_-\mathcal{S}_+\right)-\frac{27}{2}\zeta\mathcal{S}_-^2\nu\bigg)\bigg]\bigg\}\,. \qquad (6.49c)
\end{aligned}
$$

The case of the scattering angle requires particular attention. Although one can formally define a scattering angle before and after the process (denoted χ^- and χ^+, respectively), it makes physically more sense to treat this quantity globally. The total scattering angle χ, including both conservative and dissipative contributions, is in fact the average of the two aforementioned quantities, and reads:

$$\chi=\frac{\chi^+ + \chi^-}{2}=\chi^- + \frac{\Delta\chi}{2}\,. \qquad (6.50)$$

Identifying the conservative contribution as the one before radiation, $\chi^{\text{cons}}(E^-,J^-)=\chi^-$, it follows that the dissipative contribution is given by

$$\chi^{\text{diss}}=\frac{1}{2}\left[\frac{\partial\chi^{\text{cons}}}{\partial E}\Delta E+\frac{\partial\chi^{\text{cons}}}{\partial J}\Delta J\right]\,. \qquad (6.51)$$

The formula (6.51) was initially obtained in GR in (5.99) of [151], but only at leading order in the radiation-reaction, namely 2.5PN. It was later shown to hold even at 4.5PN thanks to the time-antisymmetric (or time-odd) character of the radiation-reaction force, see (3.25) of [168]. It was then shown that time-*asymmetric* contributions to the radiation-reaction force would induce an extra term, denoted Δc_ϕ in (12.17) of [78], which would only start contributing at the order of radiation-reaction squared, namely 5PN in GR (see also App. C of [78] for details). These arguments immediately translate to ST theories, such that the radiation-reaction terms are time-antisymmetric at this order, and the extra Δc_ϕ contribution can safely be ignored (it will only enter only at the order of radiation-reaction squared, namely 3PN). Thus, the complete 2.5PN scattering angle for ST theories finally reads

$$\chi = \chi^{\text{cons}} + \chi^{\text{diss}} \tag{6.52}$$

where χ^{cons} is given in (6.33) and where the dissipative contribution reads:

$$
\begin{aligned}
\chi_{\text{diss}} = m\bar{\varepsilon}^{3/2}\nu^2 \Bigg\{ & \frac{4\zeta\mathcal{S}_-^2}{\bar{\jmath}^2}\left[\frac{1}{\bar{\jmath}^{1/2}(1+\bar{\jmath})}\left(1+\frac{2}{3}\bar{\jmath}\right)+\pi-\arctan\sqrt{\bar{\jmath}}\right] \\
& + \frac{\bar{\varepsilon}}{(1+\bar{\jmath})^2}\Bigg[\frac{1}{\bar{\jmath}^{3/2}}\bigg(\frac{4976}{45}+\frac{448}{9}\bar{\beta}_+ +\frac{340}{9}\bar{\gamma}-\frac{224}{3}\bar{\beta}_+^2\bar{\gamma}^{-1}+\frac{2191}{18}\zeta\mathcal{S}_-^2-\frac{724}{9}\bar{\beta}_+\zeta\mathcal{S}_-^2+\frac{772}{9}\zeta\bar{\gamma}\mathcal{S}_-^2 \\
& \qquad -\frac{112}{9}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2+\frac{448}{3}\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2+\frac{448}{3}\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2-\frac{112}{9}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+-\frac{641}{18}\zeta\mathcal{S}_-^2\nu \\
& \qquad +\frac{896}{3}\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ +\delta\left(\frac{724}{9}\bar{\beta}_-\zeta\mathcal{S}_-^2-\frac{560}{9}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2-\frac{560}{9}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right)\bigg) \\
& + \frac{1}{\bar{\jmath}^{1/2}}\bigg(\frac{514}{9}+\frac{208}{9}\bar{\beta}_+ +\frac{359}{18}\bar{\gamma}-\frac{104}{3}\bar{\beta}_+^2\bar{\gamma}^{-1}+\frac{1189}{18}\zeta\mathcal{S}_-^2-\frac{280}{9}\bar{\beta}_+\zeta\mathcal{S}_-^2+\frac{376}{9}\zeta\bar{\gamma}\mathcal{S}_-^2 \\
& \qquad -\frac{52}{9}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2+\frac{208}{3}\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2+\frac{208}{3}\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2-\frac{52}{9}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+-\frac{395}{18}\zeta\mathcal{S}_-^2\nu \\
& \qquad +\frac{416}{3}\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ +\delta\left(\frac{280}{9}\bar{\beta}_-\zeta\mathcal{S}_-^2-\frac{260}{9}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2-\frac{260}{9}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right)\bigg) \\
& + \frac{1}{\bar{\jmath}^{5/2}}\bigg(\frac{170}{3}+\frac{80}{3}\bar{\beta}_+ +\frac{115}{6}\bar{\gamma}-40\bar{\beta}_+^2\bar{\gamma}^{-1}+\frac{190}{3}\zeta\mathcal{S}_-^2-\frac{140}{3}\bar{\beta}_+\zeta\mathcal{S}_-^2+\frac{140}{3}\zeta\bar{\gamma}\mathcal{S}_-^2 \\
& \qquad -\frac{20}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2+80\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2+80\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2-\frac{20}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+-\frac{50}{3}\zeta\mathcal{S}_-^2\nu \\
& \qquad +160\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ +\delta\left(\frac{140}{3}\bar{\beta}_-\zeta\mathcal{S}_-^2-\frac{100}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2-\frac{100}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right)\bigg) \\
& + \bar{\jmath}^{1/2}\left(\frac{16}{5}+\frac{4}{3}\bar{\gamma}+\frac{32}{3}\zeta\mathcal{S}_-^2+\frac{16}{3}\zeta\bar{\gamma}\mathcal{S}_-^2-\frac{8}{3}\zeta\mathcal{S}_-^2\nu\right)\Bigg] \\
& + \frac{\bar{\varepsilon}}{(1+\bar{\jmath})^2}(\pi-\arctan\sqrt{\bar{\jmath}})\Bigg[\frac{242}{15}+\frac{16}{3}\bar{\beta}_+ +\frac{35}{6}\bar{\gamma}-8\bar{\beta}_+^2\bar{\gamma}^{-1}+\frac{193}{6}\zeta\mathcal{S}_-^2 \\
& - 8\bar{\beta}_+\zeta\mathcal{S}_-^2+\frac{56}{3}\zeta\bar{\gamma}\mathcal{S}_-^2-\frac{4}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2+16\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2+16\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2-\frac{4}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
& + 32\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ +\delta\left(8\bar{\beta}_-\zeta\mathcal{S}_-^2-\frac{20}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2-\frac{20}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right)-\frac{47}{6}\zeta\mathcal{S}_-^2\nu \\
& + \frac{1}{\bar{\jmath}^2}\bigg(\frac{1942}{15}+\frac{176}{3}\bar{\beta}_+ +\frac{265}{6}\bar{\gamma}-88\bar{\beta}_+^2\bar{\gamma}^{-1}+\frac{953}{6}\zeta\mathcal{S}_-^2-\frac{304}{3}\bar{\beta}_+\zeta\mathcal{S}_-^2+112\zeta\bar{\gamma}\mathcal{S}_-^2 \\
& \qquad -\frac{44}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2+176\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2+176\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2-\frac{44}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+-\frac{247}{6}\zeta\mathcal{S}_-^2\nu \\
& \qquad +352\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ +\delta\left(\frac{304}{3}\bar{\beta}_-\zeta\mathcal{S}_-^2-\frac{220}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2-\frac{220}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right)\bigg) \\
& + \frac{1}{\bar{\jmath}}\bigg(\frac{1334}{15}+\frac{112}{3}\bar{\beta}_+ +\frac{185}{6}\bar{\gamma}-56\bar{\beta}_+^2\bar{\gamma}^{-1}+\frac{383}{3}\zeta\mathcal{S}_-^2-\frac{188}{3}\bar{\beta}_+\zeta\mathcal{S}_-^2+84\zeta\bar{\gamma}\mathcal{S}_-^2 \\
& \qquad -\frac{28}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2+112\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2+112\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2-\frac{28}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+-\frac{97}{3}\zeta\mathcal{S}_-^2\nu \\
& \qquad +224\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ +\delta\left(\frac{188}{3}\bar{\beta}_-\zeta\mathcal{S}_-^2-\frac{140}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2-\frac{140}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right)\bigg) \\
& + \frac{1}{\bar{\jmath}^3}\bigg(\frac{170}{3}+\frac{80}{3}\bar{\beta}_+ +\frac{115}{6}\bar{\gamma}-40\bar{\beta}_+^2\bar{\gamma}^{-1}+\frac{190}{3}\zeta\mathcal{S}_-^2-\frac{140}{3}\bar{\beta}_+\zeta\mathcal{S}_-^2+\frac{140}{3}\zeta\bar{\gamma}\mathcal{S}_-^2 \\
& \qquad -\frac{20}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2+80\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2+80\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2-\frac{20}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+-\frac{50}{3}\zeta\mathcal{S}_-^2\nu \\
& \qquad +160\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ +\delta\left(\frac{140}{3}\bar{\beta}_-\zeta\mathcal{S}_-^2-\frac{100}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2-\frac{100}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right)\bigg) \\
& + \frac{16\zeta\mathcal{S}_-^2(\pi-\arctan\sqrt{\bar{\jmath}})}{\bar{\jmath}^{5/2}}(\bar{\jmath}+1)^2\left(1-\frac{1}{3}\bar{\beta}_+ +\frac{2}{3}\bar{\gamma}+\frac{1}{3}\bar{\beta}_-\delta\right)\Bigg]\Bigg\}.
\end{aligned} \tag{6.53}
$$

