Scuola Superiore Meridionale

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PhD Thesis

Symmetrization, comparison results and their applications in shape optimization problems

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Introduction

Some historical notes

> The Kingdom you see is Carthage, the Tyrians, the town of Agenor;
> but the country around is Libya, no folk to meet in war.
> Dido, who left the city of Tyre to escape her brother,
> rules here–a long and labyrinthine tale of wrong
> is hers, but I will touch on its salient points in order... Dido, in great disquiet, organised her friends for escape.
> They met together, all those who harshly hated the tyrant
> or keenly feared him: they seized some ships which chanced to be ready...
> They came to this spot, where to-day you can behold the mighty
> battlements and the rising citadel of New Carthage,
> and purchased a site, which was named 'Bull's Hide' after the bargain
> by which they should get as much land as they could enclose with a bull's hide.

Virgilio, *Aeneid*, Book I, lines 307-372.
Translated by C. D. Lewis.

In this paragraph of Virgil's *Aeneid*, Queen Dido arrives on the coast of Africa and receives a gift from a local king: she can possess as much land as she can enclose with the hide of a bull. Virgil does not describe in detail what happens next, but Dido's strategy is well known: she cut the bull's hide into thin strips that she glues together to form a rope so that she can use it to mark the boundary of a piece of land. After this smart decision, Queen Dido must decide on the shape of the piece of land to enclose. By a mathematical point of view, this is clearly an optimization problem: we want to maximize the area enclosed by a curve having fixed perimeter.

Queen Dido's choice is famous: she opted for a semicircle with the border lying on the sea coast (see Figure 1). The choice of the boundary on the sea cost is due to the fact that Dido wanted the city to have a harbor, while the circular shape is not chosen by chance: it was known by the ancient time (already to the Greeks), but not rigorously proven, that the shape that encloses the greatest area with fixed the perimeter is the circle.

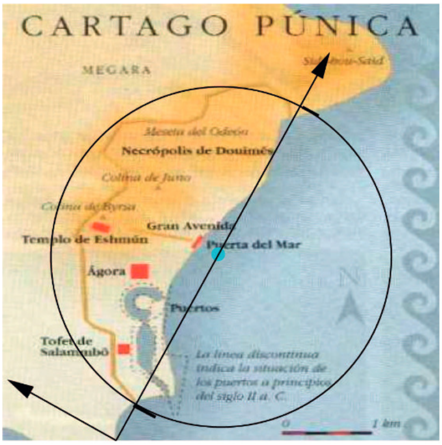

Figure 1: Ancient Carthage: notice the similarity between the boundary of the city and the semicirle. Image from [Pes12], courtesy of H.J. Pesch.

This is not the only case in which this problem, known as *isoperimetric problem*, appears in the literature. Tito Livio in *Ab Urbe Condita Libri*, tells the story of the Romanian soldier Horatius Cocles: during the war between Rome and the Etruscans, Horatius Cocles performed a heroic act by saving the city. To thanks him, the population build a statue for him and he was provided as much of the public land as he himself could plow around in one day with a yoke of oxen. In this case, there is no evidence on Horace Cocle's choice, but again we can recognise the same isoperimetric problem as Dido: which is the largest land among those having fixed perimeter?

Incidentally let us mention that Dido's problem inspires also the Danish author Saxo Grammaticus, who lived in the 12th century and tells the story of Iwar Lodbrok, son of the legendary danish king Ragnar Lodbrok. After the death of Ragnar, Iwar asked for a compensionation for the father's death and the king Ælla of Northumbria gives to Iwar a piece of land which can be demarcated by a bull's hide. Iwar adopts the same strategy as Dido, cutting the hide into thin strips, gluing them together and forming a rope, which he then uses to demarcate a circular piece of land.

A Shape Optimization problem

Clearly the three previous examples are related to the same question: *which is the planar domain that maximizes the area among sets of fixed perimeter?* or in formulas

$$\max_{\Omega\in\mathcal{A}} |\Omega| \qquad \text{where} \qquad \mathcal{A} = \{\Omega \subset \mathbb{R}^2 \colon \Omega \text{ measurable, } \mathrm{Per}(\Omega) = L\}, \tag{1}$$

for a certain $L > 0$, where $|\cdot|$ and $\mathrm{Per}(\cdot)$ denote the measure and the perimeter of the set, respectively.

Despite being known (intuitively) by Greeks, the proof of the optimality of the circle comes only in 19th century due to the works of Steiner. Let us underline that Steiner shows that the only possible maximizer of (1) is the circle without proving that problem (1) admits a maximizer, which was proven only in 1901 by Hurwitz in [Hur01].

Clearly problem (1) has a natural generalization to $n \geq 3$:

$$\max_{\Omega\in\mathcal{A}} |\Omega| \qquad \text{where} \qquad \mathcal{A} = \{\Omega \subset \mathbb{R}^n \colon \Omega \text{ measurable, } \mathrm{Per}(\Omega) = L\}, \tag{2}$$

or equivalently

$$\max_{\Omega\in\mathcal{A}} \frac{|\Omega|^{\frac{n-1}{n}}}{\mathrm{Per}(\Omega)}. \tag{3}$$

Let us underline that (3) is just the scaling invariant version of problem (2), indeed we can notice that

$$\frac{|t\Omega|^{\frac{n-1}{n}}}{\mathrm{Per}(t\Omega)} = \frac{t^{\frac{n-1}{n}}|\Omega|^{\frac{n-1}{n}}}{t^{\frac{n-1}{n}}\mathrm{Per}(\Omega)} = \frac{|\Omega|^{\frac{n-1}{n}}}{\mathrm{Per}(\Omega)} \qquad \forall t > 0.$$

In the 50s De Giorgi introduce a notion of perimeter and thanks to this, he was able to solve problem (3) proving that the optimum is the n-dimensional ball, so it holds

$$\frac{|\Omega|^{\frac{n-1}{n}}}{\mathrm{Per}(\Omega)} \leq \frac{1}{(n\omega_n)^{\frac{1}{n}}} \iff \mathrm{Per}(\Omega) \geq (n\omega_n)^{\frac{1}{n}}|\Omega|^{\frac{n-1}{n}} \qquad \forall \Omega \subseteq \mathbb{R}^n,\ \Omega \text{ measurable.} \tag{4}$$

Equation (4) is known in literature as *isoperimetric inequality.*

The problem (2) (or equivalently (3)) is an optimization problem in which the functional to maximize is defined on a class of subsets of $\mathbb{R}^n$. In the following decades, the interest in these type of problems grows up: find the shape to give to an object in order to reach the most resistant, the most efficient, the lightest or the cheapest is a interesting and challenging problem in a variety of physical and engineering situations. These applied motivations, combined with the theoretical interest after De Giorgi's work, leads to the birth of a new mathematical topic, the *Shape Optimization*.

A Shape Optimization problem can be written as follows: let $\mathcal{A}$ be a class of subsets of $\mathbb{R}^n$ and let $\mathcal{F}\colon \mathcal{A} \to \mathbb{R}$ be a functional, then we are interested in the following minimization (maximization) problem

$$\min_{\Omega\in\mathcal{A}} \mathcal{F}(\Omega) \qquad \left(\max_{\Omega\in\mathcal{A}} \mathcal{F}(\Omega)\right). \tag{5}$$

A set Ω in the class $\mathcal{A}$ is called *admissible set* or *admissible shape* and F is usually referred to *shape functional*.

By a mathematical point of view, in the study of a problem as (5) some questions naturally arise.

- Proving the existence of a solution: the existence of optimizer depends on the topological properties of the class $\mathcal{A}$ and on the continuity property of the functional $\mathcal{F}$ with respect to the topology of $\mathcal{A}$.
- Writing necessary and sufficient condition that the optimizer satisfies, using the so-called *shape derivative* of the shape functional $\mathcal{F}$.
- Finding geometric properties that the optimizer verifies: connectedness, convexity, symmetry, regularity (or singularity).
- Compute the optimal shape.

Despite the fact that (3) is canonically referred as isoperimetric inequality, nowadays, with some abuse of nomenclature, any inequality that expresses the optimality of a certain functional under some geometric constraint, e.g.

$$\mathcal{F}(\Omega) \geq \mathcal{F}(\Omega^*) \qquad \forall \Omega \in \mathcal{A}, \tag{6}$$

where Ω^* is the optimal set, is called isoperimetric inequality.

When an inequality like (6) holds, one can ask the following questions.

1) If Ω is "close" in some sense to Ω^*, can we say that $\mathcal{F}(\Omega)$ is close to $\mathcal{F}(\Omega^*)$?

2) Viceversa, if $\mathcal{F}(\Omega)$ is close to $\mathcal{F}(\Omega^*)$, can we say that Ω is "close" in some sense to Ω^*?

First of all, to properly state the 1 and 2 we need a function $\mathcal{G}\colon \mathcal{A}\times\mathcal{A} \to \mathbb{R}_+$ which "measures the distance" between two sets, i.e. that satisfies

$$\mathcal{G}(\Omega,\Omega) = 0 \qquad \text{and} \qquad \mathcal{G}(\Omega_1,\Omega_2) = \mathcal{G}(\Omega_2,\Omega_1) \qquad \forall \Omega,\Omega_1,\Omega_2 \in \mathcal{A}.$$

Question 1 can be mathematically rephrased as it follows.

Denoting with Ω^* the optimizer for a certain functional $\mathcal{F}$ in the class $\mathcal{A}$, we want to prove that for any $\Omega \in \mathcal{A}$ it holds

$$\mathcal{F}(\Omega) - \mathcal{F}(\Omega^*) < \mathcal{G}(\Omega,\Omega^*). \tag{7}$$

Dually, question 2 consists in showing that for any $\Omega \in \mathcal{A}$ it holds

$$\mathcal{F}(\Omega) - \mathcal{F}(\Omega^*) > \mathcal{G}(\Omega,\Omega^*). \tag{8}$$

Some examples

Let us present other examples of Shape Optimization problems.

In 1894, in [Ray45] Lord Rayleigh studied the vibration of certain elastic bodies. In particular he conjectured that, among all membranes fixed at the boundary of given area, the circular one has the lowest fundamental frequency of vibration. By an engineerical point of view, this conjecture ensures that, in the building of a drum, the shape with minimal surface (i.e. minimal cost of production) to achieve a certain frequency is the circular one.

By a mathematical point of view, the fundamental frequency of a membrane fixed at the boundary is the smallest $\lambda > 0$ such that the following problem admits a non zero solution in $H^1_0(\Omega)$:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where Δ denotes the Laplacian operator, $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$. Such a λ is called *first Dirichlet eigenvalue* and it is denoted as $\lambda_D(\Omega)$.

The conjecture was proven almost 30 years later indipendently by Faber, [Fab23], and Krahn, [Kra25], by means of symmetrization techniques. In a mathematical framework, it can be stated in a scaling invariant way as

$$\lambda_D(\Omega)|\Omega|^{\frac{2}{n}} \geq \lambda_D(B)|B|^{\frac{2}{n}}, \qquad \forall \Omega \subset \mathbb{R}^n,\ \Omega \text{ open},\ |\Omega| < +\infty, \tag{10}$$

where B is a ball. Moreover, equality in (10) holds if and only if Ω is equivalent to a ball.

Another relevant example of shape optimization problem is the one related to the de Saint Venant principle. Let us consider a cylindrical bar of a certain material and let us call Ω its cross section, if we apply a torque at the ends of the bar, the bar will offer a certain resistance to the torque depending on the shape of Ω. In elasticity theory, such a property is called *torsional rigidity*.

Mathematically, the deformation of a bar due to the application of a torque can be studied through the so-called *warping function*: when the cylindrical bar is twisted by couples applied at the ends, all the cross-sections of the bar (equal to Ω) are subject to a warping that is the same for each cross-section. It can be proven (see [TG51, §90]) that denoting with u the warping function, up to a multiplicative constant, it satisfies

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

and the Torsional Rigidity is given by

$$T(\Omega) = \int_\Omega u \, dx. \tag{12}$$

With the same spirit as before, one may wonder which one is the shape having the maximal torsional rigidity among sets having prescribed measure, i.e. to maximize the functional (12) in the class of sets having fixed measure. In 1856 de Saint-Venant ([Sai55]) conjectured that the optimal shape in order to maximize the torsional rigidity is the circle. This conjecture took almost one century to be solved by Polya in [Pól48] and then Makai in [Mak66] provides an alternative proof to Polya's and gives another estimate, stronger than Polya's one. In a scaling invariant form, the de Saint-Venant principle can be stated as

$$T(\Omega)|\Omega|^{-\frac{n+2}{n}} \leq T(B)|B|^{-\frac{n+2}{n}} \qquad \forall \Omega \subset \mathbb{R}^n,\ \Omega \text{ measurable}, \tag{13}$$

where B is a ball.

The eigenvalue and Torsion problem presented above are both boundary value problems of Dirichlet type, i.e. in (9) and (11) the function u has to be zero at the boundary. One can ask for different boundary conditions, for instance the *Neumann* and the *Robin boundary conditions* are respectively

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial \nu} + \beta u = 0,$$

where $\beta \in \mathbb{R}$, usually $\beta > 0$. Robin boundary conditions are a sort of intermediate case between Neumann and Dirichlet boundary conditions: indeed, for $\beta = 0$ we have Neumann boundary conditions, while formally sending $\beta \to +\infty$ we recover Dirichlet boundary conditions.

These boundary conditions are used to model different types of physical and engineering situations. The Dirichlet boundary conditions are used in the case in which the function is "fixed" at the boundary, e.g. the elastic membrane presented above. The Neumann boundary conditions are used in case there is no flux exiting from Ω, e.g. in the case u denotes a temperature, $\frac{\partial u}{\partial \nu} = 0$ means that the Ω is perfectly insulated. The Robin boundary conditions model the intermediate case among Dirichlet and Neumann in which the value of the function at the boundary depends on the normal derivative of the function itself, e.g. if u denotes the temperature, the Robin boundary conditions describe the case in which part of the heat flux is absorbed at the boundary.

Structure of the thesis

The aim of the thesis is to study some problems in shape optimization. In particular we will present some techniques to face such a kind of problem, like the *rearrangements*. We will introduce and present the properties of rearrangements in Section 1.2 but roughly speaking, given a function $f\colon \Omega \to \mathbb{R}$, a rearrangement of f is any function whose superlevel sets have the same measure as those of f; so by Cavalieri's principle, a rearrangement of f has the same L^p norm as f for any $p \geq 1$. Above all, given a function f we can construct its *Schwarz decreasing rearrangement* of f, that is the rearrangement having as level sets, balls centered at the origin.

The interest in this topic stems from the pioneering work by Talenti [Tal76], in which the author prove the following result: let $\Omega \subset \mathbb{R}^n$ a measurable set, $f \in L^2(\Omega)$, u and v denote respectively the solutions to the following problems

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta v = f^\sharp & \text{in } \Omega^\sharp \\ v = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{14}$$

where, through the whole thesis, $\Omega^\sharp$ will be the ball centered at the origin having the same measure as Ω; then it holds

$$u^\sharp(x) \leq v(x) \qquad \forall x \in \Omega^\sharp. \tag{15}$$

For sake of completeness, in [Tal76] the author consider a generic elliptic operator L and compare the solution to $Lu = f$, $u \in H_0^1$, with the solution to

$-\Delta v = f^\sharp$, $v \in H_0^1$, on the ball. Second problem in (14) is usually referred as the *symmetrized problem* of the first one and results of such a kind, in which one is able to compare some L^p norm of a problem with the same L^p norm of the symmetrized problem, are called *Talenti comparison results* or *comparison results à la Talenti.*

We underline that formula (15) also implies

$$\|u\|_{L^p(\Omega)} \le \|v\|_{L^p(\Omega^\sharp)} \qquad \forall p \ge 1.$$

So in particular, choosing $f = 1$, (15) implies the Saint-Venant inequality (13) and it is possible to show that it implies the Faber-Krahn inequality (10) too.

Because of the flexibility of the proof technique and the powerful results which it gives, a huge literature has formed on the subject. It is impossible to make a comprehensive list of all the results, let us just mention the one by Talenti himself [Tal79] in which the author deals with nonlinear operator in divergence form and the one by Alvino, Lions and Trombetti [ALT89] in which the authors study both the elliptic and the parabolic case.

The previous works regard Dirichlet boundary conditions and for more than 40 years, there were no results regarding Robin boundary conditions, up to [ANT23] when Alvino, Nitsch and Trombetti consider the following two problems

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega \end{cases} \qquad \text{and} \qquad \begin{cases} -\Delta v = f^\sharp & \text{in } \Omega^\sharp \\ \dfrac{\partial v}{\partial \nu} + \beta v = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{16}$$

where Ω is a Lipschitz domain and f is a non-negative L^2 function. They were able to establish the following comparison

$$\begin{aligned} &\|u\|_{L^{k,1}(\Omega)} \le \|v\|_{L^{k,1}(\Omega^\sharp)} && \forall\, 0 < k \le \frac{n}{n-2} \\ &\|u\|_{L^{2k,2}(\Omega)} \le \|v\|_{L^{2k,2}(\Omega^\sharp)} && \forall\, 0 < k \le \frac{n}{3n-4}, \end{aligned}$$

where $\|\cdot\|_{L^{p,q}}$ is the so-called *Lorentz norm*, see section 1.2 for its definition. Furthermore the authors prove, for $n = 2$ and $f \equiv 1$, the following pointwise comparison:

$$u^\sharp(x) \le v(x) \qquad \text{a.e. in } \Omega^\sharp.$$

One can find generalizations of [ANT23] to the anisotropic case, the Hermite operator and for mixed boundary conditions respectively in [San22; CGNT23; ACNT21].

In Chapter 2 we will discuss the generalization of the result in [ANT23] to the non-linear case.

Specifically let Ω be a a Lipschitz domain, $f \in L^{p'}(\Omega)$ be a non-negative function and $\beta > 0$. We consider the following problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{17}$$

where $\Delta_p u := \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big)$, $p \geq 2$.

We want to establish a comparison principle with the solution to the following symmetrized problem

$$\begin{cases} -\mathrm{div}(|\nabla v|^{p-2}\nabla v) = f^\sharp & \text{in } \Omega^\sharp \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \beta |v|^{p-2}v = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{18}$$

and we prove the following ([AGM22]):

Theorem. *Let u and v be the solutions to problem (17) and (18) respectively. Then we have*

$$\|u\|_{L^{k,1}(\Omega)} \leq \|v\|_{L^{k,1}(\Omega^\sharp)} \qquad \forall\, 0 < k \leq \frac{n(p-1)}{(n-1)p}, \tag{19}$$

$$\|u\|_{L^{pk,p}(\Omega)} \leq \|v\|_{L^{pk,p}(\Omega^\sharp)} \qquad \forall\, 0 < k \leq \frac{n(p-1)}{(n-2)p+n}. \tag{20}$$

We observe that from Theorem 2.1.3, $p \geq n$, it holds

$$\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\sharp)} \qquad \text{and} \qquad \|u\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega^\sharp)}.$$

Moreover, in the case $f \equiv 1$ we get a stronger comparison between $u^\sharp$ and v:

Theorem. *Assume that $f \equiv 1$ and let u and v be the solutions to (17) and (18) respectively.*

(i) If $1 \leq p \leq \dfrac{n}{n-1}$ then

$$u^\sharp(x) \leq v(x) \qquad x \in \Omega^\sharp, \tag{21}$$

(ii) if $p > \dfrac{n}{n-1}$ and $0 < k \leq \dfrac{n(p-1)}{n(p-1)-p}$, then

$$\begin{aligned} \|u\|_{L^{k,1}(\Omega)} &\leq \|v\|_{L^{k,1}(\Omega^\sharp)}, \\ \|u\|_{L^{pk,p}(\Omega)} &\leq \|v\|_{L^{pk,p}(\Omega^\sharp)}. \end{aligned} \tag{22}$$

We observe that for $f \equiv 1$, we have that

$$\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\sharp)} \qquad \text{and} \qquad \|u\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega^\sharp)}. \tag{23}$$

The first one inequality in (23) can be seen as the p-Robin version of the Saint-Venant inequality (13). We recall that, the result by Talenti implies Faber-Krahn inequalities. In the same manner, we can prove an inequality for the first Robin p-Laplacian eigenvalue, i.e.

$$\lambda_{p,\beta}(\Omega) = \min_{\substack{\omega \in W^{1,p}(\Omega)\\ \omega \neq 0}} \frac{\displaystyle\int_\Omega |\nabla \omega|^p \, dx + \beta \int_{\partial\Omega} |\omega|^p \, d\mathcal{H}^{n-1}(x)}{\displaystyle\int_\Omega |\omega|^p \, dx}. \tag{24}$$

and, as corollary of Theorem 2.1.3, we have the following

Corollary. *If $p \geq n$, it holds*

$$\lambda_{p,\beta}(\Omega) \geq \lambda_{p,\beta}(\Omega^\sharp). \tag{25}$$

In literature (25) is usually referred as *Bossel-Daners* inequality, because it was firstly proven for the Laplacian case in [Bos86; Dan06], then generalized to the p-Laplacian in [BD10]. An alternative proof to the one by Bossel and Daners can be found in [BG10] and a generalization of this result in [BG15]. Actually, the results in [BD10] are more general, since they hold for every $p > 1$, but they are obtained with completely different tools than the ones contained in our paper.

In the second part of Chapter 2 we will prove a comparison result for variable Robin boundary condition, i.e. when $\beta = \beta(x)$. Specifically, we generalize the result contained in [CGNT23] considering the following problem

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = f(x)|x|^\ell & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta(x)|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{26}$$

where Ω is a bounded and Lipschitz set such that $0 \notin \partial\Omega$. We will also assume

$$p \geq n \tag{H_1}$$

$$-n < \ell < 0 \tag{H_2}$$

$$m = \inf_{\partial\Omega} \beta(x) > 0 \quad \text{and} \quad M = \sup_{\partial\Omega} \beta(x) < +\infty \tag{H_3}$$

$$0 \leq f(x) \in L^{p'}(\Omega, |x|^\ell \, dx), \tag{H_4}$$

where $p' = \frac{p}{p-1}$.

We will discuss them in details in Section 1.3, but we can say that hypotheses (H_1) and (H_2) ensure the validity of certain isoperimetric inequalities, where two different weights (which are suitable powers of the distance from the origin) appear in the perimeter and in the area element, respectively (see, e.g., [How15; CH15; DHHT12; BBMP99; Csa15; ABCMP17] and the references therein). Indeed we need to use, in place of the more common Schwarz symmetrization, some weighted rearrangement, based on these "double density" isoperimetric inequalities.

Now we need to construct the so-called symmetrized problem. That is a radial problem, of the same type of (26), whose solution will estimate the one to problem (26).

To this aim we introduce, with some abuse of notation, the set $\Omega^\sharp$ which is the ball centered at the origin of radius $r^\sharp$, with $r^\sharp$ such that

$$|\Omega^\sharp|_\ell = \int_{\Omega^\sharp} |x|^\ell \, dx = \int_\Omega |x|^\ell \, dx = |\Omega|_\ell \, ,$$

and the function $f^\sharp(x)$ defined by the following relation

$$\big|\{x \in \Omega \colon |f(x)| > t\}\big|_\ell = \big|\{x \in \Omega \colon f^\sharp(x) > t\}\big|_\ell \quad \text{for a.e. } t \geq 0.$$

Our symmetrized problem is the following

$$\begin{cases} -\mathrm{div}\big(|\nabla v|^{p-2}\nabla v\big) = f^\sharp(x)|x|^\ell & \text{in } \Omega^\sharp \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \tilde{\beta}(r^\sharp)^{\frac{\ell}{p'}}|v|^{p-2}v = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{27}$$

where

$$\widetilde{\beta} = \inf_{\partial\Omega} \beta(x)|x|^{-\frac{\ell}{p'}} > 0, \tag{28}$$

since $0 \notin \partial\Omega$ and H_3 holds. From the (28), it immediately follows that

$$\beta(x) \geq \widetilde{\beta}|x|^{\frac{\ell}{p'}}. \tag{29}$$

So we prove the following theorems ([ACG24]), see Section 1.3 for the definition of $\|\cdot\|_{L^p(\Omega,|x|^\ell\,dx)}$.

Theorem. *Assume that hypotheses (H_1)-(H_4) are in force and let u and v be the solutions to problems (26) and (27), respectively. Then we have*

$$\|u\|_{L^1(\Omega,|x|^\ell dx)} = \int_\Omega |u(x)||x|^\ell \, dx \leq \int_{\Omega^\sharp} |v(x)||x|^\ell \, dx = \|v\|_{L^1(\Omega^\sharp,|x|^\ell dx)}, \tag{30}$$

and

$$\|u\|^p_{L^p(\Omega,|x|^\ell dx)} = \int_\Omega |u(x)|^p|x|^\ell dx \le \int_{\Omega^\sharp} |v(x)|^p|x|^\ell dx = \|v\|^p_{L^p(\Omega^\sharp,|x|^\ell dx)}. \quad (31)$$

As before, if $f(x)$ is constant we get a stronger result, namely a pointwise comparison between $u^\sharp$ and v.

Theorem. *Assume that hypotheses (H_1)-(H_4) are in force and suppose that $f(x) \equiv 1$ in Ω. If either $p = n = 2$ or if $p > 2$, $n \ge 2$ and*

$$\ell \le -n + \frac{p-n}{p-2} \quad (32)$$

then

$$u^\sharp(x) \le v(x) \qquad x \in \Omega^\sharp, \quad (33)$$

where u and v are the solutions to (2.28) and (27) respectively.

We can define the first eigenvalue of p-Laplace operator with the boundary conditions as in (2.28) as the minimum of the following Rayleigh quotient

$$\lambda_{p,\beta(x)}(\Omega) = \min_{\substack{\psi\in W^{1,p}(\Omega)\\ \psi\neq 0}} \frac{\displaystyle\int_\Omega |\nabla\psi|^p\,dx + \int_{\partial\Omega} \beta(x)|\psi|^p\,d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |\psi|^p|x|^\ell\,dx}. \quad (34)$$

Finally, we are able to prove the following Faber-Krahn inequality (see also [BGT19; Csa15; FK15; FNT13] and the references therein).

Theorem. *Assume hypotheses (H_1)-(H_3) and let $\tilde\beta$ be the constant defined in (28), then we have*

$$\lambda_{1,\beta(x)}(\Omega) \ge \lambda_{1,\tilde\beta}(\Omega^\sharp).$$

Chapter 3 is devoted to the study of a particular type of rearrangement, the gradient rearrangement. The starting point of our work is the result by Giarrusso and Nunziante [GN84]: the authors, following the idea of Talenti, consider the following Hamilton-Jacobi problems

$$\begin{cases} H(\nabla u) = f & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (35a) \qquad \begin{cases} K(|\nabla v|) = f_\sharp & \text{a.e. in } \Omega^\sharp \\ v = 0 & \text{on } \partial\Omega^\sharp \end{cases} \quad (35b)$$

where $H\colon \mathbb{R}^n \to \mathbb{R}$ and $K\colon \mathbb{R} \to \mathbb{R}$ are measurable functions, $u, v \in W^{1,p}_0(\Omega)$ and $f_\sharp$ is the increasing rearrangement of f (see Section 1.2 for its definition).

In particular, under suitable assumptions on H and K, it is proven ([GN84, Theorem 2.2]) that whenever u, v are solutions to (35a) and (35b) respectively, then

$$\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\sharp)}. \tag{36}$$

In [ALT89] the authors study the problem of maximization of the L^q norm among functions with prescribed gradient rearrangement. Precisely, the following cases are considered

- $1 \leq q \leq \frac{np}{n-p}$ if $p < n$,
- $1 \leq q < +\infty$ if $p = n$,
- $1 \leq q \leq +\infty$ if $p > n$,

and for a fixed $\varphi = \varphi^* \in L^p(0, |\Omega|)$, they define

$$I(\Omega) := \sup\left\{\|v\|_{L^q} \;:\; |\nabla v| \leq f \text{ a.e. in } \Omega,\, v \in W_0^{1,p}(\Omega),\, f \geq 0,\, f^* = \varphi^*\right\},$$

and they proved that there exists v, g spherically symmetric functions on $\Omega^\sharp$ such that $g^* = \varphi$,

$$I(\Omega^\sharp) = \|v\|_{L^q(\Omega^\sharp)},$$

hence

$$v \in W_0^{1,p}(\Omega^\sharp),\; v \geq 0,\; |\nabla v| = g \qquad \text{a.e. in } \Omega^\sharp.$$

Moreover $I(\Omega^\sharp) \geq I(\Omega)$ for all open sets Ω in $\mathbb{R}^n$ such that $|\Omega^\sharp| = |\Omega|$.

Let us mention that in [Cia96] the author proved a representation formula for the function g, the existence of which was proved by Alvino, Lions and Trombetti in [ALT89].

In the first part Chapter 2 we extend the validity of (36) for Sobolev functions with non zero trace ([AG23]).

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open and Lipschitz set and let $u \in W^{1,p}(\Omega)$ be a non-negative function. If we denote with $\Omega^\sharp$ the ball centered at the origin with same measure as Ω, then there exists a non-negative function $u^\star \in W^{1,p}(\Omega^\sharp)$ that satisfies*

$$\begin{cases} |\nabla u^\star| = |\nabla u|_\sharp(x) & \text{a.e. in } \Omega^\sharp \\ u^\star = \dfrac{\displaystyle\int_{\partial\Omega} u\, d\mathcal{H}^{n-1}}{\mathcal{H}^{n-1}(\partial\Omega^\sharp)} & \text{on } \partial\Omega^\sharp. \end{cases} \tag{37}$$

and such that

$$\|u\|_{L^1(\Omega)} \leq \|u^\star\|_{L^1(\Omega^\sharp)}. \tag{38}$$

As a consequence of the previous theorem, we have the following

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open and Lipschitz set. Let $f \in L^\infty(\Omega)$ be a function such that*

$$f^*(t) \geq \left(1 - \frac{1}{n}\right)\frac{1}{t}\int_0^t f^*(s)\, ds \qquad \forall t \in [0, |\Omega|]. \tag{39}$$

If $u \in W^{1,p}(\Omega)$ and $u^\star$ is the function given by previous theorem, then we have

$$\int_\Omega f(x)u(x)\, dx \leq \int_{\Omega^\sharp} f^\sharp(x)u^\star(x)\, dx. \tag{40}$$

While the literature concerning rearrangements in Sobolev spaces is exhaustive, to our knowledge, results on BV functions are rarer. One of the most relevant papers in this framework is [CF02] where the authors extend the validity of Polya-Szegö inequality to BV functions. More specifically, they proved that if $u \in \mathrm{BV}(\mathbb{R}^n)$, then its Schwarz rearrangement $u^\sharp$ (see Section 1.2 for its definition) belongs to $\mathrm{BV}(\mathbb{R}^n)$ and it holds [CF02, Theorem 1.3]

$$\begin{aligned} |Du^\sharp|(\mathbb{R}^n) &\leq |Du|(\mathbb{R}^n) \\ |D^{\mathrm{s}}u^\sharp|(\mathbb{R}^n) &\leq |D^{\mathrm{s}}u|(\mathbb{R}^n) \\ |D^{\mathrm{j}}u^\sharp|(\mathbb{R}^n) &\leq |D^{\mathrm{j}}u|(\mathbb{R}^n), \end{aligned} \tag{41}$$

where D^{s} and D^{j} denote respectively the singular and the jump part of the gradient (see 1.4 for their definitions). There is no analogue of (41) for the absolutely continuous and the Cantorian part of the gradient, i.e. in the symmetrization procedure the total variation of D^{a} and D^{c} can be increased, as shown in the example given in [CF02].

In [AGNT24], we introduce a symmetrization that keeps the absolutely continuous part separate from the singular part (sum of jump and Cantorian part) of the gradient. To be more precise, we define the radial function $u^\star \in W^{1,1}(\Omega^\sharp) \cap \mathrm{BV}_0(\Omega^\sharp) \cap L^\infty(\Omega^\sharp)$ such that

$$\begin{cases} |\nabla u^\star|(x) = |\nabla^a u|_\sharp(x) & \text{a.e. in } \Omega^\sharp \\ u^\star(x) = \dfrac{1}{\mathrm{Per}(\Omega^\sharp)}|D^s u|(\mathbb{R}^n) & \text{on } \partial\Omega^\sharp \end{cases}, \tag{42}$$

where $\nabla^a u$ will be defined in Section 1.4.

In second part of Chapter 3 we prove the following Theorem ([AGNT24]).

Theorem. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with finite perimeter and let $\Omega^\sharp$ be the centered ball. Assume that u is a non-negative function belonging to* $\mathrm{BV}_0(\Omega)$ *and assume that $u^\star$ is defined as in (42), then*

$$\|u\|_{L^1(\Omega)} \leq \|u^\star\|_{L^1(\Omega^\sharp)}.$$

We will also present some applications of the previous results. In particular, we will consider

- a Robin torsional rigidity problem

$$\frac{1}{T(\Omega,\beta)} = \inf_{w \in H^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla w|^2 \, dx + \beta \mathrm{Per}(\Omega) \int_{\partial\Omega} w^2 \, d\mathcal{H}^{n-1}}{\left(\displaystyle\int_\Omega w \, dx\right)^2},$$

  where $\beta > 0$;

- a penalized torsional rigidity problem

$$T_{\mathcal{F}}(\Omega,\Lambda) := -\inf_{\psi \in H_0^1(\Omega)} \left(\frac{1}{2} \int_\Omega |\nabla \psi|^2 \, dx - \int_\Omega |\psi| \, dx + \Lambda \big| \{ |\nabla \psi| \neq 0 \} \big| \right),$$

  where $\Lambda > 0$;

- a torsional rigidity related to thermal insulation

$$\frac{1}{T_{\mathcal{G}}(\Omega,m)} := \inf_{\psi \in H^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla \psi|^2 \, dx + \frac{1}{m} \left(\int_{\partial\Omega} |\psi| \, d\mathcal{H}^{n-1} \right)^2}{\left(\displaystyle\int_\Omega |\psi| \, dx\right)^2},$$

  where $m > 0$.

We will prove Saint-Venant type inequalities:

$$\begin{aligned} T(\Omega,\beta) &\leq T(\Omega^\sharp,\beta) \\ T_{\mathcal{F}}(\Omega,\Lambda) &\leq T_{\mathcal{F}}(\Omega^\sharp,\Lambda) \\ T_{\mathcal{G}}(\Omega,m) &\leq T_{\mathcal{G}}(\Omega^\sharp,m). \end{aligned}$$

In Chapter 4 we will deal with two shape optimization problems for eigenvalues where the class of admissible sets is the class of convex sets.

Firstly, in [AGM24] we face a problem related to the Robin-Laplacian eigenvalues, defined in (24), with positive and negative boundary parameter. We recall that for $\beta = 0$, we have Neumann boundary conditions.

As mentioned above, for $\beta > 0$ Bucur and Daners in [BD10] proved that among sets having prescribed measure, the ball has the minimum Robin p-Laplacian eigenvalue, i.e.

$$\lambda_{p,\beta}(\Omega) \geq \lambda_{p,\beta}(\Omega^\sharp) \qquad \forall \Omega \subset \mathbb{R}^n,\ \Omega \text{ Lipschitz}. \tag{43}$$

Once one has (43), one can ask what if we change the constraint, e.g. we fixed the perimeter of the set Ω. The following rescaling property of the Robin eigenvalue (see [BFK17])

$$\lambda_{p,\beta}(t\Omega) \leq \frac{1}{t}\lambda_{p,\beta}(\Omega) \leq \lambda_{p,\beta}(\Omega) \qquad \forall t \geq 1,$$

gives a Faber-Krahn type inequality for fixed perimeter

$$\lambda_{p,\beta}(\Omega) \geq \lambda_{p,\beta}(\Omega^\star) \qquad \forall \Omega \subset \mathbb{R}^n,\ \Omega \text{ Lipschitz}, \tag{44}$$

where $\Omega^\star$ is the ball with the same perimeter as Ω.

When an equation like (44) is in force, one can ask if a continuity bound holds, as in (7). We positively answer to this question giving the following continuity estimate in terms of the isoperimetric deficit ([AGM24]):

Theorem. *Let β be a positive parameter. Let Ω be a bounded, open and convex set in $\mathbb{R}^n$ and let $\Omega^\star$ be the ball, centered at the origin, such that $\mathrm{Per}(\Omega) = \mathrm{Per}(\Omega^\star) = \rho$. Denote by $\lambda_{p,\beta}(\Omega)$ and $\lambda_{p,\beta}(\Omega^\star)$ the first eigenvalues of the p-Laplacian operator with Robin boundary conditions respectively on Ω and $\Omega^\star$, and by v a positive eigenfunction associated to $\lambda_{p,\beta}(\Omega^\star)$, then*

$$\frac{\lambda_{p,\beta}(\Omega) - \lambda_{p,\beta}(\Omega^\star)}{\lambda_{p,\beta}(\Omega)} \leq C(n,p,\beta,\rho)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{\mathrm{Per}(\Omega)^{\frac{n}{n-1}}}\right), \tag{45}$$

where ω_n is the measure of the unitary ball in $\mathbb{R}^n$, and $C(n,p,\beta,\rho) = \dfrac{\|v\|_\infty^p|\Omega^\star|}{\|v\|_p^p}$.

Formula (45) can be seen as a generalization of the result contained in [BNT10] for the Dirichlet p-Laplacian eigenvalue.

When β is a negative parameter, the authors in [BFNT19] proved a reverse Faber-Krahn inequality for the first eigenvalue of the Robin-Laplacian among convex sets of given perimeter. In particular, they proved that among convex sets of given perimeter the ball $\Omega^\star$ maximizes the first Robin eigenvalue of the p-Laplacian, i.e.

$$\lambda_{p,\beta}(\Omega) \le \lambda_{p,\beta}(\Omega^\star). \tag{46}$$

For completeness' sake, we quote that in [AFK17], the authors already proved that the disk maximizes the first eigenvalue under a perimeter constraint, among C^2 domains in $\mathbb{R}^2$, while the question remained open in arbitrary dimension.

This question is related to the conjecture of Bareket (see [Bar77]) claiming that the ball maximizes $\lambda_{p,\beta}(\Omega)$ among all Lipschitz sets with given volume. Freitas and Krejčiřík in [FK15] proved that the conjecture is false, giving a counter-example based on the asymptotic behavior of the eigenvalues on a disk and an annulus of the same area when $\beta \to -\infty$. They also proved that among sets of area equal to 1, the conjecture is true, provided β is close to 0. For more details see [Hen17].

In [CL21] the authors proved a quantitative version of the reverse Faber-Krahn (46) in the case $p = 2$ among convex sets of fixed perimeter following the Fuglede's approach introduced in [Fug89].

In Chapter 4 we recover the result in [CL21] obtaining a quantitative version of (46) for all $p \in (1, +\infty)$, but using a different approach. Indeed, following the method introduced by Payne and Weinberger in [PW61], we establish a comparison using the so-called parallel coordinates method. In particular, we firstly prove a lower bound in terms of perimeter and measure of Ω, that is

Theorem. *Let β be a negative parameter. Let Ω be a bounded, open and convex set in $\mathbb{R}^n$ and let $\Omega^\star$ be the ball, centered at the origin, such that* $\mathrm{Per}(\Omega) = \mathrm{Per}(\Omega^\star) = \rho$. *Denote by $\lambda_{p,\beta}(\Omega)$ and $\lambda_{p,\beta}(\Omega^\star)$ the first eigenvalues of the p-Laplacian operator with Robin boundary conditions respectively on Ω and $\Omega^\star$, and by v a positive eigenfunction associated to $\lambda_{p,\beta}(\Omega^\star)$, then*

$$\frac{\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega)}{|\lambda_{p,\beta}(\Omega)|} \ge C(n,p,\beta,\rho)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{\mathrm{Per}(\Omega)^{\frac{n}{n-1}}}\right), \tag{47}$$

where ω_n is the measure of the unitary ball in $\mathbb{R}^n$, and $C(n,p,\beta,\rho) = \dfrac{v_m^p|\Omega^\star|}{\|v\|_p^p}$ *with* $v_m = \min_{\Omega^\star} v$.

Then we prove the quantitative result as in [CL21].

Theorem. *Let $n \geq 2$, $\rho > 0$ and $\beta < 0$. Then, there exist two positive constants $C(n,p,\beta,\rho) > 0$ and $\delta_0(n,p,\beta,\rho) > 0$, such that, for all $\Omega \subset \mathbb{R}^n$ bounded and convex with $\mathrm{Per}(\Omega) = \rho$ and $\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \leq \delta_0$, it holds*

$$\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \geq C(n,p,\beta,\rho) g\big(\mathcal{A}^\star_{\mathcal{H}}(\Omega)\big), \tag{48}$$

where $\Omega^\star$ is a ball with the same perimeter of Ω, $\mathcal{A}^\star_{\mathcal{H}}$ is the Hausdorff asymmetry defined in (1.23) and g is defined in (1.26).

In the second part of Chapter 4 we focus on a problem for the Neumann Laplacian. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open and convex set, we consider the following Neumann-Laplacian eigenvalue problem

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}, \tag{49}$$

where ν is the outer normal to Ω.

Because of the assumptions on Ω, the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact, hence the Neumann-Laplacian is compact, its spectrum is discrete and the eigenvalues goes up to infinity,

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots \to +\infty.$$

By classical variational methods, each eigenvalue can be characterized with the following formula

$$\mu_k(\Omega) = \min_{E_k} \max_{0 \neq u \in E_k} \frac{\displaystyle\int_\Omega |\nabla u|^2 \, dx}{\displaystyle\int_\Omega u^2 \, dx}, \tag{50}$$

where the minimum is taken over all the k-dimensional subspace of $H^1(\Omega)$ that are L^2-orthogonal to the constant functions.

Let us highlight that for $k = 1$, (50) reads as

$$\mu_1(\Omega) = \min \left\{ \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx}{\displaystyle\int_\Omega v^2 \, dx} : v \in H^1(\Omega), \int_\Omega v \, dx = 0 \right\}. \tag{51}$$

The spectrum of the Laplacian is widely studied in the literature. Indeed, since the exact values of the eigenvalues (and the corresponding eigenfunctions) are know only for a small class of domains, a central question in spectral geometry is related to find bounds on the eigenvalues in terms of geometrical quantities. The first result in this direction involves the measure of the domain. More precisely, since

$$\mu_k(t\Omega) = t^{-2}\mu_k(\Omega) \qquad \forall t > 0,$$

a natural question regards the following maximization problem

$$\sup\{\mu_1(\Omega)|\Omega|^{\frac{2}{n}} : \Omega \subset \mathbb{R}^n,\ \Omega \text{ open, bounded and Lipschitz } \}. \tag{52}$$

This problem was solved by Szegö ([Sze54]) for planar and simply connected domains while Weinberger ([Wei56]) generalizes the result to any dimension without topological restrictions, obtaining

$$\mu_1(\Omega)|\Omega|^{\frac{2}{n}} \leq \mu_1(B)|B|^{\frac{2}{n}}, \tag{53}$$

where B is the unitary ball in $\mathbb{R}^n$.

Following this path, in [GNP09] and [BH19] the authors proved an upper bound for the second (non trivial) eigenvalue of the Neumann Laplacian respectively for simply connected planar domain and in any dimension without topological restrictions, obtaining

$$\mu_2(\Omega)|\Omega|^{\frac{2}{n}} \leq \mu_2(\Omega^*)|\Omega^*|^{\frac{2}{n}}, \tag{54}$$

where Ω^* is the union of two disjoint balls. For higher eigenvalue even the existence of maximizer of the quantity $\mu_k(\Omega)|\Omega|^{\frac{2}{n}}$ is an open problem. See [BMO23] for an analysis of $\mu_k(\Omega)$ for $k \geq 3$.

In this context of optimization of Neumann eigenvalues, another relevant question regards the link between eigenvalues and diameter, see for example [Krö99; HM23] in which it it studied the following maximization problem

$$\max\left\{\operatorname{diam}(\Omega)^2\mu_k(\Omega) \colon \Omega \subset \mathbb{R}^n,\ \Omega \text{ open, bounded and convex } \right\}, \tag{55}$$

where $\operatorname{diam}(\Omega)$ denotes the diameter of Ω.

