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Novel perspectives for Horndeski gravity: from effective fluids to exact solutions

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To my grandmother, Antonietta

Abstract

This thesis delves into the study of scalar-tensor gravitational theories that extend General Relativity. Such alternative formulations arise primarily due to the shortcomings of the standard cosmological model. By introducing an additional scalar field, these theories seek to deepen our understanding of gravity and resolve both theoretical and observational challenges. The focus is mainly on the viable (or reduced) subclass of Horndeski gravity, which represents the most general non-degenerate scalar-tensor theory, characterized by manifestly second-order field equations and luminal propagation of tensor modes. Despite this observational constraint, viable models of Horndeski gravity remain quite general due to the presence of unknown functions within the theory. To gain insights and criteria for selecting the functional form of the theory, we explore novel perspectives. Framing the discussion within the effective fluid approach, we interpret the scalar-tensor contributions to the field equations as the effective stress-energy tensor of a dissipative fluid, leading to a classification of Horndeski gravity in terms of "Newtonian" and "non-Newtonian" fluids. We discuss the formal analogy with first-order thermodynamics of imperfect fluids, examining cosmological applications and possible alternative formulations. Another approach involves studying covariant effective formulations of a non-singular bounce, replacing the Big Bang initial singularity. In this framework, the selected models correspond to scalar-tensor theories without additional propagating degrees of freedom, such as cuscuton and extended cuscuton. Next, the case of Horndeski gravity as a source for non-rotating black holes embedded in a dynamic universe is explored. This again selects a non-dynamic scalar field, which is stealth at small scales near the central object but, at large scales, influences the evolution of the universe and its matter content. Finally, the last criterion analyzed is the requirement for the theory to possess a Noether symmetry. Based on this, a general classification of Horndeski gravity is provided.

Preface

In this thesis, I present part of the research conducted over the past four years.

Starting from the idea that General Relativity is not a complete and exhaustive theory, we will investigate Horndeski gravity, offering novel perspectives for its physical interpretation and in the selection of a general functional form for the theory. The dissertation is organized as follows.

The first Chapter introduces General Relativity and the current state-of-the-art in modified or extended theories of gravity, illustrated in Fig. 1.1. Specifically, in Sec.1.1, the foundations of General Relativity are outlined, along with its successes, limitations, and challenges; in Sec.1.2, an overview of gravitational theories is provided with a historical timeline of their development; Sec. 1.3 offers a concise review of Horndeski gravity and its generalizations, following some of the most recent and useful works in the literature.

In Ch. 2, the effective fluid formalism is presented, interpreting Horndeski gravity through the lens of a dissipative effective fluid and shedding light on its physical characteristics. Imposing Newtonian fluid behavior on the constitutive equations of this fluid constrains the theory to two distinct "viable" subclasses within Horndeski gravity. Consequently, when the stress-energy tensor of the Horndeski effective fluid is linear in the first derivatives of the fluid's 4-velocity, gravitational waves propagate at the speed of light. Horndeski models outside these linear subclasses are identified as exotic non-Newtonian fluids. A schematic summary is provided in Tab. 2.1. A first-order thermodynamic framework for viscous fluids is then applied to these two linear Horndeski classes, leading to additional restrictions on the theory's functional form, as reported in Tab. 2.2.

In Ch. 3, the first-order thermodynamics of Horndeski gravity is considered, with applications to cosmology. A notable analogy between the constitutive relations of the "Horndeski fluid" and those of Eckart's thermodynamic framework is shown. Then, focusing on a spatially flat, homogeneous, and isotropic spacetime, the analysis narrows to two theoretical categories: shift-symmetric and asymptotically shift-symmetric models, characterized by a non-zero braiding parameter. Unlike traditional scalar-tensor gravity, these subclasses of Horndeski gravity do not revert to General Relativity, often seen as an equilibrium state at zero temperature, but instead reach equilibrium states with non-zero viscosity. Additionally, this analysis reaffirms earlier findings that curvature singularities exhibit extreme temperature divergence, indicating that scalar-tensor deviations from General Relativity grow near spacetime singularities. Finally, as part of this investigation, a new exact cosmological solution for an asymptotically shift-symmetric model is presented.

In Ch. 4, alternative approaches to analogizing viable Horndeski gravity with Eckart's first-order thermodynamics are examined. The focus is on a specific class of mappings for the effective stress-energy tensor of the scalar field fluid which, when decomposed as an imperfect fluid, yields constitutive relations similar to those in Eckart's framework. The discussion then moves to how different couplings to Einstein's gravity within the field equations influence the overall thermodynamic structure. This leads to a coupling function that simplifies the formalism and enhances understanding of these models.

Ch. 5 investigates the feasibility of an effective Friedmann equation that replaces the classical Big Bang singularity with a non-singular bounce at a critical energy density. In a flat, homogeneous, and isotropic universe, this framework is established by introducing a function parametrically dependent on the critical energy density, serving as a measure of deviation from a benchmark theory. This benchmark theory characterizes the behavior of the effective model as the critical energy density approaches infinity. By focusing on a covariant single-field formulation within viable Horndeski gravity, it is demonstrated that both the effective and benchmark theories belong to the same scalar-tensor framework, involving no additional degrees of freedom—specifically, the cuscuton and extended cuscuton models.

In Ch. 6, McVittie and generalized McVittie solutions are investigated within Horndeski gravity, focusing on a spatially homogeneous gravitational scalar field. This field is stealthy on small scales around the central mass but, on larger scales, drives the cosmological background embedding the central inhomogeneity. Unlike previous studies, the matter is also considered here. A classification of configurations is then provided, based on the evolution of the gravitational coupling, radial energy flow, accretion onto the central mass, and the Hubble expansion rate.

In Ch. 7, the Noether Symmetries Approach is applied to Horndeski gravity, expanding the classification of scalar-tensor theories in the literature. Reviewing the minimally coupled scalar field and the first-generation scalar-tensor models, the analysis extends to include kinetic gravity braiding and Horndeski gravity. Key results are highlighted by focusing on the non-minimally coupled Gauss–Bonnet term and the extended cuscuton model. The impact of matter on Noether symmetries is also explored, revealing that the selected Horndeski functions remain unchanged compared to the vacuum scenario.

All the results are summarized in the final Chapter 8, and future investigation proposals are outlined.

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Even though it shows up right at the beginning, here we arrive at one of the most important parts of this thesis—at least for me—which comes as the conclusion of four incredible years.

It is a matter of fact that no personal growth and no meaningful gains are achieved without pain. Nothing truly worthy comes easily. For every conquest, there is always something one must be willing to sacrifice. This chapter of my life is no exception to this *law of nature*. However, it would not have been the same without the presence of several people. This is precisely one of those moments when you look around at the people in your life and realize just how lucky you truly are. While I believe I always strive to show my gratitude and love, it is also important *not to take things for granted*. Therefore, I am trying to acknowledge all the people who have left a mark on my life over the past four years and, in one way or another, contributed to helping me reach this point.

I would sincerely like to open this section of acknowledgments by expressing my deep gratitude to my PhD supervisors. I want to thank Salvatore Capozziello for his guidance and suggestions, as well as for being an invaluable role model. I am grateful for the many opportunities he provided to challenge myself and push my limits, and for granting me the freedom to make my own choices while supporting me. I wish to thank Daniele Vernieri for actively contributing to my growth and—citing his own words—for helping me navigate the jungle of the research world (sometimes acting as a guardian angel). His role was crucial in the development of precious tools.

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Publication List

This thesis is based on the following publications:

[1] M. Miranda, S. Capozziello, and D. Vernieri, *General analysis of Noether symmetries in Horndeski gravity*, Eur. Phys. J. C **84**, 771 (2024), arXiv:2408.09018 [gr-qc] .

[2] L. Gallerani, M. Miranda, A. Giusti, and A. Mentrelli, *Alternative formulations of the thermodynamics of scalar-tensor theories*, Phys. Rev. D **110**, 064087 (2024), arXiv:2405.20865 [gr-qc] .

[3] M. Miranda, *Covariant single-field formulation of effective cosmological bounces* (2024), arXiv:2405.08071 [gr-qc] .

[4] M. Miranda, S. Giardino, A. Giusti, and L. Heisenberg, *First-order thermodynamics of Horndeski cosmology*, Phys. Rev. D **109**, 124033 (2024), arXiv:2401.10351 [gr-qc] .

[5] M. Miranda, D. Vernieri, S. Capozziello, and V. Faraoni, *Fluid nature constrains Horndeski gravity*, Gen. Rel. Grav. **55**, 84 (2023), arXiv:2209.02727 [gr-qc] .

[6] M. Miranda, D. Vernieri, S. Capozziello, and V. Faraoni, *Generalized McVittie geometry in Horndeski gravity with matter*, Phys. Rev. D **105**, 124024 (2022), arXiv:2204.09693 [gr-qc]

Additional publications which are not included in this thesis are listed below:

[7] M. Miranda, M. Rinaldi, and V. Faraoni, *How the Schwarzschild–de Sitter horizons remain in thermal equilibrium at vastly different temperatures*, Phys. Rev. D **110**, 104065 (2024), arXiv:2409.12861 [gr-qc] .

[8] M. Miranda, P.-A. Graham, and V. Faraoni, *Effective fluid mixture of tensor-multi-scalar gravity*, Eur. Phys. J. Plus **138**, 387 (2023), arXiv:2211.03958 [gr-qc] .

[9] M. Miranda, D. Vernieri, S. Capozziello, and F. S. N. Lobo, *Bouncing Cosmology in Fourth-Order Gravity*, Universe **8**, 161 (2022), arXiv:2203.04918 [gr-qc] .

[10] M. Miranda, D. Vernieri, S. Capozziello, and F. S. N. Lobo, *Effective actions for loop quantum cosmology in fourth-order gravity*, Eur. Phys. J. C **81**, 975 (2021), arXiv:2107.07777 [gr-qc] .

[11] D. Dell'Aquila *et al.*, *OSCAR: A new modular device for the identification and correlation of low energy particles*, Nucl. Instrum. Meth. A **877**, 227–237 (2018)

Contents

1 Introduction

1.1 No longer enough: no modifications, no progress

Understanding the intrinsic nature of gravity is one of the most compelling goals in physics. Despite being the most familiar fundamental interaction, daily experienced, gravity remains one of the greatest enigmas of modern physics. Thus, researchers are daily driven in their relentless effort to uncover the mysteries of the force that governs the evolution of the observable universe. But what do we know—or think we know—about gravity?

From Aristotle to Galileo Galilei, and from Isaac Newton to Albert Einstein, gravity has long been a subject of study, with its formulation and comprehension evolving over time. For centuries, Newton's law of universal gravitation—describing gravity as a force of attraction acting between two distant masses—was sufficient to describe the motion of planets and moons, as illustrated by Kepler's laws, and the behavior of falling objects on Earth. However, as measurements and observations grew more precise, it became clear that Newton's formulation, while powerful, was not the full picture.

The *general theory of relativity* (GR), formulated by Albert Einstein in 1915 [12], revolutionized our understanding of gravity: gravitational interaction is described as the curvature of spacetime caused by mass and energy. This outstanding geometric interpretation of gravity replaced Newton's conception of gravity as a distance-acting force, offering a more profound understanding of the interaction between matter and spacetime. GR provided a comprehensive framework that not only addressed known gravitational effects but also predicted new, previously unobserved, phenomena.

One of the earliest and most striking confirmations of GR was its ability to explain the anomalous precession of Mercury's orbit, a problem that had long puzzled astronomers and could not be fully explained by Newtonian mechanics [13–15]. Furthermore, Einstein's theory predicted that light passing near a massive object would be deflected due to the curvature of spacetime. This prediction was famously confirmed during a solar eclipse in 1919, in an expedition led by Eddington [16]. The observation of starlight bending around the (eclipsed) Sun provided the first experimental verification of GR, and its success catapulted Einstein to worldwide fame. In modern times, gravitational lensing has become one of the most powerful tools for probing the universe, and its explanation rests firmly on the predictions of GR. Gravitational lensing, where massive objects like galaxies or galaxy clusters bend the light of more distant objects, has been observed extensively, confirming the warping of spacetime predicted by Einstein [17, 18].

GR's predictive power did not end there. It foresaw the existence of black holes—regions of spacetime where the curvature becomes so extreme that not even light can escape [19, 20]. These exotic objects, once considered purely theoretical, have since been observed through both their gravitational influence on nearby matter and, more recently, through direct imaging by the Event

Horizon Telescope [21, 22].

Another monumental prediction of GR was the existence of gravitational waves, ripples in the fabric of spacetime generated by cataclysmic astrophysical events, such as the coalescence and merging of black holes and neutron stars. Before their direct detection, gravitational waves were indirectly observed through the changing orbital period of binary pulsar systems, most notably the Hulse–Taylor binary pulsar [23], discovered in 1974. The gradual decay of the pulsar's orbit matched precisely with the energy loss predicted by GR due to gravitational wave emission. This indirect evidence provided a strong confirmation of the existence of gravitational waves and earned the 1993 Nobel Prize in Physics [24]. A century after Einstein's prediction, the first direct detection of gravitational waves was made in 2015 by the LIGO and Virgo Collaborations [25, 26], marking a new era in observational astrophysics and providing yet another powerful confirmation of the theory.

In addition to its success in explaining astrophysical phenomena, GR constitutes the cornerstone of modern cosmology. It provides the mathematical framework for the Big Bang model, describing the expansion of the universe [27–30] and predicting phenomena such as cosmic inflation [31, 32] and the formation of large-scale structures like galaxies and clusters [33, 34]. Therefore, GR underpins our understanding of the universe's evolution, from the early moments following the Big Bang to the accelerated expansion observed today [35–37].

GR's applications are not limited to theoretical physics; its principles are also integral to technologies such as the Global Positioning System (GPS), where precise adjustments based on relativistic time dilation are necessary for accurate positioning [38]. Without these relativistic corrections, GPS would be inaccurate by several kilometers each day. GPS and satellite systems represent the first significant technological applications of GR in our everyday lives. While GR has pushed technological development through experiments designed to test its predictions, its direct application to modern technology has also left an indelible mark on society, enabling precision navigation and communication systems that are now indispensable in our daily lives.

With its elegance and extraordinary predictive power, GR is one of the most beautiful and successful formulations of gravity in the history of physics, a theory that has stood the test of time and continues to deepen our understanding of the universe.

1.1.1 Some essentials of Einstein's theory

Since we have just mentioned some successes of GR, it is fitting to briefly summarize the theoretical framework. While comprehensive discussions can be found in many textbooks [39–41], this section aims to provide a concise and ready-to-use overview that will serve as a foundation for the more advanced discussions to come.

In GR, spacetime is modeled as a 4-dimensional differentiable manifold $\mathcal{M}$, equipped with a metric tensor $g_{\mu\nu}$ and a connection $\Gamma^{\lambda}{}_{\mu\nu}$ which is associated to the covariant derivative ∇_{μ}. The connection is assumed to be the Levi–Civita connection, which is *torsionless*,

$$T^{\lambda}{}_{\mu\nu} \equiv \Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu} = 2\Gamma^{\lambda}{}_{[\mu\nu]} = 0\,, \tag{1.1}$$

and *metric-compatible*,

$$Q_{\alpha\mu\nu} \equiv \nabla_{\alpha} g_{\mu\nu} = \partial_{\alpha} g_{\mu\nu} - \Gamma^{\lambda}{}_{\alpha\mu} g_{\lambda\nu} - \Gamma^{\lambda}{}_{\alpha\nu} g_{\mu\alpha} = 0\,. \tag{1.2}$$

Therefore, the connection is fully gauged by the metric tensor,

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) . \tag{1.3}$$

The metric tensor $g_{\mu\nu}$ determines both the causal structure of spacetime (*i.e.*, the behavior of light cones) and the measurement of distances and times. The connection $\Gamma^{\lambda}{}_{\mu\nu}$ defines how vectors and tensors are parallel transported on the manifold, and governs the geodesic motion of free-falling particles.

This description of gravity perfectly implements the Einstein Equivalence Principle (EEP) by requiring that the free-fall motion of test particles is described by the geodesic equation. Indeed, it implies that the gravitational field, the metric, determines the spacetime. Specifically, gravity is mediated through the curvature, *i.e.* the deviation of spacetime from flatness, and this is directly related to the matter and energy content of the universe. Mathematically, it is always possible to choose a so-called *local inertial frame* in a small neighborhood around a point, wherein $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\alpha g_{\mu\nu} = 0$, *i.e.* the gravitational effects vanish—while second derivatives of the metric tensor are non-zero in general. A local inertial frame is associated with an observer that free falls in the gravitational field, experiencing no proper acceleration.

Free-falling test particles follow a geodesic motion. Namely, given a vector field u^μ associated with a free-falling test particle, it is parallel transported (or conserved) along its integral curves. Therefore, the geodesic equations are given by $\dot{u}^\mu \equiv u^\nu \nabla_\nu u^\mu = 0$, vanishing 4-acceleration, which in components turns into

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^{\mu}{}_{\nu\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds} = 0 , \tag{1.4}$$

where, $u^\mu = dx^\mu/ds$, and s is the affine parameter of the curve, in general, or the proper time of a massive test particle.

The geodesic is also interpreted as the generalization of a *straight line* to curved space. Indeed, alternatively, timelike geodesics can be derived as extremal of the length functional $\ell = \int_A^B ds$ between two events A and B, where $ds = \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$. Varying this functional with respect to the path yields the geodesic equation.

Globally, the curvature of spacetime describes the influence of gravity, which is captured by the Riemann curvature tensor, $R^{\mu}{}_{\nu\rho\sigma} = 2 \left(\partial_{[\rho} \Gamma^{\mu}{}_{\sigma]\nu} + \Gamma^{\mu}{}_{\lambda[\rho} \Gamma^{\lambda}{}_{\sigma]\nu} \right)$. The Riemann tensor has several symmetries, $g_{\mu\alpha} R^{\alpha}{}_{\nu\rho\sigma} = R_{\mu\nu\rho\sigma} = R_{[\mu\nu]\rho\sigma} = R_{\mu\nu[\rho\sigma]} = R_{\rho\sigma\mu\nu}$, and satisfies the first and the second Bianchi identities, $R_{\mu[\nu\rho\sigma]}$=0 and $\nabla_{[\lambda|} R^{\mu}{}_{\nu|\rho\sigma]} = 0$, respectively.

By contracting the Riemann tensor, one obtains the Ricci tensor $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$ and the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$. These quantities play central roles in the Einstein field equations, which relate the curvature of spacetime to the energy-momentum content.

Einstein used an inductive approach to determine the equation governing the dynamics of spacetime. By analogy with Maxwell and Poisson equations, he looked for an equation that was as simple as possible: because of the possibility of choosing a locally inertial frame, it is necessary to involve second-order derivatives of the metric tensor, and the criterion of simplicity leads to consider a linear dependence on them. Moreover, he was looking for tensorial equations to ensure the general covariance principle.

The Riemann tensor and its contraction are the only tensors that are linear in second-order derivatives of the metric. Therefore, a natural reasoning choice is to consider in vacuum $R^{\mu}{}_{\nu\rho\sigma} = 0$, but it provides only flat spacetimes as solutions. Thus, the choice goes to $R_{\mu\nu} = 0$.

Then, to include the description of the matter dynamic and the influence of the matter on the spacetime, the Ricci tensor was replaced by the Einstein tensor, yielding to

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu}\, R = 0\,. \tag{1.5}$$

In the presence of matter, for example, field equations can be obtained using the stress-energy tensor of a perfect fluid and taking into account the classical Poisson equation for the gravitational field ($\Delta U_g = 4\pi G\rho$). In general, the Einstein field equations read as follows,

$$G_{\mu\nu} = \kappa^2\, T_{\mu\nu}\,, \tag{1.6}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the matter and the constant term $\kappa^2 = 8\pi G/c^4$ provides the correct Newtonian limit. However, throughout the thesis, we adopt the reduced Planck units $8\pi G = c = \hbar = 1$, so that $\kappa^2 = 1$.

Using a variational approach, the dynamics of spacetime can be derived from the Einstein–Hilbert action:

$$S_{\rm EH} = \frac{1}{2} \int d^4x \sqrt{-g}\, R\,, \tag{1.7}$$

where g is the determinant of the metric, R is the Ricci scalar. Then, the total action is $S = S_{\rm EH} + S^{(\rm m)}$, where $S^{(\rm m)}$ represents the action of matter fields, which are coupled to the metric through the matter Lagrangian $\mathcal{L}^{(\rm m)}$:

$$S^{(\rm m)} = \int d^4x \sqrt{-g}\, \mathcal{L}^{(\rm m)}\,[g_{\mu\nu}, \Psi], \tag{1.8}$$

with Ψ denoting the matter fields.

Performing the variation of the total action $S = S_{\rm EH} + S^{(\rm m)}$ with respect to the metric $g^{\mu\nu}$, one derives the Einstein field equations (1.6), where the energy-momentum tensor is defined as

$$T^{(\rm m)}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S^{(\rm m)}}{\delta g^{\mu\nu}}\,. \tag{1.9}$$

The Einstein field equations describe how matter and energy determine the curvature of spacetime, which in turn dictates the motion of objects and light. Importantly, these equations are consistent with the conservation of stress-energy, as the covariant divergence of the Einstein tensor vanishes due to the contracted Bianchi identity, $\nabla^\mu G_{\mu\nu} = 0$, ensuring $\nabla^\mu T^{(\rm m)}_{\mu\nu} = 0$. In particular, the latter represents the matter field equations of motion and is strictly related to the diffeomorphism invariance of the matter action.

Astrophysical solutions in vacuo

One of the most significant vacuum solutions of the Einstein field equations is the Schwarzschild solution, which describes the spacetime outside a spherically symmetric, non-rotating, and uncharged mass [19]. The Schwarzschild metric, in Schwarzschild coordinates (t, r, θ, ϕ), is given by:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{1.10}$$

where M is the mass of the central object.

This solution has several important features. Mainly, it predicts the existence of black holes, characterized by an event horizon at $r = r_s = 2M$ (the Schwarzschild radius) delimiting regions from which nothing, not even light, can escape. Moreover, it accurately describes the gravitational field of planets and stars, leading to precise predictions such as the precession of Mercury's orbit and the deflection of light by the Sun [13, 14].

The uniqueness of this solution is dictated by Birkhoff's theorem [42, 43], which states that any spherically symmetric solution of the vacuum Einstein equations must be static and asymptotically flat, and is given uniquely by the Schwarzschild solution.

There are notable vacuum astrophysical solutions that can be seen as a generalization of the Schwarzschild spacetime.

The Reissner–Nordström solution describes the spacetime outside a spherically symmetric, charged, non-rotating mass [44, 45]. The metric includes additional terms accounting for the electric charge Q of the central object:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right). \tag{1.11}$$

The Kerr solution describes the spacetime outside a rotating, uncharged mass [20]. This solution introduces frame dragging, where spacetime itself is dragged around the rotating mass. The Kerr metric in Boyer–Lindquist coordinates is:

$$\begin{aligned} ds^2 = &- \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \\ &+ \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2, \end{aligned} \tag{1.12}$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2Mr + a^2$, and $a = J/M$ is the specific angular momentum per unit mass.

The Kerr–Newman solution generalizes the Kerr and Reissner–Nordström solutions to include both rotation and electric charge [46]. The metric is given by modifying the Kerr solution to include the charge Q:

$$\begin{aligned} ds^2 = &- \left(1 - \frac{2Mr - Q^2}{\Sigma}\right) dt^2 - \frac{2(2Mr - Q^2)a\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \\ &+ \left(r^2 + a^2 + \frac{(2Mr - Q^2)a^2\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2, \end{aligned} \tag{1.13}$$

where $\Delta = r^2 - 2Mr + a^2 + Q^2$.

These solutions are essential for understanding the physics of compact objects like neutron stars and black holes, and they play a crucial role in astrophysics and gravitational wave astronomy [47, 48].

Fundamentals of standard Cosmology

In modern cosmology, one of the foundational assumptions is the *cosmological principle*, a modern generalization of the Copernican principle, according to which we do not occupy a special position in the universe. Consequently, the universe is expected to be spatially homogeneous and isotropic at sufficiently large scales: the distribution of matter and energy is uniform, and the universe appears the same in all directions when viewed from any *comoving observer*.

The "comoving observer" refers to the presence of a preferred reference frame, defined as the rest frame of the cosmic matter-energy content.

This assumption is strongly supported by observations of the Cosmic Microwave Background (CMB) [49–51], a homogeneous and isotropic radiation field with a blackbody spectrum corresponding to a temperature of $T_{\rm CMB} \simeq 2.725\,\mathrm{K}$. This temperature is interpreted as the remnant radiation from the Big Bang.

The most general metric tensor satisfying the above assumption is the Friedmann–Lemaître–Robertson–Walker (FLRW). In spherical coordinates (t, r, θ, ϕ), the FLRW line element is given by:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right], \tag{1.14}$$

where $a(t)$ is the scale factor describing the expansion of the universe, and $k = 0, \pm 1$ is the curvature parameter, distinguishing among spatially flat universe (Euclidean geometry), closed universe (spherical geometry), open universe (hyperbolic geometry), respectively.

The unknown scale factor is obtained by solving the differential equations yield by substituting the FLRW ansatz into the Einstein equations, where the matter content is described as a perfect fluid stress-energy tensor:

$$T^{(\rm m)}_{\mu\nu} = \rho\, u_\mu u_\nu + P\, h_{\mu\nu}\,, \tag{1.15}$$

being u^μ the 4-velocity of the cosmic fluid volume elements normalized as $u^\mu u_\mu = -1$, $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the induced metric onto the spatial 3-dimensional slides orthogonal to u^μ, $\rho = \rho(t)$ and $P = P(t)$ are the energy density and the isotropic pressure of the perfect fluid.

Moreover, an additional *cosmological constant* Λ modifies Einstein's field equations, accounting for the current accelerated cosmic expansion,

$$S = \frac{1}{2}\int \sqrt{-g}\,(R - 2\Lambda) + S^{(m)} \qquad \longrightarrow \qquad G_{\mu\nu} + \Lambda\, g_{\mu\nu} = T^{(\rm m)}_{\mu\nu}\,. \tag{1.16}$$

Therefore, we obtain what are called Friedmann equations,

$$H^2 = \frac{\rho}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}\,, \tag{1.17}$$

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho + 3P) + \frac{\Lambda}{3}\,, \tag{1.18}$$

where $H \equiv \dot{a}/a$ is the Hubble rate. The latter equation is also sometimes referred to as *Raychaudhury equation* or *acceleration equation*.

Before discovering the universe was expanding, Einstein introduced the cosmological constant Λ to achieve a static universe [52, 53]. Setting $\ddot{a} = 0$ and $\dot{a} = 0$ in the Friedmann equations leads to:

$$\Lambda = -\rho + 3\frac{k}{a^2}\,, \tag{1.19}$$

$$\Lambda = \frac{1}{2}(\rho + 3P)\,. \tag{1.20}$$

Assuming pressureless matter ($P = 0$), *dust*, the cosmological constant balances the gravitational attraction of matter to produce a static universe.

However, this solution is unstable to perturbations; any small deviation leads to either expansion or contraction [54]. After Hubble's observation of the expanding universe [30], Einstein reportedly referred to the introduction of Λ as his "biggest blunder."

As better discussed in the next section, the cosmological constant Λ has been reinterpreted as *dark energy* driving the current accelerated expansion of the universe [35, 36]. The presence of Λ modifies the Friedmann equations and allows for solutions where $\ddot{a} > 0$.

The contracted Bianchi identity [55] guarantees the conservation of the stress-energy tensor, which reads as a continuity equation for the cosmic fluid,

$$\dot{\rho} + 3H\,(\rho + P) = 0\,. \tag{1.21}$$

Therefore, the continuity equation is not independent of the Friedmann equations. It can be obtained by differentiating the first Friedmann equation and rewriting $\dot{H} = \ddot{a}/a - H^2$ in terms of energy density and pressure. Similarly, the second Friedmann equation can be obtained by differentiating the first Friedmann equation and using the continuity equation to rewrite $\dot{\rho}$, which yields the continuity Eq. (1.21).

The above equation system is *underdetermined* since there are three unknown functions and only two equations. The solution can be obtained only by introducing an equation of state for the matter content. The simplest assumption is a *barotropic equation of state*, $P = P(\rho)$, specifically, the linear equation $P = w\,\rho$, where w is called barotropic constant. The constant w specifies the fluid spices: $w = 0$ corresponds to a nonrelativistic dust (or pressureless) fluid, $w = 1$ represents a relativistic fluid, $w = 1/3$ is a radiation fluid, $w = -1$ corresponds to a cosmological constant. Notice that the spatial curvature contribution effectively behaves as a fluid with barotropic constant $w = -1/3$.

Using the linear barotropic assumption, it is possible to solve the Eq. (1.21) which provides

$$\dot{\rho} + 3\frac{\dot{a}}{a}(1 + w)\rho = 0 \quad \Rightarrow \quad \rho = \rho_* \left(\frac{a_*}{a}\right)^{3(1+w)}\,, \tag{1.22}$$

where the $*$-quantities indicates integration constants.

The solution for the scale factor can be easily analytically obtained in a very simple case when considering the case of a spatially flat universe, $k = 0$, and neglecting the presence of the cosmological constant, $\Lambda = 0$. Then, substituting the above equation into the first Friedmann equation (1.17), in the case of $w > -1$, the solution for the scale factor reads as follows

$$a = a_* \left[1 + \frac{3}{2}(w + 1)H_*\,(t - t_*)\right]^{\frac{2}{3(w+1)}}\,, \tag{1.23}$$

which, in terms of the shifted time $\tau = t - t_* + \frac{2}{3H_*(w+1)}$, provides

$$H = \frac{2}{3}\frac{1}{\tau(w + 1)}\,. \tag{1.24}$$

In the case of $w = -1$, corresponding to $P = -\rho = -\Lambda$, a cosmological constant, the scale factor turns into

$$a = a_* \exp\sqrt{\frac{\Lambda}{3}}\,t\,. \tag{1.25}$$

However, our current universe is filled with different *decoupled* matter species, $\rho = \sum_i \rho_i$, each of them characterized by a different scale factor dependence according to Eq. (1.22). Meanly, we can identify four principal constituents: radiation $\rho_r \propto a^{-4}$, total dust matter $\rho_m \propto a^{-3}$, curvature contribution $\rho_k \propto a^{-2}$, and the cosmological constant.

At this point, it is practical to define the concept of *critical energy density*, ρ_C representing the total energy density (at any given time) corresponding to $k = 0$, a spatially flat universe. The current critical energy density is given by the Hubble constant, H_0, which is the present-day value of the Hubble parameter: $\rho_C = 3H_0^2$. In this way, we can rewrite the first Friedmann equation as:

$$H^2 = H_0^2 \left[\Omega_m \left(\frac{a_0}{a} \right)^3 + \Omega_r \left(\frac{a_0}{a} \right)^4 + \Omega_\Lambda + \Omega_k \left(\frac{a_0}{a} \right)^2 \right], \tag{1.26}$$

where $\Omega_i = \rho_{0i}/\rho_C$ are the dimensionless density parameters, and a_0 represents the present-day scale factor.

Evaluating the above equation at the present time, we obtain

$$1 = \Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k \,. \tag{1.27}$$

Therefore, being $\Omega_{\text{total}} = \sum_i \Omega_i = \Omega_m + \Omega_r + \Omega_\Lambda$ the total density parameter, the curvature parameter k is related to Ω_{total} by:

$$\Omega_k = 1 - \Omega_{\text{total}} = -\frac{k}{(a_0 H_0)^2} \,. \tag{1.28}$$

Thus, $\Omega_{\text{total}} \lesseqgtr 1$ correspond to open universe ($k < 0$), flat universe ($k = 0$), and closed universe ($k > 0$), respectively.

The FLRW metric is underpinned by the cosmological principle and provides the framework for understanding the large-scale structure and evolution of the universe. The inclusion of different energy-matter components, such as matter, radiation, and the cosmological constant, allows us to model the history of the universe from the Big Bang to the present accelerated expansion.

1.1.2 The need for new theories

The evolution of physics is a continuous and iterative process. Scientific progress is driven by an ongoing cycle of observations, theory formulations, and technological advancements. As our observational tools become more precise, we are able to probe deeper into the universe, uncovering new phenomena that challenge existing theories. This process can begin with observations revealing previously unknown aspects of the universe, prompting scientists to formulate theories to explain the new data. As theories are developed, they are tested against further observations, leading to refinements and improvements in technologies. When existing theories are pushed to their limits or when new data cannot be reconciled with current models, the need arises for more comprehensive theories.

This cycle is evident in the history of physics. Analogously to GR, another example is the advent of Quantum Mechanics and Quantum Field Theory (QFT), which arose from the need to explain phenomena at microscopic scales that classical physics could not address.

So far, modern physics rests on two main pillars: GR and QFT. The former excels in explaining large-scale gravitating systems, while the latter is effective at describing high-energy and small-scale phenomena. However, QFT assumes a flat spacetime, known as Minkowski spacetime, and even in its extensions, such as QFT in curved spacetime, the spacetime serves as a rigid background for quantum fields [56]. In contrast, GR describes gravity as the dynamic curvature of spacetime but does not account for the quantum nature of matter. This discrepancy leads to the natural question of how quantum fields behave in strong gravitational fields and whether these two successful theories are compatible.

In our current understanding of the universe, the first evidence of missing pieces is represented by the lack of a unified picture of the four fundamental forces. QFT and GR alone are not enough to bridge the gap between the *infinitely small* and the *infinitely large*. To address this gap, physicists are pushing the boundaries of both theories, searching for a framework that can seamlessly integrate quantum mechanics with gravity [57]. This endeavor involves exploring new theoretical models and experimental approaches that might unify these disparate realms. The quest for a theory of quantum gravity, which could provide a comprehensive description of all fundamental forces within a single framework, remains one of the most significant challenges in contemporary physics.

Therefore, at this level, we could argue that GR is not a complete theory because it lacks a quantum counterpart. However, the issues with GR extend beyond its incompatibility with quantum theory. Indeed, our understanding of gravity reveals several significant shortcomings in the realms of both quantum theory and cosmology.

Although GR has passed multiple astrophysical and cosmological observational tests, as we extend our observations and theoretical insights into more extreme regimes—both at large cosmological scales and at the quantum level—Einstein's theory reveals limitations.

The *Einsteinian revolution* led to the building of the current standard model of cosmology, the Λ Cold Dark Matter (ΛCDM) model [58–60]. It is seen as a "concordance model" that combines GR at cosmological scales and the Standard Model of particle physics (SM), describing non-gravitational interactions. However, this "compromise" is still far from representing an exhaustive description of the universe, because important puzzle pieces remain missing.

The observed astrophysical and cosmological anomalies are attributed to the presence of the so-called *dark components*: dark energy and dark matter. Together, they constitute the major part of the total energy-matter content of the universe [61, 62].

For instance, dark matter is associated with the flatness of galaxy rotation curves [63–67]. It is estimated that non-baryonic dark matter represents about 25% of the total constituents of our universe. Due to its significant presence, the SM needs to be extended to account for non-baryonic dark matter particles. However, despite extensive searches, experiments at the Large Hadron Collider (LHC) have not produced any results indicating the existence of new physics beyond SM [68–70] (see also Refs. [71, 72]).

Approximately 70% of the content of the universe is constituted by dark energy, which is responsible for the late-time cosmic accelerated expansion. In the ΛCDM model, dark energy is modeled by a constant Λ (*i.e.*, the cosmological constant), which is *ad hoc* plugged into Einstein's equations without a solid theoretical explanation [73–76]. Since it does not undergo a dynamical evolution, it is conventionally associated with the energy of the vacuum in QFT. However, the vacuum energy density associated with the cosmological constant is about 121 orders of magnitude smaller than the zero-point energy predicted by QFT calculations—"the worst theoretical prediction in the history of physics" [77–79]. This enormous discrepancy is known as *the cosmological constant problem* and highlights a profound inconsistency between GR and QFT [80].

With the advancement of new technologies and high precision measurements in cosmology, the limitations of GR in describing gravity became increasingly evident. Observations of acoustic oscillations in the CMB have provided crucial insights into the early universe but also revealed discrepancies that challenge our current understanding [49–51, 61, 81]. Additionally, studies of large-scale structure formation and the distribution of galaxies have highlighted observations that diverge from GR's predictions [82–85]. Moreover, anomalous gravitational lensing further underscores the limitations of GR in explaining certain aspects of gravitational behavior [86–89].

Tensions within the model

One of the biggest issues in modern cosmology is the discrepancy between values of the Hubble constant H_0 determined by different observational methods. Since H_0 sets the scale of the universe's current expansion rate, it is extremely crucial for understanding cosmological parameters. For instance, measurements based on the CMB observations and those based on the local distance ladder yield significantly different values, leading to the so-called H_0 *tension* [90–93].

Among the early-time and late-time observations, we can distinguish five principal methods that provide competitive (and improvable) measurements of H_0.

Cepheid Variables + Type Ia Supernovae (Cosmic Distance Ladder) This late-universe method uses nearby galaxies to calibrate distance measurements and then extends that calibration to far-off galaxies using Type Ia supernovae. It gives a direct measurement of the local expansion rate of the universe.

Cepheid variables are pulsating stars with a well-defined period-luminosity relation, serving as standard candles to measure distances to nearby galaxies [94, 95]. Using this method, Ref. [95] reported $H_0 = 73.24 \pm 1.74\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

Type Ia supernovae are exploding white dwarfs that reach a consistent peak luminosity, allowing them to be used as *standardizable candles* for measuring distances to distant galaxies [35, 36]. Combined with Cepheid calibrations, they contribute to H_0 measurements.

The SH0ES (Supernovae H_0 for the Equation of State) team measures H_0 using Type Ia supernovae as standard candles, calibrated by Cepheid variable stars in nearby galaxies [92, 96]. Their latest measurement is $H_0 = (73.04 \pm 1.04)\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

An alternative calibration using the Tip of the Red Giant Branch (TRGB) yields a slightly lower value [97, 98]: $H_0 = (69.8 \pm 1.9)\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. While this reduces the tension, it does not eliminate it (see also Ref. [93]).

Gravitational Lensing Gravitational lensing occurs when the gravitational field of a massive object bends the light from a background source, creating multiple, highly distorted, or magnified images. By measuring the time delays between these images, H_0 can be inferred based on current cosmic structures. In Ref. [99], the H0LiCOW ($H0$ Lenses in COSmological MOnitoring of GRAvItational Lenses's Wellspring) collaboration reported $H_0 = (73.3^{+1.7}_{-1.8})\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. However, when reanalyzing the data with different assumptions about the mass profile of lens galaxies, the TD-COSMO (Time-Delay COSMOgraphy) collaboration [100] found $H_0 = (67.4^{+4.1}_{-3.2})\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, which is consistent with the *Planck mission*, highlighting the sensitivity of this method to modeling choices.

Gravitational Waves (Standard Sirens) This is a very recent late-universe method that uses gravitational wave events (like binary neutron star mergers) to directly measure distances [97, 101–103]. It is independent of the cosmic distance ladder and other methods but still measures H_0 in the current universe. Indeed, gravitational waves from binary neutron star mergers provide direct distance measurements. Combined with redshift information from electromagnetic counterparts, H_0 can be derived.

The LIGO/Virgo Collaboration [104] found $H_0 = 70.0^{+12.0}_{-8.0}\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. Although the uncertainty is large due to a limited number of events, future observations are expected to improve the precision significantly [105–107].

Cosmic Microwave Background Anisotropies (CMB) The CMB provides a snapshot of the universe when it was about 380,000 years old. Measurements of temperature fluctuations in the CMB are used to infer cosmological parameters [37], including H_0.

Two types of technologies are used to map the CMB anisotropies: space-based telescopes, such as the Wilkinson Microwave Anisotropy Probe (WMAP) [108, 109] and Planck [61], and ground-based experiments, like the Atacama Cosmology Telescope [110] and the South Pole Telescope [111].

The Planck satellite has provided precise measurements of the CMB anisotropies, leading to a value of the Hubble constant by fitting the data within the ΛCDM model [61]. The latest results from Planck yield $H_0 = (67.4 \pm 0.5)\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. This value is inferred from early-time observations and depends on the cosmological model assumptions.

Additionally, ground-based CMB experiments like the Atacama Cosmology Telescope (ACT) [110] have provided independent measurements which, combined with WMAP yields $H_0 = (67.6 \pm 1.1)\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. This is consistent with the Planck results, reinforcing the tension with late-time measurements.

The measurements of the Hubble constant (H_0) based on CMB data provide a precise estimate of the baryon density [37]. This estimate is consistent with the results of Big Bang Nucleosynthesis (BBN) [112–114], which accurately predicts the abundance of light elements such as deuterium and helium-4. The agreement between the observed primordial abundances and the predictions from BBN further supports the standard cosmological model and helps fix the parameters that influence the determination of H_0.

The tension between the value of H_0 obtained from local measurements and that derived from the CMB becomes more significant when considering the BBN results. The observed abundances of deuterium and helium-4 align with the H_0 value inferred from the CMB, adding weight to the lower estimate from the early universe and raising questions about why local measurements suggest a higher value.

Baryon Acoustic Oscillations BAO are periodic density fluctuations in the distribution of visible matter (baryons) in the universe, which are imprinted in the large-scale structure of galaxies [83, 85]. These fluctuations originated from sound waves (acoustic waves) that propagated through the hot plasma of the early universe before the formation of neutral atoms, around 380,000 years after the Big Bang. While this is technically observed in the late universe (by looking at galaxy distributions), BAO is a relic from the early universe that serves as a "standard ruler." When combined with CMB data or supernovae, it provides constraints on H_0 consistent with early-time measurements. For instance, [85] found $H_0 = (67.6 \pm 0.7)\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

The discrepancy between different measurements of H_0 represents a significant challenge in cosmology. While observational uncertainties may account for some differences, the persistent tension suggests the possibility of new physics. Continued efforts in observations, theoretical developments, and cross-disciplinary collaboration are essential for resolving this issue and advancing our understanding of the universe.

The Dark Energy Spectroscopic Instrument (DESI) collaboration has started its five-year survey, aiming to measure the expansion history of the universe with unprecedented precision [115]. Early data release results have begun to constrain cosmological parameters. While no definitive H_0 measurement is yet available from DESI alone, combining DESI data with other probes is expected to improve constraints and potentially shed light on the tension.

The North American Nanohertz Observatory for Gravitational Waves (NANOGrav) collaboration uses precise timing of millisecond pulsars to detect nanohertz gravitational waves [116]. In their 15-year data set, NANOGrav reported strong evidence for a stochastic gravitational-wave background [117]. While their primary goal is to observe gravitational waves from supermassive black hole binaries, their data can also be used to probe the expansion rate of the universe and potentially contribute to future measurements of H_0.

The James Webb Space Telescope (JWST) promises to refine measurements of H_0 by providing high-resolution observations of Cepheid variables and supernovae in distant galaxies [97]. Early data from JWST are expected to reduce systematic uncertainties, particularly in the calibration of distance indicators, and may help resolve the tension. Even if, some observations show that high-redshift galaxies appear more evolved than expected (suggesting that a hybrid cosmological model could alleviate the tension) [118], so far, JWST's observations confirmed the accuracy of the H_0 measurement from Cepheids and supernovae [119].

Resolving the H_0 tension is crucial for cosmology and may hint at new physics beyond the standard ΛCDM model [91, 103].

Upcoming surveys like the Vera C. Rubin Observatory's Legacy Survey of Space and Time (LSST) [120] and Euclid mission [121, 122] will provide vast amounts of data on supernovae, galaxies, and large-scale structure. Moreover, facilities like the Einstein Telescope [102] and Cosmic Explorer [123] will detect more standard sirens with greater precision. Collaborations like NANOGrav, the European Pulsar Timing Array (EPTA) [124], and the Parkes Pulsar Timing Array (PPTA) [125] are improving sensitivity, potentially detecting signals that can inform cosmological models. Meanwhile, exploring extensions to the standard cosmological model, such as early dark energy [126, 127], interacting dark energy [128], or modifications of gravity [129, 130], is fundamental for future investigations.

From the quantum side

Discrepancies between observations and theoretical predictions drive scientific research, and the gravity community has not given up on attempts to unify quantum mechanics and gravity. Although the Planck scale—the energy scale at which quantum effects of gravity become significant, approximately 1.22×10^{19} GeV—is experimentally inaccessible [131], it is extremely relevant. Indeed, one of the most profound theoretical issues in GR is the existence of singularities, points (events) in spacetime where curvature scalars (and observables) become infinite. These singularities, such as those at the centers of black holes and the Big Bang, indicate regions where GR fails as a physical theory. According to the Penrose–Hawking singularity theorems [132, 133], under reasonable physical assumptions, singularities are an inevitable outcome of gravitational collapse in GR. These singularities lead to a breakdown in the laws of physics as we understand them and suggest that a more fundamental theory of gravity, possibly incorporating quantum effects, is required to resolve these issues.

Attempts to quantize GR using conventional methods from QFT lead to non-renormalizable divergences, suggesting that GR cannot be a fundamental quantum theory [134, 135]. In QFT, forces are mediated by particles, such as photons for electromagnetism. Similarly, in a quantum theory of gravity, gravitons would mediate gravitational interactions. However, the non-renormalizability of GR implies that an infinite number of counterterms would be required to cancel out the divergences [136].

To address these issues, several alternative approaches to quantum gravity have been proposed [137]. One such approach is string theory, which posits that the fundamental building blocks of the universe are not particles but one-dimensional “strings.” The vibrations of these strings correspond to different particles, including the graviton, which mediates the gravitational force [138]. String theory naturally incorporates extra dimensions and provides a framework for unifying gravity with the other fundamental forces of nature. However, despite its theoretical appeal, string theory faces significant challenges. One major issue is the “string landscape,” which refers to the vast number of possible vacua—estimated to be on the order of 10^{500}—that the theory allows [139]. This leads to difficulties in making concrete, testable predictions, as it is unclear which version of the theory corresponds to our universe. Additionally, string theory requires extra dimensions beyond the familiar four, and the mechanisms for compactification are not fully understood or experimentally verified [140].

Another promising approach is Loop Quantum Gravity (LQG), which does not rely on the framework of string theory. Instead, LQG directly quantizes spacetime by describing it as a network of discrete loops, called spin networks. This leads to a granular structure of spacetime at the Planck scale, where the geometry of space is quantized [141, 142]. Unlike string theory, LQG does not require extra dimensions and is background-independent, meaning it does not assume a fixed spacetime geometry. LQG successfully addresses some singularity issues, such as the Big Bang singularity, which it replaces with a quantum bounce [143]. However, LQG also faces challenges, particularly in making contact with low-energy physics and in providing testable predictions. This is a problem in common with all the quantum theories of gravity since the “observable predictions” occur at energy ranges which are not experimentally accessible.

Effective field theories threads

Despite these efforts, a fully satisfactory theory of quantum gravity remains elusive. Moreover, some in the scientific community question whether the pursuit of a quantum theory of gravity is the most productive approach. They suggest that rather than seeking a direct quantization of gravity, it might be more fruitful to explore effective field theories or alternative frameworks [144] that could address the observed phenomena without necessitating a full quantum description of gravity [145].

Based on this mindset, a huge plethora of alternative theories aim to extend or modify GR to address its shortcomings [146–150]. Modified gravity theories offer potential explanations for cosmic acceleration without invoking dark energy, provide alternative candidates for dark matter, and attempt to resolve singularity problems [151]. They provide an extended framework to explore deviations from GR and to address some cosmological and theoretical issues mentioned above.

However, how does the scientific community determine which theory is *superior*? The criteria typically involve the predictive power of a theory, its falsifiability, and its simplicity. Furthermore, scientific advancements, along with the development of new technologies for gravitational experiments, enable increasingly sensitive and stringent tests, probing gravity across different scales and energetic regimes. This allows us to discriminate among theories that would otherwise be observationally degenerate [115, 116, 152, 153].

The development of new theories, along with the introduction of innovative tools and mathematical frameworks, is a fundamental part of the exploratory and cognitive process in the search for new physical theories. The formulation of these theories often leads to the design of new experiments aimed at testing their feasibility or distinguishing between competing theoretical models.

In addition to phenomenological considerations, exploring consistent modifications of gravity provides deeper insights into both GR and the nature of gravity itself. For instance, attempts to develop theories involving massive gravitons can enhance our understanding of why a massless graviton is unique in GR [154]. Similarly, studying gravity in higher or lower dimensions can clarify the special nature of four-dimensional gravity [155].

Therefore, while GR remains an extraordinarily successful framework and is widely regarded as the standard theory of gravitation, researchers continue to explore and test alternative theories of gravity. This exploration is essential not only for resolving current theoretical and observational puzzles but also for deepening our understanding of the fundamental forces that govern the universe.

1.2 Overview of alternative theories of gravity

The evolution of physics often shows that groundbreaking theories emerge as responses to the limitations of their predecessors. This iterative process reflects the ongoing quest in science to refine our understanding of the universe. GR is an emblematic example: it was born as a response to the inconsistencies of classical mechanics and the observations of the time, revolutionizing our understanding of gravity. However, as with all theories, over time observations and shortcomings have emerged which cannot be fully addressed within the theoretical framework of GR. This section aims to capture the dynamic aspect of the restless research around the mystery of gravity.

Usually, the standard approach that is adopted for the above-mentioned purpose starts by saying that the first theories aimed at modifying GR in response to the observed universe's expansion [30]—something that GR could only explain by introducing an additional constant[1]. This is because the universe's expansion was the first sign that something observable and strictly related to gravity was missing. It represents one of the main boosts in the development of alternative theories, producing the seeds from which numerous theories have been proposed to address both the evolution of the universe evolution and the dynamics of gravity.

In terms of the Friedmann equations (1.17) and (1.18), a period of accelerated expansion is described by $\ddot{a} > 0 \ \Rightarrow\ P < -\rho/3$. Therefore, it can only be achieved through a dominant component of the energy density with a negative equation of state. The simplest realization is a modified matter content described by a slow-rolling scalar field [156].

However, the most natural approach is to modify how spacetime geometry responds to regular matter sources. This response may deviate from Einstein's field equations, necessitating modifications to GR in the low-curvature regime, which could admit self-accelerating solutions in the presence of negligible regular matter.

[1] We recall that at the time of GR formulation, the prevailing belief among scientists was that the universe was static and unchanging. This view was supported by both philosophical considerations and observational data available at the time. Moreover, a static universe was a natural assumption, since Newtonian gravity predicts that a universe filled with matter would eventually collapse under its own gravitational pressure unless counteracted by some repulsive force. For this reason, Einstein wrongly introduced the cosmological constant for the first time [52].

Among the earliest of these theories were the works of P. Dirac [157, 158] and P. Jordan [159, 160], which introduced the concept of a time-varying gravitational coupling and elevated it to the role of a gravitational scalar field. Dirac had hypothesized a time-dependent gravitational coupling, keeping the other fundamental constants fixed. Building on this, P. Jordan further developed the idea, promoting the scalar field to a gravitational degree of freedom. The turning point came in 1961 when Brans and Dicke formalized these ideas in what is now known as Brans–Dicke theory [161–163]. This theory was one of the first attempts to modify GR by introducing a scalar field that is non-minimally coupled to the metric tensor, providing a more flexible framework. Motivated by Mach's principle—which suggests that the large-scale structure of the universe influences local physical laws—Brans and Dicke sought to incorporate this concept into gravitational theory. While Einstein considered Mach's principle during the formulation of GR, it was not fully integrated into the final theory.

Following a variational approach, Brans and Dicke started from the Einstein–Hilbert action

$$S_{\rm EH} = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}R\,, \tag{1.29}$$

where, for the sake of clarity, we make explicit the speed of light c and Newton's constant G. Then, they introduced the scalar field by defining $G^{-1} \to G^{-1}_{\rm eff}(\phi) = \phi$ and by adding a kinetic term,

$$S_{\rm BD} = \frac{c^4}{16\pi}\int d^4x\sqrt{-g}\left(\phi R - \frac{\omega_{\rm BD}}{\phi}\nabla_a\phi\nabla^a\phi\right), \tag{1.30}$$

with $\nabla_a\phi\nabla^a\phi = g^{\mu\nu}\partial_a\phi\partial_b\phi$ since the scalar field ϕ is a scalar function of the spacetime. The ϕ^{-1} factor next to the kinetic terms was introduced to make the constant $\omega_{\rm BD}$ dimensionless.

The associated field equations are obtained by variating with respect to the metric tensor and the scalar field, respectively,

$$G_{\mu\nu} - \frac{1}{\phi}\left(\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\Box\phi\right) - \frac{\omega_{\rm BD}}{\phi^2}\left(\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla_\rho\phi\nabla^\rho\phi\right) = \frac{8\pi\phi^{-1}}{c^4}T^{(\rm m)}_{\mu\nu}\,, \tag{1.31}$$

$$2\omega_{\rm BD}\Box\phi + \phi R - \frac{\omega_{\rm BD}}{\phi}\nabla_\alpha\phi\nabla^\alpha\phi = 0\,, \tag{1.32}$$

where $\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi$. The above action can assume more general forms, and be generalized to a class of theories, by redefining the scalar field and including an additional scalar function V as potential [164, 165],

$$S_{\rm ST} = \frac{c^4}{16\pi}\int d^4x\sqrt{-g}\left[F(\phi)R - \frac{\omega(\phi)}{2}\nabla_a\phi\nabla^a\phi - V(\phi)\right]. \tag{1.33}$$

Therefore, unlike GR, in Brans–Dicke theory gravitational interactions are governed by both the metric tensor and the scalar field. This represents the first prototype of modified theory which opened new pathways for understanding how gravity behaves on cosmological scales and laid the foundation for a broader class of theories known as scalar-tensor theories—which have continued to inspire numerous developments in modified gravity, as we will see in the next section. For these reasons, we will refer to it as "old-school" or traditional scalar-tensor theories.

In the same years, GR was shown not to be renormalizable and, therefore, it could not be conventionally quantized [166]: the only way was to add higher-order curvature terms to the

Einstein–Hilbert action to make it renormalizable but not unitary [136]. Further studies were driven by the aim of formulating a curvature-based inflationary model [167] and attempting to cure cosmological and black hole singularities [168]. To this, the observation of the accelerated expansion of the universe officially endorsed the birth of what we call $f(R)$ theories [169–171]. These theories modify the Einstein–Hilbert action by considering functions of the Ricci scalar,

$$S_{\text{EH}} = \frac{1}{2}\int d^4x\sqrt{-g}R \quad \longrightarrow \quad \frac{1}{2}\int d^4x\sqrt{-g}f(R) \tag{1.34}$$

where the convention $c = 8\pi G = 1$ has been restored. Therefore, the field equations turn out to be

$$G_{\mu\nu}f'(R) - \frac{1}{2}g_{\mu\nu}\left[f(R) - Rf'(R)\right] - \left[\nabla_\mu\nabla_\nu f'(R) - g_{\mu\nu}\Box f'(R)\right] = T^{(m)}_{\mu\nu}\,. \tag{1.35}$$

Over time, several functional forms of $f(R)$ were studied and tested, for describing inflation and late-time acceleration both individually and in a unified picture [172–178].

Brans–Dickie gravity and $f(R)$ gravity share some similarities [169, 170]: both are characterized by an additional propagating scalar mode; moreover, $f(R)$ gravity can be recast into a scalar-tensor theory with a non-minimally coupled scalar field. Thus, even if $f(R)$ gravity is classified as a fourth-order theory, it can be seen as a scalar-tensor gravity. Then, in both theories, one can eliminate the non-minimal coupling next to the Ricci scalar by performing a conformal transformation, obtaining GR plus a scalar field. As a result, the scalar field couples to the matter sector of the theory.

Thus, one can say that the Brans–Dicke gravity and $f(R)$ gravity are the first main modifications of GR, the genitors of a huge realm of extended theories of gravity, each of which is characterized by the presence of different and additional *ingredients* aiming to generalize previous theories, to cure pathologies, and to improve stability aspects. However, it can also be an extreme simplification of the immense effort behind the landscape of alternative theories.

The current complex zoology of theory is the result of numerous innovative approaches and investigations, reflecting an ongoing dialogue within the scientific community [146–150, 179]. These theories incorporate diverse concepts, from higher-dimensional constructs to novel field interactions, each contributing to a deeper understanding of gravity. As research progresses, the landscape of gravitational theories continues to evolve, offering new insights and potential solutions to longstanding questions in cosmology and fundamental physics.

Making a clear categorization of the extended theories of gravity can be hard. Not only because of the variety of the theories but also because, whatever criteria are used for the classification, there are theories belonging to different frameworks that can share multiple characteristics, such as the case of scalar-tensor gravity and $f(R)$ gravity. Nevertheless, classifying theories of gravity is essential for comprehending their differences and similarities. It enables a more precise evaluation of each theory through observational tests, allowing for effective discrimination based on their empirical predictions. Furthermore, several findings can be used to demonstrate the uniqueness of GR under quite general assumptions.

In this regard, in Fig. 1.1, we are giving a classification that solely serves to give a clearer view of where the current thesis work is collocated in the landscape of the extended theories:

- Theories in which the graviton is massive.
- Theories breaking fundamental assumptions GR is based on.
- Theories characterized by additional fields.

- Theories with modifications due to higher-order invariants.
- Theories with additional geometric elements.
- Theories aiming to a quantum formulation of gravity.

However, because many theories fall within several categories, the above descriptions are not exclusive. Different diagrams representing schematic categorizations of extended theories of gravity are present in Refs. [180, 181]. Another similar attempt to give a graphical classification can be found in Ref. [182], where the roadmap summarizing possible extensions of GR highlights the main experimental test constraining each modified theory. Nevertheless, even if such classifications are useful tools for summarizing the variety of theories, especially if one aims to focus only on a particular aspect, this is not the best choice when one is trying to give a broad overview.

Discussing all the theories of gravity in Fig. 1.1 is beyond the scope of this thesis (you can see Refs. [146–150]), but we will follow a chronological approach trying to provide an overall picture. Therefore, we will follow a timeline remarking on some main modified theories of gravity and the additional tools to GR that have been introduced.

In the history of cosmology and gravitational physics, it is possible to identify some milestones that can help categorize the development of the extended theories of gravity into distinct historical periods. These changed our understanding of the universe and provided the basis for new theories and approaches to gravity and cosmology. Therefore, the following divisions can provide a framework for discussing the theoretical advancements:

- Pre-Hubble era (1918–1929). This period marks the early years following the formulation of Einstein's GR. Initial attempts to modify gravity were motivated by the search for a static universe and early theoretical explorations.
- Hubble era (1929–1965). Following the discovery of the expanding universe by Edwin Hubble, cosmology entered a new phase where modified gravity theories aimed to address the dynamics of expansion. The shift from a static to an expanding universe led to the development of cosmological models and theories such as the Steady State and Big Bang theories. Then, the discovery of the CMB in 1965, which confirmed the Big Bang model, marked the end of this era.
- CMB era (1965–1998). The discovery of the CMB provided strong evidence for the Big Bang Theory, the hunt for dark matter, and the rise of inflationary cosmology. During this period, observations of the CMB anisotropies and the formulation of the inflationary model took center stage. The end of this era was marked by the discovery of the accelerating expansion of the universe in 1998, which introduced the need for dark energy.
- Dark energy and gravitational waves era (1998–2015–Present). With the discovery of the accelerating expansion of the universe in 1998, alternative theories emerged, driven by the need to explain dark energy. Later, the direct detection of gravitational waves in 2015 marked the beginning of a new observational frontier, providing unprecedented tests for both GR and its alternatives.