To conclude this Section, one can check that the GR limits of $\Delta\bar{e}_t$, Δb and Δv_∞ are in agreement with Eq. (5.110) of [151] and Eq. (49,50) of [216], respectively. Taking the GR limit of the various ST parameters, this expression for the scattering angle is in agreement with Eq. (5.115) of [151]. Although Ref. [205] obtained the 3PN-accurate result for the scattering angle in the *conservative* sector, the result just presented cannot be used to claim that there is now a completion of the full 3PN scattering angle. The first reason is that at 3PN, the hereditary tail appearing in the equations of motion also has a dissipative contribution, which was not computed. The second reason is that 3PN is also the order of radiation-reaction squared, where the linear response formula breaks down. The third reason is that the energy and angular momentum in the linear response formula was only needed at 1PN, so there was no need to include any Schott terms [220, 221]: these enter the energy at 1.5PN in ST theories [204], are nonvanishing for hyperbolic orbits [151] and would be thus expected to contribute to the 3PN scattering angle.

6.5 Parabolic and bremsstrahlung (or PM) limits

6.5.1 Parabolic limit

A quasiparabolic orbit, in the full dissipative problem, is defined as the orbit that is marginally bound, namely whose final energy after the scattering event vanishes ($E^+ = 0$), or equivalently, whose final eccentricity goes to one ($e_t^+ = e_r^+ = e_\phi^+ = 1$). The first goal is to determine the corresponding initial energy E^- in terms of the only degree of freedom: J^-. Since $\Delta E \sim \mathcal{O}(c^{-3})$, one can obtain $E^- = -\Delta E$ by setting $E = 0$ and $J = J^-$ in (6.47) (its expression in terms of J^+ would be identical). Hence, one has $J^+ = J^- + \Delta J$, where ΔJ is obtained using (6.47) by setting again $E = 0$ and $J = J^-$. Then ,in the parabolic limit:

$$\begin{aligned}
\Delta E|_{\text{parabolic}} = -\frac{c^2 m\pi\bar{\varepsilon}^{5/2}\nu^2}{\bar{\jmath}^{5/2}}\bigg\{4\zeta\mathcal{S}_-^2 + \frac{\bar{\varepsilon}}{\bar{\jmath}}\bigg[&\frac{170}{3} + \frac{80}{3}\bar{\beta}_+ + \frac{115}{6}\bar{\gamma} - 40\bar{\beta}_+^2\bar{\gamma}^{-1} + \frac{190}{3}\zeta\mathcal{S}_-^2 - \frac{140}{3}\bar{\beta}_+\zeta\mathcal{S}_-^2 + \frac{140}{3}\zeta\bar{\gamma}\mathcal{S}_-^2 \\
&- \frac{20}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 80\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 80\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
&- \frac{20}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 160\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ - \frac{50}{3}\zeta\mathcal{S}_-^2\nu \\
&+ \delta\left(\frac{140}{3}\bar{\beta}_-\zeta\mathcal{S}_-^2 - \frac{100}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - \frac{100}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right)\bigg]\bigg\},
\end{aligned} \tag{6.54a}$$