In the aforementioned works, the authors take advantage of the link between Neumann and the following Sturm-Liouville problem:

$$\mu_k(p) = \inf_{E_k} \sup_{0 \neq p \in E_k} \frac{\displaystyle\int_0^1 \big(\varphi'(x)\big)^2 p(x)\, dx}{\displaystyle\int_0^1 \varphi(x)^2 p(x)\, dx} \tag{56}$$

where the infimum is taken among all the k-dimensional subspace of $H^1(0,1)$ that are L^2-orthogonal to the constant functions with respect to p in $[0,1]$ and $p\colon [0,1] \to \mathbb{R}$ is a $\frac{1}{n-1}$-concave function (see Section 1.6 for its definition).

In particular they prove that for any $k, n \in \mathbb{N}$ there exists $\mu^*_{k,n}$ (see [HM23] for the explicit value) such that

$$\mathrm{diam}(\Omega)^2 \mu_k(\Omega) \leq \mu^*_{k,n} \qquad \forall \Omega \subset \mathbb{R}^n,\ \Omega \text{ open bounded and convex.} \tag{57}$$

We point out that the previous inequality is sharp but not achieved: it is asymptotically reached on a sequence of collapsing domains that converges to a segment in the sense of Hausdorff (see 1.6 for the definition of Hausdorff convergence).

Incidentally let us mention that the case $k = 1$ is also studied in [BNT16] with a different technique, in particular the authors use a suitable test function to prove the following inequality

$$\mathrm{diam}(\Omega)^2 \mu_1(\Omega) < 4 j_{0,1}^2 \qquad \forall \Omega \subset \mathbb{R}^2,\ \Omega \text{ convex and bounded domain,} \tag{58}$$

where $j_{0,1}$ is the first zero of the Bessel function J_0 and $j_{0,1} = \mu^*_{1,2}$.

We establish the following quantitative estimates of (57) (see Section 1.6 for the definition of John ellipsoid):

Theorem. *There exists a constant $C = C(k,n) > 0$ such that for any $\Omega \subset \mathbb{R}^n$ open, bounded and convex set, it holds*

$$\mu_k(\Omega) \leq \frac{\mu^*_{k,n}}{\mathrm{diam}(\Omega)^2} - C \frac{a_2(\Omega)^2}{\mathrm{diam}(\Omega)^4}, \tag{59}$$

where $a_2(\Omega)$ is the second biggest semiaxis of the John ellipsoid of Ω and $\mu^*_{k,n}$ is the optimal upper bound given in [HM23].

We will prove the previous Theorem splitting the case $k = 1$ and $k \geq 2$ because for the first eigenvalue the proof is slightly easier but the main idea of the two proofs is the same.

Moreover we prove the optimality of exponent 2 in dimension $n = 2$.

Theorem. *Exponent 2 in (59) is sharp for $n = 2$ and $k = 2$.*

Chapter 1

Notations and preliminaries

1.1 Notations

Here and in the following, $|\cdot|$ denotes the n-dimensional Lebesgue measure of a measurable set E, while $\mathcal{H}^k(\cdot)$, for $k \in [0, n)$, will stand for the k-dimensional Hausdorff measure in $\mathbb{R}^n$. Unless otherwise specified, when we refer to measure we mean Lebesgue measure.

Given $\Omega \subseteq \mathbb{R}^n$ with finite measure, we will denote with $\Omega^\sharp$ the ball centered at the origin with the same Lebesgue measure, i.e. $|\Omega| = |\Omega^\sharp|$. We will refer to $\Omega^\sharp$ just as centered ball. Unless otherwise specified, ω_n will denote the Lebesgue measure of the n dimensional ball with radius 1.

1.2 Rearrangements

1.2.1 Some basic facts about rearrangements

We now briefly recall some notions about rearrangements. We refer for instance to [HLP88; Kaw85; Kes06; Tal94a; Tal94b] for all the details.

Definition 1.2.1. Let Ω be a measurable set and let $u\colon \Omega \to \mathbb{R}$ be a measurable function, the **distribution function** of u is defined as

$$\mu\colon [0, +\infty) \to [0, +\infty) \qquad \mu(t) = \left|\left(\{x \in \Omega \,:\, |u(x)| > t\}\right)\right|.$$

It can be proved that

- μ is a decreasing function in $[0, +\infty)$;
- μ is right-continuous;
- $\mu(0) = |\operatorname{supp} u|$ and $\mu(+\infty) = 0$;
- $\mu(t^-) = \big|\{x \in \Omega \,:\, |u(x)| \geq t\}\big|$.

Definition 1.2.2. Let $u\colon \Omega \to \mathbb{R}$ be a measurable function, the **decreasing rearrangement** of u is defined as

$$u^*\colon \mathbb{R}^+ \to \mathbb{R}^+ \qquad u^*(s) = \inf\{t > 0 \,:\, \mu(t) \leq s\}\,.$$

and the **increasing rearrangement** of u as

$$u_*\colon [0, |\Omega|] \to \mathbb{R}^+ \qquad u_*(s) = u^*(|\Omega| - s).$$

It can be proved that

- u^* (u_*) is a decreasing (increasing) function in $[0, +\infty)$;
- u^* and u_* are lower semi-continuous;
- whenever $u \in L^\infty(\Omega)$ $u^*(0) = \|u\|_{L^\infty(\Omega)}$ and $u^*(t) = 0$ $\forall t \geq |\operatorname{supp} u|$;
- $u_*(|\Omega|) = \|u\|_{L^\infty(\Omega)}$ and $u_*(t) = 0$ $\forall 0 \leq t \leq |\Omega| - |\operatorname{supp} u|$;
- $u^*(\mu(t)) \leq t$ for every non-negative t, $\mu(u^*(s)) \leq s$ for every non-negative s;
- $u^*(\mu(t)^-) \geq t$ for every non-negative t, $\mu(u^*(s)^-) \geq s$ for every non-negative s.

Definition 1.2.3. Let $u\colon \Omega \to \mathbb{R}$ be a measurable function. The **Schwarz rearrangement** or the **spherically symmetric decreasing rearrangement** of u is defined as

$$u^\sharp\colon \mathbb{R}^n \to \mathbb{R}^+ \qquad u^\sharp(x) = u^*(\omega_n |x|^n),$$

where ω_n is the Lebesgue measure of the unit n-dimensional ball.

Moreover the **spherically symmetric increasing rearrangement** of u is defined as

$$u_\sharp\colon \mathbb{R}^n \to \mathbb{R}^+ \qquad u_\sharp(x) = u_*(\omega_n |x|^n).$$

It can be proved that

- $u^\sharp$ ($u_\sharp$) is non-negative, radial and radially decreasing (increasing);
- $u^*, u_*, u^\sharp$ and $u_\sharp$ have the same distribution function as u, so by Cavalieri's principle the L^p norms are equal for every p, i.e. $\forall p \geq 1$ it holds

$$\|u\|_{L^p(\Omega)} = \|u^*\|_{L^p\left(0,|\Omega|\right)} = \left\|u^\sharp\right\|_{L^p(\Omega^\sharp)} = \|u_*\|_{L^p\left(0,|\Omega|\right)} = \|u_\sharp\|_{L^p(\Omega^\sharp)}; \tag{1.1}$$

- the **Hardy-Littlewood inequality** holds: for any $u, v \colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$

$$\int_0^{|\Omega|} u_*(s)v^*(s)\, ds \leq \int_\Omega |u(v)v(x)|\, dx \leq \int_0^{|\Omega|} u^*(s)v^*(s)\, ds; \tag{1.2}$$

- the Polya-Szegö inequality holds true [PS51]: if $u \in W_0^{1,p}(\Omega)$, then $u^\sharp \in W_0^{1,p}(\Omega^\sharp)$ and

$$\|\nabla u^\sharp\|_{L^p(\Omega^\sharp)} \leq \|\nabla u\|_{L^p(\Omega)};$$

- the operator which assigns to a function its symmetric decreasing rearrangement is a contraction in L^p , see ([CP79]), i.e.

$$\|f^* - g^*\|_{L^p([0,|\Omega|])} \leq \|f - g\|_{L^p(\Omega)}. \tag{1.3}$$

Let $u, v \colon \Omega \to \mathbb{R}$. Applying the Hardy-Littlewood inequality (1.2) to the functions $\chi_{\{|u|>t\}}$, v, we get

$$\int_{\{|u|>t\}} |v(x)|\, dx \leq \int_0^{\mu(t)} v^*(s)\, ds.$$

1.2.2 Lorentz spaces

Now we introduce the Lorentz spaces.

Definition 1.2.4. Let $0 < p < +\infty$ and $0 < q \leq +\infty$. The Lorentz space $L^{p,q}(\Omega)$ is the space of those functions such that the quantity:

$$\|g\|_{L^{p,q}} = \begin{cases} p^{\frac{1}{q}} \left(\displaystyle\int_0^\infty t^q \mu(t)^{\frac{q}{p}} \frac{dt}{t} \right)^{\frac{1}{q}} & 0 < q < \infty \\ \sup\limits_{t>0} \left(t^p \mu(t)\right) & q = \infty, \end{cases}$$

is finite.

Let us observe that for $p = q$ the Lorentz space coincides with the L^p space, as a consequence of the well known *Cavalieri's Principle*

$$\int_\Omega |g|^p = p \int_0^{+\infty} t^{p-1}\mu(t)\,dt.$$

See [Tal94a] for more details on Lorentz space.

1.2.3 Pseudo-rearrangements

Other powerful tools are the pseudo-rearrangements. Let $u \in W^{1,p}(\Omega)$ and let $f \in L^1(\Omega)$, as in [AT78] $\forall s \in [0, |\Omega|]$, there exists a subset $D(s) \subseteq \Omega$ such that

1. $|D(s)| = s$;
2. $D(s_1) \subseteq D(s_2)$ if $s_1 < s_2$;
3. $D(s) = \{x \in \Omega : |u(x)| > t\}$ if $s = \mu(t)$.

So the function

$$\int_{D(s)} f(x)\,dx,$$

is absolutely continuous, therefore it exists a function F such that

$$\int_0^s F(t)\,dt = \int_{D(s)} f(x)\,dx. \tag{1.4}$$

The following lemma holds, see [AT78, Lemma 2.2].

Lemma 1.2.1. *Let $f \in L^p$ for $p > 1$ and let $D(s)$ be a family described above. If F is defined as in (1.4), then there exists a sequence $\{F_k\}$ such that F_k has the same rearrangement as f and*

$$F_k \rightharpoonup F \qquad \text{in } L^p([0, |\Omega|]).$$

If $f \in L^1$ it follows that

$$\lim_k \int_0^{|\Omega|} F_k(s)g(s)\,ds = \int_0^{|\Omega|} F(s)g(s)\,ds, \tag{1.5}$$

for each function $g \in BV([0, |\Omega|])$.

1.2.4 Rearrangement of the gradient

We recall the Theorem of Giarrusso and Nunziante ([GN84, Theorem 2.2]).

Theorem 1.2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\Omega^\sharp$ be the centered ball, let $p \geq 1$, let $f\colon \Omega \to \mathbb{R}$ be a measurable function, let $H\colon \mathbb{R}^n \to \mathbb{R}$ be measurable non-negative functions and let $K\colon [0, +\infty) \to [0, +\infty)$ be a strictly increasing real-valued function such that*

$$0 \leq K(|y|) \leq H(y) \qquad \forall y \in \mathbb{R}^n \qquad \text{and} \qquad K^{-1}(f) \in L^p(\Omega).$$

Let $v \in W_0^{1,p}(\Omega)$ be a function that satisfies

$$\begin{cases} H(\nabla v) = f(x) & \text{a.e. in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

denoting by $z \in W_0^{1,p}(\Omega^\sharp)$ the unique spherically decreasing symmetric solution to

$$\begin{cases} K(|\nabla z|) = f_\sharp(x) & \text{a.e. in } \Omega^\sharp \\ z = 0 & \text{on } \partial\Omega^\sharp, \end{cases}$$

then

$$\|v\|_{L^1(\Omega)} \leq \|z\|_{L^1(\Omega^\sharp)}.$$

In [Mer97] the author characterizes the case of equality of (1.2.2).

Theorem 1.2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $v \in W_0^{1,1}(\Omega)$ be a non-negative function. Denote by $f(x) = |\nabla v|(x)$ and by $w \in W_0^{1,1}(\Omega^\sharp)$ the decreasing spherically symmetric solution to*

$$|\nabla w| = f_\sharp.$$

If $\|v\|_{L^1} = \|w\|_{L^1}$ then there exists $x_0 \in \mathbb{R}^n$ such that $\Omega = x_0 + \Omega^\sharp$, $f = f_\sharp(\cdot + x_0)$ and $v = w(\cdot + x_0)$.

1.3 Weighted volumes

Here we recall some basic facts on weighted volumes and perimeters, and some related isoperimetric inequalities. We refer to [ABCMP17; BBMP99; CH15; Csa15; DHHT12] for more details.

Definition 1.3.1. Let Ω be a Lebesgue measurable subset of $\mathbb{R}^n$ and let $\ell \in (-n, 0)$ and $k \in (-n+1, 0)$. We define the ℓ-weighted measure of Ω as

$$|\Omega|_\ell := \int_\Omega |x|^\ell \, dx, \tag{1.6}$$

and the k-weighted perimeter of Ω as

$$\operatorname{Per}_k(\Omega) = \begin{cases} \displaystyle\int_{\partial\Omega} |x|^k \, d\mathcal{H}^{n-1} & \text{if } \Omega \text{ is } (n-1)\text{-rectifiable} \\ +\infty & \text{otherwise.} \end{cases} \tag{1.7}$$

Here we assume that

$$p \geq n \tag{H_1}$$

$$-n < \ell < 0 \tag{H_2}$$

where $p' = \dfrac{p}{p-1}$.

For this family of measures, an isoperimetric inequality holds true if k, ℓ and n are suitably related (see either [CH15, Theorem 1.3] or [ABCMP17, Theorem 1.1] case *ii*).

Theorem 1.3.1. *If assumptions* (H_1) *and* (H_2) *hold true, then*

$$\operatorname{Per}_{\frac{\ell}{p'}}(\Omega) \geq \operatorname{Per}_{\frac{\ell}{p'}}(\Omega^\sharp), \tag{1.8}$$

where $\Omega^\sharp$ is the centered ball with same ℓ-measure as Ω.

Remark 1.3.1. Since $|\Omega^\sharp|_\ell$ and $\operatorname{Per}_k(\Omega^\sharp)$ can be explicitly computed, is it possible to rewrite (1.8) in the following equivalent way

$$\operatorname{Per}_{\frac{\ell}{p'}}(\Omega) \geq \gamma_{n,\ell,p} \, |\Omega|_\ell^{\frac{\ell(p-1)+(n-1)p}{p(n+\ell)}}, \tag{1.9}$$

where

$$\gamma_{n,\ell,p} = (n\omega_n)^{\frac{\ell+p}{p(n+\ell)}} (\ell+n)^{\frac{\ell(p-1)+(n-1)p}{p(n+\ell)}},$$

and ω_n is the Lebesgue measure of the unit ball of $\mathbb{R}^n$.

Similarly to what was done in subsection 1.2, we introduce some notions starting from the measure of the superlevel set of a measurable function.

Let $u\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be a measurable function.

- The *weighted distribution function* of u, $\mu_\ell\colon [0,+\infty) \to [0,+\infty)$ defined as
$$\mu_\ell = \big|\{x \in \Omega\colon |u(x)| > t\}\big|_\ell.$$
- The *weighted decreasing rearrangement* of u, u^*, that is the distribution function of μ_ℓ;
- The *weighted rearrangement* of u, denoted with $u^\sharp$, is the function whose superlevel sets are balls with the same ℓ-measure as the superlevel sets of $|u|$; more precisely
$$u^\sharp(x) = u^*\Big(\big|B_{|x|}\big|_\ell\Big) = u^*\left(\frac{n\omega_n}{\ell+n}|x|^{\ell+n}\right),$$
where $B_{|x|}$ is the ball centered at the origin having radius $|x|$.
- If $1 \le p < +\infty$, we define the space $L^p(\Omega, |x|^\ell\, dx)$ as the space of all measurable functions such that the following quantity
$$\|u\|_{L^p(\Omega,|x|^\ell\, dx)} := \left(\int_\Omega |u|^p |x|^\ell\, dx\right)^{\frac{1}{p}},$$
is finite.
- If $0 < p < +\infty$, $0 < q \le +\infty$, we define the *weighted Lorentz space* $L^{p,q}(\Omega, |x|^\ell\, dx)$ as the space of all measurable functions such that the following quantity
$$\|u\|_{L^{p,q}(\Omega,|x|^\ell\, dx)} := \begin{cases} p^{\frac{1}{q}}\left(\displaystyle\int_0^{+\infty} t^q \mu_\ell(t)^{\frac{q}{p}}\, \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 0 < q < +\infty \\ \displaystyle\sup_{t\ge 0}\big(t^p \mu_\ell(t)\big) & \text{if } q = +\infty, \end{cases} \tag{1.10}$$
is finite. If $p = q$, by Cavalieri's principle, the space $L^{p,p}(\Omega, |x|^\ell\, dx)$ coincides with $L^p(\Omega, |x|^\ell\, dx)$, indeed it holds
$$\int_\Omega |g|^p |x|^\ell\, dx = p\int_0^{+\infty} t^{p-1}\mu(t)\, dt \qquad \forall p \ge 1.$$

If $u \in L^1(\Omega, |x|^\ell\, dx)$ is a non negative function, then we have the following Hardy-Littlewood inequality
$$\int_E u(x)|x|^\ell\, dx \le \int_0^{|E|_\ell} u^*(s)\, ds \qquad \forall E \subseteq \Omega. \tag{1.11}$$

Furthermore, the following Rellich-Kondrakov theorem holds.

Theorem 1.3.2. *If Ω is a bounded and Lipschitz domain of $\mathbb{R}^n$, $p \geq n$ and $-n < \ell < 0$ then $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega, |x|^\ell \, dx)$.*

1.4 Functions of Bounded Variation

Let us summarize some basic notions concerning BV functions. We refer for instance [AFP00; CF02; EG15] for more details.

Let Ω will be an open set of $\mathbb{R}^n$.

Definition 1.4.1. A function $u \in L^1(\Omega)$ is said to be a **function of bounded variation** in Ω if its distributional derivative is a Radon measure, i.e.

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, dx = \int_\Omega \varphi \, dD_i u \qquad \forall \varphi \in C_C^\infty(\Omega),$$

with Du a $\mathbb{R}^n$-valued measure in Ω. The total variation of Du will be denoted with $|Du|$.

The set of functions of bounded variation in Ω is denoted by $\mathrm{BV}(\Omega)$ and it is a Banach space with respect to the norm $\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega)$.

Definition 1.4.2. Let E be a Lebesgue-measurable set. The **perimeter** of E inside Ω is defined as

$$\mathrm{Per}(E, \Omega) := |D\chi_E|(\Omega),$$

and we say that E is a **set of finite perimeter** in Ω if $\chi_E \in \mathrm{BV}(\Omega)$. If $\Omega = \mathbb{R}^n$, we denote $\mathrm{Per}(E) := \mathrm{Per}(E, \mathbb{R}^n)$.

It is also worth mentioning the isoperimetric inequality for sets of finite perimeter.

Theorem 1.4.1 (Isoperimetric inequality)**.** *Let $E \subset \mathbb{R}^n$ be a bounded set of finite measure. Then it holds*

$$|E| \leq n^{-\frac{n}{n-1}} \omega_n^{-\frac{1}{n-1}} \left[\mathrm{Per}(E)\right]^{\frac{n}{n-1}}.$$

By the Lebesgue decomposition Theorem, each component of Du can be decomposed with respect to the Lebesgue measure $\mathcal{L}^n$, namely

$$D_i u = D_i^{\mathrm{a}} u + D_i^{\mathrm{s}} u \qquad \text{with} \qquad D_i^{\mathrm{a}} u \ll \mathcal{L}^n, \quad D_i^{\mathrm{s}} u \perp \mathcal{L}^n.$$

and

$$D_i^{\mathrm{a}} u = f_i \llcorner \mathcal{L}^n,$$

for some $f_i \in L^1(\Omega)$. So, defining

$$\frac{\partial u}{\partial x_i} := f_i, \qquad \nabla^{\mathrm{a}} u = \left(\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n} \right) \qquad \text{and } D^{\mathrm{s}} u = \big(D_1^{\mathrm{s}} u, \ldots, D_n^{\mathrm{s}} u \big),$$

we can write

$$dDu = \nabla^{\mathrm{a}} u \llcorner \mathcal{L}^n + dD^{\mathrm{s}} u.$$

Clearly it holds

$$|Du|(A) = |D^{\mathrm{a}} u|(A) + |D^{\mathrm{s}} u|(A) = \int_A |\nabla^{\mathrm{a}} u| \, dx + |D^{\mathrm{s}} u|(A),$$

for every Borel set $A \subseteq \Omega$.

Let us recall the **Fleming-Rishel formula** (see [FR60] or [EG15]):

Theorem 1.4.2 (Fleming-Rishel formula)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $f \in BV(\Omega)$, then for almost every $t \in (-\infty, +\infty)$ the set $\{f > t\}$ has finite perimeter in Ω and it holds*

$$|Df|(\Omega) = \int_{-\infty}^{+\infty} \mathrm{Per}(\{f > t\}, \Omega) \, dt. \tag{1.12}$$

Moreover if $f \in L^1(\Omega)$ and

$$\int_{-\infty}^{+\infty} \mathrm{Per}(\{f > t\}, \Omega) \, dt < +\infty,$$

then $f \in BV(\Omega)$ and consequently (1.12) holds.

Furthermore if f is Lipschitz and $u \colon \Omega \to \mathbb{R}$ is a measurable function, then it holds

$$\int_\Omega u(x) |\nabla f(x)| \, dx = \int_{\mathbb{R}} dt \int_{(\Omega \cap f^{-1}(t))} u(y) \, d\mathcal{H}^{n-1}(y). \tag{1.13}$$

Formula (1.13) is known in literature as **coarea formula**. We refer to [FR60; EG15] and [AFP00; Mag12] for the proofs of (1.12) and (1.13) respectively.

1.5 A Gronwall's Lemma

Lemma 1.5.1 (Gronwall). *Let $\tau_0 \in (0,+\infty)$, let $\xi(t) : [\tau_0,+\infty[\to \mathbb{R}$ be a continuous and differentiable function satisfying, for some non negative constant C, the following differential inequality*

$$\tau\xi'(\tau) \leq (p-1)\xi(\tau) + C \qquad \forall \tau \geq \tau_0 > 0.$$

Then we have

(i) $\xi(\tau) \leq \left(\xi(\tau_0) + \dfrac{C}{p-1}\right)\left(\dfrac{\tau}{\tau_0}\right)^{p-1} - \dfrac{C}{p-1} \quad \forall \tau \geq \tau_0$*;*

(ii) $\xi'(\tau) \leq \left(\dfrac{(p-1)\xi(\tau_0)+C}{\tau_0}\right)\left(\dfrac{\tau}{\tau_0}\right)^{p-2} \quad \forall \tau \geq \tau_0$*.*

Proof. Dividing both sides of the differential inequality by τ^p, we obtain

$$\left(\frac{\xi'(\tau)}{\tau^{p-1}} - (p-1)\frac{\xi(\tau)}{\tau^p}\right) = \left(\frac{\xi(\tau)}{\tau^{p-1}}\right)' \leq \frac{C}{\tau^p}.$$

Now, we integrate from τ_0 to τ and we obtain

$$\begin{aligned}\int_{\tau_0}^{\tau}\left(\frac{\xi(\tau)}{\tau^{p-1}}\right)' d\tau \leq \int_{\tau_0}^{\tau}\frac{C}{\tau^p}\,d\tau \\ \implies \xi(\tau) \leq \left(\xi(\tau_0)+\frac{C}{p-1}\right)\left(\frac{\tau}{\tau_0}\right)^{p-1} - \frac{C}{p-1},\end{aligned}$$

which gives *(i)*.
In order to obtain *(ii)*, we just take into account *(i)* in the differential inequality. □

1.6 Convex geometry

Here we recall some basic facts about convex sets. For more details we refer to [GW93; Sch13].

First of all, we say that Ω is convex if for any $x, y \in \Omega$ and $t \in [0,1]$, it holds

$$tx + (1-t)y \in \Omega.$$

Because of the convexity assumption, $\partial\Omega$ is locally the graph of a concave function, so Ω has Lipschitz boundary and it holds $\mathrm{Per}(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$.

We will denote by $\mathrm{diam}(\Omega)$ the following quantity

$$\mathrm{diam}(\Omega) = \sup_{x,y\in\Omega} |x-y|.$$

Definition 1.6.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open and convex set. The **support function** of Ω is defined as

$$g_\Omega(z) = \sup_{w\in\Omega} \langle w, z\rangle \qquad z \in \mathbb{R}^n \,. \tag{1.14}$$

When restricted to the circle $\mathbb{S}^{n-1}$, the value of the support function of a convex set in a point z represents the distance between the origin and the support hyperplane having z as outer normal. If the origin belongs to Ω, then g_Ω is non-negative and $g_\Omega(z) \le \mathrm{diam}(\Omega)$.

Given a bounded convex set $\Omega \subseteq \mathbb{R}^n$, is it possibile to define its **John ellipsoid** (see [Guz75; Joh14]) E such that

$$E \subseteq \Omega \subseteq nE.$$

Let us recall the definition of ε-cone property.

Definition 1.6.2. Let x be a point in $\mathbb{R}^n$, $\nu \in \mathbb{S}^{n-1}$ and $\varepsilon > 0$. We define the cone with vertex x, direction ν and dimension ε the following set

$$C(x,\nu,\varepsilon) = \{\, y \in \mathbb{R}^n \colon \langle x-y, \xi\rangle \ge \cos(\varepsilon)|x-y|, \text{ and } |x-y| < \varepsilon \,\}\,. \tag{1.15}$$

Definition 1.6.3. We say that an open set $\Omega \subset \mathbb{R}^n$ satisfies the **uniform ε-cone property** if

$$\forall z \in \partial\Omega\ \exists \nu_z \in \mathbb{S}^{n-1} \text{ such that } \forall x \in B(z,\varepsilon)\cap\overline{\Omega}, \text{ it holds } C(x,\nu_z,\varepsilon)\setminus\{x\} \subset \Omega.$$

1.6.1 Brunn-Minkowski theorem and profile function

An important property of convex sets is contained in the following theorem, see [GW93; Sch13].

Theorem 1.6.1 (Brunn-Minkowski inequality). *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ two compact convex sets with non empty interior, then for each $0 < \lambda < 1$ it holds*

$$\mathcal{L}^n\left(\lambda\Omega_1 + (1-\lambda)\Omega_2\right)^{\frac{1}{n}} \ge \lambda\mathcal{L}^n(\Omega_1)^{\frac{1}{n}} + (1-\lambda)\mathcal{L}^n(\Omega_2)^{\frac{1}{n}}, \tag{1.16}$$

where $\cdot + \cdot$ denotes the Minkowski sum among two sets.

Moreover equality in (1.16) holds if and only if Ω_1 and Ω_2 are homothetic.

Let Ω be a open, bounded and convex set in $\mathbb{R}^n$, $1 \le j \le n-1$ and let us define for each $x \in \mathbb{R}^j$

$$S_x = \{x' \in \mathbb{R}^{n-j} : (x, x') \in \Omega\}.$$

Let $\omega \subseteq \mathbb{R}^n$ be the projection of Ω onto the first j variables, so we can define

$$p_j(x) = \mathcal{H}^{n-j}\big(S_x \cap \Omega\big) \qquad \forall x \in \omega.$$

In case $j = 1$, the function p is usually referred as the **profile function** of Ω.

We will need the following definition.

Definition 1.6.4. Let p be a non negative bounded function, we say that p is **α-concave** if $\alpha > 0$ is the largest number for which p^α is concave (note that if p^α is concave for some $\alpha > 0$, then p^{α_1} is concave for any $0 \le \alpha_1 \le \alpha$).

As consequence of the Brunn-Minkowski inequality 1.6.1, we have the following corollary.

Corollary 1.6.2. *Let $\Omega \subset \mathbb{R}^n$ be a convex set and p_j defined as above. Then p_j is $\frac{1}{n-j}$-concave.*

1.6.2 Hausdorff distance

Now we introduce the Hausdorff distance. For more details we refer to [HP18, Section 2.2.3].

If E, F are any two convex sets in $\mathbb{R}^n$ the Hausdorff distance between E and F is defined as

$$d_{\mathcal{H}}(E, F) := \inf\left\{\varepsilon > 0 \,:\, E \subset F + \varepsilon B,\, F \subset E + \varepsilon B\right\},$$

where B is the unitary ball centered at the origin. If a sequence $\{\Omega_h\}$ converges in the sense of Hausdorff to a set Ω, we will write $\Omega_h \xrightarrow{\mathcal{H}} \Omega$.

We recall the following compactness of the Hausdorff distance ([HP18, Theorem 2.2.25]).

Theorem 1.6.3. *Let $\{\Omega_k\}$ be a sequence of convex sets contained in a fixed compact set $K \subset \mathbb{R}^n$. Then there exists a convex set Ω contained in K and a subsequence, still denoted with $\{\Omega_k\}$, that converges in the sense of Hausdorff to Ω.*

It is possible to rephrase the Hausdorff convergence in terms of support functions (see [Sch13]).

Proposition 1.6.4. *Let $\{\Omega_k\}$ be a sequence of convex sets containing the origin that converges in the sense of Hausdorff to a convex set Ω. Denote with g_{Ω_k} and g_Ω the corresponding support function, then*

$$\|g_{\Omega_k} - g_\Omega\|_{L^\infty} \to 0 \qquad \textit{(and conversely).}$$

1.6.3 Quermassintegrals

Let us recall some basic properties of quermassintegrals, for more details we refer to [Sch13]. Let $K \subset \mathbb{R}^n$ be a non-empty, bounded, convex set, let B be the unitary ball centered at the origin and $\rho > 0$. We can write the Steiner formula for the Minkowski sum $K + \rho B$ as

$$|K + \rho B| = \sum_{i=0}^{n} \binom{n}{i} W_i(K)\rho^i. \tag{1.17}$$

The coefficients $W_i(K)$ are known in the literature as **quermassintegrals** of K. In particular, $W_0(K) = |K|, nW_1(K) = \mathrm{Per}(K)$ and $W_n(K) = \omega_n$ where ω_n is the measure of B.

Incidentally let us mention that if K has C^2 boundary, the quermassintegrals can be written in terms of principal curvatures of K.

Formula (1.17) can be generalized also to the other quermassintegrals, obtaining

$$W_j(K + \rho B) = \sum_{i=0}^{n-j} \binom{n-j}{i} W_{j+i}(K)\rho^i \qquad j = 0, \ldots, n-1.$$

For $j = 1$ we have

$$\begin{aligned}\mathrm{Per}(K + \rho B_1) &= n \sum_{i=0}^{n-1} \binom{n-1}{i} W_{i+1}(K)\rho^i \\ &= \mathrm{Per}(K) + n(n-1)W_2(K)\rho + \ldots + nW_n(K)\rho^n,\end{aligned}$$

from which follows

$$\lim_{\rho\to 0} \frac{\mathrm{Per}(K + \rho B) - \mathrm{Per}(K)}{\rho} = n(n-1)W_2(K). \tag{1.18}$$

Furthermore, Aleksandrov-Fenchel inequalities hold true

$$\left(\frac{W_j(K)}{\omega_n}\right)^{\frac{1}{n-j}} \geq \left(\frac{W_i(K)}{\omega_n}\right)^{\frac{1}{n-i}} \qquad 0 \leq i < j \leq n-1, \tag{1.19}$$

where equality holds if and only if K is a ball. When $i = 0$ and $j = 1$, formula (1.19) reduces to the classical isoperimetric inequality, i.e.

$$\mathrm{Per}(K) \geq n\omega_n^{\frac{1}{n}} |K|^{\frac{n-1}{n}}.$$

In the following, we will use (1.19) for $i = 1$ and $j = 2$, that is

$$W_2(K) \geq n^{-\frac{n-2}{n-1}} \omega_n^{\frac{1}{n-1}} \mathrm{Per}(K)^{\frac{n-2}{n-1}}. \tag{1.20}$$

1.6.4 Inner parallel sets

Let $\Omega \subset \mathbb{R}^n$ be a convex set, for $t \in [0, r_\Omega]$ we denote by

$$\Omega_t = \{x \in \Omega : d(x) > t\},$$

where $d(x)$ is the distance of $x \in \Omega$ from the boundary of Ω and r_Ω is the inradius of Ω, that is to say, the radius of the biggest ball contained in Ω. By the Brunn-Minkowski Theorem 1.6.1 and the concavity of the distance function, the map

$$t \mapsto \mathrm{Per}(\Omega_t)^{\frac{1}{n-1}},$$

is concave in $[0, r_\Omega]$, hence absolutely continuous in $(0, r_\Omega)$. Moreover, there exists its right derivative at 0 and it is negative, since $\mathrm{Per}(\Omega_t)^{\frac{1}{n-1}}$ is strictly monotone decreasing, hence almost everywhere differentiable. The following two results are contained in [BNT10], we report them here for the reader's convenience.

Lemma 1.6.5. *Let Ω be a bounded, convex, open set in $\mathbb{R}^n$. Then for almost every $t \in (0, r_\Omega)$*

$$-\frac{d}{dt}\mathrm{Per}(\Omega_t) \geq n(n-1)W_2(\Omega_t), \tag{1.21}$$

and equality holds if Ω is a ball.

Proof. For every $s \in (0, t)$ it holds

$$\Omega_t + sB_1 \subset \Omega_{t-s},$$

and if Ω is a ball, the two sets coincide. The monotonicity of the perimeter with respect to the inclusion of convex sets and formula (1.18) give, for almost every $t \in (0, r_\Omega)$,

$$\begin{aligned}
-\frac{d}{dt}\mathrm{Per}(\Omega_t) &= \lim_{s\to 0^+} \frac{\mathrm{Per}(\Omega_{t-s}) - \mathrm{Per}(\Omega_t)}{s} \\
&\geq \lim_{s\to 0^+} \frac{\mathrm{Per}(\Omega_t + sB_1) - \mathrm{Per}(\Omega_t)}{s} = n(n-1)W_2(\Omega_t).
\end{aligned}$$

□

Combining the chain rule, the previous lemma and the fact that $|\nabla d(x)| = 1$ almost everywhere, we obtain

Lemma 1.6.6. *Let $f\colon [0, +\infty) \to [0, +\infty)$ be a strictly increasing C^1 function with $f(0) = 0$. Set $u(x) = f(d(x))$ and*

$$E_t = \{x \in \Omega \,:\, u(x) > t\} = \Omega_{f^{-1}(t)},$$

then

$$-\frac{d}{dt}\mathrm{Per}(E_t) \geq (n-1)\frac{W_2(E_t)}{|\nabla u|_{u=t}}. \tag{1.22}$$

In particular, we observe that the map $t \mapsto \mathrm{Per}(E_t)$ is absolutely continuous since it is obtained by composing the map $t \mapsto f^{-1}(t)$ which is a C^1 function and the map $s \mapsto \mathrm{Per}(\Omega_s)$ which is absolutely continuous.

1.6.5 Asymmetry indices

In the following we will prove some quantitative results, so we need a geometric quantity that gives us information about the shape of Ω. We will consider the Hausdorff asymmetry index, as already done in [CL21].

For any measurable set E of finite perimeter, we define two isoperimetric deficits

$$\mathcal{D}(E) := \mathrm{Per}(E) - \mathrm{Per}(E^\sharp), \quad \mathcal{M}(E) := |E^\star| - |E|,$$

where $E^\sharp$ and $E^\star$ are the balls with the same measure and the same perimeter of E, respectively.

Moreover, we will consider the following Hausdorff asymmetry indices

$$\mathcal{A}^\star_{\mathcal{H}}(E) = \min_{x\in\mathbb{R}^n}\Big\{d_{\mathcal{H}}\big(E, B_r(x)\big),\ \mathrm{Per}(\Omega) = \mathrm{Per}\big(B_r(x)\big)\Big\}, \tag{1.23}$$

and

$$\mathcal{A}^\sharp_{\mathcal{H}}(E) = \min_{x\in\mathbb{R}^n}\Big\{d_{\mathcal{H}}\big(E, B_r(x)\big),\ |\Omega| = |B_r(x)|\Big\}. \tag{1.24}$$

Lemma 2.9 in [GLPT20] tells us how these two indices are related one to the other.

Lemma 1.6.7. *Let $n \geq 2$. There exists $\delta > 0$ such that for any $E \subset \mathbb{R}^n$ be a bounded, open and convex set with $\mathcal{D}(E) \leq \delta$, then*

$$\mathcal{A}^\star_H(E) \leq C(n)\mathcal{A}^\sharp_H(E).$$

With these definitions, we can recall the quantitative isoperimetric inequality proved in [Fug89; Fus15].

Theorem 1.6.8 (Fuglede)**.** *Let $n \geq 2$, and let E be a bounded open and convex set with $|E| = \omega_n$. There exists δ_F, C, depending only on n, such that if $\mathcal{D}(E) \leq \delta_F$ then*

$$\mathcal{D}(E) \geq C g\big(\mathcal{A}^{\sharp}_{\mathcal{H}}(E)\big), \tag{1.25}$$

where g is defined by

$$g(s) = \begin{cases} s^2 & \text{if } n = 2 \\ f^{-1}(s^2) & \text{if } n = 3 \\ s^{\frac{n+1}{2}} & \text{if } n \geq 4, \end{cases} \tag{1.26}$$

and $f(t) = \sqrt{t \log(\frac{1}{t})}$ for $0 < t < e^{-1}$.

We are interested in a modified version of this theorem, in terms of $\mathcal{M}(E)$, so we have the following lemma.

Lemma 1.6.9. *Let $E \subset \mathbb{R}^n$ be a bounded, open and convex set and let $E^\star$ be the ball satisfying $\mathrm{Per}(E) = \mathrm{Per}(E^\star) = \rho$. Then, there exist δ, C, depending only on n and ρ, such that, if*

$$\mathcal{M}(E) = |E^\star| - |E| \leq \delta, \tag{1.27}$$

then

$$\mathcal{M}(E) \geq C g\big(\mathcal{A}^{\star}_{\mathcal{H}}(E)\big),$$

where g is the function defined in (1.26).

Proof. Let us start by setting $\delta < \dfrac{|E^\star|}{2}$, so

$$|E| > \frac{|E^\star|}{2}. \tag{1.28}$$

Let us divide the proof in two steps. First we assume that $|E| = \omega_n$. By the differentiability of the function $h(t) = t^{\frac{n-1}{n}}$, there exists $\xi \in (|E|, |E^\star|)$ such that

$$|E^\star|^{\frac{n-1}{n}} - |E|^{\frac{n-1}{n}} = h'(\xi)\left(|E^\star| - |E|\right) \leq \frac{n-1}{n\omega_n^{\frac{1}{n}}}\left(|E^\star| - |E|\right). \tag{1.29}$$

Moreover,

$$|E^\star|^{\frac{n-1}{n}} - |E|^{\frac{n-1}{n}} = \frac{\mathrm{Per}(E)}{n\omega_n^{\frac{1}{n}}} - |E|^{\frac{n-1}{n}} = \frac{\mathrm{Per}(E)}{n\omega_n^{\frac{1}{n}}} - \frac{\mathrm{Per}(E^\sharp)}{n\omega_n^{\frac{1}{n}}} = \frac{\mathcal{D}(E)}{n\omega^{\frac{1}{n}}}. \quad (1.30)$$

Hence, by (1.29) and (1.30), we get

$$\mathcal{D}(E) \leq (n-1)\mathcal{M}(E).$$

So, choosing $\delta \leq \dfrac{\delta_F(n)}{n-1}$, where $\delta_F(n)$ is the one provided by Theorem 1.6.8, we can apply the aforementioned Theorem to write

$$\mathrm{Per}(E) \geq n\omega_n\Big(1 + \gamma(n) g\big(\mathcal{A}^\sharp_{\mathcal{H}}(E)\big)\Big),$$

obtaining

$$\begin{aligned} |E^\star| - |E| &= \frac{\mathrm{Per}(E)^{\frac{n}{n-1}}}{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}} - |E| \geq \omega_n \left(1 + \gamma(n) g\big(\mathcal{A}^\sharp_{\mathcal{H}}(E)\big)\right)^{\frac{n}{n-1}} - \omega_n \\ &\geq \frac{n\omega_n\gamma(n)}{n-1} g\big(\mathcal{A}^\sharp_{\mathcal{H}}(E)\big), \end{aligned}$$

where in the last step we used Bernoulli's inequality

$$(1+x)^r \geq 1 + rx \qquad x \geq -1,\, r \geq 1.$$

Applying Lemma 1.6.7, we eventually have

$$|\Omega^\star| - |\Omega| \geq C(n) g\big(\mathcal{A}^\star_{\mathcal{H}}(\Omega)\big).$$

Now we treat the general case. Let us rescale the set E as

$$E_1 = \left(\frac{\omega_n}{|E|}\right)^{\frac{1}{n}} E,$$

so we have

$$\begin{aligned} \mathcal{M}(E_1) = |E_1^\star| - |E_1| &= \frac{\omega_n}{|E|}|E^\star| - \omega_n \\ &= \frac{\omega_n}{|E|}\mathcal{M}(E) \leq \frac{2\omega_n}{|E^\star|}\mathcal{M}(E), \end{aligned}$$

where the last inequality follows from (1.28).

In order to apply the previous case to the set E_1, we need

$$\mathcal{M}(E_1) \leq \frac{\delta_F}{n-1},$$

which is guaranteed if we choose

$$\mathcal{M}(E) \leq \frac{|E^\star|}{2(n-1)\omega_n}\,\delta_F = \delta(n,\rho).$$

$\square$

Chapter 2

Talenti type comparison results for p-Poisson equation with Robin boundary conditions

Let β be a positive parameter and let Ω be a bounded open set of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. Let $f \in L^{p'}(\Omega)$ be a non-negative function. We consider the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

A function $u \in W^{1,p}(\Omega)$ is a weak solution to (2.1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx + \beta \int_{\partial\Omega} |u|^{p-2}u\varphi \, d\mathcal{H}^{n-1}(x) = \int_\Omega f\varphi \, dx \quad \forall \varphi \in W^{1,p}(\Omega). \tag{2.2}$$

In Section 2.1 we will present a comparison principle with the solution to the following symmetrized problem

$$\begin{cases} -\operatorname{div}(|\nabla v|^{p-2}\nabla v) = f^\sharp & \text{in } \Omega^\sharp \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \beta |v|^{p-2}v = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{2.3}$$

where $\Omega^\sharp$ is the ball centered in the origin with the same measure of Ω and $f^\sharp$ is the Schwarz rearrangement of f

In Section 2.2, we will consider the case of Robin variable parameter and we will compare the solution to

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f(x)|x|^\ell & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta(x)|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 \notin \partial\Omega$, with the solution to

$$\begin{cases} -\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) = f^\sharp(x)|x|^\ell & \text{in } \Omega^\sharp \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \tilde{\beta}(r^\sharp)^{\frac{\ell}{p'}}|v|^{p-2}v = 0 & \text{on } \partial\Omega^\sharp, \end{cases}$$

where

$$\widetilde{\beta} = \inf_{\partial\Omega} \beta(x)|x|^{-\frac{\ell}{p'}} > 0,$$

and p, n and ℓ are suitably related.

2.1 The case of constant Robin parameter

Let u be the solution to (2.1). For $t \geq 0$, we denote by

$$U_t = \{x \in \Omega : u(x) > t\} \quad \partial U_t^{int} = \partial U_t \cap \Omega, \quad \partial U_t^{ext} = \partial U_t \cap \partial\Omega,$$

and by

$$\mu(t) = |U_t| \quad P_u(t) = \operatorname{Per}(U_t).$$

If v is the solution to (2.3), using the same notations, we set

$$V_t = \left\{x \in \Omega^\sharp : v(x) > t\right\}, \quad \phi(t) = |V_t|, \quad P_v(t) = \operatorname{Per}(V_t).$$

Because of the invariance of the p-Laplacian and of the Schwarz rearrangement of f by rotation, there exists a radial solution to (2.3) and, by uniqueness of solutions, this solution is v.