Figure 1.1: Schematic representation of the extended and modified theories of gravity landscape. The above classification is neither unique nor comprehensive. The aim is to give an idea of the variety of theories and orientate among them. See also Refs. [180–182] for alternative categorizations, viability, and constraints on the theories. Exploring all the theories of gravity lies beyond the scope of this thesis. Refs. [146–150] constitute exhaustive reviews within their references.

1.2.1 Pre-Hubble era (1918–1929)

The first attempts to modify GR were based on geometrical modifications, by considering more general connections than the Levi–Civita one and higher dimensions. These theories constitute the fundaments for metric-affine gravity, symmetries-based unifying theories, and gauge theories, quantum gravity formulations, and contributed to the study of higher-derivatives theories.

Weyl's Conformal Gravity

In 1918, only a few years after Einstein's formulation, Hermann Weyl introduced a pioneering attempt to unify gravity and electromagnetism through what is now known as *Weyl's Conformal Gravity* [183, 184]. Weyl's theory extended Riemannian geometry by incorporating a new kind of symmetry, local conformal scale transformations of the metric tensor (or Weyl transformations). These transformations allow the metric tensor $g_{\mu\nu}$ to be rescaled at each point in spacetime by a smooth, non-zero function $\Omega(x)$:

$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x) \,. \tag{1.36}$$

This means that the laws of physics remain invariant under local changes of the unit of length, introducing a gauge symmetry for scale transformations similar to how electromagnetism involves a gauge symmetry for $U(1)$ phase transformations.

Weyl's motivation was to create a unified field theory where both gravitational and electromagnetic interactions emerge from the geometry of spacetime. In his framework, electromagnetism was associated with the connection that preserves the conformal structure, rather than the metric structure:

$$Q_{\alpha\mu\nu} \equiv \nabla_\alpha g_{\mu\nu} = 2A_\alpha g_{\mu\nu} \,, \tag{1.37}$$

where the Weyl's gauge field transforms as $A_\mu \longrightarrow A_\mu + \partial_\mu \Omega$ to preserve the total connection,

$$\overline{\Gamma}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} + L^\lambda{}_{\mu\nu} \,, \tag{1.38}$$

$$L^\lambda{}_{\mu\nu} = g_{\mu\nu} A^\lambda - \delta^\lambda_\mu A_\nu - \delta^\lambda_\nu A_\mu \,. \tag{1.39}$$

The above quantity is called *disformation tensor* and, in general, it is defined in the following way,

$$L^\lambda{}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\rho} \left(-Q_{\mu\rho\nu} - Q_{\nu\rho\mu} + Q_{\rho\mu\nu} \right) = L^\lambda{}_{\nu\mu} \,. \tag{1.40}$$

The non-metricity tensor leads in general to the change of a vector's norm when it is parallelly transported.

Early criticisms, notably by Einstein, pointed out that Weyl's theory predicted variations in atomic spectral lines due to the non-integrability of length standards, conflicting with observations. Specifically, transporting a clock or a rod around a closed loop would lead to a different period or length, which is not observed experimentally.

The reason is that Weyl's covariant derivative, associated with a symmetric but non-compatible connection, did not preserve the metric and, as a consequence, the lengths of parallel transported vectors.

Although Weyl's original attempt to unify gravity and electromagnetism did not succeed, his work laid the foundation for gauge theories and conformal field theories [185–187]. Indeed, it led to the idea of conformal invariance as a more general physical symmetry, a *fundamental symmetry* *of spacetime* [188, 189]. Over time, several studies and observations pushed toward this direction. For instance, the fact that Maxwell's equations and the causal structure are invariant under conformal transformations [190], the hierarchical structure of the conformal group, Poincaré group, and Galilei group [191], the Penrose's conformal treatment of infinity [192], Hawking's exploration of the topology of curved spacetime, incorporating causal, differential, and conformal structures, having important implications for black holes and the universe's large-scale structure [40, 193]. These contributions represent some key steps in the ongoing exploration of conformal transformations as a fundamental symmetry in theoretical physics. The study of conformal invariance has not only advanced our understanding of classical gravitation but also opened new avenues for quantum gravity. For instance, a conformal theory can naturally introduce mass scaling laws for ultrarelativistic systems due to the dimensionless nature of coupling constants [194]. Moreover, conformal quantum gravity theories have the potential to eliminate singularities and horizons by pushing them to infinity in certain coordinate systems, offering a possible resolution to issues that arise in traditional approaches to gravity [195].

However, conformal theories of gravity face two significant challenges: the existence of massive particles, which inherently break conformal symmetry in the universe; the issue of the conformal anomaly, where conformal symmetry is not preserved during the quantization process, often resulting in a non-zero trace of the energy-momentum tensor [186].

Among the modern adaptations of Weyl's ideas, there is the conformal theory of gravity developed by Philip D. Mannheim, revisiting the notion of conformal invariance in gravity [196, 197]. The conformally invariant action is obtained from the Weyl-squared scalar, so that it is conformally invariant. Indeed, Rudolf Bach and Hermann Weyl demonstrated that there exists a unique conformally invariant action constructed from the Weyl tensor $C_{\mu\nu\rho\sigma}$ in four dimensions, the Weyl-Squared action:

$$S_{\rm WS} = \int d^4x\sqrt{-g}\, C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\,, \tag{1.41}$$

where the Weyl tensor represents the trace-free part of the Riemann tensor,

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{1}{3}Rg_{\mu[\rho}g_{\sigma]\nu}\,, \tag{1.42}$$

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2\,. \tag{1.43}$$

The conformal invariance of the Weyl tensor reads $\tilde{C}^{\mu}{}_{\nu\rho\sigma} = C^{\mu}{}_{\nu\rho\sigma}$. Therefore, there is a crucial dependence on the index position (*i.e.*, $\tilde{C}_{\mu\nu\rho\sigma} = \tilde{g}_{\mu\lambda}\tilde{C}^{\lambda}{}_{\nu\rho\sigma} = \Omega^2 g_{\mu\lambda}\tilde{C}^{\lambda}{}_{\nu\rho\sigma} = \Omega^2 C_{\mu\nu\rho\sigma}$).

Mannheim's Conformal Gravity aims to address galactic rotation curves without invoking dark matter [198]. Despite this, it faces issues in consistently explaining cosmological observations, including the CMB anisotropies and large-scale structure formation.

Kaluza–Klein Theory

Theodor Kaluza, in 1919, proposed an extension of GR to a five-dimensional spacetime to unify gravity and electromagnetism [199]. In Kaluza's theory, the extra fifth dimension is not directly observable but is responsible for electromagnetic phenomena. Oskar Klein later refined this idea in 1926 by suggesting that the fifth dimension is compactified on a small scale, akin to a circle of very small radius, effectively rendering it unobservable at low energies [200].

Kaluza–Klein theory introduced the concept that additional spatial dimensions could unify fundamental forces. The metric tensor in five dimensions incorporates both the gravitational potentials and electromagnetic potentials, providing a geometric interpretation of electromagnetism.

The five-dimensional Einstein–Hilbert action can be written as:

$$S_{\mathrm{KK}} = \frac{1}{2}\int d^5x\sqrt{-g}\,R^{(5)} \tag{1.44}$$

where g is the determinant of the five-dimensional metric $g_{\mu\nu}$ and $R^{(5)}$ is the corresponding Ricci scalar. By decomposing the metric $g_{\mu\nu}$ in terms of a four-dimensional metric $g_{\mu\nu}$ and a gauge field A_μ (corresponding to the electromagnetic field), and a scalar field ϕ representing the size of the fifth dimension,

$$g_{\mu\nu} = \begin{pmatrix} g_{\mu\nu} + \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & \phi^2 \end{pmatrix} \tag{1.45}$$

the action (1.44) can reproduce both the Einstein field equations and the Maxwell equations. This approach provided a geometric interpretation of electromagnetism, where the extra dimension behaves like the electromagnetic field.

This idea significantly influenced later developments in theoretical physics, especially string theory and higher-dimensional models. The notion of compact extra dimensions became a cornerstone in models aiming for unification, such as superstring theories requiring ten dimensions and M-theory requiring eleven [138, 201].

The lack of observed effects from extra dimensions at accessible energies imposes constraints on the size and geometry of these dimensions. The compactification scale must be much smaller than current experimental resolutions [202]. Additionally, stabilizing the extra dimensions (moduli stabilization) to prevent variations in their size, which would have observable consequences, remains a theoretical challenge in higher-dimensional models.

Despite these challenges, the idea of extra dimensions remains an attractive theoretical concept. Kaluza–Klein theory continues to provide insight into efforts to unify gravity with other fundamental forces. It also laid the conceptual foundation for modern frameworks like brane-world scenarios and Randall-Sundrum models [155, 203–205].

Einstein–Cartan Theory

In 1922, Élie Cartan introduced the concept of spacetime's torsion, extending differential geometry by allowing for an asymmetric connection [206, 207]. This development led to the Einstein–Cartan theory, further elaborated by Dennis Sciama and Tom W. B. Kibble in the 1960s [208, 209]. The theory generalizes GR by incorporating torsion alongside curvature, enabling spacetime to have intrinsic angular momentum or spin. Therefore, the general connection reads as follows

$$\overline{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu}\,, \tag{1.46}$$

where the second term is the contorsion tensor, defined through the torsion tensor

$$K^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2} g^{\lambda\rho} (T_{\mu\rho\nu} + T_{\nu\rho\mu} + T_{\rho\mu\nu}) \tag{1.47}$$

$$= -\frac{1}{2} g^{\lambda\rho} (T_{\mu\nu\rho} + T_{\nu\mu\rho} + T_{\rho\nu\mu}) = -K_{\nu\mu}{}^{\lambda}\,. \tag{1.48}$$

Geometrically, the torsion tensor describes the *not-closure* of an infinitesimal parallelogram obtained by the parallel transport of one vector along the direction of another vector.

In Einstein–Cartan Theory, torsion is directly related to the spin density of matter, providing a natural way to include intrinsic spin in the gravitational interaction [210–212]. This is particularly relevant for spinor fields (*e.g.*, Dirac fields) which can be consistently coupled to gravity through torsion. However, it is important to say that coupling with gravity does not necessarily require the introduction of torsion. It has been shown that spinor fields can be consistently coupled to a metric-compatible, torsion-free connection by leveraging the structure of the spin bundle and the associated gauge symmetries [213].

The theory modifies the Einstein field equations, adding terms that become significant at extremely high matter densities, such as those present in the early universe or inside neutron stars. This has implications for avoiding singularities and may lead to a natural explanation for the avoidance of cosmological and black hole singularities [214].

Since torsion effects are negligible at the densities found in the present universe, they do not contradict current experimental observations, which are consistent with GR [212]. However, no direct evidence for spacetime torsion has been observed. Experimental tests involving spin-polarized matter place constraints on torsion components [215].

Einstein–Cartan Theory remains a viable theoretical extension of GR, particularly in regimes involving extremely high densities. While it does not contradict known observations, the lack of empirical evidence for torsion limits its acceptance as a necessary modification of gravity.

Palatini formalism

The Palatini formalism was formulated by Einstein in 1925, inspired by the Attilio Palatini's article in 1919. In his work, *"Deduzione invariantiva delle equazioni gravitazionali dal principio di Hamilton"* [216], Palatini demonstrated the tensorial nature of the variation of the Christoffel symbols and introduced a method for varying the Riemann curvature tensor independent of a specific symmetric connection.

This pushed for development of a variational principle where the metric $g_{\mu\nu}$ and the affine connection $\overline{\Gamma}^{\lambda}{}_{\mu\nu}$ are treated as independent variables. This distinction is important because the connection evolves separately in the variational principle, instead of assuming that it corresponds to the Levi–Civita one.

Although Palatini's paper did not receive immediate widespread attention, Albert Einstein recognized its significance. In 1925, Einstein revisited the Palatini's idea, attempting to unify gravity and electromagnetism [217, 218]. By varying the action with respect to both $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$, Einstein aimed to find a more general theory that could encompass both gravitational and electromagnetic interactions.

Einstein applied the Palatini method to the Einstein–Hilbert action, leading to the realization that the affine connection must be compatible with the metric, *i.e.*, the connection is the Levi–Civita connection of $g_{\mu\nu}$.

Allowing the Einstein–Hilbert action to independently depend on the metric tensor and the connection, one obtains what can be called Einstein–Palatini action,

$$S_{\mathrm{EP}} = \frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\overline{R}_{\mu\nu}\left(\overline{\Gamma}\right), \tag{1.49}$$

where the Ricci tensor only depends on the connection according to its definition as indices contraction of the Riemann tensor,

$$\overline{R} = g^{\mu\nu}\overline{R}_{\mu\nu}\left(\overline{\Gamma}\right), \qquad R_{\mu\nu}\left(\overline{\Gamma}\right) = \partial_\lambda\overline{\Gamma}^\lambda{}_{\mu\nu} - \partial_\nu\overline{\Gamma}^\lambda{}_{\mu\lambda} + \overline{\Gamma}^\lambda{}_{\lambda\rho}\overline{\Gamma}^\rho{}_{\mu\nu} - \overline{\Gamma}^\lambda{}_{\nu\rho}\overline{\Gamma}^\rho{}_{\mu\lambda}\,. \tag{1.50}$$

Performing the variation with respect to both the fields, the following equations are obtained:

$$\overline{R}_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}\overline{R} = 0 \tag{1.51}$$

$$\overline{\nabla}_\alpha(\sqrt{g}g_{\mu\nu}) = 0 \tag{1.52}$$

where $\overline{R}_{(\mu\nu)}$ is the symmetric part of $\overline{R}_{\mu\nu}$ and $\overline{\nabla}$ denotes the covariant associated with $\overline{\Gamma}$.

Equation (1.52) constraints the connection, which is *a priori* arbitrary, to be the Levi–Civita connection of the metric $g_{\mu\nu}$: $\overline{\Gamma}^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu}$. Notice that this is no longer an assumption, but it is a direct consequence of the field equations. However, it is strictly due to the simple form of the action. Considering alternative Lagrangians, field equations in metric and Palatini formalisms are different, in general. Thus, it is also possible to obtain field equations equivalent to GR with a different connection.

Throughout the 1920s and 1930s, several researchers explored the implications of the Palatini formalism. However, it was not until the 1950s and 1960s that the Palatini formalism gained renewed interest. The development of *metric-affine theories* and the desire to understand the role of torsion and non-metricity in gravity led researchers like Sciama [209] and Kibble [208] to consider more general connections.

In the 1970s, the formalism became particularly relevant in the study of higher-order gravity theories, in the context of $f(R)$ and other modified gravity theories [218, 219]. These studies laid the groundwork for modern investigations into alternative theories of gravity that attempt to address cosmological and astrophysical observations without invoking dark matter or dark energy.

The Palatini formalism becomes especially useful in modified gravity theories, such as $f(R)$ gravity. In the Palatini version of $f(R)$ gravity, the independent variation with respect to the connection and metric yields second-order field equations, unlike the fourth-order equations that arise in the metric-only formulation [169, 170]. This makes the theory more tractable analytically and numerically while leading to different cosmological predictions, especially in the context of strong gravitational fields or cosmic acceleration [220].

A significant advantage of the Palatini formalism is its second-order equations, which simplify the analysis in comparison to the higher-order equations encountered in the metric version of modified gravity theories. This makes the Palatini approach particularly appealing when exploring phenomena like the accelerated expansion of the universe, without needing to introduce new fields such as dark energy [147]. However, the formalism also faces some challenges. For instance, it can lead to ambiguities in the choice of the connection, particularly in theories involving torsion or non-metricity, where the connection is no longer uniquely tied to the metric [211].

Despite these challenges, the Palatini formalism remains an important tool for extending GR, offering a wider framework for investigating gravitational theories beyond Einstein's original formulation [221–223]. This flexibility is especially useful when exploring cosmological models that aim to explain the universe's large-scale structure and dynamics, including the possibility of modified gravity theories as alternatives to dark energy [224].

1.2.2 Hubble era (1929–1965)

Brans–Dicke Theory

As mentioned at the beginning of the section, Dirac was one of the first to introduce the hypothesis of a varying gravitational constant, including it as part of his Large Numbers Hypothesis. However, Jordan's work [159] was one of the first systematic attempts to incorporate a scalar field into gravity theory that would directly affect the gravitational constant [160]. This idea was further developed and formalized by Carl H. Brans and Robert H. Dicke in 1961, leading to the Brans–Dicke Theory [161, 162]. Their motivation was to incorporate *Mach's principle*, which suggests that local inertial properties are influenced by the mass distribution of the universe.

Experiments such as lunar laser ranging and measurements [15] of the Shapiro time delay using spacecraft like Cassini [225] have placed stringent limits on the Brans–Dicke parameter $\omega_{\rm BD}$. The Cassini experiment, for instance, requires $\omega_{\rm BD} > 40,000$. Such high values of $\omega_{\rm BD}$ make the predictions of Brans–Dicke Theory practically indistinguishable from GR in the solar system.

While the theory remains mathematically consistent, the tight observational constraints imply that any deviations from GR are minimal at observable scales.

Brans–Dicke theory serves as a prototype for more general scalar-tensor theories, some of which may evade these constraints in cosmological contexts [164, 226]. This aspect will be further explored during the chapter.

Seeds for $f(R)$ gravity and higher-order curvature terms

Beginning in the 1960s, evidence emerged suggesting that modifying the gravitational action could have theoretical advantages. GR is not renormalizable and, as such, cannot be quantized in the conventional sense. In 1962, Utiyama and DeWitt showed that renormalization at one-loop requires the Einstein–Hilbert action to be supplemented by higher-order curvature terms [166]. Later, Stelle demonstrated that higher-order actions are renormalizable, although not unitary. More recent studies show that, when quantum corrections or string theory are considered, the effective low-energy gravitational action includes higher-order curvature invariants [56, 227, 228].

In the late 20th century, the scientific community's interest in higher-order gravity theories grew significantly. These theories modify the Einstein–Hilbert action by incorporating higher-order curvature invariants beyond the Ricci scalar, as discussed in various historical reviews [229]. Initially, it was believed that such terms would only be relevant in extremely strong gravitational fields, and their effects were expected to be diminished by small coupling constants, in line with effective field theory arguments. As a result, these modifications were thought to be important only near the Planck scale, particularly in contexts like the early universe or near black hole singularities.

Several studies examined these strong-field effects, such as the well-known inflationary model driven by curvature [167], and efforts to avoid cosmological and black hole singularities [230–233]. However, it was not widely expected that these corrections would have any significant impact on gravitational behavior at low energies or on large scales, such as in the late universe.

1.2.3 CMB era (1965–1998)

Lovelock Gravity

In 1971, David Lovelock generalized Einstein's theory of gravity to higher-dimensional spacetimes by deriving the most general tensor that yields second-order field equations from a Lagrangian constructed solely from the metric and its derivatives [234]. This work resulted in what is now known as *Lovelock Gravity*, which includes higher-order curvature invariants in the action.

Lovelock discovered that in higher dimensions ($D > 4$), additional terms like the *Gauss–Bonnet term* [235] and even higher-order terms can be added to the action without introducing higher-order field equations. The Lovelock action is given by:

$$S = \int d^D x \sqrt{-g} \sum_{k=0}^{[D/2]} \alpha_k \mathcal{L}_k,$$

where $\mathcal{L}_k$ are the Lovelock Lagrangians constructed from the Riemann tensor and its contractions, and α_k are coupling constants.

These higher-order terms contribute non-trivially to the field equations in dimensions greater than four, leading to richer gravitational dynamics and new solutions such as black holes with distinct properties [236–238].

In four dimensions, the Gauss–Bonnet term is a topological quantity and does not affect the field equations,

$$S_{\mathrm{GB}} = \int d^4 x \sqrt{-g}\, \mathcal{G} = \int d^4 x \sqrt{-g} \left(R^2 - 4P + Q \right). \tag{1.53}$$

where, $\mathcal{G} = R^2 - 4P + Q$ is the Gauss–Bonnet term, R is the Ricci scalar, $P = R_{\mu\nu}R^{\mu\nu}$, and $Q = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is the Kretschmann scalar.

However, when $\mathcal{G}$ is coupled to a scalar field or when a non-linear function $f(\mathcal{G})$ is taken into account, it leads to modifications that can have observational consequences [239].

Higher-dimensional theories must reconcile with the lack of observed extra dimensions, typically through compactification mechanisms.

String theory naturally includes Lovelock terms in its low-energy effective action, supporting the theoretical viability of such extensions [240].

Lovelock Gravity remains an important theoretical framework for exploring gravity in higher dimensions and understanding the possible extensions of GR. Its compatibility with string theory enhances its significance, although direct observational evidence is lacking due to the hidden nature of extra dimensions.

Horndeski's theory

Gregory W. Horndeski, in 1974, derived the most general scalar-tensor theory in four dimensions that yields second-order field equations [241]. This work led to what now is known as *Horndeski gravity*. Horndeski's formulation ensures the avoidance of Ostrogradsky instabilities, associated with higher-order derivatives [242–244].

Horndeski Lagrangian encompasses a wide range of scalar-tensor models, including those with non-minimal couplings and derivative interactions. The general action involves combinations of the scalar field ϕ, its kinetic term $X = -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi$, and curvature tensors.

The theory gained renewed attention in the 2010s with the development of *Galileon* models and their applications to cosmology and dark energy [245, 246].

The detection of gravitational waves from binary neutron star mergers (GW170817) [26] and their electromagnetic counterparts constrained the speed of gravitational waves to be equal to the speed of light to within one part in 10^{15} [153]. This observation ruled out large classes of Horndeski theories where the scalar field causes deviations in the propagation speed of gravitational waves [247].

Horndeski models that comply with the gravitational wave speed constraints remain viable. These include models where the scalar field's coupling does not affect the gravitational wave propagation (even though provides additional polarization modes [248–251]), allowing for rich phenomenology in cosmology and astrophysics.

This theory represents the main object of study of this thesis, and the next section will be entirely dedicated to Horndeski gravity and its generalizations.

Higher-derivative gravity and renormalization

K. S. Stelle, in 1977, made significant contributions to quantum gravity by investigating higher-derivative gravity theories [136]. By including quadratic curvature terms like R^2 and $R_{\mu\nu}R^{\mu\nu}$ in the gravitational action, he showed that gravity becomes power-counting renormalizable.

Stelle's work demonstrated that adding higher-order curvature invariants to the Einstein–Hilbert action improves the UV behavior of gravity. The modified action takes the form:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2}R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}\right), \tag{1.54}$$

where α and β are constant quantities. These terms lead to fourth-order field equations, which complicate the theory. Although the theory becomes renormalizable, it introduces a massive spin-2 ghost, a particle with negative norm states, leading to violations of unitarity [252].

The presence of ghosts renders the theory problematic from a physical standpoint, as unitarity is a fundamental requirement for a consistent quantum theory. Moreover, higher-derivative terms can have significant effects at high energies, but they must reduce to GR at low energies to be consistent with observations.

While Stelle's higher-derivative gravity is not viable as a fundamental theory due to unitarity issues, it has influenced approaches to quantum gravity and inspired further research into asymptotically safe gravity and effective field theory methods [253].

Starobinsky inflation and $f(R)$ gravity

In 1980, Alexei A. Starobinsky proposed an inflationary model based on quantum corrections to gravity [167]. By adding an R^2 term to the Einstein–Hilbert action, he demonstrated that the early universe could undergo exponential expansion.

The Starobinsky's action reads as follows,

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\left(R + \frac{R^2}{6M^2}\right), \tag{1.55}$$

where M is a mass scale. This model naturally leads to inflation without the need for an explicit scalar field. The R^2 term dominates at high curvatures, driving inflation, and becomes negligible at late-time.

Starobinsky's model makes specific predictions for the spectral index n_s and the tensor-to-scalar ratio r, which are consistent with observations from the Planck satellite [254]. The small value of r predicted ($r \approx 0.003$) is within current observational bounds.

Given its compatibility with observational data, Starobinsky inflation is considered one of the most successful inflationary models. It also serves as a prototype for $f(R)$ gravity theories that can explain cosmic acceleration without introducing dark energy.

For a comprehensive analysis covering various formulations of $f(R)$ gravity as well as other alternative theories of gravity, see Refs. [169, 170].

Modified Newtonian Dynamics (MOND)

Mordehai Milgrom introduced Modified Newtonian Dynamics (MOND) in 1983 as an alternative to dark matter [255]. MOND proposes that Newton's second law is modified at accelerations below a certain threshold a_0.

MOND successfully explains the flat rotation curves of galaxies without invoking dark matter by modifying the acceleration law:

$$\mu\left(\frac{a}{a_0}\right) a = a_{\text{Newton}}, \tag{1.56}$$

where $\mu(x)$ is an interpolation function satisfying $\mu(x) \approx 1$ for $x \gg 1$ and $\mu(x) \approx x$ for $x \ll 1$.

While MOND explains galactic rotation curves and the Tully–Fisher relation [256], it struggles with galaxy cluster dynamics, gravitational lensing, and cosmological observations like the CMB [257]. Relativistic extensions like TeVeS (Tensor-Vector-Scalar theory) have been proposed [258], but they face challenges in fully reproducing the successes of GR with dark matter. In addition, it has been shown that alternative frameworks can reproduce MOND-like behavior [259].

MOND remains an important alternative hypothesis that challenges the dark matter paradigm at galactic scales. However, its inability to account for observations at larger scales limits shows the need for further developments.

Quintessence

In 1988, Christof Wetterich [260] and, independently, Bharat Ratra and Paul J. E. Peebles [261], proposed models where a dynamic scalar field, dubbed quintessence, drives the accelerated expansion of the universe.

Quintessence involves a scalar field ϕ slowly rolling down its potential $V(\phi)$, leading to negative pressure and acceleration [74, 262]. Unlike a cosmological constant, quintessence evolves over time, which can alleviate the fine-tuning problems associated with the cosmological constant [263].

Observations constrain the equation of state parameter $w = p/\rho$ of dark energy to be close to -1 [61]. Quintessence models must reproduce this value while remaining consistent with structure formation and the CMB. Additionally, the absence of detectable fifth forces constrains couplings of the scalar field to matter [15].

1.2.4 Dark energy and gravitational waves era (1998–2015–Present)

k-essence

Originally developed in the context of inflation [264], k-essence was later adapted to model dark energy [265, 266]. It involves scalar fields with non-canonical kinetic terms, described by a Lagrangian $\mathcal{L} = \sqrt{-g}\,K(\phi, X)$, where $X = -\frac{1}{2}\nabla^\mu \phi \nabla_\mu \phi$.

k-essence models can exhibit tracking behavior and avoid fine-tuning by dynamically adjusting to dominate the universe's energy density at late times. The kinetic term's form allows for a varying equation of state, potentially explaining the coincidence problem [267].

Constraints arise from the requirement of stability (avoiding ghosts and Laplacian instabilities), compatibility with the observed expansion history, and sound speed considerations to prevent issues with structure formation [268].

While k-essence remains a compelling idea, constructing viable models that meet all observational constraints without fine-tuning is challenging.

Einstein–aether theory

Einstein–aether theory was developed in the early 2000s by Ted Jacobson and David Mattingly [269]. It is a vector-tensor theory that includes a dynamical, unit timelike vector field (the "aether") coupled to gravity, leading to spontaneous Lorentz symmetry breaking.

The action includes terms constructed from the vector field u^μ and its derivatives, respecting diffeomorphism invariance but not local Lorentz invariance. The theory modifies the propagation of gravitational waves and has implications for black hole physics and cosmology [270, 271].

Constraints come from tests of Lorentz invariance, including observations of high-energy cosmic rays [272], gravitational waves [273], and precision measurements in the solar system. The parameter space of the theory is tightly constrained but not entirely ruled out.

Dvali–Gabadadze–Porrati (DGP) model

In 2000, Gia Dvali, Gregory Gabadadze, and Massimo Porrati proposed the DGP model, a braneworld scenario where our four-dimensional universe is a brane embedded in a five-dimensional Minkowski bulk [204]. This model modifies gravity at cosmological scales and offers a potential explanation for the observed late-time acceleration of the universe without invoking dark energy.

The DGP model introduces a crossover length scale r_c, beyond which gravity transitions from four-dimensional to five-dimensional behavior. The action combines the four-dimensional Einstein–Hilbert term on the brane and the five-dimensional term in the bulk:

$$S_{\mathrm{DGP}} = M_5^3 \int_{\mathrm{bulk}} d^5x\sqrt{-G}R^{(5)} + M_4^2 \int_{\mathrm{brane}} d^4x\sqrt{-g}R^{(4)} + \int_{\mathrm{brane}} d^4x\sqrt{-g}\,\mathcal{L}_{\mathrm{matter}}\,, \tag{1.57}$$

where $G_{\mu\nu}$ is the bulk metric, $g_{\mu\nu}$ is the induced metric on the brane, $R^{(5)}$ and $R^{(4)}$ are the Ricci scalars in the bulk and on the brane, respectively, and M_5 and M_4 are the five-dimensional and four-dimensional Planck masses.

The DGP model predicts deviations from GR at large distances, which can affect cosmological observations. The model has two branches of solutions:

1. Self-Accelerating Branch: Explains cosmic acceleration without dark energy but suffers from the presence of a ghost (a particle with negative kinetic energy), leading to theoretical inconsistencies [274].

2. Normal Branch: Avoids the ghost issue but requires dark energy to explain acceleration, diminishing its advantage over standard cosmology.

Observational data from supernovae, CMB, and baryon acoustic oscillations have been used to test the DGP model [275]. The results indicate tension between the model's predictions and observations, particularly in the self-accelerating branch.

Due to theoretical problems (ghost instability) and discrepancies with observational data, the original DGP model is considered disfavored. However, it has inspired further research into braneworld scenarios and modified gravity theories, such as Galileon gravity [276].

Loop Quantum Gravity (LQG)

LQG is a non-perturbative and background-independent approach to quantizing gravity, developed by Abhay Ashtekar, Carlo Rovelli, Lee Smolin, and others [142, 277, 278]. LQG reformulates GR using connection variables, leading to a quantum theory where spacetime geometry is quantized.

Ashtekar introduced new variables, called *Ashtekar variables*, where the gravitational field is described by an $SU(2)$ connection and its conjugate momentum [277]. The spatial geometry is represented by spin networks, graphs labeled by representations of $SU(2)$, capturing the quantized areas and volumes [279]. According to this description, geometric operators (area, volume) have discrete spectra, implying a granular structure of space at the Planck scale. Moreover, LQG does not require a fixed spacetime background, aligning with the spirit of GR.

LQG aims to resolve singularities in GR, such as the Big Bang and black hole singularities. However, making direct contact with low-energy physics and deriving testable predictions remains challenging [280]. Efforts include exploring possible quantum gravitational effects on CMB polarization and high-energy astrophysical phenomena [281, 282].

LQG is a mathematically rigorous approach to quantum gravity with active research programs. While direct experimental evidence is lacking, the theory provides valuable insights into the quantum nature of spacetime.

Loop Quantum Cosmology (LQC)

Loop Quantum Cosmology (LQC) is an application of LQG techniques to cosmological settings, focusing on homogeneous and isotropic spacetimes [283–286]. Martin Bojowald initiated LQC, with significant contributions from Parampreet Singh and others [285, 287–289].

LQC predicts that the Big Bang singularity is replaced by a quantum "bounce," where the universe reaches a minimum but finite size before expanding again [287]. Moreover, it allowed for the development of effective equations that capture the quantum corrections while remaining computationally tractable [290].

Parampreet Singh has extensively worked on the phenomenology of LQC, studying the implications for early universe cosmology, inflation, and singularity resolution [291, 292].

While LQC provides a consistent framework for avoiding singularities, its predictions for observable effects (*e.g.*, modifications to the primordial power spectrum) are generally subtle and require precise measurements to detect [293].

Tensor-Vector-Scalar (TeVeS) theory

Jacob D. Bekenstein developed TeVeS as a relativistic theory of gravity that reproduces Modified Newtonian Dynamics (MOND) in the appropriate limit [258]. TeVeS aims to address the successes and shortcomings of MOND by providing a consistent relativistic framework.

TeVeS includes a tensor field (metric), a vector field, and a scalar field, interacting in a way that modifies gravitational dynamics. It successfully accounts for galaxy rotation curves and gravitational lensing without dark matter [294]. Furthermore, it provides cosmological models that can, in principle, reproduce key observations [295]. However, TeVeS faces challenges in explaining cosmological observations, such as the CMB anisotropies and large-scale structure formation [296–298].

Hořava–Lifshitz gravity

Hořava–Lifshitz gravity was born from Petr Hořava's works. He proposed a theory that modifies GR by introducing anisotropic scaling between space and time at high energies [299]. The theory aims to be power-counting renormalizable, potentially offering a UV-complete theory of gravity. To do that, it implements different scaling dimensions for space and time, breaking Lorentz invariance at high energies but recovering it at low energies. This leads to modifications in the propagation of gravitational and matter fields at high energies. Additionally, Hořava–Lifshitz gravity is capable of generating novel cosmological models, including bouncing and inflationary scenarios [300].

However, breaking Lorentz invariance is limited by high-precision experiments constraining deviations from Lorentz symmetry [301]. The theory is further constrained by the speed of gravitational wave measurements [302]. While, from the theoretical side, it faces issues with ghost modes, instabilities, and strong coupling at low energies [303, 304].

Kinetic Gravity Braiding (KGB)

Kinetic Gravity Braiding (KGB), introduced by Deffayet, Gao, Steer, and Zahariade [305], is a scalar-tensor theory where the kinetic term of the scalar field interacts with the metric in a non-trivial manner. The central idea behind KGB is the "braiding" of the kinetic terms, where the coupling between the scalar field's kinetic term and the metric derivatives introduces a novel interaction between the scalar and tensor sectors. This interaction leads to distinct modifications in gravitational dynamics, allowing for a richer set of behaviors compared to standard scalar-tensor theories.

Despite involving higher-order derivatives, the equations of motion in KGB remain second-order, ensuring the avoidance of Ostrogradsky instabilities, which would otherwise introduce unphysical degrees of freedom [242–244]. KGB models have found important applications in cosmology, as they can produce self-accelerating solutions without the need for a cosmological constant, and they also offer mechanisms for screening gravitational effects, such as through the Vainshtein mechanism [306, 307].

De Rham-Gabadadze-Tolley (dRGT) Massive Gravity

The concept of massive graviton dates back to Fierz and Pauli, who first proposed a linear theory of massive gravity in 1939 [308]. However, massive gravity started to be seriously taken into account only after the formulation of the De Rham-Gabadadze-Tolley (dRGT) massive gravity theory [154, 309]. The latter represents a significant advancement in modifying gravitational theory by giving the graviton a small mass while successfully avoiding the unwanted Boulware–Deser ghost instability [310]. This development marked a turning point for massive gravity models, which had historically suffered from instability issues. The key innovation in dRGT massive gravity is the presence of nonlinear interaction terms that are carefully constructed to eliminate ghost degrees of freedom while maintaining a consistent theory [311, 312].

One of the major cosmological implications of massive gravity is the possibility of self-accelerating solutions, where the expansion of the universe could be driven by the graviton's mass itself, without the need for a cosmological constant [313]. This leads to the intriguing possibility that massive gravity could explain the universe's accelerated expansion in a way that differs from the dark energy hypothesis.

Although dRGT massive gravity is ghost-free at the classical level, quantum corrections could potentially reintroduce instabilities, leaving the stability of the theory at the quantum level an open question [309]. Additionally, deviations from GR at solar system scales predicted by massive gravity theories need to be carefully suppressed to align with observations, typically requiring mechanisms such as the Vainshtein screening effect. This adds complexity to the theory and, in some cases, makes it difficult to distinguish from GR itself [307]. Furthermore, recent observations, particularly those from gravitational wave events such as GW170817, have placed strong constraints on the speed of gravitational waves. These observations significantly limit the viable parameter space for massive gravity models, as deviations from the observed speed of gravitational waves could rule out certain versions of the theory [314].

Revival and Extension of Horndeski Theories

Interest in Horndeski's work was reignited when it was connected with generalized Galileon theories, which expanded the possibilities for scalar-tensor models [315]. These developments led to the creation of Degenerate Higher-Order Scalar-Tensor (DHOST) theories, which incorporate higher-order derivatives while skillfully avoiding the Ostrogradsky instability [316]. DHOST theories represent an extension beyond Horndeski, offering new classes of scalar-tensor models that feature richer dynamics with potential cosmological applications [317]. One of the critical innovations in these extensions is the inclusion of screening mechanisms, such as the Vainshtein mechanism, to satisfy stringent solar system tests and remain consistent with local gravity experiments [318].

At the same time, Galileon theories, which are a subset of scalar-tensor theories, gained prominence due to their symmetry properties and their ability to address cosmic acceleration without invoking a cosmological constant or dark energy. These theories, first introduced in 2009, respect a Galilean symmetry, where the scalar field undergoes shifts and gradient transformations without altering the equations of motion [276]. They were initially developed as a generalization of the Dvali-Gabadadze-Porrati (DGP) model, which attempted to modify gravity on cosmological scales using an extra-dimensional approach. The Galileon theory emerged as a local effective field theory, capturing essential features of the DGP model while avoiding its pitfalls, such as the presence of ghost instabilities [154].

One of the significant contributions of Galileon theories is their ability to naturally provide self-accelerating solutions, where the dynamics of the scalar field drive the expansion of the universe. Additionally, these models incorporate screening mechanisms like the Vainshtein mechanism, allowing them to modify gravity on large scales while maintaining consistency with solar system tests [306]. Importantly, Galileon theories avoid ghost instabilities through a careful combination of higher-order derivative terms, making them well-behaved in the IR regime. Beyond cosmic acceleration, these theories have been applied to inflationary models and the formation of large-scale structures, making them versatile tools in cosmology. Theories such as the covariant Galileon and extended Galileon have been developed to address a range of cosmological issues, including the production of primordial gravitational waves and the generation of anisotropies in the CMB [319].

Observationally, one of the most significant constraints on these theories comes from the speed of gravitational waves. The detection of gravitational waves from GW170817 has imposed strict limits on modifications to the speed of gravitational waves, thereby restricting the parameter space for many scalar-tensor models [320, 321]. Additionally, any viable theory must be able to account for cosmological observations, including the universe's expansion history and the formation of large-scale structure, which requires precise modeling to align with data [322]. Despite these challenges, viable models exist within the DHOST framework that satisfy both gravitational wave constraints and cosmological data.

In order to avoid constraints coming from cosmological observation, in the last years, Horndeski has been the object of further studies. The main idea was to generalize the Horndeski framework to verify if the constrained terms could survive in the new setting, non-trivially contributing to the field equations. Therefore, there was the development of Teleparallel Horndeski gravity [323–329] and, lately, of Kaluza–Klein compactified Horndeski theory [330–333].

Non-local Gravity

Non-local gravity theories introduce modifications to gravitational dynamics through the use of inverse differential operators, such as $\Box^{-1}$, which lead to non-local interactions [334]. These theories aim to explain cosmic acceleration without the need for introducing dark energy by altering the structure of gravity on large scales. A key feature of non-local gravity models is the incorporation of terms like $R\Box^{-1}R$ into the gravitational action, which significantly affects how gravity operates at cosmological distances [335, 336]. This modification is particularly relevant in the infrared regime, where it has the potential to address both the dark energy problem and the cosmological constant issue, providing a new perspective on the cause of the universe's accelerated expansion [337].

However, non-local gravity theories must satisfy various observational and theoretical constraints to remain viable. For instance, these models need to fit the observed expansion history of the universe, align with CMB data, and accurately describe the growth of large-scale structure [338]. The formulation of non-local terms introduces several theoretical challenges as well, such as defining initial conditions for the non-local operators, ensuring causality within the framework, and avoiding potential instabilities that could arise from the non-local interactions [339].

Despite these hurdles, certain non-local gravity models continue to be explored as viable alternatives to dark energy. The success of these theories depends heavily on careful formulation to ensure that they not only align with observational data but also maintain theoretical consistency and avoid issues like instability and causality violations [340–342].

$f(R, P, Q)$ gravity

Theories that generalize $f(R)$ gravity by incorporating dependencies on higher-order curvature invariants extend the framework of modified gravity, allowing for richer gravitational dynamics. In these models, additional terms related to the curvature of spacetime are introduced, such as $P = R_{\mu\nu}R^{\mu\nu}$, which represents the Ricci tensor squared, and $Q = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, the Kretschmann scalar, involving the square of the Riemann tensor [224, 343–345]. These terms naturally emerge in higher-dimensional theories and quantum corrections to gravity, playing crucial roles in regimes where curvature becomes large, such as near black holes or during the early universe's inflationary phase.

The gravitational action in these extended theories depends not only on $f(R)$ but also on the additional curvature invariants $f(R, P, Q)$, which results in more complex dynamics. The general form allows for novel solutions that can differ significantly from GR in certain regimes, offering insights into phenomena that GR alone cannot fully explain. For instance, in the early universe, these extended theories have been explored as potential models for inflation, offering mechanisms that may better fit observed cosmic inflationary data without requiring a fine-tuned inflation field [346]. Similarly, black hole solutions in $f(R, P, Q)$ gravity can exhibit novel behavior, possibly avoiding the formation of singularities that typically arise in GR, thus addressing one of the key unresolved issues in classical gravity [347, 348].

In terms of observational constraints, these theories must recover the predictions of GR in low-curvature regimes, such as in the solar system, where GR has been extensively tested. Modified gravity models often face challenges here because any deviations from GR at these scales are highly constrained by precise measurements, such as those of planetary orbits and light deflection [349]. As a result, viable $f(R, P, Q)$ models typically incorporate mechanisms that reduce the modifications to GR-like behavior in these contexts, ensuring that the predictions align with solar system tests.

Cosmological observations also impose strong constraints on $f(R, P, Q)$ theories. The accelerated expansion of the universe, as inferred from supernovae data, the CMB, and large-scale structure, must be consistently explained within these frameworks [350]. Additionally, structure formation data, which tracks how matter clumps under gravity over cosmic time, must match the predictions of these extended models. This is a significant challenge because modifications to gravity often alter the rate at which structures form, necessitating careful tuning of the function $f(R, P, Q)$ to ensure consistency with the observed distribution of galaxies and clusters.

Despite these challenges, $f(R, P, Q)$ theories remain attractive due to their versatility and potential to resolve key issues in cosmology and gravitational physics. For example, some models have been constructed to avoid cosmological singularities, providing a smoother evolution of the universe that could eliminate the need for initial singularities such as the Big Bang [346]. Other models suggest corrections to black hole physics, leading to potential resolutions of singularities or modified thermodynamic behavior, which might be testable through future gravitational wave observations or high-precision astrophysical measurements.

The viability of $f(R, P, Q)$ theories largely depends on selecting a function f that not only satisfies observational constraints but also resolves theoretical issues such as ghost instabilities, which arise from the higher-order derivatives introduced by these curvature invariants. Some models have successfully navigated these challenges by carefully constructing $f(R, P, Q)$ functions that ensure stability and match observations. Thus, the framework of $f(R, P, Q)$ gravity continues to be an active area of research [10, 351–355], offering promising avenues for understanding phenomena beyond the reach of classical GR.

Generalized Proca theories

Generalized Proca theories are a class of vector-tensor theories that extend the original Proca theory [356]. In the 1930s, Alexandru Proca formulated a theory of a massive vector field to generalize Maxwell's electromagnetism. which describes a massive spin-1 vector field. In its simplest form, the Proca Lagrangian involves a vector field A_μ with a mass term $m^2 A_\mu A^\mu$, ensuring that the vector field propagates three degrees of freedom, corresponding to the physical polarizations of a massive spin-1 particle. However, this simple theory does not capture the range of possible self-interactions and couplings with gravity that could arise in more general settings, particularly in cosmological models. Thus, the development of generalized Proca theories began to explore how vector fields could contribute to cosmic acceleration or act as alternatives to dark energy.

These theories are constructed to avoid the Ostrogradsky instability, which typically arises from higher-order derivatives in the equations of motion, by ensuring that the equations remain second-order despite the additional complexity in the interactions of the vector field.

A crucial advancement in this direction was made when generalized Proca theories were formalized with non-linear interactions, similar to the way Galileon terms are introduced in scalar-tensor theories. These non-linear interactions between the vector field and gravity allow the theory to propagate three physical degrees of freedom without introducing additional ghost degrees of freedom, ensuring that the theory remains stable [357]. The introduction of generalized Proca theories opens up a new set of possibilities for constructing consistent modifications to gravity, particularly at large scales, where the effects of a massive vector field could contribute to the universe's accelerated expansion.

One of the distinguishing features of generalized Proca theories is the inclusion of derivative self-interactions for the vector field, such as $(\nabla_\mu A^\mu)^2$, which go beyond the standard mass term while still avoiding higher-order equations of motion. These derivative couplings allow the vector field to influence the gravitational dynamics at cosmological scales, offering potential explanations for phenomena such as dark energy. The mathematical formalism of generalized Proca theories involves writing the most general Lagrangian for a massive vector field that preserves second-order field equations. This includes terms not only for the vector field's mass and kinetic energy but also higher-order interactions and non-minimal couplings to curvature terms like the Ricci scalar R or the Ricci tensor $R_{\mu\nu}$, further enriching the dynamics [358].

The cosmological applications of generalized Proca theories are significant, particularly in explaining the universe's accelerated expansion. These theories have been explored as models for dark energy, where the massive vector field plays a role analogous to that of a cosmological constant but with more dynamic behavior [359]. By modifying gravity at large scales, generalized Proca models offer an alternative to scalar-tensor theories and provide additional degrees of freedom that could help address the shortcomings of GR in explaining cosmic acceleration. Moreover, these models can also include screening mechanisms, such as the Vainshtein mechanism, to ensure that any deviations from GR are suppressed at smaller scales, making the theory consistent with solar system observations [360].

Despite their potential, generalized Proca theories face significant observational constraints, particularly from cosmological data and gravitational wave observations. For instance, the detection of gravitational waves from the neutron star merger GW170817 has imposed stringent constraints on any modifications to the speed of gravitational waves, which directly impacts the viability of many modified gravity models, including generalized Proca theories [247]. Theories in this class must carefully balance the freedom to modify gravity at cosmological scales with the need to recover GR in well-tested regimes, such as in the solar system and during the propagation of gravitational waves.

Revival of metric-affine gravity and further explorations

In recent decades, the renewed interest in metric-affine gravity has been driven by the limitations of GR in explaining cosmological phenomena such as dark energy, dark matter, and cosmic inflation. While GR assumes that the Levi–Civita connection is the only geometric object describing spacetime, metric-affine gravity generalizes this by allowing the connection $\overline{\Gamma}^{\lambda}_{\mu\nu}$ to be independent of the metric $g_{\mu\nu}$. This leads to a richer geometric framework that incorporates not only curvature but also torsion $T^{\lambda}_{\mu\nu}$ and non-metricity $Q_{\alpha\mu\nu}$, which measures the failure of the connection to preserve the metric under parallel transport [211, 220, 361].

In fact, the connection can be in general decomposed into the following three parts

$$\overline{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}\,, \tag{1.58}$$

the Levi–Civita part, the contorsion tensor, and the information tensor.

Then, the general Riemann scalar reads as follows

$$\bar{R} = g^{\mu\nu}\bar{R}_{\mu\nu} = R + K + L + I, \tag{1.59}$$

where

$$K \equiv 2\nabla_{\lambda}T^{\lambda} - T_{\lambda}T^{\lambda} + \frac{1}{4}T_{\mu\nu\lambda}(T^{\mu\nu\lambda} + 2T^{\lambda\nu\mu}) \tag{1.60}$$

$$L \equiv \nabla_{\lambda}(Q^{\lambda} - \tilde{Q}^{\lambda}) - \frac{1}{4}Q_{\lambda}Q^{\lambda} + \frac{1}{2}Q_{\lambda}\tilde{Q}^{\lambda} + \frac{1}{4}Q_{\mu\lambda\alpha}Q^{\mu\lambda\alpha} - \frac{1}{2}Q_{\mu\lambda\alpha}Q^{\lambda\mu\alpha} \tag{1.61}$$

$$I \equiv T_{\lambda}(\tilde{Q}^{\lambda} - Q^{\lambda}) + L_{\alpha\lambda\mu}K^{\alpha\lambda\mu}. \tag{1.62}$$

being

$$Q_{\alpha} \equiv g^{\mu\nu}Q_{\alpha\mu\nu}\,, \quad \tilde{Q}_{\nu} \equiv g^{\alpha\mu}Q_{\alpha\mu\nu}\,, \quad T_{\mu} \equiv T^{\lambda}{}_{\mu\lambda}\,. \tag{1.63}$$

Irrespective of the action being considered, metric-affine gravity is characterized by the dependence of the matter Lagrangian on the connection, in general. The variation of the matter action with respect to the connection is the hypermomentum tensor,

$$\Delta^{\lambda}{}_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta\overline{\Gamma}^{\lambda}{}_{\mu\nu}}\,. \tag{1.64}$$

One of the most significant *extensions* of metric-affine gravity is teleparallel gravity, which is characterized by a vanishing metric curvature and attributes gravitational effects to either torsion or non-metricity. In particular, teleparallel gravity allows us to obtain two equivalent formulation to GR, wherein gravity is described by a "pure torsion space", giving rise to the *Teleparallel Equivalent to GR* (TEGR), or by a "pure non-metricity space", giving rise to the *Symmetric Teleparallel Equivalent to GR* (STEGR). These three formulations of gravity go under the name of *Geometrical Trinity of Gravity* [362].

TEGR and extensions Within the tetrad formalism [211, 235, 363], gravity can be reformulated entirely in terms of torsion by considering a flat spacetime equipped with a metric-compatible connection known as the Weitzenböck connection. The term teleparallel reflects this inherent flatness condition of spacetime in the theory.

In this framework, tetrad fields serve as the dynamical variables, with torsion acting as their corresponding "field strength." This perspective allows gravity to be interpreted as a gauge theory [364] for the translation group $\mathscr{T}^{(1,3)}$. The geometrical setting involves a principal bundle [235] associated with the tangent bundle of spacetime, where the translation group functions as the structure group. Transformations under the Lorentz group $\mathrm{SO}(1,3)$ are employed to account for inertial effects and facilitate working within general Lorentz frames [363, 365, 366].

The action for this theory is constructed from quadratic invariants of the torsion tensor. The most general form of such an action is:

$$S_{\mathbb{T}} = \frac{1}{2}\int d^4x \left(\sqrt{-g}\,\mathbb{T} + \lambda_\mu{}^{\nu\rho\sigma}\overline{R}^\mu{}_{\nu\rho\sigma} + \tilde{\lambda}^{\alpha\mu\nu}Q_{\alpha\mu\nu}\right), \tag{1.65}$$

where $\overline{\mathbb{T}}$ is the general torsion scalar defined by:

$$\overline{\mathbb{T}} = -c_1 T_\alpha T^\alpha + c_2\frac{1}{4}T_{\mu\nu\alpha}T^{\mu\nu\alpha} + c_3\frac{1}{2}T_{\mu\nu\alpha}T^{\alpha\nu\mu}. \tag{1.66}$$

The terms involving Lagrange multipliers $\lambda_\mu{}^{\nu\rho\sigma}$ and $\tilde{\lambda}^{\alpha\mu\nu}$ enforce the conditions of vanishing curvature and non-metricity, respectively. The Lagrange multipliers exhibit natural symmetries: $\lambda_\mu{}^{\nu\rho\sigma} = \lambda_\mu{}^{\nu[\rho\sigma]}$ is antisymmetric in its last two indices, and $\tilde{\lambda}^{\alpha\mu\nu} = \tilde{\lambda}^{\alpha(\mu\nu)}$ is symmetric in its last two indices. For convenience, they are defined as tensor densities of weight -1.

By setting the constants $c_1 = c_2 = c_3 = 1$, the action (1.65) becomes equivalent to the Einstein–Hilbert action, modulo a total derivative term, provided that the connection is both flat and metric-compatible from the outset. This equivalence can be demonstrated by considering the vanishing of the curvature tensor:

$$\begin{aligned}
0 = \overline{R} = R + K \implies R &= -K \\
&= T_\lambda T^\lambda - \frac{1}{4}T_{\mu\nu\lambda}T^{\mu\nu\lambda} - \frac{1}{2}T_{\mu\nu\lambda}T^{\lambda\nu\mu} - 2\nabla_\lambda T^\lambda \\
&= -\overline{\mathbb{T}}(c_1 = 1, c_2 = 1, c_3 = 1) - 2\nabla_\alpha T^\alpha \\
&= -\mathbb{T} - 2\nabla_\alpha T^\alpha,
\end{aligned} \tag{1.67}$$

where $\mathbb{T}$ denotes $\overline{\mathbb{T}}$ evaluated with $c_1 = c_2 = c_3 = 1$.

Consequently, varying the action with respect to the tetrad fields yields equations of motion identical to those of GR, confirming the equivalence between the two theories. This equivalence poses a challenge because it implies that teleparallel gravity and GR are experimentally indistinguishable based solely on their equations of motion. However, this degeneracy might be resolved by considering small violations of the EEP, such as those arising from interactions between torsion and particles with intrinsic spin.

An initial extension of the Teleparallel Equivalent of GR (TEGR) is the class of $f(\mathbb{T})$ theories of gravity [367–369]. Similar to how GR is generalized to $f(R)$ theories by replacing the Ricci scalar R with a function $f(R)$, $f(\mathbb{T})$ theories replace the torsion scalar $\mathbb{T}$ with a function $f(\mathbb{T})$.

The action for an $f(\mathbb{T})$ theory is given by:

$$S_{f(\mathbb{T})} = \frac{1}{2} \int d^4x \, ||e|| \, f(\mathbb{T}) \,, \tag{1.68}$$

where $||e||$ is the determinant of the tetrad $e^a{}_\mu$.

Before delving deeper into these theories, it is essential to emphasize that the TEGR Lagrangian and its field equations are invariant under local Lorentz transformations, even in the frame where the spin connection vanishes. However, when extending to $f(\mathbb{T})$ theories, this invariance is generally lost. In TEGR, terms that are boundary contributions become non-trivial in $f(\mathbb{T})$ theories. While f(R) theories encounter similar issues, $f(\mathbb{T})$ theories specifically face the problem of local Lorentz symmetry breaking.

To overcome this challenge, two main approaches have been proposed [370]:

1. Pure Tetrad Formalism: Continue working within the Weitzenböck gauge—characterized by a zero spin connection—and select, *a posteriori*, those tetrads that yield field equations whose solutions match those of GR in the limit $f(\mathbb{T}) \to \mathbb{T}$.

2. Covariant Formalism: Use a general frame with a non-vanishing spin connection and perform variations of the action with respect to both the tetrad $e^a{}_\mu$ and the spin connection $\omega^a{}_{b\nu}$. This approach preserves local Lorentz invariance but significantly complicates the calculations.

While TEGR is entirely equivalent to GR, this equivalence does not generally extend to $f(R)$ and $f(\mathbb{T})$ theories. The arbitrariness introduced by the boundary term when dealing with non-linear functions of the torsion tensor disrupts this equivalence. Nevertheless, the correspondence between TEGR and $f(\mathbb{T})$ theories is often considered more natural than that between GR and $f(R)$ theories. This is because the field equations in $f(R)$ theories are fourth-order differential equations, whereas in GR, TEGR, and $f(\mathbb{T})$ theories, the field equations remain second-order.

STEGR and extensions GR can be formulated using geometrical frameworks other than Riemannian geometry. In this section, we focus on one such alternative: the Symmetric Teleparallel Equivalent to GR (STEGR) [371–374]. In STEGR, the torsion tensor vanishes, but the non-metricity tensor ∇g remains non-zero and is responsible for mediating the gravitational interaction.

A distinctive feature of this framework is the possibility to eliminate the connection globally by selecting appropriate coordinates, rendering the spacetime trivially connected. This choice corresponds to a preferred set of frames where calculations become significantly simpler.

The action for this theory is constructed from quadratic forms of the non-metricity tensor that are even under parity transformations. The most general form of such an action is:

$$S_{\overline{\mathbb{Q}}} = \frac{1}{2} \int d^4x \left(-\sqrt{-g}\, \overline{\mathbb{Q}} + \lambda_\mu{}^{\nu\rho\sigma} \overline{R}^\mu{}_{\nu\rho\sigma} + \tilde{\lambda}\alpha^{\ \ \mu\nu} T^\alpha{}_{\ \ \mu\nu} \right), \tag{1.69}$$

where $\lambda_\mu{}^{\nu\rho\sigma} = -\lambda_\mu{}^{\nu\sigma\rho}$ and $\tilde{\lambda}\alpha^{\ \ \mu\nu} = -\tilde{\lambda}\alpha^{\ \ \nu\mu}$ are Lagrange multipliers enforcing the conditions of vanishing curvature and torsion, respectively. The scalar $\overline{\mathbb{Q}}$ is the general non-metricity scalar defined as:

$$\overline{\mathbb{Q}} = \frac{c_1}{4} Q_{\mu\nu\alpha} Q^{\mu\nu\alpha} - \frac{c_2}{2} Q_{\mu\nu\alpha} Q^{\nu\mu\alpha} - \frac{c_3}{4} Q_\lambda Q^\lambda + (c_4 - 1)\tilde{Q}\sigma \tilde{Q}^\sigma + \frac{c_5}{2} Q\rho \tilde{Q}^\rho . \tag{1.70}$$

Here, $Q_{\mu\nu\alpha}$ is the non-metricity tensor, while Q_λ and $\tilde{Q}_\lambda$ are its trace vectors. The constants c_i (with $i = 1, \ldots, 5$) define a five-parameter family of quadratic theories.

By setting all constants $c_i = 1$ and imposing that the connection is torsion-free and curvature-free from the outset, the action in equation (1.69) becomes equivalent to the Einstein–Hilbert action, differing only by a total derivative term. This equivalence is evident from the following identities:

$$\begin{aligned}
0 = \overline{R} = R + L \implies R &= -L \\
&= \frac{1}{4}Q_\lambda Q^\lambda - \frac{1}{2}Q_\lambda \tilde{Q}^\lambda - \frac{1}{4}Q_{\mu\lambda\alpha}Q^{\mu\lambda\alpha} + \frac{1}{2}Q_{\mu\lambda\alpha}Q^{\lambda\mu\alpha} - \nabla_\lambda(Q^\lambda - \tilde{Q}^\lambda) \\
&= -\overline{\mathbb{Q}}(c_i = 1) - \nabla_\lambda(\tilde{Q}^\lambda - Q^\lambda) \\
&= -\mathbb{Q} - \nabla_\lambda(\tilde{Q}^\lambda - Q^\lambda), \qquad (1.71)
\end{aligned}$$

where $\mathbb{Q} \equiv \overline{\mathbb{Q}}\,(c_i = 1)$. The term L captures contributions from non-metricity, and $\overline{R}$ is the Ricci scalar derived from the connection with zero curvature and torsion.