$$\begin{aligned}
\Delta J|_{\text{parabolic}} = -\frac{2\alpha\tilde{G}m^2\pi\bar{\varepsilon}\nu^2}{3c\bar{\jmath}}\bigg\{4\zeta\mathcal{S}_-^2 + \frac{\bar{\varepsilon}}{\bar{\jmath}}\bigg[&36 + 15\bar{\gamma} + 30\zeta\mathcal{S}_-^2 - 24\bar{\beta}_+\zeta\mathcal{S}_-^2 + 24\zeta\bar{\gamma}\mathcal{S}_-^2 + 24\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 24\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ \\
&+ \delta\left(24\bar{\beta}_-\zeta\mathcal{S}_-^2 - 24\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - 24\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right) - 12\zeta\mathcal{S}_-^2\nu\bigg]\bigg\}.
\end{aligned} \tag{6.54b}$$

Also these quantities are in full agreement with the parabolic limit ($e_t \to 1^-$) of the radiated energy and angular momentum during one radial period of an elliptic binary given in (4.11) and (4.12) of [203]. Note that in the hyperbolic case, one computes the energy loss, whereas in the elliptic case, it is the (orbit-averaged) energy *flux* which is obtained, so in order to check the two results, one should take into account a factor of the radial period P when comparing the two expressions.

Of particular interest will be the initial eccentricity of of parabolic orbit, which is found under the constraint that $e_t^+\big|_{\text{parabolic}} = 1$, and reads

$$\begin{aligned}
e_t^-\big|_{\text{parabolic}} = 1 + \frac{m\pi\bar{\varepsilon}^{3/2}\nu^2}{\bar{\jmath}^{3/2}}\bigg\{4\zeta\mathcal{S}_-^2 + \frac{\bar{\varepsilon}}{\bar{\jmath}}\bigg[&\frac{170}{3} + \frac{80}{3}\bar{\beta}_+ + \frac{115}{6}\bar{\gamma} - 40\bar{\beta}_+^2\bar{\gamma}^{-1} + \frac{214}{3}\zeta\mathcal{S}_-^2 - \frac{116}{3}\bar{\beta}_+\zeta\mathcal{S}_-^2 + \frac{140}{3}\zeta\bar{\gamma}\mathcal{S}_-^2 \\
&- \frac{20}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 80\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 80\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 - \frac{20}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ - \frac{74}{3}\zeta\mathcal{S}_-^2\nu
\end{aligned}$$

$$+ 160\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ + \delta\left(\frac{116}{3}\bar{\beta}_-\zeta\mathcal{S}_-^2 - \frac{100}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - \frac{100}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right)\Bigg]\Bigg\}. \tag{6.55}$$

In particular, this result can be useful when integrating hereditary effects over bound orbits, see Sec. VI.E.2 of Ref. [204] for more details.

6.5.2 Bremsstrahlung (or PM) limit

The bremsstrahlung limit of an hyperbolic encounter, which corresponds to the large eccentricity expansion or, alternatively, to the limit for $J^- \to \infty$, in GR see [164, 166]. It is worth noticing that this limit corresponds to the PM expansion around straight-line motion, which is precisely the limiting case of a scattering event when the two bodies are far from each other.

In this case, one can directly compute the energy and angular momentum loss, which read in the bremsstrahlung limit

$$\begin{aligned}
\Delta E|_{\text{bremsstrahlung}} = -\frac{c^2 m\pi\bar{\varepsilon}^{5/2}\nu^2}{\bar{\jmath}^{3/2}}\Bigg\{\frac{2}{3}\zeta\mathcal{S}_-^2 + \bar{\varepsilon}\Bigg[&\frac{37}{15} + \frac{8}{3}\bar{\beta}_+ + \frac{7}{12}\bar{\gamma} - 4\bar{\beta}_+^2\bar{\gamma}^{-1} + \frac{13}{6}\zeta\mathcal{S}_-^2 + \frac{2}{3}\zeta\bar{\gamma}\mathcal{S}_-^2 \\
&- \frac{10}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 + 8\bar{\beta}_-^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 + 8\bar{\beta}_+^2\zeta\bar{\gamma}^{-2}\mathcal{S}_-^2 \\
&- \frac{10}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+ + 16\bar{\beta}_-\bar{\beta}_+\zeta\bar{\gamma}^{-2}\mathcal{S}_-\mathcal{S}_+ \\
&+ \delta\left(-\frac{2}{3}\bar{\beta}_-\zeta\bar{\gamma}^{-1}\mathcal{S}_-^2 - \frac{2}{3}\bar{\beta}_+\zeta\bar{\gamma}^{-1}\mathcal{S}_-\mathcal{S}_+\right) - \frac{7}{6}\zeta\mathcal{S}_-^2\nu\Bigg]\Bigg\}, \qquad (6.56a)\\
\Delta J|_{\text{bremsstrahlung}} = -\frac{\alpha\tilde{G}m^2\bar{\varepsilon}\nu^2}{c\bar{\jmath}^{1/2}}\Bigg\{\frac{8}{3}\zeta\mathcal{S}_-^2 + \bar{\varepsilon}&\left(\frac{16}{5} + \frac{4}{3}\bar{\gamma} + \frac{17}{3}\zeta\mathcal{S}_-^2 + \frac{8}{3}\zeta\bar{\gamma}\mathcal{S}_-^2 - \frac{7}{3}\zeta\mathcal{S}_-^2\nu\right)\Bigg\}, \qquad (6.56b)
\end{aligned}$$

where the $\Delta E \sim \mathcal{O}(G^3)$ and $\Delta J \sim \mathcal{O}(G^2)$ in both the scalar and gravitational sector. These scalings are consistent with results from scattering computations in the PM framework, see [56, 168], where the leading contributions to ΔE and ΔJ are identified as 3PM and 2PM effects, respectively.

7 Conclusions

This Thesis has explored analytically a scattering event through the language of Black Hole Perturbation Theory and Self Force. A formalism valid for perturbations induced by a point particle on a generic equatorial orbit was developed and applied to the study of a scalar charge in unbound motion around a Schwarzschild black hole. In parallel, a complementary project focused on the quasi-Keplerian parametrization of scattering within Scalar-Tensor theories at Post Newtonian (PN) accuracy. With the exception of the discussions in Chapter 5, the results presented here are contained in the publications listed in the List of Publications.