Since v is radial, positive and decreasing along the radius then, for $0 \leq t \leq v_m$, $V_t = \Omega^\sharp$, while, for $v_m < t < \max_{\Omega^\sharp} v$, V_t is a ball, concentric to $\Omega^\sharp$ and strictly contained in it.

We observe that the solutions u and v to (2.1) and (2.3) respectively are both p-superharmonic and then, by the strong maximum principle in [Váz84], it follows that they achieve their minima on the boundary. Denoting by u_m and v_m the minimum of u and v respectively, thanks to the positiveness of β and the Robin boundary conditions, we have that $u_m \geq 0$ and $v_m \geq 0$. Hence u and v are strictly positive in the interior of Ω. Moreover we can observe that

$$u_m = \min_{\Omega} u \leq \min_{\Omega^\sharp} v = v_m, \tag{2.4}$$

indeed, if we consider

$$\begin{aligned} v_m^{p-1}\operatorname{Per}(\Omega^\sharp) &= \int_{\partial\Omega^\sharp} v(x)^{p-1}\, d\mathcal{H}^{n-1}(x) = \frac{1}{\beta}\int_{\Omega^\sharp} f^\sharp\, dx = \frac{1}{\beta}\int_{\Omega} f\, dx \\ &= \int_{\partial\Omega} u(x)^{p-1}\, d\mathcal{H}^{n-1}(x) \\ &\geq u_m^{p-1}\operatorname{Per}(\Omega) \geq |u|_m^{p-1}\operatorname{Per}(\Omega^\sharp). \end{aligned}$$

A consequence of (2.4) that will be used in what follows is that

$$\mu(t) \leq \phi(t) = |\Omega| \quad \forall t \leq v_m. \tag{2.5}$$

The following lemmas are the main ingredient for the comparison result we will show.

Lemma 2.1.1. *Let u and v be solutions to (2.1) and (2.3) respectively. Then for almost every $t > 0$ we have*

$$\gamma_n \mu(t)^{\left(1-\frac{1}{n}\right)\frac{p}{p-1}} \leq \left(\int_0^{\mu(t)} f^*(s)\, ds\right)^{\frac{1}{p-1}} \left(-\mu'(t) + \frac{1}{\beta^{\frac{1}{p-1}}}\int_{\partial U_t^{ext}} \frac{1}{u}\, d\mathcal{H}^{n-1}(x)\right), \tag{2.6}$$

where $\gamma_n = \left(n\omega_n^{1/n}\right)^{\frac{p}{p-1}}$.
And

$$\gamma_n \phi(t)^{\left(1-\frac{1}{n}\right)\frac{p}{p-1}} = \left(\int_0^{\phi(t)} f^*(s)\, ds\right)^{\frac{1}{p-1}} \left(-\phi'(t) + \frac{1}{\beta^{\frac{1}{p-1}}}\int_{\partial V_t^{ext}} \frac{1}{v}\, d\mathcal{H}^{n-1}(x)\right). \tag{2.7}$$

Proof. Let $t > 0$ e $h > 0$, we choose the test function

$$\varphi(x) = \begin{cases} 0 & \text{if } u < t \\ u - t & \text{if } t < u < t+h \\ h & \text{if } u > t+h. \end{cases}$$

Then,

$$\int_{U_t\setminus U_{t+h}} |\nabla u|^p\,dx + \beta h\int_{\partial U_{t+h}^{ext}} u^{p-1}\,d\mathcal{H}^{n-1}(x) + \beta\int_{\partial U_t^{ext}\setminus\partial U_{t+h}^{ext}} u^{p-1}(u-t)\,d\mathcal{H}^{n-1}(x)$$
$$= \int_{U_t\setminus U_{t+h}} f(u-t)\,dx + h\int_{U_{t+h}} f\,dx.$$

Dividing by h, using coarea formula and letting h go to 0, we have that for a. e. $t>0$

$$\int_{\partial U_t} g(x)\,d\mathcal{H}^{n-1}(x) = \int_{U_t} f\,dx,$$

where

$$g(x) = \begin{cases} |\nabla u|^{p-1} & \text{if } x\in\partial U_t^{int},\\ \beta u^{p-1} & \text{if } x\in\partial U_t^{ext}.\end{cases}$$

So, using the isoperimetric inequality, for a.e. $t>0$ we have

$$\begin{aligned}
n\omega_n^{\frac{1}{n}}\mu(t)^{\left(1-\frac{1}{n}\right)} &\le P_u(t) = \int_{\partial U_t} d\mathcal{H}^{n-1}(x)\\
&\le \left(\int_{\partial U_t} g\,d\mathcal{H}^{n-1}(x)\right)^{\frac{1}{p}}\left(\int_{\partial U_t}\frac{1}{g^{\frac{1}{p-1}}}\,d\mathcal{H}^{n-1}(x)\right)^{1-\frac{1}{p}}\\
&= \left(\int_{\partial U_t} g\,d\mathcal{H}^{n-1}(x)\right)^{\frac{1}{p}}\\
&\times\left(\int_{\partial U_t^{int}}\frac{1}{|\nabla u|}\,d\mathcal{H}^{n-1}(x) + \frac{1}{\beta^{\frac{1}{p-1}}}\int_{\partial U_t^{ext}}\frac{1}{u}\,d\mathcal{H}^{n-1}(x)\right)^{1-\frac{1}{p}}\\
&\le \left(\int_0^{\mu(t)} f^*(s)\,ds\right)^{\frac{1}{p}}\left(-\mu'(t) + \frac{1}{\beta^{\frac{1}{p-1}}}\int_{\partial U_t^{ext}}\frac{1}{u}\,d\mathcal{H}^{n-1}(x)\right)^{1-\frac{1}{p}},
\end{aligned}$$

for $t\in[0,\max_\Omega u)$.

Then (2.6) follows. We notice that if v is the solution to (2.3), than all the inequalities are verified as equalities, so we have (2.7). □

Lemma 2.1.2. *For all $\tau\ge v_m$ we have*

$$\int_0^\tau t^{p-1}\left(\int_{\partial U_t^{ext}}\frac{1}{u(x)}\,d\mathcal{H}^{n-1}(x)\right)dt \le \frac{1}{p\beta}\int_0^{|\Omega|} f^*(s)\,ds. \tag{2.8}$$

Moreover,

$$\int_0^\tau t^{p-1}\left(\int_{\partial V_t\cap\partial\Omega^\sharp}\frac{1}{v(x)}\,d\mathcal{H}^{n-1}(x)\right)dt=\frac{1}{p\beta}\int_0^{|\Omega|}f^*(s)\,ds. \tag{2.9}$$

Proof. If we integrate the quantity

$$t^{p-1}\left(\int_{\partial U_t^{ext}}\frac{1}{u(x)}\,d\mathcal{H}^{n-1}(x)\right),$$

from 0 to $+\infty$, by Fubini theorem, we obtain

$$\begin{aligned}\int_0^\infty \tau^{p-1}\left(\int_{\partial U_\tau^{ext}}\frac{1}{u(x)}\,d\mathcal{H}^{n-1}(x)\right)d\tau &= \int_{\partial\Omega}\left(\int_0^{u(x)}\frac{\tau^{p-1}}{u(x)}\,d\tau\right)d\mathcal{H}^{n-1}(x)\\ &=\frac{1}{p}\int_{\partial\Omega}u(x)^{p-1}\,d\mathcal{H}^{n-1}(x)\\ &=\frac{1}{p\beta}\int_0^{|\Omega|}f^*(s)\,ds,\end{aligned}$$

where the last equality follows from the fact that u solves (2.1).
Analogously

$$\int_0^\infty \tau^{p-1}\left(\int_{\partial V_\tau\cap\partial\Omega^\sharp}\frac{1}{v(x)}\,d\mathcal{H}^{n-1}(x)\right)d\tau=\frac{1}{p\beta}\int_0^{|\Omega|}f^*(s)\,ds.$$

Since u is positive, we obtain, $\forall t\geq 0$,

$$\int_0^t \tau^{p-1}\left(\int_{\partial U_\tau^{ext}}\frac{1}{u(x)}\,d\mathcal{H}^{n-1}(x)\right)d\tau\leq\frac{1}{p\beta}\int_0^{|\Omega|}f^*(s)\,ds,$$

on the other hand, since $\partial V_t\cap\partial\Omega^\sharp$ is empty for $t\geq v_m$, we have

$$\int_0^t \tau^{p-1}\left(\int_{\partial V_\tau\cap\partial\Omega^\sharp}\frac{1}{v(x)}\,d\mathcal{H}^{n-1}(x)\right)d\tau=\frac{1}{p\beta}\int_0^{|\Omega|}f^*(s)\,ds.$$

and the proof of lemma 2.1.2 is complete. □

Remark 2.1.1. It can be observed that, since $\partial V_t \cap \partial\Omega^\sharp$ is empty for $t \geq v_m$ and $\phi(t) = |\Omega|$ for $t \leq v_m$, for all $\delta > 0$ and for all t, we have

$$\begin{aligned}
\int_0^t \tau^{p-1}\phi(\tau)^\delta \left(\int_{\partial V_\tau \cap \partial\Omega^\sharp} \frac{1}{v(x)}\, d\mathcal{H}^{n-1}(x)\right) d\tau \\
&= \int_0^{v_m} \tau^{p-1}\phi(\tau)^\delta \left(\int_{\partial V_\tau \cap \partial\Omega^\sharp} \frac{1}{v(x)}\, d\mathcal{H}^{n-1}(x)\right) d\tau \\
&= \int_0^{+\infty} \tau^{p-1}\phi(\tau)^\delta \left(\int_{\partial V_\tau \cap \partial\Omega^\sharp} \frac{1}{v(x)}\, d\mathcal{H}^{n-1}(x)\right) d\tau \\
&= \frac{|\Omega|^\delta}{p\beta} \int_0^{|\Omega|} f^*(s)\, ds.
\end{aligned}$$

Now we are in position to prove the following comparison result.

Theorem 2.1.3. *Let u and v be the solutions to problem (2.1) and (2.3) respectively. Then we have*

$$\|u\|_{L^{k,1}(\Omega)} \leq \|v\|_{L^{k,1}(\Omega^\sharp)} \qquad \forall\, 0 < k \leq \frac{n(p-1)}{(n-1)p}, \tag{2.10}$$

$$\|u\|_{L^{pk,p}(\Omega)} \leq \|v\|_{L^{pk,p}(\Omega^\sharp)} \qquad \forall\, 0 < k \leq \frac{n(p-1)}{(n-2)p+n}. \tag{2.11}$$

Remark 2.1.2. We observe that from Theorem 2.1.3, we have that, if $p \geq n$

$$\|u\|_{L^1(\Omega)} \leq \|v\|_{L^1(\Omega^\sharp)} \qquad \text{and} \qquad \|u\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega^\sharp)}.$$

Proof of Theorem 2.1.3. Let $0 < k \leq \dfrac{n(p-1)}{p(n-1)}$, so $\delta = \dfrac{1}{k} - \dfrac{(n-1)p}{n(p-1)}$ is positive.
Multiplying (2.6) by $t^{p-1}\mu(t)^\delta$ and integrating from 0 to $\tau \geq v_m$, by the previous Lemma, we obtain

$$\begin{aligned}
\int_0^\tau \gamma_n t^{p-1}\mu(t)^{\frac{1}{k}}\, dt \leq \int_0^\tau \left(-\mu'(t)\right) t^{p-1}\mu(t)^\delta \left(\int_0^{\mu(t)} f^*(s)\, ds\right)^{\frac{1}{p-1}} dt \\
+ \frac{|\Omega|^\delta}{p\beta^{\frac{p}{p-1}}} \left(\int_0^{|\Omega|} f^*(s)\, ds\right)^{\frac{p}{p-1}}.
\end{aligned} \tag{2.12}$$

Setting $F(l) = \displaystyle\int_0^l \omega^\delta \left(\int_0^\omega f^*(s)\,ds\right)^{\frac{1}{p-1}} d\omega$, we can integrate by parts both sides of the last inequality, getting

$$\begin{aligned}
&\tau^{p-1}\left(\left(\int_0^\tau \gamma_n \mu(t)^{\frac{1}{k}}\,dt\right) + F(\mu(\tau))\right) \\
&\qquad\qquad \le (p-1)\int_0^\tau t^{p-2}\left(\left(\int_0^t \gamma_n \mu(s)^{\frac{1}{k}}\,ds\right) + F(\mu(t))\right) dt \\
&\qquad\qquad + \frac{|\Omega|^\delta}{p\beta^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^{\frac{p}{p-1}}.
\end{aligned}$$

Setting

$$\xi(\tau) := \int_0^\tau t^{p-2}\left(\int_0^t \gamma_n \mu(s)^{\frac{1}{k}}\,ds + F(\mu(t))\right) dt,\ C := \frac{|\Omega|^\delta}{p\beta^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\,ds\right)^{\frac{p}{p-1}},$$

we are in the hypothesis of Lemma 1.5.1 (Gronwall), namely

$$\tau\xi'(\tau) \le (p-1)\xi(\tau) + C,$$

so, choosing $\tau_0 = v_m$, we have

$$\tau^{p-2}\left(\int_0^\tau \gamma_n \mu(s)^{\frac{1}{k}}\,ds + F(\mu(\tau))\right) \le \left(\frac{(p-1)\xi(v_m) + C}{v_m}\right)\left(\frac{\tau}{v_m}\right)^{p-2},$$

where

$$\xi(v_m) = \int_0^{v_m} t^{p-2}\left(\int_0^t \gamma_n \mu(s)^{\frac{1}{k}}\,ds + F(\mu(t))\right) dt.$$

The previous inequality becomes an equality if we replace $\mu(t)$ with $\phi(t)$. Since $\mu(t) \le \phi(t) = |\Omega|, \quad \forall t \le v_m$, and $F(l)$ is monotone, we obtain

$$\int_0^{v_m} t^{p-2}\left(\int_0^t \gamma_n \mu(s)^{\frac{1}{k}}\,ds + F(\mu(t))\right) dt \le \int_0^{v_m} t^{p-2}\left(\int_0^t \gamma_n \phi(s)^{\frac{1}{k}}\,ds + F(\phi(t))\right) dt,$$

hence

$$\int_0^\tau \gamma_n \mu(s)^{\frac{1}{k}}\,ds + F(\mu(\tau)) \le \int_0^\tau \gamma_n \phi(s)^{\frac{1}{k}}\,ds + F(\phi(\tau)).$$

Passing to the limit as $\tau \to \infty$, we get

$$\int_0^\infty \mu(t)^{\frac{1}{k}}\,dt \le \int_0^\infty \phi(t)^{\frac{1}{k}}\,dt,$$

and hence

$$\|u\|_{L^{k,1}(\Omega)} \leq \|v\|_{L^{k,1}(\Omega^\sharp)} \qquad \forall\, 0 < k \leq \frac{n(p-1)}{p(n-1)}.$$

To prove the inequality (2.11), it is enough to show that

$$\int_0^\infty t^{p-1}\mu(t)^{\frac{1}{k}}\, dt \leq \int_0^\infty t^{p-1}\phi(t)^{\frac{1}{k}}\, dt. \tag{2.13}$$

Let us consider equation (2.12), let us integrate by parts the first term on the right-hand side from 0 to τ and then let us pass to the limit as $\tau \to \infty$, we have

$$\int_0^\infty \gamma_n t^{p-1}\mu(t)^{\frac{1}{k}}\, dt \leq (p-1)\int_0^\infty t^{p-2}F(\mu(t))\, dt + \frac{|\Omega|^\delta}{p\beta^{1+\frac{1}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\, ds\right)^{\frac{p}{p-1}}.$$

Therefore, if we show that

$$\int_0^\infty t^{p-2}F(\mu(t))\, dt \leq \int_0^\infty t^{p-2}F(\phi(t))\, dt, \tag{2.14}$$

we obtain the (2.13). To this aim, we multiply (2.6) by $t^{p-1}F(\mu(t))\mu(t)^{-\frac{(n-1)p}{n(p-1)}}$ and integrate.

First, we observe that, by the choice $k \leq \dfrac{n(p-1)}{(n-2)p+n}$, it follows that the function $h(l) = F(l)l^{-\frac{(n-1)p}{n(p-1)}}$ is non decreasing. Hence, we obtain

$$\begin{aligned}\int_0^\tau \gamma_n t^{p-1}F(\mu(t))\, dt &\leq \int_0^\tau \Big(-\mu'(t)\Big)t^{p-1}\mu(t)^{-\frac{(n-1)p}{n(p-1)}}F(\mu(t))\left(\int_0^{\mu(t)} f^*(s)\, ds\right)^{\frac{1}{p-1}} dt\\ &\quad + F(|\Omega|)\frac{|\Omega|^{-\frac{(n-1)p}{n(p-1)}}}{p\beta^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\, ds\right)^{\frac{p}{p-1}}.\end{aligned}$$

If we integrate by parts both sides of the last expression and set

$$C = F(|\Omega|)\frac{|\Omega|^{-\frac{p(n-1)}{n(p-1)}}}{p\beta^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|} f^*(s)\, ds\right)^{\frac{p}{p-1}},$$

we obtain

$$\tau\int_0^\tau \gamma_n t^{p-2}F(\mu(t))\, dt + \tau H_\mu(\tau) \leq \int_0^\tau\int_0^t r^{p-2}F(\mu(r))\, dr dt + \int_0^\tau H_\mu(t)\, dt + C, \tag{2.15}$$

where

$$H_\mu(\tau) = -\int_\tau^{+\infty} t^{p-2}\mu(t)^{-\frac{p(n-1)}{n(p-1)}}F(\mu(t))\left(\int_0^{\mu(t)} f^*(s)\,ds\right)^{\frac{1}{p-1}} d\mu(t).$$

Setting

$$\xi(\tau) = \int_0^\tau \int_0^t \gamma_n r^{p-2}F(\mu(r))\,dr + \int_0^t H_\mu(t)\,dt,$$

then (2.15) becomes

$$\tau\xi'(\tau) \le \xi(\tau) + C.$$

So lemma 1.5.1, with $\tau_0 = v_m$, gives

$$\int_0^\tau \gamma_n t^{p-2}F(\mu(t))\,dt + H_\mu(\tau) \le \left(\frac{(p-1)\displaystyle\int_0^{v_m} t^{p-2}F(\mu(t)\,dt + H_\mu(v_m) + C}{v_m}\right)\left(\frac{\tau}{v_m}\right)^{p-2}.$$

Of course, the inequality holds as an equality if we replace $\mu(t)$ with $\phi(t)$, so we get, keeping in mind that $\mu(t) \le \phi(t) = |\Omega|$ for $t \le v_m$,

$$\int_0^\tau \gamma_n t^{p-2}F(\mu(t)\,dt + H_\mu(\tau) \le \int_0^\tau \gamma_n F(\phi(t))\,dt + H_\phi(\tau).$$

Letting $\tau \to \infty$, one has

$$\int_0^\infty t^{p-2}F(\mu(t))dt \le \int_0^\infty t^{p-2}F(\phi(t))dt,$$

as $H_\mu(\tau), H_\phi(\tau) \to 0$. This proves (2.14), and hence (2.11).
The fact that both H_μ and H_ϕ go to 0 as τ goes to infinity can be easily deduced distinguishing the cases.

- If $p \ge 2$, it holds

  $$t^{p-2}\mu(t) = \int_{u>t} t^{p-2}\,dx \le \int_{u>t} u^{p-2}\,dx \le \|u\|_{L^p}^{p-2}\mu(t)^{\frac{2}{p}},$$

  so we have

  $$\begin{aligned}|H_\mu(\tau)| &= \int_\tau^{+\infty} t^{p-2}F(\mu(t))\mu(t)^{-\frac{p(n-1)}{n(p-1)}}\left(\int_0^{\mu(t)} f^*(s)\,ds\right)(-\mu'(t))\,dt\\ &\le \left(\int_0^{|\Omega|} f^*(s)\,ds\right)\|u\|_{L^p}^{p-2}\int_\tau^{+\infty} F(\mu(t))\mu(t)^{\frac{2}{p}-\frac{p(n-1)}{n(p-1)}-1}(-\mu'(t))\,dt \to 0,\end{aligned}$$

  for $\tau \to +\infty$.

- If $p < 2$ we have

$$
\begin{aligned}
|H_\mu(\tau)| &= \int_\tau^{+\infty} t^{p-2} F(\mu(t))\mu(t)^{-\frac{p(n-1)}{n(p-1)}} \left(\int_0^{\mu(t)} f^*(s)\, s\right)(-\mu'(t))\, dt \\
&\le \tau^{p-2} \int_\tau^{+\infty} F(\mu(t))\mu(t)^{-\frac{p(n-1)}{n(p-1)}} \left(\int_0^{\mu(t)} f^*(s)\, s\right)(-\mu'(t))\, dt \to 0,
\end{aligned}
$$

for $\tau \to +\infty$.

Analogously for H_ϕ, which concludes the proof. $\square$

When the source term is constant, we obtain a stronger result, as stated in the following theorem.

Theorem 2.1.4. *Assume that $f \equiv 1$ and let u and v be the solutions to (2.1) and (2.3) respectively.*

(*i*) *If $1 \le p \le \dfrac{n}{n-1}$ then*

$$
u^\sharp(x) \le v(x) \qquad x \in \Omega^\sharp, \tag{2.16}
$$

(*ii*) *if $p > \dfrac{n}{n-1}$ and $0 < k \le \dfrac{n(p-1)}{n(p-1)-p}$, then*

$$
\begin{aligned}
\|u\|_{L^{k,1}(\Omega)} &\le \|v\|_{L^{k,1}(\Omega^\sharp)} \\
\|u\|_{L^{pk,p}(\Omega)} &\le \|v\|_{L^{pk,p}(\Omega^\sharp)}.
\end{aligned} \tag{2.17}
$$

Remark 2.1.3. We explicitly observe that in the case $f \equiv 1$ from Theorem 2.1.4 we have that

$$
\|u\|_{L^1(\Omega)} \le \|v\|_{L^1(\Omega^\sharp)} \quad \text{and} \quad \|u\|_{L^p(\Omega)} \le \|v\|_{L^p(\Omega^\sharp)} \qquad \text{for } p > 1.
$$

While we have the point-wise comparison only for $p \le \dfrac{n}{n-1}$.

Proof of Theorem 2.1.4.

(*i*) Firstly, we observe that $\displaystyle\int_0^{\mu(t)} f^*(s)\, ds = \mu(t)$, so (2.6) becomes

$$
\gamma_n \mu(t)^{\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}} \le -\mu'(t) + \frac{1}{\beta^{\frac{1}{p-1}}} \int_{\partial U_t^{\text{ext}}} \frac{1}{u}\, d\mathcal{H}^{n-1}(x). \tag{2.18}
$$

Let us multiply both sides by $t^{p-1}\mu(t)^\delta$, where $\delta = -\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}$. We point out that $\delta \geq 0$ for $p \leq \frac{n}{n-1}$. Hence, integrating from 0 to $\tau \geq v_m$, we have

$$\begin{aligned}\int_0^\tau \gamma_n t^{p-1}\,dt &\leq \int_0^\tau t^{p-1}\mu(t)^\delta(-\mu'(t))\,dt + \frac{1}{\beta^{\frac{1}{p-1}}}\int_0^\tau t^{p-1}\mu(t)^\delta \int_{\partial U_t^{\text{ext}}} \frac{1}{u}\,d\mathcal{H}^{n-1} \\ &\leq \int_0^\tau t^{p-1}\mu(t)^\delta(-\mu'(t))\,dt + \frac{|\Omega|^{\delta+1}}{p\beta^{\frac{p}{p-1}}}.\end{aligned} \tag{2.19}$$

Taking into account remark 2.1.1, if we replace $\mu(t)$ with $\phi(t)$ the previous inequality holds as equality.

Hence, we get

$$\int_0^\tau t^{p-1}\mu(t)^\delta(-\mu'(t))\,dt \geq \int_0^\tau t^{p-1}\phi(t)^\delta(-\phi'(t))\,dt.$$

Then an integration by parts gives

$$\begin{aligned}-\tau^{p-1}\frac{\mu(\tau)^{\delta+1}}{\delta+1} &+ (p-1)\int_0^\tau t^{p-2}\frac{\mu(t)^{\delta+1}}{\delta+1}\,dt \\ &\geq -\tau^{p-1}\frac{\phi(\tau)^{\delta+1}}{\delta+1} + (p-1)\int_0^\tau t^{p-2}\frac{\phi(t)^{\delta+1}}{\delta+1}\,dt.\end{aligned}$$

Finally, using Gronwall's Lemma with the function

$$\xi(\tau) = \int_0^\tau s^{p-2}\left(\frac{\mu(s)^{\delta+1}-\phi(s)^{\delta+1}}{\delta+1}\right)ds,$$

we obtain

$$\tau^{p-2}\left(\frac{\mu^{\delta+1}(\tau)-\phi^{\delta+1}(\tau)}{\delta+1}\right) \leq (p-1)\frac{\tau^{p-2}}{v_m^{p-2}}\int_0^{v_m} s^{p-2}\left(\frac{\mu^{\delta+1}(s)-\phi^{\delta+1}(s)}{\delta+1}\right)ds.$$

The quantity on the right-hand side is non-positive, thanks to (2.4), so

$$\mu(\tau) \leq \phi(\tau) \quad \forall \tau \geq v_m.$$

and, remembering that

$$\mu(\tau) \leq \phi(\tau) = |\Omega| \quad \forall \tau \leq v_m,$$

we get the point-wise inequality of the functions.

(*ii*) Now we want to show that

$$\|u\|_{L^{k,1}(\Omega)} \leq \|v\|_{L^{k,1}(\Omega^\sharp)},$$

so it is enough to show

$$\int_0^{+\infty} \mu(t)^{\frac{1}{k}}\, dt \leq \int_0^{+\infty} \phi(t)^{\frac{1}{k}}\, dt. \tag{2.20}$$

We multiply (2.18) by $t^{p-1}\mu(t)^{\frac{1}{k}-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}$ and integrate from 0 to $\tau \geq v_m$. Then using Lemma 2.1.2 and Remark 2.1.1, we obtain

$$\begin{aligned} \int_0^\tau \gamma_n t^{p-1}\mu(t)^{\frac{1}{k}}\, dt \leq & \int_0^\tau t^{p-1}\mu(t)^{\frac{1}{k}-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}(-\mu'(t))\, dt \\ & + \frac{|\Omega|^{\frac{1}{k}-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}+1}}{p\beta^{\frac{p}{p-1}}}, \end{aligned} \tag{2.21}$$

and equality holds if we replace μ with ϕ. In order to be shorter, we set

$$\eta = \frac{1}{k} - \left(1 - \frac{1}{n} - \frac{1}{p}\right)\frac{p}{p-1}, \qquad C = \frac{|\Omega|^{\eta+1}}{p\beta^{\frac{p}{p-1}}}.$$

We point out that (2.21) follows by (2.19) if $\eta \geq 0$, namely

$$0 < k \leq \frac{n(p-1)}{n(p-1)-p}.$$

With these notations and keeping in mind that μ is a non increasing function, we have from (2.21) taht

$$\int_0^\tau \gamma_n t^{p-1}\mu(t)^{\frac{1}{k}}\, dt \leq \int_0^\tau -t^{p-1}\mu(t)^{\eta}\, d\mu(t) + C. \tag{2.22}$$

Let us set $G(\ell) = \displaystyle\int_0^\ell w^\eta\, dw = \frac{\ell^{\eta+1}}{\eta+1}$, let us integrate by parts both sides of (2.22) in order to obtain

$$\begin{aligned} \gamma_n \tau^{p-1}\int_0^\tau \mu(t)^{\frac{1}{k}}\, dt + \tau^{p-1}G(\mu(\tau)) \leq (p-1)\Bigg[& \int_0^\tau \gamma_n t^{p-2}\int_0^t \mu(r)^{\frac{1}{k}}\, drdt \\ & + \int_0^\tau t^{p-2}G(\mu(t))\, dt\Bigg] + C. \end{aligned} \tag{2.23}$$

Setting

$$\xi(\tau) = \int_0^\tau \left(\gamma_n t^{p-2} \int_0^t \mu(r)^{\frac{1}{k}} \, dr \right) dt + \int_0^\tau t^{p-2} G(\mu(t)) \, dt,$$

(2.23) reads as follows

$$\tau \xi'(\tau) \le (p-1)\xi(\tau) + C.$$

Hence, using Gronwall's Lemma 1.5.1 with $\tau_0 = v_m$, we get

$$\gamma_n \tau^{p-2} \int_0^\tau \mu(t)^{\frac{1}{k}} \, dt + \tau^{p-2} G(\mu(\tau)) \le \left(\frac{(p-1)\xi(v_m) + C}{v_m} \right) \left(\frac{\tau}{v_m} \right)^{p-2},$$

where

$$\xi(v_m) = \int_0^{v_m} \gamma_n t^{p-2} \int_0^t \mu(r)^{\frac{1}{k}} \, dr \, dt + \int_0^{v_m} t^{p-2} G(\mu(t)) \, dt.$$

Again, if we replace μ with ϕ, the previous inequality holds as an equality and $\xi(v_m)$ is less or equal than the same quantity obtained by replacing μ with ϕ, as (2.4) holds. Keeping in mind (2.5), we have

$$\tau^{p-2} \left(\gamma_n \int_0^\tau \mu(t)^{\frac{1}{k}} \, dt + G(\mu(\tau)) \right) \le \tau^{p-2} \left(\gamma_n \int_0^\tau \phi(t)^{\frac{1}{k}} \, dt + G(\phi(\tau)) \right).$$

Passing to the limit as $\tau \to +\infty$, we get

$$\int_0^{+\infty} \mu(t)^{\frac{1}{k}} \, dt \le \int_0^{+\infty} \phi(t)^{\frac{1}{k}} \, dt,$$

namely (2.20).

To conclude the proof, we have to show that

$$\|u\|_{L^{pk,p}(\Omega)} \le \|v\|_{L^{pk,p}(\Omega^\sharp)} \qquad \forall \, 0 < k \le \frac{n(p-1)}{n(p-1)-p},$$

that is to say

$$\int_0^{+\infty} t^{p-1} \mu(t)^{\frac{1}{k}} \, dt \le \int_0^{+\infty} t^{p-1} \phi(t)^{\frac{1}{k}} \, dt.$$

We consider (2.22), pass to the limit as $\tau \to +\infty$ and integrate by parts the first term on the right-hand side

$$\int_0^{+\infty} \gamma_n t^{p-1} \mu(t)^{\frac{1}{k}} \, dt \le (p-1) \int_0^{+\infty} t^{p-2} G(\mu(t)) \, dt + C.$$

Hence it is enough to show that

$$\int_0^{+\infty} t^{p-2} G(\mu(t))\, dt \le \int_0^{+\infty} t^{p-2} G(\phi(t))\, dt.$$

To this aim, we multiply (2.18) by $t^{p-1} G(\mu(t)) \mu(t)^{-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}$ and integrate from 0 to $\tau \ge v_m$

$$\begin{aligned}\int_0^{\tau} \gamma_n t^{p-1} G(\mu(t))\, dt \le & \int_0^{\tau} t^{p-1} G(\mu(t)) \mu(t)^{-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}\, d\mu(t) \\ & + \frac{1}{\beta^{\frac{1}{p-1}}} \int_0^{\tau} t^{p-1} G(\mu(t)) \mu(t)^{-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}} \\ & \times \left(\int_{\partial U_t^{ext}} \frac{1}{u}\, d\mathcal{H}^{n-1} \right) dt.\end{aligned}$$

Since $k \le \dfrac{n(p-1)}{n(p-1)-p}$, using Lemma 2.1.2 and the fact that the function $G(\ell)\ell^{-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}$ is non decreasing, we obtain

$$\int_0^{\tau} \gamma_n t^{p-1} G(\mu(t))\, dt \le \int_0^{\tau} t^{p-1} G(\mu(t)) \mu(t)^{-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}\, d\mu(t) + C, \tag{2.24}$$

with

$$C = \frac{1}{p\beta^{\frac{p}{p-1}}} G\big(|\Omega|\big) |\Omega|^{-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}+1}.$$

If we replace μ with ϕ the previous inequality holds as an equality, thanks to (2.1.1). Now, let us integrate by parts both sides of (2.24), obtaining

$$\begin{aligned}\tau \int_0^{\tau} \gamma_n t^{p-2} G(\mu(t))\, dt + \tau H(\tau) \le & \int_0^{\tau} \int_0^{t} \gamma_n t^{p-2} G(\mu(r))\, dr dt \\ & + \int_0^{\tau} H_\mu(t)\, dt + C,\end{aligned} \tag{2.25}$$

where

$$H_\mu(\tau) = -\int_\tau^{+\infty} t^{p-2} G(\mu(t)) \mu(t)^{-\left(1-\frac{1}{n}-\frac{1}{p}\right)\frac{p}{p-1}}\, d\mu(t).$$

Setting

$$\xi(\tau) = \int_0^{\tau} \int_0^{t} \gamma_n t^{p-2} G(\mu(r))\, dr dt + \int_0^{\tau} H_\mu(t)\, dt,$$

the (2.25) reads as follows

$$\tau\xi'(\tau) \le \xi(\tau) + C.$$

Again, using Gronwall's Lemma 1.5.1, we get

$$\int_0^\tau \gamma_n t^{p-2} G(\mu(t))\, dt + H_\mu(\tau) \le \left(\frac{(p-1)\xi(v_m) + C}{v_m}\right)\left(\frac{\tau}{v_m}\right)^{p-2},$$

with

$$\xi(v_m) = \int_0^{v_m} \int_0^t \gamma_n t^{p-2} G(\mu(r))\, dr dt + \int_0^{v_m} H_\mu(t)\, dt.$$

Keeping in mind that for ϕ the previous inequalities hold as equality and the fact that G is not decreasing, $\xi(v_m)$ is less or equal to the same quantity obtained by replacing μ with ϕ. Hence, we obtain

$$\int_0^\tau \gamma_n t^{p-2} G(\mu(t))\, dt + H_\mu(\tau) \le \int_0^\tau \gamma_n t^{p-2} G(\phi(t))\, dt + H_\phi(\tau),$$

and passing to the limit as $\tau \to +\infty$, we finally get

$$\int_0^{+\infty} t^{p-2} G(\mu(t))\, dt \le \int_0^{+\infty} t^{p-2} G(\phi(t))\, dt,$$

indeed, as in the proof of Theorem 2.1.3, $H_\mu(\tau)$ and $H_\phi(\tau)$ go to 0 as $\tau \to \infty$. That concludes the proof. □

Corollary 2.1.5. *Let u and v be the solutions to (2.1) and (2.3) respectively. Then, if $p \ge n$, we have*

$$\|u\|_{L^1(\Omega)} \le \|v\|_{L^1(\Omega^\sharp)} \quad \textit{and} \quad \|u\|_{L^p(\Omega)} \le \|v\|_{L^p(\Omega^\sharp)}.$$

Moreover in the case $f \equiv 1$, Theorem 2.1.4 gives

$$\|u\|_{L^1(\Omega)} \le \|v\|_{L^1(\Omega^\sharp)} \quad \textit{and} \quad \|u\|_{L^p(\Omega)} \le \|v\|_{L^p(\Omega^\sharp)} \qquad \forall p > 1,$$

and the point-wise comparison for $p \le \dfrac{n}{n-1}$.

Proof. If $p \ge n$ the upper bounds of k, in both cases (2.10) e (2.11), are greater than 1 and so we can choose $k = 1$. The assertion follows from the fact that

$$\|\cdot\|_{L^{p,p}(\Omega)} = \|\cdot\|_{L^p(\Omega)}.$$

Analogously if $f \equiv 1$. □

As a consequence of theorem 2.1.3 we give a new proof of the Bossel-Daners inequality for the p-Laplacian when $p \geq n$.

We recall that the first eigenvalue of p-Laplace operator with Robin boundary conditions is obtained as the minimum of the Rayleigh quotients, i.e.,

$$\lambda_{1,\beta}(\Omega) = \min_{\substack{\omega \in W^{1,p}(\Omega) \\ \omega \neq 0}} \frac{\displaystyle\int_\Omega |\nabla \omega|^p \, dx + \beta \int_{\partial\Omega} |\omega|^p \, d\mathcal{H}^{n-1}(x)}{\displaystyle\int_\Omega |\omega|^p \, dx}. \tag{2.26}$$

We can observe that if u achieves the minimum of Rayleigh quotients, so does $|u|$. From this we have that u in non-negative. Furthermore, we have if $u_{\lambda_1} \geq 0$, as a consequence of Harnack inequality, $u_{\lambda_1} > 0$.

Another important thing is that the eigenvalue is simple. Indeed, as shown in [Lin92], if Ω is smooth enough and u and v are to eigenfunctions referred to the first eigenvalue, we can choose as test function $\varphi_1 = \dfrac{u^p - v^p}{u^{p-1}}$ in

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi_1 \, dx + \beta \int_{\partial\Omega} u^{p-1} \varphi_1 \, d\mathcal{H}^{n-1}(x) = \int_\Omega \lambda_1 u^{p-1} \varphi_1 \, dx,$$

and $\varphi_2 = \dfrac{v^p - u^p}{v^{p-1}}$ in

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \varphi_2 \, dx + \beta \int_{\partial\Omega} v^{p-1} \varphi_2 \, d\mathcal{H}^{n-1}(x) = \int_\Omega \lambda_1 v^{p-1} \varphi_2 \, dx.$$

Summing the two equations, we have

$$\begin{aligned} 0 = & \int_\Omega \left\{ 1 + (p-1) \left(\frac{v}{u} \right)^p \right\} |\nabla u|^p + \left\{ 1 + (p-1) \left(\frac{u}{v} \right)^p \right\} |\nabla v|^p \\ & - \int_\Omega p \left(\frac{v}{u} \right)^{p-1} |\nabla u|^{p-2} \nabla u \nabla v + p \left(\frac{u}{v} \right)^{p-1} |\nabla v|^{p-2} \nabla v \nabla u \\ = & \int_\Omega (u^p - v^p) \left(|\nabla \log u|^p - |\nabla \log v|^p \right) \\ & - \int_\Omega p v^p |\nabla \log u|^{p-2} |\nabla \log u| \left(\nabla \log v - \nabla \log u \right) \\ & - \int_\Omega p u^p |\nabla \log v|^{p-2} |\nabla \log v| \left(\nabla \log u - \nabla \log v \right). \end{aligned}$$

Now, using the well known inequalities, which hold true for each w_1 and $w_2 \in \mathbb{R}^n$,

$$
\begin{aligned}
&|w_2|^p \geq |w_1|^p + p|w_1|^{p-2}w_1 \cdot (w_2 - w_1) + \frac{|w_2 - w_1|^p}{2^{p-1}-1} && \text{if } p \geq 2 \\
&|w_2|^p \geq |w_1|^p + p|w_1|^{p-2}w_1 \cdot (w_2 - w_1) + C(p)\frac{|w_2 - w_1|^2}{(|w_1| + |w_2|)^{2-p}} && \text{if } 1 < p < 2,
\end{aligned}
\tag{2.27}
$$

if we consider the case $p \geq 2$, we choose $w_2 = \nabla \log u$ and $w_1 = \nabla \log v$, we obtain

$$
\frac{1}{2^{p-1}-1}\int_\Omega \left(\frac{1}{v^p} + \frac{1}{u^p}\right)|v\nabla u - u\nabla v|^p = 0.
$$

Hence, we obtain that $v\nabla u = u\nabla v$ a.e. in Ω, and so there exists a constant K for which $u = Kv$. This means that λ_1 is simple.
For the proof of (2.27), we refer to [Lin92].
The following corollary of Theorem 2.1.3 holds true, that is Faber-Krahn inequality.

Corollary 2.1.6. *Let u and v be the solutions to* (2.1) *and* (2.3)*, respectively. Then , if $p \geq n$, we have*

$$
\lambda_{1,\beta}(\Omega) \geq \lambda_{1,\beta}(\Omega^\sharp).
$$

Proof. Let u an eigenfunction referred to the first eigenvalue of (2.1), then it solves

$$
\begin{cases}
-\Delta_p u = \lambda_{1,\beta}(\Omega)\,|u|^{p-2}u & \text{in } \Omega \\
|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta|u|^{p-2}u = 0 & \text{on } \partial\Omega.
\end{cases}
$$

Now, let z be a solution to the following problem

$$
\begin{cases}
-\Delta_p z = \lambda_{1,\beta}(\Omega)\,|u^\sharp|^{p-2}u^\sharp & \text{in } \Omega^\sharp \\
|\nabla z|^{p-2}\dfrac{\partial z}{\partial \nu} + \beta|z|^{p-2}z = 0 & \text{on } \partial\Omega^\sharp.
\end{cases}
$$

In that case, corollary 2.1.5 gives

$$
\int_\Omega |u|^p\,dx = \int_{\Omega^\sharp} \left|u^\sharp\right|^p dx \leq \int_{\Omega^\sharp} |z|^p\,dx,
$$

and hence, by Hölder inequality

$$
\int_{\Omega^\sharp} (u^\sharp)^{p-2}u^\sharp z\,dx \leq \left(\int_{\Omega^\sharp} \left|u^\sharp\right|^p dx\right)^{\frac{p-1}{p}} \left(\int_{\Omega^\sharp} z^p\,dx\right)^{\frac{1}{p}} \leq \int_{\Omega^\sharp} z^p\,dx.
$$

Therefore, observing that we can write the eigenvalue $\lambda_{1,\beta}(\Omega)$ in the following way, we obtain

$$\lambda_{1,\beta}(\Omega) = \frac{\displaystyle\int_{\Omega^\sharp} |\nabla z|^p \, dx + \beta \int_{\partial\Omega^\sharp} z^p \, d\mathcal{H}^{n-1}(x)}{\displaystyle\int_{\Omega^\sharp} (u^\sharp)^{p-2} u^\sharp z \, dx}$$

$$\geq \frac{\displaystyle\int_{\Omega^\sharp} |\nabla z|^p \, dx + \beta \int_{\partial\Omega^\sharp} z^p \, d\mathcal{H}^{n-1}(x)}{\displaystyle\int_{\Omega^\sharp} z^p \, dx} \geq \lambda_{1,\beta}(\Omega^\sharp).$$

□

2.2 The case of variable Robin parameter

Now we consider a generalization of the previous problem with variable boundary conditions, namely

$$\begin{cases} -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = f(x)|x|^\ell & \text{in } \Omega \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} + \beta(x)|u|^{p-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.28}$$

where $n \geq 2$, ν denotes the outer unit normal to $\partial\Omega$ and Ω is a bounded and Lipschitz domain of $\mathbb{R}^n$ such that $0 \notin \partial\Omega$. We will also assume

$$p \geq n \tag{H_1}$$

$$-n < \ell < 0 \tag{H_2}$$

$$m = \inf_{\partial\Omega} \beta(x) > 0 \quad \text{and} \quad M = \sup_{\partial\Omega} \beta(x) < +\infty \tag{H_3}$$

$$0 \leq f(x) \in L^{p'}(\Omega, |x|^\ell \, dx), \tag{H_4}$$

where $p' = \frac{p}{p-1}$.

A weak solution to problem (2.28) is a function $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi \, dx + \int_{\partial\Omega} \beta(x) \, |u|^{p-2} \, u\varphi d\mathcal{H}^{n-1}(x) = \int_\Omega f \, |x|^\ell \, \varphi \, dx, \tag{2.29}$$

for any $\varphi \in W^{1,p}(\Omega)$.

We will compare the solution to (2.28) with the solution to

$$\begin{cases} -\mathrm{div}\big(|\nabla v|^{p-2}\nabla v\big) = f^\sharp(x)|x|^\ell & \text{in } \Omega^\sharp \\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial \nu} + \tilde{\beta}(r^\sharp)^{\frac{\ell}{p'}}|v|^{p-2}v = 0 & \text{on } \partial\Omega^\sharp, \end{cases} \tag{2.30}$$

where

$$\widetilde{\beta} = \inf_{\partial\Omega} \beta(x)|x|^{-\frac{\ell}{p'}} > 0, \tag{2.31}$$

since $0 \notin \partial\Omega$, H_3 holds and $r^\sharp$ is the radius of the ball $\Omega^\sharp$ having the same ℓ measure as Ω.