A straightforward extension of this theory involves generalizing the action to an arbitrary function of the non-metricity scalar $\mathbb{Q}$, analogous to $f(R)$ and $f(\mathbb{T})$ theories. This leads to the action [375]:

$$S_{f(\mathbb{Q})} = \frac{1}{2}\int d^4x\,\sqrt{-g}\,f(\mathbb{Q})\,. \qquad (1.72)$$

This generalization broadens the class of gravitational theories based on non-metricity, potentially offering new perspectives on gravitational phenomena.

The geometrical trinity of gravity unites these three approaches—curvature-based gravity (GR), torsion-based gravity (teleparallel gravity), and non-metricity-based gravity (symmetric teleparallel gravity). While each approach uses a different geometrical framework, they are mathematically equivalent under certain conditions, meaning they describe the same gravitational phenomena.

The observational viability of these theories is being tested through data from the CMB, large-scale structure formation, and gravitational wave observations. Constraints on gravitational wave propagation, in particular, have placed strong limits on many modified gravity theories, refining the parameter space for viable models. Extensions of teleparallel and symmetric teleparallel gravity, such as $f(\mathbb{T}, \mathcal{B}_\mathbb{T})$ gravity and $f(\mathbb{Q}, \mathcal{B}_\mathbb{Q})$ gravity, introduce boundary terms that provide additional freedom for addressing singularities, cosmic inflation, and late-time acceleration [376, 377].

1.3 To Horndeski gravity. . . and beyond!

This section is intended to provide a compact reference to Horndeski gravity and its generalization. Following some of the most recent and useful reviews [378–380], this section addresses topics such as generalized Galileons, the (re)discovery of Horndeski theory, cosmological perturbations within this framework, the implications of GW170817 multi-messenger observation in Horndeski theory and its extensions.

1.3.1 Ostrogradsky's instability

In 1974, Gregory W. Horndeski extended the ideas of scalar-tensor theories by formulating the most general scalar-tensor theory that produces second-order equations of motion for both the metric and the scalar field [241]. This development was crucial because second-order equations of motion were necessary to avoid the Ostrogradsky instability [242–244, 316].

The Ostrogradsky instability is a fundamental issue that arises in theories with higher-order derivatives in the Lagrangian, leading to an unbounded Hamiltonian from below. This instability is particularly concerning in classical and quantum field theories, as it implies that the system can access arbitrarily negative energy states, making the theory physically inconsistent. For this reason, Horndeski gravity represents a significant milestone in the history of modified gravity theories.

To understand this better, let us consider a system described by a Lagrangian that contains time derivatives of a variable $q(t)$ higher than the first order. Suppose we have a Lagrangian of the form:

$$L = L(q, \dot{q}, \ddot{q}), \tag{1.73}$$

where $\ddot{q}$ represents the second time derivative of q. According to Ostrogradsky's theorem, if the Lagrangian depends non-degenerately on these higher-order derivatives, the resulting Hamiltonian will be linear in the canonical momenta associated with the higher derivatives. This linearity leads to an unbounded Hamiltonian, which implies the existence of states with arbitrarily negative energy.

To illustrate the problem, let us consider a simple example with a higher-order derivative term:

$$L = \frac{1}{2}\alpha\ddot{q}^2 - V(q)\,, \tag{1.74}$$

where α is a constant. In this case, the Euler–Lagrange equation derived from this Lagrangian will involve fourth-order time derivatives of q.

To derive the equations of motion, we use the Euler-Lagrange equation for higher-order derivatives, which is given by

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0\,, \tag{1.75}$$

yielding

$$\alpha\ddddot{q} - V'(q) = 0\,. \tag{1.76}$$

This is a fourth-order differential equation, reflecting the higher-order nature of the system due to the dependence on $\ddot{q}$. To solve this, four initial conditions are required, indicating the presence of two dynamical degrees of freedom. According to Ostrogradsky's theorem, one of these must correspond to a ghost degree of freedom.

To make the analysis more evident, we can introduce an auxiliary variable ξ, allowing the Lagrangian to be rewritten in an equivalent form:

$$L = \alpha\xi\ddot{q} - \frac{\alpha}{2}\xi^2 - V(q), \tag{1.77}$$

which can be expressed as:

$$L = -\alpha\dot{\xi}\dot{q} - \frac{\alpha}{2}\xi^2 - V(q) + \alpha\frac{d}{dt}(\xi q)\,. \tag{1.78}$$

The total derivative term does not affect the equations of motion and can be neglected for this purpose. By redefining the variables $x = (q + \xi)/\sqrt{2}$ and $y = (q - \xi)/\sqrt{2}$, we can rewrite the Lagrangian as:

$$L = -\frac{1}{2}\alpha\dot{x}^2 + \frac{1}{2}\alpha\dot{y}^2 - U(x, y). \tag{1.79}$$

This reformulated Lagrangian demonstrates that the system has two degrees of freedom, one of which has a kinetic term with the wrong sign, indicating the presence of ghost instabilities, which occur regardless of the sign of the parameter α.

The above formulation reveals the challenges of handling higher-order derivatives and the emergence of ghost-like degrees of freedom. These ghost instabilities are inherently problematic as they can lead to unbounded energy growth, a signature of the Ostrogradsky instability.

The Ostrogradsky instability is a significant problem in the context of modified gravity theories and higher-order field theories. One way to circumvent the Ostrogradsky instability is to construct degenerate higher-order Lagrangians, where the dependence on higher derivatives is such that the resulting Hamiltonian remains bounded. This approach is used in the construction of viable modified gravity theories, such as the Horndeski theory, which avoids Ostrogradsky instabilities by ensuring that the equations of motion remain second-order despite the presence of higher-order terms in the action. Another approach to avoiding the Ostrogradsky instability is to impose constraints on the theory that eliminate the problematic degrees of freedom associated with the higher-order derivatives. By introducing suitable constraints, one can reduce the number of independent degrees of freedom, thereby removing the linear dependence on the problematic canonical momenta. This strategy is used in several effective field theories, where the goal is to retain the benefits of higher-order terms while ensuring that the resulting dynamics are stable and physically meaningful.

Notice that GR does not suffer from such instability because its Lagrangian is linear in the second-order derivatives of the metric tensor, then it violates the non-degeneracy Ostrogradsky's assumption. Moreover, higher-order theories, like $f(R)$, can be re-written as second-order derivative theories with an extra scalar field [147].

1.3.2 The revival of theory

Horndeski's theory represents a comprehensive generalization of earlier scalar-tensor theories, encompassing them as particular cases while ensuring the avoidance of higher-order derivatives in the field equations.

Despite its generality and elegance, Horndeski's work remained relatively obscure for several decades. It was not until the early 21st century, with the discovery of dark energy and the accelerated expansion of the Universe, that his theory received renewed attention.

Following the publication of his seminal work, subsequently known as Horndeski gravity, Gregory Horndeski departed from the field of physics to pursue a career in painting. This artistic endeavor allowed him to explore a different form of creativity, and for many years, he remained largely unknown within the physics community. However, in the early 2000s, his theory was rediscovered and found to be of significant relevance to modern cosmology, prompting Horndeski's re-engagement with the field. This resurgence of interest, particularly in relation to dark energy and cosmic acceleration, brought his contributions back into the scientific spotlight.

In this renewed context, the generalized Galileon theories, which extend the original Galileon model proposed by Nicolis, Rattazzi, and Trincherini in 2009 [276], gained considerable attention. These theories are formulated to maintain second-order field equations and exhibit Galilean symmetry, rendering them robust against certain quantum corrections [381].

It was subsequently recognized that the generalized Galileon theories constitute specific cases of Horndeski's theory [315, 382], leading to a renewed interest in his work. This positioned Horndeski gravity as a leading candidate for a modified theory of gravity capable of explaining dark energy, inflation, and other cosmological phenomena without introducing additional degrees of freedom or succumbing to Ostrogradsky instability.

A notable feature of Horndeski gravity is its capacity to incorporate screening mechanisms, such as the Vainshtein mechanism, which mitigates deviations from GR in high-density environments [383]. This allows Horndeski models to satisfy solar system constraints while still modifying gravity on cosmological scales.

Despite the strengths of Horndeski gravity, the framework is not without challenges. Certain extensions of Horndeski gravity have been found to permit superluminal propagation of gravitational waves, raising concerns regarding the theory's consistency with causality [321]. Moreover, identifying specific models within the Horndeski framework that comply with all observational constraints remains an ongoing endeavor.

Furthermore, the recent detection of gravitational waves by LIGO and Virgo has imposed stringent constraints on the propagation speed of gravitational waves, requiring it to match the speed of light. This has excluded a significant portion of the Horndeski parameter space, necessitating the development of more refined models that are consistent with these constraints [247, 320].

Let us write Horndeski theory in its modern formulation [379] to clarify all the aspects. The action is commonly written as:

$$S_{\mathrm{H}}\left[g_{\mu\nu},\phi\right]=\frac{1}{2}\int d^4x\sqrt{-g}\,\left(\mathcal{L}_2+\mathcal{L}_3+\mathcal{L}_4+\mathcal{L}_5\right)\,, \tag{1.80}$$

where

$$\mathcal{L}_2=G_2\left(\phi,X\right)\,, \tag{1.81}$$

$$\mathcal{L}_3=-\,G_3\left(\phi,X\right)\Box\phi\,, \tag{1.82}$$

$$\mathcal{L}_4=G_4\left(\phi,X\right)R+G_{4X}\left(\phi,X\right)\left[(\Box\phi)^2-(\nabla\nabla\phi)^2\right]\,, \tag{1.83}$$

$$\mathcal{L}_5=G_5\left(\phi,X\right)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi-\frac{G_{5X}}{6}\left[(\Box\phi)^3-3\Box\phi\left(\nabla\nabla\phi\right)^2+2\left(\nabla\nabla\phi\right)^3\right]\,, \tag{1.84}$$

recalling that $X\equiv-\frac{1}{2}\nabla^\alpha\phi\nabla_\alpha\phi$, the G_i $(i=2,3,4,5)$ are regular functions of ϕ and X, while $G_{i\phi}\equiv\partial G_i/\partial\phi$ and $G_{iX}\equiv\partial G_i/\partial X$, $(\nabla\nabla\phi)^2\equiv\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi$ and $(\nabla\nabla\phi)^3\equiv\nabla_\mu\nabla_\lambda\phi\nabla^\lambda\nabla^\sigma\phi\nabla_\sigma\nabla^\mu\phi$.

Taking the functional derivatives of S_{H}, with respect to $g_{\mu\nu}$ and ϕ, one can define the following quantities

$$\mathcal{E}_{\mu\nu}\equiv\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{H}}}{\delta g^{\mu\nu}}\,,\qquad\mathcal{J}\equiv\frac{\delta S_{\mathrm{H}}}{\delta\phi}\,, \tag{1.85}$$

which can be written as the sum of the individual $\mathcal{E}^{(i)}_{\mu\nu}$ and $\mathcal{J}^{(i)}$ contributions coming from each Lagrangian $\mathcal{L}_i$.

The equations of motion of the system described by the total action $S\left[g_{\mu\nu},\phi,\psi\right]=S_{\mathrm{H}}\left[g_{\mu\nu},\phi\right]+$

$S^{(\mathrm{m})}\left[g_{\mu\nu}, \psi\right]$ then read, in general,

$$\mathcal{E}_{\mu\nu} = T^{(m)}_{\mu\nu}\,, \qquad \mathcal{J} = 0\,, \qquad \frac{\delta S^{(\mathrm{m})}}{\delta\psi} = 0\,, \tag{1.86}$$

where the latter gives the equations of motion of each matter field.

The consistency between field equations and the covariant conservation of the matter stress-energy tensor is ensured by the diffeomorphism invariance of the action, implying the following algebraic identity:

$$\mathcal{J}^{(i)}\,\nabla_\nu\phi = \nabla^\mu\mathcal{E}^{(i)}_{\mu\nu}\,. \tag{1.87}$$

By choosing the Horndeski functions properly, it is possible to replicate any second-order scalar-tensor theory. As an example, one can verify that a scalar field coupled with the Gauss–Bonnet topological invariant,

$$S_{\mathcal{G}} = \int d^4x\sqrt{-g}\,h(\phi)\,\mathcal{G}\,. \tag{1.88}$$

can be reproduced by Horndeski's theory, by taking the following functions:

$$G_5(\phi, X) = -4\,\frac{dh}{d\phi}\,\ln X\,, \tag{1.89}$$

$$G_4(\phi, X) = 4\,\frac{d^2h}{d\phi^2}\,X\,(2-\ln X)\,, \tag{1.90}$$

$$G_3(\phi, X) = 4\,\frac{d^3h}{d\phi^3}\,X\,(7-3\ln X)\,, \tag{1.91}$$

$$G_2(\phi, X) = 8\,\frac{d^4h}{d\phi^4}\,X^2\,(3-\ln X)\,. \tag{1.92}$$

The easiest way to prove this is by directly comparing their field equations, yielding from the variation with respect to the metric tensor, on a spatially flat FLRW background.

1.3.3 Cosmological background

Let us specialize the discussion of Horndeski gravity to the FLRW metric. Before delving into the details, it is useful to note that there are two main approaches to tackling this problem. The first approach involves using the usual variational method and then substituting the FLRW metric and the time-dependent scalar field $\phi = \phi(t)$ (representing the only possibility because of the homogeneity and isotropy of the metric). The second approach is directly substituting the metric and the homogeneous field into the action from the beginning, rewriting it in the $3+1$ formalism using the lapse function $N(t)$. Then, the variation of the action with respect to $N(t)$ provides the first Friedmann equation (or constraint equation) while varying with respect to the scale factor $a(t)$ yields the second Friedmann equation. Because of the many functions, especially $G_4(\phi, X)$ and $G_5(\phi, X)$, the second approach is the most indicated.

Therefore, substituting the FLRW line element and the homogeneous scalar field into the action,

$$ds^2 = -N(t)^2dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2d\Omega^2\right), \qquad \phi = \phi(t)\,, \tag{1.93}$$

varying it with respect to $N(t)$ and $a(t)$, and then setting $N = 1$, one obtains the background equations corresponding to the Friedmann and acceleration equations in the following form,

$$\mathcal{E}_0 + \mathcal{E}_k = 0, \tag{1.94}$$

$$\mathcal{P}_0 + \mathcal{P}_k = 0, \tag{1.95}$$

where $\mathcal{E}_0$ and $\mathcal{P}_0$ are independent of k and $\mathcal{E}_k$ and $\mathcal{P}_k$ are proportional to k. Their explicit writings are given by [384, 385]

$$\begin{aligned}\mathcal{E}_0 = {} & XG_{2X} - G_2 + 3X\dot{\phi}HG_{3X} - XG_{3\phi} - 3H^2G_4 \\ & + 12H^2X(G_{4X} + XG_{4XX}) - 6HX\dot{\phi}G_{4\phi X} \\ & - 3H\dot{\phi}G_{4\phi} + H^3X\dot{\phi}(5G_{5X} + 2XG_{5XX}) \\ & - 3H^2X(3G_{5\phi} + 2XG_{5\phi X}),\end{aligned} \tag{1.96}$$

$$\begin{aligned}\mathcal{P}_0 = {} & \frac{1}{2}G_2 - X(G_{3\phi} + \ddot{\phi}G_{3X}) + (3H^2 + 2\dot{H})G_4 \\ & - 6H^2XG_{4X} - 2H\dot{X}G_{4X} - 4\dot{H}XG_{4X} \\ & - 4HX\dot{X}G_{4XX} + (\ddot{\phi} + 2H\dot{\phi})G_{4\phi} + 2XG_{4\phi\phi} \\ & + 2X(\ddot{\phi} - 2H\dot{\phi})G_{4\phi X} \\ & - X(2H^3\dot{\phi} + 2H\dot{H}\dot{\phi} + 3H^2\ddot{\phi})G_{5X} \\ & - 2H^2X^2\ddot{\phi}G_{5XX} + 2HX(\dot{X} - HX)G_{5\phi X} \\ & + \left[2(HX)^{\cdot} + 3H^2X\right]G_{5\phi} + 2HX\dot{\phi}G_{5\phi\phi},\end{aligned} \tag{1.97}$$

and

$$\mathcal{E}_k = -\frac{3}{2}\mathcal{G}_T\frac{k}{a^2}, \qquad \mathcal{P}_k = \frac{1}{2}\mathcal{F}_T\frac{k}{a^2}, \tag{1.98}$$

with time-dependent coefficients

$$\mathcal{F}_T = 2\left[G_4 - X(\ddot{\phi}G_{5X} + G_{5\phi})\right], \tag{1.99}$$

$$\mathcal{G}_T = 2\left[G_4 - 2XG_{4X} - X(H\dot{\phi}G_{5X} - G_{5\phi})\right]. \tag{1.100}$$

Unlike the case where a scalar field is minimally coupled to gravity, the spatial curvature contributes to both the first Friedmann equation and the equation of motion of the scalar field because of the non-constant non-minimal coupling functions G_4 and G_5.

To describe the late-time universe within the framework of Horndeski's theory, additional matter components such as dark matter, baryons, and radiation must be included. Assuming that this matter is minimally coupled to gravity, the corresponding background equations are given by

$$\mathcal{E}_0 + \mathcal{E}_k = -\rho\,, \tag{1.101}$$

$$\mathcal{P}_0 + \mathcal{P}_k = -P\,. \tag{1.102}$$

1.3.4 Linear perturbations

Linear perturbations around an FLRW background are crucial both for assessing the stability of cosmological models and for testing modified gravity theories against observational data. To explore these aspects, we will perform a $3+1$ Arnowitt–Deser–Misner (ADM) decomposition [386]. We can write the metric tensor as

$$ds^2 = -N^2 dt^2 + g_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \tag{1.103}$$

where N is the lapse function, $N_i = g_{ij}N^j$ is the shift vector, and g_{ij} is the 3-dimensional spatial metric. Since vector perturbations are non-dynamical in both scalar-tensor theories and Einstein gravity, they are less relevant. Thus, the analysis always focuses on scalar and tensor perturbations. Moreover, because of the general covariance, we can work in the so-called *unitary gauge* which consists in choosing the time coordinate such that the scalar field perturbation is vanishing $\delta\phi = 0$.

Therefore, we can parameterize the perturbations in the following way:

$$\begin{aligned} N &= 1 + \delta n, \qquad N_i = \mathcal{D}_i \chi, \\ g_{ij} &= a^2 e^{2\zeta}\left(\gamma_{ij} + h_{ij} + \frac{1}{2}h_{ik}h^k_j\right), \end{aligned} \tag{1.104}$$

where Latin indices are raised and lowered with γ_{ij}, is the metric of maximally symmetric spatial hypersurfaces, $\mathcal{D}_i$ is the covariant derivative compatible with γ_{ij}, and the tensor perturbation h_{ij} satisfies $h^i_i = \mathcal{D}_i h^i_j = 0$. Substituting this metric to the action and expanding it to second order in perturbations, we will obtain the general quadratic actions for tensor and scalar perturbations in open and closed universes.

Substituting the metric (1.103) to the action and expanding it to second order in perturbations, one obtains

$$S^{(2)}_{\mathrm{H}} = S^{(2)}_{\mathrm{H},S} + S^{(2)}_{\mathrm{H},T} \tag{1.105}$$

Following [385], with some manipulation the general quadratic action for tensor perturbations in open and closed universes is found to be

$$S^{(2)}_{\mathrm{H},T} = \frac{1}{8}\int dt d^3x\sqrt{\gamma}a^3\left[\mathcal{G}_T \dot{h}^2_{ij} + \frac{\mathcal{F}_T}{a^2}h^{ij}(\mathcal{D}^2 - 2k)h_{ij}\right], \tag{1.106}$$

where $\mathcal{F}_T$ and $\mathcal{G}_T$ were already defined in Eqs. (1.99) and (1.100), respectively, and $\mathcal{D}^2 \equiv \gamma^{ij}\mathcal{D}_i\mathcal{D}_j$.

Therefore, it is possible to define the propagation speed of tensor perturbations as

$$c^2_{\mathrm{H},T} \equiv \frac{\mathcal{F}_T}{\mathcal{G}_T}\,. \tag{1.107}$$

This means that the effects of the spatial curvature k can be seen only in the spatial derivative operator $\mathcal{D}^2 - 2k$. In the case of $G_4 = 1$, $G_5 = 0$, the action (1.106) reduces to the standard result known in GR (see, *e.g.*, [387]).

From Eq. (1.106) we see that ghost instabilities are absent if $\mathcal{G}_T > 0$. Gradient instabilities can be avoided by requiring that $\mathcal{F}_T/\mathcal{G}_T > 0$. Therefore, the stability conditions in open and closed universes are given by

$$\mathcal{F}_T > 0, \qquad \mathcal{G}_T > 0\,. \tag{1.108}$$

The quadratic action for scalar perturbations can be obtained as

$$S^{(2)}_{\mathrm{H},S} = \int dt d^3x \sqrt{\gamma} a^3 \Big\{ -3\mathcal{G}_T \dot{\zeta}^2 - \frac{\mathcal{F}_T}{a^2}\zeta\mathcal{D}^2\zeta + \Sigma\delta n^2 - 2\Theta_{\mathrm{H}}\delta n \frac{\mathcal{D}^2\chi}{a^2} + 2\mathcal{G}_T\dot{\zeta}\frac{\mathcal{D}^2\chi}{a^2} + 6\Theta_{\mathrm{H}}\delta n\dot{\zeta}$$
$$-2\mathcal{G}_T\delta n\frac{\mathcal{D}^2\zeta}{a^2} - 3\mathcal{F}_T\zeta^2\frac{k}{a^2} - 6\mathcal{G}_T\delta n\zeta\frac{k}{a^2} - \frac{\mathcal{G}_T}{2a^4}\left[(\mathcal{D}^2\chi)^2 - (\mathcal{D}_i\mathcal{D}^j\chi)^2\right]\Big\}, \tag{1.109}$$

where

$$\Theta_{\mathrm{H}} = \Theta_0 + \Theta_k\,, \qquad \Sigma_{\mathrm{H}} = \Sigma_0 + \Sigma_k\,, \tag{1.110}$$

with

$$\Theta_0 \equiv -\dot{\phi}XG_{3X} + 2HG_4 - 8HXG_{4X} - -8HX^2G_{4XX} + \dot{\phi}G_{4\phi} + 2X\dot{\phi}G_{4\phi X}$$
$$- H^2\dot{\phi}(5XG_{5X} + 2X^2G_{5XX}) + 2HX(3G_{5\phi} + 2XG_{5\phi X})\,, \tag{1.111}$$

$$\Theta_k \equiv -\dot{\phi}XG_{5X}\frac{k}{a^2}\,, \tag{1.112}$$

$$\Sigma_0 \equiv XG_{2X} + 2X^2G_{2XX} + 12H\dot{\phi}XG_{3X} + 6H\dot{\phi}X^2G_{3XX} - 2XG_{3\phi} - 2X^2G_{3\phi X} - 6H^2G_4$$
$$+ 6\left[H^2(7XG_{4X} + 16X^2G_{4XX} + 4X^3G_{4XXX}) - H\dot{\phi}(G_{4\phi} + 5XG_{4\phi X} + 2X^2G_{4\phi XX})\right]$$
$$+ 2H^3\dot{\phi}\left(15XG_{5X} + 13X^2G_{5XX} + 2X^3G_{5XXX}\right)$$
$$- 6H^2X(6G_{5\phi} + 9XG_{5\phi X} + 2X^2G_{5\phi XX}), \tag{1.113}$$

$$\Sigma_k \equiv 6(XG_{4X} + 2X^2G_{4XX} - XG_{5\phi} - X^2G_{5\phi X} + 2H\dot{\phi}XG_{5X} + H\dot{\phi}X^2G_{5XX})\frac{k}{a^2}\,. \tag{1.114}$$

After substituting the constraint equations coming from the variation with respect to δn and χ, the quadratic action for the scalar curvature perturbation in the unitary gauge is

$$S^{(2)}_{\mathrm{H},\zeta} = \int dt d^3x \sqrt{\gamma} a^3 \left[\mathcal{G}_S\dot{\zeta}^2 + \zeta\frac{\mathcal{F}_S}{a^2}(\mathcal{D}^2 + 3k)\zeta\right], \tag{1.115}$$

where

$$\mathcal{G}_S \equiv \frac{\mathcal{D}^2 + 3k}{\mathcal{D}^2 - (\mathcal{G}_T\Sigma_{\mathrm{H}}/\Theta^2_{\mathrm{H}})k}\mathcal{G}_T\left(\frac{\mathcal{G}_T\Sigma_{\mathrm{H}}}{\Theta^2_{\mathrm{H}}} + 3\right), \tag{1.116}$$

$$\mathcal{F}_S \equiv \frac{1}{a}\frac{d}{dt}\left[\frac{\mathcal{D}^2 + 3k}{\mathcal{D}^2 - (\mathcal{G}_T\Sigma_{\mathrm{H}}/\Theta^2_{\mathrm{H}})k}\left(\frac{a\mathcal{G}^2_T}{\Theta_{\mathrm{H}}}\right)\right] - \mathcal{F}_T + \frac{\mathcal{D}^2 + 3k}{\mathcal{D}^2 - (\mathcal{G}_T\Sigma_{\mathrm{H}}/\Theta^2_{\mathrm{H}})k}\left(\frac{\mathcal{G}^3_T}{\Theta^2_{\mathrm{H}}}\frac{k}{a^2}\right). \tag{1.117}$$

This is the general quadratic action for ζ in open and closed universes. In contrast to the case of the tensor perturbations, the coefficients can depend non-trivially on k through Θ_{H} and Σ_{H}. The squared sound speed may be defined as

$$c^2_{\mathrm{H},S} = \frac{\mathcal{F}_S}{\mathcal{G}_S}\,, \tag{1.118}$$

which is also dependent on k in general.

The propagation speed of the tensor and scalar perturbations must be positive, *i.e.*, $c^2_{\mathrm{H},T} > 0$ and $c^2_{\mathrm{H},S} > 0$, because otherwise, each perturbation mode exhibits exponential growth. This is called the *gradient instability*.

1.3.5 Beyond Horndeski theories

The generalizations of Horndeski gravity are based on the fact that degenerate systems are free of the Ostrogradsky ghost and hence are healthy despite the higher-order Euler-Lagrange equations.

The degeneracy condition is a constraint equation, contained inside the field equations, allowing for a reduction of the order of the equations.

Beyond Horndeski and Degenerate Higher-Order Scalar-Tensor (DHOST) theories represent extensions of Horndeski gravity that allow for higher-order derivatives in the field equations while avoiding Ostrogradsky instabilities. These theories were developed to provide even greater flexibility in constructing viable cosmological models [316, 317, 388]. These theories were designed to preserve the key features of Horndeski gravity while complying with new observational constraints [321].

The first example of DHOST theories beyond Horndeski [389] was obtained by performing an invertible disformal transformation [390]

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + D(\phi, X)\nabla_\mu\phi\nabla_\nu\phi\,, \tag{1.119}$$

which constitutes the generalization of the more familiar conformal transformation, $\tilde{g}_{\mu\nu} = C(\phi)g_{\mu\nu}$. Additionally, the disformal coefficient must satisfy the following condition in order to be invertible:

$$C(C - XC_X + 2X^2 D_X) \neq 0. \tag{1.120}$$

In the case of both C and D depending only on ϕ, the above disformal transformation maps Horndeski theory in itself [391]. However, in general, because of the presence of scalar field derivatives, the transformed theory possesses higher-order field equations. Nevertheless, the transformation preserves the number of degrees of freedom since it is an invertible field redefinition [392–394]. Therefore, this transformation provides a wider class of healthy scalar-tensor theories than the Horndeski class[316, 395–399].

An extension of Horndeski's $\mathcal{L}_4$ Lagrangian given by

$$\mathcal{L}_{4,\,\mathrm{DHOST}} = f(\phi, X)R + \sum_{i=1}^{5} A_i(\phi, X) I_i, \tag{1.121}$$

where

$$\begin{aligned} &I_1 = (\nabla\nabla\phi)^2\,, \quad I_2 = (\Box\phi)^2\,, \quad I_3 = \Box\phi\,, \nabla^\mu\phi\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi\,, \\ &I_4 = \nabla^\mu\phi\nabla_\mu\nabla_\alpha\phi\nabla^\alpha\nabla^\nu\phi\nabla_\nu\phi\,, \quad I_5 = (\nabla^\mu\phi\nabla^\nu\phi\nabla_\mu\nabla_\nu\phi)^2. \end{aligned} \tag{1.122}$$

These five scalars constitute all the possible quadratic terms in second derivatives of ϕ. Horndeski theory is the special case of $A_2 = -A_1 = f_X$ and $A_3 = A_4 = A_5 = 0$.

It is possible to verify that the degeneracy conditions are given by three equations, usually leaving three arbitrary functions.

The degenerate theories whose Lagrangian is of the form (1.121) are called quadratic DHOST theories. They are classified into several subclasses [316, 396, 397]. The most important is the so-called *class Ia*, which contains Horndeski theory as a special case and is characterized by

$$A_2 = -A_1\,, \tag{1.123}$$

$$\begin{aligned} A_4 = \frac{1}{2(f+2XA_1)^2}\big[&8XA_1^3 + (3f+16Xf_X)A_1^2 - X^2 f A_3^2 + 2X(4Xf_X - 3f)A_1A_3 \\ &+ 2f_X(3f+4Xf_X)A_1 + 2f(Xf_X - f)A_3 + 3ff_X^2\big]\,, \end{aligned} \tag{1.124}$$

$$A_5 = -\frac{(f_X + A_1 + XA_3)\left(2fA_3 - f_XA_1 - A_1^2 + 3XA_1A_3\right)}{2(f+2XA_1)^2}\,, \tag{1.125}$$

with $f + 2XA_1 \neq 0$. The arbitrary functions are thus taken to be f, A_1, and A_3. It has been shown that cosmology within this class of DHOST theories is described by higher-order equations of motion which can be simplified to a second-order system for both the scale factor and the scalar field [322, 400].

A key feature of class Ia DHOST theories is that all their Lagrangians can be transformed into a Horndeski Lagrangian via a disformal transformation (1.119) [397]. In other words, two of the three functions of ϕ and X in the quadratic DHOST sector can be eliminated by utilizing the two functions, $C(\phi, X)$ and $D(\phi, X)$, in the disformal transformation, resulting in a 'Horndeski frame' with a single function $G_4(\phi, X)$ of quadratic order. It is important to note that class Ia DHOST theories, when coupled to minimally coupled matter, are equivalent to Horndeski with disformally coupled matter, but not to Horndeski with minimally coupled matter.

It has been shown that subclasses other than class Ia are not viable. In these cases, the gradient terms in the quadratic actions for scalar and tensor cosmological perturbations have opposite signs (*i.e.*, one of the two modes becomes unstable), or tensor perturbations become non-dynamical [399]. Consequently, only DHOST theories that are disformally related to Horndeski remain viable.

In this subsection, we have simplified the discussion by focusing on DHOST theories whose Lagrangian is a quadratic polynomial in $\nabla_\mu\nabla_\nu\phi$. A similar approach can be applied to construct cubic DHOST theories as an extension of Horndeski's $\mathcal{L}_5$ Lagrangian (*i.e.*, DHOST theories with a Lagrangian that is a cubic polynomial in $\nabla_\mu\nabla_\nu\phi$ [398], although their classification is significantly more complex. Cubic DHOST theories that are disformally disconnected from Horndeski also exhibit gradient instabilities in either tensor or scalar modes.

A generalized class of DHOST theories can be generated by going beyond the assumption of polynomial Lagrangians, using a non-invertible disformal transformation applied to non-degenerate theories. Recalling Eq. (1.120), this transformation is characterized by $D(\phi, X) = D(\phi) + C(\phi, X)/2X$, where $C(\phi, X)$ and $F(\phi)$ are arbitrary functions. This transformation closely resembles the field redefinition used in mimetic gravity [401]. The key concept is that a non-invertible field redefinition can alter the number of dynamical degrees of freedom. The resulting DHOST theories are generally not disformally linked to Horndeski. However, these "mimetic DHOST" models often face gradient instabilities in tensor or scalar modes [402, 403].

1.3.6 Constraint on the gravitational wave speed

The measurement of the gravitational wave speed $c_{\rm GW} \equiv c_T$, along with the associated electromagnetic counterpart, serves as a test for falsifying theories of gravity [247, 320].

The first simultaneous detection of gravitational waves and associated γ-ray burst was provided by the multi-messenger observation GW170817/GRB170817A [152, 153]. It represents the first observational constraint on $c_{\rm GW}$ both for scalar-tensor theories and other types of modified gravity. This event places a limit on the discrepancy between $c_{\rm GW}$ and the speed of light, which is

$$-3\times 10^{-15} < c_{\rm GW} - 1 < 7\times 10^{-16}. \tag{1.126}$$

This constraint drives the search for a viable subclass of Horndeski and DHOST theories that serve as alternatives to dark energy and satisfy $c_{\rm GW} = 1$ [251, 320].

In Horndeski theory, the propagation speed of gravitational waves $c_{{\rm H},T}$ is given by Eq. (1.107):

$$c_{\rm GW}^2 = \frac{G_4 - X(\ddot{\phi}G_{5X} + G_{5\phi})}{G_4 - 2XG_{4X} - X(H\dot{\phi}G_{5X} - G_{5\phi})}\,. \tag{1.127}$$

Requiring the above quantity to be equal to the unity irrespective of the cosmological background implies that

$$G_{4X} = 0, \quad G_5 = 0. \tag{1.128}$$

Therefore, the viable subclass within Horndeski is associated with the Lagrangian

$$\mathcal{L} = G_2(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi)R. \tag{1.129}$$

The above takes is dubbed *viable Horndeski theory*. For instance, this restriction excludes the Gauss-Bonnet term coupled with a scalar field.

Regarding DHOST theories, one can observe that any term in the cubic DHOST Lagrangians makes the theory nonviable. From performing the quadratic action for tensor perturbations, the propagation speed of gravitational waves is

$$c_{\rm GW}^2 = \frac{f}{f + 2XA_1}, \tag{1.130}$$

which requires $A_1 = 0$. Therefore, the viable subclass in DHOST theories is restricted to

$$A_2 = -A_1 = 0, \tag{1.131}$$

$$A_4 = \frac{1}{2f}[-X^2A_3^2 + 2(Xf_X - f)A_3 + 3f_X^2], \tag{1.132}$$

$$A_5 = -\frac{A_3\,(f_X + XA_3)}{f}. \tag{1.133}$$

Thus, there are two free functions $f(\phi, X)$ and $A_3(\phi, X)$, additionally to the lower order Horndeski terms $G_2(\phi, X)$ and $G_3(\phi, X)$. However, to avoid the gravitational wave decay into ϕ-dark energy preventing from their observation [404], one has also to require that $A_3 = 0$. This means that the final viable DHOST theory reads

$$A_4 = \frac{3f_X^2}{2f}, \quad A_1 = A_2 = A_3 = A_5 = 0 \tag{1.134}$$

which generally does not belong to Horndeski.

An important comment we must stress is that the alternative description of the dark energy provided by scalar-tensor theories is in general assumed to be valid on much higher frequencies than the one observed by LIGO ($\sim 100\,\mathrm{Hz} \sim 10^{-13}$ eV). In this sense, gravity theories at much higher energies than this remain actually unconstrained. Therefore, the gravitational waves constraint does not affect modified gravity in the early universe [405–407].

2 Effective fluid formalism

The multi-messenger event GW170817/GRB170817A, observed from a neutron star binary merger [26, 152, 153], imposes constraints on Horndeski gravity, limiting it to the so-called 'viable' subclass (1.129) characterized by $G_5 = G_{4X} = 0$, ensuring that the speed of gravitational waves matches $c = 1$. This subclass avoids instabilities and allows for an Einstein frame formulation [247, 320, 379, 408, 409].

Since the simplicity of the reduced Horndeski theory, we can write down the field equations for a general background:

$$\begin{aligned} & G_4\, G_{\mu\nu} - \nabla_\mu \nabla_\nu G_4 + \left[\Box G_4 - \frac{G_2}{2} - \frac{1}{2}\, \nabla_\lambda \phi \nabla^\lambda G_3 \right] g_{\mu\nu} \\ & + \frac{1}{2}\left[G_{3X}\, \Box\phi - G_{2X} \right] \nabla_\mu \phi \nabla_\nu \phi + \nabla_{(\mu}\phi \nabla_{\nu)} G_3 = T^{(\mathrm{m})}_{\mu\nu}\,, \end{aligned} \tag{2.1}$$

$$\begin{aligned} & G_{4,\phi} R + G_{2,\phi} + G_{2X} \Box\phi + \nabla_\lambda \phi \nabla^\lambda G_{2X} \\ & - G_{3X} (\Box\phi)^2 - \nabla_\lambda \phi \nabla^c G_{3X} \Box\phi - G_{3X} \nabla^\lambda \phi \Box \nabla_c \phi \\ & + G_{3X} R_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi - \Box G_3 - G_{3,\phi} \Box\phi = 0\,. \end{aligned} \tag{2.2}$$

recalling that $G_{\mu\nu}$ is the Einstein tensor and $T^{(\mathrm{m})}_{\mu\nu} \equiv -\frac{1}{\sqrt{-g}} \frac{\delta S^{(\mathrm{m})}}{\delta g^{\mu\nu}}$ is the matter stress-energy tensor.

Equation (2.1) can always be written as the effective Einstein equation

$$G_{\mu\nu} = T^{(\mathrm{eff})}_{\mu\nu} + T^{(\mathrm{m})}_{\mu\nu} / G_4\,, \tag{2.3}$$

where

$$T^{(\mathrm{eff})}_{\mu\nu} = T^{(2)}_{\mu\nu} + T^{(3)}_{\mu\nu} + T^{(4)}_{\mu\nu} \tag{2.4}$$

is the effective stress-energy tensor containing all the deviations from GR,

$$T^{(2)}_{\mu\nu} = \frac{1}{2G_4} \left(G_{2X} \nabla_\mu \phi \nabla_\nu \phi + G_2\, g_{\mu\nu} \right)\,, \tag{2.5}$$

$$T^{(3)}_{\mu\nu} = \frac{1}{2G_4} \left(G_{3X} \nabla_\lambda X \nabla^\lambda \phi - 2X G_{3\phi} \right) g_{\mu\nu}$$

$$-\frac{1}{2G_4}\left(2G_{3\phi}+G_{3X}\Box\phi\right)\nabla_\mu\phi\nabla_\nu\phi-\frac{G_{3X}}{G_4}\nabla_{(\mu}X\nabla_{\nu)}\phi\,, \tag{2.6}$$

$$T^{(4)}_{\mu\nu}=\frac{G_{4\phi}}{G_4}(\nabla_\mu\nabla_\nu\phi-g_{\mu\nu}\Box\phi)+\frac{G_{4\phi\phi}}{G_4}(\nabla_\mu\phi\nabla_\nu\phi+2X\,g_{\mu\nu})\,. \tag{2.7}$$

The equation of motion for the scalar field can be written as

$$\mathcal{J}^{(2)}+\mathcal{J}^{(3)}+\mathcal{J}^{(4)}=0\,, \tag{2.8}$$

where

$$\mathcal{J}^{(2)}=(G_{2X}g^{\mu\nu}-G_{2XX}\nabla^\mu\phi\nabla^\nu\phi)\,\nabla_\mu\nabla_\nu\phi+G_{2\phi}-2XG_{2\phi X}\,, \tag{2.9}$$

$$\begin{aligned}\mathcal{J}^{(3)}=&\,G_{3X}R_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi-2\left(G_{3X}g^{\mu\nu}g^{cd}-G_{3XX}\nabla^\mu\phi\nabla^\nu\phi\,g^{cd}\right)\nabla_{[a|}\nabla_\nu\phi\nabla_{|c]}\nabla_d\phi\\&-2\left[\left(G_{3\phi}-XG_{3\phi X}\right)g^{\mu\nu}-G_{3\phi X}\nabla^\mu\phi\nabla^\nu\phi\right]\nabla_\mu\nabla_\nu\phi+2XG_{3XX},\end{aligned} \tag{2.10}$$

$$\mathcal{J}^{(4)}=G_{4\phi}R\,. \tag{2.11}$$

To provide a clearer understanding, we begin by focusing on a specific subclass of Horndeski gravity *in vacuo* and will later generalize the results to the full Horndeski theory with matter couplings.

Given the numerous free functions and terms in the Horndeski action, comprehending its physical implications can be challenging, as much of the existing literature is highly formal. How can one classify the physical deviations from Einstein gravity inherent to these theories? Here, we present a new perspective on this class of theories: treating the field equations as effective Einstein equations, where non-Einsteinian terms take the form of a *dissipative* effective fluid when shifted to the right-hand side [410], a concept familiar from less general scalar-tensor frameworks [411]. This effective fluid approach allows for a classification of Horndeski gravity based on fluid characteristics: restricting to a Newtonian fluid—meaning the viscous stresses depend only on first derivatives of the fluid's 4-velocity—narrows the viable Horndeski subclasses. More general theories, however, correspond to exotic, non-Newtonian effective fluids. Here, *(non-)Newtonian* follows standard fluid mechanics terminology; the fluids we consider are always relativistic. In classical non-relativistic, three-dimensional fluid mechanics, ordinary fluids are Newtonian, whereas more complex behaviors are classified as non-Newtonian. As the literature lacks alternative physical classifications for Horndeski theories (aside from the distinction of whether gravitational waves propagate at light speed), the nature of the effective fluid provides a useful framework. Other physical interpretations of Horndeski gravity are not generally addressed in the relevant, extensive body of work.

As demonstrated below, the general Horndeski theory includes only two subclasses, referred to as "linear Horndeski" classes, which feature an effective fluid stress-energy tensor that is linear in the gradient of the four-velocity. These subclasses fall under the viable Horndeski class, characterized by $G_3 = G_{4\phi}\ln X$ and $G_3 = 0$, respectively. This finding is relevant within the framework of Eckart's (first-order) thermodynamics of relativistic viscous fluids, where out-of-equilibrium contributions to the effective $T_{\mu\nu}$ depend linearly on the temperature gradient, chemical potential, and four-velocity. This approach allows for constraints on the functional form of G_2, which is associated with the equilibrium pressure of the effective fluid.

The thermodynamic analogy with a first-order viscous fluid draws inspiration from Jacobson's concept, which views modified gravity as a non-equilibrium state in the context of a thermodynamic framework for gravitational theories [412, 413]. In this view, gravity emerges as a result of thermal processes, with classical General Relativity (GR) representing an equilibrium state, while modified gravity theories, such as Horndeski's, correspond to non-equilibrium, or excited, states. It is important to note that, although this perspective is similar in spirit, first-order thermodynamics in scalar-tensor theories differs fundamentally from Jacobson's thermodynamics of spacetime. For a comprehensive discussion of this topic see [414].

The analogy between alternative gravity theories and viscous fluids has been expanded in previous studies [4, 415–418].

In this chapter, we derive the heat current density, the *modified gravity temperature* associated with the effective fluid (i.e., the effective temperature of the excited state), and the viscosity coefficients by analyzing a first-order effective viscous fluid with zero chemical potential within the Eckart frame. The first linear Horndeski class features a unique temperature expression, whereas the temperature associated with the second class depends on the function $G_2(\phi, X)$ or, alternatively, on how the shear viscosity relates to the effective equilibrium pressure. Tables 2.1 and 2.2 present a summary of these findings.

2.1 Horndeski effective fluids

The stress-energy tensor of an imperfect fluid has the well-known form

$$\begin{aligned} T^{\mu\nu} &= \rho u^\mu u^\nu + q^\mu u^\nu + q^\nu u^\mu + \Pi^{\mu\nu} \\ &= \rho u^\mu u^\nu + P h^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu} \,, \end{aligned} \tag{2.12}$$

remembering that u^μ represents the fluid's 4-velocity, satisfying $u_\mu u^\mu = -1$, and $h_{\mu\nu} \equiv g_{\mu\nu} + u_\mu u_\nu$ denotes the projection tensor onto the 3-space orthogonal to u^μ ($h_{\mu\mu} u^\nu = 0$). The energy density is defined as $\rho = T^{\mu\nu} u_\mu u_\nu$, while the isotropic pressure is $P = \frac{1}{3} T^{\mu\nu} h_{\mu\nu}$. The heat flux density is given by $q^\mu = -T^{\mu\nu} u_\nu h^\mu{}_\lambda$, and the stress tensor is expressed as $\Pi^{\mu\nu} = h^\mu{}_\lambda h^\nu{}_\rho T^{\lambda\rho} = P h^{\mu\nu} + \pi^{\mu\nu}$, where $\pi^{\mu\nu}$ is the traceless part describing anisotropic stresses.

The viscous pressure and anisotropic stresses are governed by constitutive relations involving the expansion scalar $\Theta \equiv \nabla_\mu u^\mu$ and the traceless shear tensor $\sigma^{\mu\nu} \equiv \frac{1}{2}(h^{\mu\lambda}\nabla_\lambda u^\nu + h^{\nu\lambda}\nabla_\lambda u^\mu) - \frac{1}{3}\Theta h^{\mu\nu}$. Using the decomposition $\nabla_\mu = h_\mu{}^\lambda \nabla_\lambda - u_\mu u^\lambda \nabla_\lambda$, we find $\nabla_\nu u_\mu = \sigma_{\mu\nu} + \frac{\Theta}{3} h_{\mu\nu} + \omega_{\mu\nu} - \dot{u}_\mu u_\nu$, where $\omega_{\mu\nu} \equiv \nabla_{[\nu} u_{\mu]}$ is the vorticity tensor, and $\dot{u}^\mu \equiv u^\lambda \nabla_\lambda u^\mu$ is the 4-acceleration of the fluid.

In both non-relativistic and relativistic Newtonian fluids, the constitutive relations are *linear in the 4-velocity gradient* [419], given by

$$\pi^{\mu\nu} = -2\eta\, \sigma^{\mu\nu}, \qquad P = \bar{P} - \zeta\, \Theta, \tag{2.13}$$

where $\bar{P}$ represents the pressure of the inviscid fluid, and η and ζ are the shear and bulk viscosity coefficients, respectively. Consequently, the stress-energy tensor of an imperfect fluid takes the form

$$T^{\mu\nu} = \rho u^\mu u^\nu + \left(\bar{P} - \zeta\Theta\right) h^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu - 2\eta\sigma^{\mu\nu} . \tag{2.14}$$

In general, the requirement of linearity in the first derivatives of the fluid's 4-velocity implies a viscous contribution to the energy density as well. This can introduce an additional "constitutive relation" of the form

$$\rho = \bar{\rho} - \xi\,\Theta, \tag{2.15}$$

where $\bar{\rho}$ is the density in the inviscid limit, and ξ denotes an additional viscosity coefficient. Physically, this term may represent resistance to compressive or expansive forces, with the possibility of a connection between this term and the bulk viscosity coefficient.

The aim here is to classify Horndeski theories based on the physical characteristics (Newtonian or non-Newtonian) of their effective fluid equivalents, along with their thermodynamic properties.

The 4-velocity of the effective Horndeski fluid is given by [417]

$$u^\mu \equiv \frac{\nabla^\mu \phi}{\sqrt{2X}}\,, \tag{2.16}$$

under the assumption that the gradient of the scalar field is timelike, such that $\nabla_\mu\phi\nabla^\mu\phi < 0$. This definition enables the rewriting of the derivatives of ϕ and X in terms of the kinematic quantities associated with the effective fluid:

$$\nabla_\mu \phi = \sqrt{2X}\, u_\mu\,, \qquad \nabla_\mu X = -\dot{X}\, u_\mu - 2X\, \dot{u}_\mu\,, \tag{2.17}$$

$$\nabla_\mu\nabla_\nu\phi = \sqrt{2X}\,\nabla_\mu u_\nu - \frac{\dot{X}}{\sqrt{2X}}\, u_\mu u_\nu - \sqrt{2X}\dot{u}_\mu u_\nu\,, \tag{2.18}$$

where $\dot{X} \equiv u^c \nabla_c X$. Then, the effective stress-energy tensor associated with viable Horndeski gravity is

$$\begin{aligned} T^{(\mathrm{eff})}_{\mu\nu} = & \left[\frac{2XG_{2X} - G_2 - 2XG_{3\phi}}{2G_4} + \frac{\sqrt{2X}\left(G_{4\phi} - XG_{3X}\right)}{G_4}\,\Theta\right] u_\mu u_\nu \\ & + \left[\frac{G_2 + 4XG_{4\phi\phi} - 2XG_{3\phi}}{2G_4} - \frac{\left(G_{4\phi} - XG_{3X}\right)}{\sqrt{2X}\, G_4}\,\dot{X} - \frac{2\sqrt{2X}\, G_{4\phi}\Theta}{3G_4}\right] h_{\mu\nu} \\ & - \frac{2\sqrt{2X}\left(G_{4\phi} - XG_{3X}\right)}{G_4}\,\dot{u}_{(\mu}u_{\nu)} + \frac{\sqrt{2X}\, G_{4\phi}}{G_4}\,\sigma_{\mu\nu}\,, \end{aligned} \tag{2.19}$$

and the associated effective fluid quantities are

$$\rho = \frac{1}{2G_4}\left(2XG_{2X} - G_2 - 2XG_{3\phi}\right) + \frac{\sqrt{2X}}{G_4}\left(G_{4\phi} - XG_{3X}\right)\Theta\,, \tag{2.20}$$

$$P = \frac{1}{2G_4}\left(G_2 - 2XG_{3\phi} + 4XG_{4\phi\phi}\right) - \frac{\left(G_{4\phi} - XG_{3X}\right)}{G_4\sqrt{2X}}\,\dot{X} - \frac{2G_{4\phi}}{3G_4}\sqrt{2X}\,\Theta\,, \tag{2.21}$$

$$\eta = -\frac{G_{4\phi}}{2G_4}\sqrt{2X}\,, \qquad q^\mu = -\frac{\sqrt{2X}}{G_4}\left(G_{4\phi} - XG_{3X}\right)\dot{u}^\mu\,, \tag{2.22}$$

$$\pi_{\mu\nu} = \epsilon \frac{\sqrt{2X}\, G_{4\phi}}{G_4} \sigma_{\mu\nu} \,. \tag{2.23}$$

The stress-energy tensor features anisotropic stresses proportional to the shear tensor $\sigma_{\mu\nu}$, as well as a heat flux density proportional to the fluid's four-acceleration. However, from the above equations, we cannot derive all the constitutive relations for the effective fluid of viable Horndeski theories. In Eq. (2.21), $\dot{X}$ does not correspond to a purely kinematic quantity. Fortunately, the scalar field equation of motion allows us to re-express $\dot{X}$ in terms of kinematic quantities.

There are two possible scenarios for satisfying the condition that the total pressure depends *linearly* on $\nabla_\nu u_\mu$, which corresponds to a Newtonian effective fluid:

1. The total pressure is independent of $\dot{X}$, which implies
$$G_{4\phi} - XG_{3X} = 0 \quad \Rightarrow \quad G_3 = G_{4\phi} \ln X \,. \tag{2.24}$$
2. The scalar $\dot{X}$ depends solely on the scalar field and its kinetic term and is linear with respect to the expansion scalar,
$$\dot{X} = F_1\,(\phi, X) + F_2\,(\phi, X)\;\Theta \,, \tag{2.25}$$
where the functions F_1 and F_2 contribute to the isotropic pressure of a perfect fluid and the viscous pressure, respectively. These functions, $F_{1,2}$, can be determined from the scalar field equation of motion. Alternatively, $\Box\phi$ can be used by assuming $\Box\phi = \tilde{F}_1(\phi, X) + \tilde{F}_2(\phi, X)\,\Theta$. In both cases, $\Box\phi$ or $\dot{X}$ is a function of (ϕ, X) that is linear in Θ. Without loss of generality, we proceed to work with $\dot{X}$.

In the first case, the form of G_3 is explicitly specified. However, in the second case, $\dot{X}$ must be expressed in terms of kinematic quantities to ensure linearity in $\nabla_\mu u_\nu$. To achieve this, we utilize the field equations:

$$R_{\mu\nu} = T_{\mu\nu} - \frac{T}{2}\, g_{\mu\nu}, \qquad R = -T, \tag{2.26}$$

where $T = g_{\mu\nu}T^{\mu\nu}$ represents the trace of the effective stress-energy tensor. The equation of motion for the scalar field, reformulated using kinematic quantities, becomes

$$\begin{aligned}
&-\frac{4}{3}X\Theta^2 G_{3X} + 4X\sigma^2 G_{3X} - 4\dot{u}^2 X\,(G_{3X} + XG_{3XX}) - 2\dot{X}\Theta\,(G_{3X} + XG_{3XX})\\
&+\frac{\Theta\sqrt{2X}}{G_4}\left[3G_{4\phi}^2 - 2XG_{4\phi}G_{3X} - X^2G_{3X}^2 + G_4\left(G_{2X} - 2G_{3\phi} + 2XG_{3\phi X}\right)\right]\\
&+\frac{\dot{X}}{\sqrt{2X}G_4}\left\{3G_{4\phi}^2 - 6XG_{4\phi}G_{3X} + 3X^2G_{3X}^2\right.\\
&\left.+G_4\left[G_{2X} + 2XG_{2XX} - 2\left(G_{3\phi} + XG_{3\phi X}\right)\right]\right\}\\
&+\frac{1}{G_4}\left[G_2\left(-2G_{4\phi} + XG_{3X}\right) + G_4G_{2\phi} + X^2G_{3X}\left(6G_{4\phi\phi} + G_{2X} - 4G_{3\phi}\right)\right]\\
&+\frac{X}{G_4}\left[G_{4\phi}\left(-6G_{4\phi\phi} + G_{2X} + 2G_{3\phi}\right) + 2G_4\left(-G_{2\phi X} + G_{3\phi\phi}\right)\right] = 0\,,
\end{aligned} \tag{2.27}$$

which can be seen as a linear equation for $\dot{X}$ admitting the algebraic solution

$$\dot{X} = \frac{A(\phi, X) + B(\phi, X)\,\Theta + C(\phi, X)\,\Theta^2 + D(\phi, X)\,\sigma^2 + E(\phi, X)\,\dot{u}^2}{H(\phi, X) + I(\phi, X)\,\Theta}\,, \tag{2.28}$$

where

$$\begin{aligned} A(\phi, X) =& \sqrt{2X}\left[G_2\left(2G_{4\phi} - XG_{3X}\right) - G_4 G_{2\phi}\right. \\ & \left.-X^2 G_{3X}\left(6G_{4\phi\phi} + G_{2X} - 4G_{3\phi}\right)\right] \\ & - \sqrt{2X}\, X\left[G_{4\phi}\left(-6G_{4\phi\phi} + G_{2X} + 2G_{3\phi}\right)\right. \\ & \left.+2G_4\left(-G_{2\phi X} + G_{3\phi\phi}\right)\right]\,, \end{aligned} \tag{2.29}$$

$$\begin{aligned} B(\phi, X) =& 2X\left[3G_{4\phi}^2 - 2XG_{4\phi}G_{3X} - X^2 G_{3X}^2\right. \\ & \left.+G_4\left(G_{2X} - 2G_{3\phi} + 2XG_{3\phi X}\right)\right]\,, \end{aligned} \tag{2.30}$$

$$C(\phi, X) = \frac{4}{3}\sqrt{2X}\, XG_4G_{3X}\,, \tag{2.31}$$

$$D(\phi, X) = -4\sqrt{2X}\, XG_4G_{3X}\,, \tag{2.32}$$

$$E(\phi, X) = 4\sqrt{2X}\, XG_4(G_{3X} + XG_{3XX})\,, \tag{2.33}$$

$$\begin{aligned} H(\phi, X) =& 3G_{4\phi}^2 - 6XG_{4\phi}G_{3X} + 3X^2G_{3X}^2 + G_4[G_{2X} + 2XG_{2XX} \\ & - 2(G_{3\phi} + XG_{3\phi X})]\,, \end{aligned} \tag{2.34}$$

$$I(\phi, X) = -2\sqrt{2X}G_4(G_{3X} + XG_{3XX})\,, \tag{2.35}$$

and with $\sigma^2 \equiv \frac{1}{2}\,\sigma_{\mu\nu}\sigma^{\mu\nu}$, $\dot{u}^2 \equiv \dot{u}_\mu\dot{u}^\mu$.

As shown in Eq. (2.28), the exotic, non-Newtonian properties of the effective Horndeski fluid are encapsulated in $\dot{X}$ presence, which in turn influences the total pressure of the fluid. Notably, the first constitutive relation from (2.13) is automatically guaranteed.

Substituting Eq. (2.25) into Eq. (2.28) yields the system

$$F_1\, H = A\,, \tag{2.36}$$

$$F_1\, I + F_2\, H = B\,, \tag{2.37}$$

$$(F_2\, I - C)\,\Theta^2 - D\,\sigma^2 - E\,\dot{u}^2 = 0\,. \tag{2.38}$$

The first two equations yield the relatively simple expressions $F_1 = A/H$ and $F_2 = (BH - AI)/H^2$, while the third equation is a non-linear, second-order differential equation for ϕ. Since this equation does not derive from an action principle, it is generally incompatible with the second-order field equation for ϕ and thus cannot be enforced. For a similar reason, we do not consider the case $\dot{X} = 0$, which would imply $\nabla_\mu\phi\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi = 0$.

To self-consistently ensure that the effective fluid exhibits Newtonian behavior, it is necessary to impose $G_{3X} = 0$. This condition restricts G_3 to be a function of ϕ only, leading to $\mathcal{L}_3 = -G_3(\phi)\,\Box\phi$.

By integrating by parts, this term can be incorporated into G_2, effectively rewriting $-G_3(\phi)\,\Box\phi$ as $2XG_{3\phi}$ plus a total divergence in the action.

Then, the ϕ-equation of motion in terms of kinematic quantities yields

$$\begin{aligned} F_1(\phi, X) =& \frac{\sqrt{2X}\left[2G_2G_{4\phi} + XG_{4\phi}\left(6G_{4\phi\phi} - G_{2X}\right)\right]}{3G_{4\phi}^2 + G_4\left(G_{2X} + 2XG_{2XX}\right)} \\ &- \frac{\sqrt{2X}\,G_4\left(G_{2\phi} - 2XG_{2\phi X}\right)}{3G_{4\phi}^2 + G_4\left(G_{2X} + 2XG_{2XX}\right)}\,, \end{aligned} \tag{2.39}$$

$$F_2(\phi, X) = -\frac{2X\left(3G_{4\phi}^2 + G_4G_{2X}\right)}{3G_{4\phi}^2 + G_4\left(G_{2X} + 2XG_{2XX}\right)}\,. \tag{2.40}$$

In this scenario, Eq. (2.25) does not serve as an additional equation but instead coincides with the equation of motion for ϕ.