The formalism described in Chapter 3 and then applied in Chapter 4, lead to the central achievement of this work: *first systematic calculation of Self Force corrections to the scattering of a scalar charge in Schwarzschild spacetime*. Not only were the losses of energy and angular momentum computed, but also the conservative and dissipative contributions to the scattering angle were explicitly obtained. The analysis revealed novel analytic structures in the time-domain regularized field and force components, enabling, for the first time, a clean analytical matching with Effective Field Theory (EFT) results at 4PM order of the scattering angle. This matching fixed previously undetermined parameters in the EFT framework, providing a new bridge between two powerful approaches. Similar matchings have been already produced in the past, for instance in [222, 223], but only at the level of wave scattering, which do not involve, from the SF side of the computation, the integration of the Green function against the source. In addition, Chapter 5 presented the *first attempt at constructing scattering waveforms from Self Force computations*, with explicit analytical expressions for $h_+(T)$ and $h_\times(T)$ in the quadrupolar sector. Even in their preliminary form, these results highlight the feasibility of extracting waveform information directly from SF methods.

The second result presented in this Thesis is the calculation of the *QK parametrization for a scattering in Scalar Tensor Theories at 2PN*. From the QK parameters it was then possible to compute several quantities: energy and angular momentum fluxes, changes in the orbital parameters, such as the impact parameter and the time-eccentricity e_t and, more importantly, the scattering angle. The dissipative scattering angle, in particular, was computed up to 2.5PN. These results has then also been checked in the parabolic and bremsstrahlung (or Post-Minkowskian) limit. In order to increase the PN order one should be able to deal with the appearance of non-local in time effects.

While the inclusion of the first non-local effects in the dissipative sector is doable by following standard procedure in PN theory, the real challenge is to include the 3PN non-local effects in the parameterization of the conservative motion. This problem has not been solved neither in GR, where the first non-local in time effects appear at 4PN. Even if partial results exist at 4PN, there is still no consensus on how to complete the picture at this order. It is important to stress that this problem is not present only for unbound orbits but it is a general feature of the PN expanded equation of motions for a binary system. Investigating this problem could provide a very interesting step forward, especially for the analytical calculation of observables from bound orbits.

Regarding the scattering in SF, the formalism developed here is not tied to scalar perturbations at it is ready to be extended to the investigation of the gravitational scattering. This will lead in the near future to solving the gravitational Self Force problem for scattering. That will involve the reconstruction of the metric perturbed by the hyperbolic encounter, which will be particularly important to get analytical information on the *conservative* sector of the dynamics. In addition, the local loss of energy and angular momentum from the particle can be computed by integrating the forces components, providing a way to extract the horizon fluxes, i.e. how much energy/angular momentum falls into the horizon instead of going at infinity as gravitational waves.

The computation of the full dynamics is extremely interesting and it will be used to extract novel information on the tidal response of the black hole under a perturbation. This response is usually encoded in the so-called *Love numbers*. These numbers were already encountered in the last Section of Chapter 4 in Eq. (4.73) where they were introduced as the free parameters c_1 and $c_2(\bar{\mu})$ that appears in the conservative scattering angle. These numbers have been fixed to very specific value in several works that connects Effective Field Theory techniques with Black Hole Perturbation Theory results in the context of wave scattering, see for instance [98, 224], to values that are not in agreement with the results presented here, in particular c_1 is always vanishing in these results, while the value obtained comparing Eq. (4.73) with the conservative scattering angle in Self Force is different.

The reason behind this disagreement is still unclear and more investigation are required in order to shed light on this issue. Investigating the full dynamics for gravitational perturbations will be crucial to understand where the issue is.

Equally important is the challenge posed by the non-local sector of the dynamics in hyperbolic motion. *This sector presents an unprecedented challenge for SF calculation*. It is related to two main issues that are characteristic of the hyperbolic motion: the absence of orbital frequencies and the intrinsic complexity of the geodesic motion. The rich analytic structure uncovered in this Thesis - involving inverse hyperbolic functions, logarithms, and polylogarithms - arises from a combination of these two features. These functions must be integrated in order to get a final expression in time domain but also in order to extract observables, such as fluxes and/or the scattering angle. This integration problem is very well understood in the context of multi-loop calculations in high-energy physics that have been applied to scattering up to 5PM. Interestingly, by combining PM and PN expansion in SF, it is possible to get expressions at higher PM order that shows a structure that is not accessible, for now, to QFT based methods. A synergistic approach could be the only way to proceed: produce integrands with SF methods and integrate them via QFT techniques. Such progress would not only deepen our understanding of black hole tidal responses but also provide the analytical input required for accurate resummation of scattering observables.

Finally, although this work has primarily focused on Schwarzschild, the formalism was designed to be valid for perturbations of arbitrary spin-s fields on both Schwarzschild and Kerr backgrounds. Extending the results to Kerr black holes is therefore a natural next step. While computationally demanding, such an extension would yield results of direct astrophysical relevance, advancing our ability to model extreme-mass-ratio encounters and to interpret the rich phenomenology of gravitational-wave signals from hyperbolic interactions.

Appendix A

Coefficients of logarithmic Fourier transform

In this Appendix all the coefficients a_q and B_q relative to the IFT of $\log^k(-i\omega)$ are presented up to $k = 10$. As a remainder, the non-local part of the field is shown in Eq (3.92) as

$$
{}_s\psi^{\rm NL}_{\ell m}(k,j;x^\mu) = \sum_{q=0}^{k-1} \frac{i^j\, v^j}{b} \left\{ a_q \left[{\rm p.v.} \left(\frac{\log^q |\bar t' - \bar t|}{\bar t' - \bar t} \right) - \left(\frac{\log^q |\bar t' - \bar t|}{|\bar t' - \bar t|} \right)_{\bar t} \right] \right\}
$$
$$
+ B\, \partial^{(j)}_{\bar t}\ {}_s\hat{\mathcal{G}}^{\rm NL}_{\ell m;jk}(\bar r, \bar r_p(\bar t))\,, \tag{A.1}
$$

which points out that for any k, there will be $k + 1$ coefficients that needs to be determined.