From the (2.31), it immediately follows that

$$\beta(x) \geq \widetilde{\beta}|x|^{\frac{\ell}{p'}}. \tag{2.32}$$

We underline that, due to proposition 1.3.2, problems (2.28) and (2.30) admits a unique solution.

Let u be the solution to (2.28). For $t \geq 0$, we define

$$U_t = \{x \in \Omega : u(x) > t\} \quad \partial U_t^{int} = \partial U_t \cap \Omega, \quad \partial U_t^{ext} = \partial U_t \cap \partial\Omega,$$

and

$$\mu(t) = |U_t|_\ell \quad P_u(t) = \mathrm{Per}_{\frac{\ell}{p'}}(U_t).$$

Analogously, if v is the solution to (2.30), we set

$$V_t = \{x \in \Omega^\sharp : v(x) > t\}, \quad \phi(t) = |V_t|_\ell, \quad P_v(t) = \mathrm{Per}_{\frac{\ell}{p'}}(V_t).$$

As it is easy to check $v(x) \equiv v^\sharp(x)$, therefore for $0 \leq t \leq v_m$, $V_t = \Omega^\sharp$, while, for $t > v_m$ V_t is a centered ball contained in $\Omega^\sharp$.

Furthermore we have

$$u_m = \inf_{\Omega} u \leq \min_{\overline{\Omega^\sharp}} v = v_m. \tag{2.33}$$

Indeed, using the fact that $v(x) = v_m$ $\forall x \in \partial\Omega^\sharp$, taking into account the fact that f and $f^\sharp$ are equirearranged and choosing $\psi \equiv 1$ in the weak formulation of (2.28) and (2.30), respectively, we get

$$\begin{aligned} v_m^{p-1} P_{\frac{\ell}{p'}}(\Omega^\sharp) &= \int_{\partial\Omega^\sharp} |x|^{\frac{\ell}{p'}} v(x)^{p-1}\, d\mathcal{H}^{n-1} = \frac{1}{\tilde{\beta}} \int_{\Omega^\sharp} f^\sharp |x|^\ell\, dx \\ &= \frac{1}{\tilde{\beta}} \int_\Omega f|x|^\ell\, dx = \frac{1}{\tilde{\beta}} \int_{\partial\Omega} \beta(x) u(x)^{p-1}\, d\mathcal{H}^{n-1}. \end{aligned}$$

In turn, by (2.32) and using the isoperimetric inequality (1.8), we get

$$\begin{aligned}\frac{1}{\tilde\beta}\int_{\partial\Omega}\beta(x)u(x)^{p-1}\,d\mathcal{H}^{n-1} &\geq u_m^{p-1}\int_{\partial\Omega}|x|^{\frac{\ell}{p'}}\,d\mathcal{H}^{n-1}\\ &= u_m^{p-1}P_{\frac{\ell}{p'}}(\Omega)\geq u_m^{p-1}P_{\frac{\ell}{p'}}(\Omega^\sharp).\end{aligned}$$

Gathering the inequalities above, we conclude that

$$v_m^{p-1}P_{\frac{\ell}{p'}}(\Omega^\sharp)\geq u_m^{p-1}P_{\frac{\ell}{p'}}(\Omega^\sharp),$$

and the claim follows.

Furthermore inequality (2.33) implies that

$$\mu(t)\leq\phi(t)=|\Omega|_\ell\quad\forall t\leq v_m. \tag{2.34}$$

As in the case of constant Robin parameter, the following lemmas are the main ingredients for the comparison results we will show.

Lemma 2.2.1. *Let u and v be the solutions to (2.28) and (2.30) respectively. Then, for almost every $t>0$, it holds*

$$\begin{aligned}\gamma_{n,\ell,p}\,\mu(t)^{\left(\frac{\ell(p-1)+p(n-1)}{p(\ell+n)}\right)\frac{p}{p-1}} \leq{}& \left(\int_0^{\mu(t)}f^*(s)\,ds\right)^{\frac{1}{p-1}}\\ &\times\left(-\mu'(t)+\frac{1}{\tilde\beta^{\frac{1}{p-1}}}\int_{\partial U_t^{ext}}\frac{|x|^{\frac{\ell}{p'}}}{u}\,d\mathcal{H}^{n-1}\right),\end{aligned}\tag{2.35}$$

where

$$\gamma_{n,\ell,p}=(n\omega_n)^{\frac{\ell+p}{p(n+\ell)}}(\ell+n)^{\frac{(p-1)\ell+(n-1)p}{p(n+\ell)}}$$

and

$$\begin{aligned}\gamma_{n,\ell,p}\,\phi(t)^{\left(\frac{\ell(p-1)+p(n-1)}{p(\ell+n)}\right)\frac{p}{p-1}} ={}& \left(\int_0^{\phi(t)}f^*(s)\,ds\right)^{\frac{1}{p-1}}\\ &\times\left(-\phi'(t)+\frac{1}{\tilde\beta^{\frac{1}{p-1}}}\int_{\partial V_t^{ext}}\frac{|x|^{\frac{\ell}{p'}}}{v}\,d\mathcal{H}^{n-1}\right).\end{aligned}\tag{2.36}$$

Proof. Let $t>0$ e $h>0$, we use the following test function in the variational form of

$$\varphi(x)=\begin{cases}0 & \text{if } u<t\\ u-t & \text{if } t<u<t+h\\ h & \text{if } u>t+h.\end{cases}$$

We get

$$\begin{aligned}\int_{U_t\setminus U_{t+h}} |\nabla u|^p\,dx + h\int_{\partial U_{t+h}^{ext}} \beta(x)u^{p-1}\,d\mathcal{H}^{n-1}\\ + \int_{\partial U_t^{ext}\setminus\partial U_{t+h}^{ext}} \beta(x)u^{p-1}(u-t)\,d\mathcal{H}^{n-1}\\ = \int_{U_t\setminus U_{t+h}} f|x|^\ell(u-t)\,dx + h\int_{U_{t+h}} f|x|^\ell\,dx.\end{aligned}$$

Dividing by h, using coarea formula and letting h go to 0, we have for a.e. $t>0$

$$\int_{\partial U_t} g(x)\,d\mathcal{H}^{n-1} = \int_{U_t} f|x|^\ell\,dx,$$

with

$$g(x)=\begin{cases}|\nabla u|^{p-1} & \text{if } x\in\partial U_t^{int}\\ \beta(x)u^{p-1} & \text{if } x\in\partial U_t^{ext}.\end{cases} \tag{2.37}$$

Using isoperimetric inequality (1.9) and Hölder inequality, for a.e. $t>0$, we obtain

$$\begin{aligned}\gamma_{n,\ell,p}\mu(t)^{\left(\frac{\ell(p-1)+p(n-1)}{p(\ell+n)}\right)} &\le P_u(t) = \int_{\partial U_t} |x|^{\frac{\ell}{p'}}\,d\mathcal{H}^{n-1}\\ &\le \left(\int_{\partial U_t} g\,d\mathcal{H}^{n-1}\right)^{\frac1p}\left(\int_{\partial U_t}\frac{|x|^\ell}{g^{\frac{1}{p-1}}}\,d\mathcal{H}^{n-1}\right)^{1-\frac1p}.\end{aligned} \tag{2.38}$$

Since, for a.e. $t>0$ it holds that

$$-\mu'(t) = \int_{\partial U_t^{int}} \frac{|x|^\ell}{|\nabla u|}\,d\mathcal{H}^{n-1},$$

by (2.32), (1.11), (2.37) and (2.38), we get

$$\begin{aligned}\gamma_{n,\ell,p}\mu(t)^{\left(\frac{\ell(p-1)+p(n-1)}{p(\ell+n)}\right)} &\le \left(\int_{\partial U_t} g\,d\mathcal{H}^{n-1}\right)^{\frac1p}\\ &\times\left(\int_{\partial U_t^{int}}\frac{|x|^\ell}{|\nabla u|}\,d\mathcal{H}^{n-1} + \frac{1}{\tilde\beta^{\frac{1}{p-1}}}\int_{\partial U_t^{ext}}\frac{|x|^{\frac{\ell}{p'}}}{u}\,d\mathcal{H}^{n-1}\right)^{\frac{1}{p'}}\\ &\le\left(\int_0^{\mu(t)} f^*(s)\,ds\right)^{\frac1p}\left(-\mu'(t)+\frac{1}{\tilde\beta^{\frac{1}{p-1}}}\int_{\partial U_t^{ext}}\frac{|x|^{\frac{\ell}{p'}}}{u}\,d\mathcal{H}^{n-1}\right)^{\frac{1}{p'}},\end{aligned}$$

for a.e. $t \in [0, \sup_\Omega u)$.

Then (2.35) follows. Replacing u with v, the solution to (2.30), all the inequalities are verified as equalities, so we obtain (2.36). $\square$

Lemma 2.2.2. *Let u and v be the solutions to (2.28) and (2.30) respectively. Then for all $\tau \geq v_m$ we have*

$$\int_0^\tau t^{p-1} \left(\int_{\partial U_t^{ext}} \frac{|x|^{\frac{\ell}{p'}}}{u(x)} \, d\mathcal{H}^{n-1}(x) \right) dt \leq \frac{1}{p\tilde{\beta}} \int_0^{|\Omega|_\ell} f^*(s)\, ds, \tag{2.39}$$

and

$$\int_0^\tau t^{p-1} \left(\int_{\partial V_t \cap \partial\Omega^\sharp} \frac{|x|^{\frac{\ell}{p'}}}{v(x)} \, d\mathcal{H}^{n-1}(x) \right) dt = \frac{1}{p\tilde{\beta}} \int_0^{|\Omega^\sharp|_\ell} f^*(s)\, ds. \tag{2.40}$$

Proof. Firstly we show (2.39). Fubini's Theorem ensures that

$$\begin{aligned} \int_0^\infty \tau^{p-1} \left(\int_{\partial U_\tau^{ext}} \frac{|x|^{\frac{\ell}{p'}}}{u(x)} d\mathcal{H}^{n-1}(x) \right) d\tau &= \int_{\partial\Omega} \left(\int_0^{u(x)} \frac{\tau^{p-1}}{u(x)} \, d\tau \right) |x|^{\frac{\ell}{p'}} \, d\mathcal{H}^{n-1} \\ &= \frac{1}{p} \int_{\partial\Omega} u(x)^{p-1} |x|^{\frac{\ell}{p'}} \, d\mathcal{H}^{n-1}. \end{aligned} \tag{2.41}$$

Choosing $\varphi \equiv 1$ as test function in (2.29) and using (2.32), we obtain

$$\int_{\partial\Omega} \tilde{\beta} |x|^{\frac{\ell}{p'}} u^{p-1} \, d\mathcal{H}^{n-1} \leq \int_{\partial\Omega} \beta(x) u^{p-1} \, d\mathcal{H}^{n-1} = \int_\Omega f|x|^\ell \, dx. \tag{2.42}$$

From (2.41) and (2.42) we immediately get

$$\int_0^\infty \tau^{p-1} \left(\int_{\partial U_\tau^{ext}} \frac{|x|^{\frac{\ell}{p'}}}{u(x)} \, d\mathcal{H}^{n-1}(x) \right) d\tau \leq \frac{1}{p\tilde{\beta}} \int_0^{|\Omega|_\ell} f^*(s)\, ds,$$

and, arguing in the same way, we deduce

$$\int_0^\infty \tau^{p-1} \left(\int_{\partial V_\tau \cap \partial\Omega^\sharp} \frac{|x|^{\frac{\ell}{p'}}}{v(x)} \, d\mathcal{H}^{n-1}(x) \right) d\tau = \frac{1}{p\tilde{\beta}} \int_0^{|\Omega|_\ell} f^*(s)\, ds.$$

Hence for any $t \geq 0$, we have

$$\int_0^t \tau^{p-1} \left(\int_{\partial U_\tau^{ext}} \frac{|x|^{\frac{\ell}{p'}}}{u(x)} \, d\mathcal{H}^{n-1}(x) \right) d\tau \leq \frac{1}{p\tilde{\beta}} \int_0^{|\Omega|_\ell} f^*(s)\, ds.$$

Since $\partial V_t \cap \partial\Omega^\sharp$ is empty for any $t \geq v_m$, we get

$$\int_0^t \tau^{p-1} \left(\int_{\partial V_\tau \cap \partial\Omega^\sharp} \frac{|x|^{\frac{\ell}{p'}}}{v(x)} \, d\mathcal{H}^{n-1}(x) \right) d\tau = \frac{1}{p\tilde{\beta}} \int_0^{|\Omega|_\ell} f^*(s)\, ds.$$

Lemma 2.2.2 is hence proved. □

Now we are in position to prove the comparison result.

Theorem 2.2.3. *Assume that hypotheses (H_1)-(H_4) are in force and let u and v be the solutions to problems (2.28) and (2.30), respectively. Then we have*

$$\|u\|_{L^1(\Omega,|x|^\ell dx)} = \int_\Omega |u(x)||x|^\ell \, dx \leq \int_{\Omega^\sharp} |v(x)||x|^\ell \, dx = \|v\|_{L^1(\Omega^\sharp,|x|^\ell dx)}, \quad (2.43)$$

and

$$\|u\|^p_{L^p(\Omega,|x|^\ell dx)} = \int_\Omega |u(x)|^p |x|^\ell dx \leq \int_{\Omega^\sharp} |v(x)|^p |x|^\ell dx = \|v\|^p_{L^p(\Omega^\sharp,|x|^\ell dx)}. \quad (2.44)$$

In this Section we prove Theorem 2.2.3 and Theorem 2.2.4.

Proof of Theorem 2.2.3. Let

$$\delta_1 = 1 - \left(\frac{\ell(p-1) + p(n-1)}{(p-1)(\ell+n)} \right).$$

Note that, since $p \geq n$, δ_1 is a positive constant.

Multiplying both sides of (2.35) by $t^{p-1}\mu(t)^{\delta_1}$ and integrating over $(0,\tau)$ with $\tau \geq v_m$, by Lemma 2.2.2, we have

$$\begin{aligned} \gamma_{n,\ell,p} \int_0^\tau t^{p-1}\mu(t)\, dt \leq \int_0^\tau (-\mu'(t))\, t^{p-1}\mu(t)^{\delta_1} \left(\int_0^{\mu(t)} f^*(s)\, ds \right)^{\frac{1}{p-1}} dt \\ + \frac{|\Omega|_\ell^{\delta_1}}{p\tilde{\beta}^{\frac{p}{p-1}}} \left(\int_0^{|\Omega|_\ell} f^*(s)\, ds \right)^{\frac{p}{p-1}}. \end{aligned} \quad (2.45)$$

Setting

$$F(\ell) = \int_0^\ell \omega^{\delta_1} \left(\int_0^\omega f^*(s)\, ds \right)^{\frac{1}{p-1}} d\omega,$$

and, integrating by parts both sides of the last inequality, we obtain

$$\begin{aligned}\tau^{p-1}&\left(\gamma_{n,\ell,p}\int_0^\tau \mu(t)\,dt + F(\mu(\tau))\right)\\ &\le (p-1)\int_0^\tau t^{p-2}\left(\gamma_{n,\ell,p}\int_0^t \mu(s)\,ds + F(\mu(t))\right)dt\\ &+ \frac{|\Omega|_\ell^{\delta_1}}{p\tilde{\beta}^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|_\ell} f^*(s)\,ds\right)^{\frac{p}{p-1}}.\end{aligned}$$

We observe that all the hypotheses of Gronwall's Lemma 1.5.1 are satisfied with

$$\xi(\tau) = \xi_1(\tau) = \int_0^\tau t^{p-2}\left(\gamma_{n,\ell,p}\int_0^t \mu(s)\,ds + F(\mu(t))\right)dt,$$

and

$$C = C_1 = \frac{|\Omega|_\ell^{\delta_1}}{p\tilde{\beta}^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|_\ell} f^*(s)\,ds\right)^{\frac{p}{p-1}}.$$

Applying such a Lemma with $\tau_0 = v_m$, we infer that for any $\tau \ge v_m$ it holds

$$\tau^{p-2}\left(\gamma_{n,\ell,p}\int_0^\tau \mu(s)\,ds + F(\mu(\tau))\right) \le \left(\frac{(p-1)\xi_1(v_m) + C_1}{v_m}\right)\left(\frac{\tau}{v_m}\right)^{p-2}.$$

Replacing $\mu(t)$ with $\phi(t)$ the previous inequality holds as an equality. Therefore for any $\tau \ge v_m$ it holds that

$$\gamma_{n,\ell,p}\int_0^\tau \mu(s)\,ds + F(\mu(\tau)) \le \gamma_{n,\ell,p}\int_0^\tau \phi(s)\,ds + F(\phi(\tau)). \tag{2.46}$$

On the other hand, since $\mu(t) \le \phi(t) = |\Omega|_\ell$, $\forall t \le v_m$, and $F(\ell)$ is an increasing function, we have that for any $\tau \le v_m$ it holds that

$$\gamma_{n,\ell,p}\int_0^\tau \mu(s)\,ds + F(\mu(t)) \le \gamma_{n,\ell,p}\int_0^\tau \phi(s)\,ds + F(\phi(t)). \tag{2.47}$$

Combining inequalities (2.46) and (2.47) we conclude that for any $\tau \ge 0$ it holds

$$\gamma_{n,\ell,p}\int_0^\tau \mu(s)\,ds + F(\mu(\tau)) \le \gamma_{n,\ell,p}\int_0^\tau \phi(s)\,ds + F(\phi(\tau)).$$

Since

$$\lim_{\tau\to+\infty} F\left(\mu(\tau)\right) = \lim_{\tau\to+\infty} F\left(\phi(\tau)\right) = 0,$$

letting τ goes to $+\infty$, we conclude that

$$\int_0^\infty \mu(t)\,dt \le \int_0^\infty \phi(t)\,dt,$$

and hence

$$\|u\|_{L^1(\Omega,|x|^\ell\,dx)} \le \|v\|_{L^1(\Omega^\sharp,|x|^\ell\,dx)}.$$

Now let us show (2.44). Firstly let us observe that it is enough to verify that

$$\int_0^\infty t^{p-1}\mu(t)\,dt \le \int_0^\infty t^{p-1}\phi(t)\,dt. \tag{2.48}$$

Integrating by parts the first term on the right-hand side in (2.45), and then letting τ go to $+\infty$, we obtain

$$\gamma_{n,\ell,p}\int_0^\infty t^{p-1}\mu(t)\,dt \le (p-1)\int_0^\infty t^{p-2}F(\mu(t))\,dt + \frac{|\Omega|_\ell^{\delta_1}}{p\tilde{\beta}^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|_\ell} f^*(s)\,ds\right)^{p'},$$

and similarly we have

$$\gamma_{n,\ell,p}\int_0^\infty t^{p-1}\phi(t)\,dt = \ (p-1)\int_0^\infty t^{p-2}F(\phi(t))\,dt + \frac{|\Omega|_\ell^{\delta_1}}{p\tilde{\beta}^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|_\ell} f^*(s)\,ds\right)^{p'}.$$

Thus if we prove

$$\int_0^\infty t^{p-2}F(\mu(t))\,dt \le \int_0^\infty t^{p-2}F(\phi(t))\,dt, \tag{2.49}$$

we get the claim (2.48).

Let

$$\theta_1 = -\frac{\ell(p-1)+p(n-1)}{(p-1)(\ell+n)}.$$

Multiplying both sides of (2.35) by $t^{p-1}F(\mu(t))\mu(t)^{\theta_1}$ and then integrating over $(0,\tau)$, taking into account that the function $h(\ell) = F(\ell)\ell^{\theta_1}$ is non decreasing, we get

$$\begin{aligned}\gamma_{n,\ell,p}\int_0^\tau t^{p-1}F(\mu(t))\,dt &\le \int_0^\tau (-\mu'(t))t^{p-1}\mu(t)^{\theta_1}F(\mu(t))\left(\int_0^{\mu(t)} f^*(s)\,ds\right)^{\frac{1}{p-1}}dt \\ &\quad + F\big(|\Omega|_\ell\big)\frac{|\Omega|_\ell^{\theta_1}}{p\tilde{\beta}^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|_\ell} f^*(s)\,ds\right)^{\frac{p}{p-1}}.\end{aligned} \tag{2.50}$$

Let

$$C_2 = F(|\Omega|_\ell)\frac{|\Omega|_\ell^{\theta_1}}{p\tilde{\beta}^{\frac{p}{p-1}}}\left(\int_0^{|\Omega|_\ell} f^*(s)\,ds\right)^{\frac{p}{p-1}}.$$

Integrating by parts both sides of (2.50) we derive

$$\begin{aligned}\gamma_{n,\ell,p}\,\tau\int_0^\tau t^{p-2}F(\mu(t))\,dt+\tau H_\mu(\tau)\le{}&\gamma_{n,\ell,p}\int_0^\tau\int_0^t r^{p-2}F(\mu(r))\,drdt\\&+\int_0^\tau H_\mu(t)\,dt+C_2,\end{aligned}\tag{2.51}$$

where

$$H_\mu(\tau)=-\int_\tau^{+\infty}t^{p-2}\mu(t)^{\theta_1}F(\mu(t))\Big(\int_0^{\mu(t)}f^*(s)\,ds\Big)^{\frac{1}{p-1}}\,d\mu(t).$$

Setting

$$\xi_2(\tau)=\gamma_{n,\ell,p}\int_0^\tau\int_0^t r^{p-2}F(\mu(r))\,dr+\int_0^t H_\mu(t)\,dt,$$

then (2.51) reads as

$$\tau\xi_2'(\tau)\le\xi_2(\tau)+C_2.$$

Lemma 1.5.1, with $\tau_0=v_m$, ensures that the following inequality holds true for any $\tau\ge v_m$

$$\begin{aligned}\gamma_{n,\ell,p}\int_0^\tau t^{p-2}F(\mu(t))\,dt+H_\mu(\tau)\le{}&\frac{(p-1)\displaystyle\int_0^{v_m}t^{p-2}F(\mu(t))\,dt+H_\mu(v_m)+C_2}{v_m}\\&\times\left(\frac{\tau}{v_m}\right)^{p-2}.\end{aligned}$$

The previous inequality becomes an equality if we replace $\mu(t)$ with $\phi(t)$, so, recalling that $\mu(t)\le\phi(t)=|\Omega|_\ell$ for $t\le v_m$, we obtain

$$\gamma_{n,\ell,p}\int_0^\tau t^{p-2}F(\mu(t))\,dt+H_\mu(\tau)\le\gamma_{n,\ell,p}\int_0^\tau t^{p-2}F(\phi(t))\,dt+H_\phi(\tau).$$

Letting $\tau\to\infty$, we have

$$\int_0^\infty t^{p-2}F(\mu(t))\,dt\le\int_0^\infty t^{p-2}F(\phi(t))\,dt,$$

since, as we will show,

$$\lim_{\tau\to 0} H_\mu(\tau) = \lim_{\tau\to 0} H_\phi(\tau) = 0. \tag{2.52}$$

This proves (2.49), and hence (2.44).

To prove (2.52), recalling that $p \geq n$, we observe that

$$t^{p-2}\mu(t) = \int_{\{u>t\}} t^{p-2}|x|^\ell\,dx \leq \int_{\{u>t\}} u^{p-2}|x|^\ell\,dx \leq \|u\|^{p-2}_{L^p(\Omega,|x|^\ell\,dx)}\,\mu(t)^{\frac{2}{p}},$$

therefore

$$\begin{aligned}|H_\mu(\tau)| &= \int_\tau^{+\infty} t^{p-2}F(\mu(t))\mu(t)^{\theta_1}\left(\int_0^{\mu(t)} f^*(s)\,ds\right)(-\mu'(t))\,dt\\ &\leq \left(\int_0^{|\Omega|_\ell} f^*(s)\,ds\right)\|u\|^{p-2}_{L^p(\Omega,|x|^\ell dx)}\int_\tau^{+\infty} F(\mu(t))\mu(t)^{\frac{2}{p}+\theta_1-1}(-\mu'(t))\,dt.\end{aligned}$$

The claim follows by observing that the right-hand side of the above inequality goes to 0 as $\tau \to +\infty$.

□

Remark 2.2.1. We emphasize that, with the same argument in [ANT23; AGM22], it is possible to obtain a stronger comparison in Theorem 2.2.3. It can be shown

$$\|u\|_{L^{k,1}(\Omega,|x|^\ell\,dx)} \leq \|v\|_{L^{k,1}(\Omega^\sharp,|x|^\ell\,dx)} \qquad \forall\, 0 < k \leq \frac{(\ell+n)(p-1)}{\ell(p-1)+p(n-1)},$$

and

$$\|u\|_{L^{pk,p}(\Omega,|x|^\ell\,dx)} \leq \|v\|_{L^{pk,p}(\Omega^\sharp,|x|^\ell\,dx)} \qquad \forall\, 0 < k \leq \frac{(\ell+n)(p-1)}{\ell(p-1)+p(n-2)+n},$$

where $\|\cdot\|_{L^{p,q}(\Omega,|x|^\ell\,dx)}$ is defined in (1.10).

If the source term $f(x)$ is constant we get a stronger result, namely a pointwise comparison between $u^\sharp$ and v.

Theorem 2.2.4. *Assume that hypotheses (H_1)-(H_4) are in force and suppose that $f(x) \equiv 1$ in Ω. If either $p = n = 2$ or if $p > 2$, $n \geq 2$ and*

$$\ell \leq -n + \frac{p-n}{p-2}, \tag{2.53}$$

then

$$u^\sharp(x) \leq v(x) \qquad x \in \Omega^\sharp, \tag{2.54}$$

where u and v are the solutions to (2.28) and (2.30) respectively.

Proof. Let
$$\theta_2 = \frac{\ell(p-1)+p(n-1)}{p(\ell+n)},$$
and
$$\delta_2 = -\left(\theta_2 - \frac{1}{p}\right)\frac{p}{p-1}.$$
We point out that $\delta_2 \geq 0$ by assumption (2.53).

Obviously in this case we have
$$\int_0^{\mu(t)} f^*(s)\,ds = \mu(t).$$
Hence (2.35) can be written as
$$\gamma_{n,\ell,p}\,\mu(t)^{\left(\theta_2-\frac{1}{p}\right)\frac{p}{p-1}} \leq -\mu'(t) + \frac{1}{\tilde{\beta}^{\frac{1}{p-1}}}\int_{\partial U_t^{\text{ext}}} \frac{|x|^{\frac{\ell}{p'}}}{u}\,d\mathcal{H}^{n-1}. \tag{2.55}$$
Multiplying both sides of (2.55) by $t^{p-1}\mu(t)^{\delta_2}$ and, then, integrating from 0 to $\tau \geq v_m$, we obtain
$$\begin{aligned}\gamma_{n,\ell,p}\int_0^\tau t^{p-1}\,dt &\leq \int_0^\tau t^{p-1}\mu(t)^{\delta_2}(-\mu'(t))\,dt\\ &\quad+\frac{1}{\tilde{\beta}^{\frac{1}{p-1}}}\int_0^\tau t^{p-1}\mu(t)^{\delta_2}\int_{\partial U_t^{\text{ext}}}\frac{1}{u}\,d\mathcal{H}^{n-1}\\ &\leq \int_0^\tau t^{p-1}\mu(t)^{\delta_2}(-\mu'(t))\,dt + \frac{|\Omega|_\ell^{1+\delta_2}}{p\tilde{\beta}^{\frac{p}{p-1}}}.\end{aligned}\tag{2.56}$$

Taking into account of Lemma 2.2.2, if we replace $\mu(t)$ with $\phi(t)$, the previous inequality holds as equality. Hence, we get
$$\int_0^\tau t^{p-1}\mu(t)^{\delta_2}(-\mu'(t))\,dt \geq \int_0^\tau t^{p-1}\phi(t)^{\delta_2}(-\phi'(t))\,dt.$$
In turn an integration by parts yields
$$\begin{aligned}-\tau^{p-1}\frac{\mu(\tau)^{1+\delta_2}}{1+\delta_2} + (p-1)\int_0^\tau t^{p-2}\frac{\mu(t)^{1+\delta_2}}{1+\delta_2}\,dt&\\ \geq -\tau^{p-1}\frac{\phi(\tau)^{1+\delta_2}}{1+\delta_2} + (p-1)\int_0^\tau t^{p-2}\frac{\phi(t)^{1+\delta_2}}{1+\delta_2}\,dt.&\end{aligned}$$

The above inequality allows us to use claim (ii) of the Gronwall's Lemma, this time to the function

$$\xi_3(\tau) = \int_0^\tau s^{p-2}\left(\frac{\mu(s)^{1+\delta_2} - \phi(s)^{1+\delta_2}}{1+\delta_2}\right) ds.$$

Hence for any $\tau \geq v_m$ it holds

$$\frac{\mu^{1+\delta_2}(\tau) - \phi^{1+\delta_2}(\tau)}{1+\delta_2} \leq \frac{p-1}{v_m^{p-2}} \int_0^{v_m} s^{p-2}\left(\frac{\mu^{1+\delta_2}(s) - \phi^{1+\delta_2}(s)}{1+\delta_2}\right) ds.$$

Inequality (2.33) ensures us that the right-hand side of the inequality above is non-positive, therefore

$$\mu(\tau) \leq \phi(\tau) \qquad \forall \tau \geq v_m.$$

Finally, since by (2.33) we have

$$\mu(\tau) \leq \phi(\tau) = |\Omega|_\ell \quad \forall \tau \leq v_m,$$

and, therefore, the inequality above holds true in $[0, +\infty)$. Claim (2.54) is hence proven. □

Here we give an application of our previous results. Specifically, we derive a Faber-Krahn inequality for the Robin p-Laplacian operator with variable boundary parameter.

Theorem 2.2.5. *Assume hypotheses* (H_1)-(H_3) *and let* $\tilde{\beta}$ *be the constant defined in* (2.31)*, then we have*

$$\lambda_{1,\beta(x)}(\Omega) \geq \lambda_{1,\tilde{\beta}}(\Omega^\sharp),$$

where

$$\lambda_{p,\beta(x)}(\Omega) = \min_{\substack{\psi \in W^{1,p}(\Omega) \\ \psi \neq 0}} \frac{\displaystyle\int_\Omega |\nabla\psi|^p \, dx + \int_{\partial\Omega} \beta(x)|\psi|^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |\psi|^p |x|^\ell \, dx}. \tag{2.57}$$

Proof. Let u be a positive minimizer of (2.57), then

$$\begin{cases} -\Delta_p u = \lambda_{1,\beta}(\Omega)\, u^{p-1}|x|^\ell & \text{in } \Omega \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta(x)u^{p-1} = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, let z be a solution to the following problem

$$\begin{cases} -\Delta_p z = \lambda_{1,\beta}(\Omega)\,(u^\sharp)^{p-1}\,|x|^\ell & \text{in } \Omega^\sharp \\ |\nabla z|^{p-2}\dfrac{\partial z}{\partial \nu} + \tilde{\beta}(r^\sharp)^{\frac{\ell}{p'}}\,z^{p-1} = 0 & \text{on } \partial\Omega^\sharp. \end{cases} \tag{2.58}$$

Therefore, Theorem 2.2.3 ensures that

$$\int_\Omega u^p|x|^\ell\,dx = \int_{\Omega^\sharp}(u^\sharp)^p|x|^\ell\,dx \le \int_{\Omega^\sharp} z^p|x|^\ell\,dx,$$

while Hölder inequality gives

$$\begin{aligned}\int_{\Omega^\sharp}(u^\sharp)^{p-1}z|x|^\ell\,dx &\le \left(\int_{\Omega^\sharp}(u^\sharp)^p|x|^\ell\,dx\right)^{\frac{p-1}{p}}\left(\int_{\Omega^\sharp} z^p|x|^\ell\,dx\right)^{\frac{1}{p}} \\ &\le \int_{\Omega^\sharp} z^p|x|^\ell\,dx.\end{aligned}$$

Hence, writing $\lambda_{1,\beta(x)}(\Omega)$ according to (2.58), we get

$$\begin{aligned}\lambda_{1,\beta(x)}(\Omega) &= \frac{\displaystyle\int_{\Omega^\sharp}|\nabla z|^p\,dx + \int_{\partial\Omega^\sharp}\tilde{\beta}(r^\sharp)^{\frac{\ell}{p'}}z^p\;\,d\mathcal{H}^{n-1}}{\displaystyle\int_{\Omega^\sharp}(u^\sharp)^{p-1}z|x|^\ell\,dx} \\ &\ge \frac{\displaystyle\int_{\Omega^\sharp}|\nabla z|^p\,dx + \int_{\partial\Omega^\sharp}\tilde{\beta}(r^\sharp)^{\frac{\ell}{p'}}z^p\,d\mathcal{H}^{n-1}}{\displaystyle\int_{\Omega^\sharp} z^p|x|^\ell\,dx} \ge \lambda_{1,\tilde{\beta}}(\Omega^\sharp).\end{aligned}$$

□

Chapter 3

Gradient rearrangement and comparison results

In this chapter we will present the technique of the gradient rearrangement: given a function u, we will construct a radial function $u^\star$ whose modulus of the gradient is the increasing rearrangement of that of u and compare some suitable norm of u with those of $u^\star$.

Specifically, in the first section we will study the case of a Sobolev function with non zero trace while in the second section we will consider the case of a BV function. Finally in third and fourth section, we present some applications of this type of rearrangement.

3.1 The Sobolev case

Theorem 3.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open and Lipschitz set and let $u \in W^{1,p}(\Omega)$ be a non-negative function. If we denote with $\Omega^\sharp$ the ball centered at the origin with same measure as Ω, then there exists a non-negative function $u^\star \in W^{1,p}(\Omega^\sharp)$ that satisfies*

$$\begin{cases} |\nabla u^\star| = |\nabla u|_\sharp(x) & a.e.\ in\ \Omega^\sharp \\ u^\star = \dfrac{\displaystyle\int_{\partial\Omega} u \, d\mathcal{H}^{n-1}}{\mathcal{H}^{n-1}(\partial\Omega^\sharp)} & on\ \partial\Omega^\sharp. \end{cases} \tag{3.1}$$

and such that

$$\|u\|_{L^1(\Omega)} \le \|u^\star\|_{L^1(\Omega^\sharp)}. \tag{3.2}$$

Remark 3.1.1. By the explicit expression of u^* on the boundary and the Hölder inequality, we can estimate the L^p norm of the trace:

$$\begin{aligned} \mathcal{H}^{n-1}(\partial\Omega^\sharp)^{p-1}\int_{\partial\Omega^\sharp}(u^\star)^p\,d\mathcal{H}^{n-1} &= \left(\int_{\partial\Omega} u\,d\mathcal{H}^{n-1}\right)^p \\ &\le \mathcal{H}^{n-1}(\partial\Omega)^{p-1}\int_{\partial\Omega} u^p\,d\mathcal{H}^{n-1}, \end{aligned} \tag{3.3}$$

for any $p \ge 1$.

Proof of Theorem 3.1.1. Let us consider ε and $\delta := \delta_\varepsilon$ and the sets

$$\begin{aligned} \Omega_\varepsilon &= \{x \in \mathbb{R}^n \colon d(x,\Omega) < \varepsilon\} & \Sigma_\varepsilon &= \Omega_\varepsilon \setminus \Omega, \\ \Omega^\sharp_\varepsilon &= \{x \in \mathbb{R}^n \colon d(x,\Omega^\sharp) < \delta\} & \Sigma^\sharp_\varepsilon &= \Omega^\sharp_\varepsilon \setminus \Omega^\sharp, \\ |\Omega_\varepsilon| &= |\Omega^\sharp_\varepsilon| & |\Sigma_\varepsilon| &= |\Sigma^\sharp_\varepsilon|, \end{aligned} \tag{3.4}$$

where, since $\frac{|\Sigma_\varepsilon|}{\varepsilon} \to \mathcal{H}^{n-1}(\partial\Omega)$ and $\frac{|\Sigma^\sharp_\varepsilon|}{\delta} \to \mathcal{H}^{n-1}(\partial\Omega^\sharp)$ as $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0}\frac{\delta}{\varepsilon} = \frac{\mathcal{H}^{n-1}(\partial\Omega)}{\mathcal{H}^{n-1}(\partial\Omega^\sharp)}.$$

Let $d(\cdot,\Omega)$ defined as follows:

$$d(x,\Omega) := \inf_{y\in\Omega}|x-y|.$$

Then we divide the proof into four steps.

Step 1 First of all we assume Ω with $C^{1,\alpha}$ boundary, $u \in W^{1,\infty}(\Omega)$ and $u \ge \sigma > 0$ in Ω.

So we can consider the following "linear" extension of u, u_ε in Ω_ε

$$u_\varepsilon(x) = u\big(p(x)\big)\left(1 - \frac{d(x,\partial\Omega)}{\varepsilon}\right) \qquad \forall x \in \Omega_\varepsilon \setminus \Omega,$$

where $p(x)$ is the projection of x on $\partial\Omega$ (for ε sufficiently small, this definition is well posed since Ω is smooth, see [GT01]). The function u_ε, has the following properties:

(a) $u_\varepsilon|_\Omega = u$,

(b) $u_\varepsilon = 0$ on $\partial\Omega_\varepsilon$,

(c) $\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \leq |\nabla u_\varepsilon|(y)$ $\forall y \in \Sigma_\varepsilon$ for ε sufficiently small,

(d) $\displaystyle\lim_{\varepsilon\to 0^+} \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|\, dx = \int_{\partial\Omega} u\, d\mathcal{H}^{n-1}$.

Properties (a) and (b) follow immediately by the definition of u_ε, while (c) is a consequence of the regularity of u. Property (d) can be obtained by an easy calculation, indeed

$$\nabla u_\varepsilon(x) = \nabla\big(u(p(x))\big)\left[1 - \frac{d(x,\partial\Omega)}{\varepsilon}\right] - u\big(p(x)\big)\frac{\nabla d(x,\partial\Omega)}{\varepsilon}.$$

For the first term, we can notice

$$\int_{\Sigma_\varepsilon} \left|\nabla\big(u(p(x))\big)\right|\left[1 - \frac{d(x,\partial\Omega)}{\varepsilon}\right] dx \leq L\int_{\Sigma_\varepsilon} dx = L|\Sigma_\varepsilon|,$$

where L is the L^∞ norm of $\nabla u(p(x))$. Now we deal with the second term and, keeping in mind that $|\nabla d| = 1$ and using coarea formula, we have

$$\lim_{\varepsilon\to 0^+}\int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|\, dx = \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_{\Sigma_\varepsilon} u(p(x))\, dx = \lim_{\varepsilon\to 0^+}\int_0^\varepsilon dt\int_{\Gamma_t}(u\circ p)\, d\mathcal{H}^{n-1},$$

where $\Gamma_t = \{x \in \Sigma_\varepsilon : d(x,\partial\Omega) = \varepsilon\}$. By continuity of u and Lebesgue differentation theorem we get

$$\lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_0^\varepsilon dt\int_{\Gamma_t} u\circ p\, d\mathcal{H}^{n-1} = \int_{\Gamma_0}(u\circ p)\, d\mathcal{H}^{n-1} = \int_{\partial\Omega} u\, d\mathcal{H}^{n-1},$$

that proves property (d).

For every $\varepsilon > 0$, we consider the following problem

$$\begin{cases} |\nabla v_\varepsilon|(x) = |\nabla u_\varepsilon|_\sharp(x) & \text{in } \Omega_\varepsilon^\sharp \\ v_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon^\sharp, \end{cases} \tag{3.5}$$

and by Theorem 1.2.2 it holds

$$\|u_\varepsilon\|_{L^1(\Omega_\varepsilon)} \leq \|v_\varepsilon\|_{L^1(\Omega_\varepsilon^\sharp)}. \tag{3.6}$$

Moreover it exists $\bar\varepsilon$ such that for every $\varepsilon \leq \bar\varepsilon$

$$|\nabla v_\varepsilon|(x) = |\nabla u_\varepsilon|_\sharp(x) = |\nabla u|_\sharp(x) \qquad \forall x \in \Omega^\sharp. \tag{3.7}$$

We can see u_ε as a $W^{1,1}(\Omega_{\bar{\varepsilon}})$ function and we have

$$\int_{\Omega^\sharp_{\bar{\varepsilon}}} |\nabla v_\varepsilon| = \int_{\Omega_{\bar{\varepsilon}}} |\nabla u_\varepsilon| = \int_\Omega |\nabla u| + \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon| \leq \|\nabla u\|_{L^1(\Omega)} + 2\|u\|_{L^1(\partial\Omega)}. \tag{3.8}$$

by property (d).

Finally, by Poincarè and (3.8), there exists a constant $0 < C = C(n,\Omega)$ such that

$$\|v_\varepsilon\|_{W^{1,1}(\Omega^\sharp_{\bar{\varepsilon}})} \leq C\|\nabla v_\varepsilon\|_{L^1(\Omega_{\bar{\varepsilon}})} \leq C(n,\Omega)\|u\|_{W^{1,1}(\Omega)}.$$

Therefore, up to a subsequence, there exists a limit function $u^\star \in BV(\Omega^\sharp_{\bar{\varepsilon}})$ such that ([AFP00, Proposition 3.13])

$$v_\varepsilon \to u^\star \text{ in } L^1(\Omega^\sharp_{\bar{\varepsilon}}) \qquad \nabla v_\varepsilon \overset{*}{\rightharpoonup} \nabla u^\star \text{ in } \Omega,$$

namely

$$\lim_{\varepsilon\to 0} \int_{\Omega^\sharp_{\bar{\varepsilon}}} \varphi \, d\nabla v_\varepsilon = \int_{\Omega^\sharp_{\bar{\varepsilon}}} \varphi \, d\nabla u^\star \qquad \forall \varphi \in C_0(\Omega, \mathbb{R}^n).$$

Our aim is to show that $u^\star$ satisfies properties (3.1), (3.2) and (3.3).

Concerning (3.1) then $|\nabla u^\star| = |\nabla u|_\sharp$ follows from (3.7).

To find the value of $u^\star$ at the boundary, we observe that, from (3.5) and (3.7), we have

$$\int_{\Sigma_\varepsilon} |\nabla u_\varepsilon| = \int_{\Sigma^\sharp_\varepsilon} |\nabla v_\varepsilon|.$$

Now, for $t > 0$ setting $\Gamma_t = \{d(x,\Omega) = t\}$, $\Gamma^\sharp_t = \{d(x,\Omega^\sharp) = t\}$, $r = \left(\dfrac{|\Omega|}{\omega_n}\right)^{\frac{1}{n}}$ and recalling that v_ε is radially symmetric we have

$$\int_{\Sigma^\sharp_\varepsilon} |\nabla v_\varepsilon| = \int_r^{r+\delta} \int_{\Gamma^\sharp_t} |\nabla v_\varepsilon| \, d\mathcal{H}^{n-1} \, dt = \int_r^{r+\delta} -v'_\varepsilon(t)\Big(\mathcal{H}^{n-1}(\Gamma^\sharp_t)\Big) \, dt.$$

Therefore by monotonicity of $|\Gamma^\sharp_t|$ we have

$$v_\varepsilon(r)\mathcal{H}^{n-1}(\Gamma^\sharp_r) \leq \int_r^{r+\delta} -v'_\varepsilon(t)\Big(\mathcal{H}^{n-1}(\Gamma^\sharp_t)\Big) \, dt \leq v_\varepsilon(r)\mathcal{H}^{n-1}(\Gamma^\sharp_{r+\delta}),$$

and since

$$v_\varepsilon(r)\mathcal{H}^{n-1}(\Gamma_r^\sharp) = \int_{\partial\Omega^\sharp} v_\varepsilon \, d\mathcal{H}^{n-1},$$

using the fact that $v_\varepsilon \to v$ in $L^1(\Omega)$, $\nabla v_\varepsilon = \nabla u$ in Ω and the continuity embedding of $W^{1,1}(\Omega)$ in $L^1(\Omega)$, in the end we have

$$\int_{\Sigma_\varepsilon^\sharp} |\nabla v_\varepsilon| \to \int_{\partial\Omega^\sharp} u^\star \, d\mathcal{H}^{n-1}.$$

Using property (d) we obtain

$$\int_{\partial\Omega} u \, d\mathcal{H}^{n-1} = \int_{\partial\Omega^\sharp} u^\star \, d\mathcal{H}^{n-1}.$$

In the end we have that for $u^\star$ it holds

$$\begin{cases} |\nabla u^\star| = |\nabla u|_\sharp & \text{in } \Omega^\sharp \\ u^\star = \dfrac{\displaystyle\int_{\partial\Omega} u \, d\mathcal{H}^{n-1}}{\mathcal{H}^{n-1}(\partial\Omega^\sharp)} & \text{on } \partial\Omega^\sharp. \end{cases} \tag{3.9}$$

that proves (3.1).