To summarize, the requirement that the effective fluid exhibits linearity in the gradient of its 4-velocity restricts Horndeski gravity to only two viable classes. Either

$$\mathcal{L} = G_4(\phi)R + G_2(\phi, X) - G_{4\phi}\ln X\,\Box\phi\,, \tag{2.41}$$

which corresponds to an effective fluid with

$$\rho = \frac{1}{2G_4}\left(2XG_{2X} - G_2 - 2XG_{4\phi\phi}\ln X\right), \tag{2.42}$$

$$\bar{P} = \frac{1}{2G_4}\left[G_2 + 2XG_{4\phi\phi}\left(2 - \ln X\right)\right]\,, \tag{2.43}$$

$$\eta = -\frac{1}{2}\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,, \qquad \zeta = \frac{2}{3}\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,, \tag{2.44}$$

$$\xi = 0\,, \qquad q^\mu = 0\,, \tag{2.45}$$

or else

$$\mathcal{L} = G_4(\phi)R + G_2(\phi, X)\,, \tag{2.46}$$

which is instead characterized by

$$\rho = \frac{1}{2G_4}\left(2XG_{2X} - G_2\right), \tag{2.47}$$

$$\bar{P} = -\frac{G_{4\phi}^2\left(G_2 - 2XG_{2X}\right)}{2G_4\left[3G_{4\phi}^2 + G_4\left(G_{2X} + 2XG_{2XX}\right)\right]} + \frac{\left(G_2 + 4XG_{4\phi\phi}\right)\left(G_{2X} + 2XG_{2XX}\right)}{2\left[3G_{4\phi}^2 + G_4\left(G_{2X} + 2XG_{2XX}\right)\right]}$$

$$+\frac{G_{4\phi}\left(G_{2\phi}-2XG_{2\phi X}\right)}{3G_{4\phi}^2+G_4\left(G_{2X}+2XG_{2XX}\right)}, \tag{2.48}$$

$$\eta=-\frac{1}{2}\sqrt{2X}\,\frac{G_{4\phi}}{G_4}, \qquad \xi=-\sqrt{2X}\,\frac{G_{4\phi}}{G_4}, \tag{2.49}$$

$$\zeta=-\frac{\sqrt{2X}\,G_{4\phi}\left[3G_{4\phi}^2+G_4\left(G_{2X}-4XG_{2XX}\right)\right]}{3G_4\left[3G_{4\phi}^2+G_4\left(G_{2X}+2XG_{2XX}\right)\right]}, \tag{2.50}$$

$$q^\mu=-\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,\dot{u}^\mu\,. \tag{2.51}$$

Both models reduce to General Relativity when $G_4=1$. These represent two distinct classes of Horndeski gravity that are closed under disformal transformations, meaning they are separate and cannot transform into one another.

In any scenario other than these two cases, Horndeski gravity can be reformulated as an effective fluid governed by the linear constitutive equations

$$\rho=\bar{\rho}-\xi\Theta, \qquad \pi_{\mu\nu}=-2\eta\,\sigma_{\mu\nu}, \qquad q^\mu=\xi\,\dot{u}^\mu \tag{2.52}$$

and a non-Newtonian constitutive equation for the pressure, which, by applying Eqs. (2.28)–(2.35) in Eq. (2.21), can be parametrized as

$$P=\frac{\bar{P}_1-\zeta_1\,\Theta-\zeta_2\,\Theta^2-\zeta_3\,\sigma^2-\zeta_4\,\dot{u}^2}{\bar{P}_2-\zeta_5\,\Theta}. \tag{2.53}$$

For small gradients of velocity, $\nabla_\mu u_\nu$, the above constitutive equation reduces, to first order, to an effective viable Horndeski fluid with Newtonian characteristics.

To conclude this part, let us delve into some additional useful comments.

Only the first Horndeski class can admit a non-dynamical scalar field, *i.e.*, an extended cuscuton model [420] (corresponding to the subclass $G_3=G_{4\phi}\ln X$ which implies $f_3=0$ in Ref. [421]).

In the second Horndeski class, the denominators of Eqs. (2.48) and (2.50) vanish for theories with Lagrangian density

$$\mathcal{L}=G_4R+f_1(\phi)+f_2(\phi)\sqrt{2X}-\frac{3G_{4\phi}^2}{G_4}X\,, \tag{2.54}$$

which automatically excludes a non-dynamical scalar field, *i.e.*, the extended cuscuton model (corresponding to the subclass $G_3=0$, which implies $f_3=-2f_{4\phi}$ in Ref. [421]) and, in particular, pathological $\omega=-3/2$ Brans–Dicke gravity which corresponds to $f_1=f_2=0$ and $G_4=\phi$, as well as cuscuton gravity [422]. In this case, we cannot use the scalar field equation of motion to rewrite $\dot{X}$ (or $\Box\phi$) in terms of kinematic quantities because the corresponding multiplicative factor in Eq. (2.27) vanishes identically. Therefore, one cannot write Eq. (2.25).

The full action of the extended cuscuton model can be obtained by requiring the coefficients (2.34) and (2.35) to vanish. Moreover, the condition $G_{3X}+X\,G_{3XX}=0$ corresponds to the special case in which the viscous contribution to the pressure is a finite sum of terms at most quadratic in the 4-velocity gradient.

For the second linear Horndeski class, it is possible to find the relation between the bulk viscosity and the energy density transport coefficient ξ

$$\zeta = \frac{\xi}{3}\left[\frac{3G_{4\phi}^2 + G_4\left(G_{2X} - 4XG_{2XX}\right)}{3G_{4\phi}^2 + G_4\left(G_{2X} + 2XG_{2XX}\right)}\right] . \tag{2.55}$$

"First generation" scalar-tensor gravity corresponds to $\zeta = \xi/3$ and $G_4 = \phi$.

2.1.1 General Horndeski gravity *in vacuo*

We now briefly extend the previous analysis to general Horndeski gravity, including $G_4(\phi, X)$ and $G_5(\phi, X)$. The classes of theories selected by imposing the Newtonian nature of the effective fluid are again given by Eqs. (2.41) and (2.46). The linearity in $\nabla_\mu u_\nu$ implies $G_5 = G_{4X} = 0$ and then the above discussion holds. All we have to do is considering the energy density and total pressure for general Horndeski gravity and impose that all the non-linear terms vanish. The energy density is given by

$$\begin{aligned}
G_4\,\rho = {} & \frac{2}{3}\,X\sqrt{2X}\,\Theta\,\sigma^2\left(G_{5X} + XG_{5XX}\right) - \frac{2}{27}\,X\sqrt{2X}\,\Theta^3\left(G_{5X} + XG_{5XX}\right) \\
& - \frac{2}{3}\,X\sqrt{2X}\,\sigma_\mu{}^\lambda\sigma^{\mu\nu}\sigma_{\nu\lambda}\left(G_{5X} + XG_{5XX}\right) \\
& + X\sqrt{2X}\,R_{acbd}u^\mu u^\nu\sigma^{cd}G_{5X} + \frac{2}{3}\,X\Theta^2\left(G_{4X} + 2XG_{4XX} - G_{5\phi} - XG_{5\phi X}\right) \\
& + 2X\sigma^2\left(G_{5\phi} + XG_{5\phi X} - G_{4X} - 2\sqrt{2X}\,\eta\,G_{5X} - 2XG_{4XX}\right) \\
& - \frac{1}{2}\left(G_2 - 2XG_{2X} + 2XG_{3\phi}\right) + \rho\left[X\left(2G_{4X} - G_{5\phi}\right) - \frac{X}{3}\sqrt{2X}\,G_{5X}\,\Theta\right] \\
& + \sqrt{2X}\,\Theta\left(G_{4\phi} + 2XG_{4\phi X} - XG_{3X}\right) .
\end{aligned} \tag{2.56}$$

The first term in the second line of the above equation is multiplied by $R_{acbd}u^\mu u^\nu\sigma^{cd} = \dot{u}^\mu\dot{u}^\nu\,\sigma_{\mu\nu} - \frac{2}{3}\,\Theta\sigma^2 - u^\mu\sigma^{\nu\lambda}\nabla_\mu\sigma_{\nu\lambda} + \sigma_{\mu\nu}\nabla^\nu\dot{u}^\mu + u^\mu\sigma^{\nu\lambda}\nabla_\lambda\sigma_{\mu\nu}$. Since the latter represents a non-linear contribution in $\nabla_\mu u_\nu$ that cannot be cancelled by any other term, it is immediate to see that G_{5X} must vanish. Then, one can set $G_5 = 0$ because $G_5(\phi)$ can be absorbed, upon integration by parts, in the other functions G_2, G_3, and G_4 according to

$$G_2 \to G_2 - 2X^2G_{5\phi\phi\phi}\,, \quad G_3 \to G_3 - 3XG_{5\phi\phi}\,, \quad G_4 \to G_4 - XG_{5\phi}\,. \tag{2.57}$$

The result is the Lagrangian density

$$\mathcal{L}_5 \;=\; G_5G_{\mu\nu}\nabla^\mu\nabla^\nu\phi \simeq -G_{5\phi}G_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi - G_{5\phi}R_{\mu\nu}\nabla^\mu\phi\nabla^\nu\phi - XG_{5\phi}R$$

$$
\begin{aligned}
&= G_{5\phi}(\nabla_\mu \Box\phi - \nabla_\nu\nabla_\mu\nabla^\nu\phi)\nabla^\mu\phi - XG_{5\phi}R \\
&\simeq -G_{5\phi\phi}(-2X\Box\phi - \nabla_\mu\nabla_\nu\phi\nabla^\mu\phi\nabla^\nu\phi) - G_{5\phi}\left[\Box\phi^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] - XG_{5\phi}R \\
&\simeq 3X\,G_{5\phi\phi}\,\Box\phi - 2X^2 G_{5\phi\phi\phi} - G_{5\phi}\left[\Box\phi^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] - XG_{5\phi}R\,, \qquad (2.58)
\end{aligned}
$$

where $\simeq$ denotes equality up to a total divergence. The effective energy density then becomes

$$
\begin{aligned}
\rho =& \frac{2X\Theta^2\left(G_{4X}+2XG_{4XX}\right)}{3\left(G_4-2XG_{4X}\right)} - \frac{2X\sigma^2\left(G_{4X}+2XG_{4XX}\right)}{G_4-2XG_{4X}} \\
&+ \frac{\sqrt{2X}\,\Theta\left(-XG_{3X}+G_{4\phi}+2XG_{4\phi X}\right)}{G_4-2XG_{4X}} - \frac{G_2-2XG_{2X}+2XG_{3\phi}}{2G_4-4XG_{4X}} \qquad (2.59)
\end{aligned}
$$

where, in order to suppress the quadratic terms in $\nabla_\mu u_\nu$ in the first line, it is necessary that

$$
G_{4X} + 2XG_{4XX} = 0\,, \qquad (2.60)
$$

which implies that

$$
G_4\left(\phi, X\right) = G_0\left(\phi\right) + G_1\left(\phi\right)\sqrt{2X}\,. \qquad (2.61)
$$

Using now Eq. (2.26), the perfect fluid contribution to the effective isotropic pressure becomes

$$
\begin{aligned}
\bar{P} =& -\frac{2X\Theta^2 G_{4X}}{9\left(G_4-2XG_{4X}\right)} + \frac{2X\sigma^2 G_{4X}}{3\left(G_4-2XG_{4X}\right)} - \frac{4\dot{u}^2 X\left(G_{4X}+2XG_{4XX}\right)}{3\left(G_4-2XG_{4X}\right)} \\
&+ \frac{\dot{X}\left(-XG_{3X}+G_{4\phi}+2XG_{4\phi X}\right)}{\sqrt{2X}\left(-G_4+2XG_{4X}\right)} + \rho\,\frac{2XG_{4X}}{3\left(G_4-2XG_{4X}\right)} + \frac{G_2-2XG_{3\phi}+4XG_{4\phi\phi}}{2G_4-4XG_{4X}} \\
&+ \Theta\left[\zeta - \frac{2\dot{X}\left(G_{4X}+2XG_{4XX}\right)}{3\left(G_4-2XG_{4X}\right)} + \frac{2\sqrt{2X}\left(-G_{4\phi}+2XG_{4\phi X}\right)}{3\left(G_4-2XG_{4X}\right)}\right], \qquad (2.62)
\end{aligned}
$$

that, together with Eq. (2.60), requires $G_{4X}\left(\phi, X\right) = 0$ to eliminate the quadratic terms in the first line. This is necessary because, even if we assume $\dot{X}$ quadratic in $\nabla_\mu u_\nu$ to cancel the quadratic terms, $\Theta\dot{X}$ in the second line reintroduces a cubic term. The theory then reduces to viable Horndeski, and the previous discussion remains valid.

2.1.2 Horndeski gravity with matter

Finally, let us include the matter in this picture. The Newtonian behaviour of the effective fluid requires again Eqs. (2.41) and (2.46), with the only difference that the inviscid pressure (2.48) acquires the additional contribution

$$\bar{P} \to \bar{P} - \frac{G_{4\phi}^2 \, T^{(\mathrm{m})}}{G_4 \left[3G_{4\phi}^2 + G_4 \left(G_{2X} + 2XG_{2XX}\right)\right]} \,, \tag{2.63}$$

proportional to the trace of the matter energy-momentum tensor $T^{(\mathrm{m})} = g^{\mu\nu} T^{(\mathrm{m})}_{\mu\nu}$. The reason is that the presence of matter changes the field equations for $g_{\mu\nu}$ to

$$R_{\mu\nu} = T_{\mu\nu} - \frac{T}{2}\, g_{\mu\nu} + \frac{1}{G_4}\left(T^{(\mathrm{m})}_{\mu\nu} - \frac{T^{(\mathrm{m})}}{2}\, g_{\mu\nu}\right) , \qquad R = -T - \frac{T^{(\mathrm{m})}}{G_4} \,, \tag{2.64}$$

turning the function $A(\phi, X)$ of Eq. (2.28) into

$$A \to A + \sqrt{2X}\, G_{4\phi}\, T^{(\mathrm{m})} - \sqrt{2X}\left(T^{(\mathrm{m})} + 2T^{(\mathrm{m})}_{\mu\nu} u^\mu u^\nu\right) X\, G_{3X} \,. \tag{2.65}$$

2.2 Analogy with first-order general-relativistic viscous fluids

Before proceeding, let us recall the basic description of *real* dissipative fluids that will be applied to the Horndeski *effective* fluid later in this section. Dissipative fluids are out-of-equilibrium systems. The most general stress-energy tensor describing an out-of-equilibrium system has the form [423–425]

$$T^{\mu\nu} = (\varepsilon + \mathcal{A})\; u^\mu u^\nu + (p + \mathcal{B})\; h^{\mu\nu} + 2q^{(\mu} u^{\nu)} + \pi^{\mu\nu} \,, \tag{2.66}$$

where $\mathcal{A}$ and $\mathcal{B}$ represent the out-of-equilibrium corrections to ε and p, which are the equilibrium energy density and pressure, respectively. $\mathcal{A}$ and $\mathcal{B}$ vanish at equilibrium. In the first-order formulation of viscous fluids [424, 426, 427], all the quantities in Eq. (2.66) depend on the fluid 4-velocity u^μ, the temperature $\mathcal{T}$, and the chemical potential μ. In particular, the deviations from equilibrium are parametrized by the gradients of u^μ, $\mathcal{T}$, and μ. Here we work in the Eckart (or particle) frame. In the effective fluid description, we decompose the spacetime according to the effective fluid 4-velocity, therefore the fluid motion is described using the fluid's proper time. If we consider vanishing chemical potential $\mu = 0$, we can parametrize the out-of-equilibrium quantities as

$$\mathcal{A} = \chi_1 \frac{u^\mu \nabla_\mu \mathcal{T}}{\mathcal{T}} + \chi_2\, \Theta \,, \tag{2.67}$$

$$\mathcal{B} = \chi_3 \frac{u^\mu \nabla_\mu \mathcal{T}}{\mathcal{T}} + \chi_4\, \Theta \,, \tag{2.68}$$

$$q^\mu = \lambda \left(\frac{h^{\mu\nu} \nabla_\nu \mathcal{T}}{\mathcal{T}} + \dot{u}^\mu\right) , \tag{2.69}$$

$$\pi^{\mu\nu} = -2\eta\,\sigma^{\mu\nu}\,, \tag{2.70}$$

where the transport coefficients χ_i, λ, and η depend on the temperature $\mathcal{T}$. Since the chemical potential vanishes identically, the equilibrium energy density and pressure depend only on the temperature, $\varepsilon = \varepsilon(\mathcal{T})$ and $p = p(\mathcal{T})$. We parametrize λ as $\lambda = -\mathcal{K}\mathcal{T}$, where $\mathcal{K} = \mathcal{K}(\mathcal{T})$ is the thermal conductivity [411, 416, 419] of the effective fluid.

The second law of thermodynamics then yields [428]

$$\frac{dp}{d\mathcal{T}} = \frac{\varepsilon + p}{\mathcal{T}}\,, \tag{2.71}$$

which can also be written as

$$\varepsilon(\mathcal{T}) = -p(\mathcal{T}) + \mathcal{T}\,p'(\mathcal{T})\,, \tag{2.72}$$

where a prime denotes differentiation with respect to the temperature.

Let us apply now the thermodynamic fluid description to the Horndeski *effective* fluids classes (2.41) and (2.46) in the framework of first-order general-relativistic viscous fluids. This procedure allows one to discuss the "temperature of gravity" and its transport coefficients. Before discussing the individual Horndeski classes, let us make some considerations valid in both cases.

In general, $\mathcal{T} = \mathcal{T}(\phi, X)$ and

$$\nabla_\mu \mathcal{T} = \left(\sqrt{2X}\,\mathcal{T}_\phi - \mathcal{T}_X\,\dot{X}\right) u_\mu - 2X\,\mathcal{T}_X\,\dot{u}_\mu\,, \tag{2.73}$$

therefore Eqs. (2.67)–(2.69) turn into

$$\mathcal{A} = \chi_1\,\frac{-\sqrt{2X}\,\mathcal{T}_\phi + \mathcal{T}_X\,\dot{X}}{\mathcal{T}} + \chi_2\,\Theta\,, \tag{2.74}$$

$$\mathcal{B} = \chi_3\,\frac{-\sqrt{2X}\,\mathcal{T}_\phi + \mathcal{T}_X\,\dot{X}}{T} + \chi_4\,\Theta\,, \tag{2.75}$$

$$q^\mu = \lambda\left(-2X\,\frac{\mathcal{T}_X}{\mathcal{T}} + 1\right)\dot{u}^\mu\,. \tag{2.76}$$

In both classes we have

$$\eta = -\frac{\sqrt{2X}}{2}\,\frac{G_{4\phi}}{G_4}\,, \tag{2.77}$$

and the shear viscosity $\eta(\mathcal{T})$ depends on the temperature. Its derivative with respect to X is

$$\eta'(\mathcal{T})\,\mathcal{T}_X = -\frac{1}{2\sqrt{2X}}\,\frac{G_{4\phi}}{G_4}\,, \tag{2.78}$$

implying that T_X is always non-vanishing unless $G_{4\phi} = 0$. We rewrite this equation as

$$2X\,\mathcal{T}_X = \frac{\eta(\mathcal{T})}{\eta'(\mathcal{T})}\,, \tag{2.79}$$

where $2X\,\mathcal{T}_X$ still depends on the temperature. Eq. (2.71) implies

$$p_\phi = \frac{\mathcal{T}_\phi}{\mathcal{T}}\,(\varepsilon + p)\ , \qquad p_X = \frac{\mathcal{T}_X}{\mathcal{T}}\,(\varepsilon + p)\ . \tag{2.80}$$

If $p = -\varepsilon$ identically, the above relations imply constant equilibrium pressure and energy density, $p = -\varepsilon = -\Lambda$.

In both cases, we note a further constraint on the linear Horndeski classes: the functional form of G_2 is determined up to an unknown function of $F = F(\sqrt{2X}\,\frac{G_{4\phi}}{G_4})$ (see Eqs. (2.88) and (2.111)). While in the first class we obtain $\mathcal{T} = \alpha\,\sqrt{2X}\,\frac{G_{4\phi}}{G_4}$ and then $F = F\,(\mathcal{T}) \equiv p(\mathcal{T})$ (the equilibrium pressure), in the second one we can parameterize the dependence of F using the shear viscosity in Eq. (2.77), with the result that $F(\mathcal{T}) = p(\mathcal{T}) - 4\eta^2(\mathcal{T})$.

2.2.1 Class I: $G_3 = G_{4\phi} \ln X$

Comparing Eqs. (2.70), (2.74)–(2.76) with the quantities associated to the class (2.41), one obtains

$$\chi_1 = \chi_2 = \chi_3 = 0\,, \qquad 3\chi_4 = 4\eta\,, \tag{2.81}$$

$$\chi_4 = -\frac{2}{3}\,\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,, \qquad \eta = -\frac{1}{2}\,\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,, \tag{2.82}$$

$$\varepsilon = \frac{1}{2G_4},(2XG_{2X} - G_2 - 2XG_{4\phi\phi}\ln X)\,, \tag{2.83}$$

$$p = \frac{1}{2G_4}\left[G_2 + 2XG_{4\phi\phi}\,(2 - \ln X)\right]\,, \tag{2.84}$$

and

$$\lambda\left(-2X\,\frac{\mathcal{T}_X}{\mathcal{T}} + 1\right) = 0\,. \tag{2.85}$$

This equation is satisfied when the bracket is equal to zero or when $\lambda \equiv \mathcal{K}\mathcal{T} = 0$. In the first case, the solution of the differential equation is

$$\mathcal{T}(\phi, X) = \sqrt{2X}\,C(\phi) \tag{2.86}$$

and, we can find the integrating function $C(\phi)$ by imposing $\eta = \eta(\mathcal{T})$,

$$\mathcal{T}(\phi, X) = \alpha\,\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,, \tag{2.87}$$

where α is a constant.[1] Then the Horndeski function G_2 has the form

$$G_2\,(\phi, X) = 2G_4\,F\left(\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\right) - 2X\,G_{4\phi\phi}(2 - \ln X)\,, \tag{2.88}$$

[1] In order for the effective temperature to be positive-definite, G_4 must be monotonic and $\alpha\,G_{4\phi} > 0$. If an effective Newton constant decreasing in time is assumed, then $G_{4\phi} > 0$, $\alpha > 0$, and $\eta < 0$. Negative viscosity is characteristic of non-isolated systems, that exchange energy with their surroundings.

where $F = F(\mathcal{T}) \equiv p(\mathcal{T})$. The equilibrium energy density $\varepsilon(\mathcal{T})$ is automatically given by Eq. (2.72).

The situation $G_{4\phi} = 0$ corresponds to $\mathcal{T} = 0$ and $p(0) = -\varepsilon(0) = -\Lambda$. The Horndeski Lagrangian density collapses into $R - 2\Lambda$, where Λ is the cosmological constant.

In particular, a linear relation between pressure and temperature

$$p(\mathcal{T}) = -\gamma + \beta\,\mathcal{T} = -\gamma + \alpha\,\beta\,\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,, \tag{2.89}$$

identifies the extended cuscuton model with $f_1 = -2\gamma\,G_4$, $f_2 = 2\,\alpha\,\beta\,G_{4\phi}$, and $f_3 = 0$ (following the notation of Ref. [421]), where α, β, and γ are constants. The energy density turns into

$$\varepsilon(\mathcal{T}) = \gamma + \beta\,\mathcal{T} = \gamma + \alpha\,\beta\,\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,. \tag{2.90}$$

Assuming a linear barotropic equation of state $p = w\,\varepsilon$, where w is the constant equation of state parameter, we obtain the energy density

$$\varepsilon(\mathcal{T}) = \gamma\,\mathcal{T}^{\frac{w+1}{w}} \tag{2.91}$$

and

$$G_2 = 2w\,\gamma\,G_4\left(\alpha\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\right)^{\frac{w+1}{w}} - 2X\,G_{4\phi\phi}(2 - \ln X)\,. \tag{2.92}$$

If $(\mathcal{T} - 2X\,\mathcal{T}_X) \neq 0$, Eq. (2.85) is satisfied only for $\lambda = 0$. This is compatible only with constant pressure and energy density, $p = -\varepsilon = -\Lambda$, which corresponds to the form of G_2

$$G_2(\phi, X) = -2\Lambda\,G_4 - 2X\,G_{4\phi\phi}(2 - \ln X)\,. \tag{2.93}$$

The Horndeski Lagrangian coincides with a particular extended cuscuton model given by $f_1 = -2\Lambda\,G_4$, $f_2 = 0$, and $f_3 = 0$ (see again the notation of [421]). Therefore, we have a non-dynamical imperfect fluid that mimics the cosmological constant and whose inviscid/equilibrium contributions satisfy the familiar linear barotropic equation $p = -\varepsilon$.

2.2.2 Class II: $G_3 = 0$

For the second class, we obtain the system

$$\chi_1 = 0\,, \qquad \chi_2\chi_3 + \chi_3\lambda + \chi_2\lambda = 0\,, \qquad 3\chi_4 = 4\eta\,, \tag{2.94}$$

$$\chi_2 = \sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,, \qquad \chi_4 = -\frac{2}{3}\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,, \tag{2.95}$$

$$\chi_3 = -\frac{1}{\sqrt{2X}}\,\frac{G_{4\phi}}{G_4}\,\frac{\mathcal{T}}{\mathcal{T}_X}\,, \tag{2.96}$$

$$\lambda = -\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,\frac{\mathcal{T}}{\mathcal{T} - 2X\,\mathcal{T}_X}\,, \tag{2.97}$$

$$\eta = -\frac{1}{2}\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,, \tag{2.98}$$

$$\varepsilon = \frac{2X\,G_{2X} - G_2}{2G_4}\,, \tag{2.99}$$

$$p = \frac{G_2 + 4X\,G_{4\phi\phi}}{2G_4} - \frac{G_{4\phi}}{G_4}\frac{\mathcal{T}_\phi}{\mathcal{T}_X}\,. \tag{2.100}$$

In this case, Eq. (2.87) cannot be a solution for the temperature, *i.e.*, $\mathcal{T}$ cannot be proportional to η, otherwise we would have $\mathcal{T} - 2X\,\mathcal{T}_X = 0$ and vanishing heat flux.

Using Eq. (2.98), we write the partial derivatives of $\mathcal{T}$ in terms of $\eta(\mathcal{T})$,

$$\mathcal{T}_X = \frac{1}{2X}\frac{\eta(\mathcal{T})}{\eta'(\mathcal{T})}\,, \tag{2.101}$$

$$\mathcal{T}_\phi = \frac{2}{\sqrt{2X}}\frac{\eta^2(\mathcal{T})}{\eta'(\mathcal{T})} - \frac{1}{2}\frac{\sqrt{2X}}{\eta'(\mathcal{T})}\frac{G_{4\phi\phi}}{G_4}\,, \tag{2.102}$$

which yields

$$\frac{\mathcal{T}_\phi}{\mathcal{T}_X} = -2X\,\frac{G_{4\phi}}{G_4} + 2X\,\frac{G_{4\phi\phi}}{G_{4\phi}}\,. \tag{2.103}$$

Then the pressure reads

$$p = \frac{G_2}{2G_4} + 2X\,\frac{G_{4\phi}^2}{G_4^2}\,, \tag{2.104}$$

or

$$G_2 = 2G_4\,p(\mathcal{T}) - 2X\,\frac{G_{4\phi}^2}{G_4} = 2G_4\left[\,p(\mathcal{T}) - 4\eta^2(\mathcal{T})\right]\,. \tag{2.105}$$

Substituting this expression of G_2 in the equilibrium energy density (2.99), one finds

$$\begin{aligned}\varepsilon(\mathcal{T}) &= -p(\mathcal{T}) + 2X\,\mathcal{T}_X\,p'(\mathcal{T}) - 2X\frac{G_{4\phi}^2}{G_4^2}\\ &= -p(\mathcal{T}) + 2X\,\mathcal{T}_X\,p'(\mathcal{T}) - 4\eta^2(\mathcal{T})\,.\end{aligned} \tag{2.106}$$

We can rewrite $\varepsilon + p$ in the above equation using Eq. (2.71) to obtain

$$(\mathcal{T} - 2X\,\mathcal{T}_X)\,p' = -4\eta^2 \tag{2.107}$$

while, substituting $p' = (\varepsilon + p)/\mathcal{T}$ and Eqs. (2.99) and (2.104) in Eq. (2.106) yields the temperature

$$\mathcal{T} = C(\phi)\,\exp\left[\int_1^X \frac{4G_{4\phi}^2 + G_4\,G_{2Z}}{2Z\left(2G_{4\phi}^2 + G_4\,G_{2Z}\right)}\,dZ\right], \tag{2.108}$$

where $C(\phi)$ is a generic integration function of the scalar field[2] and the integration is performed with respect to Z, an auxiliary variable of the kinetic scalar. Finally, the correspondence $\lambda = -\mathcal{K}\mathcal{T}$ gives

$$(\mathcal{T} - 2X\,\mathcal{T}_X)\,\mathcal{K} = -2\eta\,, \quad \text{and} \quad p' = 2\eta\,\mathcal{K}\,. \tag{2.109}$$

The relations (2.80) and (2.103) yield

$$\frac{p_\phi}{p_X} = \frac{\mathcal{T}_\phi}{\mathcal{T}_X}\,, \tag{2.110}$$

providing the system

$$G_2 = 2G_4\,F(\eta)\,, \tag{2.111}$$

$$p(\eta) = F(\eta) + 4\,\eta^2\,, \tag{2.112}$$

$$\varepsilon(\eta) = -F(\eta) + \eta\,\frac{dF(\eta)}{d\eta}\,, \tag{2.113}$$

where, to avoid non-dynamical scalar fields, one must have $F(\eta) \neq \alpha + \beta\,\eta - 3\eta^2$ with α and β constant (see Eq. (2.54)).

Imposing Eq. (2.71) on $\varepsilon(\eta)$ and on $p(\eta)$ is equivalent to

$$4\eta^2 + \eta\frac{dF}{d\eta} = \mathcal{T}\left(8\eta + \frac{dF}{d\eta}\right)\eta'\,, \tag{2.114}$$

which is rewritten as

$$4\eta^2 + \eta\,\frac{d\mathcal{T}}{d\eta}\,Q' = \mathcal{T}\left(8\eta + \frac{d\mathcal{T}}{d\eta}\,Q'\right)\eta' \tag{2.115}$$

where $Q(\mathcal{T}) = F(\eta(\mathcal{T}))$. If η is a monotonic function of the temperature, Eq. (2.115) turns into

$$4\eta^2 + \frac{\eta}{\eta'}\,Q' = \mathcal{T}\,(8\eta\,\eta' + Q')\,. \tag{2.116}$$

If we consider a linear barotropic equation of state $p = w\,\varepsilon$, Eq. (2.72) gives

$$\varepsilon = \gamma\,T^{\frac{w+1}{w}}\,, \tag{2.117}$$

where the effective temperature is positive defined and $\gamma > 0$ and w are constant, while Eqs. (2.112) and (2.113) yield the differential equation

$$F(\eta) + 4\eta^2 = w\left(\eta\,\frac{dF(\eta)}{d\eta} - F(\eta)\right)\,. \tag{2.118}$$

[2] $C(\phi)$ can be determined *a posteriori* by imposing $\eta = \eta(\mathcal{T})$.

In terms of the new variable $x = \eta^2$, this equation reads

$$2wx\,\frac{dF(x)}{dx} - (1+w)F(x) - 4x = 0\,, \tag{2.119}$$

which is recognized as an Euler–Cauchy differential equation with solutions

$$F_1(x) = \frac{4x}{w-1} + c_1\,x^{\frac{w+1}{2w}}\,, \qquad w \neq 1\,, \tag{2.120}$$

$$F_2(x) = 2x\log(x) + c_2\,x\,, \qquad w = 1\,, \tag{2.121}$$

where $c_{1,2}$ are integration constants. In correspondence of $F_{1,2}$ we have the following expressions for the temperature

$$T_1 = \gamma^{-\frac{w}{w+1}}\left(\frac{4x}{w-1} + c_1\,\frac{x^{\frac{w+1}{2w}}}{w}\right)^{\frac{w}{w+1}}\,, \tag{2.122}$$

$$T_2 = \left[\gamma^{-1}\,x\,(4 + c_2 + 2\ln x)\right]^{1/2}\,, \tag{2.123}$$

respectively. Let us consider now the power-law form

$$\mathcal{T} = (\alpha\,\eta)^{\beta}\,, \tag{2.124}$$

where $\alpha < 0$ and $\beta \neq 0, 1$, are constant.[3] Equation (2.114) then yields

$$F(x) = c_1 - \frac{2(\beta-2)x}{\beta-1}\,, \tag{2.125}$$

where c_1 is an integration constant, and $x = \eta^2$. The equilibrium energy density and pressure become, respectively,

$$\varepsilon = -c_1 - \frac{2(\beta-2)}{\beta-1}\,x\,, \tag{2.126}$$

$$p = c_1 + \frac{2\beta}{\beta-1}\,x\,. \tag{2.127}$$

From the above expressions, for ε to be positive, it must be $c_1 \leq 0$ and $\beta < 1$ or $\beta \geq 2$.

Our last example consists of the linear relation

$$\mathcal{T} = \alpha\,\eta + \beta\,, \tag{2.128}$$

which implies

$$F(\eta) = -\frac{4\alpha}{3\beta}\,\eta^3 - 4\eta^2 + c_1\,, \tag{2.129}$$

[3] In this case, $\alpha < 0$, $\eta < 0$, and G_4 is strictly monotonic.

$$p(\eta) = -\frac{4\alpha\eta^3}{3\beta} + c_1\,, \tag{2.130}$$

$$\varepsilon(\eta) = -\frac{8\alpha\eta^3}{3\beta} - 4\eta^2 - c_1\,, \tag{2.131}$$

where c_1 is constant. In this case, $\mathcal{T}$ and G_2 become constant as $\eta \to 0$ and the energy density is positive-definite for $c_1 < 0$.

2.3 Conclusive remarks

The significance of Horndeski gravity in contemporary gravitational research cannot be overstated. Nevertheless, due to the complexity of these theories, physical insights are still catching up to formal advancements, and the effective fluid approach offers a new perspective. Here, we interpret the Horndeski effective fluid as a relativistic non-Newtonian fluid.

By requiring the effective fluid stress-energy tensor to be linear in the 4-velocity gradient, we identify two distinct classes of viable Horndeski theories, which ensure that gravitational waves propagate at the speed of light (see Table 2.1). For other forms of Horndeski gravity, it is impossible to reformulate the energy-momentum tensor in the form of Eq. (2.14) with viscosity coefficients linear in $\nabla_\mu u_\nu$, as in standard constitutive relations. This suggests that the effective fluid equivalent in such theories exhibits exotic, non-Newtonian behavior. This fluid-based classification of Horndeski gravity has received limited attention thus far, but it is plausible that other alternative gravity theories could be similarly classified. This correspondence opens a new research path aimed at identifying specific Horndeski theories that emulate particular non-Newtonian fluid properties in their effective fluids. When combined with observations, such an approach could narrow down the broad range of Horndeski theories.

In a broader sense, the fluid-gravity analogy provides a useful framework for tackling theoretical and observational challenges in gravitational research. The interest in classes that correspond to Newtonian fluids is particularly motivated by the potential to view them as limits of a more comprehensive theory for small 4-velocity gradients, similar to how classical theories serve as low-energy limits of effective field theories. This is particularly relevant in late-time cosmology, where the scalar field can be regarded as a low-energy field, offering an additional tool for distinguishing classes within Horndeski gravity.

We have applied this framework to first-order viscous fluids and explored their thermodynamic properties, which impose additional constraints on the linear Horndeski classes (see Table 2.2). In this context, theories where the extra scalar degree of freedom does not propagate, aside from the usual two spin-2 modes of GR, retain a unique position in the classification of Horndeski theories, meriting further consideration based on the Newtonian versus non-Newtonian nature of the effective fluid.

Table 2.1: Overview of viable Horndeski effective fluids.

Viable Horndeski as an effective fluid

Assumption of timelike scalar field gradient:

$$u^\mu = \frac{\nabla^\mu \phi}{\sqrt{2X}}\,, \qquad X = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi > 0\,.$$

Stress-energy tensor of a dissipative fluid:

$$T^{\mu\nu} = \rho\, u^\mu u^\nu + P\, h^{\mu\nu} + 2\, q^{(\mu}u^{\nu)} + \pi^{\mu\nu}\,,$$

ρ = total energy density, P = total isotropic pressure, q^μ = heat flux density, $\pi^{\mu\nu}$ = anisotropic stress.

Stress-energy tensor of viable Horndeski effective fluid (no assumptions):

$$T^{\mu\nu} = (\bar\rho - \xi\,\Theta)u^\mu u^\nu + \left(\bar P + \frac{4}{3}\,\eta\,\Theta + \xi\,\frac{\dot X}{2X}\right)h^{\mu\nu} + 2\,\xi\,\dot u^{(\mu}u^{\nu)} - 2\,\eta\,\sigma^{\mu\nu},$$

$$\eta = -\frac{1}{2}\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,, \qquad \xi = \sqrt{2X}\,\frac{(G_{4\phi} - X\,G_{3X})}{G_4}\,,$$

where $\bar\rho$, $\bar P$, ξ, η are functions of ϕ and X and an overbar denotes "non-viscous" contributions;
η is unambiguously interpreted as shear viscosity coefficient.

$\rho = \quad \bar\rho - \xi\,\Theta$	Linear in $\nabla_\mu u_\nu$
$P = \quad \bar P + \frac{4}{3}\eta\,\Theta + \xi\,\frac{\dot X}{2X}$	Not explicit
$q^a = \quad \xi\,\dot u^\mu$	Linear in $\nabla_\mu u_\nu$
$\pi^{\mu\nu} = \quad -2\,\eta\,\sigma^{\mu\nu}$	Linear in $\nabla_\mu u_\nu$

To understand the nature of the Horndeski effective fluid (*i.e.*, to obtain the constitutive equations of the fluid) we must be able to write the total pressure in terms of kinematic quantities.

$\boldsymbol{\xi = 0}$	$\boldsymbol{\xi \neq 0}$ (only dynamical scalar fields)
Linear Horndeski - Class I: $G_3(\phi, X) = G_{4\phi}\ln X$ Landau frame and Eckart frame coincide.	$P = \dfrac{\bar P_1 - \zeta_1\,\Theta - \zeta_2\,\Theta^2 - \zeta_3\,\sigma^2 - \zeta_4\,\dot u^2}{\bar P_2 - \zeta_5\,\Theta}\,,$ Non-Newtonian (in general) $\bar P_i$ and ζ_i functions of ϕ and X. **Linear Horndeski - Class II:** $G_3(\phi, X) = 0$
$\rho = \bar\rho\,,$ $P = \bar P - \zeta\,\Theta\,, \quad \left(\zeta = -\frac{4}{3}\eta\right)$ $q^\mu = 0\,,$ $\pi^{\mu\nu} = -2\,\eta\,\sigma^{\mu\nu}\,.$	$\rho = \bar\rho - 2\eta\,\Theta\,,$ $P = \bar P^* - \zeta^*\,\Theta = \bar P + \frac{4}{3}\eta\,\Theta + \eta\,\frac{\dot X}{2X}\,, \quad \left(\bar P^* = \frac{\bar P_1}{\bar P_2}\,, \quad \zeta^* = \frac{\zeta_1}{\bar P_2}\right)$ $q^\mu = 2\,\eta\,\dot u^\mu\,,$ $\pi^{\mu\nu} = -2\,\eta\,\sigma^{\mu\nu}\,.$

The coefficients ζ and ζ^* are the bulk viscosity coefficients for the class I and II, respectively.
The two classes above are the only linear effective fluid classes in the general Horndeski theory.

Table 2.2: Analogy between linear Horndeski effective fluids and first-order viscous fluids.

LINEAR HORNDESKI EFFECTIVE FLUIDS AS FIRST-ORDER VISCOUS FLUIDS IN ECKART FRAME

First-order thermodynamics of viscous fluids

Stress-energy tensor of a dissipative fluid in Eckart's theory (vanishing chemical potential $\mu = 0$):

$$T^{\mu\nu} = \varepsilon\, u^\mu u^\nu + (p - \zeta\,\Theta)\, h^{\mu\nu} + 2\, q^{(\mu} u^{\nu)} - 2\eta\, \sigma^{\mu\nu}\,,$$

ε = equilibrium energy density, p = equilibrium pressure, ζ = bulk viscosity, η = shear viscosity,

$$q^\mu = -\mathcal{K}\left(h^{\mu\nu}\nabla_\nu \mathcal{T} + \mathcal{T}\dot{u}^\mu\right)\,,$$

where $\mathcal{K} = \mathcal{K}(\mathcal{T})$ is the thermal conductivity and $\mathcal{T}$ is the fluid temperature.

General first-order stress-energy tensor of viscous fluids ($\mu = 0$):

$$T^{\mu\nu} = (\varepsilon + \mathcal{A})\, u^\mu u^\nu + (p + \mathcal{B})\, h^{\mu\nu} + 2\, q^{(\mu} u^{\nu)} + \pi^{\mu\nu}\,,$$

$$\mathcal{A} = \chi_1 \frac{u^\mu \nabla_\mu \mathcal{T}}{\mathcal{T}} + \chi_2\,\Theta\,, \qquad \mathcal{B} = \chi_3 \frac{u^\mu \nabla_\mu \mathcal{T}}{\mathcal{T}} + \chi_4\,\Theta\,, \qquad q^\mu = \lambda\left(\frac{h^{\mu\nu}\nabla_\nu \mathcal{T}}{\mathcal{T}} + \dot{u}^\mu\right), \qquad \pi^{\mu\nu} = -2\eta\,\sigma^{\mu\nu}\,,$$

$$p'(\mathcal{T}) = \frac{\varepsilon(\mathcal{T}) + p(\mathcal{T})}{\mathcal{T}}\,,$$

where all transport coefficients χ_i, η, and $\lambda = -\mathcal{K}\mathcal{T}$ are functions of the temperature $\mathcal{T}$.
Eckart's theory is obtained for $\chi_1 = \chi_2 = \chi_3 = 0$.

Horndeski effective fluid	First-order fluids
$\eta = -\frac{1}{2}\sqrt{2X}\,\frac{G_{4\phi}}{G_4}$	$\eta = \eta(\mathcal{T})$

$$\Rightarrow \quad \mathcal{T} = \mathcal{T}(\phi, X)\,, \quad \mathcal{T}_X \neq 0$$

$$\nabla_a \mathcal{T} = \left(\sqrt{2X}\,\mathcal{T}_\phi - \dot{X}\,\mathcal{T}_X\right) u_a - 2X\,\mathcal{T}_X\,\dot{u}_a$$

$$\mathcal{A} = \chi_1 \frac{-\sqrt{2X}\,\mathcal{T}_\phi + \dot{X}\,\mathcal{T}_X}{\mathcal{T}} + \chi_2\,\Theta\,, \qquad \mathcal{B} = \chi_3 \frac{-\sqrt{2X}\,\mathcal{T}_\phi + \dot{X}\,\mathcal{T}_X}{\mathcal{T}} + \chi_4\,\Theta\,, \qquad q^\mu = \lambda\left(-2X\,\frac{\mathcal{T}_X}{\mathcal{T}} + 1\right)\dot{u}^\mu\,.$$

Linear Horndeski - Class I	**Linear Horndeski - Class II**
$\chi_1 = \chi_2 = \chi_3 = 0, \quad 3\chi_4 = 4\eta$	$\chi_1 = 0, \quad \chi_2 = -2\eta, \quad 3\chi_4 = 4\eta\,,$ $\chi_2\chi_3 + \chi_2\lambda + \chi_3\lambda = 0$
From the condition $q^\mu = 0$ we obtain $\mathcal{T} \propto \eta$: $\mathcal{T} = \alpha\sqrt{2X}\,\frac{G_{4\phi}}{G_4}\,,$ where α is constant. Then, ε and p automatically satisfy $\varepsilon + p = \mathcal{T}\,p'$, and $G_2 = 2G_4\,p(\mathcal{T}) - 2X\,G_{4\phi\phi}(2 - \ln X)\,,$ or, equivalently, $G_2 = 2G_4\,p(\eta) - 2X\,G_{4\phi\phi}(2 - \ln X)\,.$	It is not possible obtain the explicit form of $\mathcal{T}$ without making some assumption. In full generality: $\mathcal{T} = C(\phi)\,\exp\left[\int_1^X \frac{4G_{4\phi}^2 + G_4\,G_{2Z}}{2Z\left(2G_{4\phi}^2 + G_4\,G_{2Z}\right)}\,dZ\right],$ and $G_2 = 2\,G_4\left[\,p(\eta) - 4\eta^2\right],$ $\varepsilon(\eta) = -p(\eta) + \eta\left(\frac{dp}{d\eta} - 4\eta\right),$ $\eta\left(\frac{dp}{d\eta} - 4\eta\right) = \frac{dp}{d\eta}\eta'\,\mathcal{T}\,.$

The above analogy provides a constraint on the linear Horndeski classes:
G_2 is specified up to a function of the shear viscosity $p(\eta)$, the equilibrium pressure of the effective fluid.

3 First-order thermodynamics analogy

As previously discussed, scalar fields play a crucial role in cosmology, addressing various unresolved issues related to cosmic evolution from early to late times. In the early universe, the inflationary mechanism relies on a minimally coupled scalar field rolling down a potential, while scalar fields also feature prominently in alternative models such as genesis and bouncing cosmologies [429]. At later times, modified gravity theories that incorporate a scalar field in addition to the tensorial degrees of freedom of GR, along with quintessence models (canonical scalar fields with potentials), are promising alternatives to the finely-tuned cosmological constant as dark energy [430]. Scalar fields are particularly versatile because they can drive accelerated expansion without violating isotropy, typically through a background configuration $\phi = \phi(t)$. For scalar fields to be viable as dark energy, they need to be extremely light, with a mass on the order of $m \approx 10^{-33}$ eV, so that modifications to GR become apparent only on large cosmological scales, where GR is less rigorously tested than in the Solar System [150].

The range of cosmological implications of Horndeski theories naturally situates them within the context of FLRW spacetimes.

A recent formalism of interest for Horndeski theories is the first-order thermodynamics of modified gravity, developed in [416, 431] and reviewed in [414, 432]. This formalism, inspired by the concepts in [412, 413], provides a unifying framework that includes GR and its generalizations. Originally developed for "traditional" scalar-tensor theories [431], this approach separates the scalar field ϕ contribution in the effective stress-energy tensor of the Einstein equations, yielding an imperfect fluid [8, 410, 411, 433]. A key novelty of the formalism is the application of Eckart's non-equilibrium thermodynamics, incorporating first-order constitutive relations for the dissipative quantities. This leads to an effective temperature associated with the fluid, effectively a "temperature of scalar-tensor gravity" that indicates a deviation from GR, which represents the equilibrium state at zero temperature. This temperature is positive in scalar-tensor theories, suggesting they are non-equilibrium states within a thermodynamic framework of gravitational theories. Furthermore, this temperature acts as an order parameter for the dissipative dynamics as it approaches equilibrium, described by an effective heat equation that often implies relaxation back to GR, particularly in cosmological scenarios [418].

This formalism has been extended to various scenarios [418, 434–436]. In particular, it reveals intriguing results for Horndeski theories, where the thermodynamic analogy only holds within the "viable" Horndeski subclass that predicts gravitational waves propagating at the speed of light. This connection links the formalism with observational constraints on Horndeski gravity, specifically those from the multi-messenger event GW170817/GRB170817A [152, 153].

Motivated by these insights, this chapter explores the application of first-order thermodynamics to Horndeski theories within an FLRW background, leveraging the formalism to assess its physical implications. In particular, we focus on Eckart's thermodynamics within a cosmological context, where a fixed background is assumed. In specific geometries, an effective Eckart-like fluid can be realized within a broader subclass of viable Horndeski models, which includes previously discussed classes as particular cases. Such a configuration is possible in FLRW spacetimes with a homogeneous scalar field.

3.1 First-order thermodynamics of Horndeski theories

The non-equilibrium thermodynamics developed by Eckart allows us to obtain a "thermodynamics of gravity theories", in which GR represents the equilibrium state and scalar-tensor theories are non-equilibrium states, providing a concrete realization of the ideas in [413, 437]. Eckart's thermodynamics is distilled in three constitutive relations[1] that connect the viscous pressure $P_{\rm vis}$ with the fluid expansion Θ, the heat current density q^a with the temperature $\mathcal{T}$, and the anisotropic stresses $\pi_{\mu\nu}$ with the shear tensor $\sigma_{\mu\nu}$:

$$P_{\rm tot} = P_{\rm non-visc} + P_{\rm visc}\,, \tag{3.1}$$

$$P_{\rm visc} = -\zeta\,\Theta\,, \tag{3.2}$$

$$q_a = -\mathcal{K}\left(h_{\mu\nu}\nabla^b\mathcal{T} + \mathcal{T}\dot{u}_a\right)\,, \tag{3.3}$$

$$\pi_{\mu\nu} = -2\eta\,\sigma_{\mu\nu}\,, \tag{3.4}$$

where $\mathcal{K}$, ζ and η are the thermal conductivity, bulk viscosity and shear viscosity, respectively, and we generally assume $h_{\mu\nu}\nabla^b\mathcal{T} = 0$. The temperature of Horndeski gravity (inextricably linked to the thermal conductivity) reads [417]

$$\mathcal{K}\mathcal{T} = \epsilon\frac{\sqrt{2X}(G_{4\phi} - XG_{3X})}{G_4}, \tag{3.5}$$

and reduces to GR equilibrium state characterized by $\mathcal{K}\mathcal{T} = 0$ if $\phi = \text{const}$.

The most interesting finding is that this formalism does not work for the most general Horndeski theories, because some terms in their field equations explicitly break the proportionality required by the constitutive equations [417]. These terms are precisely those that violate the equality between the propagation speeds of gravitational and electromagnetic waves. Therefore, the validity of first-order thermodynamics seems to be related to the physical viability of Horndeski theories, which is a very intriguing result. The breaking of the thermodynamic analogy is also interesting from the purely theoretical point of view: it happens for the operators which contain derivative nonminimal couplings and nonlinear contributions in the connection. This relates to the well-known and long-standing problem of separating matter and gravity degrees of freedom in a local description.

[1] In a more recent formulation of the first-order thermodynamics of real fluids [423–425], linear viscous contributions are present also in the expression of the energy density, similarly to Eq. (3.19). Notice that here we work in the Eckart (or particle) frame.

More specifically, in [417] it is shown that, whenever we try to apply the thermodynamic formalism to theories beyond the viable class, the effective stress-energy tensor contains the term

$$T^{(\phi)}_{\mu\nu} \supset \alpha(\phi, X)\, R_{acbd} \nabla^c \phi \nabla^d \phi \,, \tag{3.6}$$

where $\alpha(\phi, X)$ is a generic function. It is precisely the Riemann tensor R_{acbd} which ends up breaking the proportionality between the traceless shear tensor $\sigma_{\mu\nu}$ and the anisotropic stress tensor $\pi_{\mu\nu}$, and thus Eckart's constitutive equations (3.2), (3.3) and (3.4) no longer hold.

3.1.1 First-order thermodynamics in FLRW background

In [418], the first-order thermodynamics of "traditional" scalar-tensor theories was studied in an FLRW background, with the goal of testing the physical intuition behind the formalism on some well-known exact solutions. The main result is that the GR equilibrium state of zero temperature is almost always approached at late times $t \to +\infty$ throughout the cosmic expansion, while the behavior expected for singularities (namely $\mathcal{KT} \to +\infty$, indicating an extreme deviation of the theory from GR equilibrium) is confirmed for solutions endowed with an initial singularity. Compellingly, this result about scalar-tensor theories "relaxing" to GR in a cosmological setting echoes those of [438, 439], albeit in a very different context. We are now in a position to perform the same feat as [418] with the more general class of viable Horndeski theories.

We restrict our discussion to the spatially flat FLRW spacetime, *i.e.*, $k = 0$,

$$ds^2 = -dt^2 + a^2 \left(dr^2 + r^2 d\Omega^2 \right) \,. \tag{3.7}$$

The 4-velocity of the effective fluid in a FLRW setting becomes

$$u^\mu \equiv \epsilon \frac{\nabla^\mu \phi}{\sqrt{2X}} = \left(-\epsilon\, \mathrm{Sign}(\dot{\phi}), 0, 0, 0 \right) , \tag{3.8}$$

where we assume that ϕ is strictly monotonic in t. Then, (3.8) is future-oriented only if $\epsilon = -\mathrm{Sign}(\dot{\phi})$. As a consequence, the equation $\dot{\phi} = -\epsilon\sqrt{2X}$ holds, since $X = \frac{1}{2}\dot{\phi}^2$.

Before continuing, it is useful to rewrite Eq. (2.28) in terms of $\Box\phi$:

$$\Box\phi = \frac{A + B\,\Theta + C\Theta^2 + D\,\sigma_{ab}\sigma^{ab} + E\,\dot{u}^c\dot{u}_c}{J + K\,\Theta}, \tag{3.9}$$

where, now,

$$\begin{aligned} A =& T^{(\mathrm{m})}(G_{4\phi} - XG_{3X}) - 2\left(T^{(\mathrm{m})}_{ab} u^a u^b \right) XG_{3X} + G_2(2G_{4\phi} - XG_{3X}) - G_4 G_{2\phi} \\ & X\left[(G_{4\phi} - XG_{3X})(6G_{4\phi\phi} + G_{2X} - 4G_{3\phi}) + 2G_{4\phi}G_{3\phi} + 2G_4(G_{2\phi X} - G_{3\phi\phi}) \right] , \end{aligned} \tag{3.10}$$

$$B = \epsilon(2X)^{3/2}\left[2XG_{3X}^2 - 2G_{4\phi}G_{3X} + G_4(G_{2XX} - 2G_{3\phi X}) \right] , \tag{3.11}$$

$$C = -\frac{4}{3}XG_4(2G_{3X} + 3XG_{3XX})\,, \tag{3.12}$$

$$D = -2XG_4G_{3X}\,, \tag{3.13}$$

$$E = 4XG_4(G_{3X} + XG_{3XX})\,, \tag{3.14}$$

$$J = 3(G_{4\phi} - XG_{3X})^2 + G_4\left[G_{2X} + 2XG_{2XX} - 2(G_{3\phi} + XG_{3\phi X})\right]\,, \tag{3.15}$$

$$K = -2\epsilon\sqrt{2X}G_4(G_{3X} + XG_{3XX})\,. \tag{3.16}$$

In this way, After substituting Eq. (3.9) into Eq. (2.21) to obtain an expression for the pressure, we can also rewrite Eqs. (2.20), (2.22) and (2.23) in a compact way, making the dependence on the 4-velocity gradients apparent:

$$\rho^{(\phi)} = \rho_0 - \xi\Theta\,, \tag{3.17}$$

$$P^{(\phi)} = P_0 + \xi\left(\frac{A + B\Theta + C\Theta^2 + D\sigma^2 + E\dot{u}^2}{J' + K'\Theta}\right) - \left(\xi - \frac{4}{3}\eta\right)\Theta\,, \tag{3.18}$$

$$q_a^{(\phi)} = \xi\dot{u}_a\,, \tag{3.19}$$

$$\pi_{ab}^{(\phi)} = -2\eta\sigma_{ab}\,, \tag{3.20}$$

where $J' = \epsilon\sqrt{2X}J$, $K' = \epsilon\sqrt{2X}K$, $\rho_0 = \left(2XG_{2X} - G_2 - 2XG_{3\phi}\right)/2G_4$, $P_0 = \left(G_2 - 2XG_{3\phi} + 4XG_{4\phi\phi}\right)/2G_4$, $\xi = -\epsilon\sqrt{2X}\left(G_{4\phi} - XG_{3X}\right)/G_4$, $\eta = -\sqrt{2X}G_{4\phi}/2G_4$, $\sigma^2 = \sigma_{ab}\sigma^{ab}$, and $\dot{u}^2 = \dot{u}_c\dot{u}^c$.

As mentioned in the previous section, once we work with a fixed background, the constraint that an effective fluid is linear in $\nabla_\nu u_a$ is less stringent than in the general case with any geometry. The features of the FLRW metric allow us to find a larger subclass of viable Horndeski containing the previous classes as subcases. This is the case of FLRW universes with a homogeneous scalar field.

The expansion scalar in FLRW reads $\Theta = 3H$, and the shear tensor and the 4-acceleration vanish ($\sigma^2 = 0$, $\dot{u}^2 = 0$). Moreover, the Friedmann constraint reads

$$H^2 = \frac{1}{3}\left(\frac{\rho^{(\mathrm{m})}}{G_4} + \rho^{(\phi)}\right) = \frac{1}{3}\left(\frac{\rho^{(\mathrm{m})}}{G_4} + \rho_0 - 3H\xi\right)\,. \tag{3.21}$$

Therefore, since $\rho^{(\phi)}$ is always linear in H, *i.e.*, linear in the expansion scalar, we can rewrite the $\Theta^2 = 9H^2$ term in Eq. (3.9) and Eq. (3.18) as a linear expression in terms of $\Theta = 3H$. Then, the general expression for the pressure takes the form

$$P^{(\phi)} = P_0 + \xi\left(\frac{A' + B'\Theta}{J' + K'\Theta}\right) - \left(\xi - \frac{4}{3}\eta\right)\Theta\,. \tag{3.22}$$

At this point, Eckart's constitutive relation (3.2) can be realized by imposing $K' = 0$, which corresponds to assuming

$$G_3(\phi, X) = F(\phi)\ln(X/X_*)\,. \tag{3.23}$$

This functional form is a solution of the partial differential equation $G_{3X} + XG_{3XX} = 0$ (see Eq. (3.16)), which eliminates the non-linear contribution due to the denominator in Eq. (3.18). As mentioned above, an additional function of the scalar field, $V(\phi)$, in G_3 is neglected since it can be reabsorbed through a redefinition of G_2. Therefore, in order to deal with a linear effective fluid and apply the Eckart's thermodynamics, in the following we assume the above functional form of G_3. This particular choice is not just attractive for its simplicity, but also includes interesting applications like shift-symmetric theories exhibiting hairy black holes [440, 441], vanishing braiding theories [442], and the case $XG_{3X} \propto G_{4\phi}$ which appears favored by observational data [443].