$k = 1$

$$
a_0 = 1\,, \quad B = -\gamma_E\,. \tag{A.2}
$$

$k = 2$

$$
a_1 = 1\,, \quad a_0 = -\gamma_E\,, \quad B = \gamma_E^2 - \frac{\pi^2}{6}\,. \tag{A.3}
$$

$k = 3$

$$
a_0 = \frac{3\gamma_E^2}{2} - \frac{\pi^2}{4}\,, \quad a_1 = 3\gamma_E\,, \quad a_2 = \frac{3}{2}\,, \quad B = -\gamma_E^3 + \frac{\gamma_E \pi^2}{2} - 2\zeta(3)\,. \tag{A.4}
$$

$k = 4$

$$
\begin{aligned}
a_0 &= -2\gamma_E^3 + \gamma_E \pi^2 - 4\zeta(3)\,, \\
a_1 &= -6\gamma_E^2 + \pi^2\,, \quad a_2 = -6\gamma_E\,, \quad a_3 = -2\,, \\
B &= \gamma_E^4 - \gamma_E^2\pi^2 + \frac{\pi^4}{60} + 8\gamma_E\zeta(3)\,.
\end{aligned} \tag{A.5}
$$

$k = 5$

$$
\begin{aligned}
a_0 &= \frac{5\gamma_E^4}{2} - \frac{5\gamma_E^2\pi^2}{2} + \frac{\pi^4}{24} + 20\gamma_E\zeta(3)\,, \\
a_1 &= 10\gamma_E^3 - 5\gamma_E\pi^2 + 20\zeta(3)\,, \quad a_2 = 15\gamma_E^2 - \frac{5\pi^2}{2}\,, \\
a_3 &= 10\gamma_E\,, \quad a_4 = \frac{5}{2}\,, \\
B &= \gamma_E^5 - \frac{5\gamma_E^3\pi^2}{3} + \frac{\gamma_E\pi^4}{12} + 20\gamma_E^2\zeta(3) - \frac{10\pi^2\zeta(3)}{3} + 24\zeta(5)\,.
\end{aligned} \tag{A.6}
$$

$k = 6$

$$
\begin{aligned}
a_0 &= -3\gamma_E^5 + 5\gamma_E^3\pi^2 - \frac{\gamma_E\pi^4}{4} - 60\gamma_E^2\zeta(3) + 10\pi^2\zeta(3) - 72\zeta(5)\,, \\
a_1 &= -15\gamma_E^4 + 15\gamma_E^2\pi^2 - \frac{11\pi^4}{8} - 120\gamma_E\zeta(3)\,, \quad a_2 = -30\gamma_E^3 + 15\gamma_E\pi^2 - 60\zeta(3)\,, \\
a_3 &= -30\gamma_E^2 + 5\pi^2\,, \quad a_4 = -15\gamma_E\,, \quad a_5 = -3\,, \\
B &= \gamma_E^6 - \frac{5\gamma_E^4\pi^2}{2} + \frac{\gamma_E^2\pi^4}{4} - \frac{5\pi^6}{168} + 40\gamma_E^3\zeta(3) - 20\gamma_E\pi^2\zeta(3) + 40\zeta(3)^2 + 144\gamma_E\zeta(5)\,.
\end{aligned} \tag{A.7}
$$

$k = 7$

$$
\begin{aligned}
a_0 &= \frac{7\gamma_E^6}{2} - \frac{35\gamma_E^4\pi^2}{4} + \frac{7\gamma_E^2\pi^4}{8} - \frac{5\pi^6}{48} + 140\gamma_E^3\zeta(3) - 70\gamma_E\pi^2\zeta(3) + 140\zeta(3)^2 + 504\gamma_E\zeta(5)\,, \\
a_1 &= 21\gamma_E^5 - 35\gamma_E^3\pi^2 + \frac{7\gamma_E\pi^4}{4} + 420\gamma_E^2\zeta(3) - 70\pi^2\zeta(3) + 504\zeta(5)\,, \\
a_2 &= \frac{105\gamma_E^4}{2} - \frac{105\gamma_E^2\pi^2}{2} + \frac{7\pi^4}{8} + 420\gamma_E\zeta(3)\,, \quad a_3 = 70\gamma_E^3 - 35\gamma_E\pi^2 + 140\zeta(3)\,, \\
a_4 &= \frac{105\gamma_E^2}{2} - \frac{35\pi^2}{4}\,, \quad a_5 = 21\gamma_E\,, \quad a_6 = \frac{7}{2}\,, \\
B &= -\gamma_E^7 + \frac{7\gamma_E^5\pi^2}{2} - \frac{7\gamma_E^3\pi^4}{12} + \frac{5\gamma_E\pi^6}{24} - 70\gamma_E^4\zeta(3) + 70\gamma_E^2\pi^2\zeta(3) - \frac{7\pi^4\zeta(3)}{6} - 280\gamma_E\zeta(3)^2 \\
&\quad - 504\gamma_E^2\zeta(5) + 84\pi^2\zeta(5) - 720\zeta(7)\,.
\end{aligned} \tag{A.8}
$$