Furthermore by

$$\|u_\varepsilon\|_{L^1(D)} \to \|u\|_{L^1(D)} \quad \text{and} \quad \|v_\varepsilon\|_{L^1(D^\sharp)} \to \|u^\star\|_{L^1(D^\sharp)}.$$

we can pass to the limit $\varepsilon \to 0$ in (3.6) and we get

$$\|u\|_{L^1(\Omega)} \le \|u^\star\|_{L^1(\Omega^\sharp)}.$$

that proves (3.2).

Step 2 Now we remove the extra-assumption $u \ge \delta > 0$ defining

$$u_\sigma := u + \sigma.$$

Then u_σ is strictly positive in Ω and we can apply the previous result: there exists a function v_σ in $\Omega^\sharp$ such that

$$\begin{cases} |\nabla v_\sigma| = |\nabla u_\sigma|_\sharp = |\nabla u|_\sharp & \text{a.e. in } \Omega^\sharp \\ v_\sigma = \dfrac{\displaystyle\int_{\partial\Omega} u_\sigma \, d\mathcal{H}^{n-1}}{\mathcal{H}^{n-1}(\partial\Omega^\sharp)} = \dfrac{\displaystyle\int_{\partial\Omega} u \, d\mathcal{H}^{n-1}}{\mathcal{H}^{n-1}(\partial\Omega^\sharp)} + \sigma \dfrac{\mathcal{H}^{n-1}(\partial\Omega)}{\mathcal{H}^{n-1}(\partial\Omega^\sharp)} & \text{on } \partial\Omega^\sharp, \end{cases}$$

and

$$\|u_\sigma\|_{L^1(\Omega)} \leq \|v_\sigma\|_{L^1(\Omega^\sharp)}, \tag{3.10}$$

If we define

$$u^\star := v_\sigma - \sigma \frac{\mathcal{H}^{n-1}(\partial\Omega)}{\mathcal{H}^{n-1}(\partial\Omega^\sharp)},$$

then $u^\star$ solves

$$\begin{cases} |\nabla u^\star| = |\nabla u|_\sharp & \text{in } \Omega^\sharp \\ u^\star = \dfrac{\displaystyle\int_{\partial\Omega} u \, d\mathcal{H}^{n-1}}{\mathcal{H}^{n-1}(\Omega^\sharp)} & \text{on } \partial\Omega^\sharp, \end{cases} \tag{3.11}$$

Sending $\sigma \to 0$ in (3.10) we have

$$\|u\|_{L^1(\Omega)} \leq \|u^\star\|_{L^1(\Omega^\sharp)}.$$

Step 3 Now we remove the assumption on the regularity of Ω.

Let Ω be a bounded, open and Lipschitz set, and $u \in W^{1,\infty}(\Omega)$. Then there exists a sequence $\{\Omega_k\} \subset \mathbb{R}^n$ of open set with C^2 boundary such that $\Omega \subset \Omega_k$, $\forall k \in \mathbb{N}$ (for istance you can mollify χ_Ω and take a suitable superlevel set) and

$$|\Omega_k \triangle \Omega| \to 0 \qquad \mathcal{H}^{n-1}(\partial\Omega_k) \to \mathcal{H}^{n-1}(\partial\Omega) \qquad \text{for } k \to +\infty.$$

Let $\tilde{u}$ be an extension of u in $\mathbb{R}^n$ such that

$$\tilde{u}|_\Omega \equiv u, \qquad \|\tilde{u}\|_{W^{1,\infty}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,\infty}(\Omega)}.$$

We define

$$u_k = \tilde{u}\chi_{\Omega_k},$$

and clearly $u_k = u$ in Ω. By the previous step, we can construct $u_k^\star \in W^{1,\infty}(\Omega_k^\sharp)$ such that it is radial, $|\nabla u_k|_* = |\nabla u_k^\star|_*$ and

$$\|u_k\|_{L^1(\Omega_k)} \leq \|u_k^\star\|_{L^1(\Omega_k^\sharp)} \tag{3.12}$$

$$\int_{\partial\Omega_k} u_k \, d\mathcal{H}^{n-1} = \int_{\partial\Omega_k^\sharp} u_k^\star \, d\mathcal{H}^{n-1}, \tag{3.13}$$

Therefore, since $\|u_k\|_{W^{1,p}(\Omega_k)} \leq M$, for all p, the sequence $\{u_k^\star\}$ is equibounded in $W^{1,p}(\Omega^\sharp)$ and it has a subsequence which converges strongly in L^p and weakly in $W^{1,p}$ to a function w.

Let us prove that $|\nabla u|$ and $|\nabla w|$ has the same rearrangement.

$$\limsup_k \big\| |\nabla u_k^\star| - |\nabla u|_\sharp \big\|_{L^p(\Omega^\sharp)} \le \lim_k \big\| (f_k)_\sharp - f_\sharp \big\|_{L^p(\mathbb{R}^n)},$$

where

$$f(x) = \begin{cases} |\nabla \tilde{u}| & \text{in } \Omega \\ \|\nabla \tilde{u}\|_{L^\infty(\mathbb{R}^n)} & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \qquad f_k = \begin{cases} |\nabla u_k| & \text{in } \Omega_k \\ \|\nabla \tilde{u}\|_{L^\infty(\mathbb{R}^n)} & \text{in } \mathbb{R}^n \setminus \Omega_k, \end{cases}$$

So using (1.3) we have

$$\begin{aligned} \big\| (f_k)_\sharp - f_\sharp \big\|_{L^p(\mathbb{R}^n)} &\le \|f_k - f\|_{L^p(\mathbb{R}^n)} \\ &= \|f_k - f\|_{L^p(\Omega_k \setminus \Omega)} \le 2 \|\nabla \tilde{u}\|_{L^\infty(\mathbb{R}^n)} |\Omega_k \setminus \Omega|, \end{aligned}$$

that tends to 0 as $k \to +\infty$ by the fact that $|\Omega_k \triangle \Omega| \to 0$.

Hence, the functions ∇w and ∇u has the same rearrangement, by the uniqueness of the weak limit in $\Omega^\sharp$.

In the end, passing to limit $k \to +\infty$ in (3.12) and (3.13), we have

$$\begin{aligned} \|u\|_{L^1(\Omega)} &\le \|w\|_{L^1(\Omega^\sharp)} \\ \int_{\partial\Omega} u \, d\mathcal{H}^{n-1} &= \int_{\partial\Omega^\sharp} w \, d\mathcal{H}^{n-1}. \end{aligned}$$

Hence $w = u^\star$.

Step 4 Finally, we proceed by removing the assumption $u \in W^{1,\infty}(\Omega)$.

If $u \in W^{1,p}(\Omega)$, by Meyers-Serrin Theorem, there exists a sequence $\{u_k\} \subset C^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $u_k \to u$ in $W^{1,p}(\Omega)$. We can apply previous step to obtain $u_k^\star \in W^{1,\infty}(\Omega^\sharp)$ such that $|\nabla u_k|$ and $|\nabla u_k^\star|$ are equally distributed and

$$\|u_k\|_{L^1(\Omega)} \le \|u_k^\star\|_{L^1(\Omega^\sharp)} \qquad \forall k \in \mathbb{N} \tag{3.14}$$

$$\int_{\partial\Omega} u_k \, d\mathcal{H}^{n-1} = \int_{\partial\Omega^\sharp} u_k^\star \, d\mathcal{H}^{n-1} \qquad \forall k \in \mathbb{N}. \tag{3.15}$$

Arguing as the previous step, there exists a function w such that, up to a subsequence

$$u_k^\star \to w \text{ in } L^p(\Omega) \qquad \nabla u_k^\star \rightharpoonup \nabla w \text{ in } L^p(\Omega; \mathbb{R}^n),$$

and $|\nabla w|$ has the same rearrangement as $|\nabla u|$.

Finally, sending $k \to +\infty$ in (3.14) and (3.15), we have

$$\|u\|_{L^1(\Omega)} \leq \|w\|_{L^1(\Omega^\sharp)}$$
$$\int_{\partial\Omega} u \, d\mathcal{H}^{n-1} = \int_{\partial\Omega^\sharp} w \, d\mathcal{H}^{n-1}.$$

Hence $w = u^\star$. □

3.2 The BV case

Let us now consider the case of bounded variation functions. Let Ω be a bounded open set with finite perimeter and let us consider

$$\mathrm{BV}_0(\Omega) := \{u \in \mathrm{BV}(\mathbb{R}^n) : u \equiv 0 \text{ in } \mathbb{R}^n \setminus \Omega\}.$$

For any $u \in \mathrm{BV}_0(\Omega)$, we define

$$f(x,s) = \big(u - u^*(s)\big)_+(x) \qquad x \in \mathbb{R}^n, \ s \in [0,+\infty). \tag{3.16}$$

The function $f(\cdot,s)$ belongs to $\mathrm{BV}_0(\Omega)$ for every $s \in [0,+\infty)$ since it is a truncation of u (see [AFP00, Theorem 3.96]). Moreover, for every $s \in [0,+\infty)$ we denote by

$$G(s) = |Df(\cdot,s)|(\mathbb{R}^n) = |D^a f(\cdot,s)|(\mathbb{R}^n) + |D^s f(\cdot,s)|(\mathbb{R}^n) = G_1(s) + G_2(s), \tag{3.17}$$

where $D^a f$ and $D^s f$ are, respectively, the absolutely continuous part and singular part of the measure Df.

The following corollary holds.

Corollary 3.2.1. *Let u be a non-negative function belonging to $BV_0(\Omega)$ and let $G(s)$ be the function defined as in (3.17). Then for a.e. $s \in [0,+\infty)$:*

$$G(s) = \int_{u^*(s)}^{+\infty} \mathrm{Per}(\{u > \xi\}) \, d\xi. \tag{3.18}$$

Proof. For a.e. $s \in [0,+\infty)$, applying 1.4.2 with $E = \mathbb{R}^n$ to the function $f(\cdot,s)$ defined in (3.16), we have

$$G(s) = \left|D\Big(\big(u - u^*(s)\big)_+\Big)\right|(\mathbb{R}^n) = \int_{-\infty}^{+\infty} \mathrm{Per}\left(\left\{\big(u - u^*(s)\big)_+ > \xi\right\}\right) d\xi. \tag{3.19}$$

Moreover, we have

$$\int_{-\infty}^{+\infty} \mathrm{Per}\left(\left\{\left(u-u^*(s)\right)_+ > \xi\right\}\right) d\xi = \int_0^{+\infty} \mathrm{Per}\left(\{u-u^*(s) > \xi\}\right) d\xi,$$

and a change of variables gives (3.18). □

The following properties hold:

1. G is an increasing function on $(0,+\infty)$ by (3.18), constant in $(|\Omega|,+\infty)$, it belongs to $\mathrm{BV}_{\mathrm{loc}}([0,+\infty))$. Then, there exists a positive measure σ such that
$$G(s) = \int_{(0,s]} d\sigma(\tau) \qquad \forall s \in [0,+\infty); \tag{3.20}$$

2. $G_1(s) = \displaystyle\int_{\{u>u^*(s)\}} |\nabla^{\mathrm{a}} u|\, dx$ is increasing and AC on $[0,+\infty)$, then there exists a function F_1 belonging to $L^1([0,+\infty))$:
$$G_1(s) = \int_0^s F_1(\tau)\, d\tau \qquad \forall s \in [0,+\infty);$$

3. G_2 is an increasing function belonging to $\mathrm{BV}_{\mathrm{loc}}([0,+\infty))$, so there exists a positive measure σ_2 such that
$$G_2(s) = \int_{(0,s]} d\sigma_2(\tau) \qquad \forall s \in [0,+\infty).$$

Then, $\forall s \geq 0$

$$G(s) = \sigma((0,s]) = \int_{(0,s]} d\sigma(\tau) = \int_0^s F_1(\tau)\, d\tau + \int_{(0,s]} d\sigma_2(\tau). \tag{3.21}$$

Let us now define the following function

$$v(s) := \int_s^{+\infty} \frac{1}{n\omega_n^{\frac{1}{n}} \tau^{1-\frac{1}{n}}}\, d\sigma(\tau) \qquad \forall s \in [0,+\infty), \tag{3.22}$$

where σ is defined in (3.20). We observe that, since $\mathrm{supp}(\sigma) \subseteq [0,|\Omega|]$, v is identically 0 on $(|\Omega|,+\infty)$, hence $v \in \mathrm{BV}_0([0,|\Omega|])$.

As intermediate step towards Theorem 3.2.3, we prove the following proposition.

Proposition 3.2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with finite perimeter and assume that u is a non-negative function belonging to $BV_0(\Omega)$. If $v(s)$ is the function defined as in (3.22), then*

$$u^*(s) \le v(s) \qquad \textit{for a.e. } s \in [0, +\infty). \tag{3.23}$$

Proof. The isoperimetric inequality implies

$$n\omega_n^{\frac{1}{n}} \mu(t)^{1-\frac{1}{n}} \le \mathrm{Per}(\{u > t\}) \qquad \forall t \in [0. + \infty),$$

by (3.18) and (3.21) we have

$$G(s) = \int_{u^*(s)}^{+\infty} \mathrm{Per}(\{u > \xi\})\, d\xi = \int_{(0,s]} d\sigma(\tau) \qquad \text{for a.e. } s \in [0, +\infty).$$

Hence, for all $0 \le s_1 < s_2 < +\infty$ we have

$$\begin{aligned}
\sigma\big((s_1, s_2)\big) = \int_{s_1}^{s_2} d\sigma(\tau) &= \lim_{s \to s_2^-} G(s) - G(s_1) \\
&= \lim_{s \to s_2^-} \int_{u^\star(s)}^{u^\star(s_1)} \mathrm{Per}(\{u > \xi\})\, d\xi \\
&\ge \lim_{s \to s_2^-} \int_{u^\star(s)}^{u^\star(s_1)} n\omega_n^{\frac{1}{n}} \mu(\xi)^{1-\frac{1}{n}}\, d\xi \\
&= D\big[H(u^*)\big]\big((s_1, s_2)\big),
\end{aligned}$$

where

$$H(\tau) = \int_{\tau}^{+\infty} n\omega_n^{\frac{1}{n}} \mu(\xi)^{1-\frac{1}{n}}\, d\xi.$$

Since this holds for every open interval (s_1, s_2), we have

$$\sigma(A) \ge D\big[H(u^*)\big](A) \qquad \forall A \subseteq [0, +\infty), \text{ Borel set.} \tag{3.24}$$

Observing that H is a Lipschitz function, $D\big[H(u^*)\big]$ is given by (see [AD90])

$$D\big[H(u^*)\big] = \begin{cases} -n\omega_n^{\frac{1}{n}} s^{1-\frac{1}{n}} Du^* & \text{on } [0, +\infty) \setminus J_{u^*} \\ -n\omega_n^{\frac{1}{n}} s^{1-\frac{1}{n}} \big((u^*)^+ - (u^*)^-\big) & \text{on } J_{u^*}, \end{cases}$$

since $\mu(u^*(s)) = s$ a.e. with respect Du^* (by the properties of the rearrangements) and since for $s \in J_{u^*}$

$$H\big((u^*)^+(s)\big) - H\big((u^*)^-(s)\big) = \int_{u^*(s)}^{u^*(s^-)} n\omega_n^{\frac{1}{n}} \mu(\xi)^{1-\frac{1}{n}}\, d\xi$$
$$= -n\omega_n^{\frac{1}{n}} s^{1-\frac{1}{n}} \big((u^*)^+(s) - (u^*)^-(s)\big).$$

Then we can write

$$\frac{dD\big[H(u^*)\big]}{dDu^*} = -n\omega_n^{\frac{1}{n}} s^{1-\frac{1}{n}}. \tag{3.25}$$

Therefore, by means of (3.24), (3.25), we have

$$u^*(s) = -\int_s^{+\infty} d(Du^*)(\tau) = \int_s^{+\infty} \frac{dD\big[H(u^*)\big](\tau)}{n\omega_n^{\frac{1}{n}} \tau^{1-\frac{1}{n}}} \leq \int_s^{+\infty} \frac{d\sigma(\tau)}{n\omega_n^{\frac{1}{n}} \tau^{1-\frac{1}{n}}} = v(s).$$

□

As aforementioned, we want to introduce a symmetrization that keeps the absolutely continuous part separate from the singular part (sum of jump and Cantorian part) of the gradient. To be more precise, we define the radial function $u^\star \in W^{1,1}(\Omega^\sharp) \cap \mathrm{BV}_0(\Omega^\sharp) \cap L^\infty(\Omega^\sharp)$ such that

$$\begin{cases} |\nabla u^\star|(x) = |\nabla^a u|_\sharp(x) & \text{a.e. in } \Omega^\sharp \\ u^\star(x) = \dfrac{1}{\mathrm{Per}(\Omega^\sharp)} |D^s u|(\mathbb{R}^n) & \text{on } \partial\Omega^\sharp \end{cases}, \tag{3.26}$$

where $\nabla^a u$ and $D^s u$ will be defined in Section 1.4.

The main theorem of this section can be stated as follows.

Theorem 3.2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with finite perimeter and let $\Omega^\sharp$ be the centered ball. Assume that u is a non-negative function belonging to $BV_0(\Omega)$ and assume that $u^\star$ is defined as in (3.26), then*

$$\|u\|_{L^1(\Omega)} \leq \|u^\star\|_{L^1(\Omega^\sharp)}.$$

Proof. First of all, let us emphasize that the decreasing rearrangement of $u^\star$, defined in (3.26), is

$$(u^\star)^*(s) = \int_s^{+\infty} \frac{|\nabla^{\mathrm{a}} u|_*(t)}{n\omega_n^{\frac{1}{n}} t^{1-\frac{1}{n}}}\, dt + \frac{1}{\mathrm{Per}(\Omega^\sharp)} |D^s u|(\mathbb{R}^n)\, \chi_{[0,|\Omega|]}(s) \qquad \forall s \in [0, +\infty).$$

Now, let us integrate (3.23) between 0 and $+\infty$ and let us use Fubini's Theorem to obtain

$$\begin{aligned}\int_0^{+\infty} u^*(s)\,ds \le \int_0^{+\infty} v(s)\,ds\\ &= \frac{1}{n\omega_n^{\frac{1}{n}}}\int_0^{+\infty}\left(\int_s^{+\infty}\frac{d\sigma(t)}{t^{1-\frac{1}{n}}}\right)ds\\ &= \frac{1}{n\omega_n^{\frac{1}{n}}}\int_0^{+\infty}\left(\int_0^{t}\frac{ds}{t^{1-\frac{1}{n}}}\right)d\sigma(t)\\ &= \frac{1}{n\omega_n^{\frac{1}{n}}}\int_0^{+\infty} t^{\frac{1}{n}}\,dF(t)\\ &= \frac{1}{n\omega_n^{\frac{1}{n}}}\left[\int_0^{+\infty} t^{\frac{1}{n}}F_1(t)\,dt+\int_0^{+\infty} t^{\frac{1}{n}}\,d\sigma_2(t)\right].\end{aligned}$$

By (1.5) applied to F_1 and the Hardy-Littlewood inequality (1.2), we have

$$\begin{aligned}\int_0^{+\infty} t^{\frac{1}{n}}F_1(t)\,dt = \int_0^{|\Omega|} t^{\frac{1}{n}}F_1(t)\,dt &= \lim_k\int_0^{|\Omega|} t^{\frac{1}{n}}(F_1)_k(t)\,dt\\ &\le \int_0^{|\Omega|} t^{\frac{1}{n}}|\nabla^{\mathrm{a}}u|_*(t)\,dt = \int_0^{+\infty} t^{\frac{1}{n}}|\nabla^{\mathrm{a}}u|_*(t)\,dt,\end{aligned}$$

then

$$\begin{aligned}\int_0^{+\infty} u^*(s)\,ds &\le \frac{1}{n\omega_n^{\frac{1}{n}}}\left[\int_0^{+\infty} t^{\frac{1}{n}}|\nabla^{\mathrm{a}}u|_*(t)\,dt+\int_0^{+\infty} t^{\frac{1}{n}}\,d\sigma_2(t)\right]\\ &\le \frac{1}{n\omega_n^{\frac{1}{n}}}\left[\int_0^{+\infty} t^{\frac{1}{n}}|\nabla^{\mathrm{a}}u|_*(t)\,dt+|\Omega|^{\frac{1}{n}}\int_0^{+\infty} d\sigma_2(t)\right],\end{aligned}\tag{3.27}$$

since $F_2(A)=0$ for all $A\subset(|\Omega|,+\infty)$.

Using again Fubini's Theorem, we can compute

$$\int_0^{+\infty}|\nabla^{\mathrm{a}}u|_*(t)t^{\frac{1}{n}}\,dt=\int_0^{+\infty}\frac{|\nabla^{\mathrm{a}}u|_*(t)}{t^{1-\frac{1}{n}}}\int_0^t ds=\int_0^{+\infty}\left(\int_s^{+\infty}\frac{|\nabla^{\mathrm{a}}u|_*(t)}{t^{1-\frac{1}{n}}}\,dt\right)ds,$$

and

$$\frac{|\Omega|^{\frac{1}{n}}}{n\omega_n^{\frac{1}{n}}}\int_0^{+\infty} dF_2(t) = |\Omega|\frac{1}{\mathrm{Per}(\Omega^\sharp)}|D^s u|(\mathbb{R}^n)$$
$$= \int_0^{+\infty} \frac{1}{\mathrm{Per}(\Omega^\sharp)}|D^s u|(\mathbb{R}^n)\chi_{[0,|\Omega|]}(s)\,ds.$$

Hence, (3.27) can be written as

$$\|u\|_{L^1(\Omega)} \le \int_0^{+\infty}\left[\int_s^{+\infty} \frac{|\nabla^{\mathrm{a}} u|_*(t)}{n\omega_n^{\frac{1}{n}} t^{1-\frac{1}{n}}}\,dt + \frac{1}{\mathrm{Per}(\Omega^\sharp)}|D^s u|(\mathbb{R}^n)\chi_{[0,|\Omega|]}(s)\right] ds$$
$$= \|u^\star\|_{L^1(\Omega^\sharp)}.$$

□

Remark 3.2.1. We stress the following facts:

$$|D^{\mathrm{a}} u|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |\nabla^{\mathrm{a}} u|\,dx = \int_{\Omega^\sharp} |\nabla^{\mathrm{a}} u^\star|\,dx \quad \text{and} \quad |D^{\mathrm{s}} u|(\mathbb{R}^n) = |D^{\mathrm{s}} u^\star|(\mathbb{R}^n),$$

and then

$$|Du|(\mathbb{R}^n) = |Du^\star|(\mathbb{R}^n).$$

3.3 | A weighted L^1 comparison

Let us check how extend the result by [Tal94b] to the case of function non vanishing on the boundary.

Theorem 3.3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open and Lipschitz set. Let $f \in L^\infty(\Omega)$ be a function such that*

$$f^*(t) \ge \left(1 - \frac{1}{n}\right)\frac{1}{t}\int_0^t f^*(s)\,ds \qquad \forall t \in [0, |\Omega|]. \tag{3.28}$$

If $u \in W^{1,p}(\Omega)$ and $u^\star$ is the function given by Theorem 3.1.1, then

$$\int_\Omega f(x)u(x)\,dx \le \int_{\Omega^\sharp} f^\sharp(x)u^\star(x)\,dx. \tag{3.29}$$

Proof. If $u \in W_0^{1,p}(\Omega)$, the result is contained in [Tal94b]. We recall it, for sake of completeness.

By [GN84, eq. 2.7] it is known

$$u^*(s) \leq \frac{1}{n\omega_n^{\frac{1}{n}}} \int_s^{|\Omega|} \frac{F(t)}{t^{1-\frac{1}{n}}}\, dt, \tag{3.30}$$

where F is a function such that

$$\int_0^s F(t)\, dt = \int_{D(s)} |\nabla u|_*(s)\, ds,$$

with $D(s)$ defined in Section 1.2.3.

Setting $g(t) := \frac{1}{t^{1-\frac{1}{n}}} \int_0^t f^*(s)\, ds$, multiplying both terms of (3.30) for $f^*(s)$, integrating from 0 to $|\Omega|$ and using Fubini's Theorem we get

$$\begin{aligned} \int_0^{|\Omega|} f^*(s)u^*(s)\, ds &\leq \frac{1}{n\omega_n^{\frac{1}{n}}} \int_0^{|\Omega|} f^*(s) \left(\int_s^{|\Omega|} \frac{F(t)}{t^{1-\frac{1}{n}}}\, dt \right) ds \\ &= \frac{1}{n\omega_n^{\frac{1}{n}}} \int_0^{|\Omega|} F(t)g(t)\, dt. \end{aligned} \tag{3.31}$$

Let us suppose that $g(t)$ is non-decreasing, so $g_*(s) = g(s)$ and by Lemma 1.2.1 there exists a sequence $\{F_k\}$ such that $(F_k)_* = (\nabla u)_*$ and $F_k \rightharpoonup F$ in BV. Therefore

$$\int_0^{|\Omega|} F(t)g(t)\, dt = \lim_k \int_0^{|\Omega|} F_k(t)g(t)\, dt.$$

Using Hardy-Littlewood's inequality we have

$$\lim_k \int_0^{|\Omega|} F_k(t)g(t)\, dt \leq \int_0^{|\Omega|} |\nabla u|_*(t)g_*(t)\, dt = \int_0^{|\Omega|} |\nabla u|_*(t)g(t)\, dt.$$

Hence, by (3.31) and Fubini's Theorem, we obtain

$$\begin{aligned}\int_0^{|\Omega|} f^*(t)u^*(t)\,dt &\leq \frac{1}{n\omega_n^{\frac{1}{n}}}\int_0^{|\Omega|} |\nabla u|_*(t)\,g(t)\,dt \\ &= \frac{1}{n\omega_n^{\frac{1}{n}}}\int_0^{|\Omega|} |\nabla u|_*(t)\left(\frac{1}{t^{1-\frac{1}{n}}}\int_0^t f^*(s)\,ds\right)dt \\ &= \int_0^{|\Omega|} f^*(s)\left(\frac{1}{n\omega_n^{\frac{1}{n}}}\int_s^{|\Omega|}\frac{|\nabla u|_*(t)}{t^{1-\frac{1}{n}}}\,dt\right)ds \\ &= \int_0^{|\Omega|} f^*(s)(u^\star)^*(s)\,ds.\end{aligned}$$

Therefore, by Hardy-Littlewood inequality, we have

$$\int_\Omega f(x)u(x)\,dx \leq \int_0^{|\Omega|} f^*(t)u^*(t) \leq \int_0^{|\Omega|} f^*(s)(u^\star)^*(s)\,ds = \int_{\Omega^\sharp} f^\sharp(x)\,u^\star(x)\,dx. \tag{3.32}$$

But we have to deal with the assumption that g is non-decreasing, $g' \geq 0$, that is

$$\frac{d}{dt}\left(\frac{1}{t^{1-\frac{1}{n}}}\int_0^t f^*(s)\,ds\right) = -\frac{n-1}{n}\frac{1}{t^{2-\frac{1}{n}}}\left(\int_0^t f^*(s)\,ds\right) + \frac{1}{t^{1-\frac{1}{n}}}f^*(t) \geq 0,$$

hence, if and only if

$$f^*(t) \geq \left(1-\frac{1}{n}\right)\frac{1}{t}\int_0^t f^*(s)\,ds.$$

Now let us deal with $u \notin W_0^{1,p}(\Omega)$. Suppose that $u \in C^2(\Omega)$ is a non-negative function, that Ω has C^2 boundary and that f satisfies (3.28). Proceeding as in Step 1 of Theorem 3.1.1, for every $\varepsilon > 0$ we can construct u_ε that coincides with u in Ω and is zero on $\partial\Omega_\varepsilon$. Moreover we can extend f to Ω_ε simply defining

$$f_\varepsilon(t) = \begin{cases} f(x) & \text{in } \Omega \\ f^*(|\Omega|) & \text{in } \Omega_\varepsilon \setminus \Omega.\end{cases}$$

The rearrangement, for every $\varepsilon > 0$, is

$$f_\varepsilon^*(t) = \begin{cases} f^*(t) & \text{in } \big[0, |\Omega|\big] \\ f^*(|\Omega|) & \text{in } \big[|\Omega|, |\Omega_\varepsilon|\big],\end{cases}$$

so we just have to check (3.28) for $t \in \big[|\Omega|, |\Omega_\varepsilon|\big]$, namely

$$f_\varepsilon^*(t) \geq \left(\frac{n-1}{n}\right)\frac{1}{t}\int_0^t f_\varepsilon^*(s)\,ds. \tag{3.33}$$

Keeping in mind that f verifies (3.28), we have

$$f_\varepsilon^*(t) = f^*(|\Omega|) \geq \left(\frac{n-1}{n}\right)\frac{1}{|\Omega|}\int_0^{|\Omega|} f^*(s)\,ds.$$

If we show that

$$\frac{1}{|\Omega|}\int_0^{|\Omega|} f^*(s)\,ds \geq \left[\frac{1}{t}\int_0^{|\Omega|} f^*(s)\,ds + \frac{t-|\Omega|}{t}f^*(|\Omega|)\right] = \frac{1}{t}\int_0^t f_\varepsilon^*(s)\,ds,$$

then (3.33) is true. By direct calculations

$$\frac{t-|\Omega|}{t|\Omega|}\int_0^{|\Omega|} f^*(s)\,ds \geq \frac{t-|\Omega|}{t}f^*(|\Omega|) \iff \frac{1}{|\Omega|}\int_0^{|\Omega|} f^*(s)\,ds \geq f^*(|\Omega|).$$

that is true of the fact that f^* is decreasing.

So, $\forall \varepsilon > 0$ we can apply the first part of the Theorem obtaining

$$\int_{\Omega_\varepsilon} u_\varepsilon f_\varepsilon\,dx \leq \int_{\Omega_\varepsilon^\sharp} v_\varepsilon f_\varepsilon^\sharp\,dx.$$

Sending $\varepsilon \to 0$ we get

$$\int_\Omega uf\,dx \leq \int_{\Omega_\sharp} u^\star f^\sharp\,dx.$$

Arguing as in Theorem 3.1.1, we get (3.29). $\square$

Remark 3.3.1. Condition (3.28) implies the f is strictly positive. Moreover, if the essential oscillation of f is bounded

$$\text{ess osc}\,|f| := \frac{\operatorname*{ess\,sup}_{x\in\Omega}|f(x)|}{\operatorname*{ess\,inf}_{x\in\Omega}|f(x)|} \leq \frac{n}{n-1},$$

then (3.28) is satisfied.

Moreover we can use Theorem 3.3.1 to get a comparison between Lorentz norm of u and $u^\star$.

Corollary 3.3.2. *Let $1 \le p \le \frac{n}{n-1}$, under the assumption of Theorem 3.1.1 it holds*

$$\|u\|_{L^{p,1}(\Omega)} \le \|u^\star\|_{L^{p,1}(\Omega^\sharp)}, \tag{3.34}$$

where $u^\star$ is the function given by Theorem 3.1.1

Proof. Let us explicit the $L^{p,1}$ norm of u

$$\|u\|_{L^{p,1}(\Omega)} = \int_0^{+\infty} t^{\frac{1}{p}-1} u^*(t)\,dt = \int_0^{+\infty} t^{-\frac{1}{p'}} u^*(t)\,dt.$$

Hence by Theorem 3.3.1, it is sufficient that

$$t^{-\frac{1}{p'}} - \frac{n-1}{n}\frac{1}{t}\int_0^t s^{-\frac{1}{p'}}\,ds \ge 0. \tag{3.35}$$

If we compute

$$\frac{1}{t}\int_0^t s^{-\frac{1}{p'}}\,ds = \frac{1}{t} p\, t^{-\frac{1}{p'}+1} = p\, t^{-\frac{1}{p'}},$$

then we have

$$t^{-\frac{1}{p'}} - \frac{n-1}{n}\frac{1}{t}\int_0^t s^{-\frac{1}{p'}}\,ds = t^{-\frac{1}{p'}}\left(1 - \frac{n-1}{n}p\right) \ge 0 \iff p \le \frac{n}{n-1},$$

so (3.35) is true and we can apply Theorem 3.3.1 obtaining

$$\int_0^{+\infty} t^{-\frac{1}{p'}} u^*(t)\,dt \le \int_0^{+\infty} t^{-\frac{1}{p'}} u^\star(t)\,dt,$$

that is (3.34). □

Remark 3.3.2. We emphasize that the bound $p \le \frac{n}{n-1}$ is the best we can hope for Lorentz norm $L^{q,1}$. Indeed, if by absurd (3.34) holds for $p > \frac{n}{n-1}$, by the embedding of $L^{p,q}$ spaces, $L^{q,1}(\Omega) \subseteq L^{q,q}(\Omega) = L^q(\Omega)$, which gives a contradiction.

3.4 Application to Torsional Rigidities

Here we present some applications of this type of rearrangement.

3.4.1 A Torsional Rigidity with Robin boundary conditions

Let $\beta > 0$, let $\Omega \subset \mathbb{R}^n$ be a bounded open and Lipschitz set. We consider the following functional:

$$\mathcal{F}_\beta(\Omega, \psi) = \frac{\displaystyle\int_\Omega |\nabla \psi|^2 \, dx + \beta \mathrm{Per}(\Omega) \int_{\partial\Omega} \psi^2 \, d\mathcal{H}^{n-1}}{\left(\displaystyle\int_\Omega \psi \, dx \right)^2}, \qquad \psi \in W^{1,2}(\Omega),$$

and the associate minimum problem

$$T(\Omega, \beta) = \inf_{\psi \in W^{1,2}(\Omega)} \mathcal{F}_\beta(\Omega, \psi).$$

The minimum u is a weak solution to

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \beta \mathrm{Per}(\Omega)\, u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.36}$$

Our aim is to compare $T(\Omega, \beta)$ with

$$T(\Omega^\sharp, \beta) := \min_{v \in W^{1,2}(\Omega)} \mathcal{F}_{\Omega,\beta}(v) = \min_{v \in W^{1,2}(\Omega)} \frac{\displaystyle\int_{\Omega^\sharp} |\nabla v|^2 \, dx + \beta \mathrm{Per}(\Omega^\sharp) \int_{\partial\Omega^\sharp} v^2 \, d\mathcal{H}^{n-1}}{\left(\displaystyle\int_{\Omega^\sharp} v \, dx \right)^2},$$

where the minimum is a weak solution to

$$\begin{cases} -\Delta z = 1 & \text{in } \Omega^\sharp \\ \dfrac{\partial z}{\partial \nu} + \beta \mathrm{Per}(\Omega^\sharp)\, z = 0 & \text{on } \partial\Omega^\sharp. \end{cases} \tag{3.37}$$

In particular we get

Corollary 3.4.1. *Let $\beta > 0$, let $\Omega \subset \mathbb{R}^n$ be a bounded, open and Lipschitz set. If we denote with $\Omega^\sharp$ the ball centered at the origin with same measure as Ω, it holds*

$$T(\Omega, \beta) \geq T(\Omega^\sharp, \beta). \tag{3.38}$$

Proof. Let $w \in W^{1,p}(\Omega)$, by Theorem 3.1.1 and Remark 3.1.1 there exists $w^\star \in W^{1,\infty}(\Omega^\sharp)$ radial such that

$$\int_\Omega |\nabla w|^2\,dx = \int_{\Omega^\sharp} |\nabla w^\star|^2\,dx$$
$$\int_\Omega |w|\,dx = \int_{\Omega^\sharp} |w^\star|\,dx$$
$$\mathcal{H}^{n-1}(\partial\Omega^\sharp)\int_{\partial\Omega^\sharp}(w^\star)^2 \le \mathcal{H}^{n-1}(\partial\Omega)\int_{\partial\Omega} w^2.$$

Therefore

$$\mathcal{F}_\beta(w) \ge \mathcal{F}_\beta(w^\star).$$

Passing to the infimum on right-hand side and successively to the left-hand side, we obtain

$$T(\Omega,\beta) \ge T(\Omega^\sharp,\beta).$$

□

Remark 3.4.1. We highlight that all the arguments work also in the non-linear case, where the functional

$$\mathcal{F}_{\beta,p}(w) = \frac{\displaystyle\int_\Omega |\nabla w|^p\,dx + \beta \mathrm{Per}(\Omega)^{p-1}\int_{\partial\Omega} w^p\,d\mathcal{H}^{n-1}}{\left(\displaystyle\int_\Omega w\,dx\right)^p} \qquad \text{for } w \in W^{1,p}(\Omega). \tag{3.39}$$

is considered.

Remark 3.4.2. If f satisfied (3.28), then theorem 3.3.1 allows us to compare the minimum of

$$T_{\beta,f}(\Omega) := \min_{w\in W^{1,2}(\Omega)}\left\{\frac{1}{2}\int_\Omega |\nabla w|^2\,dx + \frac{\beta\mathrm{Per}(\Omega)}{2}\int_{\partial\Omega} w^2\,d\mathcal{H}^{n-1} - \int_\Omega wf\,dx\right\},$$

with the one of

$$T_{\beta,f}(\Omega^\sharp) := \min_{v\in W^{1,2}(\Omega^\sharp)}\left\{\frac{1}{2}\int_{\Omega^\sharp} |\nabla v|^2\,dx + \frac{\beta\mathrm{Per}(\Omega^\sharp)}{2}\int_{\partial\Omega^\sharp} v^2\,d\mathcal{H}^{n-1} - \int_{\Omega^\sharp} vf^\sharp\,dx\right\}.$$

Indeed, it holds

$$T_{\beta,f}(\Omega) \ge T_{\beta,f^\sharp}(\Omega^\sharp).$$

3.4.2 A penalized Torsional Rigidity

For a given $\Lambda > 0$, we consider the following functional

$$\mathcal{F}_\Lambda(\psi) := \frac{1}{2}\int_\Omega |\nabla\psi|^2\,dx - \int_\Omega \psi\,dx + \Lambda\big|\{|\nabla\psi| \neq 0\}\big| \qquad \psi \in H_0^1(\Omega), \quad (3.40)$$

and the associated minimum problem:

$$T_{\mathcal{F}}(\Omega,\Lambda) := -\inf_{\psi\in H_0^1(\Omega)} \mathcal{F}_\Lambda(\psi). \quad (3.41)$$

First of all, let us observe that the minimum can be found among non-negative functions. Indeed, passing from ψ to $|\psi|$ it holds $\mathcal{F}(\psi) \geq \mathcal{F}(|\psi|)$.

Assuming that problem (3.41) admits a minimum $u \in H_0^1(\Omega)$, then it is also a maximum for the torsional rigidity defined by Diaz, Polya and Weinstein in [DW48; PW50] of a multiply-connected cross-section with fixed measure of the holes, that is

$$T(\Omega) = \max_{\substack{\psi\in C_0(D)\cap C^1(\Omega)\\ \psi \text{ constant}\\ \text{in every } A_i}} \frac{\left(\displaystyle\int_D \psi\,dx\right)^2}{\displaystyle\int_D |\nabla\psi|^2\,dx},$$

where A_i are the connected component of $\{|\nabla u| = 0\}$ and $D = \Omega \cup \bigcup_i A_i$.

Functionals with penalizing terms are very common in the mathematical modelling of physical problems. The bibliography is very wide and some cornerstones are [AC81; DCL89].

However, in the literature, penalizing terms of the form $|\{|\nabla\psi| \neq 0\}|$ are quite unusual. The main difficulty in the study of (3.41) is to prove the existence of a minimizer because of the lack of the lower semicontinuity of the functional.

For this reason, we prove the existence of a minimizer in the case when Ω is a ball.

Proposition 3.4.2. *Let $\Lambda, R > 0$ and let B_R be the centered ball with radius R. Then the functional $\mathcal{F}_\Lambda$ defined in (3.40) admits a minimizer v belonging to $H_0^1(\Omega)$. Such a minimizer is unique up to a sign, it is radially symmetric and $|\nabla v|$ is radially increasing.*

Proof. We divide the proof in 3 steps.

1. Boundness from below.

   First of all, let us prove that the functional $\mathcal{F}_\Lambda$ is bounded from below for every choice of Λ and for every $R > 0$. For all $\psi \in H^1_0(B_R)$, sing Young and Poincaré inequalities, we get

$$\begin{aligned}
\mathcal{F}_\Lambda(\psi) &= \frac{1}{2}\int_{B_R} |\nabla\psi|^2\,dx - \int_{B_R} \psi\,dx + \Lambda\big|\{|\nabla\psi| \neq 0\}\big| \\
&\geq \frac{1}{2}\int_{B_R} |\nabla\psi|^2\,dx - \varepsilon\int_{B_R} \frac{\psi^2}{2} - \frac{|B_R|}{2\varepsilon} \\
&\geq \frac{1}{2}\int_{B_R} |\nabla\psi|^2\,dx - \frac{\varepsilon C(n,B_R)}{2}\int_{B_R} |\nabla\psi|^2\,dx - \frac{|B_R|}{2\varepsilon} \\
&= \frac{(1-\varepsilon C(n,B_R))}{2}\int_{B_R} |\nabla\psi|^2\,dx - \frac{|B_R|}{2\varepsilon}.
\end{aligned}$$

   Chosing ε sufficiently small such that

$$0 < \varepsilon \leq \frac{1}{C(n,B_R)},$$

   then

$$\mathcal{F}_\Lambda(\psi) \geq -\frac{|B_R|}{2C(n,B_r)} \geq -C(n,B_R) > -\infty,$$

   so

$$T(B_R,\Lambda) = -\inf_{\psi\in H^1_0(B_R)} \mathcal{F}_\Lambda(\psi) < \infty.$$

2. Compactness and semicontinuity.

   Now we consider a minimizing sequence $\{\psi_k\}$ for $T_\mathcal{F}(B_R,\Lambda)$ and we prove that it is bounded in $H^1_0(B_R)$. We can assume that $\mathcal{F}_\Lambda(\psi_k) \leq -T_\mathcal{F}(B_R,\Lambda) + 1$ and by Proposition 1.2.2 we can assume that ψ_k are radial function with $|\nabla\psi_k|$ radially symmetric increasing.

Using Young and Poincaré inequalities, we obtain

$$\begin{aligned}
\mathcal{F}_\Lambda(\psi_k) &= \frac{1}{2}\int_{B_R} |\nabla\psi_k|^2\,dx - \int_{B_R} \psi_k\,dx + \Lambda\big|\big\{|\nabla\psi_k| \neq 0\big\}\big| \\
&\geq \frac{1}{2}\int_{B_R} |\nabla\psi_k|^2\,dx - \int_{B_R} \psi_k\,dx \\
&\geq \frac{1}{2}\int_{B_R} |\nabla\psi_k|^2\,dx - \varepsilon\int_{B_R} \frac{\psi_k^2}{2} - \frac{|B_R|}{2\varepsilon} \\
&\geq \frac{1}{2}\int_{B_R} |\nabla\psi_k|^2\,dx - \frac{\varepsilon C(n,B_r)}{2}\int_{B_R} |\nabla\psi_k|^2\,dx - \frac{|B_R|}{2\varepsilon} \\
&= \frac{1-\varepsilon C(n,B_R)}{2}\int_{B_R} |\nabla\psi_k|^2\,dx - \frac{|B_R|}{2\varepsilon}.
\end{aligned}$$

Choosing $\varepsilon < \dfrac{1}{C(n,B_R)}$ we have

$$-T_{\mathcal{F}}(B_R,\Lambda) + 1 \geq \mathcal{F}_\Lambda(\psi_k) \geq \frac{1}{4}\int_{B_R} |\nabla\psi_k|^2\,dx - C(B_R),$$

then by Poincaré inequality, the sequence $\{\psi_k\}$ is bounded in $H_0^1(B_R)$.