Recalling that, in a homogeneous and isotropic background, the matter stress-energy tensor is $T^{(\mathrm{m})}_{\mu\nu} = \rho^{(\mathrm{m})} u_\mu u_\nu + P^{(\mathrm{m})} h_{\mu\nu}$, the pressure in the effective ϕ-fluid (3.18) is comprised of three contributions,

$$P^{(\phi)} = P_{\mathrm{int}} + P_{\mathrm{non-visc}} + P_{\mathrm{visc}}, \tag{3.24}$$

the interaction[2], non-viscous and viscous pressure, respectively. The viscous pressure, similarly to the case in [418], is proportional to H.[3] Taking into account Eq. (3.23), the explicit expressions of the pressures are

$$P_{\mathrm{int}} = \frac{(G_{4\phi} - F)}{G_4 \Delta} \left[G_{4\phi} \left(\rho^{(\mathrm{m})} - 3P^{(\mathrm{m})} \right) - 3F \left(\rho^{(\mathrm{m})} - P^{(\mathrm{m})} \right) \right], \tag{3.25}$$

$$\begin{aligned} P_{\mathrm{non-visc}} = \frac{1}{G_4 \Delta} \{ & 2XG_4 \left(G_{2X} + 2XG_{2XX} \right) \left(2G_{4\phi\phi} - F_\phi \ln(X/X_*) \right) \\ & + G_2 \left[G_4 \left(G_{2X} + 2XG_{2XX} \right) - 2G_4 (1 + \ln(X/X_*)) F_\phi + 4FG_{4\phi} - 3F^2 - G_{4\phi}^2 \right] \\ & + 2XG_{2X} \left(3F^2 - 4FG_{4\phi} + G_{4\phi}^2 \right) - 2G_4 \left(G_{2\phi} - 2XG_{2\phi X} \right) \left(F - G_{4\phi} \right) \\ & - 2X \left[\ln(X/X_*) \left(2G_4 F_{\phi\phi} \left(F - G_{4\phi} \right) + F_\phi \left(-2G_4 \left(F_\phi - 2G_{4\phi\phi} \right) \right.\right.\right. \\ & \left.\left.\left. -4FG_{4\phi} + 3F^2 + G_{4\phi}^2 \right) \right) + 4G_4 F_\phi G_{4\phi\phi} - 2G_4 \ln^2(X/X_*) F_\phi^2 \right] \}, \end{aligned} \tag{3.26}$$

$$\begin{aligned} P_{\mathrm{visc}} = -\frac{\epsilon\sqrt{2X}H}{G_4 \Delta} & \left\{ G_{4\phi} \left[G_4 \left(G_{2X} - 4XG_{2XX} + 2(5 - \ln(X/X_*)) F_\phi \right) + 21F^2 \right] \right. \\ & \left. -3FG_4 \left[2(1 - \ln(X/X_*)) F_\phi + G_{2X} \right] - 3 \left(5FG_{4\phi}^2 + 3F^3 - G_{4\phi}^3 \right) \right\}, \end{aligned} \tag{3.27}$$

where

$$\Delta = G_4 \left[G_{2X} + 2XG_{2XX} - 2(1 + \ln(X/X_*)) F_\phi \right] + 3 \left(F - G_{4\phi} \right)^2. \tag{3.28}$$

From the viscous component P_{visc} of the pressure, we can extract the bulk viscosity coefficient ζ as defined in Eq. (3.2), which is proportional to $\dot\phi$ similarly to scalar-tensor theories in [418] and reads

$$\begin{aligned} \zeta = \frac{\epsilon\sqrt{2X}}{3G_4 \Delta} & \left\{ G_{4\phi} \left[G_4 \left(G_{2X} - 4XG_{2XX} + 2(5 - \ln(X/X_*)) F_\phi \right) + 21F^2 \right] \right. \\ & \left. -3FG_4 \left[2(1 - \ln(X/X_*)) F_\phi + G_{2X} \right] - 3 \left(5FG_{4\phi}^2 + 3F^3 - G_{4\phi}^3 \right) \right\}. \end{aligned} \tag{3.29}$$

[2] The non-minimal coupling of the scalar field with the metric tensor can be translated into an interaction contribution between standard matter and scalar field at the level of field equations.

[3] For a detailed discussion on the splitting of viscous and non-viscous terms within this thermodynamic analogy for maximally symmetric spaces we refer the reader to [418].

Focusing only on dynamical scalar fields, we can have either a vanishing bulk viscosity, corresponding to

$$
\begin{aligned}
G_2(\phi, X) = \mu(\phi)\left(\frac{5G_{4\phi}(\phi) - 3F(\phi)}{4G_{4\phi}(\phi)}\right)^{-1} X^{\frac{5G_{4\phi}(\phi)-3F(\phi)}{4G_{4\phi}(\phi)}} + \nu(\phi) \\
- 4X\left(F_\phi(\phi) + \frac{3\left[F(\phi) - G_{4\phi}(\phi)\right]^2}{4G_4(\phi)}\right) + 2F_\phi(\phi)X\ln(X/X_*)\,,
\end{aligned}
\tag{3.30}
$$

or a vanishing interaction term, associated with $F(\phi) = G_{4\phi}(\phi)$. If one imposes both vanishing $P_{\rm int}$ and $P_{\rm visc}$, the scalar field becomes non-dynamical.

Following the argument in [418], we can still find the $\mathcal{KT}$ of Horndeski gravity in FLRW, despite the fact that the heat flux density q_a vanishes identically due to homogeneity. Indeed, the general expression for $\mathcal{KT}$ (3.5) is found in [417] for Horndeski theories without specifying to particular geometries. Then, substituting $XG_{3X} = F$ from Eq. (3.23) into Eq. (3.5), we find

$$
\mathcal{KT} = \epsilon\sqrt{2X}\frac{(G_{4\phi} - F)}{G_4}\,. \tag{3.31}
$$

The above quantity is strictly related to the *braiding* that measures the strength of kinetic mixing between tensor and scalar perturbations [246, 444]. We notice that the relationship $\zeta = \mathcal{KT}/3$, valid for "traditional" scalar-tensor theories [418] is not valid for Horndeski theories.

However, Eq. (3.31) leads to an intriguing observation: $\mathcal{KT} = 0$ both for $\dot{\phi} = 0$ (which is the usual GR equilibrium) and $F = G_{4\phi}$. The latter is a novel feature of Horndeski gravity in first-order thermodynamics that went unnoticed in [417]. It is interesting because it means there are equilibrium states at $\mathcal{KT} = 0$ in the theory that are different than GR. In general, such alternative equilibrium states are found to be unstable [434] and are therefore unable to compete with the special role of GR in the landscape of gravity theories seen through the lens of the first-order thermodynamics. The stability of such states is assessed (generally after reducing to an exact solution of the theory) through the effective heat equation that provides the precise description of the dissipative process leading from non-equilibrium to equilibrium. For Horndeski theories this equation reads [417]

$$
\frac{\mathrm{d}(\mathcal{KT})}{d\tau} = \left(\epsilon\frac{\Box\phi}{\sqrt{2X}} - \Theta\right)\mathcal{KT} + \nabla^c\phi\nabla_c\left(\frac{G_{4\phi} - XG_{3X}}{G_4}\right), \tag{3.32}
$$

where $\dfrac{d}{d\tau} \equiv u^\mu\nabla_\mu = \epsilon\dfrac{\nabla^\mu\phi}{\sqrt{2X}}\nabla_\mu$.

3.2 Exact solutions

In order to test the thermodynamic formulation detailed in the previous sections, we now turn to studying some exact FLRW solutions. In particular, we focus on background cosmologies in cubic shift-symmetric Horndeski theories with a vanishing scalar current. Since Galileons possess shift symmetry (in addition to Galilean symmetry), this class of theories has some of the most interesting and well-explored cosmological consequences. We start in subsection 3.2.1 from the shift-symmetric solution and follow the strategy in [441] to find a cosmological solution with the desired expansion behavior. From this, we are able to obtain a new, shift-symmetric-inspired solution with explicit scalar field dependence in subsection 3.2.2. This can be interpreted as a theory that asymptotically approaches its shift-symmetric formulation.

3.2.1 Shift-symmetric gravity

The shift symmetry refers to the theory being invariant under $\phi \to \phi + \phi_0$, where ϕ_0 is a constant. The shift-symmetric subclass of the Horndeski theory corresponds to the choice $G_i = G_i(X)$, *i.e.* the Lagrangian does not explicitly depend on ϕ. In this case, the theory is characterized by the presence of a Noether conserved current, J^μ, and the scalar field equation of motion assumes the form of $\nabla_\mu J^\mu = 0$. The shift-symmetric viable Horndeski scalar current is

$$J^a = \left(G_{3X}\Box\phi - G_{2X}\right)\nabla^a\phi + G_{3X}\nabla^a X\,. \tag{3.33}$$

In the spatially flat FLRW, the scalar current has only the time component, and it reads as follows

$$J^\mu = \delta^\mu_0\dot\phi\left(G_{2X} + 3HG_{3X}\dot\phi\right). \tag{3.34}$$

If we restrict the class of "viable" shift-symmetric Horndeski theories, we have that the shift-symmetry sets $G_i = G_i(X)$ while the "viability" requires the conditions $G_{4X} = G_5 = 0$. Combining the two, one finds that $G_4 =$ constant and all non-minimal couplings disappear. Then, taking the covariant divergence of Eq. (2.3), recalling Eq. (2.4) and the contracted Bianchi identity, one finds

$$\nabla^a T^{(\phi)}_{\mu\nu} = G_4^{-1}\,\nabla^a T^{(\mathrm{m})}_{\mu\nu}\,.$$

Thus, at the level of the field equations for the metric tensor, the conservation of the stress-energy tensor of matter implies that of $T^{(\phi)}_{\mu\nu}$ in this specific scenario. Another way of understanding this point consists in observing that for this class of models $\nabla^\mu T^{(\phi)}_{\mu\nu} = -\frac{1}{2}(\delta L/\delta\phi)\nabla_\nu\phi\big|_{\mathrm{on-shell}} = 0$. Hence, since $G_4 =$ constant and $T^{(\mathrm{eff})}_{\mu\nu} = G_4^{-1}\,T^{(\mathrm{m})}_{\mu\nu} + T^{(\phi)}_{\mu\nu}$, then $\nabla^\mu T^{(\mathrm{eff})}_{\mu\nu} = 0$ comes from the independent covariant conservation of both $T^{(\mathrm{m})}_{\mu\nu}$ and $T^{(\phi)}_{\mu\nu}$.

The following gravitational action describes the shift-symmetric sector of the linear model selected in the previous section

$$S_g = \frac{1}{2}\int d^4x\sqrt{-g}\,\left[\,R + G_2(X) - \lambda\ln\left(X/X_*\right)\Box\phi\,\right], \tag{3.35}$$

so that $G_2 = G_2(X)$, $G_3 = \lambda\ln\left(X/X_*\right)$, and $G_4 = 1$. The study of this choice of couplings is also motivated from a phenomenological point of view, since it provides a good fit to cosmological data from standard probes [443, 445].

Then, the associated scalar current reduces to

$$J^a = \delta^\mu_0\left(\dot\phi\,G_{2X} + 6\lambda H\right). \tag{3.36}$$

In this work, we just restrict to the solutions associated with vanishing scalar current, in order to provide some concrete examples, similarly to [441]. Excluding the trivial case of $\dot\phi = 0$ which is equivalent to GR, the vanishing scalar current entails

$$\dot\phi\,G_{2X} + 6\lambda H = 0\,. \tag{3.37}$$

Using the above equation, and assuming that the standard matter content is described by the linear barotropic equation of state $P^{(m)} = w\rho^{(m)}$, we obtain the following expressions for the scalar field energy density and pressure, respectively,

$$\rho^{(\phi)} = -\frac{1}{2}G_2\,, \tag{3.38}$$

$$P^{(\phi)} = \frac{1}{2}G_2 - \frac{XG_{2X}^2 + 3\lambda^2 G_2}{G_{2X} + 2XG_{2XX} + 3\lambda^2} - \frac{3\lambda^2(w-1)\rho^{(\mathrm{m})}}{G_{2X} + 2XG_{2XX} + 3\lambda^2}\,. \tag{3.39}$$

The effective scalar fluid temperature of shift-symmetric viable Horndeski then reads

$$\mathcal{KT} = -\lambda\epsilon\sqrt{2X}\,. \tag{3.40}$$

If we want a positive defined $\mathcal{KT}$, then $\lambda\epsilon$ must be negative, *i.e.*, $\lambda\,\mathrm{Sign}(\dot{\phi}) > 0$. Since X must be strictly positive, the above temperature will not reach the zero temperature equilibrium state associated to GR. This shows that the approach to equilibrium is not always granted, as it was also found in [418].

In order to find exact analytical solutions, we assume $P^{(\mathrm{m})} = w\rho^{(\mathrm{m})}$, H strictly monotonic (*i.e.*, we can write t as $t = t(H)$), so that Eq. (3.21) and the scalar equations of motion (3.37) respectively read as follows

$$G_2\,[H] = 2\rho^{(\mathrm{m})}\,[H] - 6H^2\,, \tag{3.41}$$

$$G_{2X}\,[H] = -\frac{6\lambda H}{\dot{\phi}\,[H]} = \frac{6\lambda\epsilon H}{\sqrt{2X[H]}}\,, \tag{3.42}$$

where $\epsilon = -\mathrm{Sign}(\dot{\phi})$. Differentiating G_2 with respect to H one gets

$$\frac{dG_2}{dH} = G_{2X}\frac{dX}{dH}\,,$$

and differentiating Eq. (3.41) with respect to H one obtains

$$\frac{dG_2}{dH} = 2\frac{d\rho^{(\mathrm{m})}}{dH} - 12H\,.$$

Combining the last two equations yields

$$2\frac{d\rho^{(\mathrm{m})}}{dH} - 12H = G_{2X}\frac{dX}{dH}\,.$$

Then, taking advantage of Eq. (3.42), the latter can be rewritten as

$$2\frac{d\rho^{(\mathrm{m})}}{dH} - 12H = \left(\frac{6\lambda H}{\epsilon\sqrt{2X}}\right)\frac{dX}{dH}\,.$$

The last equation can be *formally* integrated on both sides in H as

$$\int\left(\frac{2}{H}\frac{d\rho^{(\mathrm{m})}}{dH} - 12\right)dH = 6\lambda\epsilon\int\frac{dX}{\sqrt{2X}}\,, \tag{3.43}$$

or, equivalently,

$$\int \left(\frac{2}{H}\frac{d\rho^{(\mathrm{m})}}{dH}\right) dH - 12H = 6\lambda\epsilon\sqrt{2X}\,. \tag{3.44}$$

Inspired by the strategy used in [441], our approach consists of choosing a cosmological evolution, either power-law expansion or exponential expansion, and then solving the continuity equation for the matter perfect fluid energy density, such that we can analytically obtain the function G_{2X} by inverting the relation $X[H]$ (if possible) and integrating the vanishing scalar current condition.

Let us start by considering a power-law expanding universe,

$$a(t) = a_* \left(\frac{t}{t_*}\right)^n, \quad n > 0\,, \quad t \geq 0\,, \tag{3.45}$$

with a_* constant. As a consequence, the following equations hold

$$H(t) = \frac{n}{t} \qquad \longleftrightarrow \qquad t(H) = \frac{n}{H}\,, \tag{3.46}$$

$$\rho^{(\mathrm{m})}(t) = \rho_* \left(\frac{t_*}{t}\right)^{3n(w+1)} \qquad \longleftrightarrow \qquad \rho^{(\mathrm{m})}(H) = \rho_* \left(\frac{H}{H_*}\right)^{3n(w+1)}\,, \tag{3.47}$$

where all the $*$-quantities are constant. Using Eqs. (3.46) and (3.47) into Eq. (3.44), and performing the integration, we obtain

$$6\lambda\epsilon\sqrt{2X} = 6\lambda\epsilon\, c_X - 12H + \frac{6n\rho_*(w+1)}{H_*[3n(w+1)-1]}\left(\frac{H}{H_*}\right)^{3n(w+1)-1}\,, \tag{3.48}$$

where c_X is an integration constant, and it is associated with the non-vanishing asymptotic value of $\sqrt{2X}$ reached in correspondence with $H = 0$ (in the limit $t \longrightarrow \infty$).

The simplest case admitting an analytical solution corresponds to $n = 2/3(w+1)$, with $w \neq -1$. Then, it turns out that

$$6\lambda\epsilon\sqrt{2X} = 12H\left(\frac{\rho_*}{3H_*^2} - 1\right) + 6\lambda\epsilon\, c_X \quad \Rightarrow \quad H = \frac{\lambda\epsilon}{2}\left(\sqrt{2X} - c_X\right)\left(\frac{\rho_*}{3H_*^2} - 1\right)^{-1}\,. \tag{3.49}$$

Thus the system is analytically solvable, yielding

$$G_2(X) = \frac{3\lambda^2}{2}\left(\sqrt{2X} - c_X\right)^2\left(\frac{\rho_*}{3H_*^2} - 1\right)^{-1}\,, \tag{3.50}$$

with the scalar field having the following form

$$\phi(t) = \phi_* - \epsilon\, c_X(t - t_*) - \frac{2n}{3\lambda}\left(\frac{\rho_*}{3H_*^2} - 1\right)\ln(t/t_*)\,, \tag{3.51}$$

where ϕ_* is constant.

Therefore, the effective scalar fluid temperature reads

$$\mathcal{KT} = \frac{2n}{t}\left(1 - \frac{\rho_*}{3H_*^2}\right) - \lambda\epsilon\, c_X\,. \tag{3.52}$$

As expected for cosmological solutions with an initial singularity [418], $\mathcal{KT} \to +\infty$ for $t \to 0$, indicating an extreme deviation of Horndeski theory from the GR equilibrium state as the singularity is approached.

We can now make a sensible consideration about the sign of the constants appearing in our solution. First, let us take into account the case $\mathcal{KT} > 0$. This implies $\lambda\epsilon < 0$ because of Eq. (3.40). However, there is an additional condition to be satisfied in order to have a positive definite $\mathcal{KT}$ for any $t > 0$, namely $\left(1 - \frac{\rho_*}{3H_*^2}\right)(w+1) > 0$, which is associated with the requirement of a non-vanishing kinetic term. Then, assuming $w > -1$ (corresponding to $n > 0$), we obtain $\left(1 - \frac{\rho_*}{3H_*^2}\right) > 0$. Therefore, the term $\sqrt{2X}$ starts from an infinite positive value, corresponding to the initial singularity at $t = 0$, and approaches c_X for $t \to \infty$. This is consistent with Eqs. (3.46) and (3.49), where $H = \frac{n}{t}$ must be positive. As a consequence, G_2 is actually negative definite, and this implies $\rho^{(\phi)} > 0$ from (3.38). It is straightforward to verify that $\rho^{(\phi)} = \frac{3n^2}{t^2}\left(1 - \frac{\rho_*}{3H_*^2}\right)$ and $P^{(\phi)} = w\rho^{(\phi)}$. Therefore, $\rho^{(\phi)} + P^{(\phi)} = \rho^{(\phi)}(1+w) > 0$ is satisfied. In this sense, the definition of effective temperature obtained by generalizing the one found for "traditional" scalar-tensor first-order thermodynamics implies the weak energy condition for the effective fluid. It is interesting to notice that this consideration is independent of $\mathrm{Sign}(\dot{\phi})$, which is involved only in the condition $\lambda\epsilon < 0$. Lastly, the bulk viscosity coefficient ζ (cfr. Eq. (3.29)) yields

$$\zeta = \lambda\epsilon\left(\sqrt{2X} - c_X\frac{3H_*^2}{\rho_*}\right) = -\frac{2n}{t}\left(1 - \frac{\rho_*}{3H_*^2}\right) + \lambda\epsilon c_X\left(1 - \frac{3H_*^2}{\rho_*}\right). \tag{3.53}$$

The above equation shows that the effective fluid starts off with a negative (and diverging to $-\infty$) bulk viscosity approaching the initial singularity, then ζ vanishes as the gradient of the scalar field approaches $\sqrt{2X} = c_X\frac{3H_*^2}{\rho_*}$, or, equivalently, as the cosmological time approaches $t = -\frac{1}{\lambda\epsilon}\frac{2n\rho_*}{3c_XH_*^2}$, and finally it becomes positive as t increases from that point.

Let us now take into account the case of a spatially flat de Sitter spacetime,

$$a(t) = a_* \exp(H_* t)\,. \tag{3.54}$$

The continuity equation gives

$$\rho^{(\mathrm{m})}(t) = \rho_* \exp\left[-3H_*(w+1)t\right], \tag{3.55}$$

and, from Eq. (3.42), we obtain

$$G_2(X) = 6\lambda H_*\left(\epsilon\sqrt{2X} + c_X\right). \tag{3.56}$$

Then, from the temporal component of the field equations (3.41), we obtain

$$6\lambda H_*(\epsilon\sqrt{2X} + c_X) = 2\rho^{(\mathrm{m})} - 6H_*^2, \tag{3.57}$$

which is equivalent to

$$\dot{\phi} = c_X + \frac{H_*}{\lambda}\left(1 - \frac{\rho_*}{3H_*^2}\exp\left[-3H_*(w+1)t\right]\right). \tag{3.58}$$

Integrating the equation above, the scalar field reads

$$\phi(t) = \phi_* + t\left(c_X + \frac{H_*}{\lambda}\right) + \frac{1}{3(w+1)\lambda}\frac{\rho_*}{3H_*^2}\exp\left[-3H_*(w+1)t\right]. \tag{3.59}$$

The effective scalar fluid temperature is

$$\mathcal{K}\mathcal{T} = (\lambda\epsilon\, c_X + H_*) - \frac{\rho_*}{3H_*}\exp\left[-3H_*(w+1)t\right]. \tag{3.60}$$

Also in this case, the condition $\left(1 - \frac{\rho_*}{3H_*^2}\right) > 0$ ensures the positivity of $\rho^{(\phi)}$ and of $\mathcal{K}\mathcal{T}$, under the assumption of $\epsilon\lambda < 0$ and $c_X < -\frac{H_*}{\epsilon\lambda}\left(1 - \frac{\rho_*}{3H_*^2}\right)$. The constant c_X can be properly chosen so that the condition $\rho^{(\phi)} + P^{(\phi)} > 0$ is satisfied as well. Therefore, the effective fluid can be easily tuned to satisfy the weak energy condition, characteristic of a real fluid. Also in this case, imposing the positivity of $\mathcal{K}\mathcal{T}$ goes in the direction of recovering the weak energy condition.

3.2.2 New exact solution with asymptotic shift-symmetry

Using a heuristic approach, we can generalize the previous power-law solution, adding an explicit ϕ-dependence inside the action. In particular, we consider the case described by Eqs. (3.45)-(3.47), within $n = 2/3(w+1)$, and, inspired by Eq. (3.49), we assume the following equation

$$H(t) = \alpha\sqrt{2X} + \beta\,, \tag{3.61}$$

where, α and β are constants. The expression above provides a differential equation for the scalar field, implying

$$\dot{H}(t) = -\alpha\epsilon\,\ddot{\phi}(t)\,. \tag{3.62}$$

Since we already have fixed the scalar field time-dependence, once we use Eqs. (3.61) and (3.62), we must require the scalar field equation of motion to be identically solved by the functional form of the action,

$$S_g = \frac{1}{2}\int d^4x\sqrt{-g}\left[\, G_4(\phi)R + G_2(\phi, X) - F(\phi)\ln(X/X_*)\Box\phi\,\right]. \tag{3.63}$$

Therefore, let us write down the equation of motion of the scalar field

$$\begin{aligned}
&\ddot{\phi}\left[\, 2F_\phi \ln(X/X_*) + 2F_\phi + 6\alpha\epsilon\left(F - G_{4\phi}\right) - G_{2X} - 2XG_{XX}\,\right] \\
&\quad + X\left[6\alpha\left(\epsilon G_{2X} + 2\epsilon F_\phi - 6\alpha F + 4\alpha G_{4\phi}\right) - 2G_{2\phi X}\right] \\
&\quad + \ln(X/X_*)\left[2X\left(F_{\phi\phi} - 6\alpha\epsilon F_\phi\right) - 6\beta\epsilon\sqrt{2X}F_\phi\right] \\
&\quad + 3\beta\sqrt{2X}\left[\epsilon G_{2X} + 2\epsilon F_\phi - 12\alpha F + 8\alpha G_{4\phi}\right] \\
&\quad + G_{2\phi} + 6\beta^2\left[2G_{4\phi} - 3F\right] \qquad = 0\,.
\end{aligned} \tag{3.64}$$

First, we need to impose that the coefficient of $\ddot{\phi}$ vanishes, *i.e.*,

$$2F_\phi \ln(X/X_*) + 2F_\phi + 6\alpha\epsilon\left(F - G_{4\phi}\right) - G_{2X} - 2XG_{XX} = 0\,, \tag{3.65}$$

which can be seen as a differential equation for G_2, having as solution the following functional form,

$$G_2(\phi, X) = \mu\sqrt{2X} + \nu + 2XF_\phi \ln(X/X_*) - X\left[4F_\phi - 6\alpha\epsilon\left(F - G_{4\phi}\right)\right], \tag{3.66}$$

where $\mu = \mu(\phi)$ and $\nu = \nu(\phi)$ are integrating functions of the scalar field. Substituting Eq. (3.66) into the field equations of the scalar field and of the metric tensor yields

$$6\alpha\left[\epsilon\left(G_{4\phi\phi} - F_\phi\right) - 2\alpha G_{4\phi}\right] X + 3\alpha\left[2\beta\left(G_{4\phi} - 3F\right) + \epsilon\mu\right]\sqrt{2X} \\ +6\beta^2\left(2G_{4\phi} - 3F\right) + 3\beta\epsilon\mu + \nu_\phi = 0\,, \tag{3.67}$$

$$3\left[\epsilon\left(F - G_{4\phi}\right) + 2\alpha\left(G_4 - \frac{\rho_*}{3H_*^2}\right)\right]\left(\alpha X + \beta\sqrt{2X}\right) + 3\beta^2\left(G_4 - \frac{\rho_*}{3H_*^2}\right) + \frac{1}{2}\nu = 0. \tag{3.68}$$

The above equations appear to have the same structure with respect to the X-dependence, namely $c_1(\phi)X + c_2(\phi)\sqrt{2X} + c_3(\phi) = 0$. We can obtain the general solution associated with Eqs. (3.61) and (3.62), by requiring all coefficients $c_i(\phi)$ of the above equations to vanish. This provides the following (unknown) functions of the scalar field,

$$F(\phi) = 2\alpha\epsilon\left(\frac{\rho_*}{3H_*^2} - G_4\right) + G_{4\phi}\,, \tag{3.69}$$

$$\mu(\phi) = 4\beta\epsilon\left[3\alpha\epsilon\left(\frac{\rho_*}{3H_*^2} - G_4\right) + G_{4\phi}\right], \tag{3.70}$$

$$\nu(\phi) = 6\beta^2\left(\frac{\rho_*}{3H_*^2} - G_4\right), \tag{3.71}$$

while the effective temperature reads

$$\mathcal{KT} = 2\alpha\sqrt{2X}\left(1 - \frac{1}{G_4}\frac{\rho_*}{3H_*^2}\right). \tag{3.72}$$

Then, the only remaining free Horndeski function is G_4. When the non-minimal coupling function is equal to one, the action (3.63) reduces to that of the shift-symmetric theory, provided that

$$\alpha = \frac{\lambda\epsilon}{2}\left(\frac{\rho_*}{3H_*^2} - 1\right)^{-1}, \qquad \beta = -\frac{\lambda\epsilon}{2}c_X\left(\frac{\rho_*}{3H_*^2} - 1\right)^{-1}. \tag{3.73}$$

It is useful to understand the solution considered above as asymptotically approaching its shift-symmetric formulation. In this case, the remark about the positivity of $\mathcal{KT}$ translates to $G_4 > \dfrac{\rho_*}{3H_*^2}$. This means that G_4 cannot be chosen such that the initial singularity is removed and the original singularity of the shift-symmetric model remains unchanged.

3.3 Discussion of the findings

In this chapter, we have specialized the formalism for the first-order thermodynamics of viable Horndeski gravity to the case of spatially flat isotropic and homogeneous spacetimes, with the goal of testing the physical intuition behind the formalism in this class of theories with interesting implications.

In general, the thermodynamics of scalar-tensor gravity [416, 431] relies on the *effective fluid approach* to modified gravity theories [411, 446]. According to this framework, given an alternative theory of gravity, we can derive from its field equations a generalized Einstein equation where all the contributions of the additional scalar degree of freedom – other than the matter fields – are collected in an effective stress-energy tensor $T^{(\phi)}_{\mu\nu}$. Such an effective tensor is symmetric by construction: thus, given a timelike vector field u^a, $T^{(\phi)}_{\mu\nu}$ always admits an imperfect fluid decomposition based on u^a (this is a trivial algebraic result, though for details we refer the reader to [447]). In other words, we can always find effective energy density, pressure, heat fluxes, and anisotropic stresses associated to $T^{(\phi)}_{\mu\nu}$ and u^a. More importantly, if $\nabla^a\phi$ is timelike, we can construct a 4-velocity field $u^a \propto \nabla^a\phi$, allowing us to provide a fluid interpretation for $T^{(\phi)}_{\mu\nu}$. This fluid, with 4-velocity $u^a \propto \nabla^a\phi$, is the dubbed ϕ-fluid. Studying the kinematic quantities of such an effective fluid and comparing them to the imperfect fluid decomposition of $T^{(\phi)}_{\mu\nu}$ one can then infer the constitutive relations of the ϕ-fluid. It was then shown in the previous chapter (first demonstrated in [417], and, then, generalized in [5]), and further expanded on in this chapter, that the effective fluid representation for the viable subclass of Horndeski gravity satisfies the constitutive laws of Eckart's theory of non-equilibrium thermodynamics. This allows to define an effective temperature for the ϕ-fluid, which is positive-definite for scalar-tensor theories and represents the order parameter characterizing the approach (or lack thereof) to the GR equilibrium state at zero temperature.

Since Eckart's constitutive relations are linear in the velocity gradient $\nabla_\nu u_\mu$, specializing the analysis for cosmological backgrounds further restricts the Horndeski theory to the subclass characterized by $G_3 = F(\phi)\ln(X/X_*)$.

Contrary to "traditional" scalar-tensor theories [418], viable Horndeski gravity naturally admits zero temperature equilibrium states other than GR (which is characterized by ϕ = const.) corresponding to the condition $\sqrt{2X}(G_{4\phi} - XG_{3X}) = 0$, *i.e.* $G_3 = G_4\ln(X/X_*)$. This class of zero-temperature equilibrium states alternative to GR is characterized by non-vanishing viscosity coefficients, thus suggesting that such equilibrium states are actually *unstable*.

In flat FLRW cosmology, due to the symmetries of the background, the heat flux and the anisotropic stress vanish identically. However, the viscous contribution remains and is visible through the isotropic pressure giving rise to a non-vanishing bulk viscosity. We computed the effective bulk viscosity for such models in this scenario, while the temperature and thermal conductivity are naturally inherited from the general (background-independent) approach.

The general results for the thermodynamics of viable Horndeski cosmology were then tested against exact solutions for interesting subclasses of the general theory that are also favored by cosmological observations. The considered examples differ significantly from the results obtained for "traditional" scalar-tensor cosmologies, since they display a non-vanishing effective temperature at all times in the cosmic evolution and asymptotically approach a constant effective temperature at late times. These results have been obtained, in particular, for classes of shift-symmetric and asymptotically shift-symmetric theories (the latter being shift-symmetric as the non-minimal coupling function G_4 approaches unity), both characterized by a non-vanishing braiding parameter.

In addition to showing the existence of subclasses of viable Horndeski gravity that never relax to the GR equilibrium state, our analysis further confirms previous findings according to which curvature singularities are “hot” [431], exhibiting a diverging temperature. This suggests that the deviations of these models from GR become extreme at spacetime singularities. An additional intriguing consequence of finding the effective temperature associated to these viable Horndeski subclasses is that imposing its positivity recovers the weak energy condition for the ϕ-fluid, which is characteristic of a real fluid and was not expected to hold for an effective fluid.

Lastly, in this chapter, we have also provided a novel exact cosmological solution for an asymptotically shift-symmetric theory as a toy model for our thermodynamic analysis.

4 Alternative formulations of thermodynamics

4.1 Statement of the problem and main results

From the discussion presented so far it might appear as if the proposed thermodynamic analogy of viable Horndeski gravity with Eckart's thermodynamics (and corresponding implications) is heavily dependent on the splitting performed in the right-hand side of Eq. (2.3). In other words, it can appear as though, given a scalar-tensor theory of gravity, different identifications for the effective stress-energy tensor $T_{ab}^{(\text{eff})}$ for the same theory would lead to different constitutive laws for the corresponding effective fluid. The aim of this chapter is to address this point and determine to which extent of the constitutive laws of the ϕ-fluid are mapped onto Eckart's thermodynamics, and to investigate possible alternative formulations of this analogy.

The procedure followed in Eq. (2.3) aims at completely separating the contributions of the scalar field ϕ from the matter content of the theory. In other words, the ϕ-fluid and ordinary matter are seen as completely independent "fluids". However, a key distinction between the two is the fact that, while the matter stress-energy tensor is covariantly conserved due to the diffeomorphism invariance of the Action, the same does not hold for the effective ϕ-fluid, indeed

$$\nabla^b T_{ab}^{(\text{eff})} = \frac{G_{4\phi}}{G_4^2}\, T_{ab}^{(\text{m})}\, \nabla^b \phi\,, \tag{4.1}$$

that is non-vanishing in general. This is a direct consequence of the non-minimal coupling of ϕ to gravity. Furthermore, another important conclusion that can be drawn from Eq. (2.3) and the fluid interpretation of $T_{ab}^{(\text{eff})}$ is the fact that the ϕ-fluid has a constant coupling to Einstein's gravity whereas that of matter is gauged by the scalar field ϕ though $1/G_4(\phi)$. This somewhat contradicts the universality of the gravitational interaction in this effective Einsteinian analogy for scalar-tensor gravity. However, $T_{ab}^{(\text{eff})}$ was postulated of this form and nothing prevents us from arbitrarily defining a new effective energy-momentum tensor

$$T_{ab}^{(f)} := [f(\phi, X)]^{-1}\, T_{ab}^{(\text{eff})}\,, \tag{4.2}$$

with $f(\phi, X)$ a non-vanishing continuous function of ϕ and X, so that the effective Einstein equation can be recast as

$$G_{ab} = \frac{1}{G_4}\, T_{ab}^{(\text{m})} + f(\phi, X)\, T_{ab}^{(f)}\,. \tag{4.3}$$

Note that, without loss of generality, we can assume $f(\phi, X) > 0$. This argument leads to the following results.

Theorem 1. *In viable Horndeski gravity, if we define the effective stress-energy tensor of the ϕ-fluid as*

$$T^{(f)}_{ab} := \frac{1}{f}\left(G_{ab} - \frac{1}{G_4}\,\mathcal{E}_{ab}\right), \tag{4.4}$$

with $f = f(\phi, X)$ an arbitrary strictly positive continuous function of ϕ and X, then the constitutive laws of the ϕ-fluid can always be mapped onto those of Eckart's first-order thermodynamics. Furthermore, the corresponding thermal conductivity and temperature of gravity read

$$\mathcal{K}^{(f)}\mathcal{T}^{(f)} = [f(\phi, X)]^{-1}\,\frac{\epsilon\sqrt{2X}\,(G_{4\phi} - XG_{3X})}{G_4}, \tag{4.5}$$

whereas the evolution equation for the temperature of gravity for the ϕ-fluid reads

$$\begin{aligned}\frac{d\left[\mathcal{K}^{(f)}\mathcal{T}^{(f)}\right]}{d\tau} &= \left[\epsilon\sqrt{2X}f_\phi - (\epsilon\sqrt{2X}\Box\phi - \Theta)f_X + \left(\epsilon\frac{\Box\phi}{\sqrt{2X}} - \Theta\right)f\right]\frac{\mathcal{K}^{(f)}\mathcal{T}^{(f)}}{f} \\ &\quad + \frac{1}{f}\,\nabla^c\phi\nabla_c\left(\frac{G_{4\phi} - XG_{3X}}{G_4}\right)\end{aligned} \tag{4.6}$$

with τ the proper time of an observer comoving with the ϕ-fluid.

Remark 1. It is easy to see that for $f = 1$ we recover the standard description of the first-order thermodynamics of scalar-tensor gravity.

Corollary 1. *In viable Horndeski gravity, if we define the effective stress-energy tensor of the ϕ-fluid as in* (4.4) *with $f = 1/G_4 \equiv G_{\text{eff}}$, then*

$$\mathcal{K}^{(G_{\text{eff}})}\mathcal{T}^{(G_{\text{eff}})} = \epsilon\sqrt{2X}\,(G_{4\phi} - XG_{3X}), \tag{4.7}$$

and we also have that

$$\begin{aligned}\frac{d\left[\mathcal{K}^{(G_{\text{eff}})}\mathcal{T}^{(G_{\text{eff}})}\right]}{d\tau} &= \left[-\epsilon\sqrt{2X}\frac{G_{4\phi}}{G_4} + \left(\epsilon\frac{\Box\phi}{\sqrt{2X}} - \Theta\right)\right]\mathcal{K}^{(G_{\text{eff}})}\mathcal{T}^{(G_{\text{eff}})} \\ &\quad + G_4\,\nabla^c\phi\nabla_c\left(\frac{G_{4\phi} - XG_{3X}}{G_4}\right),\end{aligned} \tag{4.8}$$

with τ the proper time of an observer comoving with the ϕ-fluid.

These results are particularly useful since they allow a simplification of the analysis of the approach to equilibrium equation for traditional scalar-tensor theories, *i.e.*,

$$S^{(\text{g})}_{\text{st}} = \frac{1}{2}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\,\nabla^c\phi\nabla_c\phi - V(\phi)\right], \tag{4.9}$$

that represents a well-investigated subclass of viable Horndeski gravity; specifically corresponding to

$$G_4 = \phi\,, \qquad G_2 = \frac{2\,\omega(\phi)}{\phi}\,X - V(\phi)\,, \qquad G_3 = 0\,.$$

In fact, we shall show that

Theorem 2. *In traditional scalar-tensor theories, for $f = 1/\phi \equiv G_{\rm eff}$ the evolution equation for the temperature of gravity reads*

$$\frac{d\left[\mathcal{K}^{(G_{\rm eff})}\mathcal{T}^{(G_{\rm eff})}\right]}{d\tau} = -\Theta\,\mathcal{K}^{(G_{\rm eff})}\mathcal{T}^{(G_{\rm eff})} + \Box\phi\,, \tag{4.10}$$

with τ the proper time of an observer comoving with the ϕ-fluid.

This result allows for a simplified proof of the fact that *electrovacuum* scalar-tensor theories satisfying $\omega = \text{const.}$, $V(\phi) = 0$, and $\Box\phi = 0$ will produce extreme deviations from General Relativity (diverging temperature of gravity) near spacetime singularities [416, 431]. Furthermore, Eq. (4.10) allows for a streamlined investigation of peculiar fixed points, other than General Relativity, and their thermal stability [434].

4.2 Alternative fluid representations: proof of Theorem 1

In this section, we shall provide detailed proofs of **Theorem 1** and **Corollary 1**.

Again, let us consider viable Horndeski gravity with a scalar field ϕ such that its four-gradient is timelike and define the future-directed four-velocity u^a associated to ϕ as in (2.16). If we now consider the modified definition (4.2) for the effective energy-momentum tensor of the ϕ-fluid, it is easy to see that the effective Einstein equations for viable Horndeski gravity read as in Eq. (4.3). On the one hand, since the modification of the effective energy-momentum tensor of the ϕ-fluid occurs only at the level of the effective Einstein equations, this procedure does not affect the kinematic quantities. On the other hand, because of Eq. (4.2) we have that the ϕ-fluid still admits an imperfect fluid decomposition for $T^{(f)}_{ab}$, *i.e.*,

$$T^{(f)}_{ab} = \rho^{(f)}u_a u_b + P^{(f)}h_{ab} + 2q^{(f)}_{(a}u_{b)} + \pi^{(f)}_{ab}\,, \tag{4.11}$$

with fluid quantities proportional to those in Eqs. (2.20)–(2.23), specifically,

$$\rho = T^{(\rm eff)}_{ab}u^a u^b = f\,T^{(f)}_{ab}u^a u^b = f\,\rho^{(f)}\,, \tag{4.12}$$

$$q_a = -T^{(\rm eff)}_{ab}\,u^c h_a{}^d = f\,q^{(f)}_a\,, \tag{4.13}$$

$$\Pi_{ab} = Ph_{ab} + \pi_{ab} = T^{(\rm eff)}_{cd}\,h_a{}^c\,h_b{}^d = f\,\Pi^{(f)}_{ab}\,, \tag{4.14}$$

$$P = \frac{1}{3}\,g^{ab}\Pi_{ab} = \frac{1}{3}\,h^{ab}T^{(\rm eff)} = f\,P^{(f)}\,, \tag{4.15}$$

$$\pi_{ab} = \Pi_{ab} - Ph_{ab} = f\,\pi^{(f)}_{ab}\,, \tag{4.16}$$

or in a more compact form $\{\rho^{(f)}\,,\ P^{(f)}\,,\ q^{(f)}_a\,,\ \pi^{(f)}_{ab}\} = f^{-1}\{\rho\,,\ P\,,\ q_a\,,\ \pi_{ab}\}$. Eqs. (2.20)–(2.23) then imply that the constitutive relations of the ϕ-fluid still map into those of Eckart's first-order thermodynamics up to a rescaling of the temperature and viscosities by a factor $1/f$. Let us consider the constitutive relation for the heat-flux density as an illustrative example. From Eq. (2.22) and Eq. (4.13) we can conclude that

$$q^{(f)}_a = -\epsilon\frac{\sqrt{2X}\left(G_{4\phi} - XG_{3X}\right)}{f\,G_4}\dot{u}_a\,, \tag{4.17}$$

which upon comparing this expression with that of Eckart's theory [Eq. (3.3)] implies a definition of "temperature of modified gravity" that reads

$$\mathcal{K}^{(f)}\mathcal{T}^{(f)} = \frac{\epsilon\sqrt{2X}\,(G_{4\phi} - XG_{3X})}{f\,G_4}\,, \tag{4.18}$$

or, equivalently, $\mathcal{K}^{(f)}\mathcal{T}^{(f)} = \mathcal{K}\mathcal{T}/f$.

We can now investigate the evolution of this alternative definition of temperature as a function of the proper time of an observer comoving with the fluid. Specifically one has that

$$\begin{aligned}\frac{\mathrm{d}\left(\mathcal{K}^{(f)}\mathcal{T}^{(f)}\right)}{\mathrm{d}\tau} &= u^c\nabla_c\left(\mathcal{K}^{(f)}\mathcal{T}^{(f)}\right) = u^c\nabla_c(f^{-1}\mathcal{K}\mathcal{T})\\ &= \frac{\mathcal{K}\mathcal{T}}{f^2}\left(\epsilon\sqrt{2X}f_\phi - \dot{X}\,f_X\right) + \frac{u^a\nabla_a(\mathcal{K}\mathcal{T})}{f}\\ &= \frac{\mathcal{K}\mathcal{T}}{f^2}\left[\epsilon\sqrt{2X}f_\phi - (\epsilon\sqrt{2X}\Box\phi - \Theta)f_X\right] + \frac{1}{f}\frac{\mathrm{d}\,(\mathcal{K}\mathcal{T})}{\mathrm{d}\tau}\,.\end{aligned} \tag{4.19}$$

Replacing $\frac{\mathrm{d}(\mathcal{K}\mathcal{T})}{\mathrm{d}\tau}$ with the expression provided by Eq. (3.32) yields

$$\begin{aligned}\frac{\mathrm{d}\left(\mathcal{K}^{(f)}\mathcal{T}^{(f)}\right)}{\mathrm{d}\tau} &= \frac{\mathcal{K}\mathcal{T}}{f^2}\left[\epsilon\sqrt{2X}f_\phi - (\epsilon\sqrt{2X}\Box\phi - \Theta)f_X\right] +\\ &\quad + \left(\epsilon\frac{\Box\phi}{\sqrt{2X}} - \Theta\right)\frac{\mathcal{K}\mathcal{T}}{f} + \frac{1}{f}\nabla^c\phi\nabla_c\left(\frac{G_{4\phi} - XG_{3X}}{G_4}\right)\,.\end{aligned} \tag{4.20}$$

If we now recall that $\mathcal{K}^{(f)}\mathcal{T}^{(f)} = \mathcal{K}\mathcal{T}/f$, we have that

$$\begin{aligned}\frac{d\left[\mathcal{K}^{(f)}\mathcal{T}^{(f)}\right]}{d\tau} &= \left[\epsilon\sqrt{2X}f_\phi - (\epsilon\sqrt{2X}\Box\phi - \Theta)f_X + \left(\epsilon\frac{\Box\phi}{\sqrt{2X}} - \Theta\right)f\,\right]\frac{\mathcal{K}^{(f)}\mathcal{T}^{(f)}}{f}\\ &\quad + \frac{1}{f}\,\nabla^c\phi\nabla_c\left(\frac{G_{4\phi} - XG_{3X}}{G_4}\right)\,,\end{aligned} \tag{4.21}$$

and this concludes the proof of **Theorem 1**.

If we now choose $f = 1/G_4$, *i.e.* we assume that gravity couples with the same strength to both matter and the effective ϕ-fluid (somewhat implementing the weak equivalence principle at the level of the effective Einstein equations), from Eq. (4.5) we have that

$$\mathcal{K}^{(f)}\mathcal{T}^{(f)}\Big|_{f=1/G_4} = \left.\frac{\epsilon\sqrt{2X}\,(G_{4\phi} - XG_{3X})}{f\,G_4}\right|_{f=1/G_4} = \epsilon\sqrt{2X}\,(G_{4\phi} - XG_{3X})\,, \tag{4.22}$$

and similarly

$$\begin{aligned}\left.\frac{d\left[\mathcal{K}^{(f)}\mathcal{T}^{(f)}\right]}{d\tau}\right|_{f=1/G_4} &= \left[-\epsilon\sqrt{2X}\frac{G_{4\phi}}{G_4} + \left(\epsilon\frac{\Box\phi}{\sqrt{2X}} - \Theta\right)\right]\left(\mathcal{K}^{(f)}\mathcal{T}^{(f)}\Big|_{f=1/G_4}\right)\\ &\quad + G_4\,\nabla^c\phi\nabla_c\left(\frac{G_{4\phi} - XG_{3X}}{G_4}\right)\,,\end{aligned} \tag{4.23}$$

thus concluding the proof of **Corollary 1**.

4.3 Traditional scalar-tensor theories: proof of Theorem 2

In order to prove the statement in **Theorem 2** we have to specialize our analysis of viable Horndeski gravity to its subclass that coincides with traditional scalar-tensor theories, *i.e.*, the subclass such that

$$G_4 = \phi\,, \quad G_2 = \frac{2\,\omega(\phi)}{\phi}\,X - V(\phi)\,, \quad G_3 = 0\,,$$

with $\phi > 0$ over the spacetime manifold.

Taking advantage of **Corollary 1** we can easily infer that

$$\mathcal{K}^{(f)}\mathcal{T}^{(f)}\Big|_{f=\frac{1}{\phi}\,,\,\mathrm{st}} = \frac{\epsilon\sqrt{2X}\,(G_{4\phi} - XG_{3X})}{f\,G_4}\Bigg|_{f=\frac{1}{\phi}\,,\,\mathrm{st}} = \epsilon\sqrt{2X}\,. \tag{4.24}$$

Furthermore, from Eq. (4.6) specialized to traditional scalar-tensor theories we have that

$$\begin{aligned}
\frac{d\big[\mathcal{K}^{(f)}\mathcal{T}^{(f)}\big]}{d\tau}\Bigg|_{f=\frac{1}{\phi}\,,\,\mathrm{st}} &= \left[-\frac{\epsilon\sqrt{2X}}{\phi} + \left(\epsilon\frac{\Box\phi}{\sqrt{2X}} - \Theta\right)\right]\left(\mathcal{K}^{(f)}\mathcal{T}^{(f)}\Big|_{f=\frac{1}{\phi}\,,\,\mathrm{st}}\right) + \phi\,\nabla^c\phi\nabla_c\left(\frac{1}{\phi}\right)\\
&= \left[-\frac{\epsilon\sqrt{2X}}{\phi} + \left(\epsilon\frac{\Box\phi}{\sqrt{2X}} - \Theta\right)\right]\epsilon\sqrt{2X} + \frac{2X}{\phi}\\
&= \left(\epsilon\frac{\Box\phi}{\sqrt{2X}} - \Theta\right)\epsilon\sqrt{2X} = \Box\phi - \Theta\,\epsilon\sqrt{2X}\\
&= \Box\phi - \Theta\left(\mathcal{K}^{(f)}\mathcal{T}^{(f)}\Big|_{f=\frac{1}{\phi}\,,\,\mathrm{st}}\right)\,,
\end{aligned} \tag{4.25}$$

which concludes the proof of **Theorem 2**.

4.4 An effective fluid with covariantly conserved stress-energy tensor

As pointed out in Section 4.1, the effective stress-energy tensor (2.4) is not covariantly conserved due to the nonminimal coupling of the scalar field to Einstein's gravity through $G_4(\phi)$ [Eq. (4.1)]. Nonetheless, one could try to look for alternative definitions for the effective stress-energy tensor that are covariantly conserved and see their effects on the thermodynamic analogy.

Turning our attention to viable Horndeski gravity, the simplest way of constructing a conserved rank-two tensor consists in considering linear combinations of conserved quantities. For instance, we know that $\mathcal{E}_{ab}$, as defined in Eq. (1.85), and G_{ab} are divergence-free tensors (on-shell), thus a simple proposal for an alternative covariantly-conserved stress-energy tensor reads

$$\hat{T}_{ab} := G_{ab} - \mathcal{E}_{ab}\,, \tag{4.26}$$

leading to the following form of the effective Einstein equation

$$G_{ab} = \hat{T}_{ab} + T^{(\mathrm{m})}_{ab}\,. \tag{4.27}$$

A nice feature of this definition is that both matter and the "effective fluid" are equally coupled to Einstein's gravity. However, this is done at the price of mixing matter and scalar field contributions, and this leads to complications in the thermodynamic analogy. Indeed, the conserved effective stress-energy tensor $\hat{T}_{ab}$ can be written in terms of the *minimal* one $T^{(\text{eff})}_{ab}$ [Eq. (2.4)] as

$$\hat{T}_{ab} = T^{(\text{eff})}_{ab} + \left(\frac{1-G_4}{G_4}\right) T^{(\text{m})}_{ab} \,. \tag{4.28}$$

Then, assuming that the scalar field has a timelike gradient and defining the effective fluid 4-velocity as in Eq. (2.16) we can perform an imperfect fluid decomposition of $\hat{T}_{ab}$ which yields

$$\hat{\rho} = \rho + \left(\frac{1-G_4}{G_4}\right) T^{(\text{m})}_{ab} u^a u^b \,, \tag{4.29}$$

$$\hat{P} = P + \frac{1}{3}\left(\frac{1-G_4}{G_4}\right) T^{(\text{m})}_{ab} h^{ab} \,, \tag{4.30}$$

$$\hat{q}_a = q_a - \left(\frac{1-G_4}{G_4}\right) T^{(\text{m})}_{cd} u^c h^d{}_a \,, \tag{4.31}$$

$$\hat{\pi}_{ab} = \pi_{ab} + \left(\frac{1-G_4}{G_4}\right) T^{(\text{m})}_{ab} \left(h^{ac} h^{bd} - \frac{1}{3} h^{cd} h^{ab}\right) \,, \tag{4.32}$$

where ρ, P, q_a, and π_{ab} as in Eq. (2.20)–(2.23).

Because of the presence of matter in the definition of the effective stress-energy tensor the fluid quantities read as the ones resulting from a mixture of tilted fluids [8, 448]. Only in the case of a minimally coupled scalar field, *i.e.* $G_4 = 1$ corresponding to the so-called kinetic gravity braiding, the proposed definition is equivalent to that of $T^{(\text{eff})}_{ab}$ [Eq. (2.4)].

It is now crucial to note that, in light of the above observations, the thermodynamic analogy with Eckart's first-order thermodynamics holds only for $T^{(\text{eff})}_{ab}$ and $T^{(f)}_{ab}$, for which the associated quantities do not explicitly contain matter contributions, and not for the alternative definition in Eq. (4.26). Hence, such a definition would have limited use within the thermodynamics of scalar-tensor gravity.

4.5 Main insights

In this chapter, we examined the foundations of the formalism underlying the Thermodynamics of Scalar-Tensor gravity. The exploration of the properties of the imperfect fluid decomposition for modified theories of gravity led us to a class of alternative formulations of the thermodynamic analogy connecting scalar-tensor theories with Eckart's first-order thermodynamics. In particular, we made clear the need for the identification of an effective stress-energy tensor such that standard matter does not mix with the additional (non-minimally coupled) scalar field contribution. The only additional free parameter, that does not spoil the thermodynamic analogy, turns out to be the "coupling" $f(\phi, X)$ of this effective stress-energy tensor for the ϕ-fluid to Einstein's gravity.

More in detail, in Section 4.2 we have shown that if we reformulate the thermodynamic formalism for an effective stress-energy tensor for the ϕ-fluid defined as in Eq. (4.2), then this identification preserves the form of the constitutive relations for the ϕ-fluid. This allows for the identification of the fluid quantities analogous to the standard formalism, up to a rescaling of a factor $[f(\phi, X)]^{-1}$, and to a slightly altered evolution equation for the temperature of gravity. Furthermore, when restricted to the traditional class of scalar-tensor theories the identification $f(\phi) = 1/\phi$ allows for a streamlined implementation of the original thermodynamic analogy, of the investigation of the evolution of the temperature of gravity, and it also has the additional perk that the ϕ-fluid couples to Einstein's gravity with the same strength as the matter fields.

These results not only reinforce the connection between alternative theories of gravity and non-equilibrium thermodynamics, but they also open new avenues for exploring peculiar thermal states and the thermal stability of these gravitational theories. Future work will focus on investigating the implications of this thermodynamic analogy further, particularly in the context of cosmological and astrophysical scenarios, where the effects of scalar fields could play a role.

5 Covariant non-singular bounces

According to standard cosmology, our Universe originated from an initial singularity known as the Big Bang. This classical picture posits that as we approach the singularity, the volume of the Universe tends to zero, while the energy density and temperature diverge within the curvature scalars. However, it is widely accepted that classical singularities and divergences represent limitations in our current understanding of physics, suggesting the need for a more comprehensive theory. They may be resolved by considering the effects of quantum gravity, which becomes dominant over the other interactions in extreme scenarios. Another approach to resolving the initial singularity is through a classical non-singular bounce occurring at an energy density sufficiently below the Planck scale. In this case, either a stable form of stress-energy that violates the null energy condition or a modification of Einstein's General Theory of Relativity (GR) that bypasses the null convergence condition is necessary. Although this may lead to instabilities in some scenarios [449], there are examples that have successfully avoided all known instabilities [450–453].

One of the most compelling frameworks suggests that quantum geometry might give rise to a short-range repulsive force in the Planck regime (otherwise negligible), thereby transforming the Big Bang singularity into a so-called quantum bounce. This is the case for theories such as Loop Quantum Cosmology (LQC), in which the bounce is described by an effective Friedmann equation [283, 285, 454].

In this context, we are interested in studying the viability of an effective Friedmann equation, describing the dynamics of a spatially flat isotropic and homogeneous bouncing universe. The standard equation is modified by assuming a maximum energy density value, ρ_c, and we question whether an effective formulation can remove the classical initial singularity. Moreover, we want to investigate covariant formulations of non-singular bounces wherein the modifications are sourced by a scalar field in second-order theories of gravity (extending GR).

The Universe is modeled by a spatially flat FLRW metric, $g_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2)$. Then, the *overdot* indicates the time derivative. Remember that we are using the reduced Planck units, $c = 8\pi G = \hbar = 1$.

Let us provide a brief overview of the initial singularity issue in GR. Evaluating Einstein's field equations on an FLRW background, one obtains the (first) Friedmann, $H^2 = \rho/3$, and the Raychaudhuri equation (or second Friedmann equation), $\ddot{a}/a = -(\rho + 3P)/6$, with $H \equiv \dot{a}/a$ denoting the Hubble rate, and ρ and $p = -2\dot{H} - 3H^2$ representing the homogeneous and isotropic energy density and pressure of a perfect fluid, respectively. The contracted Bianchi identity [55] guarantees the conservation of the stress-energy tensor[1], which yields the continuity equation $\dot{\rho} + 3H(\rho + P) = 0$. To solve the field equations, it is necessary to use an equation of state characterizing the matter content. The simplest assumption (used to describe standard matter) is a linear barotropic equation of state, $p = w\rho$, where w is the barotropic constant. Then, for $w \neq -1$, the continuity equation yields $\rho = \rho_* (a_*/a)^{3(1+w)}$ where the quantities with the *subscript* '$*$' are integration constants. Substituting the above equation into the first Friedmann equation, considering the case of $w > -1$, there exists a cosmological time, t_{BB}, such that the scale factor vanishes and, therefore, the energy density diverges to infinity.

For instance, assuming the existence of a quantum short-range repulsive force occurring at the Planck scale could solve the initial singularity. The general conditions for the occurrence of a bounce are $H = 0$ and $\ddot{a}/a > 0$, taking place when $\rho = \rho_c$ (or, equivalently, $t = t_c$). However, the standard Friedmann equation does not satisfy these conditions. Moreover, for the bounce to occur in GR, the matter must violate the null energy condition [455, 456], *i.e.*, $\rho + P > 0$. Thus, one aims to modify the Friedmann equation to mimic the bounce at early cosmological time and trace back the standard matter behavior at late cosmological time.

There are different ways and frameworks in which one can obtain modifications of the Friedmann equation leading to a bounce. In particular, for a flat FLRW universe, a simple effective Friedmann equation being suitable for a bouncing interpretation is achieved in LQC, $H^2 = \rho(1 - \rho/\rho_c)/3$, where the matter content satisfies the continuity equation $\dot{\rho} + 3H(\rho + P) = 0$ [286, 288–290]. The bounce takes place at $\rho = \rho_c$, corresponding to $H^2 = 0$. The quadratic ρ^2 term is relevant only at very high energy density values, $\rho \lesssim \rho_c$, in the proximity of the bounce. Otherwise, for $\rho \ll \rho_c$, the correction is negligible and the standard Friedmann equation is recovered. The GR limit is realized for $\rho_c \to \infty$, where the effective Friedmann equation is reduced to the standard one[2].