$k = 8$

$$
\begin{aligned}
a_0 &= -4\gamma_E^7 + 14\gamma_E^5\pi^2 - \frac{7\gamma_E^3\pi^4}{3} + \frac{5\gamma_E\pi^6}{6} - \frac{14}{3}\left(60\gamma_E^4 - 60\gamma_E^2\pi^2 + \pi^4\right)\zeta(3) \\
&\quad - 1120\gamma_E\zeta(3)^2 + 336\left(-6\gamma_E^2 + \pi^2\right)\zeta(5) - 2880\zeta(7)\,, \\
a_1 &= -28\gamma_E^6 + 70\gamma_E^4\pi^2 - 7\gamma_E^2\pi^4 + \frac{5\pi^6}{6} - 560\gamma_E\left(2\gamma_E^2 - \pi^2\right)\zeta(3) - 1120\zeta(3)^2 \\
&\quad - 4032\gamma_E\zeta(5)\,, \\
a_2 &= -7\left(12\gamma_E^5 - 20\gamma_E^3\pi^2 + \gamma_E\pi^4 + 40\left(6\gamma_E^2 - \pi^2\right)\zeta(3) + 288\zeta(5)\right)\,, \\
a_3 &= -\frac{7}{3}\left(60\gamma_E^4 - 60\gamma_E^2\pi^2 + \pi^4 + 480\gamma_E\zeta(3)\right)\,, \quad a_4 = -140\gamma_E^3 + 70\gamma_E\pi^2 - 280\zeta(3)\,, \\
a_5 &= -84\gamma_E^2 + 14\pi^2\,, \quad a_6 = -28\gamma_E\,, \quad a_7 = -4\,, \\
B &= \gamma_E^8 - \frac{14\gamma_E^6\pi^2}{3} + \frac{7\gamma_E^4\pi^4}{6} - \frac{5\gamma_E^2\pi^6}{6} - \frac{67\pi^8}{720} + \frac{28}{3}\gamma_E\left(12\gamma_E^4 - 20\gamma_E^2\pi^2 + \pi^4\right)\zeta(3) \\
&\quad + \frac{560}{3}\left(6\gamma_E^2 - \pi^2\right)\zeta(3)^2 + 672\left(2\gamma_E^3 - \gamma_E\pi^2\right)\zeta(5) + 2688\zeta(3)\zeta(5) + 5760\gamma_E\zeta(7)\,. \qquad \text{(A.9)}
\end{aligned}
$$

$k = 9$

$$
\begin{aligned}
a_0 &= \frac{9\gamma_E^8}{2} - 21\gamma_E^6\pi^2 + \frac{21\gamma_E^4\pi^4}{4} - \frac{15\gamma_E^2\pi^6}{4} - \frac{67\pi^8}{160} + 42\gamma_E\left(12\gamma_E^4 - 20\gamma_E^2\pi^2 + \pi^4\right)\zeta(3) \\
&\quad + 840\left(6\gamma_E^2 - \pi^2\right)\zeta(3)^2 + 3024\left(2\gamma_E^3 - \gamma_E\pi^2\right)\zeta(5) + 12096\zeta(3)\zeta(5) + 25920\gamma_E\zeta(7) \\
a_1 &= 36\gamma_E^7 - 126\gamma_E^5\pi^2 + 21\gamma_E^3\pi^4 - \frac{15\gamma_E\pi^6}{2} + 2520\gamma_E^4\zeta(3) - 2520\gamma_E^2\pi^2\zeta(3) + 42\pi^4\zeta(3) \\
&\quad + 10080\gamma_E\zeta(3)^2 + 18144\gamma_E^2\zeta(5) - 3024\pi^2\zeta(5) + 25920\zeta(7)\,, \\
a_2 &= 126\gamma_E^6 - 315\gamma_E^4\pi^2 + \frac{63\gamma_E^2\pi^4}{2} - \frac{15\pi^6}{4} + 5040\gamma_E^3\zeta(3) - 2520\gamma_E\pi^2\zeta(3) + 5040\zeta(3)^2 \\
&\quad + 18144\gamma_E\zeta(5)\,, \\
a_3 &= 252\gamma_E^5 - 420\gamma_E^3\pi^2 + 21\gamma_E\pi^4 + 5040\gamma_E^2\zeta(3) - 840\pi^2\zeta(3) + 6048\zeta(5)\,, \\
a_4 &= 315\gamma_E^2(\gamma_E^2 - \pi^2) + \frac{21\pi^4}{4} + 2520\gamma_E\zeta(3)\,, \quad a_5 = 252\gamma_E^3 - 126\gamma_E\pi^2 + 504\zeta(3)\,, \\
a_6 &= 126\gamma_E^2 - 21\pi^2\,, \quad a_7 = 36\gamma_E\,, \quad a_8 = \frac{9}{2}\,, \\
B &= -\gamma_E^9 + 6\gamma_E^7\pi^2 - \frac{21\gamma_E^5\pi^4}{10} + \frac{5\gamma_E^3\pi^6}{2} + \frac{67\gamma_E\pi^8}{80} \\
&\quad - \left(168\gamma_E^6 - 420\gamma_E^4\pi^2 + 42\gamma_E^2\pi^4 - 5\pi^6\right)\zeta(3) - 1680\gamma_E\left(2\gamma_E^2 - \pi^2\right)\zeta(3)^2 - 2240\zeta(3)^3 \\
&\quad - \frac{252}{5}\left(60\gamma_E^4 - 60\gamma_E^2\pi^2 + \pi^4\right)\zeta(5) - 24192\gamma_E\zeta(3)\zeta(5) - 4320\left(6\gamma_E^2 - \pi^2\right)\zeta(7) \\
&\quad - 40320\zeta(9)\,. \qquad \text{(A.10)}
\end{aligned}
$$