This implies that there exists a subsequence (still denoted by ψ_k) and a function $v \in H_0^1(B_R)$ such that $\psi_k \to v$ strongly in $L^2(\Omega)$, a.e. in Ω and $\nabla\psi_k \rightharpoonup \nabla v$ weakly in L^2. Let us show that v is a minimum for $\mathcal{F}_\Lambda$.

The lower semicontinuity of the norms gives

$$\liminf_k \left[\frac{1}{2}\int_{B_R} |\nabla\psi_k|^2\,dx - \int_{B_R} \psi_k\,dx\right] \geq \frac{1}{2}\int_{B_R} |\nabla v|^2\,dx - \int_{B_R} v\,dx. \tag{3.42}$$

Let us deal with the last term of $\mathcal{F}_\Lambda$ and let us prove that

$$\liminf_k \big|\big\{|\nabla u_k| \neq 0\big\}\big| \geq \big|\big\{|\nabla u| \neq 0\big\}\big|.$$

Denoting by r_k the radius of the ball where $|\nabla\psi_k| = 0$, we can assume that r_k converges to some $r \geq 0$. Therefore

$$\liminf_k\, |\{|\nabla\psi_k| \neq 0\}| = \lim_k\,[\omega_n(R^n - r_k^n)] = \omega_n(R^n - r^n).$$

So we have just to prove that $|\nabla v| = 0$ in B_r. Since $\{\psi_k\}$ are radial functions, obviously v is radial too.

If $r = 0$ there is nothing to prove.

If $r > 0$, assume by contradiction that there exists $A \subset B_r$ with $|A| > 0$ and that $|\nabla v| \neq 0$ in A. Clearly there exists $\varepsilon > 0$ such that $|A \cap B_{r-\varepsilon}| > 0$.

Since $r_k \to r$ if we choose a function $g \in C_C^\infty(B_R, \mathbb{R}^n)$ with support included in $A \cap B_{r-\varepsilon}$ we have

$$\int_{B_R} \langle \nabla v, g \rangle \, dx = \lim_k \int_{B_R} \langle \nabla \psi_k, g \rangle \, dx = 0.$$

Since this must be true for every $g \in C_C^\infty(A \cap B_{r-\varepsilon}, \mathbb{R}^n)$, we get a contradiction.

Then in any case

$$\liminf_k |\{|\nabla \psi_k| \neq 0\}| \geq |\{|\nabla v| \neq 0\}|. \tag{3.43}$$

By (3.42) and(3.43), we get

$$-T_{\mathcal{F}}(B_R, \Lambda) = \liminf_k \mathcal{F}_\Lambda(\psi_k) \geq \mathcal{F}_\Lambda(v) \geq -T_{\mathcal{F}}(B_R, \Lambda),$$

so v is a minimum of $\mathcal{F}_\Lambda$ in B_R.

3. Uniqueness.

Let us suppose that v is a minimum of $\mathcal{F}_\Lambda(\psi)$. By Theorem 1.2.2, it exists $\overline{v} \in H_0^1(B_R)$ such that

$$\mathcal{F}_\Lambda(v) \geq \mathcal{F}_\Lambda(\overline{v}),$$

and since v is minimum, it holds

$$\mathcal{F}_\Lambda(v) = \mathcal{F}_\Lambda(\overline{v}).$$

Since $|\nabla v|$ is equally distributed with $|\nabla \overline{v}|$, the previous equality implies

$$\|v\|_{L^1} = \|\overline{v}\|_{L^1},$$

so Theorem 1.2.3 gives that $|v| = \overline{v}$.

□

Remark 3.4.3. We highlight that Theorem 1.2.2 ensures us that the minimum when Ω is a ball has gradient equal to zero only in a ball B_r centered at the origin with $0 \leq r \leq R$.

Now, as already above mentioned, we prove a Saint-Venant type inequality for $T_{\mathcal{F}}(\Omega, \Lambda)$.

Corollary 3.4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with finite perimeter and let $\Omega^\sharp$ be the centered ball. If $\Lambda > 0$, then*

$$T_{\mathcal{F}}(\Omega, \Lambda) \leq T_{\mathcal{F}}(\Omega^\sharp, \Lambda).$$

Proof. For every function $\psi \in H_0^1(\Omega)$, by Theorem 1.2.2 or 3.2.3, there exists $\overline{\psi} \in H_0^1(\Omega^\sharp)$ that satisfies

$$\mathcal{F}_\Lambda(\psi) \geq \mathcal{F}_\Lambda(\overline{\psi}) \geq -T_{\mathcal{F}}(\Omega^\sharp, \Lambda),$$

and then

$$T_{\mathcal{F}}(\Omega, \Lambda) \leq T_{\mathcal{F}}(\Omega^\sharp, \Lambda).$$

$\square$

3.4.3 A Torsional Rigidity related to thermal insulation

Now we deal with the functional

$$\mathcal{G}(\psi) := \frac{\displaystyle\int_\Omega |\nabla \psi|^2 \, dx + \frac{1}{m} \left(\int_{\partial\Omega} |\psi| \, d\mathcal{H}^{n-1} \right)^2}{\displaystyle\left(\int_\Omega |\psi| \, dx \right)^2} \qquad \psi \in H^1(\Omega),$$

with $m > 0$.

The interest in this type of functional is related to the problem of optimal insulation in a given domain. Indeed, the minimum of $\mathcal{G}$ gives the long-time distribution of temperature of the domain Ω and the displacement around Ω of a thin layer of insulator with total mass equal to m. We refer to [BBN17] for more details.

If Ω is a Lipschitz domain, $\mathcal{G}(\psi)$ achieves its minimum among all $H^1(\Omega)$ functions. So we define

$$\frac{1}{T_{\mathcal{G}}(\Omega, m)} := \min_{\psi \in H^1(\Omega)} \mathcal{G}(\psi).$$

It is easy to check that the Euler–Lagrange equation of this functional is

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} + \dfrac{1}{m} \displaystyle\int_{\partial\Omega} |u| \, d\mathcal{H}^{n-1} = 0 & \text{on } \partial\Omega. \end{cases}$$

So Theorem 3.2.3 gives us the following Saint-Venant type inequality for $T_{\mathcal{G}}(\Omega)$.

Corollary 3.4.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with finite perimeter and let $\Omega^\sharp$ be the centered ball. If $m > 0$, then*

$$T_{\mathcal{G}}(\Omega, m) \leq T_{\mathcal{G}}(\Omega^\sharp, m).$$

Proof. For every function $\psi \in H^1(\Omega)$, by 3.2.3, there exists $\overline{\psi} \in H^1(\Omega^\sharp)$ that satisfies

$$\mathcal{G}(\psi) \geq \mathcal{G}(\overline{\psi}) \geq \frac{1}{T_{\mathcal{G}}(\Omega^\sharp, m)},$$

and then

$$T_{\mathcal{G}}(\Omega, m) \leq T_{\mathcal{G}}(\Omega^\sharp, m).$$

□

Chapter 4

Two shape optimization problems for eigenvalues

In this Chapter we study two shape shape functional in the class of convex sets: the Robin p-Laplacian eigenvalue and the Neumann Laplacian.

In Section 4.1 we will consider the eigenvalue of the p-Laplacian operator with Robin boundary conditions among convex sets having prescribed perimeter and give continuity and quantitative estimate for this functional, respectively in the case $\beta > 0$ and $\beta < 0$, in terms of isoperimetric ratio.

In Section 4.2 we will focus on the eigenvalue of the Laplacian operator with Neumann boundary conditions among convex sets having prescribed diameter. Recalling that the Kröger inequalities are sharp but only asymptotically achieved by suitable sequence of collapsing domains, we will give upper bound for these inequalities in terms of the second semiaxis of the John ellipsoid.

4.1 Estimates for Robin p-Laplacian eigenvalues

Let Ω be a bounded, open and convex set in $\mathbb{R}^n$. We recall that the fundamental eigenvalue of the Robin Laplacian on Ω is defined by

$$\lambda_{p,\beta}(\Omega) = \min_{\substack{w \in W^{1,p}(\Omega) \\ w \neq 0}} \frac{\displaystyle\int_\Omega |\nabla w|^p \, dx + \beta \int_{\partial\Omega} |w|^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |w|^p \, dx}. \tag{4.1}$$

To prove our results, we will need some properties of the eigenfunction of $\lambda_{p,\beta}$ on the ball.

4.1.1 Monotonicity of eigenfunctions

It is clear from the definition of the eigenvalue (4.1) that for fixed $1 < p < +\infty$ and $\Omega \subset \mathbb{R}^n$, the map

$$\beta \in \mathbb{R} \mapsto \lambda_{p,\beta}(\Omega),$$

is increasing and it holds (see for instance [Lê06])

$$\mu_p < \lambda_{p,\beta} < \lambda_{p,D}, \qquad \forall \beta \in (0, +\infty),$$

and

$$\lim_{\beta \to 0^+} \lambda_{p,\beta} = \mu_p = 0, \qquad \lim_{\beta \to +\infty} \lambda_{p,\beta} = \lambda_{p,D},$$

where μ_p and $\lambda_{p,D}$ denotes the first eigenvalues of the Neumann p-Laplacian and Dirichlet p-Laplacian respectively.

Now we want to observe that, if we consider a ball B_R and we fix the value of the eigenfunction in the origin, the map

$$\beta \mapsto v_{p,\beta}, \qquad \text{such that } v_{p,\beta(0)} = C,$$

is decreasing, in the sense that

$$\text{if } \beta_1 < \beta_2 \Rightarrow v_{p,\beta_1}(x) \geq v_{p,\beta_2}(x), \qquad \forall x \in B_R. \tag{4.2}$$

With this aim, let us recall that the first eigenvalue of p-Laplacian is simple and that the corresponding eigenfunction is radially symmetric, i.e. there exists

$h\colon [0,R] \to \mathbb{R}$ such that $v_{p,\beta}(x) = h(|x|)$. In the following, we will denote with $v_{p,\beta}$ both the eigenfunction and the function h.

In order to prove the monotonicity in (4.2), we first prove an intermediate lemma. This result is not new in the literature (see for instance [BD10, Proposition 4.2]), but it is not usually stated as proposition or lemma.

Lemma 4.1.1. *Let $0 < r < R$, $1 < p < +\infty$, $\beta \in \mathbb{R}$. Let us denote by $\lambda_{p,\beta}$ the first eigenvalue of the p-Laplacian on the ball B_R defined in (4.1) with boundary parameter β and let $v_{p,\beta}$ be the corresponding eigenfunction. Then, $v_{p,\beta}|_{B_r}$ is the first eigenfunction of the p-Laplacian on the ball B_r with boundary parameter*

$$\gamma = -\frac{|v'_{p,\beta}|^{p-2}(r)v'_{p,\beta}(r)}{v_{p,\beta}^{p-1}(r)}.$$

Proof. Let us suppose $p = 2$, the general case is analogous. For sake of simplicity, we denote by $\lambda_\beta := \lambda_{2,\beta}$ and $v_\beta := v_{2,\beta}$.

By the radiality of v_β, we can infer that it is a solution to

$$\begin{cases} -\Delta v_\beta = \lambda_\beta v_\beta & \text{in } B_r \\ \dfrac{\partial v_\beta}{\partial \nu} + \gamma v_\beta = 0 & \text{on } \partial B_r. \end{cases}$$

Let us suppose by contradiction that λ_β is not the first eigenvalue of the Robin Laplacian with boundary parameter γ. So we can choose

$$\lambda_\gamma < \lambda_\beta, \tag{4.3}$$

the first Robin eigenvalue with boundary parameter γ and w_γ the associated eigenfunction, that is

$$\lambda_\gamma = \frac{\displaystyle\int_{B_r} |\nabla w_\gamma|^2\,dx + \gamma \int_{\partial B_r} w_\gamma^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_{B_r} w_\gamma^2\,dx} = \min_{\psi \in W^{1,2}(B_r)} \frac{\displaystyle\int_{B_r} |\nabla \psi|^2\,dx + \gamma \int_{\partial B_r} \psi^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_{B_r} \psi^2\,dx}. \tag{4.4}$$

We know that w_γ is unique up to a multiplicative constant, so we can choose the constant such that $v_\beta = w_\gamma$ on ∂B_r.

Let us consider the function

$$f(x) = \begin{cases} w_\gamma(x) & \text{if } x \in B_r \\ v_\beta(x) & \text{if } x \in B_R \setminus B_r, \end{cases}$$

we can use it as a test function in the definition of λ_β. In particular, we have

$$\lambda_\beta \le \frac{\displaystyle\int_{B_R} |\nabla f|^2 + \beta \int_{\partial B_r} f^2 \, d\mathcal{H}^{n-1}}{\displaystyle\int_{B_R} f^2 \, dx} = \frac{\displaystyle\int_{B_r} |\nabla w_\gamma|^2 + \int_{B_R\setminus B_r} |\nabla v_\beta|^2 + \beta \int_{\partial B_R} v_\beta^2 \, d\mathcal{H}^{n-1}}{\displaystyle\int_{B_r} w_\gamma^2 \, dx + \int_{B_R\setminus B_r} v_\beta^2 \, dx}.$$

If we add and subtract $\gamma \displaystyle\int_{\partial B_r} w_\gamma^2 \, d\mathcal{H}^{n-1}$ and $\displaystyle\int_{B_r} |\nabla v_\beta|^2 \, dx$, and we recall that v_β achieves (4.1) on B_R with value λ_β and w_γ achieves (4.1) on B_r with value λ_γ, we have

$$\lambda_\beta \le \frac{\displaystyle\lambda_\gamma \int_{B_r} w_\gamma^2 \, dx + \lambda_\beta \int_{B_R} v_\beta^2 \, dx - \gamma \int_{\partial B_r} w_\gamma^2 \, d\mathcal{H}^{n-1} - \int_{B_r} |\nabla v_\beta|^2 \, dx}{\displaystyle\int_{B_r} w_\gamma^2 \, dx + \int_{B_R\setminus B_r} v_\beta^2 \, dx}.$$

Since $v_\beta = w_\gamma$ on ∂B_r and v_β is an eigenfunction on B_r with eigenvalue λ_β, we get

$$\begin{aligned}
\lambda_\beta &\le \frac{\displaystyle\lambda_\gamma \int_{B_r} w_\gamma^2 \, dx + \lambda_\beta \int_{B_R} v_\beta^2 \, dx - \lambda_\beta \int_{B_r} v_\beta^2 \, dx}{\displaystyle\int_{B_r} w_\gamma^2 \, dx + \int_{B_R\setminus B_r} v_\beta^2 \, dx} \\
&= \frac{\displaystyle\lambda_\gamma \int_{B_r} w_\gamma^2 \, dx + \lambda_\beta \int_{B_R\setminus B_r} v_\beta^2 \, dx}{\displaystyle\int_{B_r} w_\gamma^2 \, dx + \int_{B_R\setminus B_r} v_\beta^2 \, dx} \\
&< \lambda_\beta \frac{\displaystyle\int_{B_r} w^2 \, dx + \int_{B_R\setminus B_r} w^2 \, dx}{\displaystyle\int_{B_r} w^2 \, dx + \int_{B_R\setminus B_r} w^2 \, dx} = \lambda_\beta,
\end{aligned}$$

where in the last formula we use (4.3), and we get a contradiction.

$\square$

Now we are in position to prove (4.2).

Proposition 4.1.2. *Let $R > 0$, $1 < p < +\infty$ and $\beta_1 < \beta_2$. Let us denote by λ_{p,β_1} and λ_{p,β_2} the eigenvalues defined in (4.1) and let v_{p,β_1} and v_{p,β_2} be the corresponding eigenfunctions normalized such that $v_{p,\beta_1}(0) = v_{p,\beta_2}(0) > 0$, then*

$$v_{p,\beta_1}(x) \geq v_{p,\beta_2}(x) \qquad \forall x \in B_R. \tag{4.5}$$

Proof. Let us suppose $p = 2$, the general case is analogous. For sake of simplicity, we denote by $\lambda_{\beta_i} := \lambda_{2,\beta_i}$ and $v_{\beta_i} := v_{2,\beta_i}$.

Since both v_{β_1} and v_{β_2} are radial, we can write the Laplacian in polar coordinates, that is

$$r^{n-1}\Delta u(r) = \left(r^{n-1}u'(r)\right)' \qquad r \in [0, R]. \tag{4.6}$$

Therefore function u_{β_i}, for $i = 1, 2$, satisfies

$$v'_{\beta_i}(r) = -\frac{1}{r^{n-1}}\left[\int_0^r s^{n-1}\lambda_{\beta_i} v_{\beta_i}\, ds\right]. \tag{4.7}$$

Since $\lambda_{\beta_1} < \lambda_{\beta_2}$ and $v_{\beta_1}(0) = v_{\beta_2}(0)$, by continuity there exists $\sigma > 0$ such that

$$\lambda_{\beta_1} v_{\beta_1}(s) < \lambda_{\beta_2} v_{\beta_2}(s) \qquad \forall s \in (0, \sigma),$$

and by (4.7)

$$v'_{\beta_1}(s) > v'_{\beta_2}(s) \qquad s \in (0, \sigma).$$

Hence, integrating the previous inequality and recalling $v_{\beta_1}(0) = v_{\beta_2}(0)$, we obtain

$$v_{\beta_1}(s) > v_{\beta_2}(s) \qquad s \in (0, \sigma). \tag{4.8}$$

Let us define

$$A = \{r > \sigma \,:\, v_{\beta_1}(r) = v_{\beta_2}(r)\},$$

we want to prove that A is empty, so by (4.8) we get the claim. Let us suppose by contradiction that $A \neq \varnothing$, hence there exists

$$t = \inf A.$$

By continuity $v_{\beta_1}(t) = v_{\beta_2}(t)$, and this, combined with (4.8), leads to

$$v'_{\beta_1}(t) < v'_{\beta_2}(t). \tag{4.9}$$

Let us set

$$\gamma_i = -\frac{v'_{\beta_i}(t)}{v_{\beta_i}(t)},$$

by (4.9) we have $\gamma_1 > \gamma_2$.
By Lemma 4.1.1, v_{β_1} and v_{β_2} are the first eigenfunctions of the Robin Laplacian in B_t with eigenvalue λ_{γ_1} and λ_{γ_2} with parameter γ_1 and γ_2 respectively. Therefore by monotonicity of eigenvalues with respect to the boundary parameter, we get

$$\lambda_{\beta_1} = \lambda_{\gamma_1} > \lambda_{\gamma_2} = \lambda_{\beta_2},$$

which is a contradiction. □

Remark 4.1.1. We highlight that, if we fix the value $v_{p,\beta}(0)$, (4.5) implies that

$$\beta \mapsto \|v_{p,\beta}\|_p,$$

is non-increasing, while the map

$$\beta \mapsto C(n,p,\beta,\rho) = \frac{v_{p,\beta}(0)|\Omega^\star|}{\|v_{p,\beta}\|_p^p},$$

is non-decreasing, for all $\beta \in \mathbb{R}$.

4.1.2 The case of positive boundary parameter

Let us prove the continuity estimate in the case of a positive boundary parameter β.

Theorem 4.1.3. *Let β be a positive parameter. Let Ω be a bounded, open and convex set in $\mathbb{R}^n$ and let $\Omega^\star$ be the ball, centered at the origin, such that* $\mathrm{Per}(\Omega) = \mathrm{Per}(\Omega^\star) = \rho$. *Denote by $\lambda_{p,\beta}(\Omega)$ and $\lambda_{p,\beta}(\Omega^\star)$ the first eigenvalues of the p-Laplacian operator with Robin boundary conditions respectively on Ω and $\Omega^\star$, and by v a positive eigenfunction associated to $\lambda_{p,\beta}(\Omega^\star)$, then*

$$\frac{\lambda_{p,\beta}(\Omega) - \lambda_{p,\beta}(\Omega^\star)}{\lambda_{p,\beta}(\Omega)} \leq C(n,p,\beta,\rho)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{\mathrm{Per}(\Omega)^{\frac{n}{n-1}}}\right), \tag{4.10}$$

where ω_n is the measure of the unitary ball in $\mathbb{R}^n$, and $C(n,p,\beta,\rho) = \dfrac{\|v\|_\infty^p|\Omega^\star|}{\|v\|_p^p}$.

Proof of Theorem 4.1.3. The quantity

$$\frac{\lambda_{p,\beta}(\Omega) - \lambda_{p,\beta}(\Omega^\star)}{\lambda_{p,\beta}(\Omega)},$$

is bounded from above by 1, so inequality (4.10) is trivial when

$$\left(1-\frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{\mathrm{Per}(\Omega)^{\frac{n}{n-1}}}\right)\geq\frac{1}{C(n,p,\beta,\rho)}=\frac{\|v\|_\infty^p|\Omega^\star|}{\|v\|_p^p}.$$

We can assume

$$\left(1-\frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{\mathrm{Per}(\Omega)^{\frac{n}{n-1}}}\right)<\frac{1}{C(n,p,\beta,\rho)}. \tag{4.11}$$

Let v be the solution to

$$\begin{cases}-\Delta_p v=\lambda_{p,\beta}(\Omega^\star)|v|^{p-2}v & \text{in } \Omega^\star\\ |\nabla v|^{p-2}\dfrac{\partial v}{\partial\nu}+\beta|v|^{p-2}v=0 & \text{on } \partial\Omega^\star,\end{cases} \tag{4.12}$$

as we have already remarked, v is positive and radially symmetric.

So the function

$$g(t)=|\nabla v|_{v=t},$$

is well defined for all $t\in(v_m,v_M)$, where $v_m=\min\limits_{\Omega^\star}v$ and $v_M=\max\limits_{\Omega^\star}v$. Let us define the following function

$$G^{-1}(t)=\int_{v_m}^t\frac{1}{g(s)}\,ds.\qquad v_m<t<v_M. \tag{4.13}$$

Since G^{-1} is strictly increasing, its inverse G is well defined and it is an increasing function satisfying $G(0)=v_m$. So we can construct $u(x)=G(d(x))$ for $x\in\Omega$. By construction, $u\in W^{1,p}(\Omega)$ and

- $\min\limits_{\Omega}u=G(0)=v_m$, being G increasing;
- $\|u\|_\infty\leq\|G\|_\infty\leq v_M$, by formula (4.13);
- the chain rule formula implies

$$|\nabla u|_{u=t}=|G'(d(x))|_{u=t}=\Big|g\Big(G\big(d(x)\big)\Big)\Big|_{u=t}=g(t)=|\nabla v|_{v=t}.$$

Let us set

$$E_t=\{x\in\Omega:\ u(x)>t\},\ \ B_t=\{x\in\Omega^\star:\ v(x)>t\}.$$

By Lemma 1.6.6 and formula (1.20), for almost every $t \in [0, \|u\|_\infty]$, we have

$$-\frac{d}{dt}\mathrm{Per}(E_t) \geq (n-1)\frac{W_2(E_t)}{g(t)} \geq (n-1)n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}}\frac{(\mathrm{Per}(E_t))^{\frac{n-2}{n-1}}}{g(t)},$$

while for v it holds

$$-\frac{d}{dt}\mathrm{Per}(B_t) = (n-1)\frac{W_2(B_t)}{g(t)} = (n-1)n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}}\frac{(\mathrm{Per}(B_t))^{\frac{n-2}{n-1}}}{g(t)},$$

and $\mathrm{Per}(E_0) = \mathrm{Per}(B_0)$. By direct computations, it holds

$$-\frac{1}{\mathrm{Per}(B_t)^{\frac{n-2}{n-1}}}\frac{d}{dt}\mathrm{Per}(B_t) \leq -\frac{1}{\mathrm{Per}(E_t)^{\frac{n-2}{n-1}}}\frac{d}{dt}\mathrm{Per}(E_t) \qquad \text{a.e. } t \in \big[0, \|u\|_\infty\big],$$

therefore, integrating and recalling that $\mathrm{Per}(E_t)$ and $\mathrm{Per}(B_t)$ are absolutely continuous functions, we get

$$\mathrm{Per}(E_t) \leq \mathrm{Per}(B_t), \quad 0 \leq t \leq \|u\|_\infty. \tag{4.14}$$

Denoting by $\mu(t) = |E_t|$ and by $\nu(t) = |B_t|$, the coarea formula (1.13) ensures us that

$$\begin{aligned}-\mu'(t) &= \int_{u=t}\frac{1}{|\nabla u|}\,d\mathcal{H}^{n-1} = \frac{\mathrm{Per}(E_t)}{g(t)} \leq \frac{\mathrm{Per}(B_t)}{g(t)}\\ &= \int_{v=t}\frac{1}{|\nabla v|}\,d\mathcal{H}^{n-1} = -\nu'(t),\end{aligned}$$

for almost every $t \in (0, \|u\|_\infty)$. The first equality holds true since $|\nabla u| \neq 0$ in $\{v_m < u < \|u\|_\infty\}$ (see [BZ88]). So the function $\nu - \mu$ is decreasing in $[0, \|u\|_\infty]$, and

$$\begin{aligned}\int_\Omega u^p\,dx &= \int_0^{\|u\|_\infty} pt^{p-1}\mu(t)\,dt = \int_0^{v_M} pt^{p-1}\mu(t)\,dt\\ &= \int_0^{v_M} pt^{p-1}\nu(t)\,dt - \int_0^{v_M} pt^{p-1}(\nu(t)-\mu(t))\,dt\\ &\geq \int_{\Omega^\star} v^p\,dx - v_M^p\big(|\Omega^\star| - |\Omega|\big).\end{aligned}$$

Moreover, by (4.14), we get

$$\int_{u=t}|\nabla u|^{p-1}\,d\mathcal{H}^{n-1} = g(t)^{p-1}\mathrm{Per}(E_t) \leq g(t)^{p-1}\mathrm{Per}(B_t) = \int_{v=t}|\nabla v|^{p-1}\,d\mathcal{H}^{n-1},$$

so, if we integrate over Ω,

$$\int_{\Omega} |\nabla u|^p \, dx \leq \int_0^{\|u\|_\infty} \int_{v=t} |\nabla v|^{p-1} \, d\mathcal{H}^{n-1} \, dt \leq \int_{\Omega^\star} |\nabla v|^p.$$

We observe that by construction both u and v are constant on $\partial\Omega$, so

$$\beta \int_{\partial\Omega} u^p \, d\mathcal{H}^{n-1} = \beta v_m^p \mathrm{Per}(\Omega) = \beta \int_{\partial\Omega^\star} v^p \, d\mathcal{H}^{n-1}.$$

We eventually get

$$\begin{aligned} \lambda_{p,\beta}(\Omega) &\leq \frac{\displaystyle\int_{\Omega} |\nabla u|^p \, dx + \beta \int_{\partial\Omega} u^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_{\Omega} u^p \, dx} \leq \frac{\displaystyle\int_{\Omega^\star} |\nabla v|^p \, dx + \beta \int_{\partial\Omega^\star} v^p \, d\mathcal{H}^{n-1}}{\displaystyle\int_{\Omega^\star} v^p \, dx - v_M^p(|\Omega^\star| - |\Omega|)} \\ &= \lambda_{p,\beta}(\Omega^\star) \frac{1}{1 - C(n,p,\beta,\rho)\left(1 - \dfrac{|\Omega|}{|\Omega^\star|}\right)}. \end{aligned} \tag{4.15}$$

The claim follows from (4.11) as the quantity

$$1 - C(n,p,\beta,\rho)\left(1 - \frac{|\Omega|}{|\Omega^\star|}\right),$$

is non-negative. □

Remark 4.1.2. The constant $C(n,p,\beta,\rho) = \dfrac{\|v\|_\infty^p |\Omega^\star|}{\|v\|_p^p}$ depends on the perimeter of the set Ω and on β. Thanks to Proposition 4.1.2 and Remark 4.1.1, it is possible to bound the constant $C(n,p,\beta,\rho)$ from above with a constant independent of the perimeter and β. Indeed, $\forall\beta > 0$, if we denote by $v_{p,\infty}$ the first Dirichlet eigenfunction normalized in such a way $v_{p,\beta}(0) = v_{p,\infty}(0)$, we have

$$C(n,p,\beta,\rho) \leq \frac{v_{p,\infty}(0)|\Omega^\star|}{\|v_{p,\infty}\|_p^p} =: C(n,p),$$

that is independent of the perimeter thanks to the rescaling properties of the Dirichlet p-Laplacian eigenfunction.

4.1.3 The case of negative boundary parameter

Let us now move to the case of a negative boundary parameter β and let us prove the quantitative estimate with respect to the isoperimetric deficit.

Theorem 4.1.4. *Let β be a negative parameter. Let Ω be a bounded, open and convex set in $\mathbb{R}^n$ and let $\Omega^\star$ be the ball, centered at the origin, such that $\mathrm{Per}(\Omega) = \mathrm{Per}(\Omega^\star) = \rho$. Denote by $\lambda_{p,\beta}(\Omega)$ and $\lambda_{p,\beta}(\Omega^\star)$ the first eigenvalues of the p-Laplacian operator with Robin boundary conditions respectively on Ω and $\Omega^\star$, and by v a positive eigenfunction associated to $\lambda_{p,\beta}(\Omega^\star)$, then*

$$\frac{\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega)}{|\lambda_{p,\beta}(\Omega)|} \geq C(n,p,\beta,\rho)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{\mathrm{Per}(\Omega)^{\frac{n}{n-1}}}\right), \tag{4.16}$$

where ω_n is the measure of the unitary ball in $\mathbb{R}^n$, and $C(n,p,\beta,\rho) = \dfrac{v_m^p|\Omega^\star|}{\|v\|_p^p}$ with $v_m = \min\limits_{\Omega^\star} v$.

In this case, the constant $C(n,p,\beta,\rho)$ cannot be replaced by a constant independent of β and the perimeter, as it is shown in Remark 4.1.3.

Proof of Theorem 4.1.4. Let v be a positive eigenfunction associated to $\lambda_{p,\beta}(\Omega^\star)$, then v is a p sub-harmonic function as $\lambda_{p,\beta}(\Omega^\star) < 0$. We denote by $v_m = v(0) = \min_{\Omega^\star} v$ and by $v_M = \max_{\Omega^\star} v$.

Let us consider the function $\tilde{v} = v_M - v$, which is a positive function with zero trace. The function

$$g(t) = |\nabla \tilde{v}|_{\tilde{v}=t},$$

is well-defined for all $t \in (0, v_M - v_m)$. Let us define the following function

$$G^{-1}(t) = \int_0^t \frac{1}{g(s)}\,ds. \qquad 0 < t < v_M - v_m. \tag{4.17}$$

Since G^{-1} is strictly increasing, its inverse G is well defined and it is an increasing function satisfying $G(0) = 0$. So we can construct $\tilde{u}(x) = G(d(x))$ for $x \in \Omega$ and $u = v_M - \tilde{u}$. By construction, $\tilde{u}, u \in W^{1,p}(\Omega)$ and

- $u_M = \max\limits_{\Omega} u = \max\limits_{\Omega}(v_M - \tilde{u}) = v_M$, since $\tilde{u}$ is a positive function in $W_0^{1,p}(\Omega)$;
- by formula (4.17), it holds

$$u_m = \min_{\Omega} u = v_M - \max_{\Omega} \tilde{u} \geq v_M - \max_{\Omega^\star} \tilde{v} = \min_{\Omega^\star} v = v_m,$$

- the chain rule formula implies, for $u_m < t < u_M$

$$|\nabla u|_{u=t} = |\nabla \tilde{u}|_{\tilde{u}=v_M-t} = \left| g\Big(G\big(d(x)\big)\Big)\right|_{\tilde{u}=v_M-t}$$
$$= |\nabla \tilde{v}|_{\tilde{v}=v_M-t} = |\nabla v|_{v=t} = g(t).$$

Let us set

$$\widetilde{E}_t = \{x \in \Omega \,:\, \tilde{u}(x) > t\}, \qquad \widetilde{B}_t = \{x \in \Omega^\star \,:\, \tilde{v}(x) > t\},$$
$$E_t = \{x \in \Omega \,:\, u(x) > t\} = \Omega \setminus \overline{\widetilde{E}}_{v_M-t}, \quad B_t = \{x \in \Omega^\star \,:\, v(x) > t\} = \Omega^\star \setminus \overline{\widetilde{B}}_{v_M-t}.$$

By Lemma 1.6.6 and formula (1.20) we have

$$-\frac{d}{dt}\mathrm{Per}(\widetilde{E}_t) \geq (n-1)\frac{W_2(\widetilde{E}_t)}{g(t)} \geq (n-1)n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}}\frac{(\mathrm{Per}(\widetilde{E}_t))^{\frac{n-2}{n-1}}}{g(t)},$$

while for $\tilde{v}$ it holds

$$-\frac{d}{dt}\mathrm{Per}(\widetilde{B}_t) = (n-1)\frac{W_2(\widetilde{B}_t)}{g(t)} = (n-1)n^{-\frac{n-2}{n-1}}\omega_n^{\frac{1}{n-1}}\frac{(\mathrm{Per}(\widetilde{B}_t))^{\frac{n-2}{n-1}}}{g(t)},$$

and $\mathrm{Per}(\widetilde{E}_0) = \mathrm{Per}(\widetilde{B}_0)$. By direct computations, it holds

$$-\frac{1}{\mathrm{Per}(\widetilde{B}_t)^{\frac{n-2}{n-1}}}\frac{d}{dt}\mathrm{Per}(\widetilde{B}_t) \leq -\frac{1}{\mathrm{Per}(\widetilde{E}_t)^{\frac{n-2}{n-1}}}\frac{d}{dt}\mathrm{Per}(\widetilde{E}_t) \qquad \text{a.e. } t \in [0, v_M - v_m],$$

therefore, integrating and recalling that $\mathrm{Per}(\tilde{E}_t)$ and $\mathrm{Per}(\tilde{B}_t)$ are absolutely continuous functions, we have

$$\mathrm{Per}(\widetilde{E}_t) \leq \mathrm{Per}(\widetilde{B}_t), \quad 0 \leq t \leq v_M - v_m.$$

Hence

$$\mathrm{Per}(E_t) = \mathrm{Per}(\widetilde{E}_{v_M-t}) \leq \mathrm{Per}(\widetilde{B}_{v_M-t}) = \mathrm{Per}(B_t), \quad v_m \leq t \leq v_M. \tag{4.18}$$

Denoting by $\tilde{\mu}(t) = |\widetilde{E}_t|$ and $\tilde{\nu}(t) = |\widetilde{B}_t|$, the coarea formula (1.13) ensures us that

$$-\tilde{\mu}'(t) = \int_{\tilde{u}=t}\frac{1}{|\nabla\tilde{u}|}\,d\mathcal{H}^{n-1} = \frac{\mathrm{Per}(\widetilde{E}_t)}{g(t)} \leq \frac{\mathrm{Per}(\widetilde{B}_t)}{g(t)}$$
$$= \int_{\tilde{v}=t}\frac{1}{|\nabla\tilde{v}|}\,d\mathcal{H}^{n-1} = -\tilde{\nu}'(t), \qquad 0 \leq t < v_M - v_m.$$

Moreover, setting $\mu(t)=|E_t|=|\Omega|-\tilde\mu(v_M-t)$ and by $\nu(t)=|B_t|=|\Omega^\star|-\tilde\nu(v_M-t)$, we have $-\mu'(t)\le-\nu'(t)$ in $[v_m,v_M]$. So the function $\nu-\mu$ is decreasing in $[v_m,v_M]$, and

$$\begin{aligned}\int_\Omega u^p\,dx&=\int_0^{u_M}pt^{p-1}\mu(t)\,dt=\int_0^{v_M}pt^{p-1}\nu(t)\,dt-\int_0^{v_M}pt^{p-1}(\nu(t)-\mu(t))\,dt\\&=\int_0^{v_M}pt^{p-1}\nu(t)\,dt-\int_0^{v_m}pt^{p-1}\big(\nu(t)-\mu(t)\big)\,dt-\int_{v_m}^{v_M}pt^{p-1}\big(\nu(t)-\mu(t)\big)\,dt\\&\le\int_{\Omega^\star}v^p\,dx-v_m^p(|\Omega^\star|-|\Omega|)=\int_{\Omega^\star}v^p\,dx\left[1-\frac{v_m^p}{\|v\|_p^p}(|\Omega^\star|-|\Omega|)\right].\end{aligned}$$

By (4.18), we get

$$\int_{\tilde u=t}|\nabla\tilde u|^{p-1}\,d\mathcal{H}^{n-1}=g(t)^{p-1}\mathrm{Per}(\widetilde E_t)\le g(t)^{p-1}\mathrm{Per}(\widetilde B_t)=\int_{\tilde v=t}|\nabla\tilde v|^{p-1}\,d\mathcal{H}^{n-1},$$

so, if we integrate over Ω,

$$\begin{aligned}\int_\Omega|\nabla u|^p\,dx=\int_\Omega|\nabla\tilde u|^p\,dx&=\int_0^{\|\tilde u\|_\infty}\int_{\tilde u=t}|\nabla\tilde u|^{p-1}\,d\mathcal{H}^{n-1}\,dt\\&\le\int_0^{\|\tilde u\|_\infty}\int_{\tilde v=t}|\nabla\tilde v|^{p-1}\,d\mathcal{H}^{n-1}\,dt\\&=\int_{\Omega^\star}|\nabla\tilde v|^p\,dx=\int_{\Omega^\star}|\nabla v|^p\,dx.\end{aligned}$$

We also observe that by construction both u and v are constant on $\partial\Omega$, so

$$\beta\int_{\partial\Omega}u^p\,d\mathcal{H}^{n-1}=\beta u_M^p\mathrm{Per}(\Omega)=\beta v_M^p\mathrm{Per}(\Omega)=\beta\int_{\partial\Omega^\star}v^p\,d\mathcal{H}^{n-1}.$$

We eventually get

$$\begin{aligned}\lambda_{p,\beta}(\Omega)&\le\frac{\displaystyle\int_\Omega|\nabla u|^p\,dx+\beta\int_{\partial\Omega}u^p\,d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega u^p\,dx}\le\frac{\displaystyle\int_{\Omega^\star}|\nabla v|^p\,dx+\beta\int_{\partial\Omega^\star}v^p\,d\mathcal{H}^{n-1}}{\displaystyle\int_{\Omega^\star}v^p\,dx\Big[1-\frac{v_m^p}{\|v\|_p^p}(|\Omega^\star|-|\Omega|)\Big]}\\&=\lambda_{p,\beta}(\Omega^\star)\,\frac{1}{1-\dfrac{v_m^p}{\|v\|_p^p}\,(|\Omega^\star|-|\Omega|)}.\end{aligned}\tag{4.19}$$

Hence, by direct calculation

$$\frac{\lambda_{p,\beta}(\Omega^\star)-\lambda_{p,\beta}(\Omega)}{|\lambda_{p,\beta}(\Omega)|}\geq\frac{v_m^p}{\|v\|_p^p}(|\Omega^\star|-|\Omega|). \tag{4.20}$$

$\square$

Remark 4.1.3. Unlike the constant in Theorem 4.1.3, the constant $C(n,p,\beta,\rho)=\dfrac{v_m^p|\Omega^\star|}{\|v\|_p^p}$ cannot be bounded from below with a constant independent of the perimeter and β. Indeed, for example if $n=p=2$ and $\mathrm{Per}(\Omega)=2\pi$, we have that

$$v_\beta(x)=I_0\left(\sqrt{-\lambda_\beta(B_1)}\,|x|\right),$$

where I_0 is the modified Bessel function.

We recall that for z sufficiently large (see [AS64, Section 9.7]), we have

$$I_0(z)\sim\frac{e^z}{\sqrt{2\pi z}}\left[1+\frac{1}{8z}+\frac{9}{2!(8z)^2}+\dots\right],$$

therefore

$$\|v_\beta(x)\|_2\sim\frac{e^{-\beta}}{\beta^2}\xrightarrow{\beta\to-\infty}+\infty$$

and

$$C(2,2,\beta,2\pi)=\frac{(v_\beta)_m^2|B_1|}{\|v_\beta\|_2^2}\sim\frac{\beta^2}{e^{-\beta}}\xrightarrow{\beta\to-\infty}0.$$

Remark 4.1.4. We want to highlight that the constant C depends actually just on n,p and $\rho^{\frac{p-1}{n-1}}\beta$. Indeed, for all $\Omega\subset\mathbb{R}^n$ bounded and convex set with $\mathrm{Per}(\Omega)=\rho$, we can consider

$$\Omega_1=\left(\frac{n\omega_n}{\rho}\right)^{\frac{1}{n-1}}\Omega,\qquad\Omega_1^*=\left(\frac{n\omega_n}{\rho}\right)^{\frac{1}{n-1}}\Omega^\star,\qquad t:=\left(\frac{\rho}{n\omega_n}\right)^{\frac{1}{n-1}},$$

so $\mathrm{Per}(\Omega_1) = \mathrm{Per}(\Omega_1^\star) = n\omega_n$ and we have

$$\begin{aligned}
\frac{\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega)}{-\lambda_{p,\beta}(\Omega)} &= \frac{\lambda_{p,\beta}(t\Omega_1^\star) - \lambda_{p,\beta}(t\Omega_1)}{-\lambda_{p,\beta}(t\Omega_1)} \\
&= \frac{\lambda_{p,t^{p-1}\beta}(\Omega_1^*) - \lambda_{p,t^{p-1}\beta}(\Omega_1)}{\lambda_{p,t^{p-1}\beta}(\Omega_1)} \\
&\geq C\big(n, p, n\omega_n, t^{p-1}\beta\big)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{\mathrm{Per}(\Omega)^{\frac{n}{n-1}}}\right) \\
&= C\left(n, p, n\omega_n, \Big(\frac{\rho}{n\omega_n}\Big)^{\frac{p-1}{n-1}}\beta\right)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{\mathrm{Per}(\Omega)^{\frac{n}{n-1}}}\right) \\
&= C\big(n, p, \rho^{\frac{p-1}{n-1}}\beta\big)\left(1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{\mathrm{Per}(\Omega)^{\frac{n}{n-1}}}\right).
\end{aligned}$$

Eventually, let us prove the quantitative estimate in the spirit of [CL21].

Theorem 4.1.5. *Let $n \geq 2$, $\rho > 0$ and $\beta < 0$. Then, there exist two positive constants $C(n, p, \beta, \rho) > 0$ and $\delta_0(n, p, \beta, \rho) > 0$, such that, for all $\Omega \subset \mathbb{R}^n$ bounded and convex with $\mathrm{Per}(\Omega) = \rho$ and $\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \leq \delta_0$, it holds*

$$\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \geq C(n, p, \beta, \rho) g\big(\mathcal{A}_{\mathcal{H}}^\star(\Omega)\big). \tag{4.21}$$

where $\Omega^\star$ is a ball with the same perimeter of Ω, $\mathcal{A}_{\mathcal{H}}^\star$ is the Hausdorff asymmetry defined in (1.23) and g is defined in (1.26).

Proof of Theorem 4.1.5. If we choose $w = 1$ in the definition of $\lambda_{p,\beta}(\Omega)$ (4.1), we get

$$|\lambda_{p,\beta}(\Omega)| \geq |\beta| \frac{\mathrm{Per}(\Omega)}{|\Omega|},$$

and if we combine it with (4.20) we have

$$\begin{aligned}
\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) &\geq |\lambda_{p,\beta}(\Omega)| \frac{v_m^p}{\|v\|_p^p}\big(|\Omega^\star| - |\Omega|\big) \\
&\geq |\beta| \frac{\mathrm{Per}(\Omega)}{|\Omega|} \frac{v_m^p}{\|v\|_p^p}\big(|\Omega^\star| - |\Omega|\big) \\
&\geq |\beta| \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}}{\rho^{\frac{1}{n-1}}} \frac{v_m^p}{\|v\|_p^p}\big(|\Omega^\star| - |\Omega|\big).
\end{aligned} \tag{4.22}$$

Now, if we suppose that $\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \leq \delta_0$, by (4.22) we have

$$|\Omega^\star| - |\Omega| \leq K(n, \rho, p, \beta)\delta_0.$$

So by Lemma 1.6.9 we conclude

$$\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \geq C(n, \rho, p, \beta)\, g(\mathcal{A}^\star_{\mathcal{H}}(\Omega)).$$

$\square$

Remark 4.1.5. Our result, Theorem 4.1.5, applies only when

$$\lambda_{p,\beta}(\Omega^\star) - \lambda_{p,\beta}(\Omega) \leq \delta_0.$$

It is possible to remove this hypothesis, obtaining a weaker result.