However, the quadratic correction is just one example of an effective Friedmann equation that can be obtained in LQC. For instance, looking for higher-order quantum corrections to the field strength [457], it is possible to get a different modification with a critical density value shifted to the higher energies, while, modifying the handling of the Lorentzian term inside the Hamiltonian it is possible to obtain an asymmetric bounce scenario where the evolution of the universe is described by two different branches [458–460], before and after the bounce. The presence of multiple modifications to the standard Friedmann equation results from the lack of a systematic derivation from the Loop Quantum Gravity framework and quantization ambiguities [461–463]. Finally, in any LQC description, one can always obtain a modified Raychaudhuri equation by differentiating the modified Friedmann equation and using the continuity equation [464–466]. In the simpler case of the quadratic correction, it yields $\ddot{a}/a = -\frac{1}{6}(\rho + 3P)\left(1 - 2\,\frac{2\rho+3P}{\rho+3P}\,\rho/\rho_c\right)$. Then, one can identify the effective energy density and the effective pressure, $\rho_{(\text{eff})} = \rho(1 - \rho/\rho_c)$ and $P_{(\text{eff})} = P(1 - 2\rho/\rho_c) - \rho^2/\rho_c$, for which the usual continuity equation holds, $\dot{\rho}_{(\text{eff})} + 3H(\rho_{(\text{eff})} + P_{(\text{eff})}) = 0$.

Inspired by the depicted framework, one can make purely classical considerations to assess the feasibility of an effective Friedmann equation in removing the initial singularity through a classical non-singular bounce.

The novel effective approach is based on the assumption that the effective theory can be written perturbatively in terms of another UV-divergent (benchmark) theory and an intrinsic energy scale,

[1] It is well known that the continuity equation can be obtained by differentiating the first Friedmann equation and rewriting $\dot{H} = \ddot{a}/a - H^2$ in terms of energy density and pressure. Similarly, the second Friedmann equation can be obtained by differentiating the first Friedmann equation and using the continuity equation to rewrite $\dot{\rho}$.

[2] In LQC, it is obtained that $\rho_c \sim c^5/G^2\hbar$ in agreement with the quantum gravitational origin of the bounce. Then, the classical GR limit corresponds to a vanishing Planck length, *i.e.*, $\hbar \to 0 \Rightarrow \rho_c \to \infty$.

ρ_c. The latter represents the maximal value reachable by the energy density in the benchmark theory. The critical energy density constitutes the parameter that explicitly modifies the Friedmann equation. The effective correction must ensure the finiteness of all observables when evaluated at the critical point leading to a finite curvature value. Therefore, the benchmark theory and the effective theory only differ in the presence of a maximal energy density value. The standard equations correspond to the limit as ρ_c approaches infinity. We refer to *standard quantities* as parametrically independent of ρ_c and are associated with the benchmark theory, denoted by the subscript '0'.

In Sec. 5.1, we derive the necessary properties holding for a general effective modification of the Friedmann equation encoding a bouncing behavior. The effective approach imposes some constraints on the effective model in the proximity of the bounce, implying that the standard energy density and the pressure satisfy the null energy condition.

In the context of modified theories of gravity [146–148, 150, 169, 467, 468], there are many realizations of repulsive effects generating a non-singular bounce (not necessarily associated with a quantum gravity origin) which are usually characterized by additional degrees of freedom compared to GR (see also [429, 450, 451, 469–480]). Moreover, there are different methods to reconstruct the modified theory mimicking an LQC-like bounce. Some of them are characterized by what is called the *order reduction* technique [481, 482] applied to metric theories of gravity [10, 352, 483–486], and others are developed in the context of Palatini theories [487–490]. However, the novel analysis can open new prospects and criticisms of the realization of cosmic bounces in effective theories.

In particular, we analyze the possibility of reproducing a non-singular cosmological bounce in a covariant formulation through a single classical scalar field. The covariant formulation must encode a bouncing behavior compatible with the effective modification to the first Friedman equation. The scalar field is introduced into the model through modifications of the Einstein–Hilbert action associated with the k-essence model [265, 436, 491, 492] or general scalar-tensor theories [241, 305, 315, 382, 493]. The modifications can be seen as effectively describing an exotic matter sector or departures from GR dominating at early cosmological time so that other matter species filling the universe can be neglected. The additional contributions to the GR action should automatically guarantee a bouncing mechanism. This means that the effective theory $S_{(\text{eff})}$ is obtained from the benchmark one $S_{(0)}$ through a *minimal modification* preserving the scalar field equation of motion, analogously to LQC equations, such that $\lim_{\rho_c\to\infty} S_{(\text{eff})} = S_{(0)}$. Therefore, the benchmark theory is, in general, a smaller subclass of theories contained within the class to which the effective theory belongs, $S_{(0)} \subseteq S$. However, the request for a covariant effective bouncing behavior imposes the benchmark theory to belong to the same scalar-tensor class of the effective theory. For this reason, it is possible, and more convenient, to consider separately the cases of minimal and non-minimal coupled scalar fields.

It is important to stress that the results obtained from this analysis directly arise from the primary assumption of the presented approach, *i.e.*, that the non-singular effective theory can be written perturbatively in terms of a benchmark theory, for which the standard equations hold, and an intrinsic energy scale, ρ_c. This assumption, along with the requirement of a covariant formulation for the effective theory, constrains the possible realizations of the effective cosmological bounce.

In Sec. 5.2, we discuss the case of a general k-essence in describing the bounce. As a result, covariant effective bounces can be formulated only by a non-dynamical scalar field, *i.e.*, with no propagating degrees of freedom, the cuscuton model [422, 494, 495]. In Sec. 5.3, we extend our treatment to a non-minimally coupled scalar field, showing that, also in this case, a covariant description of the effective bounce selects a non-dynamical scalar field representing a particular case of the extended cuscuton model [420, 421, 496, 497]. We highlight some final comments in

Sec. 5.4.

5.1 Effective cosmological bounces

To introduce and study the properties of the effective cosmological bounce, we denote the quantities associated with the standard Friedmann equation by the subscript '0'. These quantities are parametrically independent of ρ_c. For example, the energy density and the pressure of the benchmark theory are $\rho_{(0)}$ and $P_{(0)}$, respectively.

The effective Friedmann equation must approach the standard form $H^2_{(0)} = \rho_{(0)}/3$ in the benchmark limit. Saying that the effective UV-complete theory can be written as a perturbative expansion of another UV-divergent benchmark theory implies that the standard variables evolve according to the standard continuity equation, $\dot{\rho}_{(0)} + 3H_{(0)}(\rho_{(0)} + P_{(0)}) = 0$, as it represents the zeroth perturbative order, and the equations should indeed be satisfied order-by-order. Thus, the modified Friedmann equation we are searching for must admit the following parameterization,

$$H^2 = \frac{1}{3}\rho_{(0)} f(x)\,, \tag{5.1}$$

where f is a generic function of the dimensionless variable $x \equiv 1 - \rho_{(0)}/\rho_c \in [0, 1]$. Moreover, let f be differentiable and positive in $(0, 1)$, such that $\lim_{x\to 0^+} f(x) = 0^+$. Then, the benchmark limit corresponds to $x \to 1$ ($\rho_c \to \infty$), while the bounce is reached at the limit of $x \to 0^+$ ($\rho_{(0)} \to \rho_c$).

Notice that in the case of an asymmetric bounce, one would have, for instance, f_- and f_+ representing the situation before and after the bounce, respectively. Here, we are not interested in distinguishing different branches because the following discussion must be valid for both, independently of the matching conditions.

Differentiating the left-hand side of Eq. (5.1) gives

$$\frac{d(H^2)}{dt} = 2H\dot{H} = 2H_0\sqrt{f(x)}\,\dot{H}\,, \tag{5.2}$$

because of $H = H_{(0)}\sqrt{f(x)}$; while, from the right-hand side, using the standard continuity equation, one obtains

$$\begin{aligned} \frac{1}{3}\frac{d\left(\rho_0 f(x)\right)}{dt} &= \frac{1}{3}\left[f(x)\dot{\rho}_0 - \frac{\rho_0}{\rho_c}f'(x)\dot{\rho}_0\right] \\ &= \frac{\dot{\rho}_0}{3}\left[f(x) - (1-x)f'(x)\right] \\ &= -H_0(\rho_0 + P_0)\left[f(x) - (1-x)f'(x)\right]\,, \end{aligned} \tag{5.3}$$

where the prime represents the derivative with respect to the variable x. Then, solving for $\dot{H}$, it turns out

$$\dot{H} = -\frac{1}{2}\left(\rho_{(0)} + P_{(0)}\right)\frac{f(x) - (1-x)\,f'(x)}{\sqrt{f(x)}}\,, \tag{5.4}$$

Thus, the modified Raychaudhuri equation associated with Eq. (5.1) reads

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\left[3\left(\rho_{(0)} + P_{(0)}\right)\frac{f(x) - (1-x)\,f'(x)}{\sqrt{f(x)}} - 2\rho_{(0)}f(x)\right]. \tag{5.5}$$

Analogously to LQC, it is possible to define an effective energy density and effective pressure to recast the modified Friedman in the standard form,

$$\rho_{(\text{eff})} = \rho_{(0)} f(x), \tag{5.6}$$

$$P_{(\text{eff})} = \left(\rho_{(0)} + P_{(0)}\right) \frac{f(x) - (1-x) f'(x)}{\sqrt{f(x)}} - \rho_{(0)} f(x), \tag{5.7}$$

for which the continuity equation $\dot{\rho}_{(\text{eff})} + 3H(\rho_{(\text{eff})} + P_{(\text{eff})}) = 0$ holds. Then, the limit $x \to 1$ provides the asymptotic benchmark behavior, $\rho_{(\text{eff})} \to \rho_{(0)}$ and $P_{(\text{eff})} \to P_{(0)}$.

In correspondence of $\rho_{(0)} = \rho_c$, our system must satisfy the conditions $H = 0$ and $0 < \ddot{a}/a < \infty$ to have a viable cosmological bounce. Since the first condition is satisfied by definition, the only condition to impose is the second one, or equivalently, $0 < \dot{H} < \infty$ at the bounce. In this regard, let us analyze the contribution due to the presence of f in Eq. (5.4). Approaching the bounce,

$$\lim_{x\to 0} \frac{f(x) - (1-x) f'(x)}{\sqrt{f(x)}} = -\lim_{x\to 0} \frac{f'(x)}{\sqrt{f(x)}}\,. \tag{5.8}$$

It is possible to prove that the above limit is always non-positive. Since $f(x)$ is positive-definite on $(0, 1]$ and approaches zero as x approaches 0^+, its first derivative must be positive in the immediate proximity of zero, while it can vanish at the bounce[3], namely, $\lim_{x\to 0^+} f'(x) \geq 0$. Consequently, the above limit can only be zero, finite, or vanishing, corresponding to $f'(x) \sim o(\sqrt{f(x)})$, $f'(x) \sim \sqrt{f(x)}$, and $f'(0^+) > 0$, respectively.

To ensure $\dot{H}$ is finite at the bounce, we require that $f'(x) \sim \sqrt{f(x)}$, so that

$$\lim_{x\to 0} \frac{f'(x)}{\sqrt{f(x)}} = \ell\,, \tag{5.9}$$

where ℓ is a constant. In Eq. (5.9), we are evaluating the ratio of two positive quantities in the limit as both the numerator and the denominator approach zero. This implies that the only case in which the above limit is finite corresponds to $\ell > 0$. Then, the defining properties of f imply that Eq. (5.4) is compatible with an effective cosmological bounce only for

$$f\,:\, \lim_{x\to 0^+} \frac{f'(x)}{\sqrt{f(x)}} = \ell > 0\,. \tag{5.10}$$

Assuming f is sufficiently regular, differentiating the numerator and denominator, one obtains

$$\ell = \lim_{x\to 0^+} \frac{f'(x)}{\sqrt{f(x)}} = \lim_{x\to 0^+} 2 f''(x) \frac{\sqrt{f(x)}}{f'(x)} = \lim_{x\to 0^+} \frac{2}{\ell} f''(x)$$

$$\Rightarrow \quad \lim_{x\to 0^+} f''(x) = \frac{\ell^2}{2} > 0\,. \tag{5.11}$$

[3] If $f'(x)$ were negative in the immediate proximity of zero—meaning there is no arbitrarily small neighborhood $(0, \epsilon]$ where $f'(x)$ is positive—then $f(x)$ would take on negative values, contradicting the positive-definiteness hypothesis.

Therefore, approaching the critical energy density, the conditions ensuring a viable cosmological bounce described by a general effective Friedmann equation (5.1) are the following:

$$\begin{cases} f(x) \sim 0\,, \\ f'(x) \sim \ell\sqrt{f(x)}\,, \\ f''(x) \sim \ell^2/2\,. \end{cases} \tag{5.12}$$

Integrating the above asymptotic behavior yields

$$f(x) \sim \frac{\ell^2}{4}x^2\,. \tag{5.13}$$

This analysis has an important consequence: the benchmark theory must satisfy the null energy condition at the bounce. Indeed, at the bounce, Eq. (5.4) reads

$$\dot{H} = \frac{1}{2}\left(\rho_c + P_c\right)\ell\,, \tag{5.14}$$

where $P_c = P_{(0)}\big|_{t=t_c}$ is the critical value of the pressure, with t_c being the time at which the bounce occurs. Knowing from the above discussion that ℓ is positive, the condition $\dot{H} > 0$ at the bounce implies that the standard matter must satisfy the null energy condition, $\rho_{(0)} + P_{(0)} > 0$. This general result naturally comes from the request that Eq. (5.1) describes a general effective cosmological bounce (admitting a perturbative approach) and holds for any benchmark theory[4]. It is worth stressing that, while it is well known that $\rho_{(\text{eff})}$ and $P_{(\text{eff})}$ must violate the null energy condition to have a bounce, here, the standard variables are the ones that must satisfy it (at the bounce).

From Eq. (5.13), a class of regular solutions is characterized by $f(x) = h(x)\,x^2$ where $h(x)$ is a regular positive-definite function[5] in $x \in [0,1]$ such that $\ell = 2\sqrt{h(0)}$. The simplest case is for h to be constant. In particular, let us consider the case $h(x) = 1$ corresponding to $f(x) = x^2$ and $\ell = 2$. Then, effective energy density and pressure turn out to be

$$\rho_{(\text{eff})} = \rho_{(0)}\left(1 - \frac{\rho_{(0)}}{\rho_c}\right)^2\,, \tag{5.15}$$

$$P_{(\text{eff})} = \left(\rho_{(0)} + P_{(0)}\right)\left(1 - 3\frac{\rho_{(0)}}{\rho_c}\right) - \rho_{(0)}\left(1 - \frac{\rho_{(0)}}{\rho_c}\right)^2\,, \tag{5.16}$$

and the modified Friedmann equations read

$$H^2 = \frac{1}{3}\rho_{(0)}\left(1 - \frac{\rho_{(0)}}{\rho_c}\right)^2\,, \tag{5.17}$$

[4] Notice that considering the pathological cases $f'(0^+) > 0$ and $f'(x) \sim o(\sqrt{f(x)})$, the sum $\rho_{(0)} + P_{(0)}$ should properly go to zero and diverge, respectively, to guarantee the condition $\dot{H} > 0$. One could include those cases by requiring $\rho_{(0)} + P_{(0)} \geq 0$. However, they are out of our interest since it is possible to demonstrate that they do not admit a covariant single-field formulation of an effective cosmological bounce.

[5] If $f(x)$ is analytic, it is always possible to write $f(x) = h(x)x^2$ where $f(x) = \sum_{n=0}^{\infty} f^{(n+2)}(0)x^n/(n+2)!$.

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\left[3\left(\rho_{(0)} + P_{(0)}\right)\left(1 - 3\frac{\rho_{(0)}}{\rho_c}\right) - 2\,\rho_{(0)}\left(1 - \frac{\rho_{(0)}}{\rho_c}\right)^2\right]. \tag{5.18}$$

The above one is the simplest effective cosmological bounce realization. In the limit for $\rho_c \to \infty$ it reproduces the standard Friedmann equations, and for $\rho_{(0)} \to \rho_c$ it turns out $\rho_{(\text{eff})} = 0$ and $P_{(\text{eff})} = -2\,(\rho_c + P_c)$. Then, the condition $(\rho_c + P_c) > 0$ guarantees the bounce, associated with a negative effective pressure.

From Eq. (5.17), the effective Hubble rate can be written as follows,

$$H = H_{(0)}\left(1 - \frac{3H_{(0)}^2}{\rho_c}\right). \tag{5.19}$$

Then, assuming a linear barotropic equation of state for the standard matter, $P_{(0)} = w\rho_{(0)}$, one takes back the usual GR results: *i.e.*, for $w \neq -1$, according to the continuity equation, it is found that

$$\dot{\rho}_{(0)} + 3\frac{\dot{a}_{(0)}}{a_{(0)}}(1+w)\rho_{(0)} = 0 \quad \Rightarrow \quad \rho_{(0)} = \rho_*\left(\frac{a_*}{a_{(0)}}\right)^{3(1+w)}, \tag{5.20}$$

and, from the standard Friedmann equation, the scale factor reads

$$a_{(0)} = a_*\left[1 + \frac{3}{2}(w+1)H_*\,(t - t_*)\right]^{\frac{2}{3(w+1)}}. \tag{5.21}$$

Finally, in terms of the shifted time $\tau = t - t_* + \frac{2}{3H_*(w+1)}$, the benchmark energy density and the Hubble rate turn out to be

$$\rho_{(0)} = \frac{4}{3\tau^2(w+1)^2}, \tag{5.22}$$

$$H = \frac{2}{3}\frac{1}{\tau(w+1)}\left(1 - \frac{4}{3\tau^2(w+1)^2\rho_c}\right), \tag{5.23}$$

showing that the bounce takes place at $\tau = \tau_c$, where

$$\tau_c = \frac{2}{\sqrt{3\rho_c}}\frac{1}{(w+1)}, \tag{5.24}$$

which approaches zero, the classical initial singularity, in the limit $\rho_c \to \infty$.

It is important to stress that the effective model (5.17) is used just as an example, being one of the simplest analytical solutions of an effective viable bounce within the framework under consideration. However, even if this particular choice ($h = 1 \Rightarrow \ell = 2$) lacks a compelling physical motivation, it encodes the asymptotic behavior (5.13). Looking for f being a third-order polynomial, the simplest realization generalizing the previous example is $h(x) = x + (1-x)\ell^2/4$.

The effective bounce model inherits stability features from the asymptotic benchmark theory. The perturbation of the effective part decays as the universe moves away from the bounce, indicating that the bouncing behavior is stable against small perturbations. The easiest way to check the stability property is by perturbing the system $\rho_{(0)} \to \rho_{(0)} + \delta\rho_{(0)}$ close to the solution (5.22) and verifying that the perturbed equations describe a non-singular bounce in correspondence with $\tau = \tau_c + \delta\tau$. The additional terms in the Friedmann equation can be neglected as soon as we move from the bounce.

To concretely appreciate this property, we can take into account the solution corresponding to a linear barotropic fluid $P_{(0)} = w\rho_{(0)}$,

$$\rho_{(0)} = \frac{4}{3\tau^2(w+1)^2} \tag{5.25}$$

where τ is the shifted-time (*i.e.*, $\tau = 0$ corresponds to the standard initial singularity). Then, perturbing the continuity equation $\dot{\rho}_{(0)} + 3H_{(0)}\left(P_{(0)} + \rho_{(0)}\right) = 0$ at the first order, $\rho_{(0)} \to \rho_{(0)} + \delta\rho_{(0)}$, one obtains the following equation for the perturbation,

$$\delta\dot{\rho}_{(0)} + \frac{3\delta\rho_{(0)}}{\tau} = 0\,. \tag{5.26}$$

The solution of the above ordinary differential equation is

$$\delta\rho_{(0)} = \frac{\lambda}{\tau^3}\,, \tag{5.27}$$

where λ is a dimensional integration constant. Then, the perturbed effective Friedmann equation turns out to be

$$H^2 = \frac{1}{3}\left(\frac{4}{3\tau^2(w+1)^2} + \frac{\lambda}{\tau^3}\right)\left(1 - \frac{4}{3\tau^2(w+1)^2\rho_c} - \frac{\lambda}{\tau^3\rho_c}\right)^2. \tag{5.28}$$

Perturbing around the bounce, the integration constant λ can be appropriately set such that H vanishes for $\tau = \tau_c + \delta\tau$, where

$$\tau_c = \frac{2}{\sqrt{3\rho_c}}\frac{1}{(w+1)}\,. \tag{5.29}$$

In this way, one obtains

$$\lambda = \delta\tau^3\rho c + \frac{2\delta\tau^2\sqrt{3\rho_c}}{w+1} + \frac{8\delta\tau}{3(w+1)^2}\,. \tag{5.30}$$

Then, in correspondence with $\tau = \tau_c + \delta\tau$, the perturbed equation for $\dot{H}$ reads

$$\dot{H} = (1+w)\rho_c > 0\,, \tag{5.31}$$

with $w > -1$ because the benchmark theory must satisfy the NEC at the bounce. The independence from $\delta\tau$ means that the bouncing behavior is stable against small perturbations.

The presented framework is also compatible with LQC formulations. One needs to remember the only assumption on which the discussion is based: the effective bouncing theory can be written as a UV-diverging benchmark theory. Therefore, for instance, the effective Friedmann equation $H^2 = \rho\,(1 - \rho/\rho_c)/3$ must be rewritten to make explicit $\rho_{(0)}$ and ρ_c dependencies. Indeed, the energy density ρ does parametrically depend on the critical energy because it is associated with the continuity equation $\dot{\rho} + 3H(\rho + P) = 0$ where $H = H_{(0)}\sqrt{f(x)}$, in our framework. This implies that LQC energy density must admit a parameterization of the following form,

$$\rho = \rho_{(0)}\,\varepsilon(x)\,, \tag{5.32}$$

where $\varepsilon(x)$ is a positive-definite continuous function in $[0,1]$, with $\lim_{x\to 1^-}\varepsilon(x)=1$, to obtain the right standard limit, and $\lim_{x\to 0^+}\varepsilon(x)=1$, such that ρ_c is the maximal value of energy density[6]. Then, in the case of the quadratic correction, $f(x)=\varepsilon(x)\,[\,1-(1-x)\,\varepsilon(x)\,]$, imposing the effective cosmological bounce conditions (5.12) and (5.13), it turns out that $\lim_{x\to 0^+}\varepsilon'(x)=1$ and $\lim_{x\to 0^+}\varepsilon''(x)=2-\ell^2/2$. To give an example, assuming $\ell=2$, a simple analytical realization of the quadratic LQC correction is $\varepsilon(x)=1+x-x^3$ which corresponds to $f(x)=x^7-x^6-2x^5+2x^3+x^2$. Keeping general the value of ℓ, the simplest case is $\varepsilon(x)=1-(1-x)x\left[x\left(\ell^2-8\right)-4\right]/4$.

Finally, using the continuity equation $\dot\rho+3H(\rho+P)=0$, it is possible to obtain the expression of the pressure p associated with ρ, which, at the bounce, reads as $p\big|_{t=t_c}=\ell\,P_c+(\ell-1)\,\rho_c$. Consequently, at the bounce, $\rho+P=\ell\,(\rho_c+P_c)>0$, meaning that LQC variables also satisfy the null energy condition in this framework.

5.2 Minimally coupled scalar field

In this section, we study the possibility of implementing a general effective cosmological bounce in a covariant way, by taking into account a general k-essence model [492],

$$S=\int d^4x\sqrt{-g}\left[\frac{1}{2}R+L(\phi,X)\right], \tag{5.1}$$

with $X=-\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi$ and $\sqrt{-g}$ being the square root of the metric determinant.

The field equations of the above theory are given by

$$G_{\mu\nu}=L_X\nabla_\mu\phi\nabla_\nu\phi+L\,g_{\mu\nu}\,, \tag{5.2}$$

coming from the variation of (5.1) with respect to the metric tensor, while the variation of the action relative to the scalar field gives

$$\left(L_Xg^{\alpha\beta}-L_{XX}\nabla^\alpha\phi\nabla^\beta\phi\right)\nabla_\alpha\nabla_\beta\phi-2XL_{X\phi}+L_\phi=0\,, \tag{5.3}$$

where $L_{,\phi}\equiv\partial L/\partial\phi$ and $L_X\equiv\partial L/\partial X$.

For a spatially flat FLRW spacetime with a homogeneous scalar field, $\phi=\phi(t)$ and $X=\frac{1}{2}\dot\phi^2$, the scalar field is described by a perfect fluid stress-energy tensor [436]. It is possible to interpret the right-hand side of Eq. (5.2) as the stress-energy tensor, $T^{(\phi)}_{\mu\nu}$, associated with the scalar field,

$$T^{(\phi)}_{\mu\nu}=\rho^{(\phi)}\,u_\mu u_\nu+P^{(\phi)}\,h_{\mu\nu}\,, \tag{5.4}$$

where $h_{\mu\nu}=g_{\mu\nu}+u_\mu u_\nu$, and

$$\rho^{(\phi)}=2X\,L_X-L\,,\qquad P^{(\phi)}=L\,. \tag{5.5}$$

with $u^\mu=\nabla^\mu\phi/\sqrt{2X}=-\,\mathrm{Sign}(\dot\phi)\,\delta^\mu_t$. Hence, the partial differential equation (5.3) is hyperbolic and ϕ describes a propagating degree of freedom if the inequality $L_X+2XL_{XX}>0$ is satisfied.

[6]This condition can be relaxed by assuming that $\lim_{x\to x_c^+}\varepsilon(x)>0$ is finite, so that $\rho=\rho_{(0)}\,\varepsilon\to\rho_c$ corresponds to the bounce. Therefore, as ρ approaches ρ_c, the variable $x=1-\rho_{(0)}/\rho_c$ approaches $x_c=1-\rho_{(0)c}/\rho_c$, where $\rho_{(0)c}$ and x_c represent the maximum values of $\rho_{(0)}$ and $x\in[0,x_c]$, respectively, and $\varepsilon(x)$ tends to $\rho_c/\rho_{(0)c}$. The final result does not change with respect to our purpose.

The equation of motion for the scalar field is linked to the conservation of the stress-energy tensor; indeed, the covariant derivative of the scalar field stress-energy tensor is proportional to the gradient of the scalar field through the left-hand side of Eq. (5.3), *i.e.*, $\nabla^\mu T^{(\phi)}_{\mu\nu} = -\frac{1}{2}(\delta L/\delta\phi)\nabla_\nu\phi$ which is vanishing on-shell.

In order to model the effective bounce through $L(\phi, X)$ representing the effective theory, let ϕ be a scalar field independent of ρ_c and associated with $L_{(0)}(\phi, X)$, the asymptotic benchmark theory[7]. The quantities $\rho^{(\phi)}$ and $P^{(\phi)}$ correspond to the effective energy density and pressure in Eqs. (5.6) and (5.7), respectively. The standard quantities are associated with the Lagrangian $L_{(0)}(\phi, X)$ and read as follows,

$$\rho_{(0)} = 2X\, L_{(0)X} - L_{(0)}\,, \qquad P_{(0)} = L_{(0)}\,. \tag{5.6}$$

Therefore, we investigate the functional form of $L(\phi, X)$ such that $\lim_{\rho_c\to\infty} L(\phi, X) = L_{(0)}(\phi, X)$ and leading to an effective cosmological bounce. Thus, it is necessary to impose that the energy density and pressure (5.5) behave as the effective energy density and pressure, while (5.6) are the standard one.

Using the above standard quantities, from Eqs. (5.6), (5.7) and (5.5), one can derive the following equations,

$$2X\, L_X - L = \rho_{(0)}\, f(x)\,, \tag{5.7}$$

$$L = 2X\, L_{(0)X}\, \frac{f(x) - (1-x)\, f'(x)}{\sqrt{f(x)}} - \rho_{(0)}\, f(x), \tag{5.8}$$

where $x = 1 - \rho_{(0)}/\rho_c$ and $2X\, L_{(0)X} = \rho_{(0)} + P_{(0)}$. Now, substituting Eq. (5.8) into Eq. (5.7) yields

$$L_X = L_{(0)X}\, \frac{f(x) - (1-x)\, f'(x)}{\sqrt{f(x)}}\,. \tag{5.9}$$

The above equation must be equal to the derivative with respect to X of Eq. (5.8), giving a consistency equation that reads as follows,

$$\begin{aligned} &\left\{ \frac{X\, L_{(0)X}}{\rho_c}\, \frac{(x-1)\, f'(x)^2 - f(x)\left[\, 2\,(x-1)\, f''(x) + 3 f'(x)\,\right]}{f(x)^{3/2}} \right. \\ &\qquad \left. - \frac{(x-1)\ f'(x) + f(x)}{\sqrt{f(x)}/(\sqrt{f(x)} - 1)} \right\} \left(L_{(0)X} + 2X\, L_{(0)XX} \right) = 0. \end{aligned} \tag{5.10}$$

Apparently, the consistency equation seems to constrain both $L_{(0)}$ and f. It is satisfied if and only if one of the two brackets is vanishing.

In particular, the case of

$$L_{(0)X} + 2X\, L_{(0)XX} = 0\,, \tag{5.11}$$

corresponds to

$$L_{(0)} = \mu(\phi)\, \sqrt{2X} - V(\phi)\,, \tag{5.12}$$

[7] We can always write the effective theory in terms of the standard variables.

where μ and V are two arbitrary functions of the scalar field, representing a non-dynamical scalar field usually called *cuscuton*[8] [420–422, 494–497].

The first curly bracket in Eq. (5.10) is meaningful only if

$$(1+w)\, L_{(0)} - 2wXL_{(0)X} = 0\,, \tag{5.13}$$

where $w \neq -1$ is constant. The last equation is equivalent to requiring that $2X\, L_{(0)X} = (1+w)\,\rho_{(0)}$, which comes from the fact that the linear term in $1/\rho_c$ must be proportional to $1-x$. In this way, Eq. (5.10) is a well-defined differential equation for $f(x)$. Moreover, solving Eq. (5.13), it turns out

$$L_{(0)} = \mu(\phi)\, X^{\frac{w+1}{2w}}\,. \tag{5.14}$$

In this case, the scalar field is associated with the barotropic equation of state $P_{(0)} = w\rho_{(0)}$, where w must be greater than -1 to guarantee $\rho_{(0)} + P_{(0)} > 0$ (a necessary condition for the bounce occurrence as shown in the previous section)[9]. Then the differential equation for f reads as follows,

$$\begin{aligned}&(w+1)(1-x)^2\frac{f'(x)^2 - f(x)\left[2f''(x) + 3f'(x)/(1-x)\right]}{f(x)^{3/2}}\\&+2\left(\sqrt{f(x)}-1\right)\frac{(1-x)\, f'(x) + f(x)}{\sqrt{f(x)}} = 0\,.\end{aligned} \tag{5.15}$$

The above differential equation must be defined in the entire interval $x \in [0,1]$, and, in particular, in correspondence with the bounce. Evaluating the above equation for x approaching 0^+, according to Eqs. (5.12) and (5.13), one obtains

$$(3w+5)\,\ell = 0 \quad \Rightarrow \quad w = -\frac{5}{3} \quad \Rightarrow \quad \rho_{(0)} + P_{(0)} < 0\,. \tag{5.16}$$

Thus, the Lagrangian (5.14) cannot describe any effective bounce because it violates the null energy condition necessary for removing the singularity in Eq. (5.4). Then, only the cuscuton field can provide a covariant effective formulation. Then, the limiting curvature mechanism is realized by the Lagrangian (5.8) where $\rho_{(0)}$ and $L_{(0)}$ are given respectively by Eqs. (5.6) and (5.12), respectively.

5.3 Non-minimal coupling case

We can repeat a similar analysis to the one in the previous section, taking into account the viable (or reduced) Horndeski gravity [241, 315, 382], where the adjective "viable" is used to indicate Horndeski subclass in which tensor perturbations propagate at the speed of light [247, 320, 408, 409].

Using the modern notation of the Horndeski functions, the action reads as follows,

$$S = \frac{1}{2}\int d^4x\sqrt{-g}\,\left[G_4(\phi)R + G_2(\phi,X) - G_3(\phi,X)\Box\phi\right], \tag{5.1}$$

where $\Box\phi = \nabla_\alpha\nabla^\alpha\phi$, the function G_2 represents the k-essence contribution, G_3 is the kinetic braiding term, and G_4 is the non-minimal coupling function. The field equations associated with

[8] It is worth highlighting that the cuscuton model is not dynamical only on a cosmological background where the scalar field is assumed to have a time-like gradient (see also [5]).

[9] For $w = 1$ and redefining ϕ, Eq. (5.14) can be reduced to the standard massless scalar field.

the action (5.1) can be written in terms of effective Einstein equations, $G_{\alpha\beta} = T^{(\phi)}_{\alpha\beta}$, where the scalar field stress-energy tensor is given by the sum of the individual contributions associated with G_i functions,

$$T^{(2)}_{\alpha\beta} = \frac{1}{2G_4}\left(G_{2X}\nabla_\alpha\phi\nabla_\beta\phi + G_2\, g_{\alpha\beta}\right) , \tag{5.2}$$

$$\begin{aligned} T^{(3)}_{\alpha\beta} =& \frac{\left(G_{3X}\nabla_\gamma X\nabla^\gamma\phi - 2XG_{3\phi}\right)}{2G_4} g_{\alpha\beta} - \frac{G_{3X}}{G_4}\nabla_{(\alpha}X\nabla_{\beta)}\phi \\ &- \frac{\left(2G_{3\phi} + G_{3X}\Box\phi\right)}{2G_4}\nabla_\alpha\phi\nabla_\beta\phi \end{aligned} \tag{5.3}$$

$$\begin{aligned} T^{(4)}_{\alpha\beta} =& \frac{G_{4\phi}}{G_4}\left(\nabla_\alpha\nabla_\beta\phi - g_{\alpha\beta}\Box\phi\right) \\ &+ \frac{G_{4\phi\phi}}{G_4}\left(\nabla_\alpha\phi\nabla_\beta\phi + 2X\, g_{\alpha\beta}\right) . \end{aligned} \tag{5.4}$$

where, parentheses encompassing indices indicate the symmetrization, *i.e.*, $V_{(\alpha\beta)} = (V_{\alpha\beta} + V_{\beta\alpha})/2$. In addition, the equation of motion of the scalar field guarantees the conservation of the effective stress-energy tensor, $\nabla^\nu T^{(\phi)}_{\mu\nu} = -\frac{1}{2}(\delta L/\delta\phi)\nabla_\mu\phi\big|_{\text{on-shell}} = 0$.

Because of the presence of G_4 and G_3, the effective tensor $T^{(\phi)}_{\mu\nu}$ behaves as an imperfect fluid [305, 411, 433, 446] characterized by dissipative (viscous) contributions inside the effective energy density and the effective pressure [4, 5, 417]. Specializing the discussion to the spatially flat FLRW geometry, the energy density and the pressure associated with the scalar field read as follows,

$$\begin{aligned} \rho^{(\phi)} &= \bar{\rho} + 3H\xi , \\ P^{(\phi)} &= \bar{p} - 3H\zeta + \xi\,\ddot{\phi}/\sqrt{2X} \end{aligned} \tag{5.5}$$

where,

$$\bar{\rho} = \frac{\left(2XG_{2X} - G_2 - 2XG_{3\phi}\right)}{2G_4} , \tag{5.6}$$

$$\bar{p} = \frac{\left(G_2 - 2XG_{3\phi} + 4XG_{4\phi\phi}\right)}{2G_4} , \tag{5.7}$$

$$\xi = \frac{\sqrt{2X}\left(G_{4\phi} - XG_{3X}\right)}{G_4} , \tag{5.8}$$

$$\zeta = \frac{2}{3}\frac{\sqrt{2X}\, G_{4\phi}}{G_4} . \tag{5.9}$$

Analogously to the previous section, we want to use the functions G_i to model a general effective cosmological bounce. Therefore, Eq. (5.5) corresponds to the effective quantities in

Eqs. (5.6) and (5.7). Then, the standard energy density read

$$\rho_{(0)} = \bar{\rho}_{(0)} + 3H_{(0)}\xi_{(0)} ,$$
$$P_{(0)} = \bar{p}_{(0)} - 3H_{(0)}\zeta_{(0)} + \xi_{(0)}\, \ddot{\phi}/\sqrt{2X} \tag{5.10}$$

but with the subscript '0' meaning that they are associated with the benchmark functions $G_{(0)i} = \lim_{\rho_c \to \infty} G_i$. Using the above quantities, and imposing Eqs. (5.5) and (5.10) satisfy Eqs. (5.6) and (5.7), it is possible to obtain the functional form of the effective theory.

At this point, the analysis can be performed through two alternative ways, bringing us to the same conclusion: requiring that the different contributions to effective and standard quantities are connected according to Eqs. (5.6) and (5.7); rewriting the $\ddot{\phi}$ contribution in terms of the Hubble rate and Horndeski functions by using the field equation of the scalar field [5]. In both cases, it turns out

$$\xi = 0 \quad \Rightarrow \quad G_3 = G_{4\phi}\, \ln \overline{X} , \tag{5.11}$$

$$\xi_{(0)} = 0 \quad \Rightarrow \quad G_{(0)3} = G_{(0)4\phi}\, \ln \overline{X} , \tag{5.12}$$

where $\overline{X} = X/X_*$ is used to make the logarithm argument dimensionless[10], and,

$$\frac{G_{4\phi}}{G_4} = \frac{G_{(0)4\phi}}{G_{(0)4}} \left(\frac{f(x) - (1-x)\, f'(x)}{f(x)} \right) . \tag{5.13}$$

Since G_4 and $G_{(0)4}$ are only functions of the scalar field, the above equation implies that x must be independent of the kinetic term, namely, $\rho_{(0)}$ depends only on the scalar field. Imposing this condition yields the following differential equation,

$$G_{(0)2X} + 2XG_{(0)2XX} = 2\left(1 + \ln \bar{X}\right) G_{(0)4\phi\phi} , \tag{5.14}$$

which admits the general solution

$$G_{(0)2} = \mu_{(0)}(\phi)\sqrt{2X} - V_{(0)}(\phi) + 2X\left(\ln \bar{X} - 2\right) G_{(0)4\phi\phi} . \tag{5.15}$$

Then, the effective theory is described by a similar functional form,

$$G_2 = \mu(\phi)\sqrt{2X} - V(\phi) + 2X\left(\ln \bar{X} - 2\right) G_{4\phi\phi} , \tag{5.16}$$

where,

$$\mu(\phi) = \mu_{(0)}(\phi) \frac{G_4}{G_{(0)4}} \frac{f(x) - (1-x)\, f'(x)}{\sqrt{f(x)}} , \tag{5.17}$$

$$V(\phi) = V_{(0)}(\phi) \frac{G_4}{G_{(0)4}} f(x) , \tag{5.18}$$

[10] We are not considering pure scalar field contributions in G_3 since they can be eliminated by integrating by parts, namely $-F(\phi)\,\Box\phi = 2X\,F_\phi(\phi)$ up to a total derivative. Therefore, $G_3 = F(\phi)$ is equivalent to considering $\widetilde{G}_3 = 0$ and $\widetilde{G}_2 = G_2 + 2X\,F_\phi$.

while G_4 is written in terms of the benchmark theory according to Eq. (5.13).

Eqs. (5.15) and (5.12) correspond to a subclass[11] of the extended cuscuton formulation [420, 421, 496, 497]. Therefore, also in the case of viable Horndeski, the effective cosmological bounce selects a non-dynamical scalar field. It is straightforward to verify that for the coupling function approaching a constant value, one traces back the result of the previous section[12].

5.4 Highlights

Considering a spatially flat, homogeneous, and isotropic universe, and assuming the existence of a critical energy density ρ_c, we analyze the conditions characterizing a general effective cosmological bounce.

In the used approach, the effective Friedmann equation takes the form of Eq. (5.1), where the standard energy density $\rho_{(0)}$ is parametrically independent of ρ_c, while the effective modification f is a function of $x = 1 - \rho_{(0)}/\rho_c$. From Eq. (5.1), using the standard continuity equation of the benchmark theory allows us to obtain a modified Raychaudhuri equation (5.5). Although the modified Friedmann equation does not diverge at $\rho_{(0)} = \rho_c$ by definition, this is not guaranteed for the modified Raychaudhuri equation. Therefore, it is necessary to impose the bounce condition $\ddot{a}/a > 0$ (equivalent to $\dot{H} > 0$) in correspondence with the critical energy density, as shown in Eq. (5.10). Consequently, the condition for a viable effective cosmological bounce takes the form of Eq. (5.13). Moreover, since the limit (5.10) is positive, this framework imposes the benchmark theory to satisfy the null energy condition, $\rho_{(0)} + P_{(0)} > 0$, at the bounce (while the effective quantities violate it). Eq. (5.17) is the simplest analytical example of an effective cosmological bounce reproducing the required behavior (5.13). Then, the bounce model is such that the effective theory inherits stability features from the asymptotic benchmark theory. However, at this level, it is not possible to make any additional claims about $f(x)$ beyond the constraints on the limit values of its derivatives approaching the bounce. The global modification to the Friedmann equation must adhere to some well-motivated non-singular bounce or a quantum theory, as in the case of LQC. As shown in Eq.(5.32), LQC quantities can be contextualized in this framework by explicitly characterizing the roles of standard variables and effective corrections.

We then moved on to analyze covariant realizations of a general effective bounce in second-order theories characterized by an additional classical scalar field. This is done by searching for two actions describing the effective theory $S_{(\text{eff})}$ (depending on ρ_c) and the asymptotic benchmark $S_{(0)}$. The effective theory must automatically guarantee an effective bouncing behavior and must approach the asymptotic one in the limit of $\rho_c \to \infty$. Distinguishing between the effective and the benchmark theory, it is possible to impose the conditions (5.6), (5.7), and (5.9), holding for a viable bounce.

First, we focused on the compatibility of an effective bounce with a general k-essence model. We demonstrated that the only possibility for the scalar field is the cuscuton. Thus, a dynamical k-essence model is incompatible with any bouncing model. This is because a dynamical model would have a benchmark limit violating the null energy condition, required for the bounce. In fact, only the theory (5.14) yields a well-posed consistency equation (5.10). It is characterized by the barotropic equation of state $P_{(0)} = w\rho_{(0)}$, where w must be greater than -1 to ensure

[11] Also assuming from the beginning an extended cuscuton model, it turns out that $\rho_{(0)}$ must not depend on the kinetic term. This condition corresponds to Eqs. (5.12) and (5.15).

[12] It is worth noting that, also in the case of the cuscuton (5.12) the energy density does not depend on the kinetic term. In fact, $\rho_{(0)X} = 0$ corresponds to Eq. (5.11).

$\rho_{(0)} + P_{(0)} > 0$. However, the consistency equation provides $w = -5/3$ which implies the standard matter violates the null energy condition at the bounce. Next, in the last section, we reiterated a similar discussion for a general viable Horndeski theory. Even in this case, the effective bouncing behavior constrains the theory in a non-dynamical scalar field described by Eqs.(5.12) and (5.15) representing a particular case of the extended cuscuton model.

The novel effective approach offers an innovative perspective to investigate bounce corrections. It opens up possibilities for studying theories that can reproduce or mimic effective cosmological bounces and exploring the limits and physical conditions required for their realization. At this point, one could extend the discussion to more general scalar-tensor theories, but the resulting theory would always be associated with a non-dynamical scalar field. The reason is that the effective theory and the benchmark limit must reproduce the same scalar field equation of motion (associated with the continuity equation of the standard variables). This strongly constrains $S_{(\text{eff})}$ and $S_{(0)}$. Nevertheless, it is interesting to notice that not all the theories with no extra dynamical degrees of freedom (in addition to the two tensor modes) are compatible with an effective bounce. Moreover, adding an explicit standard matter contribution to the current framework, the conclusion would likely be the same because the standard continuity equation implies the conservation of the effective quantities, and *vice versa*. In this sense, the adopted approach preserves the number of total degrees of freedom, and, in the specific case of an additional single scalar field, it selects among theories having only two tensor modes [498]. Remarkably, these findings align with prior research. In Ref. [499], it has been shown that the cuscuton model allows for an effective violation of the null energy condition in FLRW backgrounds while the standard matter sources satisfy it. Studies on the cuscuton power spectrum and the absence of instabilities can be found in Refs. [500–502] (see also Refs. [503, 504]).

In this work, we focused only on scalar-tensor theories. Other modified theories of gravity would need a different *ad hoc* analysis. It would be interesting to specialize this approach to alternative theories of gravity. Moreover, one can generalize the effective bounce to different cosmologies than FLRW. We leave this discussion to the near future, addressing this issue in forthcoming works.

6 Cosmological black holes

The McVittie solution of the Einstein field equations, sourced by a fluid [505–509], describes a central object embedded in a Friedmann–Lemaître–Robertson–Walker (FLRW) universe and it is a special case of the Kustaanheimo–Qvist family of shear-free solutions [510]. McVittie chose to forbid explicitly the accretion of cosmic fluid onto the central object [505]. This cosmic fluid has a homogeneous energy density $\rho(t)$ and an inhomogeneous, radially-dependent, pressure $P(t,r)$ (unless the cosmic fluid is replaced by a cosmological constant, in which case the McVittie solution degenerates into a Schwarzschild–de Sitter/Kottler geometry [505]). In addition to relevant previous literature, in the past decade, the McVittie solution has been the subject of much work, mostly devoted to understanding the nature of the time-dependent apparent horizon covering a singularity at a finite radius [511–530]. This apparent horizon has been shown to depend on the form of the scale factor $a(t)$ of the FLRW universe where the object is embedded [530]. The McVittie geometry has also been used to study the interplay between local dynamics and cosmological expansion [518, 531], to model cosmological wormholes [532], and generalized McVittie geometries allowing for cosmic fluid accretion have been introduced [531] and studied [518, 533, 534].

Since a minimally coupled scalar field is equivalent to an irrotational perfect fluid, it is natural to ask whether a scalar field can source the McVittie metric in Einstein gravity. The answer is negative but a gravitational scalar field, nonminimally coupled to the curvature (a signature of alternative gravity) has been shown to be a possible effective source for McVittie and generalized McVittie geometries [422, 535]. This context is different: theories of gravity alternative to general relativity have seen a huge resurgence of interest with the goal of explaining the current acceleration of the universe discovered in 1998 with type Ia supernovae without advocating an *ad hoc* dark energy [147, 156, 536]. It is then natural to ask whether scalar-tensor gravity, possibly in its most general formulations given by Horndeski and DHOST gravity, admits (generalized) McVittie solutions. This question was answered affirmatively in Refs. [422, 422, 535, 535], but there are severe restrictions on the class of theories in which this is possible. McVittie spacetimes are also exact solutions of shape dynamics of interest for quantum gravity [537], and are non-deformable solutions for extended $f(T)$ teleparallel gravity, where T is the torsion scalar [538] (see also Ref. [369] for a general discussion on teleparallel gravity).

It is time to revisit McVittie and generalized McVittie spacetimes in the most general scalar-tensor theories for three reasons. First, our understanding of the cuscuton that admits McVittie geometries as solutions has improved in the last decade and, very recently, the "extended cuscuton" has been introduced in research devoted to avoiding cosmological singularities [420, 497, 539]. Second, the previous analyses were restricted to vacuum but, ultimately, a realistic description of a cosmological setting must include matter, which is what we do in this article. Third, there has been growing attention to stealth solutions of scalar-tensor gravity with motivations coming mainly from the possibility that black hole hair arising in stealth solutions could be detected directly in gravitational wave observations. Such a detection would discriminate between General Relativity (GR) and scalar-tensor gravity. Both "first generation" scalar-tensor and more modern Horndeski theories admit stealth solutions that circumvent the no-hair theorems and for which the scalar field does not gravitate. They include the Schwarzschild geometry with a non-vanishing scalar field $\phi = \phi_0 t$. Then, it makes sense to look for "locally stealth" McVittie solutions of Horndeski theories with a gravitational scalar field, the geometry being McVittie and the scalar being homogeneous, $\phi = \phi(t)$. This scalar field would be a stealth with respect to the central object but would contribute to creating the FLRW "background" (indeed, in the first part of the present chapter, it will be the only effective cosmic fluid). Horndeski gravity is the most general scalar-tensor theory with second-order equations of motion, which automatically avoids the notorious Ostrogradsky instability encountered for higher-order field equations [242, 243]. Continuing, the generalized McVittie geometries with a radial energy flow offer more possibilities for modeling but have not yet received sufficient attention because of their higher complexity with respect to "standard" McVittie geometries.

Last but not least, McVittie and generalized McVittie spacetimes offer toy models for cosmological inhomogeneities which can be exploited to investigate the current problem of Hubble tension in relation to local inhomogeneities [91, 540–542].

We search for McVittie-compatible solutions within the Horndeski field equations.

The layout of the chapter is the following. Sec. 6.1 is devoted to a general discussion of McVittie geometry in Horndeski theory. In particular, we discuss the McVittie scalar field as a cuscuton. Generalized McVittie solutions in Horndeski gravity are discussed in Sec. 6.2, with attention to various cases with respect to the functions G_2, G_3, and G_4 entering the Lagrangian (1.129). Horndeski-McVittie as an extended cuscuton model is studied in Sec. 6.3, while McVittie solutions in extended cuscuton models in the presence of matter are considered in Sec. 6.4 with various combinations of the above functions G_i . Conclusions are drawn in Sec. 6.5.

6.1 McVittie geometry in Horndeski theory

Let us take into account the most general class of Horndeski theories in which the McVittie geometry is a solution of the field equations. The McVittie line element in isotropic coordinates $(t, r, \vartheta, \varphi)$ is [505]

$$ds^2 = -\frac{\left(1 - \dfrac{m}{2a(t)r}\right)^2}{\left(1 + \dfrac{m}{2a(t)r}\right)^2} dt^2 + a(t)^2 \left(1 + \frac{m}{2a(t)r}\right)^4 \left(dr^2 + r^2 d\Omega_{(2)}^2\right) , \tag{6.1}$$

where the mass parameter m is a positive constant, $a(t)$ is the scale factor of the FLRW "background", and $d\Omega_{(2)}^2 \equiv d\vartheta^2 + \sin^2\vartheta \, d\varphi^2$ is the line element on the unit 2-sphere. Following the recent literature, and in agreement with cosmological observations, we restrict ourselves to spatially flat FLRW universes. In the spirit of looking for "locally stealth" solutions discussed above, we consider a *homogeneous* scalar field $\phi(t)$, for which

$$X(t, r) = \frac{1}{2} \left(\frac{2a(t)r + m}{2a(t)r - m}\right)^2 \dot{\phi}^2(t) \tag{6.2}$$

in the geometry (6.1), where an overdot denotes differentiation with respect to the comoving time t of the FLRW "background". The only independent field equations that are not identically satisfied are the (t,t), (t,r), (r,r) and (r,θ) ones. The homogeneous scalar field $\phi(t)$ is equivalent to a *perfect* fluid. This is not true for a general Brans–Dicke-like or Horndeski scalar field which depends on the spatial position, which is instead equivalent to an *imperfect* fluid with heat conduction, shear, bulk viscosity, and anisotropic stresses [411, 416, 417, 431, 433]. The imperfect fluid structure is a consequence of the explicit nonminimal coupling of ϕ to gravity and disappears only in very special spacetimes, such as the FLRW ones where shear and currents would violate spatial isotropy [411], or in special theories (such as the cuscuton [497]). The assumption that the scalar ϕ is spatially homogeneous has the important consequence of making it possible for the McVittie solution to satisfy the Horndeski field equations. In fact, the latter are equivalent to effective Einstein equations with extra effective fluid in their right-hand side and this fluid is dissipative if ϕ depends on the spatial position [411, 416, 417, 431, 433], while the McVittie geometry is the only perfect fluid solution of the Einstein equations which is simultaneously[1] spherically symmetric, shear-free, and asymptotically FLRW with vanishing (radial) energy current [543] (see also Ref. [509]). Therefore, if $\phi = \phi\left(t,\vec{x}\right)$, the corresponding effective fluid in scalar-tensor or Horndeski gravity is necessarily a dissipative one [411, 416, 417, 431, 433] and the McVittie geometry cannot be a solution.

In order to find the functional forms of G_2, G_3, and G_4 such that the McVittie geometry (6.1) is a solution of Horndeski field equations, we substitute the line element (6.1) in Eq. (2.3) in absence of matter, and impose that G_2, G_3, and G_4 solve it. The (t,r) equation becomes

$$\frac{4m\, a(t)\, \dot{\phi}(t)}{m^2-4a^2(t)r^2}\left[X(t,r)G_{3X}(\phi,X)-G_{4\phi}(\phi)\right]=0\,, \tag{6.3}$$

and, assuming $\dot{\phi}(t)\neq 0$, it gives

$$G_{3X}(\phi,X)=\frac{G_{4\phi}(\phi)}{X}\,, \tag{6.4}$$

which integrates to

$$G_3(\phi,X)=G_{4\phi}(\phi)\ln X+F(\phi)\,. \tag{6.5}$$

Inserting this form of $G_3(\phi,X)$, the Horndeski field equations radial component provides

$$\begin{aligned} G_2(\phi,X) \;&=\; 2X\left[G_{4\phi\phi}(\phi)\,(\ln X-2)+F_\phi(\phi)\right]-4\,\frac{2a(t)r+m}{2a(t)r-m}\,\dot{\phi}(t)\,H(t)\,G_{4\phi}(\phi)\\ &+G_4(\phi)\left[-6H^2(t)-4\,\dot{H}(t)\,\frac{2ra(t)+m}{2a(t)r-m}\right]. \end{aligned} \tag{6.6}$$

Let us further assume that $\phi(t)$ is monotonic, $\dot{\phi}(t)\gtrless 0$, while $m<2a(t)r$. Eq. (6.2) then yields

$$\dot{\phi}(t)=\pm\frac{2a(t)r-m}{2a(t)r+m}\sqrt{2X(t,r)}\,, \tag{6.7}$$

while Eq. (6.6) becomes

$$G_2^{(\pm)}(\phi,X) \;=\; 2X\left[G_{4\phi\phi}(\phi)\Big(\ln X-2\Big)+F_\phi(\phi)\right]\mp 4\sqrt{2}\,\sqrt{X}\,H(t)\,G_{4\phi}(\phi)$$

[1] The proof of this statement is the actual derivation of the line element (6.1) in the original McVittie article [505].

$$+G_4(\phi)\left[-6H^2(t) \mp 4\sqrt{2}\,\sqrt{X}\,\frac{\dot{H}(t)}{\dot{\phi}(t)}\right]. \tag{6.8}$$

Since ϕ is monotonic, one can regard $H(t) \equiv S(\phi)$ as a function of ϕ and then $\dot{H}(t) = S_\phi(\phi)\,\dot{\phi}(t)$. In order for $G_2^{(\pm)}(\phi, X)$ to depend only on $\phi(t)$ and $X(t,r)$, it must be

$$\begin{aligned} G_2^{(\pm)}(\phi, X) &= 2X\left[G_{4\phi\phi}(\phi)\,(\ln X - 2) + F_\phi(\phi)\right] \mp 4\sqrt{2}\,\sqrt{X}\,S(\phi)\,G_{4\phi}(\phi) \\ &\quad + G_4(\phi)\left[-6S^2(\phi) \mp 4\sqrt{2}\,\sqrt{X}\,S_\phi(\phi)\right]. \end{aligned} \tag{6.9}$$

Assuming the four-velocity

$$u^\mu_{(\phi)} \equiv \frac{\nabla^\mu \phi}{\sqrt{2X}} \tag{6.10}$$

of the perfect fluid equivalent to the scalar field to be future-oriented, *i.e.*,

$$u^0_{(\phi)} = -\left(\frac{2a(t)r + m}{2a(t)r - m}\right)^2 \frac{\dot{\phi}(t)}{\sqrt{2X}} > 0\,, \tag{6.11}$$

or

$$\dot{\phi}(t) = -\frac{2\,a(t)r - m}{2\,a(t)r + m}\,\sqrt{2X(t,r)}\,, \tag{6.12}$$

the Horndeski functions that solve Eqs. (2.1) and (2.2) are

$$\begin{aligned} G_2(\phi, X) &= 2X\left[G_{4\phi\phi}(\phi)\,(\ln X - 2) + F_\phi(\phi)\right] + 4\sqrt{2X}\,S(\phi)\,G_{4\phi}(\phi) \\ &\quad + G_4(\phi)\left[-6S^2(\phi) + 4\sqrt{2}\,\sqrt{X}\,S_\phi(\phi)\right], \end{aligned} \tag{6.13}$$

$$G_3(\phi, X) = G_{4\phi}(\phi)\ln X + F(\phi)\,, \tag{6.14}$$

where $S(\phi) = H(t)$ and $G_4(\phi)$ are free functions. In general, $G_4(\phi)$ is completely unconstrained. Therefore, any set $\left\{G_4(\phi), \phi(t), m(t), a(t)\right\}$ is a solution of the theory (1.129) for which Eqs. (6.13) and (6.14) hold with $S(\phi) = H(t)$.

The function $F(\phi)$ appearing in Eqs. (6.13) and (6.14) can be safely neglected because, when inserted in the action, it only appears in the total divergence $2X\,F_\phi(\phi) - F(\phi)\Box\phi = -\nabla_\mu[F(\phi)\nabla^\mu\phi]$, which generates a boundary term and does not contribute to the field equations.

If $m = 0$, we obtain the same set of functions and X reduces to $X = \dot{\phi}^2(t)/2$.

In the special situation $G_3 \equiv 0$ it is $G_{4\phi}(\phi) = 0$ and $G_4(\phi) = \text{const.}$; then, as shown in the next section, the scalar field ϕ is not dynamical.

6.1.1 McVittie scalar field as a cuscuton

Let us consider now the situation in which $G_4 = 1/2$, which implies $G_3 = 0$ (keeping in mind that $F(\phi)$ only contributes a boundary term $-\int d^4x\sqrt{-g}\,\nabla_\mu[F(\phi)\,\nabla^\mu\phi]$ to the action), according to Eq. (6.14). Using Eqs. (6.14) and (6.13), the Horndeski action becomes $S = S_g + S_\phi$, where

$$S_g = \frac{1}{2}\int d^4x\,\sqrt{-g}\,R\,, \tag{6.15}$$

$$\begin{aligned} S_\phi &= \frac{1}{2}\int d^4x\,\sqrt{-g}\,L(\phi,X) \equiv \int d^4x\,\sqrt{-g}\,G_2(\phi,X) \\ &= \frac{1}{2}\int d^4x\,\sqrt{-g}\,\left[A(\phi)+B(\phi)\sqrt{X(t,r)}\right]\,, \end{aligned} \tag{6.16}$$

with

$$A(\phi) = -6\,S^2(\phi)\,, \tag{6.17}$$

$$B(\phi) = 4\sqrt{2}\,S_\phi(\phi)\,. \tag{6.18}$$

This action belongs to the class of k-essence theories of gravity and its variation with respect to ϕ produces the well-known equation of motion for ϕ

$$\frac{\delta S}{\delta\phi} = \left(L_X g^{\alpha\beta} - L_{,XX}\nabla^\alpha\phi\nabla^\beta\phi\right)\nabla_\alpha\nabla_\beta\phi - 2XL_{,X\phi} + L_\phi = 0\,. \tag{6.19}$$

When the inequality

$$L_X + 2XL_{,XX} > 0 \tag{6.20}$$

is satisfied, the partial differential equation (6.19) is hyperbolic and ϕ describes a propagating degree of freedom. In the case under consideration the inequality (6.20) is violated because

$$L_X = \frac{B(\phi)}{2\sqrt{X}}\,, \tag{6.21}$$

$$2XL_{,XX} = -\frac{B(\phi)}{\sqrt{X}}\,, \tag{6.22}$$

and $L_X + 2XL_{,XX} = 0$, which is the condition for the scalar field ϕ to be a cuscuton [422, 494–496, 535]. It is always possible to redefine the scalar field $\phi \longrightarrow \tilde{\phi}(\phi)$ so that $B(\phi)\,d\tilde{\phi}/d\phi = \pm\mu^2$ and to rewrite the cuscuton Lagrangian as

$$L(\phi,X) = \frac{\sqrt{-g}}{2}\left[\pm\mu^2\sqrt{X} - V(\phi)\right]\,, \tag{6.23}$$

where $V(\phi) = -A(\phi)$. As a consequence, $V(\phi) = 6H^2(t)$ and $\dot{H}(t) = \mp\,\mu^2\,\dot{\phi}(t)/\left(4\sqrt{2}\right)$, which yield

$$H(t) = C \pm \frac{\mu^2\phi(t)}{4\sqrt{2}}\,, \tag{6.24}$$

$$V(\phi) \;=\; 6\left[C \pm \frac{\mu^2\phi(t)}{4\sqrt{2}}\right]^2 , \tag{6.25}$$

where C is a constant of integration.