$k = 10$

$$
\begin{aligned}
a_0 &= -5\gamma_E^9 + 30\gamma^7\pi^2 - \frac{21\gamma_E^5\pi^4}{2} + \frac{25\gamma_E^3\pi^6}{2} + \frac{67\gamma_E\pi^8}{16} - 8400\left(2\gamma_E^3 - \gamma_E\pi^2\right)\zeta(3)^2 \\
&\quad - 11200\zeta(3)^3 - 5\left(168\gamma_E^6 - 420\gamma_E^4\pi^2 + 42\gamma_E^2\pi^4 - 5\pi^6\right)\zeta(3) - 120960\gamma_E\zeta(3)\zeta(5) \\
&\quad - 252\left(60\gamma_E^4 - 60\gamma_E^2\pi^2 + \pi^4\right)\zeta(5) - 21600\left(6\gamma_E^2 - \pi^2\right)\zeta(7) - 201600\zeta(9) \\
a_1 &= -45\gamma_E^8 + 210\gamma_E^6\pi^2 - \frac{105\gamma_E^4\pi^4}{2} + \frac{75\gamma_E^2\pi^6}{2} + \frac{67\pi^8}{16} - 420\gamma_E\left(12\gamma_E^4 - 20\gamma_E^2\pi^2 + \pi^4\right)\zeta(3) \\
&\quad - 8400\left(6\gamma_E^2 - \pi^2\right)\zeta(3)^2 - 30240\left(2\gamma_E^3 - \gamma_E\pi^2\right)\zeta(5) - 120960\zeta(3)\zeta(5) - 259200\gamma_E\zeta(7)\,, \\
a_2 &= -180\gamma_E^7 + 630\gamma_E^5\pi^2 - 105\gamma_E^3\pi^4 + \frac{75\gamma_E\pi^6}{2} - 12600\gamma_E^4\zeta(3) + 12600\gamma_E^2\pi^2\zeta(3) \\
&\quad - 210\pi^4\zeta(3) - 50400\gamma_E\zeta(3)^2 - 90720\gamma_E^2\zeta(5) + 15120\pi^2\zeta(5) - 129600\zeta(7)\,, \\
a_3 &= -420\gamma_E^6 + 1050\gamma_E^4\pi^2 - 105\gamma_E^2\pi^4 + \frac{25\pi^6}{2} - 16800\gamma_E^3\zeta(3) + 8400\gamma_E\pi^2\zeta(3) \\
&\quad - 16800\zeta(3)^2 - 60480\gamma_E\zeta(5)\,, \\
a_4 &= -630\gamma_E^5 + 1050\gamma_E^3\pi^2 - \frac{105\gamma_E\pi^4}{2} - 12600\gamma_E^2\zeta(3) + 2100\pi^2\zeta(3) - 15120\zeta(5)\,, \\
a_5 &= -630\gamma_E^4 + 630\gamma_E^2\pi^2 - \frac{21\pi^4}{2} - 5040\gamma_E\zeta(3)\,, \\
a_6 &= -420\gamma_E^3 + 210\gamma_E\pi^2 - 840\zeta(3)\,, \quad a_7 = -180\gamma_E^2 + 30\pi^2\,, \quad a_8 = -45\gamma_E\,, \quad a_9 = -5 \\
B &= \gamma_E^{10} - \frac{15\gamma_E^8\pi^2}{2} + \frac{7\gamma_E^6\pi^4}{2} - \frac{25\gamma_E^4\pi^6}{4} - \frac{67\gamma_E^2\pi^8}{16} - \frac{225\pi^{10}}{352} \\
&\quad + 10\left(24\gamma_E^7 - 84\gamma_E^5\pi^2 + 14\gamma_E^3\pi^4 - 5\gamma_E\pi^6\right)\zeta(3) + 140\left(60\gamma_E^4 - 60\gamma_E^2\pi^2 + \pi^4\right)\zeta(3)^2 \\
&\quad + 22400\gamma_E\zeta(3)^3 + 504\gamma_E\left(12\gamma_E^4 - 20\gamma_E^2\pi^2 + \pi^4\right)\zeta(5) + 120960\gamma_E^2\zeta(3)\zeta(5) - 20160\pi^2\zeta(3)\zeta(5) \\
&\quad + 72576\zeta(5)^2 + 43200\left(2\gamma_E^3 - \gamma_E\pi^2\right)\zeta(7) + 172800\zeta(3)\zeta(7) + 403200\gamma_E\zeta(9)\,. \qquad \text{(A.11)}
\end{aligned}
$$

Appendix B
Alternative derivation for non-local integrals

In this Appendix an alternative procedure for computing the integrals from Eq. (3.65) is presented. This procedure can be iterated for any value of j and k, here only $j = 0$ and $k = 1$ will be discussed for simplicity.

The integral that needs to be computed can be schematized as follows

$$2\pi I^{\mathrm{NL},\epsilon}(t) = \int_{-\infty}^{+\infty} dt'\,\phi(t') \int_{-\infty}^{+\infty} d\omega\, e^{-\epsilon|\omega|} \log(-i\omega) e^{i\omega(t'-t)}\,, \tag{B.1}$$

with $\omega \in \mathbb{R}$ and where the barred notation on the rescaled variables has been removed to simplify the notation. The function $\phi(t')$ is used as a placeholder for all the terms that does not contain the Fourier frequency.

The ω integral can be straighforwardly computed as

$$\begin{aligned}
\mathcal{F}^{-1}\left\{\log(-i\omega)\right\}(s) = \lim_{\epsilon\to 0^+} \mathcal{F}^{-1}&\left\{e^{-\epsilon|\omega|}\log(-i\omega)\right\}(s) = \\
&= \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} d\omega\, e^{-\epsilon|\omega|} e^{is\omega} \log(-i\omega) \\
&= \lim_{\epsilon\to 0^+} \left\{ \frac{\pi s}{s^2+\epsilon^2} - \frac{2\gamma_E \epsilon}{s^2+\epsilon^2} - \frac{2s\arctan s/\epsilon}{s^2+\epsilon^2} - \frac{\epsilon\log\left(s^2+\epsilon^2\right)}{s^2+\epsilon^2} \right\} \\
&= \lim_{\epsilon\to 0^+} G_\epsilon^{(1)}(s) + G_\epsilon^{(2)}(s) + G_\epsilon^{(3)}(s) + G_\epsilon^{(4)}(s),
\end{aligned} \tag{B.2}$$

where $s = t' - t$ and the notation $\mathcal{F}^{-1}\{\cdot\}$ is used to identify the Inverse Fourier Transform. The functions $G_\epsilon^{(i)}(s)$ are respectively

$$G_\epsilon^{(1)}(s) = \frac{\pi s}{s^2+\epsilon^2},\; G_\epsilon^{(2)}(s) = -\frac{2\gamma_E\epsilon}{s^2+\epsilon^2},\; G_\epsilon^{(3)}(s) = -\frac{2s\arctan s/\epsilon}{s^2+\epsilon^2},\; G_\epsilon^{(4)}(s) = -\frac{\epsilon\log\left(s^2+\epsilon^2\right)}{s^2+\epsilon^2}\,,$$

where γ_E is the Euler constant. The action of the various $G_\epsilon^{(i)}$ against the test function $\phi(t')$ will be studied, in order to determine their distributional behavior as

$$\lim_{\epsilon\to 0^+} \left\langle G_\epsilon^{(i)}, \phi \right\rangle(t) = \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} ds\,\phi(s+t)\, G_\epsilon^{(i)}(s)\,. \tag{B.3}$$

$G_\epsilon^{(1)}(s)$

The first term that needs to be computed is

$$\lim_{\epsilon\to 0^+}\left\langle G_\epsilon^{(1)},\phi\right\rangle(t)=\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty}ds\frac{\pi s}{s^2+\epsilon^2}\phi(s+t)=\pi\int_{-\infty}^{+\infty}ds\frac{\phi(s+t)}{s}$$
$$=\pi\int_0^{+\infty}ds\frac{\phi(s+t)-\phi(-s+t)}{s}=\pi\,\mathrm{p.v.}\left(\frac{1}{\mathrm{s}}\right)\,, \tag{B.4}$$

which is the standard definition for the principal value of an integral.