In order to obtain it, we need the quantitative version of the isoperimetric inequality proved in [FMP08]

$$\mathrm{Per}(\Omega) \geq \mathrm{Per}(\Omega^\sharp)\big(1 + \gamma(n)\alpha(\Omega)^2\big), \tag{4.23}$$

where

$$\alpha(\Omega) = \min\left\{ \frac{|\Omega \triangle B_r|}{|\Omega|} \, : \, |B_r| = |\Omega| \right\},$$

and the following result [EFT05, Lemma 4.2]: there exists a constant $C(n)$ such that if C, W are open and convex sets such that $|C| = |W|$ and $|C \triangle W| < \frac{|C|}{2}$, it holds

$$d_{\mathcal{H}}(C, W) \leq C(n)\big[\mathrm{diam}(C) + \mathrm{diam}(W)\big]\left(\frac{|C \triangle W|}{|C|}\right)^{\frac{1}{n}}. \tag{4.24}$$

Moreover, we have to recall that

$$|\Omega^\star| = \frac{\mathrm{Per}(\Omega)^{\frac{n}{n-1}}}{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}} \qquad \text{and} \qquad \mathrm{Per}(\Omega^\sharp) = n\omega_n^{\frac{1}{n}}|\Omega|^{\frac{n-1}{n}},$$

so we have

$$\begin{aligned}
1 - \frac{|\Omega|}{|\Omega^\star|} &= 1 - \frac{n^{\frac{n}{n-1}}\omega_n^{\frac{1}{n-1}}|\Omega|}{\mathrm{Per}(\Omega)} \\
&\geq 1 - \frac{1}{\big(1 + \gamma(n)\alpha^2(\Omega)\big)^{\frac{n}{n-1}}} \\
&\geq 1 - \frac{1}{\big(1 + \gamma(n)\alpha^2(\Omega)\big)} \\
&= \frac{\gamma(n)\alpha^2(\Omega)}{1 + \gamma(n)\alpha^2(\Omega)},
\end{aligned}$$

where we used Bernoulli's inequality

$$(1+x)^r \geq 1 + rx \qquad \forall x \geq -1,\, \forall r \geq 0.$$

Since $0 < \alpha^2(\Omega) < 4$ we have

$$1 - \frac{|\Omega|}{|\Omega^\star|} \geq \frac{\gamma(n)}{1+4\gamma(n)}\,\alpha^2(\Omega) = C(n)\alpha^2(\Omega).$$

If $\Omega^\sharp$ is a ball that realizes the minimum in (4.23), then using (4.24) we obtain

$$C(n)\alpha^2(\Omega) \geq C(n)\,\frac{d_{\mathcal{H}}(\Omega,\Omega^\sharp)^{2n}}{\left[\mathrm{diam}(\Omega)+\mathrm{diam}(\Omega^\sharp)\right]^{2n}},$$

and so

$$\frac{\lambda_{p,\beta}(\Omega^\star)-\lambda_{p,\beta}(\Omega)}{|\lambda_{p,\beta}(\Omega)|} \geq C(n,p,\beta\rho)C(n)\frac{d_{\mathcal{H}}(\Omega,\Omega^\sharp)^{2n}}{\left[\mathrm{diam}(\Omega)+\mathrm{diam}(\Omega^\sharp)\right]^{2n}}.$$

4.2 | A quantitative Kröger inequality

Now we consider the Neumann case. Let Ω be an open, bounded and convex set in $\mathbb{R}^n$ and $k \in \mathbb{N}$. We recall that the k-th Neumann-Laplacian eigenvalue is defined by

$$\mu_k(\Omega) = \min_{E_k}\max_{0\neq u\in E_k} \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\int_\Omega u^2\,dx}, \tag{4.25}$$

where the minimum is taken over all the k-dimensional subspace of $H^1(\Omega)$ that are L^2-orthogonal to the constant functions.

Let us mention the following continuity property for the Neumann eigenvalues (see [FH22, Theorem 2.4] or [Che75]).

Theorem 4.2.1. *Let $B \subset \mathbb{R}^n$ be a fixed ball and let $\{\Omega_k\}$ be a sequence of open sets contained in a fixed ball B and assume that $\{\Omega_h\}$ satisfies the ε-cone property (for the same ε). Let us assume that there exists an open set Ω such that the sequence of the corresponding characteristic function χ_{Ω_h} converges in $L^1(B)$ to χ_Ω. Then*

$$\mu_k(\Omega_h) \to \mu_k(\Omega) \qquad \forall k \in \mathbb{N}.$$

In particular, this continuity result for Neumann eigenvalues holds true for a sequence of convex domains that converges in the Hausdorff sense to an open convex set.

Definition 4.2.1. Let $\omega \subseteq \mathbb{R}^j$ be an open, bounded and convex set, $p\colon \omega \to \mathbb{R}$ be a non negative $\frac{1}{m}$-concave function. We define the following quantity

$$\mu_k(\omega, p) = \inf_{E_k} \sup_{0 \neq u \in E_k} \frac{\displaystyle\int_\omega |\nabla \varphi(x)|^2 p(x)\, dx}{\displaystyle\int_\omega \varphi(x)^2 p(x)\, dx}, \tag{4.26}$$

where the infimum is taken among all the k-dimensional subspace of $H^1(\omega)$ that are L^2-orthogonal to the constant functions with respect to p on ω.

Because of the compact embedding of $W^{1,2}(\omega, p)$ into $L^2(\omega, p)$ (see [ABF24, Lemma 7], we have the existence of eigenfunction and eigenvalue of the following problem

$$\begin{cases} -\mathrm{div}\big(p(x)\nabla u(x)\big) = \mu_k(\omega,p)p(x)u(x) & \text{in } \omega \\ p(x)\dfrac{\partial u}{\partial \nu}(0) = 0 & \text{on } \partial\omega. \end{cases} \tag{4.27}$$

If $j = 1$, i.e. $\omega \subseteq \mathbb{R}$, and the infimum is achieved, problem (4.27) is known as Sturm-Liouville problem.

From the variational characterization of $\mu_k(\Omega)$ and choosing function depending only on the first j variable, we get the following link between $\mu_k(\Omega)$ and $\mu_k(\omega, p)$ (see [HM23, Lemma 2.2] for the case $j = 1$).

Proposition 4.2.2. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and Lipschitz domain. Denote with $\omega \subset \mathbb{R}^j$ the projection of Ω on the first j variable, $1 \leq j < n$, and with $p\colon \omega \to \mathbb{R}$ the function*

$$p(x) = \mathcal{H}^{n-j}\big(\{x' \in \mathbb{R}^{n-j} : (x, x') \in \Omega\}\big) \qquad \forall x \in \omega.$$

Then

$$\mu_k(\Omega) \leq \mu_k(\omega, p). \tag{4.28}$$

Incidentally, let us mention that for $j = 1$ and $\omega = (0, 1)$ in [HM23], the authors also prove the following result

Proposition 4.2.3. *If $0 \not\equiv p \in L^\infty(0,1)$ is a non negative $\frac{1}{m}$-concave function, there exists a sequence of domain Ω_h with profile function $\dfrac{p}{h^{n-1}}$ such that*

$$\lim_h \operatorname{diam}(\Omega_h)^2 \mu_k(\Omega_h) = \mu_k\big((0,1),p\big).$$

For sake simplicity, in case $\omega = (0,1)$ we will just write $\mu_k(p)$.

In [HM23] the authors use the previous proposition to study the problem of maximizing the Neumann eigenvalues with diameter constraint, indeed proposition 4.2.3 gives

$$\sup S_1 = \sup S_2,$$

where

$$S_1 = \left\{\mu_k(p) : p \in L^\infty(0,1),\, p \geq 0,\, p \text{ is } \frac{1}{n-1} \text{ concave and } p \not\equiv 0\right\},$$

$$S_2 = \big\{\operatorname{diam}(\Omega)^2 \mu_k(\Omega) : \Omega \subset \mathbb{R}^n \text{ open, bounded and convex}\big\}.$$

Moreover, the authors characterize the optimal p and the corresponding eigenfunction of the maximization problem at left hand-side. In particular, they prove the following

Theorem 4.2.4. *For each $k, n \in \mathbb{N}$, $k, n \geq 1$, the following problem*

$$\sup\left\{\mu_k(p) :\ p \in L^\infty(0,1),\, p \geq 0,\, p \text{ is } \frac{1}{n-1} \text{ concave and } p \not\equiv 0\right\}, \tag{4.29}$$

admits a solution. Let us denote with $p^*_{k,n}$ the function that achieves the maximum in (4.29).

- *If $k = 1$, then the graph of $(p^*_{1,n})^{\frac{1}{n-1}}$ is an isosceles triangle having vertex in $x = \frac{1}{2}$.*
- *If $k \geq 2$, then $(p^*_{k,n})^{\frac{1}{n-1}}$ is a piecewise affine function.*

Remark 4.2.1. We underline that structure of $(p^*_{k,n})^{\frac{1}{n-1}}$ is precisely given in [HM23]: for $k \geq 2$

- if $n = 3$, any function such that $p^*_{k,3}(0) = p^*_{k,3}(1) = 0$ and the graph of $(p^*_{k,3})^{\frac{1}{2}}$ is made by at most $k+1$ segments;
- if $n \neq 3$, the graph of $(p^*_{k,n})^{\frac{1}{n-1}}$ is an isosceles trapezium.

4.2.1 Collapsing domains

In this subsection we will assume that $\{\Omega_h\}$ is a sequence of bounded, open and convex sets in $\mathbb{R}^n$ with $\mathrm{diam}(\Omega_h) = 1$. We can consider E_h the John's ellipsoid (see [Guz75]) of Ω_h, and up to rotation and translation, we can assume that for each $h \in \mathbb{N}$

1. $E_h \subseteq \Omega \subseteq nE_h$;
2. E_h is centered at the origin;
3. E_h is aligned with the axes;
4. the semi-axis satisfy
$$a_h^1 \geq a_h^2 \geq \ldots \geq a_h^n > 0.$$

Since they are bounded sequence in $\mathbb{R}$, up to multiple extractions we can assume that
$$a_h^i \to a^i \qquad \forall i \in \{1, \ldots, n\}.$$
Let us assume that the sequence is collapsing, i.e. for some $j \in \mathbb{N}$, $1 \leq j \leq n$, we have
$$a^1 \geq \ldots \geq a^j > 0 = a^{j+1} = \ldots = a^n.$$
Moreover, since $\{\Omega_h\}$ are bounded, thanks to the compactness of the Hausdorff distance, up to another extraction, we can also assume

$$\Omega_h \xrightarrow{\mathcal{H}} \omega \times \{0\}^{n-j}, \tag{4.30}$$

where $\omega \subseteq \mathbb{R}^j$ is an open, convex set of diameter 1.

Moreover, for each h we can define

$$T_h \colon \mathbb{R}^n \to \mathbb{R}^n, \qquad T_h(x_1, \ldots, x_n) = \left(x_1, \ldots, x_j, \frac{x_{j+1}}{a_h^{j+1}}, \ldots, \frac{x_n}{a_h^n}\right). \tag{4.31}$$

Essentially, when applied to Ω_h, this function rescales the last $n-j$ components in such a way that $T_h(\Omega_h)$ has as semiaxis of the John's ellipsoid

$$a_h^1, \ldots, a_h^j, 1, \ldots, 1.$$

So in particular, $T_h(\Omega_h)$ is a non degenerating sequence of convex sets.

We have the following Theorem ([ABF24, Proposition 8]).

Theorem 4.2.5. *There exists $p\colon \omega \to \mathbb{R}_+$ $\frac{1}{n-j}$-concave if $j < n$ or $p \equiv 1$ if $j = n$ such that*

$$\mu_k(\Omega_h) \to \mu_k(\omega, p),$$

where $\mu_k(\omega, p)$ is the k-th eigenvalue of the following problem

$$\begin{cases} -\operatorname{div}(p\nabla u) = \mu_k(\omega, p)up & \text{in } \omega \\ p\dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\omega. \end{cases} \tag{4.32}$$

Proof. We will not show the proof in details but just the main idea that will be used also in the following.

Let us consider T_h defined in (4.31). $\{T_h(\Omega_h)\}$ is a sequence of bounded convex sets so, up to a subsequence, it converges in the sense of Hausdorff to a bounded, open and convex set Ω,

$$T_h(\Omega_h) \xrightarrow{\mathcal{H}} \Omega.$$

Furthermore, by (4.30), the projection of Ω on the first j variable, is equal to ω.

Let us consider the following function

$$p\colon \omega \to \mathbb{R} \qquad p(x) = \mathcal{H}^{n-j}(S_x \cap \Omega), \qquad \forall x \in \omega.$$

Corollary 1.6.2 ensures that p is a $\frac{1}{n-j}$-convex function. Let us denote with u_k^h the k-th Neumann eigenfunction of Ω_h, we can define

$$v_k^h\colon T_h(\Omega_h) \to \mathbb{R}, \qquad v_k^h(x_1, \dots, x_n) = c_h u_k^h\big(x_1, \dots, x_j, a_h^{j+1}x_{j+1}, \dots, a_h^n x_n\big),$$

where $c_h = \sqrt{\prod_{i=j+1}^n a_h^i}$.

With a change of variable, it holds

$$\left\|v_k^h\right\|^2_{L^2(T_h(\Omega_h))} = \left\|u_k^h\right\|_{L^2(\Omega_h)},$$

and

$$\int_{T_h(\Omega_h)} \left(\frac{\partial v_k^h}{\partial x_i}\right) dx = (a_h^i)^2 \int_{\Omega_h} \left(\frac{\partial u_k^h}{\partial x_i}\right) dx. \tag{4.33}$$

Since $T_h(\Omega_h)$ are uniformly convex and $v_{h,k} \in H^1\big(T_h(\Omega_h)\big)$ are uniformly bounded in H^1 we can assume that there exists $\overline{v}_k \in H^1(\Omega)$ such that (see [HP18, Section 3.7])

$$\begin{aligned} v_k^h \mathbb{1}_{\Omega_h} &\to \overline{v}_k \mathbb{1}_\Omega && \text{in } L^2 \\ \big(\nabla v_k^h\big)\mathbb{1}_{\Omega_h} &\to \nabla \overline{v}_k \mathbb{1}_\Omega && \text{in } L^2. \end{aligned}$$

By (4.33), we can conclude

$$\overline{v}_k(x, x') = v(x),$$

that is $\overline{v}_k$ is constant in x'. Following [ABF24, Proposition 8], it can be shown that $\overline{v}_k$ is the k-th eigenfunction of problem (4.32) and that

$$\lim_{h \to +\infty} \mu_k(\Omega_h) = \mu_k(\omega, p) \qquad \square.$$

Remark 4.2.2. Assume $j = 1$, i.e. $\omega = I \times \mathbb{R}^{n-1}$ where $I \subseteq \mathbb{R}$ is an interval of length 1. Then a sequence of diameters of Ω_h converges in Hausdorff to ω. Up to another rotation and translation we can assume that Ω_h has exactly diameter ω.

Let us introduce the following notation: if $\{\Omega_h\}$ is a sequence of open, bounded and convex sets converging in the sense of Hausdorff to $\omega \times \{0\}^{n-j}$ where ω is a bounded, open and convex set of $\mathbb{R}^j$ and p is the function given by theorem 4.2.5, we will write

$$\Omega_h \to (\omega, p).$$

4.2.2 Extension of Kröger inequalities to weighted eigenvalues

Now we extend (57).

Lemma 4.2.6. *Let $n \geq 2$, $1 \leq j < d$ and let $\omega \subseteq \mathbb{R}^j$ be an open, bounded and convex set such that the diameter is placed along the x_1 axis and $p \colon \omega \to \mathbb{R}_+$ a function $\frac{1}{n-j}$-concave. Then*

$$\mu_k(\omega, p) \leq \frac{\mu^*_{k,n}}{\mathrm{diam}(\omega)^2}. \tag{4.34}$$

Moreover equality occurs only when $j = 1$ and $p = p^*_{k,n}$ given by [HM23].

Proof. First of all let us notice that if $j = 1$, then p is a $\frac{1}{n-1}$-concave function and so

$$\mu_k\big([0, \mathrm{diam}(\omega), p]\big) \leq \frac{\mu^*_{k,n}}{\mathrm{diam}(\omega)^2},$$

and the equality occurs if and only if $p = p^*_{k,n}$.

Now we discuss the case $j \geq 2$. Firstly we prove that inequality (4.34) holds true.

Let us write $x \in \mathbb{R}^j$ as $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{j-1}$ and let us define

$$S_{x_1}(\omega) = \big\{(x, y) \colon x = x_1, (x, y) \in \omega\big\}$$
$$S_{x'}(\omega) = \big\{(x, y) \colon y = x', (x, y) \in \omega\big\},$$

and

$$\tilde{p}(x_1) = \int_{S_{x_1}(\omega)} p(x, y)\, dy \qquad \forall x_1 \in [0, \mathrm{diam}(\omega)].$$

Then it holds

$$\mu_k(\omega, p) \leq \mu_k\big([0, \mathrm{diam}(\omega)], \tilde{p}\big) \leq \mu_k\Big(\big[0, \mathrm{diam}(\omega)\big], p^*_{k,n}\Big) = \frac{\mu^*_{k,n}}{\mathrm{diam}(\omega)^2}, \quad (4.35)$$

where the first inequality comes from the variational formulation of $\mu_k(\omega, p)$ choosing function depending only on the first variable while the second is given by [HM23]. So inequality occurs, let us study the equality case.

Assume by contradiction

$$\mu_k(\omega, p) = \frac{\mu^*_{k,n}}{\mathrm{diam}(\omega)^2},$$

then by (4.35), we have

$$\tilde{p} = p^*_{k,n},$$

and from the fact that $p^*_{k,n}$ is piecewise affine, the equality case of Brunn-Minkowski (1.16) implies that the set

$$\Omega = \{(a, b) \in \mathbb{R}^j \times \mathbb{R}^{n-j} \colon a \in \omega, \|b\| < p(a)^{\frac{1}{n-j}}\},$$

is conical. Moreover the eigenfunction u of $\mu_k\big([0, \mathrm{diam}(\omega)], \tilde{p}\big)$ is also eigenfunction of single variable x_1 for (ω, p).

We use the information that $u = u_k(x_1)$ is the k-th eigenfunction for (ω, p) in the weak form of the equation with test function of the form $\varphi(x_1)\psi(x')$ where $x' \in S_{x_1}(\omega)$.

Let us start with $\psi(x')$:

$$0 = \int_\omega \nabla u \nabla \psi \, dx = \int_\omega u(x_1)\psi(x')p(x_1, x')\, dx.$$

So we can consider a sequence ψ_ε of functions converging in L^2 to $\frac{1}{\omega_{j-1} r^{j-1}} \mathbb{1}_{B'(x', r)}$ where $B'(x', r)$ is the ball in $\mathbb{R}^{j-1}$ centered at x' and with radius r, hence passing to the limit for $\varepsilon \to 0$, we get

$$\frac{1}{\omega_{j-1} r^{j-1}} \int_\omega u(x_1) \mathbb{1}_{B'(x', r)} p(x_1, x')\, dx = 0,$$

and then, passing to the limit for $r \to 0$, we get for all x'

$$\int_{S_{x'(\omega)}(\omega)} u(x_1)p(x_1, x')\, dx_1 = 0. \tag{4.36}$$

We do the same for $\varphi(x_1)\psi_\varepsilon(x')$ where $\psi_\varepsilon \to \frac{1}{\omega_{j-1}r^{j-1}}\mathbb{1}_{B'(x',r)}$ and with the same arguments as before, we get

$$\int_{S_{x'}(\omega)} u'(x_1)\varphi'(x_1)p(x_1, x')\, dx_1 = \mu^*_{k,d} \int_{S_{x'}(\omega)} u(x_1)\varphi(x_1)p(x_1, x')\, dx_1.$$

If $\varphi \in C^\infty_C\big(S_{x'}(\omega)\big)$ we can integrate by parts (because of the regularity of u and p by [HM23]) obtaining

$$-u''(x_1) - u'(x_1)\frac{p'(x_1, x')}{p(x_1, x')} = \mu^*_{k,n} u(x_1) \qquad \text{in } S_{x'}(\omega). \tag{4.37}$$

Therefore the function

$$\frac{p'(x_1, x')}{p(x_1, x')},$$

does not depend on x' since u does not depend on x' in (4.37). Hence p cannot vanish on $\partial\omega$ except on $x_1 = 0$ or $x_1 = \text{diam}(\omega)$. Finally, the weak form and the fact that u is smooth give

$$u'(x_1)p(x_1, x') = 0 \qquad \text{on } \partial S_{x'}(\omega),$$

and since p vanish only for $x_1 = 0$ or $x_1 = D_\omega$, we get $u'(x_1) = 0$ on $S_{x'}(\omega)$. From the structure of u by [HM23], this is a contradiction. □

4.2.3 Rigidity of the collapsing

Let us assume that Ω is an open, bounded and convex set such that $D_\Omega = 1$ whose profile function has the following form

$$p(x)^{\frac{1}{n-1}} = \begin{cases} c_1 x & \text{if } x \in [0, \alpha] \\ ax + b & \text{if } x \in [\alpha, \beta] \\ c_2(1 - x) & \text{if } x \in [\beta, 1], \end{cases} \tag{4.38}$$

where α, β are such that $p(x)$ is continuous and $\frac{1}{n-1}$-concave (figure 4.1).

In the following we will need to estimate some integrals on the the section of Ω, where Ω is a convex domain of $\mathbb{R}^n$ having p as profile function so let us analyze the section of Ω.

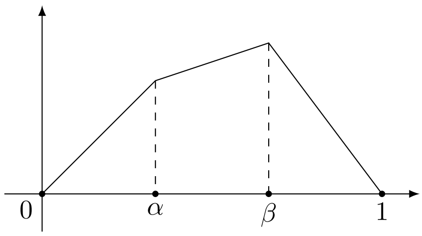

Figure 4.1

The linearity of $p^{\frac{1}{n-1}}$ implies a strong rigidity on Ω: indeed this fact gives the equality in the Brunn-Minkowski inequality (1.16) for all the section in $[0,\alpha]$, $[\alpha,\beta]$ and $[\beta,1]$. The characterization of equality case in Brunn-Minkowski inequality (see [Gar02]) gives that all the sections of Ω, between $[0,\alpha]$, $[\alpha,\beta]$ and $[\beta,1]$ are homothetic. Taking into account the convexity of Ω, we claim that in each stripe $[0,\alpha]\times\mathbb{R}^{n-1}$, $[\alpha,\beta]\times\mathbb{R}^{n-1}$ and $[\beta,1]\times\mathbb{R}^{n-1}$, Ω is the truncation of a cone (figure 4.2).

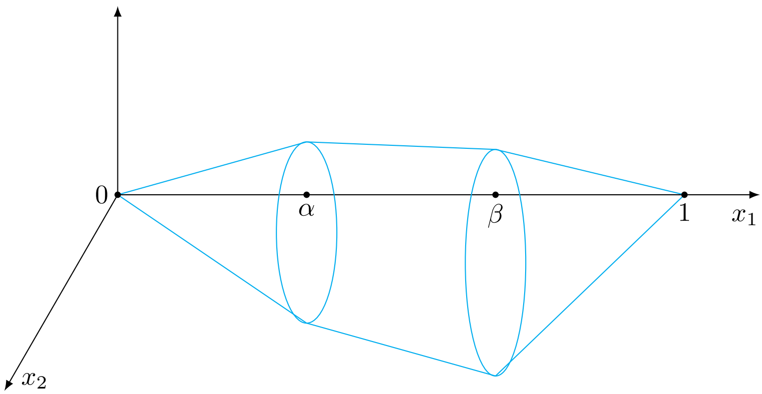

Figure 4.2

Indeed, for example in $[\alpha,\beta]$, because of the homothecity of $S_\beta\cap\Omega$ and $S_\alpha\cap\Omega$ there exist $k\in\mathbb{R}$ and $m\in\mathbb{R}^{n-1}$ such that

$$S_\beta\cap\Omega = kS_\alpha\cap\Omega + m.$$

Because of the expression of p, we deduce

$$k = \left(\frac{p(\beta)}{p(\alpha)}\right)^{\frac{1}{n-1}}.$$

Moreover, since it is convex, the set $([\alpha,\beta]\times\mathbb{R}^{n-1})\cap\Omega$ contains the cone passing through $S_\alpha\cap\Omega$ and $S_\beta\cap\Omega$ and because of the linearity of the profile function, the sets must coincide.

Let us underline that the previous analysis guarantees that for each $x_1 \in [\alpha,\beta]$ it holds

$$S_{x_1}\cap\Omega = \left[\frac{p(x_1)}{p(\alpha)}\right]^{\frac{1}{n-1}}\big(S_\alpha\cap\Omega\big) + \frac{x_1-\alpha}{\beta-\alpha}m, \qquad \forall x_1\in[\alpha,\beta]\subset[0,1]. \tag{4.39}$$

Now let us define the following function

$$c(x_1) := \fint_{S_{x_1}\cap\Omega} x_2\, d\mathcal{H}^{n-1},$$

where $\fint$ denotes the mean integral. We claim that c is a piecewise affine function, indeed e.g. for $x_1\in[\alpha,\beta]$ with a change of variable, by (4.39), we can write

$$\begin{aligned}\int_{S_{x_1}\cap\Omega} x_2\, d\mathcal{H}^{n-1} &= \frac{p(x_1)}{p(\alpha)}\int_{S_\alpha\cap\Omega}\left[\left(\frac{p(x_1)}{p(\alpha)}\right)^{\frac{1}{n-1}} y_2 + \left(\frac{x_1-\alpha}{\beta-\alpha}m_2\right)\right] d\mathcal{H}^{n-1}\\ &= p(x_1)\left(\frac{1}{p(\alpha)}\int_{S_\alpha\cap\Omega}\frac{y_2}{p(\alpha)^{\frac{1}{n-1}}}\, d\mathcal{H}^{n-1}\right)p(x_1)^{\frac{1}{n-1}} + m_2\left(\frac{x_1-\alpha}{\beta-\alpha}\right)p(\alpha),\end{aligned}$$

therefore

$$c(x_1) = c(\alpha)\left(\frac{p(x_1)}{p(\alpha)}\right)^{\frac{1}{n-1}} + m_2\left(\frac{x_1-\alpha}{\beta-\alpha}\right), \tag{4.40}$$

is an affine function since $p(x)^{\frac{1}{n-1}}$ is an affine function. Analogously for $[0,\alpha]$ and $[\beta,1]$.

Now we claim that there exists K such that

$$\int_{S_{x_1}\cap\Omega}\big(x_2-c(x_1)\big)^2\, d\mathcal{H}^{n-1} = Kp(x_1)^{1+\frac{2}{n-1}} \qquad \forall x_1\in[\alpha,\beta], \tag{4.41}$$

analogously in $[0,\alpha]$ and $[\beta,1]$.

First of all we notice that (4.41) can be written as

$$\fint_{S_{x_1}\cap\Omega} \big(x_2 - c(x_1)\big)^2 \, d\mathcal{H}^{n-1} = Kp(x_1)^{\frac{2}{n-1}},$$

and we can write

$$\begin{aligned}\fint_{S_{x_1}\cap\Omega} \big(x_2 - c(x_1)\big)^2 \, d\mathcal{H}^{n-1} &= \fint_{S_{x_1}\cap\Omega} x_2^2 \, d\mathcal{H}^{n-1} - 2c(x_1)\fint_{S_{x_1}\cap\Omega} x_2 \, d\mathcal{H}^{n-1} + c(x_1)^2 \\ &= \fint_{S_{x_1}\cap\Omega} x_2^2 \, d\mathcal{H}^{n-1} - c(x_1)^2,\end{aligned}$$

where we used the definition of $c(x_1)$.

Let us now compute $\fint_{S_{x_1}\cap\Omega} x_2^2 \, d\mathcal{H}^{n-1}$ with the same change of variable as before:

$$\begin{aligned}\fint_{S_{x_1}\cap\Omega} x_2^2 \, d\mathcal{H}^{n-1} &= \frac{1}{p(\alpha)}\int_{S_\alpha\cap\Omega} \left[\left(\frac{p(x)}{p(\alpha)}\right)^{\frac{1}{n-1}} y_2 + \left(\frac{x_1-\alpha}{\beta-\alpha} m_2\right)\right]^2 d\mathcal{H}^{n-1} \\ &= \left(\frac{p(x_1)}{p(\alpha)}\right)^{\frac{2}{n-1}} \fint_{S_{x_1}\cap\Omega} y_2^2 \, d\mathcal{H}^{n-1} + m_2^2\left(\frac{x_1-\alpha}{\beta-\alpha}\right)^2 \\ &\quad + 2m_2\left(\frac{x_1-\alpha}{\beta-\alpha}\right)\left(\frac{p(x_1)}{p(\alpha)}\right)^{\frac{1}{n-1}} c(\alpha).\end{aligned}$$

Hence, using (4.40), we get

$$\begin{aligned}\fint_{S_{x_1}\cap\Omega} \big(x_2 - c(x_1)\big)^2 \, d\mathcal{H}^{n-1} &= \left(\frac{p(x_1)}{p(\alpha)}\right)^{\frac{2}{n-1}} \left[\fint_{S_\alpha\cap\Omega} y_2^2 \, d\mathcal{H}^{n-1} - c(\alpha)^2\right] \\ &= Kp(x_1)^{\frac{2}{n-1}},\end{aligned}$$

where

$$K := \frac{1}{p(\alpha)^{\frac{2}{n-1}}}\left[\fint_{S_\alpha\cap\Omega} y_2^2 \, d\mathcal{H}^{n-1} - c(\alpha)^2\right],$$

so (4.41) holds true.

4.2.4 Proof of main results

We split the main theorem in case $k = 1$ and $k \geq 1$.

Theorem 4.2.7. *There exists a constant $C = C(n) > 0$ such that for any $\Omega \subset \mathbb{R}^n$ open bounded convex set, it holds*

$$\mu_1(\Omega) \leq \frac{\mu^*_{1,n}}{\operatorname{diam}(\Omega)^2} - C\frac{a_2(\Omega)^2}{\operatorname{diam}(\Omega)^4}, \tag{4.42}$$

where $a_2(\Omega)$ and $\mu^*_{1,n}$ are respectively the diameter, the second biggest semiaxis of the John ellipsoid of Ω and the optimal upper bound for $\mu_1(\Omega)\operatorname{diam}(\Omega)^2$ given in [HM23].

Proof. Thanks to the scaling property of Neumann eigenvalue, without loss of generality we can assume that $D_\Omega = 1$.

The strategy of the proof consists in a contradiction argument and it will use the relation between Neumann and Sturm-Liouville eigenvalue.

Let us argue by contradiction: assume that there exists $\{\,\Omega_h\,\}_{n\in\mathbb{N}}$ a sequence of open convex sets with $D_{\Omega_h} = 1$ such that

$$\mu_1(\Omega_h) \geq \mu^*_{1,n} - \frac{1}{h}a_2(\Omega_h)^2. \tag{4.43}$$

Since $\{\Omega_h\}$ is a bounded sequence, up to a subsequence, it will converges to $\omega \subseteq \mathbb{R}^n$ a bounded convex set in the sense of Hausdorff.

We claim that ω is a segment, so we firstly that Ω_h should collapse and then that it collapse into a segment.

- Assume by contradiction that Ω_h converges to ω a bounded open and convex set. Recalling that the Hausdorff convergence of convex sets implies uniform convergence of the support functions (Proposition 1.6.4), it follows that $a_n(\Omega_h) > 0$ for n sufficiently large and hence the family $\{\Omega_h\}$ satisfies the uniform ε-cone property. Therefore, by continuity of the Neumann eigenvalue with respect to the Hausdorff convergence, Theorem 4.2.1, we will have

  $$\mu_1(\Omega) \geq \mu^*_{1,n},$$

  which is a contradiction since such a set does not exists. So $\{\Omega_h\}$ collapses.

- Now let us show that $\{\Omega_h\}$ converges to a segment. Assume by contradiction that ω is a subset of $\mathbb{R}^j$ with $j \geq 2$ then, as seen in Subsection 4.2.1, it holds

  $$\mu_1(\omega, p) = \mu^*_{1,n},$$

  that is a contradiction because of Theorem 4.2.6.

Therefore ω is a segment and, since $D_{\Omega_h} = 1$ and the diameter is placed along the x_1 axis, $\omega \subseteq \mathbb{R}$, $\omega = (0,1)$ and $D_{\Omega_h} = (0,1) \times \{0\}^{n-1}$.

Now we discuss the convergence of the sequence $\tilde{\Omega}_h := T_h(\Omega_h)$.

With the same notations and reasonings as in Subsection 4.2.1, up to a subsequence, there exists $\tilde{\Omega}$ such that $T_h(\Omega_h) \to \tilde{\Omega}$ and we can define

$$p(x_1) = \mathcal{H}^{n-1}(S_{x_1} \cap \tilde{\Omega}).$$

Since $\mu_1(\omega, p) = \mu^*_{1,n}$, we have

$$p(x)^{\frac{1}{n-1}} = \left(p^*_{1,n}\right)^{\frac{1}{n-1}} = \begin{cases} 2x_1 & \text{if } x \in \left[0, \dfrac{1}{2}\right] \\ 2(1-x_1) & \text{if } x \in \left[0, \dfrac{1}{2}\right]. \end{cases}$$

As seen in Subsection 4.2.3, this implies that

$$\tilde{\Omega} \cap S_{x_1} = \begin{cases} x_1 \cdot S & \text{on } x_1 \in \left[0, \dfrac{1}{2}\right] \\ (1-x_1) \cdot S & \text{on } x_1 \in \left[\dfrac{1}{2}, 1\right], \end{cases}$$

where S is a convex set in $\{0\} \times \mathbb{R}^{n-1}$ containing the origin (figure 4.3).

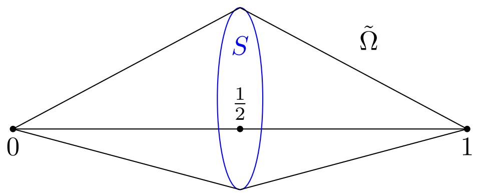

Figure 4.3

To get a contradiction, we use the weak formulation of $\mu_1(\Omega_h)$ with suitable test function. Let us consider

$$w_h(x_1, x') = u_h(x_1)\left[1 + \tau\big(x_2 - c(x_1)a_2(\Omega_h)\big)^2\right] + d_h, \qquad x_1 \in S_{x_1} \cap \Omega_h,$$

where u_h is the first eigenfunction of the following Sturm-Liouville problem

$$\begin{cases} -\dfrac{d}{dx_1}\big(u_h'(x_1)p_h(x_1)\big) = \mu_{1,n}u_h(x_1)p_h(x_1) & \text{for } x \in (0,1) \\ u_h'(x_1)p_h(x_1) = 0 & \text{on } \{0,1\}, \end{cases} \qquad p_h(x_1) = \mathcal{H}^{n-1}(S_{x_1} \cap \Omega_h),$$

normalized as follows

$$\int_{\Omega_h} u_h(x_1)^2\,dx = \int_0^1 u_h(x_1)^2 p_h(x_1)\,dx_1 = 1, \tag{4.44}$$

c is defined as

$$c(x_1) = \fint_{S_{x_1}\cap\tilde\Omega} x_2\,d\mathcal{H}^{n-1},$$

where $\tilde\Omega$ is the convex set previously defined such that $T_h(\Omega_h)\to\tilde\Omega$, d_h is a constant such that w_h has zero average and τ will be chosen later.

For $(x_1,\dots,x_n)\in\tilde\Omega_h = T_h(\Omega_h)$ we define

$$\begin{aligned} &\tilde w_h(x_1,x') = k_h w_h\big(x_1, a_2(\Omega_h)x_2,\dots,a_n(\Omega_h)x_n\big) = \tilde u_h(x_1)\Big[1+\tau\big(x_2-c(x_1)\big)^2 a_2(\Omega_h)^2\Big],\\ &\tilde p_h(x_1) = \frac{p_h(x_1)}{k_h}, \end{aligned} \tag{4.45}$$

where

$$\tilde u_h(x_1) = k_h u_h(x_1), \qquad k_h = \sqrt{\prod_{j=2}^n a_j(\Omega_h)}.$$

Hence, with the same reasonings and computations as in Subsection 4.2.1, we can write

$$\begin{aligned} \frac{\displaystyle\int_{\Omega_h}|\nabla w_h|^2\,dx}{\displaystyle\int_{\Omega_h} w_h^2\,dx} &= \frac{\displaystyle\int_{\tilde\Omega_h}\left(\frac{\partial\tilde w_h}{\partial x_1}\right)^2 + \sum_{i=2}^n\left(\frac{1}{a_i(\Omega_h)^2}\frac{\partial\tilde w_h}{\partial x_i}\right)^2 dx}{\displaystyle\int_{\tilde\Omega_h}\tilde w_h^2\,dx}\\ &= \frac{\displaystyle\int_{\tilde\Omega_h}\left(\frac{\partial\tilde w_h}{\partial x_1}\right)^2 + \frac{1}{a_2(\Omega_h)^2}\left(\frac{\partial\tilde w_h}{\partial x_2}\right)^2 dx}{\displaystyle\int_{\tilde\Omega_h}\tilde w_h^2\,dx}, \end{aligned}$$

where we used the fact that w_h depends only on thw first two variables.

Taking into account (4.45), (4.57) becomes

$$\int_{\tilde\Omega_h}\tilde w_h(x_1)^2\,dx = \int_0^1 \tilde u_h(x_1)^2\tilde p_h(x_1)\,dx_1 = 1$$

Let us compute d_h:

$$\begin{aligned}\int_{\Omega_h} w_h\,dx = 0 \iff & \int_{\Omega_h}\Big[u_h(x_1)\Big(1+\tau\big(x_2-c(x_1)a_2(\Omega_h)\big)^2\Big)-d_h\Big]\,dx_1 = 0\\ \iff & d_h = \frac{\tau}{|\Omega_h|}\int_{\Omega_h} u_h(x_1)\big(x_2-c(x_1)a_2(\Omega_h)\big)^2\,dx\\ \iff & d_h = \frac{\tau}{|\tilde\Omega_h|}\left(\int_{\tilde\Omega_h}\tilde u_h(x_1)\big(x_2-c(x_1)\big)^2\,dx\right)a_2(\Omega_h)^2,\end{aligned} \tag{4.46}$$

since u_h has zero average on Ω_h.

Since $\tilde\Omega_h \xrightarrow{\mathcal{H}} \tilde\Omega$

$$\big(\tilde p_h\big)^{\frac{1}{n-1}} \to p^*_{1,n},$$

up to a multiplicative constant.

So the weak formulation of $\mu_1(\Omega_h)$ gives

$$\mu^*_{1,n}-\frac{1}{h}a_2(\Omega_h)^2 \le \mu_1(\Omega_h) \le \frac{\displaystyle\int_{\Omega_h}|\nabla w_h|\,dx}{\displaystyle\int_{\Omega_h} w_h^2\,dx} = \frac{\displaystyle\int_{\tilde\Omega_h}\left(\frac{\partial\tilde w_h}{\partial x_1}\right)^2+\frac{1}{a_2(\Omega_h)^2}\left(\frac{\partial\tilde w_h}{\partial x_2}\right)^2 dx}{\displaystyle\int_{\tilde\Omega_h}\tilde w_h^2\,dx}. \tag{4.47}$$

Since at left hand-side we have a term in order 2 in $a_2(\Omega_h)$, we will perform a Taylor approximation of the right hand-side at order 2.

Let us study numerator and denominator of (4.47).

- Estimate of the numerator.

  First of all we notice that

$$\begin{aligned}\frac{\partial\tilde w_h}{\partial x_1} &= \tilde u_h'(x_1)\Big[1+\tau\big(x_2-c(x_1)\big)^2a_2(\Omega_h)^2\Big]-2\tau\tilde u_h(x_1)\big(x_2-c(x_1)\big)c'(x_1)a_2(\Omega_h)^2\\ \frac{\partial\tilde w_h}{\partial x_2} &= 2\tau\tilde u_h(x_1)\big(x_2-c(x_1)\big)a_2(\Omega_h)^2.\end{aligned}$$

so,

$$\begin{aligned}
\left(\frac{\partial \tilde{w}_h}{\partial x_1}\right)^2 &= \big(\tilde{u}_h'(x_1)\big)^2 \Big[1+\tau\big(x_2-c(x_1)\big)^2 a_2(\Omega_h)^2\Big]^2 \\
&+ 4\tau^2 \big(\tilde{u}_h(x_1)\big)^2 \big(x_2-c(x_1)\big)^2 c'(x_1)^2\, a_2(\Omega_h)^4 \\
&- 4\tau \tilde{u}_h'(x_1)\tilde{u}_h(x_1)\Big[1+\tau\big(x_2-c(x_1)\big)^{"} a_2(\Omega_h)^2\Big]\,\big(x_2-c(x_1)\big)c'(x_1)a_2(\Omega_h)^2 \\
\left(\frac{\partial \tilde{w}_h}{\partial x_2}\right)^2 &= 4\tau^2\big(\tilde{u}_h(x_1)\big)^2\,\big(x_2-c(x_1)\big)^2 a_2(\Omega_h)^4,
\end{aligned}$$

therefore we have

$$\begin{aligned}
\int_{\tilde{\Omega}_h} \left(\frac{\partial \tilde{w}_h}{\partial x_1}\right)^2 + \frac{1}{a_2(\Omega_h)^2}\left(\frac{\partial \tilde{w}_h}{\partial x_2}\right)^2 dx &= \int_{\Omega_h} \big(\tilde{u}_h'(x_1)\big)^2\, dx \\
&+ 2\tau\left(\int_{\tilde{\Omega}_h} \big(\tilde{u}_h'(x_1)\big)^2 \big(x_2-c(x_1)\big)^2\, dx\right) a_2(\Omega_h)^2 \\
&- 4\tau\left(\int_{\tilde{\Omega}_h} \tilde{u}_h'(x_1)\tilde{u}_h(x_1)\big(x_2-c(x_1)c'(x_1)\,dx\big)\right) a_2(\Omega_h)^2 \\
&+ 4\tau^2\left(\int_{\tilde{\Omega}_h} \tilde{u}_h(x_1)^2\big(x_2-c(x_1)\big)^2\, dx\right) a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big) \\
&= \mu_1(I,p_h) + 2\tau\left(\int_{\tilde{\Omega}_h} \tilde{u}_h'(x_1)^2\big(x_2-c(x_1)\big)^2\, dx\right) a_2(\Omega_h)^2 \\
&- 4\tau\left(\int_{\tilde{\Omega}_h} \tilde{u}_h'(x_1)\tilde{u}_h(x_1)\big(x_2-c(x_1)c'(x_1)\,dx\big)\right) a_2(\Omega_h)^2 \\
&+ 4\tau^2\left(\int_{\tilde{\Omega}_h} \tilde{u}_h(x_1)^2\big(x_2-c(x_1)\big)^2\, dx\right) a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big).
\end{aligned}$$

Recalling that $\mu_1(I, p_h) \le \mu_{1,n}^*$, we get

$$\begin{aligned}
\int_{\tilde{\Omega}_h} \left(\frac{\partial \tilde{w}_h}{\partial x_1}\right)^2 + \frac{1}{a_2(\Omega_h)^2}\left(\frac{\partial \tilde{w}_h}{\partial x_2}\right)^2 dx \le \mu_{1,n}^* + 2\tau \left(\int_{\tilde{\Omega}_h} \big(\tilde{u}_h'(x_1)\big)^2 \big(x_2 - c(x_1)\big)^2 \, dx \right) a_2(\\
- 4\tau \left(\int_{\tilde{\Omega}_h} \tilde{u}_h'(x_1)\tilde{u}_h(x_1)\big(x_2 - c(x_1)\big)\big(c(x_1)\big)\, dx \right) a_2(\Omega_h)^2 \\
+ 4\tau^2 \left(\int_{\tilde{\Omega}_h} \tilde{u}_h(x_1)^2 \big(x_2 - c(x_1)\big)^2 \, dx \right) a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big).
\end{aligned} \tag{4.48}$$

- Estimate of the denominator.

$$\begin{aligned}
\int_{\tilde{\Omega}_h} \tilde{w}_h(x)^2 \, dx &= \int_{\Omega_h} \tilde{u}_h(x_1)^2 \Big[1 + \tau \big(x_2 - c(x_1)\big)^2 a_2(\Omega_h)^2 \Big]^2 dx + d_h^2 |\Omega_h| \\
&\quad - 2 d_h \tau \int_{\Omega_h} \tilde{u}_h(x_1) \big(x_2 - c(x_1) a_2(\Omega_h)^2\big)^2 \, dx \\
&= \int_{\Omega_h} \tilde{u}_h(x_1)^2 \Big[1 + \tau \big(x_2 - c(x_1)\big)^2 a_2(\Omega_h)^2 \Big]^2 dx - d_h^2 |\Omega_h| \\
&= \int_{\tilde{\Omega}_h} \tilde{u}_h(x_1)^2 \, dx + 2\tau \left(\int_{\tilde{\Omega}_h} \tilde{u}_h(x_1)^2 \big(x_2 - c(x_1)\big) \, dx \right) a_2(\Omega_h)^2 \\
&\quad + o\big(a_2(\Omega_h)^2\big).
\end{aligned} \tag{4.49}$$

where we used (4.46).