From the physical point of view, the cuscuton is equivalent to a perfect fluid and it is incompressible since its sound speed is found to be infinite [422, 494–496, 535, 544, 545]. This feature, which expresses the facts that perturbations do not propagate and the cuscuton is not dynamical, is reminiscent of the property of the McVittie universe that the cosmic fluid is not allowed to accrete onto the central object but somehow expands rigidly, given that the energy density $\rho(t)$ is perfectly homogeneous [505]. This "McVittie condition" is removed in generalized McVittie geometries by allowing radial energy flux and, correspondingly, the class of Horndeski theories admitting generalized McVittie spacetimes as solutions is wider than cuscuton gravity.

6.2 Generalized McVittie solutions of Horndeski gravity

Let us turn now to generalized McVittie geometries [531, 533, 534] characterized by a radial spacelike energy flux onto (or away from) the central object, which was instead forbidden explicitly by McVittie in his solution of the Einstein equations [505]. The corresponding line element in isotropic coordinates reads

$$ds^2 \;=\; -\frac{\left(1-\dfrac{m(t)}{2a(t)r}\right)^2}{\left(1+\dfrac{m(t)}{2a(t)r}\right)^2}dt^2 + a^2(t)\left(1+\frac{m(t)}{2a(t)r}\right)^4\left(dr^2+r^2d\Omega^2_{(2)}\right) , \tag{6.26}$$

where $m(t) > 0$ is now a function of time, while it was constant in the McVittie solution [505]. This time-dependence of the mass parameter of the central object embedded in the FLRW universe is due to the non-vanishing radial energy flux [531, 533, 534], therefore generalized McVittie geometries are substantially different from the original McVittie ones. Here we determine the Horndeski theories that admit generalized McVittie solutions. The method is similar to the one adopted in the previous section: we substitute the line element (6.26) in Eq. (2.1) and impose that G_2, G_3, and G_4 solve it. Then we will compare our result with Ref. [544]. Let us consider again a strictly monotonic homogeneous scalar field $\phi(t)$ with $\dot{\phi}(t) < 0$ and $m(t) < 2a(t)r$, which leads to

$$X(t,r) = \frac{1}{2}\left(\frac{2\,a(t)r+m(t)}{2\,a(t)r-m(t)}\right)^2\dot{\phi}^2(t) , \tag{6.27}$$

or

$$\dot{\phi}(t) = -\frac{2\,a(t)r-m(t)}{2\,a(t)r+m(t)}\sqrt{2X(t,r)} . \tag{6.28}$$

The (t,r) component of the Horndeski field equations yields

$$G_{3\phi}(\phi,X) = \frac{G_{4\phi}(\phi)+2G_0(\phi)\,G_4(\phi)}{X} , \tag{6.29}$$

where

$$G_0(\phi) = \frac{\dot{m}(t)}{m(t)\,\dot{\phi}(t)} = \frac{W(t)}{\dot{\phi}(t)} , \tag{6.30}$$

and with $W(t) \equiv \dot{m}(t)/m(t)$.
Eq. (6.29) is integrated with respect to X obtaining

$$G_3(\phi, X) = \left[2G_0(\phi)\, G_4(\phi) + G_{4\phi}(\phi)\right]\, \ln X + G_1(\phi)\,, \tag{6.31}$$

where $G_1(\phi)$ is an arbitrary integration function of the scalar field which can be neglected because it produces a boundary term, as we will see. By imposing the generalized McVittie geometry (6.26), the radial component of the field equations (2.1) in conjunction with Eq. (6.31) gives

$$G_2(\phi, X) \;=\; \mathcal{A}(\phi, X)\, G_4(\phi) + \mathcal{B}(\phi, X)\, G_{4\phi}(\phi) - 2X\,(1 - \ln X)\, G_{4\phi\phi}(\phi) + 2XG_{1\phi}(\phi)\,, \tag{6.32}$$

where $\mathcal{A}(\phi, X)$ and $\mathcal{B}(\phi, X)$ depend on ϕ and X . Using some auxiliary functions, it is possible to write the explicit functional form of G_2 as

$$\begin{aligned} G_2(\phi, X) \;=\;& 2G_4(\phi)\left[F_1(\phi) + F_2(\phi)\sqrt{X} - 2X\Big(G_{0\phi}(\phi)\,(2 - \ln X) + 3G_0^2(\phi)\Big)\right] \\ & -4G_{4\phi}(\phi)\left[G_0(\phi)X\,(2 - \ln X) + S_0(\phi)\sqrt{X}\right] - 2XG_{4\phi\phi}\,(2 - \ln X) + 2XG_{1\phi}(\phi)\,, \end{aligned} \tag{6.33}$$

where

$$G_0(\phi) \;=\; \frac{1}{\dot{\phi}(t)}\frac{\dot{m}(t)}{m(t)} = \frac{W(t)}{\dot{\phi}(t)}\,, \tag{6.34}$$

$$S_0(\phi) \;=\; \sqrt{2}\Big(H(t) - W(t)\Big)\,, \tag{6.35}$$

$$F_1(\phi) \;=\; -3\Big(H(t) - W(t)\Big)^2 = -\frac{3}{2}\, S_0^2(\phi)\,, \tag{6.36}$$

$$\begin{aligned} F_2(\phi) \;=\;& \frac{2\sqrt{2}}{\dot{\phi}(t)}\left[\dot{H}(t) - \dot{W}(t) + 3W(t)\Big(H(t) - W(t)\Big)\right] \\ =\;& 2\Big[S_{0\phi}(\phi) + 3G_0(\phi)S_0(\phi)\Big]\,. \end{aligned} \tag{6.37}$$

These equations show that $G_1(\phi)$ is completely negligible because, together with Eq. (6.31), it produces the total divergence $2X\, G_{1\phi}(\phi) - G_1(\phi)\Box\phi = -\nabla_\mu[G_1(\phi)\nabla^\mu\phi]$ in the action integral, and therefore an irrelevant boundary term.

If $G_0(\phi) = 0$, then m is constant and we recover the function G_2 associated with the McVittie metric

$$\begin{aligned} G_2(\phi, X) \;=\;& 2G_4(\phi)\left[F_1(\phi) + F_2(\phi)\sqrt{X}\right] \\ & -4G_{4\phi}(\phi)S_0(\phi)\sqrt{X} - 2XG_{4\phi\phi}\,(2 - \ln X) + 2XG_{1\phi}(\phi)\,, \end{aligned} \tag{6.38}$$

where

$$S_0(\phi) \;=\; \sqrt{2}\, H(t)\,, \tag{6.39}$$

$$F_1(\phi) = -3H^2(t) = -\frac{3}{2}S_0^2(\phi)\,, \tag{6.40}$$

$$F_2(\phi) = 2\sqrt{2}\,\frac{\dot{H}(t)}{\dot{\phi}(t)} = 2\,S_{0\phi}(\phi)\,. \tag{6.41}$$

In general, $G_4(\phi)$ is unconstrained.

If $m(t)$ vanishes identically, $G_0(\phi) = 0$ and one recovers the above set of functions with $X = \dot{\phi}(t)^2/2$.

The case $\dot{\phi}(t) > 0$ is obtained by changing the sign of the terms with $\sqrt{X}$. However, fixing the sign of $\dot{\phi}(t)$ does not cause loss of generality because rewriting every quantity in terms of t leads to the same expression for $G_2(t)$ and $G_3(t)$.

Therefore, the class of viable Horndeski theories that admit generalized McVittie solutions is characterized by G_2 and G_3 given in Eqs. (6.33) and (6.31) respectively, together with Eqs. (6.34)–(6.37). Then, any quadruple $\left\{G_4(\phi), \phi(t), m(t), a(t)\right\}$ corresponds to a particular solution of the Horndeski theory (1.129).

The results of Ref. [544] are recovered for $G_4(\phi) = 1/2$; this special case in conjunction with $G_3(\phi, X) = 0$ reproduces Sec. 6.1.1 with $m = \text{const}$.

A particular subclass of the theories found, corresponding to $G_3 = 0$, is discussed below. This subclass is of physical interest for the reasons discussed in Ref. [546] and the vanishing of G_3 has significant consequences.

6.2.1 Vanishing $G_3(\phi, X)$

In the special subclass of Horndeski theories with $G_3 = 0$, Eq. (6.31) gives

$$2G_0(\phi)\,G_4(\phi) + G_{4\phi}(\phi) = 0 \tag{6.42}$$

and

$$G_4(\phi) = C\,\exp\left(-2\int_1^{\phi} G_0(\xi)\,d\xi\right)\,, \tag{6.43}$$

where C is an integration constant. Due to the monotonicity of ϕ, one can write $m(t) = \tilde{m}(\phi)$ and $G_0(\phi) = \dfrac{d\ln\tilde{m}(\phi)}{d\phi}$ and, redefining the constant C,

$$G_4(\phi) = \frac{C}{\tilde{m}^2(\phi)} \tag{6.44}$$

or

$$\tilde{G}_4(t) = \frac{C}{m^2(t)}\,. \tag{6.45}$$

Thus, $G_4(\phi)$ is constant if and only if $m(t)$ is.

Let us restore explicitly the speed of light c and the Newton coupling assuming it to be time-dependent, $G = G(t)$. In this way, the McVittie mass coefficient can be rewritten as

$$m(t) = \frac{G(t)M(t)}{c^2}\,, \tag{6.46}$$

where the function $M(t)$ has the dimensions of a mass, while the Horndeski nonminimal coupling reads

$$G_4(\phi) = \frac{c^4}{16\pi G(t)} \,. \tag{6.47}$$

Thus, Eq. (6.44), together with Eqs. (6.46) and (6.47), can be written as

$$G(t) = G_N \frac{M_0^2}{M(t)^2} \,, \tag{6.48}$$

and it results

$$m(t) = \frac{G_N M_0}{c^2} \frac{M_0}{M(t)} \tag{6.49}$$

that, in turn, give

$$\frac{\dot{m}(t)}{m(t)} = -\frac{\dot{M}(t)}{M(t)} \,, \tag{6.50}$$

where G_N is the present value of the gravitational coupling and $M_0 \equiv M(t_0)$ is the present value of $M(t)$.

6.3 Horndeski–McVittie as an extended cuscuton model

Here we show that the class of viable Horndeski theories admitting McVittie and generalized McVittie solutions in Eqs. (6.1) and (6.26) respectively, is a particular case of what in the literature is called extended cuscuton model [420, 539], a generalization of the cuscuton seen in Sec. 6.1.1 in which the scalar ϕ remains non-dynamical.

The cuscuton (6.23) has always received much attention ([420, 422, 494–496, 535, 544], see also Refs. [499, 500, 545, 547–551]), representing (in the unitary gauge, $\phi = \phi(t)$) the unique subclass of k-essence theory with only two propagating degrees of freedom. This fact is related to two closely related features of this model:

- The equation of motion for the scalar ϕ is of first order in the case of FLRW cosmology;
- The kinetic term of scalar cosmological perturbations vanishes.

However, the cuscuton is not the most general Horndeski theory propagating only two degrees of freedom. The theory with this feature is the extended cuscuton model, characterized by the generalization of the first statement to

- The system composed of the dynamical equations (2.1) and (2.2) is degenerate in FLRW (see Ref. [420]).

The above statement implies that the functions $G_i(\phi, X)$ in the action (1.129) must satisfy the conditions

$$\begin{aligned} G_2(\phi, X) \quad &= \quad f_1(\phi) + f_2(\phi)\sqrt{2X} - \left(2f_{3\phi}(\phi) + 4f_{4\phi\phi}(\phi) + \frac{3{f_3}^2(\phi)}{4f_4(\phi)}\right) X \\ &\quad +\Big(f_{3\phi}(\phi) + 2f_{4\phi\phi}(\phi)\Big) X \ln X \,, \end{aligned} \tag{6.51}$$

$$G_3(\phi, X) = \left(\frac{1}{2} f_3(\phi) + f_{4\phi}(\phi)\right) \ln X\,, \tag{6.52}$$

$$G_4(\phi) = f_4(\phi)\,, \tag{6.53}$$

where the $\{f_i\}$ are arbitrary functions of $\phi(t)$, so that the scalar field does not propagate at both the background and the perturbative levels.

Eqs. (6.51)–(6.53) coincide with Eqs. (6.31)–(6.33) under the identifications

$$f_1(\phi) = 2\,G_4(\phi)F_1(\phi) = -3G_4(\phi)S_0^2(\phi)\,, \tag{6.54}$$

$$\begin{aligned} f_2(\phi) &= \sqrt{2}\left[G_4(\phi)F_2(\phi) + 2S_0(\phi)\,G_{4\phi}(\phi)\right] \\ &= 2\sqrt{2}\left[G_4(\phi)(S_{0\phi}(\phi) + 3G_0(\phi)S_0(\phi)) + S_0(\phi)G_{4\phi}(\phi)\right], \end{aligned} \tag{6.55}$$

$$f_3(\phi) = 4\,G_0(\phi)\,G_4(\phi)\,, \tag{6.56}$$

where

$$G_0(\phi) = \frac{W(t)}{\dot{\phi}(t)}\,, \tag{6.57}$$

$$S_0(\phi) = \sqrt{2}\left(H(t) - W(t)\right). \tag{6.58}$$

Fixing $f_3(\phi) = 0$, or $G_0 = 0$, Eqs. (6.51)–(6.56) are equivalent to the set of functions (6.13) and (6.14). Therefore, also the standard (*i.e.*, m = const.) McVittie geometry in Horndeski theory represents an extended cuscuton model. It is well-known that this kind of theory is transformed into the standard cuscuton model [422, 494–496, 535] by a particular disformal transformation [420, 544].

Extended cuscuton models are used to describe dark energy, mimicking the ΛCDM model in late-time cosmology [539, 552] with a time-dependent gravitational coupling, or as a mechanism to remove the cosmological singularity through a bounce [497]. These works deal with a particular choice of the functions $\{f_i\}$ which provide different cosmological models. In Refs. [497, 539, 552], they discuss the extended cuscuton action in the presence of matter while, until now in the present chapter we have considered only the vacuum Horndeski action (*i.e.*, G_2 and G_3 describe *effective* matter). Therefore, we are able to find a functional form of the viable Horndeski action such that the McVittie geometry satisfies the field equations but there is freedom in choosing the functions (6.34) and (6.35) and one cannot discuss the dynamics of the system until these are fixed. For this reason, in the next section we improve the generality by adding a matter fluid to the extended cuscuton model, Eqs. (6.51)–(6.53), and classifying all the resulting possibilities.

6.4 McVittie in the extended cuscuton model with matter

Without loss of generality, let us parametrize the functions $\{f_i\}$ of Eqs. (6.51)–(6.53) in the more convenient way

$$f_1(\phi) \to 2f_1(\phi)G_4(\phi)\,, \tag{6.59}$$

$$f_2(\phi) \to \sqrt{2}f_2(\phi)G_4(\phi)\,, \tag{6.60}$$

$$f_3(\phi) \to 4f_3(\phi)G_4(\phi)\,. \tag{6.61}$$

This is just a redefinition of the functions $\{f_i\}$ because f_1, f_2, and G_0 are still free. Moreover, we assume the cosmic fluid to be described by an imperfect fluid stress-energy tensor $T_{\mu\nu}$, allowing *a priori* a non-vanishing radial flow,

$$T_{\mu\nu} = (P+\rho)u_\mu u_\nu + Pg_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu\,, \tag{6.62}$$

where the fluid four-velocity u^μ is normalized to $u^\mu u_\mu = -1$, the purely spatial vector q^μ describes a radial energy flow, and $\rho(t,r)$ and $P(t,r)$ denote the energy density and pressure of the fluid, respectively. It follows that

$$T^t{}_t = -\rho(t,r)\,, \quad T^r{}_r = P(t,r)\,. \tag{6.63}$$

Now, the (t,r) field equation gives

$$f_3(\phi) = \frac{1}{\dot\phi(t)}\left[W(t) + \frac{\Big(2a(t)r - m(t)\Big)^3 T^t{}_r(t,r)}{8a(t)m(t)G_4(\phi)\Big(2a(t)r + m(t)\Big)}\right], \tag{6.64}$$

where $W(t) = \dot m(t)/m(t)$. Since, by definition, f_3 is a function of $\phi(t)$, we can introduce a generic function of time $Q(t)$ and rewrite $T^t{}_r$ as

$$T^t{}_r = \frac{8a(t)\,m(t)\,G_4(\phi)\Big(2a(t)r + m(t)\Big)}{\Big(2a(t)r - m(t)\Big)^3}\,Q(t)\,, \tag{6.65}$$

The flux $T^t{}_r$ vanishes if $m(t) = 0$, and also as $r \to +\infty$. Therefore, we have

$$f_3(\phi) = \frac{W(t) + Q(t)}{\dot\phi(t)}\,. \tag{6.66}$$

Using this form of $f_3(\phi)$, we must now impose that the equation of motion Eq. (2.2) of ϕ is satisfied, thus obtaining constraints on f_1 and f_2 that can be used in the field equations. However, it is not possible to follow this procedure for arbitrary $G_4(\phi)$, $Q(t)$, $W(t)$, and $H(t)$, and it is necessary to make some assumptions on these four functions in order to solve the algebraic expression coming from the equation of motion of ϕ . This situation is related with the fact that this equation of motion sets constraints on f_1 and f_2 which are obtained in specific domains. We provide a classification of all the possible cases in which Eq. (2.2) is satisfied by reasoning on $\left\{G_{4\phi}(\phi), Q(t), W(t), H(t)\right\}$ according to whether one or more of them vanishes identically. To anticipate the results, when $Q(t) \neq 0$, $W(t) = 0$, and $H(t) \neq 0$, the equation of motion cannot be satisfied, as well as as when $G_{4\phi}(\phi) \neq 0$, $Q(t) \neq 0$, $W(t) \neq 0$, and $H(t) = 0$. All the possible cases are characterized by constraints on $\left\{G_{4\phi}(\phi), Q(t), W(t), H(t)\right\}$ and are summarized in the following list:

- Case 1: $Q(t) = 0$ and $W(t) = 0$;
- Case 2: $Q(t) = 0$ and $W(t) \neq 0$;
- Case 3: $Q(t) \neq 0$, $W(t) = 0$, and $H(t) = 0$;
- Case 4: $G_{4\phi} = 0$, and $Q(t) = -W(t) \neq 0$;
- Case 5: $G_{4\phi} = 0$, $Q(t) \neq 0$ and $W(t) = H(t) \neq 0$;
- Case 6: $G_{4\phi} \neq 0$, $Q(t) \neq 0$ and $Q(t) \neq -H(t)$, and $W(t) = H(t) \neq 0$.

6.4.1 Case 1: $Q(t) = 0$ and $W(t) = 0$

We consider a generic function $G_4(\phi)$, which could be constant, vanishing radial flow, and a standard McVittie geometry. This situation occurs also in the limit of asymptotically flat spacetime $H(t) = 0$. From the equation of motion for ϕ one obtains

$$f_{1\phi}(\phi) = -\left[f_1(\phi) - 3H^2(t)\right]\frac{G_{4\phi}(\phi)}{G_4(\phi)} - \frac{3}{\sqrt{2}}H(t) f_2(\phi)\,, \tag{6.67}$$

$$f_2 = f_2(\phi)\,, \tag{6.68}$$

leaving the function f_2 unconstrained. The corresponding field equations provide the energy density and the pressure of the fluid

$$\rho(t) = G_4(\phi)\left[f_1(\phi) + 3H^2(t)\right]\,, \tag{6.69}$$

$$P(r,t) = -\,G_4(\phi)\left[f_1(\phi) + 3H^2(t) + \frac{2a(t)r + m}{2a(t)r - m}\left(\sqrt{2}\, f_2(\phi)\, \dot{\phi}(t) - 4\,\dot{H}(t)\right)\right]$$

$$-\,2\,\frac{2a(t)r + m}{2a(t)r - m}\, H(t)\, \dot{\phi}(t)\, G_{4\phi}(\phi)\,. \tag{6.70}$$

6.4.2 Case 2: $Q(t) = 0$ and $W(t) \neq 0$

In this case there is no radial flow and there is a central inhomogeneity with non-constant mass parameter: the scalar field "compensates" the time variation of $m(t)$ as it happens in the case of generalized McVittie geometries in the vacuum extended cuscuton model.

The equation of motion of the scalar field yields the constraints

$$f_{1\phi}(\phi) \quad = \quad -\left[f_1(\phi) + 3\Big(H(t) - W(t)\Big)^2\right]\frac{G_{4\phi}(\phi)}{G_4(\phi)} - \frac{6}{\dot{\phi}(t)}\Big(H(t) - W(t)\Big)\left(\dot{H}(t) - \dot{W}(t)\right),$$

(6.71)

$$f_2(\phi) \;=\; \frac{2\sqrt{2}}{\dot{\phi}(t)}\left[\dot{H}(t)-\dot{W}(t)+3W(t)\Big(H(t)-W(t)\Big)+\Big(H(t)-W(t)\Big)\frac{\dot{\phi}(t)G_{4\phi}(\phi)}{G_4(\phi)}\right]. \tag{6.72}$$

As a consequence, from the field equations, we obtain that the only possible cosmological fluid is a perfect fluid with a linear barotropic coefficient equal to -1:

$$\begin{aligned} P(t) &= -\,G_4(\phi)\left[f_1(\phi)+3\Big(H(t)-W(t)\Big)^2\right] \\ &= -\,\rho(t)\,. \end{aligned} \tag{6.73}$$

In this case, f_2 has a fixed functional form, which corresponds to Eq. (6.55) and remains constrained in the limit $W(t) \to 0$. In general, it does not give back the previous case.

This result is consistent with what we obtained in the vacuum case. Indeed, for vanishing P and ρ, it is $f_1 = 3[H(t)-W(t)]^2$, corresponding to Eq. (6.54).

6.4.3 Case 3: $Q(t) \neq 0$, $W(t) = 0$, and $H(t) = 0$

This case describes a Schwarzschild black hole embedded in an imperfect fluid with radial flow. The scalar field equation of motion gives

$$f_1(\phi) = \frac{\lambda}{G_4(\phi)}\,, \qquad f_2 = f_2(\phi)\,, \tag{6.74}$$

$$Q(t) = \frac{Q_0}{\sqrt{G_4(\phi)}}\,, \tag{6.75}$$

where λ and Q_0 are integration constants. The fluid is characterized by

$$\rho(t,r) = \lambda + 3\left(\frac{2a(t)r+m}{2a(t)r-m}\right)^2 Q_0^2\,, \tag{6.76}$$

$$\begin{aligned} P(t,r) =& -\lambda + 3\left(\frac{2a(t)r+m}{2a(t)r-m}\right)^2 Q_0^2 + \frac{1}{\sqrt{2}}\,\frac{2a(t)r+m}{2a(t)r-m}\, f_2(\phi)\,\dot{\phi}(t)\,G_4(\phi) \\ &+ \left(\frac{2a(t)r+m}{2a(t)r-m}\right)^2 \frac{G_{4\phi}(\phi)\,\dot{\phi}(t)}{\sqrt{G_4(\phi)}}\,Q_0\,. \end{aligned} \tag{6.77}$$

6.4.4 Case 4: $G_{4\phi} = 0$, and $Q(t) = -W(t) \neq 0$

Now we have a generalized McVittie metric with generic scale factor and constant gravitational coupling. The solution of the equation of motion of ϕ is

$$f_{1\phi}(\phi) = 0\,, \quad f_2(\phi) = 0\,, \quad Q(t) = -W(t)\,, \tag{6.78}$$

where the last equality $Q(t) = -W(t)$ implies $f_3 = 0$ and, in conjunction with $G_{4\phi} = 0$, it has the consequence that $G_3 = 0$.

The energy density and pressure of the imperfect fluid surrounding the central object are

$$\rho(t,r) = G_4\left[f_1 + 3\left(H(t) + \frac{2m(t)}{2a(t)r - m(t)}W(t)\right)^2\right], \tag{6.79}$$

$$\begin{aligned} P(t,r) = G_4\Bigg[&- f_1 - 3H^2(t) + \frac{4m(t)\left(3m^2(t) - 8m(t)a(t)r - 4a^2(t)r^2\right)}{\left(2a(t)r - m(t)\right)^3} W^2(t) \\ &- \frac{4m(t)\left(4a(t)r - m\right)\left(2a(t)r - 3m(t)\right)}{\left(2a(t)r - m(t)\right)^3} W(t)H(t) \\ &- \frac{4m(t)\left(2a(t)r + m(t)\right)}{\left(2a(t)r - m(t)\right)^2} W'(t) - 2\,\frac{2a(t)r - m(t)}{2a(t)r - m(t)} H'(t)\Bigg]. \end{aligned} \tag{6.80}$$

The homogeneous limit $r \to \infty$ reproduces the usual perfect fluid Friedmann equation

$$\rho(t) \simeq G_4\left[f_1 + 3H^2(t)\right], \tag{6.81}$$

$$P(t) \simeq -\,G_4\left[f_1 + 3H^2(t) - 2\dot{H}(t)\right], \tag{6.82}$$

where f_1 plays the role of a cosmological constant. Therefore, this case is equivalent to the generalized McVittie metric of GR with a cosmological constant.

6.4.5 Case 5: $G_{4\phi} = 0$, $Q(t) \neq 0$ and $W(t) = H(t) \neq 0$

The condition $W(t) = H(t)$ automatically makes the generalized McVittie solution a non-rotating Thakurta geometry which we are going to discuss below. When one solves the equation of motion of ϕ imposing $G_{4\phi} = 0$, $Q(t) \neq 0$, $W(t) \neq 0$, and $H(t) \neq 0$ and assuming $Q(t) \neq -W(t)$, one obtains

$$W(t) = H(t) \quad \Rightarrow \quad m(t) = m_0\, a(t)\,, \tag{6.83}$$

$$Q(t) = \frac{Q_0}{a^3(t)}\,, \tag{6.84}$$

where m_0 and Q_0 are integration constants. In an expanding universe, the equation $W(t) = H(t)$ implies a growing black hole mass. In addition, the condition $Q(t) \neq -W(t)$ becomes $Q(t) \neq -H(t)$.

In this case, the cosmic fluid has energy density and pressure

$$\rho(t,r) = G_4 \left[3 \left(\frac{2a(t)r + m(t)}{2a(t)r - m(t)} \right)^2 \frac{Q_0^2}{a^6(t)} + f_1 \right], \tag{6.85}$$

$$P(t,r) = G_4 \left[3 \left(\frac{2a(t)r + m(t)}{2a(t)r - m(t)} \right)^2 \frac{Q_0^2}{a^6(t)} - f_1 \right]. \tag{6.86}$$

As $r \longrightarrow +\infty$, the cosmic fluid becomes a perfect fluid and ρ and P become

$$\rho(t) \simeq G_4 \left(\frac{3\,Q_0^2}{a^6(t)} + f_1 \right), \tag{6.87}$$

$$P(t) \simeq G_4 \left(\frac{3\,Q_0^2}{a^6(t)} - f_1 \right), \tag{6.88}$$

with f_1 acting again as a cosmological constant. Therefore, the cosmic fluid is made up of a cosmological constant contribution and a stiff fluid with a linear barotropic coefficient equal to 1.

6.4.6 Case 6: $G_{4\phi} \neq 0$, $Q(t) \neq 0$ and $Q(t) \neq -H(t)$, and $W(t) = H(t) \neq 0$

If none of $G_{4\phi}(\phi)$, $Q(t)$, $W(t)$, $H(t)$ vanishes, the equation of motion of the scalar field yields

$$f_1(\phi) = \frac{\lambda}{G_4(\phi)}\,, \qquad f_2(\phi) = 0\,, \tag{6.89}$$

$$W(t) = H(t)\,, \qquad Q(t) \neq -H(t)\,, \tag{6.90}$$

$$\begin{aligned} \dot{Q}(t) &= -\,3\,Q(t)\,H(t) - \frac{Q(t)}{2}\,\frac{\dot{\phi}(t)\,G_{4\phi}(\phi)}{G_4(\phi)} \left(\frac{Q(t) + 2H(t)}{Q(t) + H(t)} \right) \\ &= -\,3\,Q(t)H(t) - \frac{Q(t)}{2}\,\frac{\dot{G}_4(t)}{G_4(t)} \left(\frac{Q(t) + 2H(t)}{Q(t) + H(t)} \right), \end{aligned} \tag{6.91}$$

where λ is an integration constant.

In this case $W(t) = H(t) \Rightarrow m(t) = m_0\,a(t)$ does not mean that an expanding universe corresponds to a growing mass, because $m(t)$ is a mass parameter containing a non-constant gravitational coupling, then we have to take into account the whole evolution of the function which is not a mere change of mass.

Indeed, with the parameterization of Sec. 6.2.1, it is

$$\frac{\dot{M}(t)}{M(t)} - \frac{\dot{G}_4(t)}{G_4(t)} = H(t)\,, \tag{6.92}$$

while Eq. (6.91) can be seen as a sort of continuity equation. Then, the field equations give

$$\rho(t,r) = \lambda + 3G_4(\phi)\,Q^2(t)\left(\frac{2a(t)r + m(t)}{2a(t)r + m(t)}\right)^2, \tag{6.93}$$

$$P(t,r) = -\lambda + \left[3\,G_4(t)\,Q^2(t) + \frac{\dot{G}_4(t)\,Q^2(t)}{Q(t) + H(t)}\right]\left(\frac{2a(t)r + m(t)}{2a(t)r + m(t)}\right)^2. \tag{6.94}$$

The homogeneous limit $r \longrightarrow \infty$ reduces these quantities to

$$\rho(t) \simeq \lambda + 3G_4(t)Q^2(t)\,, \tag{6.95}$$

$$P(t) \simeq -\lambda + 3G_4(t)\,Q^2(t) + \frac{\dot{G}_4(t)\,Q^2(t)}{Q(t) + H(t)}\,. \tag{6.96}$$

Even if, to solve the scalar field equation, we impose the non-vanishing of $G_{4\phi}(\phi)$, at the end of the day this case generalizes the previous one. Indeed, imposing $G_{4\phi}(\phi) = 0$ one obtains the same Friedmann equations and the same expression for $Q(t)$ in terms of $a(t)$.

The current case is the most interesting one: to investigate it, let us consider a matter-dominated cosmological era and let us assume the barotropic equation of state $P(t) = w\,\rho(t)$, $w =$ const. In particular, for dust, we can neglect λ and obtain

$$H(t) = -Q(t) - \frac{\dot{G}_4(t)}{3\,G_4(t)}\,, \tag{6.97}$$

$$\dot{Q}(t) = \frac{3}{2}\,Q^2(t) \quad \Rightarrow \quad Q(t) = -\frac{2}{2c_1 + 3t}\,, \tag{6.98}$$

where c_1 is an integration constant, and

$$G_4(t) = \frac{c_2\,(2c_1 + 3t)^2}{a^3(t)}\,, \tag{6.99}$$

$$\rho(t) = \frac{12\,c_2}{a^3(t)}\,, \tag{6.100}$$

$$M(t) = \frac{c_3\,(2c_1 + 3t)^2}{a^2(t)}\,, \tag{6.101}$$

where $c_{2,3}$ are integration constants. Therefore, the density of dust scales as $\rho \propto a^{-3}$ as in GR. We can generalize this treatment to a generic equation of state constant parameter w. Assuming the linear barotropic equation of state $P(t) = w\rho(t)$ leads to the system of equations (when $w \neq 1$ and $\dot{G}_4(t) \neq 0$)

$$\frac{\dot{G}_4(t)}{G_4(t)} = 3\,(w-1)\left(H(t) + Q(t)\right), \tag{6.102}$$

$$\dot{Q}(t) = -\frac{3Q(t)}{2}\left(2wH(t) + (w-1)Q(t)\right), \tag{6.103}$$

yielding

$$Q(t) = \frac{a^{-3w}(t)}{c_1 + \frac{3}{2}\int_1^t (w-1)a^{-3w}(\tilde{t})\,d\tilde{t}}, \tag{6.104}$$

$$G_4(t) = c_2\, a^{3(w-1)}\left[c_1 + \frac{3}{2}\int_1^t (w-1)\,a^{-3w}(\tilde{t})\,d\tilde{t}\right]^2, \tag{6.105}$$

with $c_{1,2}$ are integration constants. Then, the energy density is

$$\rho(t) = \frac{3c_2}{a^{3(1+w)}(t)}, \tag{6.106}$$

while the black hole mass is

$$M(t) = c_3\, a^{3w-2}(t)\left[c_1 + \frac{3}{2}\int_1^t (w-1)a^{-3w}(\tilde{t})\,d\tilde{t}\right]^2, \tag{6.107}$$

where c_3 is another integration constant and the scale factor is unknown. At first sight, this result seems to hold also for $w = -1$, the simplest dark energy model, and for $w = 1$, the stiff matter case, but the latter value of the equation of state parameter is forbidden and can be taken into account only in the case $\dot{G}_4 = 0$ (the previous case).

It is worth asking what happens if the change in the effective mass is due exclusively to the change of gravitational coupling. Adopting the parametrization of Sec. 6.2.1, $W(t) = -\dot{G}_4(t)/G_4(t) \rightarrow \dot{G}_4(t)/G_4(t) = -H(t) \Rightarrow G_4(t) = c_G/a(t)$. The consequence is that

$$Q(t) = -\frac{(3w-2)}{3(w-1)}H(t), \tag{6.108}$$

$$H(t) = \frac{2}{t(3w+2) - 2c_1}, \tag{6.109}$$

$$a(t) = c_2\left[t(3w+2) - 2c_1\right]^{\frac{2}{3w+2}}, \tag{6.110}$$

$$\rho(t) = \frac{4c_G(2-3w)^2}{3c_2(w-1)^2}\left[t(3w+2)-2c_1\right]^{-\frac{6(w+1)}{3w+2}} , \tag{6.111}$$

where $\{c_i\}$ are integration constants. Therefore, in this model, the barotropic coefficient is constrained to values $w > -2/3$.

A final comment about the condition $W(t) = H(t) \Rightarrow m(t) = m_0\, a(t)$ is mandatory: when the latter holds, the generalized McVittie line element assumes the form

$$ds^2 = -\frac{\left(1-\frac{m_0}{2r}\right)^2}{\left(1+\frac{m_0}{2r}\right)^2}\, dt^2 + a^2(t)\left(1+\frac{m_0}{2r}\right)^4\left(dr^2 + r^2 d\Omega^2_{(2)}\right) . \tag{6.112}$$

This special case of generalized McVittie metrics is the non-rotating Thakurta solution [553], see also Refs. [518, 554, 555], which is conformal to the Schwarzschild metric with conformal factor $a(\eta)$, where η is the conformal time of the FLRW "background" defined by $dt = a(\eta)d\eta$. Using the "conformal Schwarzschild radius"

$$R = r\left(1+\frac{m_0}{2r}\right)^2 , \tag{6.113}$$

as the radial coordinate, the metric (6.112) can be rewritten as

$$\begin{aligned} ds^2 &= -\left(1-\frac{2m_0}{R}\right)dt^2 + a^2(t)\left(1-\frac{2m_0}{R}\right)^{-1} dR^2 + a^2(t)R^2 d\Omega^2_{(2)} \\ &= a^2(\eta)\left[-\left(1-\frac{2m_0}{R}\right)d\eta^2 + \left(1-\frac{2m_0}{R}\right)^{-1} dR^2 + R^2 d\Omega^2_{(2)}\right] . \end{aligned} \tag{6.114}$$

The metric (6.112) has a pleasant characteristic that is missing in the standard McVittie metric: it provides the usual Hubble law $v = H(t)d$. Unlike the McVittie universe, the Hubble law holds exactly and not just asymptotically. This aspect prevents the possibility that the current H_0 tension be due to proximity to an inhomogeneity, at least in this model. This could be a straightforward theoretical solution for the Hubble tension problem.

6.5 Synthesis of findings

There are several motivations to revisit McVittie and generalized McVittie solutions in the context of Horndeski gravity after the progress recently made in the study of these theories. Furthermore, there are now theoretical indications for, and the experimental capability to detect, scalar hair around black holes. This hair could vary on cosmological scales, and the McVittie solutions of scalar-tensor gravity offer a context for this possibility. While it was shown that McVittie geometries solve cuscuton theory and generalized McVittie geometries solve more general Horndeski theories, previous studies were limited to vacuum. We have now included matter in the picture, which is essential in order to achieve a realistic cosmological background where to embed the McVittie central object. Moreover, the extended cuscuton scenario introduced very recently has not been considered before in relation to (generalized) McVittie spacetimes. As in the "old" cuscuton, the scalar field of the extended cuscuton does not propagate degrees of freedom. Here we found conditions under which generalized McVittie geometries are solutions of the extended cuscuton model.

The requirement that the scalar field be homogeneous is essential for the McVittie and generalized McVittie geometries to be solutions of the Horndeski field equations. Without this property, the field equations, rewritten as effective Einstein equations, exhibit an effective imperfect fluid stress-energy tensor in their right-hand sides [411, 433]. The characteristic quantities of an imperfect fluid, *i.e.*, heat current density, shear, and viscous pressure are due to the nonminimal coupling of ϕ with gravity [411, 433] and preclude (generalized) McVittie from solving the field equations because these geometries are shear-free [510, 543]. The (generalized) McVittie geometry is inhomogeneous, due to the presence of the central object. This inhomogeneity dies off asymptotically as the metric becomes FLRW at large distances from the central object. Therefore, the homogeneous scalar field does not affect directly the gravitational field of the central object because it sources only the FLRW "background" universe in which the latter is embedded and, in the generalized McVittie case in vacuo, it determines the energy current onto it. In both cases, $\phi(t)$ controls the evolution of the mass parameter $m(t)$. It can legitimately be said that the scalar $\phi(t)$ is "locally stealth" on small scales near the central object but is not stealth with respect to the large-scale FLRW universe (indeed, it sources it). In this context, we restricted ourselves to the subclass of viable Horndeski theories.

We found that, in the extended cuscuton model in the presence of matter, a special generalized McVittie geometry appears as a solution, the non-rotating Thakurta metric. This geometry is peculiar since, contrary to all other (generalized) McVittie solutions, it is conformally equivalent to the Schwarzschild one where $m(t) = m_0 a(t)$ is scaling as a length in the FLRW background. More important, it was shown in Ref. [533] that the generalized McVittie solutions of GR with a fluid have the non-rotating Thakurta geometry as a late-time attractor. The proof of this statement in Ref. [533] depends only on the functional form of the metric and not on the field equations and can be transposed without change to the extended cuscuton model, provided that fluid is present. Therefore, non-rotating Thakurta becomes the generic solution at late times in the class of generalized McVittie geometries. We stress that the presence of the fluid is essential: in Horndeski models without matter, where the scalar field acts as the fluid for the FLRW background, the late-time limit may make the scalar ϕ constant and make the fluid disappear. In this case, one should not expect a non-rotating Thakurta limit for the generalized McVittie solution. Indeed, this is the case for the generalized McVittie solution of the "Cuscuta-Galileon" theory found in Ref. [544] which could have other interesting aspects, also from an observational point of view, in the Galileon model [556]. A more general study of the late-time behavior of the solutions for various specific models is deferred to future work.

Several aspects need to be analyzed in greater detail, such as the presence of late-time attractors for (generalized) McVittie spaces, other geometries describing objects embedded in cosmological spacetimes, anisotropic fluids, and the use of McVittie toy models to explore the current Hubble tension problem in relation with local inhomogeneities. These subjects will be examined in future research.

7 The Noether symmetry approach

The presence of so many free unknown functions appearing in the Horndeski action makes it challenging to grasp the physical meaning of the theory, as well as to establish a classification for all the possible subclasses. Moreover, these free functions are usually set *ad hoc* to face different problems, making it even more challenging to orient oneself among the infinite possibilities.

The goal of this chapter is to classify the Horndeski models according to the Noether Symmetry Approach [557, 558], extending previous works of the literature. On one side, the existence of symmetries allows us to solve exactly the dynamics; on the other, the Noether charge can always be related to some observable quantity. Unlike precedent works [559, 560], a Lagrange multiplier will be used to keep the braiding function general and, at the same time, deal with point-like Lagrangian depending at most on the time first derivative of the configuration space[1].

Since the presence (or absence) of Horndeski functions influences the result of the Noether Symmetry Approach in selecting the Lagrangian functional form, a brief review of all the scalar-tensor subclasses included in Horndeski gravity is provided. It allows us to appreciate the hierarchical structure of the scalar-tensor theories and the relation with the infinitesimal generators of the symmetries. For consistency, the modern Horndeski gravity *nomenclature* is used throughout the entire chapter: G_2 for the k-essence contribution, G_3 for the kinetic braiding term, G_4 and G_5 for the non-minimally coupling functions of the Ricci scalar and the Einstein tensor, respectively.

It is worth highlighting that this approach reverses the usual application of the Noether theorem (see also Refs. [565–567]). The Noether Symmetry Approach constitutes a criterion to select the functional forms of the arbitrary Horndeski functions, by assuming the invariance under Noether point symmetries. Usually, the Noether theorem is used to obtain integrals of motion corresponding to transformations leaving the action invariant. An exhaustive and general discussion is presented in Ref. [558]. This method provided several exact solutions of the gravitational field equations, describing the time evolution of a spatially flat FLRW universe for the scalar-tensor and Gauss–Bonnet theory.

Let us stress that throughout this chapter, the following conventions are adopted: $g_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2)$, being $a = a(t)$ the scale factor, the *over-dot* represents the (total) time derivative, the scalar field and the kinetic term are denoted by $\phi = \phi(t)$ and $X = \frac{1}{2}\dot{\phi}^2$, respectively, and $8\pi G = c = \hbar = 1$ (reduced Planck units). Moreover, to align the discussion to the already existing literature about the Noether symmetry approach, in this chapter, derivatives are

[1] The use of Lagrange multiplier in finding Noether symmetries is present in different modified theories of gravity, for instance, $f(R)$ gravity [561], $f(\mathcal{G})$ gravity [562, 563], and non-local gravity [564].

explicitly written, while the subscript notation is just used for labeling quantities.

In Sec. 7.1, the Noether Symmetry Approach is summarized and applied to the k-essence model. In Sec. 7.2, we review the Noether classification of scalar-tensor theories with the *traditional* non-minimally coupled scalar field to the Ricci scalar. In Sec. 7.3, we present the Noether Symmetry Approach specifically for kinetic braiding gravity (constituting our first achievement in generalizing the Noether analysis) and generalized in Sec. 7.4 to Horndeski gravity. In Sec. 7.5, particular modified theories of gravity, Gauss–Bonnet gravity, and the extended cuscuton model, are analyzed in this framework. The form of Lagrangians is chosen to guarantee the existence of a Noether symmetry. Next, in Sec. 7.6, we discuss the classification of Noether symmetries in the presence of matter. The final discussion is presented in Sec. 7.7.

7.1 k-essence model

Let us start this section by considering the case of a canonical kinetic term,

$$S = \int d^4x\sqrt{-g}\left(R - \frac{1}{2}\nabla_a\phi\nabla^a\phi - V(\phi)\right). \tag{7.1}$$

Evaluating the above action for the spatially flat FLRW universe, for which the Ricci scalar is $R = 6\left(\frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}\right)$, the point-like Lagrangian turns out to be

$$\begin{aligned}\mathcal{L} &= 6a\dot{a}^2 + 6a^2\ddot{a} + \frac{1}{2}a^3\dot{\phi}^2 - a^3V \\ &= -6a\dot{a}^2 + \frac{1}{2}a^3\dot{\phi}^2 - a^3V\,,\end{aligned} \tag{7.2}$$

by performing integration by parts to eliminate the second derivative of the scale factor[2].

The infinitesimal generator of the Noether symmetry is written as follows

$$\chi = \xi(t,a,\phi)\partial_t + \eta_a(t,a,\phi)\partial_a + \eta_\phi(t,a,\phi)\partial_\phi\,, \tag{7.3}$$

then the *first prolongation* corresponds to

$$\chi^{[1]} = \chi + \left(\dot{\eta}_a - \dot{a}\,\dot{\xi}\,\right)\partial_{\dot{a}} + \left(\dot{\eta}_\phi - \dot{\phi}\,\dot{\xi}\,\right)\partial_{\dot{\phi}}\,. \tag{7.4}$$

The existence of a Noether symmetry for the point-like Lagrangian (7.2) is ensured by the following identity:

$$\chi^{[1]}\mathcal{L} + \dot{\xi}\mathcal{L} = \dot{\zeta}(t,a,\phi)\,, \tag{7.5}$$

where $\dot{\zeta}$ is a generic function corresponding to the gauge freedom of the symmetry, and it can be safely set to zero (*i.e.*, ζ = cost), without losing generality. Then, using the Hamiltonian constraint corresponding to the first Friedmann equation,

$$\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\dot{a}}\dot{a} - \frac{\partial\mathcal{L}}{\partial\dot{\phi}}\dot{\phi} = 0\,, \tag{7.6}$$

[2]The integration by parts is not necessary, in general. This step is equivalent to considering a *second prolongation* of the infinitesimal generator of the Noether symmetry. However, while here the integration is sufficient to rewrite the point-like Lagrangian in a first-order canonical form (*i.e.*, depending only on the first derivatives), it is not the case of kinetic gravity braiding and Horndeski gravity.

the associated conserved quantity reads as follows,

$$\mathcal{J} = \zeta - \eta_a \frac{\partial \mathcal{L}}{\partial \dot{a}} - \eta_\phi \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \,. \tag{7.7}$$

Thus, the identity Eq.(7.5) becomes a set of equations for the functions ξ, η_a, η_ϕ, and V, by setting to zero the coefficients obtained by factorizing all the time-derivative terms, *i.e.*, $\dot{\phi}\,\dot{a}$, $\dot{\phi}^2\,\dot{a}$, $\dot{\phi}\,\dot{a}^2$, $\dot{\phi}^i$, $\dot{a}^i$, with $i = 1, 2, 3$. This yields the following configurations:

$$\xi(t) = \xi_1 t + \xi_2 \,, \quad \eta_a(a,\phi) = \frac{a}{3}\xi_1 + \frac{c_a(\phi)}{\sqrt{a}} \,, \quad \eta_\phi(a,\phi) = \xi_0 - 8\frac{c_a(\phi)}{a^{3/2}} \,, \tag{7.8}$$

with

$$(\,\mathrm{I}\,): \begin{cases} c_a(\phi) = 0 \\ \xi_0 \neq 0 \neq \xi_1 \\ V(\phi) = V_0 \exp\left(-\frac{2\xi_1}{\xi_0}\phi\right) \end{cases} \tag{7.9}$$

$$(\,\mathrm{II}\,): \begin{cases} c_a(\phi) = c_1 \exp\left(\frac{\sqrt{3}}{4}\phi\right) + c_2 \exp\left(-\frac{\sqrt{3}}{4}\phi\right) \\ V(\phi) = V_0 \exp\left(-\frac{2\xi_1}{\xi_0}\phi\right) \\ \begin{cases} c_1 = 0 \neq \xi_1 \\ \xi_0 = \frac{4}{\sqrt{3}}\xi_1 \end{cases} \quad \vee \quad \begin{cases} c_2 = 0 \neq \xi_1 \\ \xi_0 = -\frac{4}{\sqrt{3}}\xi_1 \end{cases} \end{cases} \tag{7.10}$$

or, in the case of internal symmetries[3],

$$(\,\mathrm{III}\,): \begin{cases} c_a(\phi) = c_1 \exp\left(\frac{\sqrt{3}}{4}\phi\right) + c_2 \exp\left(-\frac{\sqrt{3}}{4}\phi\right) \\ V(\phi) = V_0 \exp\left(-\frac{\sqrt{3}}{2}\phi\right)\left[c_2 - c_1 \exp\left(\frac{\sqrt{3}}{2}\phi\right)\right]^2 \\ \xi_0 = 0 = \xi_1 \end{cases} \tag{7.11}$$

where, $\xi_{0,1,2}$, $c_{1,2}$, and V_0 are arbitrary constants.

It is possible to verify that a phantom scalar field (*i.e.*, in the case of a negative sign in front of the kinetic term) admits a Noether point symmetry only in correspondence with the first set of parameters.

[3] Notice that, in general, internal symmetries correspond to $\xi(t) = 0$. However, at the level of the Noether identity (7.5), there is no difference between a vanishing and a constant ξ. This is because the Lagrangian does not explicitly depend on the time coordinate. Then, χ does not have the ξ component, and $\chi^{[1]}$ depends only on $\dot{\xi}$. Therefore, both $\xi = 0$ and $\dot{\xi} = 0$ correspond to internal symmetries. Moreover, in cosmology, due to the energy constraint (7.6), ξ is not present in the expression of the conserved scalar current associated with the Noether symmetry (7.7).

In the case of a general (unknown) kinetic dependence (like the k-essence model),

$$S = \int d^4x\sqrt{-g}\,[R + G_2(\phi, X)]\,, \tag{7.12}$$

where, using the modern Horndeski gravity notation, G_2 is a generic function of the scalar field and the kinetic term, $X = -\frac{1}{2}\nabla_a\phi\nabla^a\phi$. In this case, the kinetic term can be treated as a new additional variable, and its definition must be included in the theory by using a Lagrange multiplier. Adding a Lagrange multiplier does not change the field equations, and it is perfectly equivalent to considering the definition of X from the beginning.

To apply the Noether Symmetry Approach it is necessary to write down the point-like Lagrangian of the theory in a canonical way and to split the identity guaranteeing the existence of the symmetry (7.5) in a set of equations, by collecting all the time-derivative terms. The using of the Lagrange multiplier makes the process easier. The discussion is analogous to the one made for other theories of gravity [561–563]. Thus, the point-like Lagrangian reads

$$\mathcal{L} = -6a\dot{a}^2 + a^3G_2(\phi, X) + a^3\lambda\left(X - \frac{1}{2}\dot{\phi}^2\right). \tag{7.13}$$

The variation with respect to X gives $\lambda = -\partial_X G_2(\phi, X)$. Therefore, the previous Lagrangian can be rewritten as follows

$$\mathcal{L} = -6a\dot{a}^2 + a^3G_2(\phi, X) - a^3\partial_X G_2(\phi, X)\left(X - \frac{1}{2}\dot{\phi}^2\right). \tag{7.14}$$

The infinitesimal generator of the Noether symmetry is provided by

$$\chi = \xi(t, a, \phi, X)\partial_t + \eta_a(t, a, \phi, X)\partial_a + \eta_\phi(t, a, \phi, X)\partial_\phi + \eta_X(t, a, \phi, X)\partial_X\,, \tag{7.15}$$

and, the *first prolongation* corresponds to

$$\chi^{[1]} = \chi + \left(\dot{\eta}_a - \dot{a}\,\dot{\xi}\,\right)\partial_{\dot{a}} + \left(\dot{\eta}_\phi - \dot{\phi}\,\dot{\xi}\,\right)\partial_{\dot{\phi}} + \left(\dot{\eta}_X - \dot{X}\,\dot{\xi}\,\right)\partial_{\dot{X}}\,. \tag{7.16}$$

Following the same procedure described above, setting to zero the coefficients obtained by factorizing all the time-derivative terms, the Noether Symmetry Approach yields the following configurations:

$$\xi(t) = \xi_1 t + \xi_2\,,\quad \eta_a(a) = \frac{a}{3}\xi_1\,,\quad \eta_\phi = \eta_\phi(\phi)\,,\quad \eta_X(\phi, X) = 2X(\partial_\phi\eta_\phi - \xi_1)\,, \tag{7.17}$$

$$(\mathrm{I}):\quad \begin{cases} \xi_1 \neq 0 \\ \eta_\phi(\phi) = \xi_1\phi + \xi_0 \\ G_2(\phi, X) = \dfrac{g_2(X)}{\xi_1\phi + \xi_0} \end{cases} \tag{7.18}$$

$$(\,\mathrm{II}\,):\quad\begin{cases}\xi_1\neq 0\\[2ex] \eta_\phi(\phi)\neq\xi_1\phi+\xi_0\\[2ex] G_2(\phi,X)=\exp\left(-\int_1^\phi\dfrac{2\xi_1}{\eta_\phi(\varphi)}d\varphi\right)g_2\left(X\exp\left(\int_1^\phi\dfrac{2\left(\xi_1-\partial_\varphi\eta_\phi(\varphi)\right)}{\eta_\phi(\varphi)}d\varphi\right)\right)\end{cases}\tag{7.19}$$

or, in the case of internal symmetries,

$$(\,\mathrm{III}\,):\quad\begin{cases}\xi_1=0\\[2ex] G_2(\phi,X)=g_2\left(\dfrac{X}{\eta_\phi^2(\phi)}\right)\end{cases}\tag{7.20}$$

where, $\xi_{0,1,2}$ are constants, and g_2 is an arbitrary function of X times a factorized scalar field dependence (eventually constant). Notice that g_2 can be a linear function of its variable, *a posteriori*. However, compared to the analysis done on the linear dependence from the beginning (7.2), the above results are over-constraining the theory.

The above characterizations can be rewritten so that the kinetic dependence is fully factorized from the pure scalar field one, $G_2(\phi,X)=h(\phi)\,g_2(X)$, by redefining the scalar field. Then, one obtain $\eta_\phi(\phi)=\xi_1\phi+\xi_0$ or $\eta_\phi(\phi)=\xi_0$. However, imposing a factorized form of G_2 with a general ϕ-component of the infinitesimal generator, one obtain $g_2(X)=c_0\,X^{c_\phi}$, $h(\phi)=\left(\xi_1-\partial_\phi\eta_\phi\right)^{-1}$, where η_ϕ is implicitly defined by a differential equation, $2\left(\xi 1-\partial_\phi\eta_\phi\right)\left[(c_\phi-1)\xi_1-c_\phi\partial_\phi\eta_\phi\right]-\eta_\phi\partial_\phi^2\eta_\phi=0$, with $c_\phi\neq 1$ being a constant. For instance, the last equation is satisfied for $\eta_\phi=\left(\frac{c_\phi-1}{c_\phi}\right)\xi_1\phi+\xi_0$.

7.2 Traditional non-minimally coupled scalar field

The action of the first-generation scalar-tensor theory is

$$S=\int d^4x\sqrt{-g}\left[G_4(\phi)R-\frac{\omega(\phi)}{2}\nabla_a\phi\nabla^a\phi-V(\phi)\right],\tag{7.21}$$

where, the non-minimal coupling G_4 is an arbitrary function of ϕ. The corresponding to the following point-like action is

$$\begin{aligned}\mathcal{L}=&\,6a\dot{a}^2G_4(\phi)+6a^2\ddot{a}\,G_4(\phi)+a^3\frac{\omega(\phi)}{2}\dot{\phi}^2-a^3V(\phi)\\ =&-6a\dot{a}^2G_4(\phi)-6a^2\dot{a}\dot{\phi}\,\partial_\phi G_4(\phi)+a^3\frac{\omega(\phi)}{2}\dot{\phi}^2-a^3V(\phi).\end{aligned}\tag{7.22}$$

As done in the previous section, the configurations obtained turn out to be:

$$\xi(t)=\xi_1t+\xi_2\,,\quad \eta_a(a,\phi)=-\frac{a}{3}\left(\xi_1+\frac{\partial_\phi V(\phi)}{V(\phi)}\eta_\phi(a,\phi)\right),\quad \eta_\phi=\eta_\phi(a,\phi)\tag{7.23}$$

$$
(\,\mathrm{I}\,):\quad\begin{cases}
V(\phi) = V_0\, G_4(\phi) \\[2ex]
\omega(\phi) = \omega_0 \dfrac{\left(\partial_\phi G_4(\phi)\right)^2}{G_4(\phi)} \\[2ex]
\eta_\phi(a,\phi) = \dfrac{G_4(\phi)}{\partial_\phi G_4(\phi)} \left[3\xi_1 \Big(2\ln(a) + (\omega_0+4)\ln(G_4(\phi))\Big) + \xi_0\right] \\[2ex]
\xi_1 = 0 \quad \vee \quad \omega_0 = -\dfrac{8}{3} \quad \vee \quad \omega_0 = -3
\end{cases} \tag{7.24}
$$

$$
(\,\mathrm{II}\,):\quad\begin{cases}
V(\phi) = V_0\, G_4(\phi)^{\,3/2} \\[2ex]
\omega(\phi) = \omega_0 \dfrac{\left(\partial_\phi G_4(\phi)\right)^2}{G_4(\phi)} \\[2ex]
\eta_\phi(\phi) = -4\,\xi_1 \dfrac{G_4(\phi)}{\partial_\phi G_4(\phi)} \\[2ex]
\xi_1 \neq 0
\end{cases} \tag{7.25}
$$

$$
(\,\mathrm{III}\,):\quad\begin{cases}
V(\phi) = V_0\, G_4(\phi)^{\,\gamma}\,, \quad \gamma \neq 1\,, \tfrac{3}{2} \\[2ex]
\omega(\phi) = \omega_0 \dfrac{\left(\partial_\phi G_4(\phi)\right)^2}{G_4(\phi)} \\[2ex]
\eta_\phi(a,\phi) = \dfrac{2\xi_1}{(1-\gamma)} \dfrac{G_4(\phi)}{\partial_\phi G_4(\phi)} + \xi_0 \dfrac{G_4(\phi)^\alpha}{\partial_\phi G_4(\phi)}\, a^\beta \\[2ex]
\alpha = \dfrac{2\gamma^2 + 3(\gamma-1)\omega_0}{2(3-2\gamma)^2}\,, \quad \beta = \dfrac{3(1-\gamma)}{3-2\gamma}\gamma \\[2ex]
\xi_0 = 0 \quad \vee \quad \omega_0 = -3 \quad \vee \quad \omega_0 = \dfrac{4}{3}\,\gamma\,(\gamma-3)
\end{cases} \tag{7.26}
$$

where, $\xi_{0,1,2}$, α, β, γ, ω_0, and V_0 are arbitrary constants.