$G_\epsilon^{(2)}(s)$

The second term has also a simple distributional interpretation

$$\lim_{\epsilon\to 0^+}\left\langle G_\epsilon^{(2)},\phi\right\rangle(t)=-2\gamma_E\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty}ds\frac{\epsilon}{s^2+\epsilon^2}\phi(s+t)$$
$$=-2\gamma_E\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty}dx\frac{1}{1+x^2}\phi(\epsilon x+t)=-2\pi\gamma_E\phi(t)\,, \tag{B.5}$$

hence the same integral can be written in terms of the Dirac delta as

$$\left\langle G^{(2)},\phi\right\rangle(t)=-2\pi\gamma_E\int_{-\infty}^{+\infty}ds\,\phi(s+t)\delta(s)\,. \tag{B.6}$$

$G_\epsilon^{(3)}(s)$

The third coefficient is the most subtle and it must be computed carefully

$$\lim_{\epsilon\to 0^+}\left\langle G_\epsilon^{(3)},\phi\right\rangle(t)=-2\lim_{\epsilon\to 0^+}\int_{-\infty}^{+\infty}ds\frac{s\arctan s/\epsilon}{s^2+\epsilon^2}\phi(s+t)$$
$$=-2\lim_{\epsilon\to 0^+}\left\{\int_{|s|\le 1}ds\frac{s\arctan s/\epsilon}{s^2+\epsilon^2}\phi(s+t)+\int_{|s|\ge 1}ds\frac{s\arctan s/\epsilon}{s^2+\epsilon^2}\phi(s+t)\right\}$$
$$=-2\lim_{\epsilon\to 0^+}\left\{\int_{|s|\le 1}ds\frac{s\arctan s/\epsilon}{s^2+\epsilon^2}\phi(t)+\int_{|s|\le 1}ds\frac{s\arctan s/\epsilon}{s^2+\epsilon^2}\left(\phi(s+t)-\phi(t)\right)+\right.$$
$$\left.+\int_{|s|\ge 1}ds\frac{s\arctan s/\epsilon}{s^2+\epsilon^2}\phi(s+t)\right\}$$
$$=-2\lim_{\epsilon\to 0^+}\left\{\phi(t)\int_{|s|\le 1}ds\frac{s\arctan s/\epsilon}{s^2+\epsilon^2}+\frac{\pi}{2}\int_{|s|\le 1}ds\frac{\phi(s+t)-\phi(t)}{|s|}\right.$$
$$\left.+\frac{\pi}{2}\int_{|s|\ge 1}ds\frac{\phi(s+t)}{|s|}\right\}$$
$$=(2\pi\log\epsilon+2\pi\log 2)\phi(t)+o(\epsilon)$$

$$- \pi \left\{ \int_{|s|\leq 1} ds \frac{\phi(s+t) - \phi(t)}{|s|} + \int_{|s|\geq 1} ds \frac{\phi(s+t)}{|s|} \right\} , \tag{B.7}$$

where the procedure must be revised carefully. As a first step, one should divide the integration domain between $|s| \leq 1$ and $|s| \geq 1$: the first one contains a divergence in $s = 0$, while the second is finite. In order to isolate the divergence, in the third line has been added the a term explicitly divergent that produces the $\log \epsilon$ term that it is contained in the last lines.

Similarly to what was written for $G_\epsilon^{(2)}$, the final expression can be written as

$$\left\langle G^{(3)}, \phi \right\rangle (t) = (2\pi \log \epsilon + 2\pi \log 2) \int_{-\infty}^{+\infty} ds\, \delta(s)\, \phi(s+t) - \pi \left(\frac{1}{|s|} \right)_1 , \tag{B.8}$$

where another distribution has been defined as

$$\left(\frac{1}{|s|} \right)_1 = \int_{|s|\leq 1} ds \frac{\phi(s+t) - \phi(t)}{|s|} + \int_{|s|\geq 1} ds \frac{\phi(s+t)}{|s|} . \tag{B.9}$$

$G_\epsilon^{(4)}(s)$

The fourth coefficient leads to

$$\begin{aligned}
\lim_{\epsilon\to 0^+} \left\langle G_\epsilon^{(4)}, \phi \right\rangle (t) &= - \lim_{\epsilon\to 0^+} \int_{-\infty}^{+\infty} ds \frac{\epsilon \log\left(s^2+\epsilon^2\right)}{s^2+\epsilon^2} \phi(s+t) = \\
&= \lim_{\epsilon\to 0^+} \left\{ -2\log\epsilon \int_{-\infty}^{\infty} dx \frac{\phi(\epsilon x + t)}{1+x^2} - \int_{-\infty}^{\infty} dx \frac{\log\left(1+x^2\right)}{1+x^2} \phi(\epsilon x + t) \right\} = \\
&= -(2\pi\log\epsilon + 2\pi \log 2)\phi(t) + o(\epsilon),
\end{aligned} \tag{B.10}$$

which has again a simple interpretation

$$\left\langle G^{(4)}, \phi \right\rangle (t) = -(2\pi \log \epsilon + 2\pi \log 2) \int_{-\infty}^{+\infty} ds\, \delta(s)\, \phi(s+t) . \tag{B.11}$$

Final result

In conclusion one write

$$\begin{aligned}
2\pi I^{\text{NL}}(t) &= \int_{-\infty}^{+\infty} dt' \phi(t')\, \mathcal{F}^{-1} \left\{ \log(-i\omega) \right\} (t'-t) \\
&= \int_{-\infty}^{+\infty} dt' \left(-2\pi\gamma_E\right) \delta(t'-t) \phi(t') + \pi \left[\text{p.v.} \left(\frac{1}{t'-t} \right) - \left(\frac{1}{|t'-t|} \right)_1 \right] ,
\end{aligned} \tag{B.12}$$

which can be immediately rewritten as

$$2\pi I^{\text{NL}}(t) = -2\pi\gamma_E \phi(t) + 2\pi \left(\int_{-\infty}^{t-\alpha} dt' \frac{\phi(t')}{t'-t} - \phi(t) \log \alpha \right) , \tag{B.13}$$

which agrees with what was derived in Chapter 3.

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