So (4.47) becomes

$$
\begin{aligned}
&\mu^*_{1,n} - \frac{1}{h} a_2(\Omega_h)^2 \\
&\quad \le \frac{\mu^*_{1,n} + 2\tau \left(\displaystyle\int_{\tilde{\Omega}_h} \big(\tilde{u}'_h(x_1)\big)^2 \big(x_2 - c(x_1)\big)^2 \, dx \right) a_2(\Omega_h^2)}{1 + 2\tau \left(\displaystyle\int_{\tilde{\Omega}_h} \tilde{u}_h(x_1)^2 \big(x_2 - c(x_1)\big)^2 \, dx \right) a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big)} \\
&\quad - \frac{4\tau \left(\displaystyle\int_{\tilde{\Omega}_h} \tilde{u}'_h(x_1) \tilde{u}_h(x_1) \big(x_2 - c(x_1)\big) c'(x_1) \, dx \right) a_2(\Omega_h)^2}{1 + 2\tau \left(\displaystyle\int_{\tilde{\Omega}_h} \tilde{u}_h(x_1)^2 \big(x_2 - c(x_1)\big)^2 \, dx \right) a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big)} \\
&\quad + \frac{4\tau^2 \left(\displaystyle\int_{\tilde{\Omega}_h} \tilde{u}_h(x_1)^2 \big(x_2 - c(x_1)\big)^2 \, dx \right) a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big)}{1 + 2\tau \left(\displaystyle\int_{\tilde{\Omega}_h} \tilde{u}_h(x_1)^2 \big(x_2 - c(x_1)\big)^2 \, dx \right) a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big)}.
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
-\frac{1}{h} a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big) &\le 2\tau \Bigg[\int_{\tilde{\Omega}_h} \Big(\big(\tilde{u}'_h(x_1)\big)^2 - \mu^*_{1,n} \tilde{u}_h(x_1)^2 \Big) \big(x_2 - c(x_1)\big)^2 \, dx \\
&\quad - 2 \int_{\tilde{\Omega}_h} \tilde{u}'_h(x_1) \tilde{u}_h(x_1) \big(x_2 - c(x_1)\big) c'(x_1) \, dx \Bigg] a_2(\Omega_h)^2 \\
&\quad + 4\tau^2 \left(\int_{\tilde{\Omega}_h} \tilde{u}_h(x_1)^2 \big(x_2 - c(x_1)\big)^2 \, dx \right) a_2(\Omega_h)^2 \\
&=: 2\gamma_h a_2(\Omega_h)^2 \tau + 4\beta_h a_2(\Omega_h)^2 \tau^2.
\end{aligned} \tag{4.50}
$$

where

$$\gamma_h = \int_{\tilde{\Omega}_h} \left(\left(\tilde{u}_h'(x_1) \right)^2 - \mu_{1,n}^* \tilde{u}_h(x_1)^2 \right) \left(x_2 - c(x_1) \right)^2 dx \\ - 2 \int_{\tilde{\Omega}_h} \tilde{u}_h'(x_1) \tilde{u}_h(x_1) \big(x_2 - c(x_1) \big) c'(x_1)\, dx,$$

$$\beta_h = \int_{\tilde{\Omega}_h} \tilde{u}_h(x_1)^2 \left(x_2 - c(x_1) \right)^2 dx.$$

We notice that the term at right-hand side in (4.50) has order 2 in $a_2(\Omega_h)$. Moreover, it is a second order polynomial in τ, so the contradiction is achieved choosing $\tau < 0$ sufficiently small provided that

$$\lim_{h \to +\infty} \gamma_h \neq 0,$$

that is

$$\lim_{h \to +\infty} \left\{ \int_{\tilde{\Omega}_h} \left[\left(\tilde{u}_h'(x_1) \right)^2 - \mu_{1,n}^* \tilde{u}_h(x_1)^2 \right] \left(x_2 - c(x_1) \right)^2 dx \right. \\ \left. - 2 \int_{\tilde{\Omega}_h} \tilde{u}_h'(x_1) \tilde{u}_h(x_1) \big(x_2 - c(x_1) \big) c'(x_1)\, dx \right\} \neq 0.$$

We can rewrite the term in brackets in the following way

$$\int_{\tilde{\Omega}_h} \left[\left(\tilde{u}_h'(x_1) \right)^2 - \mu_{1,n}^* \tilde{u}_h(x_1)^2 \right] \left(x_2 - c(x_1) \right)^2 dx \\ = \int_0^1 \left[\left(\tilde{u}_h'(x_1) \right)^2 - \mu_{1,n}^* \tilde{u}_h(x_1)^2 \right] \left(\int_{S_{x_1} \cap \tilde{\Omega}_h} \left(x_2 - c(x_1) \right)^2 d\mathcal{H}^{n-1} \right) dx_1,$$

and

$$\int_{\tilde{\Omega}_h} \tilde{u}_h'(x_1) \tilde{u}_h(x_1) \big(x_2 - c(x_1) \big) c'(x_1)\, dx \\ = \int_0^1 \left[\tilde{u}_h'(x_1) \tilde{u}_h(x_1) c'(x_1) \right] \left(\int_{S_{x_1} \cap \tilde{\Omega}_h} x_2 - c(x_1)\, d\mathcal{H}^{n-1} \right) dx_1.$$

We recall that $\tilde{\Omega}_h$ converges to $\tilde{\Omega}$ in the sense of Hausdorff and $\{\tilde{u}_h\}$ is a bounded sequence in $H^1(0,1)$ so, up to a subsequence, there exists $\tilde{u} \in H^1(\tilde{\Omega})$ (Theorem 4.2.5) such that

$$\begin{aligned} \tilde{u}_h &\to \tilde{u} && \text{in } L^2(0,1) \\ \nabla \tilde{u}_h &\to \nabla \tilde{u} && \text{in } L^2(0,1), \end{aligned}$$

hence we have

$$\lim_{n\to+\infty}\int_0^1\left[\left(\tilde{u}_h'(x_1)\right)^2-\mu_{1,n}^*\tilde{u}_h(x_1)^2\right]\left(\int_{S_{x_1}\cap\tilde{\Omega}_h}\left(x_2-c(x_1)\right)^2\,d\mathcal{H}^{n-1}\right)dx_1$$
$$=\int_0^1\left[\left(\tilde{u}'(x_1)\right)^2-\mu_{1,n}^*\tilde{u}(x_1)^2\right]\left(\int_{S_{x_1}\cap\tilde{\Omega}}\left(x_2-c(x_1)\right)^2\,d\mathcal{H}^{n-1}\right)dx_1,$$

and

$$\lim_{n\to+\infty}\int_0^1\left[\tilde{u}_h'(x_1)\tilde{u}_h(x_1)c'(x_1)\right]\left(\int_{S_{x_1}\cap\tilde{\Omega}_h}x_2-c(x_1)\,d\mathcal{H}^{n-1}\right)dx_1$$
$$=\int_0^1\tilde{u}'(x_1)\tilde{u}(x_1)c'(x_1)\left[\int_{S_{x_1}\cap\tilde{\Omega}}x_2-c(x_1)\,d\mathcal{H}^{n-1}\right]dx_1=0,$$

because of the definition of c.

For reader's convenience, we will omit the tilde in $\tilde{u}$.

Recalling that for all n, it holds

$$p_{\tilde{\Omega}}^{\frac{1}{n-1}}(x_1)=\begin{cases}2x_1 & \text{in }\left[0,\dfrac{1}{2}\right]\\[2ex] 2(1-x_1) & \text{in }\left[\dfrac{1}{2},1\right]\end{cases}$$

and

$$u(x_1)=-u(1-x_1),u'(x_1)=u'(1-x_1)\qquad\forall x_1\in\left[0,\frac{1}{2}\right],$$

and using (4.41) there exists $K>0$ such that

$$\begin{aligned}&\int_0^1\left[\left(u'(x_1)\right)^2-\mu_{1,n}^*u(x_1)^2\right]\left(\int_{S_{x_1}\cap\tilde{\Omega}}\left(x_2-c(x_1)\right)^2\,d\mathcal{H}^{n-1}\right)dx_1\\&\qquad=2K\int_0^{\frac{1}{2}}\left[\left(u'(x_1)\right)^2-\mu_{1,n}^*u(x_1)^2\right]p(x_1)^{1+\frac{2}{n-1}}\,dx_1.\end{aligned}\tag{4.51}$$

Since u is an eigenfunction of the Sturm-Liouville problem related to $p(x_1)$ and taking into account the regularity of u and p (see [HM23]), it holds

$$-u''(x_1)p(x_1)-\mu_{1,n}^*u(x_1)p(x_1)=u'(x_1)p'(x_1)\qquad x_1\in\left(0,\frac{1}{2}\right),\tag{4.52}$$

so, multiplying both terms for $u(x_1)p(x_1)^{\frac{2}{n-1}}$, integrating from 0 to $\frac{1}{2}$ and integrating by parts the first term we get

$$\int_0^{\frac{1}{2}} \Big[u'(x_1)^2 - \mu^*_{1,n} u(x_1)^2\Big]\, p(x_1)^{1+\frac{2}{n-1}}\, dx_1 = -\frac{2}{n-1}\int_0^{\frac{1}{2}} u'(x_1)u(x_1)p(x_1)^{\frac{2}{n-1}}\, p'(x_1)\, dx_1,$$

where we used [HM23, Lemma 3.10].

Plug the previous equality into (4.51) we get

$$\begin{aligned}\int_0^1 \Big[\big(u'(x_1)\big)^2 - \mu^*_{1,n} u(x_1)^2\Big]\left(\int_{S_{x_1}\cap\tilde\Omega} \big(x_2 - c(x_1)\big)^2\, d\mathcal{H}^{n-1}\right) dx_1 \\ = -\frac{4K}{n-1}\int_0^{\frac{1}{2}} u'(x_1)u(x_1)\, p(x_1)^{\frac{2}{n-1}} p'(x_1)\, dx_1.\end{aligned} \tag{4.53}$$

Let us show that the last term is different from zero. Integrating by parts, we get

$$\int_0^{\frac{1}{2}} u'(x_1)u(x_1)\, p(x_1)^{\frac{2}{n-1}} p'(x_1)\, dx_1 = -\int_0^{\frac{1}{2}} \frac{u(x_1)^2}{2}\frac{d}{dx_1}\Big(p(x_1)^{\frac{2}{n-1}} p'(x_1)\Big)\, dx_1,$$

and recalling that $p^{\frac{1}{n-1}}$ is affine in $\big[0, \frac{1}{2}\big]$, we know that there exists $\tilde a > 0$

$$\begin{aligned}\frac{d}{dx_1}\big(p^{\frac{1}{n-1}}(x_1)\big) = \tilde a \iff \frac{1}{n-1}\frac{p'(x_1)}{p(x_1)^{1-\frac{1}{n-1}}} = \tilde a \\ \iff p'(x_1) = \tilde a(n-1)p(x_1)^{1-\frac{1}{n-1}},\end{aligned}$$

therefore

$$\begin{aligned}\frac{d}{dx_1}\Big(p(x_1)^{\frac{2}{n-1}} p'(x_1)\Big) &= \tilde a(n-1)\frac{d}{dx_1}\Big(p(x_1)^{1+\frac{1}{n-1}}\Big) \\ &= \tilde a(n-1)\left(1+\frac{1}{n-1}\right)p(x_1)^{\frac{1}{n-1}} p'(x_1) \\ &= \tilde a^2 n(n-1)p(x_1),\end{aligned}$$

so (4.53) becomes

$$\int_0^1 \left[\left(u'(x_1)\right)^2 - \mu_{1,n}^* u(x_1)^2\right]\left(\int_{S_{x_1}\cap\tilde{\Omega}} \left(x_2 - c(x_1)\right)^2 d\mathcal{H}^{n-1}\right) dx_1$$
$$= 2\tilde{a}^2 Kn \int_0^{\frac{1}{2}} u(x_1)^2 p(x_1).$$

We conclude the proof since the previous integral is strictly positive. □

The proof for $k \geq 2$ follows the same idea of Theorem 4.2.7, we just adapt the choice of the test functions.

Theorem 4.2.8. *There exists $C = C(n,k) > 0$ such that for any $\Omega \subset \mathbb{R}^n$ open bounded convex set, it holds*

$$\mu_k(\Omega) \leq \frac{\mu_{k,n}^*}{\operatorname{diam}(\Omega)^2} - C\frac{a_2(\Omega)^2}{\operatorname{diam}(\Omega)^4}, \tag{4.54}$$

where $a_2(\Omega)$ and $\mu_{k,n}^*$ are respectively the second biggest semiaxis of the John ellipsoid of Ω and the optimal upper bound for $\mu_k(\Omega)\operatorname{diam}(\Omega)^2$ given in [HM23].

Proof. The first part of the proof is totally analogous, we assume by contradiction that there exists a sequence $\{\Omega_h\}$ of open and convex sets with diameter equal to 1, that violates (4.54), namely

$$\mu_k(\Omega_h) > \mu_{k,n}^* - \frac{1}{h}a_2(\Omega_h)^2. \tag{4.55}$$

Reasoning as in the previous theorem, one can prove that, up to subsequence, Ω_h converges to a segment and up to rotation and translation, we can assume that $\Omega_h \to \big((0,1),p\big)$ with p a $\frac{1}{n-1}$-concave function.

Moreover, with the same notations as before, we can define

$$\begin{aligned} p_h(x_1) &= \mathcal{H}^{n-1}\big(S_{x_1}\cap\tilde{\Omega}_h\big) && \forall x_1 \in (0,1),\ \forall n \in \mathbb{N} \\ p(x_1) &= \mathcal{H}^{n-1}\big(S_{x_1}\cap\tilde{\Omega}\big) && \forall x_1 \in (0,1), \end{aligned}$$

where $\tilde{\Omega}_h = T_h(\Omega_h)$. Up to multiple extractions, we can assume that $\tilde{\Omega}_h \xrightarrow{\mathcal{H}} \tilde{\Omega}$, $\mu_i(\Omega_h) \to \mu_i(I,p)$ and $\mu_i(I,p_h) \to \mu_i(I,p)$ where $I = (0,1)$. Let us highlight that

$$\mu_{k-1}(I,p) < \mu_k(I,p), \tag{4.56}$$

Let us consider the first $k+1$ eigenfunctions of (I, p_h) (u_0 is the constant function)

$$u_0^h(x_1), u_1^h(x_1), \dots, u_k^h(x_1),$$

normalized in the following way

$$\int_0^1 \big(u_i^n(x_1)\big)^2 p_h(x_1)\,dx_1 = \int_{\Omega_h} \big(u_i^n(x_1)\big)^2\,dx = 1 \qquad \forall i \in \{0,\dots,k\}, \quad (4.57)$$

and let V be

$$V = \operatorname{span}\Big\{u_0^h(x_1), u_1^h(x_1), \dots, u_{k-1}^h(x_1), u_k^h(x_1)\Big[1+\tau\big(x_2-c(x_1)a_2(\Omega_h)\big)^2\Big]\Big\} \subseteq H^1(\Omega_h),$$

where

$$c(x_1) = \fint_{S_{x_1}\cap\tilde\Omega} x_2\,d\mathcal{H}^{n-1}.$$

In the following, for reader's sake we will omit the dependence on n of α_i^h and u_i^n.

By the variational formulation of μ_k, (50), and using (4.55) we know

$$\begin{aligned}\mu_{k,n}^* - \frac{1}{h}a_2(\Omega_h)^2 \le \mu_k(\Omega_h) &\le \max_{v\in V}\frac{\displaystyle\int_{\Omega_h}|\nabla v|^2\,dx}{\displaystyle\int_{\Omega_h} v^2\,dx}\\ &= \max_{\alpha_i\in\mathbb{R}}\frac{\displaystyle\int_{\Omega_h}\Big|\sum_{i=0}^{k-1}\alpha_i\nabla v_i+\alpha_k\nabla v_k\Big|^2\,dx}{\displaystyle\int_{\Omega_h}\Big(\sum_{i=0}^{k-1}\alpha_i v_i+\alpha_k v_k\Big)^2\,dx} =: \mu,\end{aligned} \qquad (4.58)$$

where

$$\begin{aligned}&v_i(x_1,x') = u_i(x_1) && \forall i\in\{0,\dots,n-1\}\\ &v_k(x_1,x') = u_k(x_1)\Big[1+\tau\big(x_2-c(x_1)a_2(\Omega_h)\big)^2\Big].\end{aligned}$$

Taking into account the orthogonality of the eigenfunction, i.e.

$$\int_{\Omega_h} u_iu_j\,dx = \int_{\Omega_h}\nabla u_i\,\nabla u_j\,dx = 0 \qquad \forall i\neq j,$$

we can rewrite (4.58) as

$$\mu = \max_{\alpha_i \in \mathbb{R}} \frac{\sum_{i=0}^{k-1} \alpha_i^2 \int_{\Omega_h} |\nabla u_i|^2 \, dx + \alpha_k^2 \int_{\Omega_h} |\nabla \tilde{u}_k|^2 \, dx + 2 \sum_{i=0}^{k-1} \alpha_i \alpha_k \int_{\Omega_h} \nabla u_i \, \nabla \tilde{u}_k \, dx}{\sum_{i=0}^{k-1} \alpha_i^2 \int_{\Omega_h} u_i^2 \, dx + \alpha_k^2 \int_{\Omega_h} \tilde{u}_k^2 \, dx + 2 \sum_{i=0}^{k-1} \alpha_i \alpha_k \int_{\Omega_h} u_i \, \tilde{u}_k \, dx}. \tag{4.59}$$

Let us rewrite the previous term: $\forall i \in \{0, \dots, k-1\}$ we define

$$\tilde{v}_i(x_1, x') = k_h v_i\big(x_1, a_2(\Omega_h)x_2, \dots, a_n(\Omega)x_n\big) = \tilde{u}_i(x_1)$$
$$\tilde{v}_k(x_1, x') = k_h v_k\big(x_1, a_2(\Omega_h)x_2, \dots, a_n(\Omega_h)x_n\big) = \tilde{u}_k(x_1)\Big[1 + \tau\big(x_2 - c(x_1)\big)a_2(\Omega_h)^2\Big]$$
$$\tilde{p}_h(x_1) = \frac{p_h(x_1)}{k_h},$$

where

$$\tilde{u}_i(x_1) = k_h u_i(x_1) \qquad \forall i \in \{0, \dots, k\} \qquad \text{and} \qquad k_h = \sqrt{\prod_{j=2}^{n} a_j(\Omega_h)},$$

with a change of variable, (4.59) becomes

$$\mu = \max_{\alpha_i \in \mathbb{R}} \left(\frac{\sum_{i=0}^{k-1} \alpha_i^2 \int_{\tilde{\Omega}_h} \big(\tilde{v}_i'(x_1)\big)^2 \, dx + \alpha_k^2 \int_{\tilde{\Omega}_h} \left[\left(\frac{\partial \tilde{v}_k}{\partial x_1}\right)^2 + \frac{1}{a_2(\Omega_h)^2} \left(\frac{\partial \tilde{v}_k}{\partial x_2}\right)^2 \right] dx}{\sum_{i=0}^{k-1} \alpha_i^2 \int_{\tilde{\Omega}_h} \tilde{v}_i^2 \, dx + \alpha_k^2 \int_{\tilde{\Omega}_h} \tilde{v}_k^2 \, dx + 2 \sum_{i=0}^{k-1} \alpha_i \alpha_k \int_{\tilde{\Omega}_h} \tilde{v}_i \, \tilde{v}_k \, dx} + \frac{2 \sum_{i=0}^{k-1} \alpha_i \alpha_k \int_{\tilde{\Omega}_h} \tilde{v}_i'(x_1) \left(\frac{\partial \tilde{v}_k}{\partial x_1}\right) dx}{\sum_{i=0}^{k-1} \alpha_i^2 \int_{\tilde{\Omega}_h} \tilde{v}_i^2 \, dx + \alpha_k^2 \int_{\tilde{\Omega}_h} \tilde{v}_k^2 \, dx + 2 \sum_{i=0}^{k-1} \alpha_i \alpha_k \int_{\tilde{\Omega}_h} \tilde{v}_i \, \tilde{v}_k \, dx} \right). \tag{4.60}$$

The maximum exists and it is attained on some $(\alpha_0, \dots, \alpha_k)$ such that $\sum_{i=0}^{k} \alpha_i^2 = 1$, $\alpha_k > 0$.

We want to prove that $\alpha_i^h = O\big(a_2(\Omega_h)^2\big)$ $\forall i \in \{0, \dots, k-1\}$ while $\alpha_k = 1 - O\big(a_2(\Omega_h)^2\big)$.

For any $i \in \{0, \ldots, k-1\}$ the optimality condition for (4.60) reads as

$$2\alpha_i \int_{\tilde{\Omega}_h} \big(\tilde{v}_i'(x_1)\big)^2\, dx + 2\alpha_k \int_{\tilde{\Omega}_h} \tilde{v}_i'(x_1) \left(\frac{\partial \tilde{v}_k}{\partial x_1}\right) dx = \mu \left[2\alpha_i \int_{\tilde{\Omega}_h} \tilde{v}_i^2\, dx + 2\alpha_k \int_{\tilde{\Omega}_h} \tilde{v}_i\, \tilde{v}_k\, dx\right], \tag{4.61}$$

and using that for any $i \in \{0, \ldots, k\}$

$$\begin{aligned}
\int_{\tilde{\Omega}_h} \big(\tilde{v}_i'(x_1)\big)^2\, dx &= \int_{\tilde{\Omega}_h} \big(\tilde{u}_i'(x_1)\big)^2\, dx = \mu_i(I, p_h) \\
\int_{\Omega_h} \tilde{v}_i'(x_1) \left(\frac{\partial \tilde{v}_k}{\partial x_1}\right) dx &= \int_{\tilde{\Omega}_h} \tilde{u}_1'(x_1)\tilde{u}_k'(x_1)\, dx \\
&\quad + \tau\left(\int_{\tilde{\Omega}_h} \tilde{u}_i'(x_1)\tilde{u}_k'(x_1)\big(x_2 - c(x_1)\big)^2\, dx\right) a_2(\Omega_h)^2 \\
&\quad - 2\tau\left(\int_{\tilde{\Omega}_h} \tilde{u}_i'(x_1)\tilde{u}_k(x_1)\big(x_2 - c(x_1)\big)c'(x_1)\, dx\right) a_2(\Omega_h)^2 \\
&= \tau\left(\int_{\tilde{\Omega}_h} \tilde{u}_i'(x_1)\tilde{u}_k'(x_1)\big(x_2 - c(x_1)\big)^2\, dx\right) a_2(\Omega_h)^2 \\
&\quad - 2\tau\left(\int_{\tilde{\Omega}_h} \tilde{u}_i'(x_1)\tilde{u}_k(x_1)\big(x_2 - c(x_1)\big)c'(x_1)\, dx\right) a_2(\Omega_h)^2, \\
\int_{\tilde{\Omega}_h} \tilde{v}_i^2\, dx &= \int_{\tilde{\Omega}_h} \tilde{u}_i^2\, dx = 1, \\
\int_{\tilde{\Omega}_h} \tilde{v}_i \tilde{v}_k\, dx &= \int_{\tilde{\Omega}_h} \tilde{u}_i(x_1)\tilde{u}_k(x_1)\, dx \\
&\quad + \tau\left(\int_{\tilde{\Omega}_h} \tilde{u}_i(x_1)\tilde{u}_k(x_1)\big(x_2 - c(x_1)\big)^2\, dx\right) a_2(\Omega_h)^2 \\
&= \tau\left(\int_{\tilde{\Omega}_h} \tilde{u}_i(x_1)\tilde{u}_k(x_1)\big(x_2 - c(x_1)\big)^2\, dx\right) a_2(\Omega_h)^2,
\end{aligned}$$

we get (here we make explicit the dependence on n)

$$2\alpha_i^h\left[\mu^h - \mu_i(I,p_h)\right] = 2\alpha_k^h\tau\Bigg[\int_{\tilde{\Omega}_h}\big(\tilde{u}_i^h(x_1)\big)'\,\big(\tilde{u}_k^h(x_1)\big)'\big(x_2-c(x_1)\big)^2\,dx$$
$$-\int_{\tilde{\Omega}_h}\big(\tilde{u}_i^h(x_1)\big)'\big(\tilde{u}_k^h(x_1)\big)\big(x_2-c(x_1)\big)c'(x_1)\,dx$$
$$-\,2\mu^h\int_{\tilde{\Omega}_h}\tilde{u}_i^h(x_1)\,u_k^h(x_1)\big(x_2-c(x_1)\big)^2\,dx\Bigg]a_2(\Omega_h)^2.$$

Let us denote with A_h^i and B_h^i respectively the term in square braket at left hand-side and right hand-side. By (4.58), $\mu^h \geq \mu_k(I,p_h)$, so passing to the $\liminf$ and using (4.56) we have

$$\liminf_n A_h^i \geq \mu_k(I,p) - \mu_{k-1}(I,p) > 0 \qquad\qquad \forall i\in\{0,\dots,k-1\}.$$

The term at right hand-side has order 2 in $a_2(\Omega_h)$, so also the term at right hand-side has the same order; hence, taking into account the fact that $\sum_{i=0}^k(\alpha_i^h)^2 = 1$, we obtain

$$\alpha_i^h = O\big(a_2(\Omega_h)^2\big) \qquad \forall i\in\{0,\dots,k-1\} \qquad \text{and} \qquad \alpha_k^h = 1 - O\big(a_2(\Omega_h)^2\big).$$

Let us now estimate the quotient in (4.60): we call N and D respectively the numerator and denominator related to the maximum.

- Numerator. Recalling that

$$\int_{\Omega_h}\big((\tilde{u}_i^h)'\big)^2\,dx = \mu_i(I,p_h) \qquad\qquad \forall i\in\{0,\dots,k\}$$
$$\alpha_i^h = O\big(a_2(\Omega_h)^2\big)$$

and by the estimates used in (4.61), we have

$$
\begin{aligned}
N = {} & \sum_{i=0}^{k-1}(\alpha_i^h)^2\mu_i(I,p_h) + (\alpha_k^h)^2\Bigg\{\int_{\tilde\Omega_h}\big((\tilde u_k^h)'\big)^2\left[1+\tau\big(x_2-c(x_1)\big)^2 a_2(\Omega_h)^2\right]^2 dx \\
& - 4\tau\left(\int_{\tilde\Omega_h}\tilde u_k'(x_1)\tilde u_k(x_1)\big(x_2-c(x_1)\big)c'(x_1)\,dx\right)a_2(\Omega_h)^2 \\
& - 4\tau^2\left(\int_{\tilde\Omega_h}\tilde u_k'(x_1)\tilde u_k(x_1)\big(x_2-c(x_1)\big)^3c'(x_1)\right)a_2(\Omega_h)^4 \\
& + 4\tau^2\left(\int_{\tilde\Omega_h}\big(\tilde u_k^h(x_1)\big)^2\big(x_2-c(x_1)\big)^2c'(x_1)^2\,dx\right)a_2(\Omega_h)^4 \\
& + 4\tau^2\left(\int_{\tilde\Omega_h}\big(\tilde u_k^h(x_1)\big)^2\big(x_2-c(x_1)\big)^2\,dx\right)a_2(\Omega_h)^2\Bigg\} \\
& + 2\alpha_k^h\tau\sum_{i=0}^{k-1}\alpha_i^h\left(\int_{\tilde\Omega_h}\big(\tilde u_i^h(x_1)\big)'\big(\tilde u_k^h(x_1)\big)'\,\big(x_2-c(x_1)\big)^2\,dx\right)a_2(\Omega_h)^2 \\
& - 4\alpha_k^h\tau\sum_{i=0}^{k-1}\alpha_i^h\left(\int_{\tilde\Omega_h}\big(\tilde u_i^h(x_1)\big)'\tilde u_k^h(x_1)\big(x_2-c(x_1)\big)c'(x_1)\,dx\right)a_2(\Omega_h)^2,
\end{aligned}
$$

that is

$$
\begin{aligned}
N = {} & \sum_{i=0}^{k}(\alpha_i^h)^2\mu_i(I,p_h) + 2\tau\big(\alpha_k^h\big)^2\left(\int_{\tilde\Omega_h}\big((\tilde u_k^h)'(x_1)\big)^2\big(x_2-c(x_1)\big)^2\,dx\right)a_2(\Omega_h)^2 \\
& - 4\big(\alpha_k^h\big)^2\tau\left(\int_{\tilde\Omega_h}\big(\tilde u_k^h(x_1)\big)'\tilde u_k^h(x_1)\big(x_2-c(x_1)\big)c(x_1)\,dx\right)a_2(\Omega_h)^2 \\
& + 4\tau^2\big(\alpha_k^h\big)^2\left(\int_{\tilde\Omega_h}\big(\tilde u_k^h(x_1)\big)^2\big(x_2-c(x_1)\big)^2\right)a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big).
\end{aligned}
$$

Recalling that

$$
\mu_i(I,p_h)\le\mu_{k,n}^*,\qquad \sum_{i=0}^{k}(\alpha_i^h)^2=1,\qquad (\alpha_k^h)^2 = 1-O\big(a_2(\Omega_h)^2\big),
$$

we obtain

$$
\begin{aligned}
N \leq \mu^*_{k,n} + \Bigg[& 2\tau \int_{\tilde{\Omega}_h} \big((\tilde{u}^h_k)'(x_1)\big)^2 \, \big(x_2 - c(x_1)\big)^2 \, dx \\
& - 4\tau \int_{\tilde{\Omega}_h} \big(\tilde{u}^h_k(x_1)\big)' \big(\tilde{u}^h_k(x_1)\big) \big(x_2 - c(x_1)\big) c'(x_1) \, dx \\
& + 4\tau^2 \int_{\tilde{\Omega}_h} \big(\tilde{u}^h_k\big)^2 \, \big(x_2 - c(x_1)\big)^2 \, dx \Bigg] a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big).
\end{aligned}
$$

- Denominator. Recalling that $\int_{\Omega_h} (u^n_i)^2 \, dx = 1$ for each $i \in \{0, \ldots, k\}$, and arguing as for the numerator, we get

$$
\begin{aligned}
D = & \sum_{i=0}^{k-1} (\alpha^h_i)^2 + (\alpha^h_k)^2 \int_{\tilde{\Omega}_h} \big(\tilde{u}^h_k(x_1)\big)^2 \Big[1 + \tau\big(x_2 - c(x_1)\big)^2 a_2(\Omega_h)^2\Big]^2 \, dx \\
& + 2 \sum_{i=0}^{k-1} \alpha^h_i \alpha^h_k \tau \int_{\tilde{\Omega}_h} \tilde{u}^h_i(x_1) \, u^h_k(x_1) \, \big(x_2 - c(x_1)\big)^2 a_2(\Omega_h)^2 \, dx \\
& = \sum_{i=0}^{k} (\alpha^h_i)^2 + 2\tau (\alpha^h_k)^2 \left(\int_{\tilde{\Omega}_h} \big(\tilde{u}^h_k(x_2)\big)^2 \big(x_2 - c(x_1)\big)^2 \, dx \right) a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big) \\
& = 1 + 2\tau \left(\int_{\tilde{\Omega}_h} \big(\tilde{u}^h_k(x_1)\big)^2 \big(x_2 - c(x_1)\big)^2 \, dx \right) a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big).
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
&\mu^*_{k,n} - \frac{1}{h} a_2(\Omega_h)^2 \\
&\quad \le \frac{\mu^*_{k,n} + 2\tau\left(\int_{\tilde\Omega_h} \big((\tilde u^h_k)'(x_1)\big)^2 \big(x_2 - c(x_1)\big)^2\,dx\right) a_2(\Omega_h)^2}{1 + 2\tau\left(\int_{\tilde\Omega_h} \big(\tilde u^h_k(x_1)\big)^2 \big(x_2 - c(x_1)\big)^2\,dx\right) a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big)} \\
&\quad - \frac{4\tau\left(\int_{\tilde\Omega_h} \big(\tilde u^h_k(x_1)\big)' \big(\tilde u^h_k(x_1)\big) \big(x_2 - c(x_1)\big) c'(x_1)\,dx\right) a_2(\Omega_h)^2}{1 + 2\tau\left(\int_{\tilde\Omega_h} \big(\tilde u^h_k(x_1)\big)^2 \big(x_2 - c(x_1)\big)^2\,dx\right) a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big)} \\
&\quad + \frac{4\tau^2\left(\int_{\tilde\Omega_h} \big(\tilde u^h_k(x_1)\big)^2 \big(x_2 - c(x_1)\big)^2\,dx\right) a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big)}{1 + 2\tau\left(\int_{\tilde\Omega_h} \big(\tilde u^h_k(x_1)\big)^2 \big(x_2 - c(x_1)\big)^2\,dx\right) a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big)}.
\end{aligned}
\tag{4.62}
$$

With the same computations as in the proof for $k = 1$, we have

$$
\begin{aligned}
&\left(\mu^*_{k,n} - \frac{1}{h} a_2(\Omega_h)^2\right)\left(1 + 2\tau\left[\int_{\tilde\Omega_h} \big(\tilde u^h_k(x_1)\big)^2 \big(x_2 - c(x_1)\big)^2\,dx\right] a_2(\Omega_h)^2 \right. \\
&\qquad \left. + o\big(a_2(\Omega_h)^2\big)\right) \\
&\qquad \le \mu^*_{k,n} + \left[2\tau \int_{\tilde\Omega_h} \big((\tilde u^h_k)'(x_1)\big)^2 \big(x_2 - c(x_1)\big)^2\,dx \right. \\
&\qquad - 4\tau \int_{\tilde\Omega_h} \big(\tilde u^h_k(x_1)\big)' \tilde u^h_k(x_1) \big(x_2 - c(x_1)\big) c'(x_1)\,dx \\
&\qquad \left. + 4\tau^2 \int_{\tilde\Omega_h} \big(\tilde u^h_k(x_1)\big)^2 \big(x_2 - c(x_1)\big)^2\,dx\right] a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big)
\end{aligned}
$$

that is

$$
\begin{aligned}
-\frac{1}{h}a_2(\Omega_h)^2 + o\big(a_2(\Omega_h)^2\big) \leq 2\tau\Bigg[&\int_{\tilde\Omega_h}\Big(\big((\tilde u_k^h)(x_1)'\big)^2 - \mu_{k,n}^*\big(\tilde u_k^h(x_1)\big)^2\Big)\big(x_2 - c(x_1)\big)^2\,dx \\
&- 2\int_{\tilde\Omega_h}\big(\tilde u_k^h(x_1)\big)'\tilde u_k^h(x_1)\,\big(x_2 - c(x_1)\big)c'(x_1)\,dx\Bigg]a_2(\Omega_h)^2 \\
&+ 4\tau^2\left(\int_{\tilde\Omega_h}\big(\tilde u_k^h(x_1)\big)^2\big(x_2 - c(x_1)\big)^2\,dx\right)a_2(\Omega_h) \\
&=: 2\gamma_h a_2(\Omega_h)^2\tau + 4\beta_h a_2(\Omega_h)^2\tau^2,
\end{aligned}
\tag{4.63}
$$

where

$$
\begin{aligned}
\gamma_h = &\int_{\tilde\Omega_h}\Big(\big((\tilde u_k^h)'(x_1)\big)^2 - \mu_{k,n}^*\big(\tilde u_k^h(x_1)\big)^2\Big)\big(x_2 - c(x_1)\big)^2\,dx \\
&- 2\int_{\tilde\Omega_h}\big(\tilde u_k^h(x_1)\big)'\tilde u_k^h\big(x_2 - c(x_1)\big)c'(x_1)\,dx, \\
\beta_h = &\int_{\tilde\Omega_h}\big(\tilde u_k^h(x_1)\big)^2\big(x_2 - c(x_1)\big)^2\,dx.
\end{aligned}
$$

As in the case $k = 1$, we notice that the term at right hand side in (4.63) has order 2 in $a_2(\Omega_h)$ and it is a second order polynomial in τ. Hence the contradiction is achieved choosing $\tau < 0$ sufficiently small provided that

$$
\lim_{h\to+\infty}\gamma_h \neq 0,
$$

that is

$$
\begin{aligned}
\lim_{h\to+\infty}\int_{\tilde\Omega_h}&\Big[\big((\tilde u_k^h)'(x_1)\big)^2 - \mu_{k,n}^*\big(\tilde u_k^h(x_1)\big)^2\Big]\big(x_2 - c(x_1)\big)^2\,dx \\
&- 2\int_{\tilde\Omega_h}\big(\tilde u_k^h(x_1)\big)'u_k^h\big(x_2 - c(x_1)\big)c'(x_1)\,dx
\end{aligned}
$$

Arguing as in the case $k = 1$, up to a subsequence, $\tilde u_k^h$ converges to $\tilde u_k$, the k-th eigenfunction of the Sturm-Liouville problem related to $p_{\tilde\Omega}$, and it holds

$$
\lim_{h\to+\infty}\gamma_h = \int_0^1\Big[\big(\tilde u_k'(x_1)\big)^2 - \mu_{k,n}^*\tilde u_k(x_1)^2\Big]\left(\int_{S_{x_1}\cap\tilde\Omega}\big(x_2 - c(x_1)\big)^2\,d\mathcal{H}^{n-1}\right)dx_1.
\tag{4.64}
$$

For reader's convenience, we will omit k and the tilde in the following.

Theorem 4.2.4 ensures that $p_{\tilde{\Omega}}^{\frac{1}{n-1}}$ is made by segments, so there exists $j \in \mathbb{N}$ such that $p_{\tilde{\Omega}}^{\frac{1}{n-1}}$ is an affine function in $\left[\frac{i}{j}, \frac{i+1}{j}\right]$ for $i \in \{0, \dots, j-1\}$. So by (4.41), for each $i \in \{0, \dots, j-1\}$ there exists $K_i > 0$ such that

$$\int_{S_{x_1} \cap \tilde{\Omega}} \left(x_2 - c(x_1)\right)^2 d\mathcal{H}^{n-1} = K_i p(x_1)^{1+\frac{2}{n-1}} \qquad \forall x_1 \in \left[\frac{i}{j}, \frac{i+1}{j}\right],$$

and (4.64) becomes

$$\int_0^1 \left[(\tilde{u}'(x_1))^2 - \mu_{k,n}^* \tilde{u}(x_1)^2\right] \left(\int_{S_{x_1} \cap \tilde{\Omega}} \left(x_2 - c(x_1)\right)^2 d\mathcal{H}^{n-1}\right) dx_1$$
$$= \sum_{i=0}^{j-1} K_i \int_{\frac{i}{j}}^{\frac{i+1}{j}} \left[\left(\tilde{u}'(x_1)\right)^2 - \mu_{k,n}^* \tilde{u}(x_1)^2\right] p(x_1)^{\frac{2}{n-1}}.$$

With the same reasoning as in proof for $k = 1$, there exist $\tilde{a}_i$ for each $i \in \{0, \dots, j-1\}$ such that it holds

$$\sum_{i=0}^{j-1} K_i \int_{\frac{i}{j}}^{\frac{i+1}{j}} \left[\left(\tilde{u}'(x_1)\right)^2 - \mu_{k,n}^* \tilde{u}(x_1)^2\right] p(x_1)^{1+\frac{2}{n-1}} = \sum_{i=0}^{j-1} K_i n \tilde{a}_i^2 \int_{\frac{i}{j}}^{\frac{i+1}{j}} \tilde{u}(x_1)^2 p(x_1)\, dx_1,$$

that is strictly positive, which concludes the proof. □

4.2.5 Optimality of the exponent

To prove the optimality of the exponent, we will need the definition of superequilateral triangle.

Definition 4.2.2. Let T_α be an isosceles triangle having aperture $\alpha \in (0, \pi)$ and equal sides of length l such that the basis is placed along the x axis (Figure 4.4), i.e.

$$T_\alpha = \left\{(x, y) \in \mathbb{R}^2 \ : \ -l \sin\frac{\alpha}{2} \le x \le l \sin\frac{\alpha}{2}, 0 \le y \le l \cos\frac{\alpha}{2} - \cot\frac{\alpha}{2}\,|x|\right\}.$$

If $\alpha \in \left(0, \frac{\pi}{3}\right)$, we say T_α is subequilateral, otherwise superequilateral.

We recall the following estimate for the Neumann eigenvalue for isosceles triangles proved in [LS17, Proposition 6.42]:

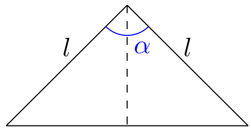

Figure 4.4: The triangle T_α

Proposition 4.2.9. *Let T_α be a superequilateral triangle, then it holds*

$$\sin^2\left(\frac{\alpha}{2}\right) \leq \frac{\mu_1(T_\alpha)D^2(T_\alpha)}{4j_{0,1}^2} < 1. \tag{4.65}$$

Now we are in position to prove the optimality of exponent 2 in dimension $n = 2$.

Theorem 4.2.10. *Exponent 2 in (4.42) is sharp for $n = 2$.*

Proof of theorem 4.2.10. Let us consider Ω an open, bounded and convex set with $\operatorname{diam}(\Omega) = 1$ rotated in such a way that the diameter is placed along the x axis. First of all let us observe that in dimension 2, it holds

$$a_2(\omega) \leq w_\Omega(e_2) \leq 2a_2(\Omega), \tag{4.66}$$

where $w_\Omega(e_2)$ is the width in direction orthogonal to the diameter. Because of formula (4.66), we will use the width in the following and for sake of brevity we will denote it by w.

Let us consider a superequilateral triangle T_α having the segment $[0,1]\times\{0\}$ as basis. Formula (4.65) gives

$$\mu_1(T_\alpha) \geq 4j_{0,1}^2 \sin^2\left(\frac{\alpha}{2}\right),$$

where α is the obtuse angle in T_α (Figure 4.5).

Since $w = \ell\cos\frac{\alpha}{2}$ we can write using (4.65)

$$\mu_1(T_w) \geq 4j_{0,1}^2 \sin^2\left(\frac{\alpha}{2}\right) = 4j_{0,1}^2\left(1 - \frac{w^2}{\ell^2}\right).$$

When $w \to 0$, $\ell \to \frac{1}{2}$ so using the previous inequality combined with (4.42), we can write

$$4j_{0,1}^2 - 16j_{0,1}^2 w^2 + o(w^2) \leq \mu_1(T_w) \leq 4j_{0,1}^2 - Cw^2, \tag{4.67}$$

therefore the exponent 2 is optimal. $\square$

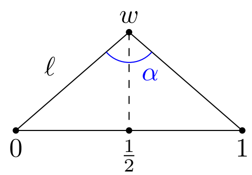

Figure 4.5

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