Let us now consider a more general scalar-tensor theory,

$$
S = \int d^4x\sqrt{-g}\,[G_4(\phi)R + G_2(\phi,X)]\,. \tag{7.27}
$$

where the non-minimal coupling function G_4 is assumed to be not constant.

As done in the previous section, excluding the linear case, the kinetic term can be treated as an additional variable, by including a Lagrange multiplier. After solving the Lagrange multiplier,

$\lambda = -\partial_X G_2(\phi, X)$, the point-like Lagrangian reads as

$$\mathcal{L} = -6a\dot{a}^2 G_4(\phi) - 6a^2\dot{a}\dot{\phi}\,\partial_\phi G_4(\phi) + a^3 G_2(\phi, X) - a^3\partial_X G_2(\phi, X)\left(X - \frac{1}{2}\dot{\phi}^2\right), \tag{7.28}$$

corresponding to the following classification,

$$\xi(t) = \xi_1\, t + \xi_2\,, \quad \eta_a(a) = \frac{a}{3}(\xi_1 - \phi_0)\,, \quad \eta_\phi(\phi) = \phi_0\frac{G_4(\phi)}{\partial_\phi G_4(\phi)}\,, \quad \eta_X(\phi, X) = 2X\left(\partial_\phi\eta_\phi - \xi_1\right) \tag{7.29}$$

with

$$(\,\mathrm{I}\,):\quad \begin{cases} \xi_1 \neq 0 \\ \eta_\phi(\phi) = \xi_1\phi + \xi_0 \\ G_4(\phi) = c_\phi\,(\xi_1\phi + \xi_0)^{\phi_0/\xi_1} \\ G_2(\phi, X) = g_2(X)(\xi_1\phi + \xi_0)^{\frac{\phi_0}{\xi_1} - 2} \end{cases} \tag{7.30}$$

$$(\,\mathrm{II}\,):\quad \begin{cases} \eta_\phi(\phi) \neq \xi_1\phi + \xi_0 \\ G_2(\phi, X) = G_4(\phi)^{1-\frac{2\xi_1}{\phi_0}}\, g_2\left(X\dfrac{\partial_\phi G_4(\phi)^2}{G_4(\phi)^{2-\frac{2\xi_1}{\phi_0}}}\right) \end{cases} \tag{7.31}$$

where, $\xi_{0,1,2}$, c_ϕ, and ϕ_0 are constants, and g_2 is an arbitrary function of X times a factorized scalar field dependence (eventually constant). Notice that g_2 can be, in general, a linear function of its variable but the above results are over-constraining the theory compared to the analysis done on the linear dependence from the beginning.

The latter system can also be rewritten in terms of η_ϕ, in the following way

$$(\,\mathrm{II}\,):\quad \begin{cases} \eta_\phi(\phi) \neq \xi_1\phi + \xi_0 \\ G_4(\phi) = c_0\,\int_1^\phi \exp\left(\int_1^{\varphi_2}\frac{\phi_0 - \partial_{\varphi_1}\eta_\phi(\varphi_1)}{\eta_\phi(\varphi_1)}d\varphi_1\right)d\varphi_2 \\ G_2(\phi, X) = \exp\left(\int_1^\phi -\frac{2\xi_1 - \phi_0}{\eta_\phi(\varphi)}d\varphi\right) g_2\left(X\exp\left(\int_1^\phi\frac{2(\xi_1 - \partial_\varphi\eta_\phi(\varphi))}{\eta_\phi(\varphi)}d\varphi\right)\right) \end{cases} \tag{7.32}$$

Unlike the result obtained for the k-essence model, η_ϕ is fully determined by the non-minimal coupling function. Moreover, the presence of the non-constant G_4 implies the absence of the pure shift-symmetry with respect to the scalar field. It is replaced by a *generalized shift-symmetry* corresponding to $\eta_\phi = \xi_0$, $\xi(t) = \xi_1\, t + \xi_2$, $\eta_a = a(\xi_1 - \phi_0)/3$, and $G_4(\phi) = \exp\left(\frac{\phi_0}{\xi_0}\phi\right)$. Allowing G_4 to be constant, the shit-symmetry is obtained in correspondence with $\phi_0 = 0$ and $\xi_1 = 0$. Notice that, using the parameterization (7.30), the case of the minimally-coupled scalar field (7.18) is obtained by setting $\phi_0 = 0$. Indeed, as it will be clear in the next sections, ϕ_0 is always associated with the ϕ dependence of the non-minimal coupling function G_4.

When, for the former case, the factorization of ϕ and X dependence is imposed, $G_2(\phi, X) = h(\phi) g_2(X)$, irrespectively of η_ϕ (*i.e.*, assuming $\eta_\phi \neq \xi_1 \phi + \xi_0$ and $\eta_\phi \neq \xi_0$), it yields

$$G_2(\phi, X) = c_0 \left(\partial_\phi G_4(\phi)\right)^{2c_\phi} \; G_4(\phi)^{\frac{2(c_\phi - 1)\xi_1}{\phi_0} + 1 - 2c_\phi} \; X^{c_\phi} \,, \tag{7.33}$$

where $c_\phi \neq 1$, since we are excluding the linear case.

7.3 Kinetic gravity braiding

Let us turn on the braiding term $G_3(\phi, X)$ inside the action, proportional to $\Box\phi = \nabla_a \nabla^a \phi$,

$$S = \int d^4x \sqrt{-g} \left[R + G_2(\phi, X) - G_3(\phi, X)\Box\phi\right] . \tag{7.34}$$

The nomenclature *kinetic braiding* refers to the fact that the function G_3 must depend on the kinetic term to contribute in a non-trivial way compared to the k-essence part. Therefore, we are focusing on the case of $\partial_X G_3 \neq 0$. Indeed, if $G_3 = G_3(\phi)$ the action (7.34) can be recast into the k-essence model by integrating by parts: $G_3(\phi)\Box\phi = -2X\partial_\phi G_3(\phi) \subseteq G_2(\phi, X)$, up to a total divergence.

In the past, this term yielded only a partial classification according to the Noether symmetries by making some assumptions on X dependence of G_3 [560]. This lies in the presence of the D'Alambertian of the scalar field. Indeed, substituting $\Box\phi = -\left(\ddot\phi + 3\frac{\dot a}{a}\dot\phi\right)$, the point-like Lagrangian associated with the braiding term is

$$\mathcal{L}_3 = a^3\, G_3(\phi, X) \left(\ddot\phi + 3\frac{\dot a}{a}\dot\phi\right) . \tag{7.35}$$

The term $a^3\, G_3(\phi, X)\ddot\phi$ cannot be transformed in a first-order Lagrangian by integrating by parts in general. However, the Noether symmetries analysis of this theory can be performed by taking into account the *second prolongation* of the infinitesimal generator, $\chi^{[2]} = \chi^{[1]} + (\ddot\eta_\phi - \dot\phi\,\ddot\xi - 2\ddot\phi\,\dot\xi)\partial_{\ddot\phi}$, and implementing the Noether identity (7.5). The only way to recast the Lagrangian in a canonical form is by considering X as a new independent variable of our point-like action (adding its definition by using a Lagrange multiplier), using the *first prolongation* of the infinitesimal generator. Otherwise, one should use the generalized Euler-Lagrange equations for Lagrangian depending up to the second derivative in time, $\frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\phi} + \frac{d^2}{dt^2}\frac{\partial\mathcal{L}}{\partial\ddot\phi} = 0$. We will adopt the former approach, *i.e.*, the Lagrange multiplier, as done in the previous section. This simplifies the resolution of the Noether identity, allowing us to use the relation $\ddot\phi = \dot X/\dot\phi$.

After solving the Lagrange multiplier $\lambda = \left(\frac{3\dot a}{a\dot\phi} - \frac{\dot X}{2X\dot\phi}\right) G_3 - \frac{\dot\phi}{a}\partial_X G_3 + \partial_\phi G_3 - \partial_X G_2$, the point-like Lagrangian turns into

$$\begin{aligned}\mathcal{L} =& -6a\dot a^2 + a^3 G_2(\phi, X) - a^3\partial_X G_2(\phi, X) + a^3\, G_3(\phi, X)\left(\frac{\dot X}{\dot\phi} + 3\frac{\dot a}{a}\dot\phi\right)\\ &+ a^3\left(X - \frac{1}{2}\dot\phi^2\right)\left[\left(\frac{3\dot a}{a\dot\phi} - \frac{\dot X}{2X\dot\phi}\right) G_3(\phi, X) - \frac{\dot\phi}{a}\partial_X G_3(\phi, X) + \partial_\phi G_3(\phi, X) - \partial_X G_2(\phi, X)\right] .\end{aligned} \tag{7.36}$$

Then, in full generality, one obtains:

$$\xi(t) = \xi_1\, t + \xi_0\,, \quad \eta_a(a) = \frac{a}{3}\,\xi_1\,, \quad \eta_\phi = \eta_\phi(\phi)\,, \quad \eta_X = 2X\left(\partial_\phi \eta_\phi - \xi_1\right)\,, \tag{7.37}$$

with the following selected configurations,

$$(\,\mathrm{I}\,):\quad \begin{cases} \xi_1 \neq 0 \\ \eta_\phi(\phi) = \xi_1\phi + \xi_0 \\ G_3(\phi, X) = \dfrac{g_3(X)}{\xi_1\phi + \xi_0} \\ G_2(\phi, X) = \dfrac{g_2(X)}{(\xi_1\phi + \xi_0)^2} \end{cases} \tag{7.38}$$

$$(\,\mathrm{II}\,):\quad \begin{cases} \eta_\phi(\phi) \neq \xi_1\phi + \xi_0 \\ G_3(\phi, X) = \exp\left(-\int_1^\phi \frac{\partial_\varphi \eta_\phi(\varphi)}{\eta_\phi(\varphi)} d\varphi\right) g_3\left(X \exp\left(\int_1^\phi \frac{2\left(\xi_1 - \partial_\varphi \eta_\phi(\varphi)\right)}{\eta_\phi(\varphi)} d\varphi\right)\right) \\ G_2(\phi, X) = \exp\left(\int_1^\phi -\frac{2\xi_1}{\eta_\phi(\varphi)} d\varphi\right) g_2\left(X \exp\left(\int_1^\phi \frac{2\left(\xi_1 - \partial_\varphi \eta_\phi(\varphi)\right)}{\eta_\phi(\varphi)} d\varphi\right)\right) \\ \qquad -2X \exp\left(\int_1^\phi \frac{-2\partial_\varphi \eta_\phi(\varphi)}{\eta_\phi(\varphi)} d\varphi\right) g_3\left(X \exp\left(\int_1^\phi \frac{2\left(\xi_1 - \partial_\varphi \eta_\phi(\varphi)\right)}{\eta_\phi(\varphi)} d\varphi\right)\right) \\ \qquad \times \int_1^\phi \exp\left(\int_1^{\varphi_2} \frac{\partial_{\varphi_1} \eta_\phi(\varphi_1)}{\eta_\phi(\varphi_1)} d\varphi_1\right) \frac{\partial^2_{\varphi_2} \eta_\phi(\varphi_2)}{\eta_\phi(\varphi_2)} d\varphi_2 \end{cases} \tag{7.39}$$

where, $\xi_{0,1,2}$ are constants, and $g_{2,3}$ are arbitrary functions of X times a factorized scalar field dependence (eventually constant). Notice that, in the case of internal symmetries the theory is manifestly shift-symmetric, *i.e.*, $G_i = G_i(X)$.

7.4 Horndeski gravity

The Horndeski action reads as follows,

$$S = \int d^4x\sqrt{-g}\ \left(L_2 + L_3 + L_4 + L_5\right), \tag{7.40}$$

where,

$$L_2 = G_2\left(\phi, X\right)\,, \qquad L_3 = -G_3\left(\phi, X\right)\Box\phi\,,$$

$$L_4 = G_4\left(\phi, X\right) R + \partial_X G_4(\phi, X)\left[(\Box\phi)^2 - (\nabla\nabla\phi)^2\right]\,,$$

$$L_5 = G_5(\phi,X)\, G_{ab}\nabla^a\nabla^b\phi - \frac{1}{6}\,\partial_X G_5(\phi,X)\left[(\Box\phi)^3 - 3\Box\phi\,(\nabla\nabla\phi)^2 + 2\,(\nabla\nabla\phi)^3\right], \tag{7.41}$$

where $(\nabla\nabla\phi)^2 = \nabla_a\nabla_b\phi\nabla^a\nabla^b\phi$ and $(\nabla\nabla\phi)^3 = \nabla_a\nabla_c\phi\nabla^a\nabla^b\phi\nabla^c\nabla_b\phi$.

Unlike the kinetic braiding function, the presence of G_4 and G_5 does not introduce an explicit dependence on second derivatives of the scalar field. All the factors having second derivatives can be integrated by parts yielding a first-order Lagrangian:

$$\begin{aligned}\mathcal{L}_4 &= 6a^2\ddot{a}\,G_4(\phi,X) + 6a^2\dot{a}\dot{\phi}\ddot{\phi}\,\partial_X G_4(\phi,X) + 6a\dot{a}^2\left[\dot{\phi}^2\partial_X G_4(\phi,X) + G_4(\phi,X)\right]\\ &= -\,6a^2\dot{a}\dot{\phi}\,\partial_\phi G_4(\phi,X) + 6a\dot{a}^2\left[\dot{\phi}^2\partial_X G_4(\phi,X) - G_4(\phi,X)\right],\end{aligned} \tag{7.42}$$

$$\begin{aligned}\mathcal{L}_5 &= 3a\dot{a}^2\ddot{\phi}\left[\dot{\phi}^2\,\partial_X G_5(\phi,X) + G_5(\phi,X)\right] + 6a\dot{a}\ddot{a}\dot{\phi}\,G_5(\phi,X) + \dot{a}^3\dot{\phi}\left[\dot{\phi}^2\,\partial_X G_5(\phi,X) + 3G_5(\phi,X)\right]\\ &= \dot{a}^2\dot{\phi}^2\left[\dot{a}\dot{\phi}\,\partial_X G_5(\phi,X) - 3a\,\partial_\phi G_5(\phi,X)\right].\end{aligned} \tag{7.43}$$

Then, introducing the Lagrange multiplier, the point-like Lagrangian reads as follows,

$$\begin{aligned}\mathcal{L} = &-6a^2\dot{a}\dot{\phi}\partial_\phi G_4 + 6a\dot{a}^2\left(2X\partial_X G_4 - G_4\right) + a^3 G_2 + \frac{a^3\dot{X}G_3}{\dot{\phi}} + 3a^2\dot{a}\dot{\phi}G_3\\ &+ 2X\dot{a}^2\left(\dot{a}\dot{\phi}\partial_X G_5 - 3a\partial_\phi G_5\right) + a^3\left(X - \frac{1}{2}\dot{\phi}^2\right)\Bigg[-\partial_X G_2 + \partial_\phi G_3\\ &-\frac{3\dot{a}\dot{\phi}\partial_X G_3}{a} + \frac{\left(6X\dot{a} - a\dot{X}\right)G_3}{2aX\dot{\phi}} + \frac{6\dot{a}\dot{\phi}\partial_{\phi X}G_4}{a} - \frac{6\dot{a}^2\partial_X G_4}{a^2} - \frac{12X\dot{a}^2\partial_X^2 G_4}{a^2}\\ &-\frac{2\dot{a}^3\dot{\phi}\partial_X G_5}{a^3} - \frac{2X\dot{a}^3\dot{\phi}\partial_X^2 G_5}{a^3} + \frac{6\dot{a}^2\partial_\phi G_5}{a^2} + \frac{6X\dot{a}^2\partial_{\phi X}G_5}{a^2}\Bigg].\end{aligned} \tag{7.44}$$

From the Noether Symmetry Approach, one obtains the following configurations:

$$\xi(t) = \xi_1\,t + \xi_2\,,\quad \eta_a(a) = \frac{a}{3}(\xi_1 - \phi_0)\,,\quad \eta_\phi = \eta_\phi(\phi)\,,\quad \eta_X(\phi,X) = 2X\left(\partial_\phi\eta_\phi - \xi_1\right) \tag{7.45}$$

with,

$$(\mathrm{I}):\quad \begin{cases}\xi_1 \neq 0\\ \eta_\phi(\phi) = \xi_1\phi + \xi_0\\ G_5(\phi,X) = \alpha\,\phi + g_5(X)\,(\xi_1\phi + \xi_0)^{\frac{\phi_0}{\xi_1}+1}\\ G_4(\phi,X) = \alpha X + \beta\sqrt{2X} + g_4(X)\,(\xi_1\phi + \xi_0)^{\phi_0/\xi_1}\\ G_3(\phi,X) = g_3(X)\,(\xi_1\phi + \xi_0)^{\frac{\phi_0}{\xi_1}-1}\\ G_2(\phi,X) = g_2(X)\,(\xi_1\phi + \xi_0)^{\frac{\phi_0}{\xi_1}-2}\end{cases} \tag{7.46}$$

$$
(\mathrm{II}):\;\begin{cases}
\eta_\phi(\phi)\neq\xi_1\phi+\xi_0\\[2mm]
G_5(\phi,X)=\alpha\,\phi+\exp\left(\int_1^\phi\frac{4\xi_1+\phi_0-3\partial_\varphi\eta_\phi(\varphi)}{\eta_\phi(\varphi)}d\varphi\right)\times\\
\qquad\qquad\times\int_1^X g_5\left(Z\exp\left(-\int_1^\phi-\frac{2\left(\xi_1-\partial_\varphi\eta_\phi(\varphi)\right)}{\eta_\phi(\varphi)}d\varphi\right)\right)dZ\\[2mm]
G_4(\phi,X)=\alpha X+\beta\sqrt{2X}\\
\qquad\qquad+\int_1^\phi\exp\left(\int_1^{\varphi_2}\frac{\phi_0-\partial_{\varphi_1}\eta_\phi(\varphi_1)}{\eta_\phi(\varphi_1)}d\varphi_1\right)g_4\left(X\exp\left(\int_1^{\varphi_2}\frac{2\left(\xi_1-\partial_{\varphi_1}\eta_\phi(\varphi_1)\right)}{\eta_\phi(\varphi_1)}d\varphi_1\right)\right)d\varphi_2\\[2mm]
G_3(\phi,X)=\exp\left(\int_1^\phi\frac{\phi_0-\partial_\varphi\eta_\phi(\varphi)}{\eta_\phi(\varphi)}d\varphi\right)g_3\left(X\exp\left(\int_1^\phi\frac{2\left(\xi_1-\partial_\varphi\eta_\phi(\varphi)\right)}{\eta_\phi(\varphi)}d\varphi\right)\right)\\[2mm]
G_2(\phi,X)=\exp\left(\int_1^\phi-\frac{2\xi_1}{\eta_\phi(\varphi)}d\varphi\right)g_2\left(X\exp\left(\int_1^\phi\frac{2\left(\xi_1-\partial_\varphi\eta_\phi(\varphi)\right)}{\eta_\phi(\varphi)}d\varphi\right)\right)\\
\qquad\qquad-2X\exp\left(\int_1^\phi\frac{-2\partial_\varphi\eta_\phi(\varphi)}{\eta_\phi(\varphi)}d\varphi\right)g_3\left(X\exp\left(\int_1^\phi\frac{2\left(\xi_1-\partial_\varphi\eta_\phi(\varphi)\right)}{\eta_\phi(\varphi)}d\varphi\right)\right)\\
\qquad\qquad\times\int_1^\phi\exp\left(\int_1^{\varphi_2}\frac{\partial_{\varphi_1}\eta_\phi(\varphi_1)}{\eta_\phi(\varphi_1)}d\varphi_1\right)\frac{\partial^2_{\varphi_2}\eta_\phi(\varphi_2)}{\eta_\phi(\varphi_2)}d\varphi_2
\end{cases}
\tag{7.47}
$$

where, $\xi_{0,1,2}$, α, β, and ϕ_0 are constants, and $g_{2,3,4,5}$ are arbitrary functions of X times a factorized scalar field dependence (eventually constant).

The constants α and β can be set to zero. Indeed, it is straightforward to notice that L_4 vanishes in the case of $G_4(\phi,X)=\beta\sqrt{2X}$. This represents a spurious solution of the Noether approach. Moreover, it is well-known that in the case of $G_5=G_5(\phi)$, one can reabsorb the G_5 dependence by redefining $G_{2,3,4}$ as follows,

$$
G_2\to G_2-2X^2\partial^3_\phi G_5\,,\quad G_3\to G_3-3X\partial^2_\phi G_5\,,\quad G_4\to G_4-X\partial_\phi G_5\,,\tag{7.48}
$$

because it turns out that

$$
\begin{aligned}
L_5 &= G_5G_{ab}\nabla^a\nabla^b\phi\simeq-\partial_\phi G_5G_{ab}\nabla^a\phi\nabla^b\phi=-\partial_\phi G_5R_{ab}\nabla^a\phi\nabla^b\phi-X\partial_\phi G_5R\\
&= \partial_\phi G_5(\nabla_a\Box\phi-\nabla_b\nabla_a\nabla^b\phi)\nabla^a\phi-X\partial_\phi G_5R\\
&\simeq -\partial^2_\phi G_5(-2X\Box\phi-\nabla_a\nabla_b\phi\nabla^a\phi\nabla^b\phi)-\partial_\phi G_5\left[\Box\phi^2-(\nabla_a\nabla_b\phi)^2\right]-X\partial_\phi G_5R\\
&\simeq 3X\,\partial^2_\phi G_5\,\Box\phi-2X^2\partial^3_\phi G_5-\partial_\phi G_5\left[\Box\phi^2-(\nabla_a\nabla_b\phi)^2\right]-X\partial_\phi G_5R\,,
\end{aligned}\tag{7.49}
$$

where $\simeq$ means equality up to a total divergence. In particular, from Eq. (7.48), it is possible to notice that, $G_5=\alpha\phi$ is equivalent to $G_4=-\alpha X$. For this reason, it is possible to set $\alpha=0$ in Eqs. (7.46) and (7.47), without losing generality. Equivalently, one can verify that $\mathcal{L}_4+\mathcal{L}_5=0$ if $G_4=\alpha\,X$ and $G_5=\alpha\,\phi$.

The Noether symmetries classification of viable Horndeski gravity, $\partial_X G_4=0$ and $G_5=0$, can be obtained by considering $g_4\to$ const and $g_5\to 0$. Then, in the case of external symmetries (7.46) the parameter ϕ_0 is associated with the presence of a non-minimal coupling $G_4 = G_4(\phi)$, while, in the case of external symmetry, ϕ_0 represents the breaking parameter of the shift-symmetry.

From the performed analysis of Noether symmetries, it is possible to notice that, in the more general scalar-tensor theories, the selected models can be written such that the ϕ and X dependence is factorized:

$$G_i(\phi, X) = h_i(\phi)\, g_i(X) \qquad \begin{cases} h_i(\phi) = \partial_\phi^{(6-i)}(\xi_1\phi + \xi_0)^{2+\phi_0/\xi_1}\,, & \text{if} \quad \xi_1 \neq 0 \\ h_i(\phi) = \exp\left(\frac{\phi_0}{\xi_0}\phi\right)\,, & \text{if} \quad \xi_1 = 0 \end{cases} \tag{7.50}$$

In particular, for the symmetries having $\xi_1 \neq 0$, the scalar field can be redefined so that $\eta_\phi = \xi_1\phi + \xi_0$; then, it turns out that $h(\phi) = (\xi_1\phi + \xi_0)^{2+\phi_0/\xi_1}$. In the case of internal symmetries, which are characterized by $\xi_1 = 0$, the scalar field can be redefined so that $\eta_\phi = \xi_0$; then, it turns out that $h(\phi) = \exp\left(\frac{\phi_0}{\xi_0}\phi\right)$. Finally, notice that the shift-symmetric class is characterized by $G_i = G_i(X)$, corresponding to $\xi_1 = 0 = \phi_0$. Less general subclasses can be obtained from the above system by manually setting the functions g_i to 1 and/or $\phi_0 = 0$, except for the linear minimally coupled scalar field and the first-generation scalar-tensor theory, characterized by a more complex substructure.

7.5 Particular cases

The Noether Symmetry Approach almost fully determines the ϕ dependence, while the $g_i(X)$ functions are unconstrained. Then, each possible set $\{g_i\}_{i=2,3,4,5}$ corresponds to a particular model contained in Horndeski gravity. Therefore, models admitting Noether symmetries are characterized by the X dependence of the g_i functions. Once the model has been selected, *i.e.*, the X dependence, the request of a Noether symmetry sets the remaining free functions of ϕ. To clarify this point, let us discuss the cases of two different theories assuming the existence of a Noether symmetry.

7.5.1 Non-minimal coupling to the Gauss–Bonnet term

The Gauss–Bonnet topological invariant is a combination of second-order curvature invariants defined as

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\,. \tag{7.51}$$

The corresponding action is a topological term, which can be written as a total derivative (*i.e.*, a boundary term), representing the four-dimensional case of the Chern–Gauss–Bonnet theorem [235]. It states that the Euler characteristic of an oriented closed even-dimensional Riemannian manifold is equal to the integral of a certain polynomial of its curvature [239, 344, 468]. Due to its topological nature, the Gauss–Bonnet invariant is often considered a tool to reduce the dynamics. However, to make its contribution not trivial in four dimensions, it is usually either coupled to a dynamical scalar field or included in the Einstein–Hilbert action[4] as a generic function $f(\mathcal{G})$. Thus, as the scalar

[4] One of the concerns about the presence of $\mathcal{G}$ into the gravitational action is the impossibility of imposing gravitational waves traveling at the speed of light in a covariant way. This represents another open issue in modified theories of gravity. However, this topic is beyond the scope of this article.

curvature is predominant at local scales, the Gauss–Bonnet correction might provide corrections at cosmological scales. Let us take into account the former case,

$$S_{\mathcal{G}} = \int d^4x\sqrt{-g}\, h(\phi)\, \mathcal{G}\,. \tag{7.52}$$

In this regard, it is well-known that Horndeski's theory can reproduce the non-minimal coupling to the Gauss–Bonnet term, since it represents the most general theory in four dimensions of ϕ, g_{ab}, and their derivatives, giving the second-order field equations [379]. The corresponding Horndeski contributions are as follows:

$$G_5(\phi, X) = -\,4\,\partial_\phi h(\phi)\ln X\,, \tag{7.53}$$
$$G_4(\phi, X) = 4\,\partial^2_\phi h(\phi)\, X\,(2 - \ln X)\,, \tag{7.54}$$
$$G_3(\phi, X) = 4\,\partial^3_\phi h(\phi)\, X\,(7 - 3\ln X)\,, \tag{7.55}$$
$$G_2(\phi, X) = 8\,\partial^4_\phi h(\phi)\, X^2\,(3 - \ln X)\,. \tag{7.56}$$

The easiest way to prove this is by directly comparing the equations yielding from the variation with respect to the metric tensor on the spatially flat FLRW background.

It is straightforward to see that the above set of functions G_i is compatible with any of the selected Noether symmetries; not only is the hierarchical derivative dependence the same as the first model, but it is also the same as the function redefinition to absorb $G_5 = G_5(\phi)$. Then, depending on the Noether symmetry, $h(\phi) = (\xi_1\phi + \xi_0)^{2+\phi_0/\xi_1}$ or $h(\phi) = (\xi_1\phi + \xi_0)^{2+\phi_0/\xi_1}$ or $h(\phi) = \exp\left(\frac{\phi_0}{\xi_0}\phi\right)$.

It is possible to analyze the non-minimal coupling to the Gauss–Bonnet terms in the Noether framework. Consistently with the Noether symmetries, the three simplest actions that one can take into account are of the following form

$$S = \int d^4x\sqrt{-g}\,[f(\phi)R + v(\phi)\,(\omega_0\, X - V_0) + h(\phi)\,\mathcal{G}]\,, \tag{7.57}$$

where,

$$\begin{cases} h = c_h\,(\xi_1\phi + \xi_0)^{2+\phi_0/\xi_1}\,, \quad f = c_f\,\partial^2_\phi h\,, \quad v = c_v\,\partial^4_\phi h\,, \quad \text{if} \quad \left\{\xi = \xi_1\, t,\ \eta_a = \frac{a}{3}(\xi_1 - \phi_0),\ \eta_\phi = \xi_1\,\phi + \xi_0\right\} \\ h = c_h \exp\left(\frac{\phi_0}{\xi_0}\phi\right), \quad f = c_f \exp\left(\frac{\phi_0}{\xi_0}\phi\right), \quad v = c_v \exp\left(\frac{\phi_0}{\xi_0}\phi\right), \quad \text{if} \quad \left\{\xi = 0,\ \eta_a = -\frac{a}{3}\,\phi_0,\ \eta_\phi = \xi_0\right\} \\ h = c_h\,\phi\,, \quad f = c_f\,, \quad v = c_v\,, \quad \text{if} \quad \{\xi = 0,\ \eta_a = 0,\ \eta_\phi = \xi_0\} \end{cases} \tag{7.58}$$

The corresponding Horndeski model is

$$G_5(\phi, X) = -\,4\,\partial_\phi h(\phi)\ln X\,, \tag{7.59}$$
$$G_4(\phi, X) = f(\phi) + 4\,\partial^2_\phi h(\phi)\, X\,(2 - \ln X)\,, \tag{7.60}$$
$$G_3(\phi, X) = 4\,\partial^3_\phi h(\phi)\, X\,(7 - 3\ln X)\,, \tag{7.61}$$
$$G_2(\phi, X) = v(\phi)\,(\omega_0\, X - V_0) + 8\,\partial^4_\phi h(\phi)\, X^2\,(3 - \ln X)\,. \tag{7.62}$$

Field equations together with the Hamiltonian constraint (the first Friedmann equation) read, respectively,

$$\frac{\partial \mathcal{L}}{\partial a} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{a}} = 0 \quad \longrightarrow \quad 3X\left(4\partial_\phi^2 f + \omega_0 v\right) + 12H\dot{\phi}\left(\partial_\phi f + 4\dot{H}\partial_\phi h + 4H^2\partial_\phi h\right)$$
$$+ 6\ddot{\phi}\left(\partial_\phi f + 4H^2\partial_\phi h\right) + 6H^2\left(3f + 8X\partial_\phi^2 h\right) + 12f\dot{H} - 3V_0 v = 0\,, \tag{7.63}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0 \quad \longrightarrow \quad \dot{\phi}\left[6\dot{H}\left(\partial_\phi f + 4H^2\partial_\phi h\right) + 12H^2(\partial_\phi f + 2H^2\partial_\phi h) - \partial_\phi v\,(\omega_0 X + V_0)\right]$$
$$- 6\omega_0 HXv - \omega_0 v\dot{\phi}\ddot{\phi} = 0\,, \tag{7.64}$$

$$\frac{\partial \mathcal{L}}{\partial X} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{X}} = 0 \quad \longrightarrow \quad X = \frac{1}{2}\dot{\phi}^2\,, \tag{7.65}$$

and

$$\dot{a}\frac{\partial \mathcal{L}}{\partial \dot{a}} + \dot{\phi}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} + \dot{X}\frac{\partial \mathcal{L}}{\partial \dot{X}} - \mathcal{L} = 0 \quad \longrightarrow \quad \dot{\phi}\left[v\,(\omega_0 X + V_0) - 6H^2 f\right] - 12HX(\partial_\phi f + 4H^2\partial_\phi h) = 0\,, \tag{7.66}$$

Taking into account the above energy constraint, the conserved scalar current turns out to be

$$\begin{aligned}
&\mathcal{J} = \zeta - \eta_a\frac{\partial \mathcal{L}}{\partial \dot{a}} - \eta_\phi\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - 2X\left(\partial_\phi\eta_\phi - \xi_1\right)\frac{\partial \mathcal{L}}{\partial \dot{X}} \\
\Rightarrow\quad &\dot{\phi}\left\{2(\xi_1 - \phi_0)\partial_\phi f + 8H^2(\xi_1 - \phi_0)\partial_\phi h - \omega_0\eta_\phi v\right. \\
&\left. + 4X\left[\eta_\phi\partial_\phi^4 h(\ln X - 3) + \partial_\phi^3 h\left(4\xi_1 + \partial_\phi\eta_\phi(3\ln X - 7) - (2\xi_1 + \phi_0)\ln X + 3\phi_0\right)\right]\right\} \\
&+ 2H\left[3\eta_\phi\partial_\phi f + 2(\xi_1 - \phi_0)f\right] + 8H^3\eta_\phi\partial_\phi h = \frac{\ell}{a^3}\,.
\end{aligned} \tag{7.67}$$

where, ℓ is a constant introduced for practical reasons, by redefining ζ. Then, it is possible to find exact solutions as shown in [558, 560, 562, 563].

7.5.2 Extended cuscuton model

The extended cuscuton model [420, 539] is a generalized formulation of the cuscuton field [422, 494, 495], which is not dynamic at the background and perturbation levels. The action of the model corresponds to the following choice of the Horndeski functions,

$$G_4(\phi) = f_4(\phi)\,, \tag{7.68}$$

$$G_3(\phi, X) = \left(\frac{1}{2}f_3(\phi) + \partial_\phi f_{4,\phi}\right)\ln X\,, \tag{7.69}$$

$$G_2(\phi, X) = f_1(\phi) + f_2(\phi)\sqrt{2X} - \left(2\partial_\phi f_3(\phi) + 4\partial_\phi^2 f_4(\phi) + \frac{3f_3(\phi)^2}{4f_4(\phi)}\right)X$$

$$+\,2\left(\frac{1}{2}\partial_\phi f_3(\phi)+\partial^2_\phi f_4(\phi)\right)X\ln X\,. \tag{7.70}$$

$$\tag{7.71}$$

As it is possible to notice for the above equations, the model has an explicit X dependence, while the ϕ dependence is parameterized by the presence of the function f_i. However, imposing the existence of Noether symmetries, our previous analysis provides a criterion to select them according to the following scheme:

$$\begin{cases} f_4 = c_4\,(\xi_1\phi+\xi_0)^{\phi_0/\xi_0}\,, \quad f_3 = c_3\,(\xi_1\phi+\xi_0)^{\frac{\phi_0}{\xi_0}-1}\,, \quad f_{1,2} = c_{1,2}\,(\xi_1\phi+\xi_0)^{\frac{\phi_0}{\xi_0}-2} \quad \text{if} \quad \xi\neq 0 \\ f_i = c_i\exp\left(\frac{\phi_0}{\xi_0}\phi\right)\,, \quad \text{if} \quad \xi = 0 \end{cases} \tag{7.72}$$

where c_i are generic constants. Then, one can write down field equations and the energy condition,

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial a}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot a}=0 \quad \longrightarrow \quad & f_1+\sqrt{2X}f_2-2X\partial_\phi f_3-\frac{3f_3^2}{4f_4}X-f_3\ddot\phi \\ & +4H\dot\phi\partial_\phi f_4+2(2\dot H+3H^2)f_4=0\,,\end{aligned} \tag{7.73}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\phi}=0 \quad \longrightarrow \quad & \dot\phi\left[\frac{\partial_\phi f_1}{3}-X\frac{f_3}{4f_4^2}\left(f_3\partial_\phi f_4-2f_4\partial_\phi f_3\right)-H^2\left(3f_3+2\partial_\phi f_4\right)-f_3\dot H\right] \\ & +H\left(3X\frac{f_3^2}{2f_4}-\sqrt{2X}f_2\right)+\frac{f_3^2}{4f_4}\dot\phi\ddot\phi=0\,,\end{aligned} \tag{7.74}$$

$$\frac{\partial\mathcal{L}}{\partial X}-\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot X}=0 \quad \longrightarrow \quad X=\frac{1}{2}\dot\phi^2\,, \tag{7.75}$$

and

$$\dot a\frac{\partial\mathcal{L}}{\partial\dot a}+\dot\phi\frac{\partial\mathcal{L}}{\partial\dot\phi}+\dot X\frac{\partial\mathcal{L}}{\partial\dot X}-\mathcal{L}=0 \quad \longrightarrow \quad \dot\phi\left(\frac{f_1}{3}+X\frac{f_3^2}{4f_4}+2H^2f_4\right)-2HXf_3=0\,, \tag{7.76}$$

while the *on-shell* conserved scalar current is

$$\begin{aligned}\mathcal{J} &= \zeta-\eta_a\frac{\partial\mathcal{L}}{\partial\dot a}-\eta_\phi\frac{\partial\mathcal{L}}{\partial\dot\phi}-2X\left(\partial_\phi\eta_\phi-\xi_1\right)\frac{\partial\mathcal{L}}{\partial\dot X} \\ \Rightarrow \quad & \dot\phi\Bigg[-\frac{4f_2\eta_\phi}{\sqrt{2X}}-2\eta_\phi(\ln X-2)\left(\partial_\phi f_3+2\partial_\phi\partial_\phi f_4\right)+\frac{3f_3^2\eta_\phi}{f_4}+2f_3\ln X\left(\phi_0-\partial_\phi\eta_\phi\right) \\ & +4\partial_\phi f_4\left(2\xi_1-\partial_\phi\eta_\phi\ln X+\phi_0\ln X-2\phi_0\right)\Bigg]+4H[4(\xi_1-\phi_0)f_4-3f_3\eta_\phi]=\frac{\ell}{a^3}\,,\end{aligned} \tag{7.77}$$

where, ℓ is a constant introduced for practical reasons, by redefining ζ.

The Noether symmetry allows us to find all general solutions for the extended cuscuton model. In particular, in the case of the external symmetries, it turns out that, for $\ell = 0$,

$$H = \left(\dot{\phi}\frac{c_3}{4c_4} - \frac{c_2}{3c_3 - 4c_4(\xi_1 - \phi_0)}\right)(\xi_0 + \xi_1\phi)^{-1}\,, \tag{7.78}$$

with $c_1 = -6c_2^2 c_4(3c_3 - 4c_4\xi_1 + 4c_4\phi_0)^{-2}$, while, for $\ell \neq 0$, it yields $c_1 = c_2 = 0$, $c_3 = \frac{4}{3}c_4(\xi_1 - \phi_0)$, and

$$H = \frac{(\xi_1 - \phi_0)}{3(\xi_0 + \xi_1\phi)}\dot{\phi} \quad \Rightarrow \quad a(t) = c_a\,(\xi_1\,\phi + \xi_0)^{\frac{\xi_1 - \phi_0}{3\xi_1}}\,, \tag{7.79}$$

where c_a is a constant depending on the other free parameters.

In the case of internal symmetries, the exact solutions are, for $\ell = 0$,

$$H = \frac{c_3}{4c_4}\dot{\phi} - \frac{c_2\xi_0}{3c_3\xi_0 + 4c_4\phi_0} \quad \Rightarrow \quad a(t) = c_a \exp\left(\frac{c_3}{4c_4}\,\phi - \frac{c_2\xi_0}{3c_3\xi_0 + 4c_4\phi_0}\,t\right), \tag{7.80}$$

with $c_1 = -6c_4c_2^2\xi_0^2(3c_3\xi_0 + 4c_4\phi_0)^{-2}$, while, for $\ell \neq 0$, it provides $c_1 = c_2 = 0$, $c_3 = -c_4\frac{4}{3}\frac{\phi_0}{\xi_0}$, and

$$H = -\frac{\phi_0}{3\xi_0}\dot{\phi} \quad \Rightarrow \quad a(t) = c_a \exp\left(-\frac{1}{3}\frac{\phi_0}{\xi_0}\phi\right) \tag{7.81}$$

where c_a is a constant depending on the other free parameters.

7.6 Noether symmetries with matter

So far, we have conducted our analysis neglecting the presence of the matter. Indeed, the modifications to GR field equations usually aim to describe a different behavior of gravity at early or late cosmic time, or in correspondence with very high/low-energy scales when the standard matter contribution can be left out of the treatment. In this framework, modified GR is often used to obtain a dynamical formulation of dark energy, instead of the cosmological constant [568]. However, generally, one has to deal with situations where the matter content cannot be neglected since it plays a crucial role. For instance, this is the case of several cosmological tests on modified theories of gravity, or, in general, observational cosmology [380, 383, 569]. Therefore, it is necessary to include the matter in this analysis. Let us consider a point-like Lagrangian accounting also the presence of the matter,

$$\mathcal{L} = \mathcal{L}^{(\mathrm{g})} + \mathcal{L}^{(\mathrm{m})}\,, \tag{7.82}$$

where, $\mathcal{L}^{(\mathrm{g})}$ represents a general scalar-tensor theory describing the gravitational part, and $\mathcal{L}^{(\mathrm{m})}$ is associated with the matter. In cosmology, the different species of standard matter are usually described by linear barotropic equations of state $P_i = w_i\rho_i$, where P_i and ρ_i are the isotropic pressure and the energy density, respectively, w_i is the barotropic coefficient, and i labels the different species such as radiation ($w = 1/3$), dust ($w = 0$), etc. However, sometimes more complex equations of state are also used to consider interactions or to model exotic dark matter (not simply as a dust fluid). Therefore, there is no univocal way to take into account the matter content and its description changes depending on the cosmic era being considered and the complexity of the model. The linear barotropic equation of state is the simplest way to describe the different species that fill the universe.

As shown in the previous sections, the Noether Symmetry Approach is a powerful tool to obtain functional forms of $\mathcal{L}^{(\mathrm{g})}$ under the *caveat* that our theory possesses a Noether symmetry. However, it is an exact mathematical computation that can be affected by the way $\mathcal{L}^{(m)}$ is modeled. An explicit example is the case of matter content characterized by a linear barotropic equation of state. It can be included in the point-like Lagrangian by considering $\mathcal{L}^{(\mathrm{m})} = \rho_0\, a^{-3w}$, with ρ_0 constant. Including this contribution, it is possible to verify that symmetries strictly survive in correspondence with $\xi = 0$ and $w = 0$: only internal symmetries admit the presence of (dust) matter. Therefore, the matter Lagrangian can strongly constrain the Noether Symmetry Approach. A possible solution to overcome this problem is to consider an additional constraint equation that removes the presence of the matter from the Noether identity, ensuring the existence of symmetries, and modifying the usual approach (as done in [561]). However, one could also accept that the matter content breaks the Noether symmetry, characterizing the theory only in vacuo.

In this section, we propose an alternative treatment that allows us to keep all the Noether classifications obtained so far, safely including the presence of matter. It is based on an additional scalar field ψ describing the matter content, minimally coupled to gravity: $\mathcal{L}^{(\mathrm{m})} = \mathcal{L}^{(\mathrm{m})}[g_{ab}, \psi, \dot{\psi}]$. For simplicity, let us consider the linear case

$$\mathcal{L}^{(\mathrm{m})} = a^3 \left(\frac{1}{2}\dot{\psi}^2 - U \right), \tag{7.83}$$

where $U = U(\psi)$ is a general potential of the scalar field (whose form is constrained by the existence of Noether symmetries).

Using an additional scalar field to effectively describe the matter content is a reasonable choice. Indeed, in this way, one has a general effective description, and a simplified parameterization can be introduced *a posteriori* in the cosmological analysis. Then, it turns out $\sum_i \rho_i = \frac{1}{2}\dot{\psi}^2 + U$ and $\sum_i P_i = \frac{1}{2}\dot{\psi}^2 - U$, or equivalently, $\dot{\psi}^2 = \sum_i(\rho_i + P_i)$ and $U = \frac{1}{2}\sum_i(\rho_i - P_i)$. In doing this, it is possible to verify that the Noether classification for scalar-tensor theories is unchanged: the selected functional forms of the action are the same as in our previous analysis; the only differences are the symmetries, *i.e.* the components of the infinitesimal generators of the symmetries (depending on ψ also, in general). It represents a formal proof of the validity of the Noether Symmetry Approach in the presence of matter. Perhaps the simplest way to demonstrate this is by assuming the functional forms selected by the Noether Symmetry Approach and verifying the existence of the symmetries given by the (non-vanishing) infinitesimal generator,

$$\chi = \xi(t,a,\phi,\psi)\partial_t + \eta_a(t,a,\phi,\psi)\partial_a + \eta_\phi(t,a,\phi,\psi)\partial_\phi + \eta_\psi(t,a,\phi,\psi)\partial_\psi\,. \tag{7.84}$$

Then, the above components of the infinitesimal generator given by the Noether Symmetry Approach for Horndeski gravity turn out to be

$$\xi(t) = \xi_1 t + \xi_2\,, \quad \eta_a(a,\phi) = \frac{a}{3}\left(\xi_1 - \phi_0\right)\,, \quad \eta_\psi(\psi) = \frac{1}{2}(\phi_0\,\psi + \xi_3)\,, \tag{7.85}$$

while the potential U must satisfy the following differential equation,

$$(\phi_0 - 2\xi_1)U(\psi) - \frac{1}{2}(\psi\,\phi_0 + \xi_3)U'(\psi) = 0\,, \tag{7.86}$$

selecting $U = U_0\,(\phi_0\,\psi + \xi_3)^{2-\frac{4\xi_1}{\phi_0}}$ for $\phi_0 \neq 0$, or $U = U_0 \exp\left(-\frac{4\xi_1}{\xi_3}\psi\right)$ for $\phi_0 = 0$, together with the same classification of Sec. 7.4. However, Eq. (7.83) represents the simplest Lagrangian to consider. Alternatively, it is possible to leave unspecified the Lagrangian $\mathcal{L}^{(\mathrm{m})}$ of the matter sector, or consider multiple scalar fields [8]. For instance, a generalization of the previous case is given by the following multi-field Lagrangian,

$$\mathcal{L}^{(\mathrm{m})} = a^3 \left(\frac{1}{2} \sum_{I=1}^{N} \dot{\psi}_I^2 - U(\psi_J) \right) , \tag{7.87}$$

where the potential U depends in general on the multiple $\psi_J = \{\psi_1, \ldots, \psi_N\}$. Consequently, one obtains an analogous result of Eq. (7.85) for each matter field,

$$\xi(t) = \xi_1 t + \xi_2 \,, \quad \eta_a(a,\phi) = \frac{a}{3}\left(\xi_1 - \phi_0\right) \,, \quad \eta_{\psi_I}(\psi_I) = \frac{1}{2}(\phi_0 \psi_I + \xi_{3,I}) \,, \tag{7.88}$$

while Eq. (7.86) turns into

$$(\phi_0 - 2\xi_1)U - \frac{1}{2}\sum_{I=1}^{N}(\psi_I \phi_0 + \xi_{3,I})\partial_I U = 0 \,, \tag{7.89}$$

corresponding to $U = U_0 \left(\phi_0 \psi_J + \xi_{3,J}\right)^{2-\frac{4\xi_1}{\phi_0}} F(\eta_{\psi_I}/\eta_{\psi_J})$ for $\phi_0 \neq 0$, or $U = U_0 \exp\left(-\frac{4\xi_1}{\xi_{3,J}}\psi_J\right) F(\psi_I - \psi_J \xi_{3,I}/\xi_{3,J})$ for $\phi_0 = 0$, where F is a generic $(N-1)$–dimensional function, with $I \neq J$.

Discussing all possible matter Lagrangians goes beyond the scope of this work. The main result of this section is that introducing a matter field (or multiple fields) leaves the Noether symmetries of Horndeski gravity unchanged compared to the vacuum case.

7.7 Closing remarks

Noether symmetries represent a powerful tool for simplifying and solving dynamical systems. Applying the Noether Symmetry Approach in cosmology constitutes a method to select the functional form of effective Lagrangians. In addition, the presence of a conserved scalar charge associated with the symmetry reduces the dynamics, helping to find exact cosmological solutions.

We focused our analysis on Horndeski gravity and its subclasses, providing a general classification. Reversing the usual Noether theorem and assuming the invariance under Noether point symmetries, we selected the functional forms of G_i, the Horndeski functions. This result is achieved by using a Lagrange multiplier to treat the kinetic term as a new variable for the system. The Lagrange multiplier allows us to keep the braiding function G_3 general and turns the point-like Lagrangian into a Lagrangian of the first order in time derivatives. This is because the point-like Lagrangian depends on the second derivative of the scalar field which cannot be removed by integrating by parts due to the presence of a (general) braiding term, G_3. However, once the braiding function has an explicit X dependence, we can always transform $\mathcal{L}_3$ into a point-like Lagrangian of the first order. For this reason, the equations of motion are associated with second-order Euler-Lagrange equations. Then, an equivalent alternative approach is to consider the *second prolongation* of the infinitesimal generator of the Noether symmetry.

From the Eq. (7.50) it is possible to see that the Noether symmetry almost fully determines the dependence of the functions on ϕ, while the X dependence is factorized in unconstrained $g_i(X)$ functions. This means that models admitting Noether symmetries are determined by the X

dependence of the G_i functions. Once the model has been selected, *i.e.*, the X dependence is fixed, requiring the existence of a Noether symmetry sets the ϕ dependence. To highlight this aspect, we discussed the case of non-minimal coupling to the Gauss–Bonnet term and the extended cuscuton model.

The general ϕ-dependence of the selected Horndeski functions is mainly characterized by two free parameters, ξ_1 and ϕ_0. The former is associated with transformations of the coordinate time (*i.e.*, $\xi_1 \neq 0$ external symmetry, $\xi_1 = 0$ internal symmetry). The latter is connected with transformations of the scale factor. However, the analysis of Eqs. (7.46) and (7.50) provides an additional interpretation of the parameter ϕ_0. Restricting to viable subclasses of Horndeski gravity, $G_4 = G_4(\phi)$ and $G_5(\phi, X) = 0$, in the case of external symmetries, a non-vanishing ϕ_0 is related with the presence of a non-minimal coupling G_4, while, in the case of internal symmetry, it represents the breaking parameter of the shift-symmetry.

Finally, we extended our analysis by taking into account a point-like Lagrangian describing the matter sector. Since the Noether Symmetry Approach is an exact mathematical computation, the classification obtained in the vacuum can strongly be affected. Indeed, parameterizing the matter Lagrangian as $\mathcal{L}^{(\mathrm{m})} = \rho_0\, a^{-3w}$ describing a single linear barotropic perfect fluid, the only symmetries surviving for Horndeski gravity are in correspondence with $\xi_1 = 0$ and $w = 0$. Moreover, since there is no unique and general parameterization of the matter Lagrangian, but it depends on the particular cosmic era, the Noether symmetries analysis cannot be done properly. For this reason, we proposed an alternative treatment of the matter sector. Introducing an additional homogeneous scalar field ψ describing the matter, $\mathcal{L}^{(\mathrm{m})}[g_{ab}, \psi, \dot{\psi}]$, the general classification of the scalar-tensor theories is preserved, *i.e.* we obtain the same classification as obtained in the vacuum case (7.50). The same result is achieved by considering a canonical multi-field Lagrangian.

In forthcoming work, we will use the classification of viable Horndeski to implement cosmological screenings on these scalar-tensor classes characterized by the Noether symmetries and constrain the remaining free parameters.

8 Conclusion

In this thesis, we investigate novel approaches to interpret the unknown Horndeski functions and to obtain new criteria for selecting the functional form of the theory. Indeed, focusing on viable Horndeski gravity, a scalar-tensor subclass that possesses tensor modes propagating at the speed of light, the theory is not completely determined but presents three undetermined functions, the non-minimal coupling function $G_4(\phi)$, the braiding function $G_3(\phi, X)$, and the kinetic function $G_2(\phi, X)$.

The findings, detailed in each Chapter, contribute to the understanding of viable Horndeski models, with implications for both cosmology and astrophysics.

Below, we summarize the major results in the following areas: the effective fluid formalism, scalar-tensor thermodynamics, applications to cosmological bounces and black holes, and the Noether Symmetry Approach.

A central achievement of this thesis is the analysis of the effective fluid approach within Horndeski gravity in Ch. 2. The entire framework is based on the assumption of a timelike 4-velocity for the fluid, defined via the scalar field gradient $\nabla_\mu\phi$. By interpreting the scalar-tensor contributions to the gravitational field equations as an effective stress-energy tensor, we have shown that Horndeski gravity can be classified in terms of "Newtonian" and "non-Newtonian" effective fluids, defined as linear and non-linear with respect to the 4-velocity gradients $\nabla_\mu u_\nu$, respectively. As shown in Tab. 2.1, requiring that the effective fluid is linear in $\nabla_\mu u_\nu$ selects only two possible classes of (viable) Horndeski gravity, either $G_3(\phi, X) = 0$ or $G_3(\phi, X) = G_{4\phi}(\phi) \ln X$. This classification provides a novel perspective on the behavior of modified gravity theories, allowing us to treat the scalar field contributions as constituents of a dissipative fluid. The effective fluid formalism offers several advantages. First, it enables a more intuitive understanding of scalar-tensor gravity by drawing direct analogies with fluid dynamics, which is a well-studied field. Second, this formalism opens pathways for comparing Horndeski gravity with observational data on astrophysical phenomena that can be modeled as fluids, such as dark matter halos and large-scale cosmic flows. This approach provides a framework to examine which models are physically viable and offers criteria for excluding or prioritizing certain functional forms of scalar-tensor gravity. In general, the analogy between fluids and gravity serves as a practical approach to tackling theoretical and observational challenges. The specific interest in classes that correspond to Newtonian fluids is motivated by the potential to represent the limit of a more comprehensive theory under small 4-velocity gradients, similar to how classical theories emerge as low-energy limits of effective field theories. Particularly within the context of late-time cosmology, where the scalar field can be considered a low-energy phenomenon, this perspective provides an additional tool for distinguishing different classes of Horndeski gravity.

A significant result from this thesis is the demonstration of a thermodynamic analogy within the context of first-order imperfect fluid thermodynamics, presented in Chs. 3 and 4. By comparing Horndeski gravity's equations with first-order thermodynamic relations, we have shown that the scalar field can be interpreted as a component of a system that behaves similarly to a dissipative fluid in a state of non-equilibrium. In particular, this formalism provides a concrete realization of Jacobson's idea, enhancing the possibility of finding a general formulation of a *thermodynamics of gravitational theories*. In this view, GR is seen as a classical equilibrium state of gravity, while modified gravity is associated with a non-equilibrium state. Specifically, Einstein's theory possesses a zero effective temperature, whereas scalar-tensor gravity generally corresponds to a non-zero temperature. Beyond the formal analogy, this framework offers a broader context for embedding GR within the vast landscape of alternative gravity theories. This type of embedding not only enhances our understanding of gravitational theories but may also offer hints for addressing GR's limitations.

Moving to a different framework, in Ch. 5, we explored the feasibility of an effective Friedmann equation in removing the classical Big Bang initial singularity and the covariant single-field formulation within viable Horndeski gravity. Thus, the initial singularity is replaced by a non-singular bounce occurring at a specific critical energy density. This density parameterizes a function that implements the effective theory bounce in a spatially flat, homogeneous, and isotropic universe. This function measures the deviation from the benchmark theory, which is recovered as the critical energy density approaches infinity.

The analysis of a covariant realization in viable Horndeski gravity shows that both the effective and benchmark theories must belong to the same scalar-tensor theory. In particular, the selected theories are characterized by no additional propagating degrees of freedom, with the cuscuton $G_4(\phi) = 1$, $G_3(\phi, X) = 0$, and $G_2 = \mu(\phi)X - V(\phi)$, as well as a subclass of extended cuscuton models with $G_3(\phi, X) = G_{4\phi}(\phi) \ln X$.

Next, in Ch. 6, the non-dynamical extended cuscuton model has been shown to be a solution for McVittie and generalized McVittie spacetimes, describing central inhomogeneity embedded in a general FLRW universe. Unlike previous studies, this work also includes matter. This allowed us to classify possible configurations based on the time dependence of the gravitational coupling, the radial energy flow, the accretion rate onto the central object, and the Hubble rate.

Finally, in Ch. 7, we analyzed the Noether symmetries of Horndeski gravity, extending the classification of general scalar-tensor theories. After reviewing the minimally coupled scalar field and the first-generation scalar-tensor gravity, we generalized the discussion to kinetic gravity braiding and Horndeski gravity. We highlight the main findings by focusing on the non-minimally coupled Gauss-Bonnet term and the extended cuscuton model. Lastly, we discuss how the presence of matter can influence Noether symmetries, demonstrating that the selected Horndeski functions remain unchanged compared to the vacuum case.

As result, applying the Noether Symmetry Approach selects the functional form of Horndeski Lagrangians, which turn out to be $G_i(\phi, X) = h_i(\phi)\, g_i(X)$, where $h_i(\phi) = \partial_\phi^{(6-i)} (\xi_1\phi + \xi_0)^{2+\phi_0/\xi_1}$ in the case of $\xi_1 \neq 0$ external symmetries, and $h_i(\phi) = \exp(\alpha\,\phi)$ with $\alpha = \phi_0/\xi_0$, for the internal one. In addition, the conserved scalar charge associated with the symmetry reduces the dynamics, helping to find exact cosmological solutions. Therefore, the Noether Symmetry Approach provides a factorization of ϕ and X dependence.

To summarize, this thesis advances our understanding of the functional forms within Horndeski gravity and provides novel criteria of investigation.

Future Developments

Nevertheless, the outlined frameworks present limitations in their fields of applicability.

From the effective fluid perspective, the introduction of non-viable Horndeski contributions exponentially increases the number of terms, making their interpretation and the applicability of the formalism more challenging. As shown in Chs. 2 and 3, the formalism breaks down beyond viable Horndeski. However, there is the possibility that, in the context of DHOST theories, the presence of higher-order additional contributions within a degenerate theory allows us to cancel out unwanted terms and potentially extend the formalism to new classes of effective fluids.

Moreover, this approach relies on scalar fields with timelike gradients to establish a well-defined fluid interpretation (for an effective stress-energy tensor). This is a major limitation for the formalism, as it excludes typical static and stationary black hole scenarios that can imply a spacelike nature of the scalar field gradient. Furthermore, for the sake of completeness, it would be interesting to investigate what happens if one tries to extend this formalism to null gradients, especially since inhomogeneous solutions of the Einstein equations sourced by null fluids play an important role in both GR and its extensions.

As a side effect, this further study would allow us to extend the thermodynamic analogy as well. Since scalar-tensor thermodynamics is based on the fluid interpretation of the scalar field stress-energy tensor, any generalization of the effective fluid formalism would open up new possibilities for implementing and understanding thermodynamic analogies. However, scalar-tensor thermodynamics is currently limited to the first-order imperfect fluid formalism. Moving beyond Horndeski and considering other modified theories could require taking into account second-order or higher-order formalisms. Therefore, investigating the general-relativistic description of higher-order thermodynamics would increase the possibility of finding a generalized thermodynamic framework, allowing for a broader description of modified theories of gravity.

Further explorations would involve the study of additional solutions of cosmological black holes, for instance, characterized by non-vanishing spin or stealth scalar fields that depend on both time and radial coordinates. In this context as well, the generalization to spatial and null scalar field gradients is fundamental. In this way, one would not only generalize the presented framework but also build a new tool for finding new black hole solutions, studying the presence of black hole hair in the context of thermodynamics, and confirming the interpretation of GR as a special equilibrium state among modified theories.

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